



# The Structure of Reality

Recognition, Necessity,  
and the Physical Foundation

Jonathan Washburn

## **The Structure of Reality**

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Recognition Science

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This monograph develops the foundation from distinction and records through comparison, hierarchy, spatial linking, completion, physical readings and primitive-event scale. The hypotheses and physical identifications of each result are stated with its argument.

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# The question this book answers

Nature already distinguishes one occurrence from another. A comparison uses information about both. A record can carry that information forward, and examining the record is another occurrence. Recognition Science asks what structure these activities require, and which physical laws follow when the required comparisons and records are preserved.

This book constructs records, updates, comparison signals, reference readings and retained paths. It states what their operations preserve and proves which alternatives those requirements exclude, with the physical conditions beside the claims they support.

Consider two positive populations. Their difference is not yet a dimensionless comparison: doubling both populations doubles the difference. A physical comparator must specify both its residual and its reference. For the apparatus developed here, their measured ratio is forced to have the normalized form

$$J(x) = \frac{1}{2} \left( x + \frac{1}{x} \right) - 1 = \frac{(x - 1)^2}{2x}, \quad x > 0.$$

An arbitrary continuous report of the specified apparatus must equal this expression in the selected normalization. The construction supplies the numerator, denominator and normalization.

The hierarchy argument has the same discipline. A successor that preserves all positive additive comparisons must scale their reading uniformly. A first costly binary join then fixes that scale to the golden ratio. The spatial argument begins with an embedded closed trace and an additive, deformation-invariant observation of paths in its complement. A nonzero closed reading forces dimension three in the stated ambient class. Each result removes freedom through an explicit requirement.

The first chapter establishes the operational floor. Later chapters introduce additional physical objects, supply constructions and prove their consequences. Each connection is addressed where it enters: balanced accounting to conserved energy, a state count to elapsed duration, and a mathematical weight to observed events.

## **Reading the argument**

The six parts follow the proof dependencies: the distinction floor; arithmetic and event accounting; the comparison law; the recognizing hierarchy; space, completion and physical readings; and the native scale result with the cumulative foundation statement.

The main spine requires proofs, elementary linear algebra and real analysis. The spatial chapters use embeddings, homotopy and singular homology. The path-to-homology argument is given in the text; the complement proof and polygon construction are in the appendices. Hatcher supplies the surrounding topology [2]. The first two chapters state the ordinary mathematical language being used.

A forcing theorem shows that its requirements leave only the displayed result. A construction shows that requirements can hold together. An obstruction identifies a requirement a construction cannot meet. A physical identification states which actual operation or reading supplies a mathematical premise. Each has its place in the argument.

## **The endpoint**

The book reaches the foundation traditionally labeled T-2 or T-1 through T10: the operational floor, record arithmetic, comparison, hierarchy, spatial linking, completion, measure and primitive-event scale. The chapter numbers follow the exposition rather than identifying one theorem with each traditional label.

The final chapter distinguishes the connected results already established from the exact remaining event and scale identifications. In particular, the native action formula is proved under its physical reading laws; identifying that assignment with the energy and duration of a complete dynamical preparation remains a separate obligation. External SI reporting is then a conversion with its stated anchor. The book ends at that foundation and its physical boundary.

# Notation and conventions

Each chapter introduces its objects locally. Natural numbers include zero; positivity is stated separately. Inner products are conjugate-linear in the first argument and linear in the second. The symbols  $\phi$  and  $\varphi$  both denote  $(1 + \sqrt{5})/2$ .

| Symbol             | Use in this book                                                                                      |
|--------------------|-------------------------------------------------------------------------------------------------------|
| $A, B, z$          | Real operand amplitudes and a complex unit phase in the comparator.                                   |
| $N, E$             | Signal size and energy reading on the specified carrier.                                              |
| $D(A, B, z)$       | Measured residual energy. The separate $D$ in the spatial proof is dimension.                         |
| $R_p, \kappa$      | The measured reference and its positive cost normalization.                                           |
| $J(x)$             | Canonical dimensionless reciprocal comparison cost, for $x > 0$ .                                     |
| $T, A, L$          | State successor, additive update and additive value in the hierarchy.                                 |
| $s_n$              | Positive reading of the $n$ th state of that hierarchy.                                               |
| $\rho$             | A local positive multiplier; in the weight argument, the one-step reciprocal weight.                  |
| $P(\gamma)$        | Additive observation of a path in the specified complement.                                           |
| $Q_d$              | Binary patterns on $d$ coordinates in the clock argument.                                             |
| $W(n), w_H(n)$     | Hierarchy block weight or reciprocal hierarchy reading, with domain specified locally.                |
| $E(e), T(e)$       | Energy and duration assigned to the same event. The scale chapter states their physical reading laws. |
| $t_*, \ell_*, j_*$ | External duration, length and energy conversion factors in the single-anchor calibration.             |

Equality of two numerical formulas does not itself identify their physical carriers. The maps that connect records, signals, readings and events are part of the corresponding statements.



PART I

# The starting point



# What an actual comparison requires

A question about reality is an occurrence within reality. To compare two records, the comparison must have information about both. To examine what has already happened, another examination must occur. These observations give the foundation its starting objects: distinctions, available records and ordered occurrences.

The first task is to state their necessity precisely. A sentence denying distinction can itself be distinguished from the sentence it denies. That fact concerns the language in which the denial is expressed. An actual comparison adds an operational fact: something happened, and its result depended on the records made available to it. We will specify both levels, construct an example, and prove the connection between them.

The construction is deliberately small. It needs finite records, three operations and a rule saying which finite histories are permitted. No distance, energy, probability or physical duration is assigned at this stage. Those quantities will enter when their defining operations and physical requirements are available.

## 1.1 A distinction within a domain

Let  $K$  be a domain of objects. A *nontrivial specification* on  $K$  consists of a predicate  $P$ , an object  $x$  for which  $P(x)$  holds, and an object  $y$  for which  $P(y)$  fails. The specification therefore has both an included object and an excluded object.

**Theorem 1.1** (Specification and distinction). *On an inhabited domain  $K$ , the following conditions are equivalent: there are objects  $x, y \in K$  with  $x \neq y$ ; and there is a nontrivial specification on  $K$ .*

*Proof.* Suppose first that  $P(x)$  and  $\neg P(y)$ . If  $x = y$ , substitution would give  $P(y)$ , a contradiction. Hence  $x \neq y$ .

Conversely, suppose  $x \neq y$ . Define  $P(z)$  by  $z = x$ . Then  $P(x)$  holds by reflexivity. The statement  $P(y)$  would give  $y = x$ , whose symmetry contradicts  $x \neq y$ . Thus  $P(y)$  fails. This constructs the required specification.  $\square$

The result identifies the content of a nontrivial specification. An account that identifies something as included and something as excluded has already distinguished objects. If no such specification exists, two unequal objects would construct one by the second half of the proof; therefore every two objects are equal.

There is an operational version of the same argument. Suppose a reading  $b : K \rightarrow \{0, 1\}$  takes the value 1 at  $x$  and 0 at  $y$ . Equality  $x = y$  would force  $b(x) = b(y)$ , contradicting the different readings. The readings supply a distinction in the objects on which they are defined. Their physical availability will matter below.

## 1.2 The logical floor

The labels T-2 and T-1 locate the beginning of the forcing argument. They do not label successive eras of the universe. T-1 names the distinction already present in a nontrivial domain of discourse. T-2 gives a concrete witness: the empty object differs from an object with one inhabitant.

Write **0** for a type with no inhabitants and **1** for a type with the single inhabitant  $\star$ . Here a type specifies the objects that can be its inhabitants. The surrounding mathematical language supplies formation of these types, equality and transport along equality.

**Proposition 1.2** (An explicit witness).  $\mathbf{0} \neq \mathbf{1}$ .

*Proof.* If the types were equal, transport along that equality would turn  $\star \in \mathbf{1}$  into an inhabitant of **0**. The empty type has no inhabitant. Thus the assumed equality is impossible.  $\square$

The one-inhabitant type is convenient, but the argument can use still less construction. The function type  $\mathbf{0} \rightarrow \mathbf{0}$  has an inhabitant: the identity function. If  $\mathbf{0} = (\mathbf{0} \rightarrow \mathbf{0})$ , transport of that function would again inhabit **0**. Hence

$$\mathbf{0} \neq (\mathbf{0} \rightarrow \mathbf{0}). \quad (1.1)$$

This witness uses an empty type, function types and equality with transport, all within one universe of types. It does not require an infinite tower of universes or a separate one-inhabitant type.

The proposition that is always true also differs from the proposition that is always false. Identifying them would transport a proof of the former into a proof of the latter. More generally, in the classical logic used here a proposition  $P$  differs from its negation  $\neg P$ . To prove this, suppose  $P = \neg P$  and separate the two cases. If  $P$  holds, the equality supplies  $\neg P$ , a contradiction. If  $\neg P$  holds, the reverse equality supplies  $P$ , again a contradiction.

Apply this argument to the proposition

$$D_K : \quad \text{there are } x, y \in K \text{ with } x \neq y.$$

It proves  $D_K \neq \neg D_K$ . It does not prove  $D_K$  for every  $K$ : the language can distinguish two claims about a domain whose objects are all equal. A distinction between propositions and a distinction between the objects they describe must be kept separate.

Inequality itself has a logical requirement. The notation  $a \neq b$  means that  $a = b$  implies falsity. The preceding proofs consequently use a language in which equality, implication and falsity are available. Their starting point is explicit. They establish a distinction within that language; they do not describe an act that first creates the language from an absence of all logical structure.

For an inhabited  $K$ , the logical and object conditions can be collected as follows: the surrounding language distinguishes propositions, and  $K$  admits a nontrivial specification. The first condition has just been proved. By the specification theorem, the second is equivalent to the existence of two unequal objects of  $K$ . The two-element domain  $\{0, 1\}$  supplies a concrete realization of this combined floor.

### 1.3 Records and the operations on them

We now give the occurrence of a question a precise representation. A *record* is a finite word of bits. The empty word is allowed. Word equality asks whether the lengths and corresponding bits agree. It is determined by a finite comparison.

An account has three components:

$$L = (R, O, n).$$

Here  $R$  is a finite ordered list of records,  $O$  is a finite ordered list of comparison outcomes, and  $n$  is its stage. An outcome 1 says the compared records agree; an outcome 0 says they differ. The empty account is  $([], [], 0)$ .

There are three operations. Posting a record  $v$  appends it to  $R$ . Retaining advances the stage while keeping the records and outcomes. Comparing positions  $i, j$  reads the records at those positions, if both exist, and appends their equality result to  $O$ . Every operation advances the stage by one.

Write  $R \parallel [v]$  for appending  $v$  to a list. The first two operations are therefore

$$\text{post}_v(R, O, n) = (R \parallel [v], O, n + 1), \quad (1.2)$$

$$\text{retain}(R, O, n) = (R, O, n + 1). \quad (1.3)$$

Positions start at zero. If  $R[i] = a$  and  $R[j] = b$  are both available, comparison gives

$$\text{compare}_{i,j}(R, O, n) = (R, O \parallel [1_{a=b}], n + 1). \quad (1.4)$$

If either record is absent, it gives  $(R, O, n + 1)$ : the attempted action has occurred, but no comparison outcome has been registered. In particular, an absent outcome and the outcome 0 are different.

A *history* is a finite list of operations. Its execution starts at the empty account and applies the operations in order. Denote the result by  $\text{run}(h)$ . The records used in a comparison are those in this executed account, rather than records merely named in a sentence about it.

This finite notation uses counting to describe the construction. The next chapter constructs the arithmetic of finite repetitions and proves agreement between its realizations. Using list positions here does not assert that an actual apparatus admits indefinitely many operations.

## 1.4 Representing a question

The sentences needed for the argument have a small grammar. There is a false sentence  $\perp$ , three sentences named Dist, Avail and Later, and a sentence  $\text{Complete}(r)$  for each finite history  $r$ . Negation and conjunction form further sentences. Their operational interpretation will be given after the representation has been specified.

A coding  $c$  assigns a finite record to each sentence. We require

$$c(P) \neq [], \quad c(P) \neq c(\neg P) \quad \text{for every sentence } P. \quad (1.5)$$

These requirements concern the records, not the truth of the sentence. They do not require that the code of every two distinct sentences differs. Separation of a sentence from its negation is sufficient.

Here is a concrete coding. Let  $u(k)$  be  $k$  copies of the bit 1 followed by 0. Encode operations by

$$\begin{aligned} a(\text{post}_v) &= 00 u(|v|) v, \\ a(\text{retain}) &= 01, \\ a(\text{compare}_{i,j}) &= 10 u(i) u(j). \end{aligned}$$

Juxtaposition denotes concatenation of words. Encode a history by  $H(h) = u(|h|)$  followed by the encodings of its operations in order. Define the sentence codes recursively:

$$\begin{aligned} c(\perp) &= 000, & c(\text{Dist}) &= 001, \\ c(\text{Avail}) &= 010, & c(\text{Later}) &= 011, \\ c(\text{Complete}(r)) &= 100 H(r), \\ c(\neg P) &= 101 c(P), \\ c(P \wedge Q) &= 110 c(P) c(Q). \end{aligned}$$

Each code begins with a three-bit tag, so it is nonempty. The code of  $\neg P$  has length  $3 + |c(P)|$ , strictly larger than  $|c(P)|$ . Equal words have equal lengths, so  $c(P) \neq c(\neg P)$ . Thus the coding requirements have an explicit realization. No truth evaluation was used to construct it.

## 1.5 Permitted histories and actual events

The operation grammar says what a history can look like. A *world* in this operational description says which histories are permitted. Its predicate  $W(h)$  must permit the empty history and must be closed under taking prefixes: if  $g$  is permitted and  $h$  is an initial part of  $g$ , then  $h$  is permitted.

These two conditions allow a world with only finitely many operations. They also allow the world that permits every finite history. Extension of a particular history is a separate matter.

An event  $e = (h, a)$  consists of a past history  $h$  and its next operation  $a$ . Its complete history and its stage are

$$h_e = h \parallel [a], \quad s(e) = |h| + 1. \quad (1.6)$$

It is *allowed* when  $W(h_e)$  holds. The event includes its position in a history; performing the same operation after a different past gives a differently represented event.

For a comparison event, define  $\text{res}(e)$  to be the equality result of the two available records in  $\text{run}(h)$ . If the operation is not a comparison, or if either record is absent, set  $\text{res}(e) = \emptyset$ , meaning no result. Thus

$$\text{res}(e) \in \{\emptyset, 0, 1\}.$$

This result is exactly what the account operation appends: nothing when it is absent, and the single outcome when it is present. The claim follows by considering posting, retaining and comparing. In the comparison case, the two record lookups give the same branches in both definitions.

Call an event *successful* if its result is 0 or 1, and a *distinction event* if its result is 0. Success here means that a comparison returned a result. It does not mean that the sentence under consideration is true.

**Proposition 1.3** (Available operands). *Every successful event compares two records that are both present in the executed account of its past.*

*Proof.* Posting and retaining return no comparison result, so a successful event must be a comparison. If its first lookup is absent, the comparison definition returns no result. If the first is present but the second is absent, it again returns no result. Both cases contradict success. The remaining case supplies the two positions and the two records required by the statement.  $\square$

To represent consideration of  $P$ , require an allowed event that compares two positions containing  $c(P)$  and  $c(\neg P)$  in its executed past. Denote this condition by  $\text{Query}(e, P)$ . It is a condition on an operation and its inputs. It makes no assumption about the truth of  $P$  or the operation's output.

**Proposition 1.4** (A represented query distinguishes). *If  $\text{Query}(e, P)$  holds, then  $e$  is a distinction event.*

*Proof.* The query condition gives the comparison operation and both successful lookups. The equality test therefore compares  $c(P)$  with  $c(\neg P)$ . They differ by (1.5), so the result is 0.  $\square$

The resulting distinction is between the two represented terms. The theorem does not decide which of those terms expresses a true claim.

## 1.6 An earlier record and a later examination

For a finite history  $r$ , let  $E(r)$  be its list of events. Its event at position  $i$ , for  $0 \leq i < |r|$ , consists of the first  $i$  operations as its past and the operation at position  $i$  as its action. Its stage is  $i + 1$ . Consequently

$$d \in E(r) \implies s(d) \leq |r|. \quad (1.7)$$

An event  $e = (h, a)$  is itself in  $E(h_e)$ : choose position  $|h|$  in the appended history. Its prefix is exactly  $h$  and its last operation is exactly  $a$ .

The argument that excludes a later examination from an earlier record uses only order.

**Lemma 1.5** (Chronological exclusion). *Suppose every event in a finite record has a stage at most  $t$ , while an event  $e$  has stage strictly greater than  $t$ . Then  $e$  is absent from that record.*

*Proof.* Membership would give  $s(e) \leq t$  by the first hypothesis, contradicting  $t < s(e)$ .  $\square$

Now suppose  $r$  is a prefix of the past  $h$  of an event  $e$ , and that  $e$  represents a query about  $\text{Complete}(r)$ . Prefix length gives  $|r| \leq |h|$ . Equation (1.6) then gives

$$|r| \leq |h| < |h| + 1 = s(e).$$

Every member of  $E(r)$  has stage at most  $|r|$  by (1.7). Chronological exclusion therefore gives  $e \notin E(r)$ . The query proposition also gives that  $e$  is a distinction event, and the appended-history construction gives  $e \in E(h_e)$ .

**Theorem 1.6** (The examination extends the earlier record). *If  $r$  is a prefix of an event's past and the event is a represented query about  $\text{Complete}(r)$ , then its completed history contains a distinction event absent from  $E(r)$ .*

*Proof.* The examining event itself is the witness, by the three conclusions just established.  $\square$

The prefix condition does real work. Talking about a record, without a temporal relation between that record and the act, would not give the strict stage inequality. Nor does the theorem say that an earlier record cannot contain a prediction or a description of the later event. Its conclusion concerns occurrence membership beyond a stated cutoff. The later examination is an additional occurrence.

## 1.7 The operational foundation theorem

We can now interpret the three foundation sentences. The sentence *Dist* says that an allowed distinction event exists. *Avail* says that an allowed successful event exists and that every allowed successful event has both operands present. *Later* says that every represented query about the completeness of an earlier prefix supplies a distinction event in the completed history that is absent from the earlier prefix.

The sentence *Complete*( $r$ ) says that every allowed event is already in  $E(r)$ . The sentence  $\perp$  is false, and negation and conjunction receive their ordinary logical interpretations. These interpretations are separate from the coding used by the operations.

**Theorem 1.7** (Foundation of a represented query). *Let  $c$  satisfy (1.5), and let  $W$  be a prefix-closed world permitting the empty history. If an allowed represented query exists, then *Dist*, *Avail* and *Later* all hold.*

*Proof.* Choose a query event  $e$  and its sentence  $P$ . The query proposition shows that  $e$  has result 0, and it is allowed by the query condition. It witnesses *Dist*.

The same event is successful, supplying the existence part of *Avail*. For any other allowed successful event, the available-operands proposition supplies the two present records. This proves the remaining universal part.

Finally, consider any represented query about a prefix  $r$  of its past. The examination theorem supplies the missing distinction event in its completed history. This proves *Later* for every such query.  $\square$

The final component is conditional on an examination occurring. It already holds for every world with this representation, including one in which no such examination is permitted. The existence premise in the joint theorem supplies an actual query and prevents the first two components from being merely empty universal statements.

## 1.8 An explicit query, including a false sentence

The existence premise can be realized without placing a true sentence in the query. Let  $W_{\text{all}}$  permit every finite history. For any sentence  $P$ , take the past

$$h_P = [\text{post}_{c(P)}, \text{post}_{c(\neg P)}, \text{retain}]$$

and the event  $e_P = (h_P, \text{compare}_{0,1})$ . Starting at the empty account, the successive accounts before the comparison are

$$\begin{aligned} ([], [], 0) &\mapsto ([c(P)], [], 1) \\ &\mapsto ([c(P), c(\neg P)], [], 2) \\ &\mapsto ([c(P), c(\neg P)], [], 3). \end{aligned}$$

Both lookup positions therefore contain the required records. The complete history is permitted by  $W_{\text{all}}$ , so this is a represented query. The comparison appends 0 and reaches stage 4.

Set  $P = \perp$ . Its query is still allowed and returns a distinction, while  $\perp$  remains false under its logical interpretation. This example establishes that represented consideration of a sentence does not assume its truth. The query's existence, its record inequality and the truth of its subject are three different claims.

## 1.9 A query does not supply unlimited continuation

There is also a finite realization. Given a finite history  $g$ , let  $W_g$  permit exactly its prefixes. It permits the empty history, and a prefix of a prefix is a prefix, so it satisfies the world conditions.

Choose  $g$  to be the four-operation query history for  $P = \text{Complete}([])$ . This history is permitted in  $W_g$ . The empty history is a prefix of its past, and the two required records are present. Hence the represented examination of the empty history occurs in this finite world. The examination theorem supplies the distinction absent from that earlier history.

**Proposition 1.8** (Finite termination is compatible with querying). *In this world, the query occurs, but no operation can be appended to its completed history as a permitted extension.*

*Proof.* If  $g \parallel [a]$  were permitted for an operation  $a$ , it would be a prefix of  $g$ . A prefix cannot be longer than the list of which it is a prefix, so  $|g| + 1 \leq |g|$ , a contradiction.  $\square$

For comparison, the world permitting only the empty history allows no event at all. Every event's complete history has length  $|h| + 1$ , so it cannot be a prefix of the empty history. This world does not satisfy the query-existence premise.

The examples settle two different questions. A query is possible within the stated representation, including a query of a false sentence. Its occurrence does not force a further permitted action, let alone an infinite physical history. Any use of continued operations must therefore exhibit the needed history or supply an extension condition.

## **1.10 What the starting point carries forward**

The argument has established the operational content of the starting facts. A represented query gives distinguishable records and a distinction event. Successful comparison requires both operands in the executed account. An examination after a recorded prefix is an occurrence beyond that prefix's cut-off. The concrete coding and four-operation history show that the conditions can hold together.

When applied to a physical comparison, the record model requires a physical interpretation: which states carry the records, which operation makes their information available, what registers the outcome, and how the earlier cutoff is fixed. Once those identifications are supplied, the proofs apply to those same objects and operations. The finite words make the requirements explicit; writing the words does not by itself build a physical recorder.

The next step is to construct the arithmetic carried by finite repetition and composition of records. Its operations must preserve what the records count. A quantitative comparison will require further structure: a carrier on which the relevant combinations can be made, a reading of those combinations, and the physical conditions the reading preserves. The following chapters supply and analyze those requirements at the point where they are used.

PART II

# Records and composition



# Arithmetic carried by records

A retained comparison gives an occurrence that can be counted. Retaining another gives a second occurrence. The marks used to count them may be strokes, list entries or positions in a posting history. What matters is whether the representation preserves the operations performed on the records.

This chapter constructs the arithmetic of finite repetition, then its signed differences and ratios. Two independently defined representations will agree on every arithmetic formula in the stated language. Finally, we connect the construction to a finite posting ledger: its actual update law supplies a count that distinguishes its stages. The physical requirements of that last step will remain visible.

## 2.1 Finite repetition

Let  $\mathcal{N}$  be the inductively generated carrier with an initial element  $0_\delta$  and a constructor  $S : \mathcal{N} \rightarrow \mathcal{N}$ . Its elements are the finite expressions

$$0_\delta, \quad S0_\delta, \quad SS0_\delta, \quad \dots$$

The surrounding mathematical language supplies inductive formation, recursion and elimination. Distinct constructors have disjoint images, and a constructor retains its argument. Consequently

$$0_\delta \neq Sa, \quad Sa = Sb \implies a = b. \quad (2.1)$$

Every element has a finite construction from these two clauses. To prove a property of every element, it therefore suffices to prove it for  $0_\delta$  and to prove that it passes from  $a$  to  $Sa$ . This is the structural induction principle of the construction.

This identifies what supports the arithmetic. The constructor laws and induction are consequences of the specified inductive carrier. A pair of unequal objects, without an available repetition operation, does not already

supply that carrier as a physical apparatus. In particular, describing every finite repetition does not grant one device unlimited execution or storage.

Write  $1_\delta = S0_\delta$ . Define addition and multiplication by recursion on the second argument:

$$a + 0_\delta = a, \quad a + Sb = S(a + b), \quad (2.2)$$

$$a \cdot 0_\delta = 0_\delta, \quad a \cdot Sb = a \cdot b + a. \quad (2.3)$$

Addition continues a tally by another tally. Multiplication repeats the first tally once for each mark of the second. Their laws must now be proved from the recursions.

## 2.2 The arithmetic laws

**Lemma 2.1** (Addition). *Addition on  $\mathcal{N}$  is commutative and associative. Zero is its identity, and a common summand can be cancelled.*

*Proof.* First prove  $0_\delta + b = b$  by induction on  $b$ . The zero case is the definition. The successor case is  $0_\delta + Sb = S(0_\delta + b) = Sb$ . Next prove the useful companion identity

$$Sa + b = S(a + b). \quad (2.4)$$

At  $b = 0_\delta$  both sides are  $Sa$ . If it holds at  $b$ , then

$$Sa + Sb = S(Sa + b) = SS(a + b) = S(a + Sb).$$

Commutativity follows by induction on  $a$ . Its zero case uses the two zero identities. At a successor,

$$Sa + b = S(a + b) = S(b + a) = b + Sa.$$

Associativity follows by induction on  $c$ . The zero case is immediate; the successor step is

$$(a + b) + Sc = S((a + b) + c) = S(a + (b + c)) = a + (b + Sc).$$

For cancellation it is useful to give an explicit numeral display. Define  $v : \mathcal{N} \rightarrow \mathbb{N}$  and  $u : \mathbb{N} \rightarrow \mathcal{N}$  recursively by

$$\begin{aligned} v(0_\delta) &= 0, & v(Sa) &= v(a) + 1, \\ u(0) &= 0_\delta, & u(n + 1) &= S(u(n)). \end{aligned}$$

Induction gives  $v(u(n)) = n$  and  $u(v(a)) = a$ . For example, the second successor step is  $u(v(Sa)) = u(v(a) + 1) = S(u(v(a))) = Sa$ . Thus  $v$  is injective. Induction on  $b$ , using the addition recursion, also gives

$$v(a + b) = v(a) + v(b). \quad (2.5)$$

Applying  $v$  to  $a + b = a + c$ , cancelling  $v(a)$  in the natural numbers and applying  $u$  gives  $b = c$ . Right cancellation follows by commutativity.  $\square$

The display is an explicit isomorphism, rather than a renaming of an undefined carrier. Its use of ordinary natural-number arithmetic is a stated part of the comparison with familiar notation. The direct addition proofs above used only structural induction and the recursive equations.

**Lemma 2.2** (Multiplication). *Multiplication is commutative and associative, has identity  $1_\delta$ , and has no zero divisors. The numeral display preserves multiplication.*

*Proof.* Induction on  $b$  gives  $0_\delta b = 0_\delta$ , since the successor step adds  $0_\delta$ . A second induction gives

$$(Sa)b = ab + b. \quad (2.6)$$

At zero both sides are zero. At  $Sb$ , the induction hypothesis and the addition laws give

$$\begin{aligned} (Sa)(Sb) &= (Sa)b + Sa = (ab + b) + Sa \\ &= S(ab + b + a) = S(ab + a + b) = a(Sb) + Sb. \end{aligned}$$

Now induct on  $a$  to prove commutativity. At zero use the zero laws; at a successor use

$$(Sa)b = ab + b = ba + b = b(Sa).$$

The unit equation follows directly:  $a1_\delta = a0_\delta + a = 0_\delta + a = a$ . Commutativity supplies the other side.

For the display equation, induction on  $b$  gives

$$\begin{aligned} v(a(Sb)) &= v(ab + a) = v(ab) + v(a) \\ &= v(a)v(b) + v(a) = v(a)v(Sb). \end{aligned}$$

The zero case is zero on both sides. Thus

$$v(ab) = v(a)v(b). \quad (2.7)$$

Both  $(ab)c$  and  $a(bc)$  consequently have the same display; injectivity of  $\nu$  proves associativity. If  $ab = 0_\delta$ , the display equation gives  $\nu(a)\nu(b) = 0$ . One displayed factor is zero, so its inverse image is  $0_\delta$ . This proves the last assertion.  $\square$

For example, the complete calculation of  $1_\delta + 1_\delta$  is

$$S0_\delta + S0_\delta = S(S0_\delta + 0_\delta) = SS0_\delta.$$

No multiplication, limit or physical calibration enters this identity.

## 2.3 An independent tally representation

Let  $\mathcal{L}$  consist of finite lists whose entries have the sole value  $\star$ . Define its operations directly:

$$0_L = [], \quad S_L x = \star :: x, \quad x +_L y = x ++ y,$$

where  $++$  denotes concatenation, and

$$x \cdot_L [] = [], \quad x \cdot_L (\star :: y) = (x \cdot_L y) ++ x.$$

These list operations are specified without first evaluating numerals.

**Theorem 2.3** (Tally equivalence). *Length  $\ell : \mathcal{L} \rightarrow \mathbb{N}$  is a bijection. Its inverse  $r(n)$  is the list of  $n$  copies of  $\star$ . Length preserves zero, successor, addition and multiplication.*

*Proof.* Induction on a list shows  $x = r(\ell(x))$ . The empty case is immediate. If  $x = \star :: y$ , the induction hypothesis gives

$$x = \star :: r(\ell(y)) = r(\ell(y) + 1) = r(\ell(x)).$$

Conversely, induction on  $n$  gives  $\ell(r(n)) = n$ . This proves both inverse identities and, in particular, that equal lengths imply equal lists.

The zero and successor equations follow directly from length. For concatenation, induction on  $x$  gives  $\ell(x ++ y) = \ell(x) + \ell(y)$ : at an added head, both sides increase by one. Induction on  $y$  then proves the product equation. Its successor step is

$$\begin{aligned} \ell(x \cdot_L (\star :: y)) &= \ell(x \cdot_L y) + \ell(x) \\ &= \ell(x)\ell(y) + \ell(x) = \ell(x)\ell(\star :: y). \end{aligned}$$

The empty case gives zero on both sides.  $\square$

The two arithmetic representations therefore agree through explicit maps. The list representation also satisfies separation directly: an empty list cannot be a nonempty list, and equality of two lists with the same head gives equality of their tails. List induction supplies induction on the representation.

A one-element carrier with a named zero and a named successor fails this requirement. Its zero is necessarily its own successor, contrary to (2.1). Merely giving operations familiar names is insufficient.

## 2.4 Preserving arithmetic statements

Representation invariance can be stated for entire formulas. Use terms formed from variables, 0, S, + and  $\cdot$ . Formulas are formed from equality of terms, falsity, conjunction, disjunction, implication, and universal and existential quantification over the arithmetic carrier. This specifies the language being compared.

An interpretation supplies a value for each variable and an operation for each term constructor. Inspecting the outer constructor and recursively interpreting its arguments defines a map from terms to the chosen carrier.

**Lemma 2.4** (Unique structural interpretation). *Once the target values and operations are supplied, exactly one interpretation preserves all the term constructors.*

*Proof.* The recursive interpretation supplies existence. For uniqueness, compare a constructor-preserving map  $F$  with that interpretation. They agree on variables and zero by the required equations. Agreement on  $t$  implies agreement on  $St$  by preservation of successor. Agreement on  $t, u$  implies agreement on  $t + u$  and  $t \cdot u$  by preservation of the two binary operations. Structural induction on terms covers every case.  $\square$

This is a uniqueness result for a supplied signature and target operations. It does not require those operations to satisfy the arithmetic laws; that is a further property of an arithmetic interpretation, established above for our two carriers.

An environment assigns values to variables. Call a numeral environment  $\rho_N$  and a list environment  $\rho_L$  *related* when  $\ell(\rho_L(i)) = \rho_N(i)$  for every variable  $i$ . Evaluate terms directly in each carrier, using that carrier's own operations.

**Theorem 2.5** (Invariance of arithmetic formulas). *Under related environments, every formula of the displayed language has the same truth value in the numeral and list interpretations.*

*Proof.* First induct on terms. Variables have equal readings by the environment condition. Zero and successor use their length equations. Sums and products use the preceding theorem and the induction hypotheses for their two arguments. Thus, for every term  $t$ ,

$$\ell(\text{ev}_L(t)) = \text{ev}_N(t). \quad (2.8)$$

Now induct on formulas. For equality, equal lists have equal lengths, while equal lengths imply equal lists. Equation (2.8) therefore preserves and reflects equality of the two evaluated terms. Falsity agrees in both interpretations. The equivalences for conjunction and disjunction follow by applying the induction hypotheses to each component. For implication, suppose it holds in the list interpretation and its premise holds in the numeral interpretation. Transfer the premise to lists, apply the implication, and transfer its conclusion back. The reverse direction uses the same two equivalences in reverse order.

For a universal quantifier, suppose first that its body holds for every list. Given a numeral  $n$ , choose the explicit list  $r(n)$ . The extended environments remain related, so the induction hypothesis gives the body at  $n$ . Conversely, if the body holds at every numeral, take any list  $x$  and apply it at  $\ell(x)$ ; transfer back to the body at  $x$ . For an existential quantifier, a list witness transfers to its length, and a numeral witness transfers to its repeated list. The extended environments are related in each case. This covers all formula constructors.  $\square$

In particular, validity under every environment agrees in the two representations: from a list environment take all its lengths, and from a numeral environment take all its repeated lists. This is agreement for the whole stated language, including quantified statements.

## 2.5 Signed differences of counts

A count records how many marks there are. A signed count records a difference between two counts. Begin with pairs  $(p, n)$  in  $\mathcal{N} \times \mathcal{N}$  and define

$$(p, n) \sim (q, m) \iff p + m = q + n. \quad (2.9)$$

The definition uses addition on the constructed carrier. A signed count is an equivalence class for this relation.

**Lemma 2.6** (Balance). *The relation  $\sim$  is an equivalence relation. Two pairs balance exactly when their displayed integer differences agree.*

*Proof.* Reflexivity is  $p + n = p + n$ , and symmetry reverses the equality. For transitivity, suppose  $p + m = q + n$  and  $q + s = r + m$ . Adding and rearranging gives

$$(p + s) + m = (r + n) + m.$$

Cancellation gives  $p + s = r + n$ , as required. For the display characterization, apply the additive, injective map  $v$  to (2.9). In ordinary integer notation the resulting equation is equivalent to

$$v(p) - v(n) = v(q) - v(m).$$

The same rearrangement and injectivity of  $v$  prove the converse.  $\square$

On representatives set

$$(p, n) + (q, m) = (p + q, n + m), \quad (2.10)$$

$$-(p, n) = (n, p), \quad (2.11)$$

$$(p, n)(q, m) = (pq + nm, pm + nq). \quad (2.12)$$

Define the display  $v(p, n) = v(p) - v(n)$ . The addition and multiplication display equations already proved give

$$\begin{aligned} v(a + b) &= v(a) + v(b), & v(-a) &= -v(a), \\ v(ab) &= v(a)v(b). \end{aligned}$$

For the product, this is the expansion

$$(v(p) - v(n))(v(q) - v(m)) = v(p)v(q) + v(n)v(m) - v(p)v(m) - v(n)v(q).$$

Replacing an input by a balanced pair leaves its display unchanged. Each output display therefore remains unchanged, and the balance lemma makes the corresponding output pairs equivalent. All three operations are consequently well-defined on signed classes.

**Theorem 2.7** (Integer representation). *The signed classes are in bijection with  $\mathbb{Z}$ . The bijection preserves zero, one, addition, negation and multiplication. In particular, their operations obey the commutative ring laws.*

*Proof.* The balance lemma makes  $v$  well-defined on classes and injective: equal displayed differences give balanced representatives and hence the same class. Given  $z \in \mathbb{Z}$ , take the class of

$$(u(\max(z, 0)), u(\max(-z, 0))),$$

where the two nonnegative integer parts are read as natural numbers. Its display is  $z$ . Thus every integer has a preimage. The displayed operation equations prove preservation, including the representatives  $(0_\delta, 0_\delta)$  and  $(1_\delta, 0_\delta)$  for zero and one.

To see the algebraic consequence explicitly, both sides of each ring law have equal displays. For associativity these are  $(\nu(a) + \nu(b)) + \nu(c)$  and  $\nu(a) + (\nu(b) + \nu(c))$ , or  $(\nu(a)\nu(b))\nu(c)$  and  $\nu(a)(\nu(b)\nu(c))$ . Commutativity exchanges the two displayed summands or factors. The zero, unit and additive-inverse laws display as  $z + 0 = z$ ,  $z \cdot 1 = z$  and  $z + (-z) = 0$ . Distributivity displays as  $x(y + z) = xy + xz$ , with the other side supplied by commutativity. Injectivity transfers each equality back to the signed classes.  $\square$

Passing from a pair to its signed class intentionally retains only the difference. The two separate counts remain available in the original pair. A later physical calculation that needs both must keep that pair; the quotient is adequate for precisely those operations that respect balance.

## 2.6 Ratios with a nonzero reference

To compare relative counts, take a signed pair  $a$  and an orbit count  $b \neq 0_\delta$ . A ratio representative is  $(a, b)$ . Its denominator is positive under  $\nu$ , because it is a nonzero natural count. Multiplying a signed pair by a count means multiplying both its components by that count. Define

$$(a, b) \approx (c, d) \iff ad \sim cb. \quad (2.13)$$

Thus equality of ratios is specified by cross-multiplication before any rational display is used.

Put  $A = \nu(a)$ ,  $B = \nu(b)$ , and similarly  $C, D$  for the other representative. The balance lemma and the product display show that (2.13) is equivalent to  $AD = CB$ . Because  $B, D > 0$ , it is also equivalent to  $A/B = C/D$  in  $\mathbb{Q}$ .

**Lemma 2.8** (Ratio equivalence). *The relation  $\approx$  is an equivalence relation. The display of its classes in  $\mathbb{Q}$  is injective.*

*Proof.* Reflexivity and symmetry follow directly from cross-multiplication. For transitivity, write the third representative as  $(e, f)$  with displays  $E, F$ . If  $AD = CB$  and  $CF = ED$ , multiplying these equations by  $F$  and  $B$  gives

$$ADF = CBF = EDB.$$

Cancel the nonzero  $D$  to obtain  $AF = EB$ . The characterization above proves transitivity. It also shows that equivalent representatives have equal rational displays, so the display is well-defined on classes. Conversely, equal rational displays give cross-equivalent representatives and hence the same class.  $\square$

Define the operations on representatives by

$$(a, b) + (c, d) = (ad + cb, bd), \quad (2.14)$$

$$-(a, b) = (-a, b), \quad (2.15)$$

$$(a, b)(c, d) = (ac, bd). \quad (2.16)$$

The product denominator is nonzero by the no-zero-divisor result. These operations display respectively as  $(AD + CB)/(BD)$ ,  $-A/B$ , and  $AC/(BD)$ .

Here is the representative-independence calculation. Suppose  $AB' = A'B$  and  $CD' = C'D$ . For addition, the difference between the two cross-products is

$$\begin{aligned} (AD + CB)B'D' - (A'D' + C'B')BD \\ = DD'(AB' - A'B) + BB'(CD' - C'D) = 0. \end{aligned}$$

For multiplication it is

$$ACB'D' - A'C'BD = CD'(AB' - A'B) + A'B(CD' - C'D) = 0.$$

Negation reverses the sign of the cross-product equality. Each operation therefore descends to ratio classes.

The additive and multiplicative laws can now be checked by common denominators. For three ratios  $A/B, C/D, E/F$ , the two associations of addition have the same numerator  $ADF + CBF + EBD$  over  $BDF$ . The two associations of multiplication have numerator  $ACE$  over  $BDF$ . Exchanging two summands or factors preserves these expressions. Zero and one are represented by  $0/1$  and  $1/1$ ; the negative  $-A/B$  adds to zero. For distributivity, both

$$\frac{A}{B} \left( \frac{C}{D} + \frac{E}{F} \right) \quad \text{and} \quad \frac{AC}{BD} + \frac{AE}{BF}$$

display as  $(ACF + AED)/(BDF)$  after cancelling a nonzero common factor in the second expression. Injectivity of the rational display proves the corresponding class equality. Right distributivity follows by commutativity.

A reciprocal must keep the denominator positive. Compute the sign of a signed pair by comparing its two counts, and compute its absolute difference

by removing their common part. Their displays are the sign and absolute value of  $A$ . For a nonzero numerator, define the reciprocal by

$$(a, b)^{-1} = \begin{cases} (+b, |a|), & A > 0, \\ (-b, |a|), & A < 0. \end{cases} \quad (2.17)$$

Here  $+b$  is the signed pair  $(b, 0_\delta)$ ,  $-b$  is its swap, and  $|a|$  is the orbit count of the absolute difference. It is nonzero exactly when the numerator does not balance zero. The operations in this construction use count comparison and subtraction on finite counts;  $A$  labels their integer display.

**Theorem 2.9** (Reciprocal cancellation). *Every ratio class different from zero has a well-defined reciprocal, and its product with that reciprocal is one.*

*Proof.* If two representatives are cross-equivalent, their rational displays are equal. Their numerators are either both zero or both nonzero. In the nonzero case the denominators are positive, so the numerators have the same sign. The reciprocal in (2.17) displays as  $B/A$  in both sign cases. Equal original displays therefore give equal reciprocal displays; the equivalence characterization makes the reciprocal well-defined.

For cancellation, choose a representative of a nonzero class. A balanced-zero numerator would give the zero class, so its  $A$  is nonzero. If  $A > 0$ , the product is represented by numerator  $AB$  and denominator  $BA$ , and hence equals  $1/1$ . If  $A < 0$ , its numerator is  $-AB$  and its denominator is  $B(-A)$ , which again agree. These are exactly the cross-product equations for equality with one. Commutativity gives cancellation in the other order.  $\square$

One may extend reciprocal to zero by defining  $0^{-1} = 0$ . This total-operation convention adds no cancellation law at zero. The nonzero reference remains a necessary condition of division.

## 2.7 Every calculation has a finite stage

For a natural budget  $n$ , let

$$\mathcal{N}_{\leq n} = \{a \in \mathcal{N} : v(a) \leq n\}.$$

The numeral display restricts to a bijection between this stage and the positions  $0, \dots, n$ . Its inverse is the same  $u$  with the stated bound. If  $n \leq m$ , the inclusion into  $\mathcal{N}_{\leq m}$  keeps the identical orbit element. These inclusions compose without changing the element.

Every  $a$  belongs to its own finite stage  $\mathcal{N}_{\leq v(a)}$ . The operation displays give exact sufficient bounds:

$$a \in \mathcal{N}_{\leq n}, b \in \mathcal{N}_{\leq m} \implies a + b \in \mathcal{N}_{\leq n+m}, \quad ab \in \mathcal{N}_{\leq nm}. \quad (2.18)$$

Indeed,  $v(a + b) = v(a) + v(b) \leq n + m$  and  $v(ab) = v(a)v(b) \leq nm$ . Enlarging an input budget changes neither operation's value.

**Theorem 2.10** (Compatible finite descriptions). *Suppose a map  $F_n : \mathcal{N}_{\leq n} \rightarrow X$  is given at every finite stage, and the maps agree whenever a stage is included in a larger one. There is exactly one map  $F : \mathcal{N} \rightarrow X$  whose restriction to each stage is  $F_n$ .*

*Proof.* Define  $F(a) = F_{v(a)}(a)$ , using  $a$  at its own budget. If  $a \in \mathcal{N}_{\leq n}$ , compatibility under the inclusion  $v(a) \leq n$  gives  $F(a) = F_n(a)$ . Thus this definition has every required restriction. Any other such map must have the same value at  $a$  by restricting it to  $a$ 's own stage. This proves uniqueness.  $\square$

There is also a finite record encoding at each stage. Record  $a$  as a word of  $v(a)$  copies of a fixed mark. Read a supplied word of length  $k$  by returning  $u(k)$  if  $k \leq n$ , and otherwise reject it. An encoded stage element has length  $v(a) \leq n$ , so reading it returns  $u(v(a)) = a$ . Preparation uses at most  $n$  marks. This is an explicit finite encoding and decoding rule; its physical implementation requires a recorder and reader that enact those operations.

## 2.8 A posting ledger supplies a faithful stage count

We can prove more once a concrete update law is supplied. Let a ledger have finitely many accounts  $k = 1, \dots, d$ . For each account retain two integer entries  $D_k$  and  $C_k$ , called debit and credit. The names specify two recorded sides. A *unit post* adds one to one side of one account and leaves every other entry unchanged. No zero initial ledger is required.

Define the size of a change from  $A$  to  $B$  by

$$L(A, B) = \sum_k |D_k(B) - D_k(A)| + \sum_k |C_k(B) - C_k(A)|.$$

Call the change *legal and atomic* when every entry is nondecreasing and  $L(A, B) = 1$ .

**Lemma 2.11** (Atomicity is one post). *A ledger change is legal and atomic if and only if it is a unit post.*

*Proof.* A unit post has one change equal to one and all others zero. It is nondecreasing and has  $L = 1$ . Conversely, for a nondecreasing change all the entry differences are nonnegative integers. Their sum is one. At least one difference is nonzero, and so is at least one. It must equal one: a larger difference would make the sum exceed one. Every other difference is zero for the same reason. The changed entry identifies its account and side, giving exactly the unit post.  $\square$

Keep both sides when counting posts. Define the census

$$N(A) = \sum_k (D_k(A) + C_k(A)). \quad (2.19)$$

Under either kind of unit post, exactly one summand increases by one. Hence  $N(B) = N(A) + 1$  for every legal atomic change. The signed potential  $D_k - C_k$  instead increases for a debit and decreases for a credit. It serves a different purpose from the census.

Let  $p = (A_0, \dots, A_n)$  be a supplied finite ledger history with every consecutive change legal and atomic.

**Theorem 2.12** (Finite posting faithfulness). *For each  $0 \leq i \leq n$ ,*

$$N(A_i) = N(A_0) + i. \quad (2.20)$$

*The stage map  $i \mapsto A_i$  is injective. Relative to this history's initial ledger and budget, its stage records have an exact decoder into  $\mathcal{N}_{\leq n}$ .*

*Proof.* Equation (2.20) follows by finite induction. At zero it is the identity. If it holds at  $i$ , the one-post census law gives  $N(A_{i+1}) = N(A_i) + 1 = N(A_0) + (i+1)$ . If  $A_i = A_j$ , apply  $N$  to obtain  $N(A_0) + i = N(A_0) + j$ . Integer cancellation gives  $i = j$ .

For a supplied ledger  $A$ , define the index

$$I_p(A) = \max\{N(A) - N(A_0), 0\},$$

read as a natural number. At an actual stage  $A_i$  the difference is exactly  $i$ , so  $I_p(A_i) = i$ . Encode  $a \in \mathcal{N}_{\leq n}$  by the ledger  $R_p(a) = A_{v(a)}$ . Define the decoder to return  $u(I_p(A))$  when  $I_p(A) \leq n$ , and to reject otherwise. For the encoded ledger it returns

$$u(I_p(R_p(a))) = u(v(a)) = a.$$

The bound is satisfied because  $v(a) \leq n$ . The encoding is injective either by the index identity or by injectivity of the stage map. This proves all assertions.  $\square$

The hypothesis supplies the legal finite history; injectivity is a conclusion. The initial census can be any integer. Its subtraction removes the initial offset, so no assumption that the ledger begins at zero is needed.

The decoder has a specific guarantee: it reads records from the supplied history correctly. It does not test that every external ledger with an acceptable census actually belongs to that history. Nor does the endpoint alone reconstruct the order of its earlier posts. Those would require additional information and different claims.

Combining this theorem with (2.18) gives bounded arithmetic readback. Given  $a \in \mathcal{N}_{\leq n}$ ,  $b \in \mathcal{N}_{\leq m}$  and a legal history long enough for the sum budget, encode  $a + b$  at its stage. The same decoder returns  $a + b$ . A history long enough for the product budget likewise reads back  $ab$ . This proves faithful representation of the calculated outputs. Producing those outputs by an autonomous physical arithmetic device requires operations that perform the calculation and acquire the indicated records.

## **2.9 What the construction supplies to comparison**

We now have finite counts, signed differences and ratios, together with explicit representation maps and their proofs. The posting construction connects finite stage positions to the same retained ledger whose unit updates it counts. A physical instance must supply that ledger, its legal transitions and access to the entries used by the census. The arithmetic theorem supplies the consequence of those conditions.

Nothing in the ratio construction says that every physical magnitude is rational. Later comparison laws use continuity over positive real readings. Their use of rational ratios as intermediate bounds will be justified by those later requirements. The next step is to specify which physical occurrences are being compared and which operations the comparison preserves.



# Events, balances and comparison

A record can tell us how much remains without telling us how much happened. If matching positive and negative entries cancel in a balance, both entries still belong to the history. This distinction becomes essential when we pass from arithmetic to a physical comparison. A residual can vanish even though the comparison required a nonempty operation.

The preceding chapter constructed signed counts from pairs of finite populations. We now retain those pairs long enough to distinguish their performed count from their balance. Cancellation then supplies a genuine minimum over all representations of a balance. Applied to the existing signed product, that minimum gives a squared comparison residual.

There is also an event account in which each comparison is accompanied by its reciprocal. We first prove exactly what that account conserves. The later signal construction connects the arithmetic residual to a measured component of the same encoded inputs. Its conservation requirement is stated separately from the conservation of a signed account.

## 3.1 Reciprocal events

An addressed event is a triple

$$e = (s, t, r), \quad r > 0.$$

The addresses  $s, t$  identify its two participants. Natural-number labels suffice for the finite records considered here; the labels carry no distance or geometry. The positive ratio  $r$  is part of the event data.

Define reversal by

$$e^* = (t, s, r^{-1}).$$

Reversing twice restores the event:

$$(e^*)^* = (s, t, (r^{-1})^{-1}) = e.$$

Consequently reversal is a bijection. In particular,

$$x^* = e \iff x = e^*.$$

These statements use the positivity of the recorded ratio to keep every reversed event in the same domain.

A finite event list  $L$  is *reciprocally balanced* when

$$\#_L(e) = \#_L(e^*) \quad \text{for every event } e. \quad (3.1)$$

Here  $\#_L(e)$  counts occurrences, including repetitions. Balance concerns multiplicity. It does not require the two occurrences to occupy adjacent positions in the list.

**Proposition 3.1** (Balance is preserved by paired posting). *The empty list is balanced. If  $L$  is balanced, then adjoining  $e$  and  $e^*$  gives another balanced list.*

*Proof.* Every multiplicity in the empty list is zero. For the second statement, let  $[P]$  be 1 when a proposition  $P$  holds and 0 otherwise. The new multiplicity of an arbitrary  $x$  is

$$\#_L(x) + [x = e] + [x = e^*].$$

Its reversed multiplicity is

$$\#_L(x^*) + [x^* = e] + [x^* = e^*].$$

The old terms agree by balance. Involutivity gives

$$[x^* = e] = [x = e^*], \quad [x^* = e^*] = [x = e].$$

Thus the two added terms are exchanged, and the new multiplicities agree.  $\square$

This construction supplies balanced lists. The condition in (3.1) remains the condition under which the conservation argument below applies. Defining a reciprocal event does not establish that an arbitrary history contains it. A physical use of this account must specify the operation that supplies both records.

### 3.2 What reciprocal balance conserves

For an address  $u$ , define the incident logarithmic contribution

$$f_u(e) = \begin{cases} \log r, & u = s \text{ or } u = t, \\ 0, & \text{otherwise.} \end{cases} \quad (3.2)$$

An event whose two addresses coincide contributes once under this definition. The same logarithm is assigned at either incident address; its sign reverses when the whole event is reversed.

Indeed, the set of incident addresses is unchanged by reversal, while

$$\log(r^{-1}) = -\log r.$$

Thus

$$f_u(e^*) = -f_u(e). \quad (3.3)$$

**Theorem 3.2** (Conservation of incident log flow). *For every finite reciprocally balanced list and every address  $u$ ,*

$$\sum_{e \in L} f_u(e) = 0.$$

*Proof.* Temporarily forget the order of the list and retain its multiset  $M$ . Reversal is a bijection, so the multiplicity of  $x$  in the reversed multiset  $M^*$  is the multiplicity of  $x^*$  in  $M$ . Balance therefore gives  $M = M^*$ .

Write  $Q = \sum_{e \in M} f_u(e)$ . Reindexing this finite sum by reversal and then using (3.3) gives

$$Q = \sum_{e \in M^*} f_u(e) = \sum_{e \in M} f_u(e^*) = - \sum_{e \in M} f_u(e) = -Q.$$

Hence  $2Q = 0$ , and  $Q = 0$ .

This also proves the statement for sequential accumulation along the original list. If an accumulator starts at  $a$  and each event adds  $f_u(e)$ , induction on the list gives its final value as  $a + \sum_{e \in L} f_u(e)$ . Taking  $a = 0$  identifies the accumulator with  $Q$ .  $\square$

The proof has two ingredients: an invariant collection of events and an antisymmetric reading. It establishes conservation of this reading. The logarithm records the additive change associated with a positive ratio; it is not being identified here with the nonnegative energy expended by a device. The event account has not yet specified a signal space, an energy function or a physical evolution.

### 3.3 Performed count and signed balance

Return to a signed population  $z = (p, n)$ , with  $p, n$  finite nonnegative counts. Its balance and gross count are

$$b(z) = p - n, \quad g(z) = p + n.$$

The signed class  $[z]$  retains the balance. The positive posting account retains both multiplicities, with one unit of count for each posting. Its total count is  $g(z)$ .

The addition and multiplication already constructed for signed pairs are

$$\begin{aligned} (p, n) + (q, m) &= (p + q, n + m), \\ (p, n)(q, m) &= (pq + nm, pm + nq). \end{aligned}$$

Before cancellation their gross counts satisfy

$$g(z + w) = g(z) + g(w), \tag{3.4}$$

$$g(zw) = g(z)g(w). \tag{3.5}$$

For multiplication, both sides are the same four contributions:

$$pq + nm + pm + nq = (p + n)(q + m).$$

Combining (3.4) and (3.5) shows separate additivity of the raw product count in each input.

The signed quotient does not preserve this gross count. Adding  $(k, k)$  preserves balance while adding  $2k$  postings:

$$[z + (k, k)] = [z], \quad g(z + (k, k)) = g(z) + 2k.$$

For any prescribed finite bound, choosing a larger  $k$  gives a representative of the same signed class whose gross count exceeds that bound.

**Proposition 3.3** (Performed count does not descend to signed classes). *There is no function of the signed class alone whose value equals the gross posting count of every representative.*

*Proof.* The representatives  $(0, 0)$  and  $(1, 1)$  have the same signed class. Their gross counts are respectively 0 and 2. A function on that one class cannot equal both values.  $\square$

This is why the full posting records in the preceding chapter were retained when counting actual successive operations. Passing to a signed quotient answers a different question. The quotient preserves the remaining imbalance; the positive account preserves the performed multiplicities.

### 3.4 Cancellation attains the minimum

There is nevertheless a count determined by a signed class: the smallest gross count among all its representatives. We can construct a representative that attains it.

For  $z = (p, n)$ , cancel matching populations:

$$c(z) = ((p - n)_+, (n - p)_+), \quad x_+ = \max(x, 0). \quad (3.6)$$

The differences in this formula are ordinary integer differences; their positive parts are nonnegative finite counts. Equivalently, the construction uses the truncated subtraction of the preceding chapter.

**Lemma 3.4** (The cancelled representative). *Cancellation preserves the signed class, leaves at least one side empty, and has gross count*

$$g(c(z)) = |p - n|.$$

*Proof.* If  $p \geq n$ , then  $c(z) = (p - n, 0)$ . Its balance is  $p - n$ , its negative side is empty and its gross count is  $p - n = |p - n|$ . If  $p \leq n$ , then  $c(z) = (0, n - p)$ , with the same three conclusions. The two descriptions agree at equality. Equality of balances gives equality of signed classes by the construction of the quotient.  $\square$

**Theorem 3.5** (Least representative count). *For a signed class with balance  $b$ , the set of gross counts of all its representatives has a least element. That element is  $|b|$ , attained by cancellation.*

*Proof.* Choose any representative  $z = (p, n)$ . The lemma supplies a representative of the same class with gross count  $|b|$ .

Now let  $w = (q, m)$  be any other representative of that class. It has  $q - m = b$ . If  $q \geq m$ , then

$$|b| = q - m \leq q + m = g(w).$$

If  $q \leq m$ , then

$$|b| = m - q \leq q + m = g(w).$$

Thus the constructed count is no larger than that of any competitor.  $\square$

Giving every posting unit weight makes the positive ledger cost equal to its gross count. The theorem therefore also gives the least such cost over all representatives. Define this residual count by

$$W([z]) = \min\{g(w) : [w] = [z]\}.$$

The construction proves both that this minimum exists and that

$$W([z]) = |b(z)|. \quad (3.7)$$

There is no assumed minimizer hidden in the definition.

This is a minimization over representations of a fixed balance. It does not remove events from an already performed history. Keeping that distinction allows the minimum to be used as a residual without misreporting the work that produced it.

### 3.5 The signed square supplies a comparison residual

Let two populations have counts  $a, b \in \mathbb{N}$ , and represent their difference by  $z = (a, b)$ . Apply the existing signed multiplication to  $z$  twice:

$$z^2 = (a^2 + b^2, 2ab). \quad (3.8)$$

The positive side contains the two same-sign products. The negative side contains the two cross products. Consequently

$$g(z^2) = (a + b)^2, \quad (3.9)$$

$$b(z^2) = (a - b)^2. \quad (3.10)$$

Both follow by expanding the two components of (3.8).

Because the balance in (3.10) is nonnegative, the least-representative theorem gives

$$W([z^2]) = (a - b)^2. \quad (3.11)$$

Thus the square appears through the signed product and its attained representation minimum. It was not assigned as an unexplained cost of the original pair.

For  $a = b = 1$ , the raw product has four postings and zero balance. Its residual minimum is zero. More generally, for adjacent populations  $a = n + 1$ ,  $b = n$ , the residual minimum is always 1, while the raw count is  $(2n + 1)^2$ . These examples concern the same constructor and the same signed class; their different counts answer different questions.

For positive populations, the actual negative cross count  $2ab$  supplies a nonzero reference. Define the normalized residual reading

$$R(a, b) = \frac{W([(a, b)^2])}{2ab}. \quad (3.12)$$

Then

$$\begin{aligned} R(a, b) &= \frac{(a - b)^2}{2ab} \\ &= \frac{1}{2} \left( \frac{a}{b} + \frac{b}{a} \right) - 1. \end{aligned} \quad (3.13)$$

The first equality uses the minimum just proved. The second expands  $(a - b)^2 = a^2 - 2ab + b^2$  and divides each term by  $2ab$ .

The denominator in (3.12) is attached to the raw square, before cancellation. The numerator is the minimum of its signed class. Their different roles are part of the specified comparison procedure. The identity does not claim that every possible device must choose this procedure. The next chapters determine the conditions under which a physical comparator and its report are forced to give the same law.

### 3.6 The same inputs in a signal comparison

The arithmetic result can be connected to a concrete signal without identifying unlike quantities merely because they share a formula. The relevant connection preserves the two input counts, constructs their signal, and compares its residual with their actual signed square.

Use an eight-entry complex register and the normalized adjacent pulse

$$\ell(a) = \frac{a}{\sqrt{2}}(e_0 - e_1).$$

If  $S$  is its cyclic shift, the two-input comparison is

$$C(a, b) = \ell(a) - S^4 \ell(b).$$

In coordinate order this is

$$C(a, b) = \frac{1}{\sqrt{2}}(a, -a, 0, 0, -b, b, 0, 0).$$

For this paragraph the finite register and its operations are supplied. Their role in the connected apparatus is developed later in the book.

Let  $H = S^4$  be the half-turn. The retained projector and its complement are

$$P = \frac{I - H}{2}, \quad I - P = \frac{I + H}{2}.$$

The half-turn squares to the identity. For this comparison, the complementary signal is explicitly

$$(I - P)C(a, b) = \frac{1}{2\sqrt{2}}(a - b, -a + b, 0, 0, a - b, -a + b, 0, 0).$$

Adding the four nonzero squared moduli gives

$$\|(I - P)C(a, b)\|_2^2 = \frac{|a - b|^2}{2}. \quad (3.14)$$

Likewise, adding the squared moduli of the original signal gives

$$\|C(a, b)\|_2^2 = |a|^2 + |b|^2.$$

The one-operand signal  $C(1, 0)$  therefore has squared norm 1.

These are identities for the explicit signal and projector. To interpret the square norm as an energy reading, we use the conservation argument of the next chapter. That argument starts with an unknown positive homogeneous size, its stated register symmetries, an unknown positive exponent, and a positive-gain sum-and-difference operation. Conservation for all input pairs forces the energy to have the form

$$E(f) = c\|f\|_2^2, \quad c > 0.$$

The exponent and gain are derived there. The same scalar  $c$  applies to all signals, including the one-operand reference.

Let

$$D(a, b) = E((I - P)C(a, b)), \quad U = E(C(1, 0)).$$

Then  $U = c$ , and for natural input counts the two constructions satisfy

$$2D(a, b) = UW([(a, b)^2]). \quad (3.15)$$

Indeed, (3.14) gives  $D(a, b) = c(a - b)^2/2$ , while (3.11) gives  $W([(a, b)^2]) = (a - b)^2$ .

Equation (3.15) is a correspondence between the actual input constructions. Its factor of two comes from the normalized pulse and projector. It does not assert an equality between the gross work of producing the square and the energy in the discarded signal.

For positive  $a, b$ , using the shared reference gives

$$\frac{D(a, b)}{Uab} = \frac{(a - b)^2}{2ab} = R(a, b).$$

The reference protocol is explicit here. Later, the comparison apparatus supplies a measured reference and the meter law, so that a desired denominator need not simply be assigned to a device.

### **3.7 The physical step that remains to be supplied**

We now have three distinct results with a precise connection. Reciprocal balance conserves incident logarithmic flow. Signed cancellation attains the least representation count of an imbalance. The conserved signal apparatus reads the square residual of the same encoded population inputs.

Their premises must travel with them. The event list supplies reciprocal multiplicities. The signed construction supplies positive posting counts and the permitted equivalence between representations. The signal construction supplies a complex carrier, linear operations and the conservation requirement on its size reading. No step in the finite bookkeeping proof alone selects all of those signal operations.

The next chapter therefore begins with the actual comparison requirements: both output channels, the allowed size function, conservation, a residual operation and a reference made from its records. It proves which choices those requirements remove. The joint apparatus chapter then constructs the comparison, hierarchy and spatial relations on the same finite source. This order exposes the physical conditions before using their consequences as part of the foundation.



PART III

# The comparison law



# What the comparison measures

Place two signals at the inputs of a device. Allow them to combine, separate the part that remains compatible with the device's recognition operation, and read the residual. The residual says how the two inputs failed to coincide. Its size alone does not yet say how large the failure was *relative to what was compared*. For that, the device needs a reference.

This distinction carries the argument. A large residual can accompany large inputs. A small residual can accompany small inputs. A statement about relative mismatch must say what the reference contains, how it was obtained, and why the same reference rule applies when the inputs change. Otherwise a proposed cost curve can be installed in the denominator.

We will construct the reference from records produced by the comparison itself. We will also leave the meter's response initially unknown. The result is a chain of constraints on an operation, an energy reading, a measured reference and a dimensionless report. At its end, the canonical reciprocal cost will appear as the report of this arrangement.

## 4.1 The signal carrier and the operation

For the present argument a signal is a vector with complex entries. The construction used here has eight registers,

$$V = \mathbb{C}^{\{0,\dots,7\}}.$$

The register count is a specified carrier here. It is not the conclusion of this chapter's conservation argument. The spatial and completion arguments later explain the separate roles of dimension and register count. Keeping those questions separate lets us examine the comparator without deriving its premises from an apparatus that already contains the desired hierarchy.

A size function  $N$  assigns a nonnegative real number to each signal. Its basic laws say that only the zero signal has size zero, scaling a signal scales

its size by the absolute value of the scalar, and combining signals satisfies the triangle inequality. The real-coordinate formulation also requires invariance under coordinate permutations and independent coordinate sign changes. The complex formulation supplies the corresponding coordinate phase symmetry. These requirements make the register addresses and their phase conventions physically interchangeable for the size reading.

The comparison operation has two outputs:

$$(\psi, \chi) \mapsto (\lambda(\psi + \chi), \lambda(\psi - \chi)), \quad \lambda > 0. \quad (4.1)$$

Both the sum and difference matter. Reading only the difference would discard the part of the account carried by the other output.

Suppose the conserved quantity is some positive power of size. We do not choose that power in advance. Write it as  $N(\psi)^q$ , with  $q > 0$ , and require

$$N(\lambda(\psi + \chi))^q + N(\lambda(\psi - \chi))^q = N(\psi)^q + N(\chi)^q \quad (4.2)$$

for all input pairs. Equation (4.2) is the conservation premise for this operation. It is a substantive restriction on which size laws and devices qualify. Calling the device a comparator does not by itself establish that restriction.

## 4.2 Two configurations determine the exponent

The first deduction needs only one nonzero signal and two ways of supplying it. This is a useful example of how a forcing argument works: the competing possibilities are removed by applying one law to configurations that the same device must accept.

**Theorem 4.1** (Two-port normalization). *Let  $N$  be a positive definite, absolutely homogeneous size function, let  $\psi \neq 0$ , and let  $\lambda, q > 0$ . If conservation (4.2) holds at  $(\psi, 0)$  and  $(\psi, -\psi)$ , then*

$$q = 2, \quad \lambda = \frac{1}{\sqrt{2}}.$$

*Proof.* At  $(\psi, 0)$ , both outputs are  $\lambda\psi$ . Homogeneity and positivity permit cancellation of  $N(\psi)^q$ , leaving

$$2\lambda^q = 1. \quad (4.3)$$

At  $(\psi, -\psi)$ , the sum port is empty and the difference port carries  $2\lambda\psi$ . The same cancellation gives

$$(2\lambda)^q = 2. \quad (4.4)$$

Using  $\lambda^q = 1/2$  from (4.3) in (4.4) yields  $2^q = 4$ . Strict monotonicity of real powers at base 2 gives  $q = 2$ . Positivity of  $\lambda$ , together with  $\lambda^2 = 1/2$ , selects  $\lambda = 1/\sqrt{2}$ .  $\square$

The argument leaves more freedom than a specification that declares the balanced device to be an involution. It leaves the gain free and obtains the usual normalization from the two conservation requirements. Its hypotheses still specify the sum-and-difference operation and homogeneous size. An arbitrary nonlinear device is outside this competitor class.

Once  $q = 2$ , conservation for every pair becomes the parallelogram identity:

$$N(\psi + \chi)^2 + N(\psi - \chi)^2 = 2N(\psi)^2 + 2N(\chi)^2. \quad (4.5)$$

The Jordan–von Neumann characterization [1] supplies an inner product whose quadratic form is  $N^2$ . The contribution of the register symmetries is more specific: independent sign changes remove the off-diagonal terms in the real-coordinate form, and permutations make the diagonal terms equal. The complex counterpart yields

$$E(f) := N(f)^q = c \sum_t |f_t|^2, \quad c > 0. \quad (4.6)$$

Conservation determines the quadratic geometry. Register symmetry makes that geometry isotropic in the specified register basis. A symmetry-free version of the result retains the inner product while leaving its coordinate matrix general.

The positive constant  $c$  is still present. Nothing so far has assigned an absolute energy unit. That remaining scale will cancel from a relative measurement when the numerator and reference come from the same apparatus.

The next steps concern three different objects. The residual is an energy record produced by the operation. The reference is another measured quantity assembled from that operation's records. The meter assigns a number to a residual against a reference. We first determine the meter, then the residual and reference. This order makes the use of a zero reading precise.

### 4.3 Why the meter reports a ratio

The reference and residual are measured quantities. We must also say what their numerical readout means. Let  $m(d, r)$  be the reading of a nonnegative record  $d$  against a positive reference  $r$ . Require

$$\begin{aligned} m(d, r) &\geq 0 & (d \geq 0, r > 0), \\ m(d + e, r) &= m(d, r) + m(e, r) & (d, e \geq 0, r > 0), \\ m(r, r) &= 1 & (r > 0). \end{aligned} \tag{4.7}$$

The addition law pools independently obtained energy records. It does not assert that the energy of a coherent superposition is the sum of its component energies. Interference between coherent signals is accounted for before their energy records are pooled.

**Proposition 4.2** (The meter law). *Under (4.7),*

$$m(d, r) = \frac{d}{r} \quad (d \geq 0, r > 0).$$

*Proof.* Fix  $r > 0$ . Additivity gives  $m(0, r) = 0$ . If  $e \geq d \geq 0$ , then

$$m(e, r) = m(d, r) + m(e - d, r) \geq m(d, r),$$

so the reading is monotone in its first argument.

Write  $f(d) = m(d, r)$  and  $a = f(1) \geq 0$ . For nonnegative integers  $p$  and positive integers  $n$ , repeated addition gives

$$nf(p/n) = f(p) = pa.$$

Thus  $f(q) = aq$  at every nonnegative rational  $q$ .

We now determine  $f(d)$  for any  $d \geq 0$ . If  $a = 0$ , choose a rational  $q > d$ . Positivity and monotonicity give  $0 \leq f(d) \leq f(q) = 0$ , so  $f(d) = ad$ .

Suppose  $a > 0$ . If  $f(d) > ad$ , choose a rational  $q$  strictly between  $d$  and  $f(d)/a$ . Then

$$f(d) \leq f(q) = aq < f(d),$$

a contradiction. If  $f(d) < ad$ , choose a rational  $q$  strictly between  $f(d)/a$  and  $d$ . Since  $f(d) \geq 0$ , this rational is nonnegative, and

$$f(d) < aq = f(q) \leq f(d),$$

another contradiction. Hence  $f(d) = ad$  on the nonnegative ray.

Finally, self-reference gives  $1 = f(r) = ar$ . Since  $r > 0$ ,  $a = 1/r$ , and therefore  $m(d, r) = d/r$ .  $\square$

No smoothness or differentiability assumption is needed for this meter theorem. Positivity supplies monotonicity, and monotonicity rules out non-linear additive pathologies. The result depends on these meter laws; it does not say that every detector response has them.

#### 4.4 The residual before its shape is selected

Let  $u, v \in V$  describe the two operand encodings. For real amplitudes  $A, B$  and a unit complex phase  $z$ , form

$$e(A, B, z) = Au + zBv. \quad (4.8)$$

Let  $T : V \rightarrow V$  be the complex-linear map whose output is measured as the residual. At this stage we do not assume that the encoded operands become opposite vectors under  $T$ . The measured energy is

$$D(A, B, z) = E(T(e(A, B, z))). \quad (4.9)$$

This formulation accommodates an actual discarded component, but its selection theorem only needs the measured residual map to be linear. The projection interpretation requires the corresponding physical construction.

#### 4.5 What a phase contrast supplies

Choose two fixed unit phases  $z, w$  and form the measured contrast

$$R_{z,w}(A, B) = D(A, B, z) - D(A, B, w). \quad (4.10)$$

The phases belong to the protocol; they do not change with the operand amplitudes. When  $B = 0$ , both measurements have exactly the same encoded input  $Au$ . Hence their difference is zero. This empty-operand law requires no conservation argument and no assumption about a reported cost function.

Conservation supplies a further fact: a single positive balanced reference controls the sign throughout the positive input domain.

**Proposition 4.3** (A measured contrast is bilinear). *For the conserved quadratic energy (4.6), the complex-linear residual map  $T$ , and fixed unit phases  $z, w$ ,*

$$R_{z,w}(A, B) = AB R_{z,w}(1, 1) \quad (A, B \in \mathbb{R}). \quad (4.11)$$

*In particular,  $R_{z,w}(1, 1) > 0$  implies  $R_{z,w}(A, B) > 0$  whenever  $A, B > 0$ .*

*Proof.* Put  $U = Tu$  and  $V = Tv$ . Linearity and the conserved energy give

$$D(A, B, z) = c \sum_t |AU_t + zBV_t|^2.$$

Expand each squared modulus. Its same-operand terms are  $A^2|U_t|^2$  and  $B^2|V_t|^2$ , since  $|z| = 1$ . They are unchanged when  $z$  is replaced by  $w$ , so they cancel in the contrast. Every remaining mixed term has the factor  $AB$ . After factoring it out, the remaining sum is precisely the contrast at  $A = B = 1$ . This proves (4.11). Positivity follows by multiplying the three positive factors on its right.  $\square$

This proof has not used  $Tv = -Tu$ , balanced zero, or the reciprocal cost formula. It can therefore supply reference positivity before those conclusions are established. The actual contrast must have a positive balanced reading; conservation does not assert that every choice of two phases has that sign.

## 4.6 Balanced zero selects a difference

Now impose a balanced observation: equal in-phase populations report zero cost, and the associated reference is positive. The meter law just proved says that its reading is the residual energy divided by the positive reference. Thus a zero reading forces the measured residual energy itself to be zero. By positive definiteness,

$$T(u + v) = 0.$$

Linearity therefore forces

$$Tv = -Tu. \quad (4.12)$$

The difference shape is a conclusion about the encoder images. It has not been put into the definition of  $e$ .

Substituting (4.12) into the residual gives

$$T(e(A, B, z)) = (A - zB)Tu.$$

Using the quadratic energy law,

$$D(A, B, z) = k|A - zB|^2, \quad k = E(Tu). \quad (4.13)$$

A positive reference constructed from energies of this same encoded pair excludes  $Tu = 0$ ; otherwise every record in the reference would vanish. Thus

$k > 0$ . This single fact prevents the purported comparison from being an invisible encoder with arbitrary labels attached.

For a unit phase,

$$D(A, B, z) = k(A^2 + B^2 - 2AB \operatorname{Re} z). \quad (4.14)$$

The in-phase residual is  $k(A - B)^2$ . The cross term measures the interference between the two encoded operands. Its dependence on both operands will determine the reference.

## 4.7 A reference made from measured records

A reference program is a finite expression built from three operations: record  $D(A, B, z)$  at a unit phase; add two records; and multiply a record by a real coefficient. Negative coefficients are permitted because a contrast between two records is permitted. The requirement is that the *evaluated reference* be positive on the positive populations where a ratio is reported.

This grammar gives a definite class of competitors. It includes the phase contrast that we will use, while also allowing references with same-operand contributions. We do not declare the correct reference to be  $AB$  and then recover the desired cost from that declaration.

For each program  $p$ , two coefficients,  $s_p$  and  $t_p$ , are obtained recursively from its syntax. A single phase record has

$$s_p = \frac{1}{2}, \quad t_p = -\frac{1}{2} \operatorname{Re} z;$$

addition and real scaling act on the coefficients in the same way as on the program. Induction on that syntax, using (4.14), proves

$$R_p(A, B) = 2k[s_p(A^2 + B^2) + 2t_p AB]. \quad (4.15)$$

The symmetric quadratic expression is therefore the evaluated form of a measured program, rather than an interpretation imposed afterward.

What distinguishes a comparison reference from a record of the whole posting activity? Consider removing the second operand. A reference that counts only cross exposure must vanish when that operand is absent:

$$R_p(1, 0) = 0. \quad (4.16)$$

In (4.15),  $k > 0$ , so this observation forces  $s_p = 0$ . Positivity of  $R_p(a, b)$  at positive native populations then forces  $t_p > 0$ . With

$$\kappa = 4t_p > 0,$$

the reference becomes

$$R_p(A, B) = \kappa k AB. \quad (4.17)$$

The empty-input condition concerns the reference. The residual numerator need not vanish when an operand is absent. Indeed, the native signed square construction retains a positive same-population posting at  $(1, 0)$ , while its negative cross posting is zero. Confusing the full posting ledger with the cross reference would erase this distinction and invalidate the selection argument.

The phase contrast (4.10) supplies the empty-input condition directly. Its positive balanced reference, together with (4.11), supplies the positivity condition on every positive native population. The reference selection therefore has an explicit measured supplier: two fixed phase records and one positive balanced reference.

Alternatively, if a reference is separately additive in the second population, including the empty population, then  $R_p(1, 0) = R_p(1, 0) + R_p(1, 0)$ , which forces the same zero. A native pair kernel supplies such a law for its own cross exposure. Applying it to a measured reference still requires the correspondence between that exposure and that reference. The distinction matters whenever a physical implementation is changed.

## 4.8 The relative reading

Combining the selected residual, the selected reference and the meter law gives

$$m(D(A, B, 1), R_p(A, B)) = \frac{(A - B)^2}{\kappa AB}, \quad A, B > 0. \quad (4.18)$$

The common factor  $k$  has canceled because both records came from the same residual operation. This cancellation is a physical accounting statement with a mathematical proof. It is not a permission to cancel unrelated calibrations from different instruments.

## 4.9 The concrete phase scan

For the normalized source pulse and recognition projector in the native construction, the discarded energy has the form

$$D_{\text{src}}(A, B, z) = \frac{c}{2}(A^2 + B^2 - 2AB \operatorname{Re} z).$$

Read it first at  $z = 1$ , and then at  $z = i$ . Define the reference by the increase in the measured residual:

$$R_{\text{scan}}(A, B) = D_{\text{src}}(A, B, i) - D_{\text{src}}(A, B, 1). \quad (4.19)$$

Calculation from the same energy law gives

$$R_{\text{scan}}(A, B) = cAB > 0 \quad (A, B > 0).$$

Thus the product reference is reconstructed from two records. The protocol did not ask the meter to multiply the operand amplitudes. In the notation of (4.17), this is  $k = c/2$  and  $\kappa = 2$ .

The dimensionless phase-dependent report is

$$\frac{D_{\text{src}}(A, B, z)}{R_{\text{scan}}(A, B)} = J(A/B) + 1 - \text{Re } z, \quad J(x) = \frac{1}{2} \left( x + \frac{1}{x} \right) - 1. \quad (4.20)$$

Its in-phase value is  $J(A/B)$ . The choice of reference phase fixes its unit. Using the full contrast between opposed and in-phase records doubles this reference and halves the reading. The energy law and meter are unchanged. Thus conservation fixes the quadratic geometry, while the stated reference protocol fixes the normalization of the dimensionless cost. Changing both amplitudes by the same nonzero real factor scales both energy records by the square of that factor and leaves their ratio unchanged. The report retains relative mismatch and phase while discarding common amplitude scale.

This is already a concrete result about a comparator. It includes an operation, an energy law, an actual reference program, and a meter. The next question is stronger: if an arbitrary function claims to report this device, how much freedom remains in that function? That is where a construction becomes a uniqueness argument.



# Why the cost has this form

The comparator has produced a dimensionless quantity. For positive amplitudes  $A, B$ , its selected in-phase report is

$$\frac{(A - B)^2}{\kappa AB}.$$

Writing  $x = A/B$  shows that the report depends on one ratio. Multiplying it by the reference normalization  $\kappa/2$  gives

$$J(x) = \frac{(x - 1)^2}{2x} = \frac{1}{2} \left( x + \frac{1}{x} \right) - 1, \quad x > 0. \quad (5.1)$$

The question is no longer whether this expression can be made to appear. It is whether another function can report the same physical comparison under the same requirements.

There are two useful routes to the answer. The first begins with a class of apparatuses and selects their residual, reference and cost together. The second begins with a constructed apparatus and leaves only its reported function arbitrary. The second route has fewer premises about the reported function because the construction has already supplied the geometric and reference facts. Neither route should borrow the other's conclusion while concealing its premises.

## 5.1 What a native report asserts

A positive native input consists of two positive natural populations,  $a, b \in \mathbb{N}$ . Their ratio is a positive rational number. Let  $F$  be an arbitrary real-valued function. Literal native reporting asserts

$$F(a/b) = m(D(a, b, 1), R_p(a, b)) \quad (a, b > 0). \quad (5.2)$$

Everything on the right belongs to the apparatus: the size function, energy exponent, operand pair, linear residual map, meter and reference program. The same tuple must be used throughout the assertion.

Equation (5.2) is stronger than agreement with a handful of observations. It holds for every positive pair of native populations. It is also narrower than equality on every positive real argument. The native inputs determine positive rational ratios directly. A statement about all positive reals needs a completion principle.

The role of reporting is easy to misstate. It does not declare  $F = J$ . It declares that the number assigned by  $F$  to the population ratio is the number read from the specified device. The proof then determines that number. Whether a particular experimental system enacts the specified device remains a physical identification that must be established for that system.

## 5.2 Selection from operational conditions

The broad empty-operand theorem uses the following conditions on the comparison described in Chapter 4. The positive-gain sum-and-difference device conserves a positive power of its size. The reference is a finite linear program of unit-phase records and is positive at positive native populations. The reference vanishes at the input  $(1, 0)$ . The arbitrary cost  $F$  reports the device, satisfies  $F(1) = 0$ , is reciprocal on positive inputs, and is monotone as the positive log-ratio increases from zero.

For clarity, the last two shape conditions are

$$F(x) = F(1/x) \quad (x > 0), \quad t \mapsto F(e^t) + 1 \text{ is nondecreasing on } [0, \infty).$$

They are premises of this broad selection theorem. They are not premises of the later minimal reporting theorem.

**Theorem 5.1** (Empty-operand selection). *Under these conditions, the operand images are opposite,  $Tv = -Tu$ . The reference has no same-operand term and has a positive cross term. There exist  $k, \kappa > 0$ , with  $\kappa = 4t_p$ , such that*

$$\begin{aligned} D(A, B, z) &= k|A - zB|^2, \\ R_p(A, B) &= \kappa kAB, \\ \frac{\kappa}{2}F(x) &= J(x) \quad (x > 0). \end{aligned} \tag{5.3}$$

*Proof.* The preceding chapter establishes the first two conclusions. For every positive native pair, the meter law consequently gives

$$F(a/b) = \frac{(a - b)^2}{\kappa ab}.$$

Put  $K(x) = (\kappa/2)F(x)$ . Then  $K(q) = J(q)$  for every positive rational  $q$ . We must establish the same equality at irrational arguments without assuming continuity of  $F$ .

First consider  $x \geq 1$ . Log-order makes  $K$  nondecreasing on  $[1, \infty)$ : the logarithm preserves the order there, and  $F(e^{\log x}) = F(x)$ . The positive normalization factor preserves that order as well. At  $x = 1$ , rational agreement gives the result.

Suppose  $x > 1$  and  $K(x) > J(x)$ . The explicit function  $J$  is continuous on the positive ray. Choose a rational  $q > x$  close enough to  $x$  that  $J(q) < K(x)$ . Then monotonicity and rational agreement give

$$K(x) \leq K(q) = J(q) < K(x),$$

which is impossible. If instead  $K(x) < J(x)$ , choose a rational  $q$  with  $1 < q < x$ , close enough that  $K(x) < J(q)$ . The opposite contradiction is

$$K(x) < J(q) = K(q) \leq K(x).$$

Hence  $K(x) = J(x)$  for every  $x \geq 1$ . Only continuity of the known function  $J$  was used in this squeeze.

For  $0 < x < 1$ , reciprocal symmetry and the result above one give

$$K(x) = K(1/x) = J(1/x) = J(x).$$

This proves the last conclusion on the entire positive domain.  $\square$

When the reference is a contrast of two fixed unit-phase records, its empty-input value vanishes by construction. A positive balanced reference supplies all positive native reference values by (4.11). In this instance, neither the empty-reference law nor infinitely many separate positivity observations need to be added after the protocol is specified.

The result selects more than a convenient expression for  $F$ . It selects the anti-alignment of the measured operand images and the cross-only form of the reference. A competitor that changes the denominator to include same-operand exposure does not satisfy the empty-reference condition. A competitor that keeps the same apparatus but changes its rational reports does not satisfy literal reporting.

The scale in (5.3) deserves equal attention. For every positive  $\kappa$ , there is an explicit reference program and its corresponding cost:

$$F_\kappa(x) = \frac{2}{\kappa} J(x), \quad p_\kappa = \frac{\kappa}{2} p_{\text{scan}}. \quad (5.4)$$

Here scaling a program scales its evaluated reference. It does not change the meter law. Doubling a reference halves the dimensionless reading of a fixed residual. Thus the invariant conclusion is  $(\kappa/2)F_\kappa = J$ . The calibration  $\kappa = 2$  makes the reported cost itself equal to  $J$ .

This distinction separates a freely chosen reference unit from a freely chosen functional shape. The theorem removes the latter freedom within its competitor class. It retains the former explicitly.

### 5.3 A constructed witness removes shape assumptions

Suppose an actual apparatus has a cost witness  $G$  satisfying the empty-operand premises. This is more than the assumption that some desired function exists. The source construction supplies a witness: a concrete signal carrier, source pair, residual, conserved size, ratio meter and phase reference satisfy the premises together. The construction is available for every positive reference gauge.

Now let  $F$  be any other function that literally reports *that same apparatus*. We do not impose balanced zero, reciprocity or log-order on  $F$ . Assume only that it is continuous on the positive real domain.

**Theorem 5.2** (Uniqueness of continuous native reporting). *Let  $G$  be an empty-operand cost witness for a fixed apparatus. If  $F$  satisfies the native reporting relation (5.2) for that apparatus and is continuous on  $(0, \infty)$ , then*

$$F(x) = G(x) \quad (x > 0).$$

Consequently,

$$\frac{\kappa}{2}F(x) = J(x) \quad (x > 0)$$

in the gauge selected by the witness.

*Proof.* For every positive pair  $a, b$ , the two reports equal the same meter reading:

$$F(a/b) = m(D(a, b, 1), R_p(a, b)) = G(a/b).$$

The witness selection theorem has already established  $G = (2/\kappa)J$  on the positive domain. Since  $\kappa > 0$  and  $J$  is continuous there,  $G$  is continuous there.

Fix  $x > 0$ . Rational density gives positive rational numbers  $r_n$  converging to  $x$ ; choose each between  $x$  and  $x + 1/(n + 1)$ . Each positive rational can be written as  $a_n/b_n$ , with positive natural numerator and denominator. Thus the

reporting relation applies at every  $r_n$ , and  $F(r_n) = G(r_n)$ . Continuity of both functions at the positive point  $x$  gives

$$F(x) = \lim_n F(r_n) = \lim_n G(r_n) = G(x).$$

The normalization conclusion follows from the witness.  $\square$

The density argument applies to any two continuous functions agreeing on positive population ratios. Continuity of the witness follows from the selected cost, so the argument applies to the two reports.

Balanced zero, reciprocity and log-order now become conclusions for  $F$ , inherited from  $G$ . This is the precise sense in which the minimal route assumes no cost law of the reported function. It does assume a particular apparatus, the reporting relation and continuity. Removing a premise from the public function interface does not remove the work performed by the apparatus construction.

The combined native theorem later applies this result to the actual Gray relative-formation residual and the original census hierarchy. That construction also supplies the retained spatial witness. Its appearance later in the book matters: we first derive the hierarchy from additive comparison preservation, then use it to construct the golden apparatus. We do not use a device already containing  $\phi$  to establish the independent argument that selects  $\phi$ .

## 5.4 The alternative that continuity excludes

The completion assumption has a sharp test. It is possible to preserve every literal native report while changing the function at a positive irrational argument.

**Proposition 5.3** (Rational reports alone leave freedom). *Let  $G$  satisfy the native reporting relation, and let  $x_0 > 0$  be irrational. Define*

$$F(x) = \begin{cases} G(x) + 1, & x = x_0, \\ G(x), & x \neq x_0. \end{cases}$$

*Then  $F$  has exactly the same native reports as  $G$ , but  $F(x_0) \neq G(x_0)$ .*

*Proof.* Every ratio  $a/b$  of positive natural populations is rational, hence differs from  $x_0$ . Therefore  $F(a/b) = G(a/b)$  at every input appearing in the reporting relation. At  $x_0$ , the definition changes the value by one.  $\square$

This counterexample explains why a proof restricted to native reports cannot silently assert equality on all positive reals. When the witness  $G$  is the selected continuous cost, the altered function fails the required continuity at  $x_0$ . The completion theorem excludes it for that exact reason.

The distinction is also useful scientifically. Finite experimental observations have still less reach than the universal native reporting relation. An empirical identification needs an account of the device and its operating domain, together with evidence that the relation applies. The theorem states what follows once that identification is in place; it does not turn finite data into all-input identity by definition.

## 5.5 What has been fixed

For  $x > 0$ , equation (5.1) makes the basic geometry explicit. The cost is nonnegative, vanishes exactly at  $x = 1$ , and is unchanged by replacing  $x$  with  $1/x$ . Equality of the compared positive quantities is the unique zero-cost case. Swapping the two operands preserves the cost.

These properties are consequences of the selected formula, with positivity, reciprocity and the zero characterization made explicit by the two expressions in (5.1). They will let us distinguish a genuine join between unequal levels from duplication of one level in the next chapter.

The construction has fixed a relative comparison cost and its reference dependence. It has not yet assigned an absolute physical energy scale, a dimensional constant, or a law for every possible material apparatus. Those are further identifications and constructions in the same physical account. The advance here is specific and substantial: an initially arbitrary report is forced to have one normalized shape by the stated operation, reference and reporting requirements.

PART IV

# The recognizing hierarchy



# The join that forces golden scaling

A comparison can be followed by another comparison. Records are combined, an update acts on them, and the resulting state becomes an input to the next operation. If the update changes which recorded difference is positive, it changes what the comparison says. If it preserves every positive additive comparison, its freedom is much smaller.

This chapter derives that restriction before introducing a golden apparatus. The starting carrier is an arbitrary additive commutative group. Its update is merely additive. There is no chosen basis, no assumed two-dimensional algebra, no assumed inverse update and no supplied hierarchy. The first result forces uniform scaling of the readout. A separate condition on the first costly join then fixes that scale to the golden ratio.

The separation is decisive. Order preservation alone determines a positive multiplier but does not choose its value. A binary join alone does not force uniform scaling. Golden scaling follows when the two requirements hold on the same realized framework.

## 6.1 Additive readings of physical states

Let  $S$  be the state carrier of a closed observable framework. Write  $T : S \rightarrow S$  for its successor operation and  $r : S \rightarrow \mathbb{R}_{>0}$  for its positive reading. Choose an additive commutative group  $G$ , an additive update  $A : G \rightarrow G$ , a reading map  $i : S \rightarrow G$ , and an additive value map  $L : G \rightarrow \mathbb{R}$ . The correspondence requires

$$\begin{aligned} L(i(s)) &= r(s), \\ i(Ts) &= A(i(s)). \end{aligned} \tag{6.1}$$

The first equation says that the additive reading reproduces the framework's numerical readout. The second says that the native update reproduces its

successor on those readings. Without the second equation, a convenient sequence of numbers in  $G$  could be unrelated to the states actually being evolved.

The signed group is useful because a difference of two records represents a comparison. If  $x, y \in G$ , then

$$L(x - y) = L(x) - L(y).$$

A positive value of  $x - y$  says that  $x$  has the larger reading. Allowing signed differences does not require a physical population to contain negative objects. It supplies the arithmetic in which two permitted populations can be compared.

The preservation condition is

$$L(x) > 0 \implies L(Ax) > 0 \quad \text{for every } x \in G. \quad (6.2)$$

Its quantifier matters. Positivity of a few basic records, or positivity of the displayed orbit, is weaker. Equation (6.2) preserves every strict comparison represented by a signed group element, including differences formed from arbitrarily many copies of the records.

The framework's positive reading supplies a point  $e = i(s_0)$  with  $L(e) > 0$ . That point will serve as a reference in the proof. No coordinate basis is required.

## 6.2 Preserving comparisons forces proportionality

We first state the result for two additive readings  $L, M : G \rightarrow \mathbb{R}$ . It is the mathematical core of the scaling argument.

**Theorem 6.1** (Proportional additive readings). *Suppose  $L(e) > 0$  for some  $e \in G$ , and*

$$L(x) > 0 \implies M(x) > 0 \quad (x \in G).$$

*Then*

$$M(x) = \frac{M(e)}{L(e)} L(x) \quad (x \in G). \quad (6.3)$$

*In particular, the proportionality factor is positive.*

*Proof.* The preservation hypothesis gives  $M(e) > 0$ . It is enough to prove

$$\frac{M(x)}{M(e)} = \frac{L(x)}{L(e)}$$

for every  $x$ .

Suppose first that the left-hand ratio is larger. Rational density provides  $q = p/n$ , with  $p \in \mathbb{Z}$  and  $n \in \mathbb{N}$ ,  $n > 0$ , strictly between the two:

$$\frac{L(x)}{L(e)} < \frac{p}{n} < \frac{M(x)}{M(e)}.$$

Form the actual group element

$$y = pe - nx.$$

Additivity gives

$$L(y) = pL(e) - nL(x) > 0, \quad M(y) = pM(e) - nM(x) < 0.$$

This contradicts preservation of positive comparisons.

If the left-hand ratio is smaller, choose a rational strictly between the ratios and use  $y = nx - pe$ . Once again  $L(y) > 0$  and  $M(y) < 0$ , a contradiction. Both strict inequalities are impossible, so the ratios coincide. Multiplying by  $M(e)$  proves (6.3).  $\square$

The proof uses rational density in the real codomain, but it does not complete  $G$  to a real vector space. The expressions  $pe$  and  $nx$  use only integer group operations. It assumes no continuity of  $L$  or  $M$ , and no scalar multiplication by arbitrary real numbers on the carrier.

This is why the result is stronger than checking that a proposed update looks approximately like multiplication along a few examples. A different normalized ratio can always be separated by a rational comparison, and that comparison reverses sign between the two readings. Preserving *all* positive comparisons rules out every such difference at once.

Apply the theorem to  $M = L \circ A$ . We obtain a number

$$\rho = \frac{L(Ae)}{L(e)} > 0 \tag{6.4}$$

such that

$$L(Ax) = \rho L(x) \quad (x \in G). \tag{6.5}$$

Using the two correspondence equations (6.1),

$$r(Ts) = \rho r(s) \quad (s \in S). \tag{6.6}$$

Thus the scaling law holds on the entire framework, not just on the chosen orbit from  $s_0$ .

Nothing in the argument makes  $A$  invertible or  $L$  injective. Uniform scaling of the observable value is not an assertion that every microscopic distinction has been recovered. Those stronger properties require their own hypotheses. The hierarchy construction needs the observable scaling just proved.

### 6.3 The first costly join

Let

$$s_n = r(T^n s_0)$$

be the positive orbit readings. Equation (6.6) gives

$$s_n = s_0 \rho^n. \quad (6.7)$$

This geometric form is an output of preserved additive comparisons. We have not assumed a geometric ladder.

Now suppose the first new level is produced by a binary join of earlier levels:

$$s_2 = s_a + s_b, \quad a \leq b < 2. \quad (6.8)$$

Require that this join have strictly positive recognition cost in the source's join-cost operation. This condition is called an initial costly join. It includes both the sum in (6.8) and the positive cost of that actual join.

Why is the cost condition needed? The index restrictions allow a join to duplicate one earlier level. Joining a level with itself has ratio one and cost  $J(1) = 0$ . A strictly positive-cost join therefore cannot have  $a = b$ . Since the only earlier indices are zero and one, the remaining possibility is

$$a = 0, \quad b = 1, \quad s_2 = s_1 + s_0. \quad (6.9)$$

This small exclusion is the point at which recognition matters. A binary sum of two slots does not guarantee that the slots represent distinct levels.

The source also gives a supplier for positive cost from its deformation-erasure principle applied to the actual rigid join kinematics. That result still takes the join equation and its kinematic premise explicitly. Bare closure of a state system does not supply an initial costly join. The hierarchy theorem should be read with the sum and the recognition condition attached, rather than as a result from the word *successor* alone.

**Theorem 6.2** (Recognizing-join hierarchy). *Suppose a closed observable framework has the additive reading (6.1), preserves all positive comparisons as in (6.2), and has an initial costly join on its positive orbit from  $s_0$ . Then*

$$r(Ts) = \phi r(s) \quad (s \in S), \quad \phi = \frac{1 + \sqrt{5}}{2}.$$

*The actual orbit levels form the realized hierarchy*

$$s_n = s_0 \phi^n.$$

*Proof.* Comparison preservation has already supplied a uniform  $\rho > 0$ . The initial costly join has supplied  $s_2 = s_1 + s_0$ . Substituting (6.7),

$$s_0 \rho^2 = s_0 \rho + s_0.$$

Since  $s_0 > 0$ , cancellation gives

$$\rho^2 = \rho + 1. \tag{6.10}$$

The two real roots are  $(1 + \sqrt{5})/2$  and  $(1 - \sqrt{5})/2$ . Positivity selects the first. Equation (6.6) therefore becomes multiplication by  $\phi$  on every state, and (6.7) gives the stated hierarchy.  $\square$

The construction uses these very orbit levels as the levels of the resulting hierarchy. Their positivity comes from the framework, their uniform ratio from additive comparison preservation, their first additive posting from the costly join, and their growth from  $\phi > 1$ . It does not request a previously constructed hierarchy as an input.

## 6.4 A census that makes the restriction visible

An abstract proof becomes more useful when the failed alternative can be seen in actual records. The census framework supplies such an example. Its additive extension is  $\mathbb{Z} \times \mathbb{Z}$ , with update and value

$$A(a, b) = (a + b, a), \tag{6.11}$$

$$L_\rho(a, b) = a\rho + b, \quad \rho > 0. \tag{6.12}$$

The underlying allowed censuses have nonnegative counts. Signed pairs are their additive comparison envelope.

This update is the one adopted in the source census construction. The correspondence theorem shows that its readings and successors match the actual ledger framework. It does not derive the update from an arbitrary unspecified physical system.

At the golden price,

$$\begin{aligned} L_\phi(A(a, b)) &= (a + b)\phi + a \\ &= a(\phi + 1) + b\phi \\ &= a\phi^2 + b\phi \\ &= \phi L_\phi(a, b). \end{aligned}$$

Thus every positive comparison stays positive. Conversely, preservation of all positive signed comparisons forces  $\rho = \phi$ . The equivalence holds in both directions.

Consider the tempting alternative  $\rho = 2$ . Its atom prices are positive, and the source's initial join is costly. Those two facts might look sufficient if we inspected only individual records. Yet the signed comparison  $(2, -3)$  has

$$L_2(2, -3) = 1, \quad L_2(A(2, -3)) = 0.$$

A positive distinction has been erased.

The same failure can be expressed without any negative census. Compare the allowed records  $(2, 0)$  and  $(0, 3)$ . Their readings before the update are 4 and 3. After the update they are

$$A(2, 0) = (2, 2), \quad A(0, 3) = (3, 0),$$

and both have reading 6. Two initially unequal records become equal in value. This is the concrete content of the failed signed comparison.

The counterexample separates the premises. Positive atom prices do not imply preservation of every comparison. A costly initial join does not repair that failure. The golden value is selected because it preserves the full additive comparison structure of this update.

There is also a simultaneous positive witness. The native golden framework has the additive reading, positive comparison preservation and the initial costly join together. The hierarchy constructor therefore applies to an exhibited source rather than to a collection of mutually incompatible assumptions.

## 6.5 Necessity and realization

The argument now has two complementary forms. For an arbitrary additive carrier with the displayed correspondence and recognition conditions, it forces golden scaling. For the adopted census update, it proves that preservation of all positive comparisons holds exactly at the golden price, and it exhibits a non-golden price that loses an actual comparison.

The first is a general conditional necessity result. The second is a realized instance and a concrete discrimination among prices. Neither is an empirical measurement of an unspecified material system. Applying the result to a physical process requires identifying its records, additive reading and successor, then establishing the preserved-comparison and costly-join conditions for those objects.

This is also the correct order for constructing the native apparatus. The hierarchy is obtained here without choosing a golden matrix for the comparator. A later construction can use the resulting  $\phi$ , show how its operation acts on the actual source pair, and transport the generic comparison cost through that action. Such a transport is legitimate because the source, residual and reference remain explicit. Replacing those correspondences with numerical resemblance would not establish the same result.

The continuing comparison has now acquired an evolution law for its additive readings. We know the normalized mismatch cost of the specified comparator and the unique scaling of a successor that preserves the stated comparisons while realizing its initial costly join. To locate this activity in space and to understand its completion time, we must next study what a retained linking record requires.



PART V

# Space, completion and measure



# Linking and three dimensions

A trace can return to its starting point while retaining an observable relation to something it has gone around. To make that statement useful, we must say what the trace avoids, which deformations preserve its reading, and how readings combine when paths are joined. Those requirements lead to a precise dimension theorem. For an embedded circle in a sphere, a nonzero additive reading of a closed complement path is possible only in three dimensions.

The proof has three steps. First, the observation rules make triangle boundaries read zero. This constructs an observation of first homology from an observation of paths. Second, complement topology excludes every dimension other than three. Third, a concrete pair of circles supplies a nonzero reading and connects it to the realized hierarchy. Each step uses the same path on which the reading is taken.

## 7.1 What the spatial premise contains

Let  $D$  be a nonnegative integer and let

$$f : S^1 \longrightarrow S^D$$

be a continuous topological embedding. An embedding identifies its domain with its image without changing the topology. The excluded trace is  $f(S^1)$ , and the available space for the second trace is its complement

$$X_f = S^D \setminus f(S^1). \quad (7.1)$$

A path in  $X_f$  cannot cross the excluded trace. A deformation within  $X_f$  cannot cross it either.

The sphere has dimension  $D$ . Writing it as the unit sphere in  $\mathbb{R}^{D+1}$  supplies coordinates; it does not add another intrinsic spatial dimension. The theorem below concerns this ambient class. Applying it to a physical carrier requires identifying the retained traces and their allowed motions with these objects. That identification belongs with the premise.

Write  $I = [0, 1]$ . A path observation assigns a real number  $P(\gamma)$  to every continuous map  $\gamma : I \rightarrow X_f$  subject to two conditions:

$$P(\gamma) = P(\delta) \quad \text{for endpoint-homotopic paths,} \quad (7.2)$$

$$P(p * q) = P(p) + P(q) \quad \text{for composable paths.} \quad (7.3)$$

Paths are composable when the end of  $p$  is the start of  $q$ . Here  $p * q$  traverses  $p$  during the first half of the interval and  $q$  during the second. Endpoint-homotopic paths are related by a continuous deformation during which their two endpoints stay fixed.

These conditions specify the changes that the observation ignores and the way successive readings combine. They apply to open paths too. This matters because the individual edges of a triangle are usually open. A label assigned only to closed traces, even one invariant under their deformations, has not yet supplied the second condition.

Finally, suppose there is a path satisfying

$$\gamma(1) = \gamma(0), \quad P(\gamma) \neq 0. \quad (7.4)$$

This is the retained distinction used by the argument. The trace returns to its starting point, but its reading is nonzero. Neither the observation rules nor this condition names an ambient dimension.

## 7.2 From an observed path to a homology class

The bridge to topology can be constructed directly. A singular one-simplex is a continuous path, with the interval written in simplex coordinates. A singular two-simplex is a continuous map from a filled triangle into the space. These maps need not be injective. Allowing all of them is what lets homology describe continuous spaces without choosing a particular mesh.

**Lemma 7.1** (Triangle cancellation). *Let  $F : \Delta^2 \rightarrow X_f$  be a singular two-simplex, and let  $Fp_{ij}$  denote its oriented edge from vertex  $i$  to vertex  $j$ . Then*

$$P(Fp_{12}) - P(Fp_{02}) + P(Fp_{01}) = 0. \quad (7.5)$$

*Proof.* Within the filled triangle, the broken path  $p_{01} * p_{12}$  and the direct path  $p_{02}$  have the same endpoints and are homotopic with those endpoints fixed. Composing this homotopy with  $F$  gives such a homotopy in  $X_f$ . The observation rules therefore give

$$P(Fp_{01}) + P(Fp_{12}) = P(Fp_{01} * Fp_{12}) = P(Fp_{02}).$$

Rearranging proves the stated alternating sum.  $\square$

Let  $C_n(X_f; \mathbb{Z})$  be the group of finite integer combinations of singular  $n$ -simplices. Its elements are called chains. The boundary of an oriented edge is its final point minus its initial point. The boundary of a triangle is its alternating edge sum:

$$\partial_2 F = F p_{12} - F p_{02} + F p_{01}.$$

The boundary of a boundary is zero. Thus the subgroup of one-boundaries  $B_1 = \text{im } \partial_2$  lies in the subgroup of one-cycles  $Z_1 = \ker \partial_1$ . First homology is the quotient

$$H_1(X_f; \mathbb{Z}) = Z_1 / B_1. \quad (7.6)$$

Two cycles represent the same class when their difference bounds a finite two-chain.

Because the one-chain group is free on its singular edges, the path reading has a unique integer-linear extension:

$$P_1 \left( \sum_{j=1}^m n_j \sigma_j \right) = \sum_{j=1}^m n_j P(\sigma_j), \quad n_j \in \mathbb{Z}. \quad (7.7)$$

The notation  $P(\sigma_j)$  uses the interval path corresponding to that simplex. The triangle lemma proves  $P_1(\partial_2 F) = 0$  for every generator of the two-chain group. Linearity then proves it for every two-chain.

**Proposition 7.2** (The derived homology observation). *The formula*

$$\bar{P}([z]) = P_1(z) \quad (7.8)$$

*defines a homomorphism  $\bar{P} : H_1(X_f; \mathbb{Z}) \rightarrow \mathbb{R}$ . For the class of each closed path  $\gamma$ , it satisfies*

$$\bar{P}([\gamma]) = P(\gamma). \quad (7.9)$$

*Proof.* If  $[z] = [z']$ , there is a two-chain  $w$  with  $z - z' = \partial_2 w$ . Hence

$$P_1(z) - P_1(z') = P_1(\partial_2 w) = 0.$$

The value is independent of the chosen representative, and additivity follows from the additivity of  $P_1$ . A closed path has equal final and initial points, so its one-simplex has zero boundary and defines a cycle. On that single simplex the linear extension has exactly the original path value.  $\square$

If  $P(\gamma) \neq 0$ , equation (7.9) shows that  $[\gamma] \neq 0$ . In particular,

$$H_1(X_f; \mathbb{Z}) \neq 0. \quad (7.10)$$

The homology map was obtained from the observation rules. No homology factorization was assumed, and the selected path was not exchanged for another representative with an unrelated physical reading.

### 7.3 How complement topology selects the dimension

We now need a theorem about the space in (7.1).

**Theorem 7.3** (Circle-complement dimension). *If a topological embedding  $f : S^1 \rightarrow S^D$  has*

$$H_1(S^D \setminus f(S^1); \mathbb{Z}) \neq 0,$$

*then  $D = 3$ . Such an embedding exists in dimension three.*

The proof for arbitrary topological embeddings cannot rely on straightening the circle in a smooth drawing. Its central ingredient is that the complement of an embedded arc has zero first homology. Appendix A proves that ingredient by bisection and compact support, and gives the full dimension exclusion. Here is the mechanism.

Cut the embedded circle into two closed arcs  $K_+$  and  $K_-$ . Their intersection consists of the two endpoint images  $p$  and  $q$ . Set

$$U = S^D \setminus K_+, \quad V = S^D \setminus K_-.$$

These sets are open, and

$$\begin{aligned} U \cap V &= S^D \setminus f(S^1), \\ U \cup V &= S^D \setminus \{p, q\}. \end{aligned}$$

The relevant part of the Mayer–Vietoris sequence for this open cover is

$$H_2(U \cup V; \mathbb{Z}) \longrightarrow H_1(U \cap V; \mathbb{Z}) \longrightarrow H_1(U; \mathbb{Z}) \oplus H_1(V; \mathbb{Z}). \quad (7.11)$$

Exactness says that the kernel of the second arrow is the image of the first. The arc result makes the right-hand group zero. For  $D \geq 2$ , removing two points from  $S^D$  leaves a space homotopy equivalent to  $S^{D-1}$ . Its second homology is zero unless  $D - 1 = 2$ . When  $D \neq 3$ , the left-hand group in (7.11) is therefore zero as well. Exactness then forces the middle group to vanish.

Dimensions zero and one have direct exclusions. The zero-sphere is a two-point space. In dimension one a continuous injection of the circle into itself is surjective, leaving an empty complement. Their first homology cannot carry the detected class. The appendix supplies the surjectivity argument rather than assuming a drawing of a circle proves it.

This explains the particular number three. The observed path supplies a class in degree one. Cutting the excluded circle makes its obstruction come from degree two of a two-point complement. That complement has the homotopy type of a sphere one dimension lower than the ambient sphere. The required degree matches only when the ambient dimension is three.

## 7.4 Constructing the observation in three dimensions

An exclusion theorem needs a realization of its premises. Use the unit three-sphere in  $\mathbb{R}^4$ , and define

$$f(z_0, z_1) = (z_0, z_1, 0, 0), \quad (7.12)$$

$$c(w_0, w_1) = (0, 0, w_0, w_1). \quad (7.13)$$

Both input pairs lie on the unit circle. The first map is an embedding, and the second has image in its complement. If  $y$  is in that complement, its last two coordinates cannot both be zero: otherwise its first two coordinates would have unit norm and place it on  $f(S^1)$ . Consequently

$$r(y) = \frac{(y_2, y_3)}{\sqrt{y_2^2 + y_3^2}} \quad (7.14)$$

is a well-defined continuous map to  $S^1$ . It satisfies  $r(c(w)) = w$ .

To read paths, use the angular covering

$$q : \mathbb{R} \longrightarrow S^1, \quad q(\theta) = (\cos \theta, \sin \theta).$$

Every continuous path  $\eta : I \rightarrow S^1$  has a continuous real lift  $\tilde{\eta}$  with  $q \circ \tilde{\eta} = \eta$ , once an initial lift value is chosen. Define its angular displacement by

$$d(\eta) = \tilde{\eta}(1) - \tilde{\eta}(0). \quad (7.15)$$

The choice of lift does not affect this difference. Indeed, two initial lift values differ by  $2\pi k$  for an integer  $k$ . Shifting one entire lift by that amount gives a

lift with the same initial value as the other. Uniqueness of path lifting makes them equal throughout the interval, so their endpoint differences agree.

The displacement has precisely the two required properties. For composable paths, choose the second lift to start where the first ends. The endpoint differences add:

$$(\tilde{p}(1) - \tilde{p}(0)) + (\tilde{q}(1) - \tilde{p}(1)) = \tilde{q}(1) - \tilde{p}(0).$$

This is the displacement of the concatenation. For a homotopy with fixed endpoints, lift all paths from the same initial value. The homotopy-lifting property makes their final lift values agree. Their displacements are therefore equal as well.

Pull this observation back to the complement:

$$P(\gamma) = d(r \circ \gamma). \quad (7.16)$$

Composition with  $r$  preserves endpoint homotopies and concatenation. Thus  $P$  is an observation on all complement paths with both required properties. No path in this construction is allowed to pass through the first circle.

For the once-around path  $\eta(t) = q(2\pi t)$ , the function  $\tilde{\eta}(t) = 2\pi t$  is an explicit lift. Hence

$$d(\eta) = 2\pi, \quad P(c \circ \eta) = 2\pi \neq 0.$$

The path is closed because its endpoints differ by one full angular turn. This proves existence in Theorem 7.3 by the path-to-homology construction itself.

More generally, a closed circle path has lift endpoints in the same fiber of  $q$ , so

$$d(\eta) = 2\pi k \quad \text{for some } k \in \mathbb{Z}. \quad (7.17)$$

This integer is its winding number. The integer comes from the fibers of the covering map; it is not a separately assigned label.

## 7.5 The same trace in the realized hierarchy

The hierarchy's recognition and credit loops can be placed in this construction. Keep the excluded debit circle  $f(S^1)$  fixed, and place the credit loop along  $c(S^1)$ . Retraction by  $r$  recovers the original credit loop exactly. Its established winding is  $-1$ , so the pulled-back observation has value

$$P(\gamma_{\text{credit}}) = -2\pi \neq 0. \quad (7.18)$$

The sign records orientation. Nonvanishing is the property required by the dimension theorem.

This supplies the embedding, the closed complement trace and its nonzero additive observation for the same realized hierarchy. It demonstrates that the hierarchy and spatial requirements can hold together. The arbitrary-dimension argument remains independent of choosing this three-dimensional witness: it first proves what any realization of the stated spatial requirements must satisfy.

## 7.6 The resulting forcing statement

**Theorem 7.4** (Dimension from an additive path observation). *Let  $f : S^1 \rightarrow S^D$  be a topological embedding. Suppose  $P$  is a real-valued observation on every continuous path in its complement, invariant under homotopies with endpoints fixed and additive under concatenation. If one closed complement path has nonzero  $P$ -value, then  $D = 3$ .*

*Proof.* Proposition 7.2 constructs the homology observation with the same value on the selected closed path. Its nonzero reading gives a nonzero homology class. Theorem 7.3 therefore gives  $D = 3$ .  $\square$

The physical content is specific. The retained record must be represented by an embedded closed trace; the second trace and its allowed deformations must stay in the complement; and the physical observation must obey the two path rules while reading some closed trace nontrivially. Once those requirements hold, another ambient dimension cannot preserve them.

Conservation alone does not supply the nonzero reading. A conserved total may vanish both on a paired configuration and on a separated one. The spatial argument needs an observation that distinguishes them. Nor does an invariant defined only on closed traces automatically supply additivity on open paths. The angular construction supplies both properties explicitly for the realized pair.

The sphere model and the path observation remain visible in the theorem's premises. The result does not classify arbitrary ambient manifolds. Its exact conclusion is necessity of three dimensions for this retained linking structure, together with a realization compatible with the hierarchy. The next foundational step concerns completion: which occupation and process requirements turn the available distinctions into a least complete pass.



# One apparatus for the connected argument

The cost, hierarchy and spatial arguments impose conditions on operations and records. To connect them physically, those conditions must coexist in one construction. We must be able to identify the census in the comparison signal, perform its join, measure the resulting change and transport the retained signal around the spatial record.

This chapter constructs those maps and proves their compatibility. It also proves a converse: starting with an arbitrary positive census price, two specified atom images under one operation force the golden value. The construction then makes the measured comparison residual the relative change of that operation. The function reporting the result remains arbitrary until literal reporting and continuity determine it.

There are two distinct accomplishments here. The selection proof excludes other prices under its stated operational conditions. The construction shows those conditions can hold together. Identifying an actual prepared source with this complete apparatus requires the physical operations and records specified below.

## 8.1 The same comparison on an even register

Let  $N > 1$  be even and index a signal by  $\mathbb{Z}/N\mathbb{Z}$ . Write  $Sf(j) = f(j + 1)$  for the cyclic shift. With  $e_j$  the unit vector at address  $j$ , define

$$\ell(a) = \frac{a}{\sqrt{2}}(e_0 - e_1), \quad (8.1)$$

$$C_N(a, b) = \ell(a) - S^{N/2}\ell(b). \quad (8.2)$$

This is the adjacent input pulse and half-register subtraction already used by the comparison apparatus. Its Fourier coefficients will expose how the census is carried.

Use the unnormalized transform

$$\hat{f}(k) = \sum_j f(j)\chi(jk), \quad \chi(k) = e^{2\pi i k/N}.$$

The shift has eigenvalue  $r_k = \chi(-k)$ . Put  $d_k = (1 - \chi(k))/\sqrt{2}$ . Direct evaluation at the two nonzero addresses gives  $\widehat{\ell(a)}(k) = d_k a$ . Iterating the shift multiplies a coefficient by  $r_k^n$ , so

$$\widehat{C_N(a, b)}(k) = d_k(a - r_k^{N/2}b). \quad (8.3)$$

Since  $(r_k^{N/2})^2 = r_k^N = 1$ , the half-turn character is either 1 or  $-1$ . Define

$$K = \{k : r_k^{N/2} = -1\}, \quad \widehat{P}f(k) = \mathbf{1}_K(k)\hat{f}(k).$$

The retained sector sees  $a + b$ ; its complement sees  $a - b$ . The primitive mode  $k = 1$  belongs to  $K$ , and  $d_1 \neq 0$ :  $\chi(1) \neq 1$  when  $N > 1$ .

At  $N = 8$ ,  $K$  is exactly the odd-frequency sector of the original comparator. Its projector is  $P = (I - S^4)/2$ . The identification between the cyclic eight-address carrier and the eight-entry vector carrier is the literal address map. It introduces neither a Fourier normalization nor an additional scaling of the operands.

## 8.2 Encoding both census coordinates

The census join is

$$A(u, v) = (u + v, u).$$

Its two coordinates count categories of records; they are not automatically the comparator's two amplitudes. For a real parameter  $\rho$ , set

$$p_\rho(u, v) = \rho u + v, \quad (8.4)$$

$$s_\rho(u, v) = u - \rho v, \quad (8.5)$$

$$E_\rho(u, v) = C_N(p_\rho + s_\rho, p_\rho - s_\rho). \quad (8.6)$$

The first coordinate is the additive census reading. The second retains the independent information needed to describe the full census.

**Proposition 8.1** (The two records). *On the eight-entry carrier,*

$$PE_\rho(x) = C_8(p_\rho(x), p_\rho(x)), \quad (8.7)$$

$$(I - P)E_\rho(x) = C_8(s_\rho(x), -s_\rho(x)), \quad (8.8)$$

$$\|E_\rho(u, v)\|^2 = 2(\rho^2 + 1)(u^2 + v^2). \quad (8.9)$$

The census coordinates can be recovered from its price and side readings:

$$u = \frac{\rho p_\rho + s_\rho}{\rho^2 + 1}, \quad v = \frac{p_\rho - \rho s_\rho}{\rho^2 + 1}. \quad (8.10)$$

*Proof.* The half-turn decomposition replaces the two operands by their common mean in the retained component and their opposite half-difference in the complement. In (8.6) these are precisely  $p_\rho$  and  $s_\rho$ . The explicit eight-entry pulse has norm  $\|C_8(a, b)\|^2 = |a|^2 + |b|^2$ . Consequently

$$\|E_\rho(x)\|^2 = (p_\rho + s_\rho)^2 + (p_\rho - s_\rho)^2 = 2(p_\rho^2 + s_\rho^2).$$

Expanding the last two squares cancels their mixed terms and gives  $p_\rho^2 + s_\rho^2 = (\rho^2 + 1)(u^2 + v^2)$ . Finally,  $\rho p_\rho + s_\rho = (\rho^2 + 1)u$  and  $p_\rho - \rho s_\rho = (\rho^2 + 1)v$ . The common denominator is positive.  $\square$

The two independent census atoms therefore have equal energy scales. Their sum has the sum of their squared norms, as follows by substituting  $(1, 0)$ ,  $(0, 1)$  and  $(1, 1)$  into the norm identity. No direction-dependent energy metric has been chosen to privilege the price.

For any conserved energy reading  $\mathcal{E}(f) = c\|f\|^2$ ,  $c > 0$ , the actual retained record gives

$$\frac{\mathcal{E}(PE_\rho(u, v))}{\mathcal{E}(PE_\rho(0, 1))} = p_\rho(u, v)^2. \quad (8.11)$$

Indeed, the numerator is  $2cp_\rho^2$ , and the denominator is  $2c$ . When the price is nonnegative, its positive square root is the additive reading itself. This identifies the physical reading as an amplitude obtained from an energy ratio, rather than equating price with the total conserved energy.

### 8.3 Two atom images force the price

The following selection argument starts before choosing  $\rho$ . Its decisive inputs are one linear operation and its action on the two encoded census atoms.

**Theorem 8.2** (Formation selection). *Let  $N > 1$  be even and  $\rho > 0$ . Suppose a complex linear operation  $T$  commutes with the cyclic shift and, for a common nonzero real gain  $g$ , satisfies*

$$TE_\rho(0, 1) = gE_\rho(A(0, 1)), \quad (8.12)$$

$$TE_\rho(1, 0) = gE_\rho(A(1, 0)). \quad (8.13)$$

Then  $\rho = \varphi$ .

*Proof.* The cyclic shift has distinct Fourier eigenvalues. A commuting linear map preserves each of its one-dimensional eigenspaces, so for some primitive response  $\beta$ ,  $\widehat{Tf}(1) = \beta\hat{f}(1)$ . Equation (8.3) gives

$$\widehat{E_\rho(x)}(1) = 2d_1 p_\rho(x).$$

The base atom has price 1, and its joined image has price  $\rho$ . The other atom has price  $\rho$ , and its joined image has price  $\rho + 1$ . Taking primitive coefficients in (8.13) and cancelling  $2d_1 \neq 0$  yields

$$\beta = g\rho, \quad \beta\rho = g(\rho + 1).$$

Eliminate  $\beta$  and cancel  $g \neq 0$ . Thus  $\rho^2 = \rho + 1$ . Its positive root is  $\varphi$ .  $\square$

Conservation on the primitive mode can supply the nonzero gain. If  $T$  preserves that mode's squared norm, its scalar has  $|\beta| = 1$ . The base-atom identity  $\beta = g\rho$  then excludes  $g = 0$ . The selector also holds with commutation restricted to the retained update, provided the primitive mode is retained. Full shift commutation is the convenient version used here.

These conditions concern the same operation and the same encoded atoms. The conclusion would not follow from allowing a separate map or a separate gain for each atom. Nor does a conserved total energy alone impose the two image equations.

## 8.4 Constructing the selected formation

At the selected value, the join acts independently on the two coordinates:

$$p_\varphi(Ax) = \varphi p_\varphi(x), \quad s_\varphi(Ax) = -\varphi^{-1} s_\varphi(x). \quad (8.14)$$

For the first identity, expand  $\varphi(u + v) + u$  and use  $\varphi^2 = \varphi + 1$ . For the second, expand  $u + v - \varphi u$  and use  $1 - \varphi = -\varphi^{-1}$ .

Define a spectral operator

$$\widehat{Gf}(k) = \begin{cases} \varphi\hat{f}(k), & k \in K, \\ -\varphi^{-1}\hat{f}(k), & k \notin K. \end{cases}$$

In the retained modes,  $E_\varphi(x)$  has coefficient  $2d_k p_\varphi(x)$ ; in the other modes it has coefficient  $2d_k s_\varphi(x)$ . Equation (8.14) therefore proves, at every mode,

$$GE_\varphi(x) = E_\varphi(Ax).$$

Equality of all Fourier coefficients is equality of the complete signals.

Normalize by the advancing hierarchy unit and write  $B = \varphi^{-1}G$ . The resulting operator has the exact form

$$Bf = Pf - \varphi^{-2}(I - P)f, \quad (8.15)$$

$$BE_\varphi(x) = \varphi^{-1}E_\varphi(Ax). \quad (8.16)$$

It fixes the retained channel and contracts the complementary amplitude by  $-\varphi^{-2}$ . Both  $G$  and  $B$  commute with the shift, because their Fourier multipliers commute with its multipliers. The construction supplies all the atom images used by the selector.

The full signal and its energy must still be accounted for. In particular,  $B$  is not a norm isometry on the whole uncommitted register. The next construction supplies a concrete second output for the encoded census.

## 8.5 Where the transferred energy is recorded

Let

$$t(u, v) = \left( \frac{u + v}{\varphi}, \frac{u}{\varphi} \right), \quad r(u, v) = \frac{u - \varphi v}{\varphi \sqrt{\varphi}}. \quad (8.17)$$

The first output is the normalized census join. The second is an explicit signed amplitude, whose square measures the energy transferred out of the principal coordinates.

**Proposition 8.3** (Complete census energy). *For every  $x = (u, v)$ ,*

$$\|t(x)\|_{\mathbb{R}^2}^2 + r(x)^2 = \|x\|_{\mathbb{R}^2}^2. \quad (8.18)$$

*For every real encoding parameter  $\rho$  and the same conserved signal-energy reading,*

$$\mathcal{E}(E_\rho(t(x))) + \mathcal{E}(E_\rho(r(x), 0)) = \mathcal{E}(E_\rho(x)). \quad (8.19)$$

*Proof.* Expansion, with  $\varphi^2 - \varphi - 1 = 0$ , gives

$$\varphi[\varphi^2(u^2 + v^2) - ((u + v)^2 + u^2)] = (u - \varphi v)^2.$$

Divide by  $\varphi^3 > 0$ . The first two terms become  $\|x\|^2 - \|t(x)\|^2$ , and the right side is  $r(x)^2$ . This proves (8.18). The code norm (8.9) multiplies each coordinate energy by the same  $2c(\rho^2 + 1)$ , proving (8.19).  $\square$

Thus both outputs can use the same comparison encoder. No missing-energy number has been appended without a corresponding signal. At  $\rho = \varphi$ , the principal output is exactly  $BE_\varphi(x)$ , by linearity and (8.16).

The principal coordinate map also has the explicit inverse

$$(y_1, y_2) \mapsto (\varphi y_2, \varphi(y_1 - y_2)).$$

Substitution of  $y = t(x)$  recovers  $x$ . Contraction of its energy therefore does not erase the distinction between its inputs. The additional output accounts for energy, rather than repairing a failure of injectivity.

Equation (8.19) holds for every  $\rho$ . It consequently cannot select the golden price on its own. Selection came from the specified retained reading and atom transitions. Physically preparing these two outputs and controlling their interactions remains a separate requirement from proving their exact algebraic energy balance.

## 8.6 Reading the primitive response from a real operation

The operation need not be introduced as a complex device. Begin with a real linear map  $L : \mathbb{R}^N \rightarrow \mathbb{R}^N$  that commutes with the real cyclic shift. Represent a primitive real signal by

$$e(z)_j = \operatorname{Re}(z\chi(-j)), \quad z \in \mathbb{C}.$$

Define its response from two actual input readings at address zero:

$$\beta = (Le(1))_0 - i(Le(i))_0. \quad (8.20)$$

**Proposition 8.4** (Derived scalar action). *For every  $z$ ,*

$$Le(z) = e(\beta z).$$

*If  $L$  preserves the sum-of-squares energies of  $e(1)$  and  $e(i)$ , then  $|\beta|^2 = 1$ .*

*Proof.* Real linearity gives  $e(z) = \operatorname{Re}(z)e(1) + \operatorname{Im}(z)e(i)$ , and hence  $(Le(z))_0 = \operatorname{Re}(\beta z)$ . Shift commutation transports this address-zero response to every address:

$$(Le(z))_j = (LS^j e(z))_0 = \operatorname{Re}(\beta r_1^j z) = e(\beta z)_j.$$

For each address, the squares of the real and imaginary quadratures sum to  $|z|^2$ , because  $|\chi(-j)| = 1$ . Summing gives

$$\|e(z)\|^2 + \|e(iz)\|^2 = N|z|^2.$$

Apply this identity at  $z = \beta$  and at  $z = 1$ . The two energy preservation assumptions equate the sums, so cancellation of  $N > 0$  gives  $|\beta|^2 = 1$ .  $\square$

This proves preservation of the primitive real plane; it is not an extra plane-preservation assumption. Extending  $L$  complex linearly now provides the operator used in the formation selector. If its two real census atoms have the common gain  $g$ , the selector gives

$$\rho = \varphi, \quad \beta = g\varphi, \quad |\beta| = 1.$$

Since  $g$  is real,  $\beta$  is real, and therefore

$$\beta = 1 \quad \text{or} \quad \beta = -1. \quad (8.21)$$

Both branches belong to the result. No assumption of positive formation gain has been inserted to remove the second one.

## 8.7 The two atoms determine every native input

The relative-change argument does not require  $L$  to agree with  $B$  on every vector in the register. Its two atom images suffice on the inputs that the comparator actually receives.

Given real operands  $a, b$ , choose

$$p = (a + b)/2, \quad s = (a - b)/2,$$

and recover  $x$  from (8.10). Substitution gives  $p_\rho(x) = p$ ,  $s_\rho(x) = s$ , and consequently

$$E_\rho(x) = C_N(a, b). \quad (8.22)$$

Moreover, linearity of the encoder gives

$$E_\rho(u, v) = uE_\rho(1, 0) + vE_\rho(0, 1).$$

Two linear maps agreeing on these atoms thus agree on every encoded census, including every real native comparison.

Let  $L_{\mathbb{C}}$  denote the complex extension of the real operation. Under the preceding selection conditions, its atom images agree with those of  $gG = \beta B$ . Therefore

$$L_{\mathbb{C}}C_N(a, b) = \beta BC_N(a, b).$$

At  $N = 8$ , define the relative residual from the measured primitive response:

$$T_{\text{rel}} = \beta I - L_{\mathbb{C}}.$$

Equation (8.15) gives

$$T_{\text{rel}}C_8(a, b) = \beta(1 + \varphi^{-2})(I - P)C_8(a, b). \quad (8.23)$$

For a real input this is literally the complex inclusion of  $\beta f - Lf$ , since  $\beta$  is real. Taking real parts has not discarded an unreported component.

There is a concrete reason to keep this conclusion on the native input space. Every native comparison has zero constant Fourier coefficient:  $d_0 = 0$ . Adding any multiple of the constant-mode projector to an operation leaves all native comparisons unchanged. Their responses cannot determine that extra response. Equation (8.23) asserts exactly the agreement that the records establish.

## 8.8 The reference uses the same relative change

Write  $s_0 = \beta(1 + \varphi^{-2})$ . It is nonzero by (8.21). Every residual energy on the native operand span is multiplied by  $|s_0|^2$  when  $I - P$  is replaced by  $T_{\text{rel}}$ . Every leaf of the finite reference program is a record from that same span. Induction over its additions and scalar weights therefore gives the same multiplier for the complete reference. The ratio meter cancels it.

A real protocol makes this cancellation particularly explicit. For the constructed formation  $B$ , let

$$D(a, b) = \mathcal{E}((I - B)C_8(a, b)).$$

It follows from (8.23), with  $\beta = 1$ , that

$$D(a, b) = \frac{c(1 + \varphi^{-2})^2}{2}(a - b)^2.$$

Choose the positive reference unit  $\kappa$  and form two real records:

$$R_{\kappa}(a, b) = \frac{\kappa}{4}[D(a, -b) - D(a, b)]. \quad (8.24)$$

The second operand is sign-reversed in the first record. Expanding the squares gives

$$R_{\kappa}(a, b) = \frac{\kappa c(1 + \varphi^{-2})^2}{2}ab.$$

It is positive when  $a, b > 0$ , and the meter reads

$$\frac{D(a, b)}{R_\kappa(a, b)} = \frac{(a - b)^2}{\kappa ab} = \frac{2}{\kappa} J(a/b). \quad (8.25)$$

This opposed-sign reference has exactly the same evaluated value as the quarter-phase reference program of the earlier comparison argument in the same unit. It supplies a real-input implementation of that reference value. The equality concerns actual records produced by the same residual; independently chosen denominator values would not justify the cancellation.

## 8.9 A real family on the whole spatial complement

The spatial witness must carry these operations away from the formation point. Let  $\Gamma$  be the original three-bit Gray polygon, embedded in a fixed chart of  $S^3$ . Appendix B constructs an ambient homeomorphism carrying  $\Gamma$  to the round unknot. Restriction gives

$$h : S^3 \setminus \Gamma \longrightarrow S^3 \setminus \text{unknot}.$$

It includes the compactification point. The source polygon has not been redefined as its straightened image.

Let  $r$  retract the round complement to its core circle, as in the spatial argument, and put  $z(x) = r(h(x))$ . We identify this circle with unit complex phases. On a register divisible by four define the signed integer charge

$$q(k) = \begin{cases} k, & k \in K, 0 \leq k < N/2, \\ k - N, & k \in K, N/2 < k < N, \\ 0, & k \notin K. \end{cases}$$

Here  $k$  denotes the representative in  $\{0, \dots, N-1\}$ . The half-turn sector consists of odd indices; zero and the half-register index are excluded when  $4 \mid N$ . Thus  $q(-k) = -q(k)$ , and  $q(k) \equiv k \pmod{N}$  on retained modes.

For a unit phase  $z$ , define

$$\widehat{U_z f}(k) = z^{q(k)} \hat{f}(k).$$

This map is continuous in  $z$  and commutes with the shift. Every multiplier has modulus one, so Parseval's identity gives  $\|U_z f\|^2 = \|f\|^2$  for every signal. Real signals have conjugate Fourier coefficients at opposite indices. The relation

$q(-k) = -q(k)$  preserves that relation, and hence  $U_z$  maps real signals to real signals throughout the continuous motion.

At the native phases  $z_n = e^{-2\pi i n/N}$ , the retained-mode multiplier is

$$z_n^{q(k)} = r_k^n.$$

Consequently  $U_{z_n} f = S^n f$  for every signal supported in  $K$ . The equality covers all retained components, not just the primitive one.

For  $N = 8$ , define the family on the Gray complement by

$$L_x f = U_{z(x)} B f \quad \text{on real inputs.} \quad (8.26)$$

Both factors preserve reality. The family is continuous at every input, commutes with the real shift, and preserves the energy of every primitive real signal:  $B$  fixes that plane and  $U_z$  is an isometry. At the basepoint  $x_0 = h^{-1}(\text{core}(1))$ ,  $z(x_0) = 1$ , so

$$L_{x_0} E_\varphi(x) = \varphi^{-1} E_\varphi(Ax).$$

This is the same census formation used above.

Let a supplied uniformly repeated source tick generate the closed phase path  $\alpha(t)$  around eight ticks, and choose the complementary path

$$\gamma(t) = h^{-1}(\text{core}(\alpha(t))).$$

Since  $r \circ \text{core}$  is the identity,  $z(\gamma(t)) = \alpha(t)$ . A committed comparison is  $f_c = \text{SPC}_8(a, b)$ ; it lies entirely in  $K$ , so  $B f_c = f_c$ . Equation (8.26) now proves

$$L_{\gamma(t)} f_c = U_{\alpha(t)} f_c. \quad (8.27)$$

At its native tick samples this is the literal cyclic shift of the whole committed comparison. For positive  $a, b$ , its primitive coefficient is nonzero. The phase read from the same operation therefore detects the closed trace's nonzero displacement. The allowed repeated-tick winding branches are retained; no particular interpolation branch is inferred from the sampled values alone.

The complementary path lies in the polygon's complement and is not the polygon itself. The construction supplies one continuous operation family with formation, real full-state clock transport and the required linking record. Its conserved committed channel and its contracting uncommitted channel remain distinct. A source evolution law, preparation protocol and resource account for repeated physical use are further physical suppliers.

## 8.10 An arbitrary report on the constructed apparatus

Fix  $\kappa > 0$ . Use the real basepoint operation  $L_{x_0}$ , its computed relative residual, the original operands, squared-norm energy, the finite reference program and the ratio meter. Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be arbitrary. Suppose it literally reports this apparatus for all positive native population pairs, and suppose it is continuous on positive inputs.

Equation (8.25) fixes  $F$  at every positive ratio of natural counts. Those ratios are dense in the positive real line. The apparatus value  $(2/\kappa)J$  is continuous there. For any  $x > 0$ , take positive rational ratios converging to  $x$ . The two functions agree on the sequence; continuity gives equality at its limit. Thus

$$\frac{\kappa}{2}F(x) = J(x), \quad x > 0. \quad (8.28)$$

At  $\kappa = 2$ ,  $F = J$ . Balanced zero, reciprocity and the required logarithmic order follow for this report. They were not extra shape assumptions imposed on  $F$ .

The same selected price supplies the original census hierarchy, and the same global formation family supplies the spatial witness just constructed. The uniqueness argument and the existence argument therefore meet on explicit objects and maps. The apparatus does not choose a curve first and rename it as a measured cost.

The eight-entry carrier in this witness is specified. The separate completion argument determines the least complete binary pass from its posting and coverage conditions. Likewise, this construction realizes the spatial conditions at dimension three; the necessity of that dimension comes from the retained linking argument. These are the precise points at which the connected foundation uses the preceding results.



# Completion and the Clock

A complete operation must include everything its completion criterion requires. If a spatial item has eight distinguishable positions, a procedure that has visited only seven has not completed that item. This gives a lower bound on the operation. To turn the bound into a clock, we must also construct a procedure that attains it and determine what repeats when the procedure continues.

There are three quantities to keep in view. A complete pass has a number of positions or acts. A process has a period if its observations repeat after a fixed number of steps. An elapsed duration counts whatever the chosen clock counts, including intervals in which the observed pattern does not change. The argument below connects these quantities under stated conditions. Each connection has its own proof.

## 9.1 The positions in one item

We begin with the integer-coordinate realization of a binary spatial floor. Write a position as  $x = (x_1, \dots, x_d) \in \mathbb{Z}^d$ . Integer division gives

$$x_i = 2a_i + p_i, \quad a_i \in \mathbb{Z}, \quad p_i \in \{0, 1\}. \quad (9.1)$$

Here  $a_i = \lfloor x_i/2 \rfloor$  and  $p_i = x_i - 2a_i$ . This convention applies also to negative coordinates: the remainder is always zero or one. Let

$$Q_d = \{0, 1\}^d, \quad N = 2^d.$$

The vector  $a$  is the address of the larger item;  $p$  specifies a position within it. The maps

$$x \mapsto (p, a), \quad (p, a) \mapsto 2a + p$$

are inverse. Indeed, division reconstructs every coordinate of  $x$ , and division of  $2a_i + p_i$  recovers  $a_i$  and  $p_i$  because  $0 \leq p_i < 2$ . Thus, at every fixed address, the positions are in bijection with  $Q_d$ . There are exactly  $N$  of them.

This is a statement about the specified realization. The spatial dimension determines  $d$ ; the integer coordinates and binary grouping specify how its positions are read. The dimension theorem alone does not select an integer lattice, a lattice spacing or this grouping. Those physical identifications belong with the realization whenever it is used.

Consider a finite pass  $q$  with  $L$  presented patterns,

$$q : \{0, \dots, L-1\} \rightarrow Q_d.$$

At a fixed address  $a$ , say that the pass *occupies the item* if, for every position  $x$  with address  $a$ , some presentation equals its pattern  $p(x)$ . This describes occupation of the full item, rather than a visit to one chosen part of it.

**Proposition 9.1** (Occupation is complete coverage). *A pass occupies the item at address  $a$  if and only if  $q$  is onto  $Q_d$ . In particular the condition is independent of the address.*

*Proof.* Suppose the item is occupied. Given any  $p \in Q_d$ , form  $x = 2a + p$ . Its address is  $a$  and its pattern is  $p$ , so the occupation condition supplies an index  $t$  with  $q(t) = p$ . Hence  $q$  is onto.

Conversely, suppose  $q$  is onto. Any  $x$  at address  $a$  has some pattern  $p(x) \in Q_d$ . Surjectivity supplies  $t$  with  $q(t) = p(x)$ , which is precisely the required occupation. The resulting condition mentions only  $Q_d$ , so changing  $a$  does not change it.  $\square$

Coverage is the completion criterion used throughout this chapter. The criterion tells us when an operation has finished. It does not itself make a process move or ensure that every required position will be visited.

## 9.2 What makes a change elementary

The distance between two patterns is the number of coordinates in which they differ:

$$h(p, q) = \#\{i : p_i \neq q_i\}.$$

An intermediate pattern  $m$  subdivides the change from  $p$  to  $q$  if

$$m \neq p, \quad m \neq q, \quad h(p, m) + h(m, q) = h(p, q).$$

The last equality says that the subdivision introduces no detour. Call a nonzero change *finest* when it has no such intermediate pattern.

**Proposition 9.2** (Finest changes). *A change is finest if and only if it changes exactly one coordinate.*

*Proof.* If  $h(p, q) = 1$ , any subdivision would express one as the sum of two positive integers. That is impossible.

If  $h(p, q) \geq 2$ , choose a coordinate  $i$  in which  $p$  and  $q$  differ and change only that coordinate of  $p$  to its value in  $q$ . The resulting  $m$  differs from  $p$ , still differs from  $q$ , and satisfies

$$h(p, m) = 1, \quad h(m, q) = h(p, q) - 1.$$

It is an intermediate pattern. The only nonzero distance that admits no subdivision is therefore one.  $\square$

This proves what a finest tick means for the binary pattern reader. Applying it to a physical process requires that the reader resolves these distinctions and that the counted steps are finest at this resolution.

### 9.3 The least complete cyclic pass

A *cyclic pass* presents  $q(0), \dots, q(L-1)$  and then returns to  $q(0)$ . For a finest cyclic pass, consecutive patterns differ in one coordinate, including the closing change from  $q(L-1)$  to  $q(0)$ . A complete cyclic pass also covers  $Q_d$ .

**Theorem 9.3** (Minimum complete pass). *Every complete pass has  $L \geq N$ . Equality holds precisely when each pattern is presented once. For every  $d > 0$  there is a finest complete cyclic pass of length  $N$ . Consequently the least such length is  $2^d$ , and is eight when  $d = 3$ .*

*Proof.* Surjectivity from  $L$  positions onto a set of  $N$  elements gives  $L \geq N$ . If  $L = N$ , a repeated value would leave fewer than  $N$  different values, so the pass is injective. Conversely a complete injective pass is a bijection and has  $L = N$ .

For existence, construct the binary reflected Gray ordering recursively. Let  $G_0$  be the list containing the empty pattern. Given  $G_d$ , obtain  $G_{d+1}$  by appending a zero to each pattern in  $G_d$ , then appending a one to each pattern in the reversed list  $G_d$ . Each half lists all old patterns once; their new last coordinates differ. Thus the combined list contains every  $(d+1)$ -coordinate pattern exactly once.

For  $d = 1$  the list is 0, 1 and both cyclic changes affect one coordinate. In the inductive step, changes within a half affect one old coordinate by

the induction hypothesis; reversing the order does not change the distance. At the join, both patterns have the same old coordinates, namely the last pattern of  $G_d$ , and differ only in the appended coordinate. At cyclic closure, the old coordinates on both sides are the first pattern of  $G_d$ , and again only the appended coordinate changes. All required changes are therefore finest. The length doubles at each step, giving  $2^d$ .  $\square$

In three dimensions one such cyclic ordering, with a fixed convention for displaying the coordinates, is

$$\begin{aligned} &000, 001, 011, 010, \\ &110, 111, 101, 100. \end{aligned} \tag{9.2}$$

The final change  $100 \rightarrow 000$  closes the cycle. There are eight presentations and eight changes including closure. Counting the initial pattern and its returned copy as two different positions would instead give a list with nine entries; that list describes the same eight changes. We use the half-open convention that lists the starting presentation and omits its returned copy.

The lower bound and its construction do different work. The bound excludes shorter complete passes. The construction establishes that the bound can be attained while respecting finest changes. It supplies no unique order of visitation; rotations, reversals and coordinate relabellings already give other admissible presentations.

## 9.4 From a cyclic pass to a continuing process

A cyclic description can generate an infinite observation sequence by repetition:

$$w(n) = q(n \bmod L), \quad n \in \mathbb{N}.$$

It follows directly that  $L$  is a period. To say that  $L$  is the *least* period requires a further argument.

**Proposition 9.4** (The period of a least pass). *If  $q$  is a complete pass of length  $N$ , then its repeated process has least positive period  $N$ .*

*Proof.* The construction gives period  $N$ . Let  $p > 0$  be any period of  $w$ . Repeatedly subtracting  $p$  from an index shows that  $w(n) = w(n \bmod p)$ . Since  $w$  reaches every pattern, the first  $p$  presentations must also reach every pattern. Hence  $p \geq N$  by the coverage bound.  $\square$

A displayed window length is not necessarily a least period. Writing the eight-entry ordering twice gives a complete finest cyclic pass with sixteen entries. Repeating that longer pass generates exactly the original process, whose least period remains eight. Thus the word “cyclic” supplies a return at the chosen window length; it does not certify that the length is minimal.

## 9.5 Completion without assuming a cycle

We now start with an arbitrary process  $w : \mathbb{N} \rightarrow Q_d$ . No periodicity is assumed. Its window of length  $L$  from index  $n$  is

$$w(n), w(n+1), \dots, w(n+L-1).$$

Call it complete if it covers  $Q_d$ . Say the process *completes* if a complete finite window exists from every starting index. Tail completion is stronger than visiting everything once at the beginning and then becoming confined to a smaller set.

For a completing process, let  $\ell(n)$  be the least length of a complete window starting at  $n$ . The natural numbers have a least member in every nonempty set, so this quantity is defined. The coverage bound gives  $\ell(n) \geq N > 0$ . Define the successive completion boundaries by

$$c_0 = 0, \quad c_{k+1} = c_k + \ell(c_k). \quad (9.3)$$

These boundaries strictly increase, and  $c_k \geq Nk$  follows by induction. The completed windows are the half-open intervals  $[c_k, c_{k+1})$ .

**Proposition 9.5** (Least completed items). *If no pattern repeats inside any completed window  $[c_k, c_{k+1})$ , then every such window has length  $N$  and  $c_k = Nk$ .*

*Proof.* The window is onto by the definition of  $\ell$ , and is injective by the no-repeat hypothesis. It is therefore a bijection with  $Q_d$ , giving  $\ell(c_k) = N$ . Substitute this equality into (9.3) and induct on  $k$ .  $\square$

An equivalent convenient presentation specifies complete aligned blocks  $[Nk, N(k+1))$  for every  $k$ . Each block has exactly  $N$  entries, so coverage makes it bijective. Such a process completes from every index: after any starting point, a later aligned block supplies all patterns. At each aligned boundary, the least complete length is  $N$ , since a shorter window cannot cover  $Q_d$ . Its completion boundaries are therefore exactly  $Nk$ .

Completion by itself need not attain the bound. For example, the cyclic list

$$\begin{array}{c} 000, 001, 000, 001, 011, \\ 010, 110, 111, 101, 100 \end{array}$$

has finest changes, including its closing change, and reaches all eight patterns. The last new pattern first appears at index nine. The first complete window consequently has length ten. The revisits are exactly what prevents attainment of the lower bound.

Even least completed blocks need not give return of the observed pattern. In two dimensions, repeat the list

$$00, 01, 11, 10 \mid 11, 01, 00, 10. \quad (9.4)$$

Every change, including the two joins, affects one coordinate. Each aligned block of four visits all four patterns once. Thus  $c_k = 4k$ , yet  $w(4) = 11 \neq 00 = w(0)$ . The completion boundaries are regularly spaced while the pattern has not returned after one completed item.

## 9.6 When completion forces a period

The missing distinction concerns what the present pattern retains about the next step. Say that the process has a *successor determined by its current pattern* when

$$w(i) = w(j) \implies w(i+1) = w(j+1) \quad \text{for all } i, j \in \mathbb{N}. \quad (9.5)$$

This is a condition on the observed pattern, not just on a more detailed underlying state. A deterministic evolution of the underlying state can present the same pattern twice and then present different next patterns.

**Theorem 9.6** (Completion and an autonomous pattern successor). *Let  $w : \mathbb{N} \rightarrow Q_d$  complete from every starting index. The following are equivalent:*

1. *The current pattern determines the next pattern as in (9.5).*
2. *There is a map  $f : Q_d \rightarrow Q_d$  with  $f(w(n)) = w(n+1)$  for every  $n$ .*
3.  *$N$  is a period of  $w$ .*

*When these conditions hold, the least positive period is  $N$ , and the aligned complete blocks all have length  $N$ . In dimension three the least positive period is eight.*

*Proof.* Completion makes  $w$  onto. Under the first condition, define  $f(q)$  by choosing any occurrence  $w(n) = q$  and setting  $f(q) = w(n + 1)$ . The condition makes this independent of the occurrence chosen. This gives the second condition. Conversely, the equality  $f(w(n)) = w(n + 1)$  immediately gives the first.

The map  $f$  is onto. To find a predecessor of  $q$ , use completion from index one to choose an occurrence  $w(t + 1) = q$ . Then  $f(w(t)) = q$ . A surjection of a finite set to itself is a bijection. Its orbits are cycles, with no transient part. By induction,

$$f^n(w(0)) = w(n).$$

This orbit visits every pattern because  $w$  is onto. It is therefore a single cycle containing all  $N$  patterns, so  $N$  is a period and the first  $N$  entries are a bijection with  $Q_d$ .

For the reverse implication, suppose  $N$  is a period. Every pattern occurs somewhere, and reduction of its occurrence index modulo  $N$  puts it in the first  $N$  entries. Those entries are therefore bijective. If  $w(i) = w(j)$ , their residues modulo  $N$  must coincide. The residues of  $i + 1$  and  $j + 1$  then coincide too, giving  $w(i + 1) = w(j + 1)$ .

Finally, every positive period  $p$  presents all patterns within its first  $p$  entries, so  $p \geq N$ . Every aligned block of length  $N$  presents the whole cycle, giving the asserted completion length.  $\square$

The theorem does not assume finest changes or assume that complete items already have minimum length. Completion and the well-defined successor force the single cycle. If finest changes also hold, that cycle additionally respects the binary adjacency required for the spatial pass.

The example (9.4) pinpoints why the successor condition matters. The pattern 11 occurs at indices two and four. Its respective successors are 10 and 01. Its current pattern has discarded a distinction that affects the next observation.

## 9.7 A spatial supplier of the successor law

The successor condition has a concrete supplier on the integer-coordinate carrier already used to construct the item. In three dimensions, let  $p : \mathbb{Z}^3 \rightarrow Q_3$  be the coordinate-parity map. Two positions have the same pattern exactly when they differ by an even lattice vector:

$$p(x) = p(y) \iff y = x + 2v \text{ for some } v \in \mathbb{Z}^3.$$

For the forward direction, equality of the remainders in (9.1) gives  $y - x = 2(a(y) - a(x))$ . For the reverse direction, adding an even integer leaves every remainder unchanged.

Let  $T : \mathbb{Z}^3 \rightarrow \mathbb{Z}^3$  be a successor on positions. The relevant covariance condition is

$$T(x + 2v) = T(x) + 2v \quad (x, v \in \mathbb{Z}^3). \quad (9.6)$$

It says that the law acts the same way in every translate of the binary cell, with the whole resulting position translated along with the input.

**Proposition 9.7** (A spatial clock supplier). *Suppose  $T$  satisfies (9.6). Then parity has a well-defined successor  $S : Q_3 \rightarrow Q_3$  satisfying*

$$p(Tx) = S(p(x)).$$

*If the parity process  $w(n) = p(T^n x_0)$  completes from every starting index, its least positive period is eight.*

*Proof.* Read a binary pattern  $q$  as its representative in  $\{0, 1\}^3 \subset \mathbb{Z}^3$  and put  $S(q) = p(Tq)$ . Every  $x$  has the form  $x = q + 2a$  with  $q = p(x)$ . Covariance gives  $T(x) = T(q) + 2a$ , and parity removes the even translation. Thus  $p(Tx) = p(Tq) = S(p(x))$ .

Along the orbit, this identity becomes  $w(n + 1) = S(w(n))$ . Theorem 9.6 applies and gives least period eight.  $\square$

The hypotheses can hold together on this very carrier. Given an autonomous pattern successor  $S$ , define

$$T(x) = 2a(x) + S(p(x)). \quad (9.7)$$

Division shows directly that

$$a(Tx) = a(x), \quad p(Tx) = S(p(x)).$$

Also  $a(x + 2v) = a(x) + v$  and  $p(x + 2v) = p(x)$ , so this  $T$  satisfies (9.6). Induction gives its entire orbit:

$$T^n x = 2a(x) + S^n(p(x)).$$

If  $w$  is any completing process with an autonomous pattern successor, choose its successor  $S$  from Theorem 9.6 and set  $x_0 = w(0)$  at address zero. Then  $T^n x_0 = w(n)$  as an integer-coordinate position. This realizes the whole trace,

not only its period or number of states. In particular, taking  $S$  around the reflected Gray cycle realizes completion, finest changes and least period eight together. The full position orbit in this construction has the same least period: its address is fixed, and its pattern runs through the eight distinct positions.

The covariance required here must be stated exactly. If position is the entire state and one instead requires  $T(x + v) = T(x) + v$  for *every* integer vector  $v$ , putting  $x = 0$  gives

$$T(v) = v + T(0).$$

The law is then a constant displacement. Two iterations add  $2T(0)$ , which leaves parity unchanged. Its parity process has period two and cannot cover eight patterns. Full translation covariance on position alone therefore cannot supply the completing clock.

Equation (9.6) is a sufficient physical condition, and (9.7) constructs dynamics satisfying it. To apply the period theorem to an independently specified evolution, one must establish that this evolution respects the even translations and completes the item. The existence of the lift does not select that evolution.

## 9.8 Retaining exactly what future observations distinguish

There is a direct construction of a state space on which observation and evolution fit together. Let  $X$  be a carrier,  $F : X \rightarrow X$  its actual successor and  $\beta : X \rightarrow C$  its current readout. Define

$$x \sim y \iff \beta(F^n x) = \beta(F^n y) \text{ for every } n \in \mathbb{N}. \quad (9.8)$$

Two states are identified exactly when this readout cannot distinguish them now or after any number of successor steps. The observations used here are precisely the family  $\beta, \beta \circ F, \beta \circ F^2, \dots$ . It is the least family containing  $\beta$  and closed under composition with  $F$ : any such family contains each iterate by induction.

**Proposition 9.8** (The quotient retained by future observations). *Relation (9.8) is an equivalence relation. It preserves the current readout and is preserved by  $F$ . On its quotient  $\mathcal{Q} = X/\sim$  there are well-defined maps*

$$\bar{F}([x]) = [F(x)], \quad \bar{\beta}([x]) = \beta(x).$$

*The first is the unique successor commuting with projection from  $X$ . Moreover, every relation that preserves the readout and is preserved by  $F$  relates only future-equivalent states.*

*Proof.* Reflexivity, symmetry and transitivity hold separately for equality at each iterate and therefore for their conjunction. The case  $n = 0$  preserves the current readout. If  $x \sim y$ , the equalities at  $n + 1$  show  $F(x) \sim F(y)$ . These facts establish the two maps' independence of representatives. Every quotient element has a representative, so the required action on  $[x]$  also determines the successor uniquely.

Let  $R$  be a relation preserving readout and successor, and suppose  $R(x, y)$ . Induction gives  $R(F^n x, F^n y)$  for every  $n$ . Readout preservation then gives  $\beta(F^n x) = \beta(F^n y)$  for every  $n$ , which is  $x \sim y$ .  $\square$

Thus the quotient makes the fewest distinctions needed to retain this whole family of future observations. Its readout and successor satisfy

$$\bar{\beta}([x]) = \beta(x), \quad \bar{F}^n([x]) = [F^n x].$$

In particular, it supplies an exact connection between return and observed period:

$$\bar{F}^p([x]) = [x] \iff \beta(F^{n+p}x) = \beta(F^n x) \quad (9.9)$$

for every  $n \in \mathbb{N}$ .

To prove it, replace the left side by  $[F^p x] = [x]$  and apply the definition of future equivalence. Composition of iterates gives the right side.

For a process  $w$ , take  $X = \mathbb{N}$ ,  $F(n) = n + 1$  and  $\beta = w$ . Its quotient then identifies two indices precisely when their entire future observation sequences coincide. The current readout  $\bar{\beta}$  is injective exactly when the current pattern determines its successor. Indeed, injectivity turns equality of current patterns into equality of quotient states, whose successors also agree. Conversely, under (9.5), induction turns equality of current patterns into equality at all future steps.

If  $w$  completes, the quotient readout is onto  $Q_d$ . If the quotient is finite, it follows that

$$|\mathcal{Q}| = N \iff \text{the current pattern determines its successor.} \quad (9.10)$$

For a surjection between finite sets of the same size is injective, and an injective surjection is a bijection. Additional quotient states record distinctions with different future observations but the same present pattern.

## 9.9 A complete eight-step item with sixteen phases

Let  $g_0, \dots, g_7$  be the eight patterns in (9.2). Repeat the following list of indices into that order:

$$\begin{array}{c} 0, 1, 2, 3, 4, 5, 6, 7 \mid \\ 6, 5, 4, 3, 2, 1, 0, 7. \end{array} \quad (9.11)$$

Each block of eight contains every index once. Consecutive indices describe neighbors on the Gray cycle, including both block joins and cyclic closure. The resulting process has finest changes and completion boundaries  $c_k = 8k$ . But its pattern at index eight is  $g_6$ , rather than its initial  $g_0$ .

All sixteen ordered pairs consisting of a current pattern and its next pattern are distinct. To see this directly, the first block uses the forward edges  $0 \rightarrow 1$  through  $6 \rightarrow 7$ , then  $7 \rightarrow 6$ . The second block uses the backward edges  $6 \rightarrow 5$  through  $1 \rightarrow 0$ , then  $0 \rightarrow 7$  and  $7 \rightarrow 0$ . Together these are the sixteen distinct directed edges of the eight-vertex cycle.

Consequently any observation period must accommodate sixteen distinct current–next pairs within one period, and has length at least sixteen. Sixteen is a period by construction, so it is the least positive period. Equality of all future observations implies equality of the current–next pair and hence equality of phase modulo sixteen. Conversely equal phases give identical futures. The quotient retained by future observations thus has exactly sixteen states, although there are only eight possible current patterns.

This distinction persists in an autonomous hierarchy. Take the carrier  $n \in \mathbb{N}$ , successor  $n \mapsto n + 1$ , observable  $\phi^n$ , and the fixed pattern readout specified by (9.11). Since  $\phi > 1$ , the observable separates the carrier states. The readout is therefore a function of that observable, yet it merges states whose next patterns differ. Retaining the finite history does not change those different next patterns. Thus an autonomous carrier and a separating hierarchy observable do not, without a further condition on the pattern readout, supply (9.5).

## 9.10 The integer multiplicity that completion does force

Even when one completed item is not a period, least complete blocks constrain every period that does exist.

**Theorem 9.9** (Completion divides every period). *Suppose every aligned block  $[Nk, N(k+1))$  of  $w$  contains all  $N$  patterns. If  $p$  is a global period, then  $N$  divides*

*p*. Thus in three dimensions every positive period is  $8m$  for some positive integer  $m$ .

*Proof.* Fix one pattern  $q$  and let  $a(n)$  be one when  $w(n) = q$  and zero otherwise. Each aligned block has  $N$  entries and covers  $N$  patterns, so it contains  $q$  exactly once. Count occurrences of  $q$  in the first  $Np$  entries. Partitioning these entries into  $p$  blocks of length  $N$  gives

$$\sum_{n=0}^{Np-1} a(n) = p.$$

Partitioning the same interval into  $N$  blocks of length  $p$ , and using periodicity in each block, gives

$$\sum_{n=0}^{Np-1} a(n) = N \sum_{t=0}^{p-1} a(t).$$

Equating the two expressions proves  $p = N \sum_{t=0}^{p-1} a(t)$ , the required divisibility. When  $p > 0$ , the integer factor is positive.  $\square$

Neither finest changes nor an autonomous pattern successor is needed for this counting argument. It uses complete aligned blocks and a global period. It constrains the extra phase count without asserting that the extra count is one.

The same conclusion applies to the size of a finite future-observation quotient when its successor is onto. Every state of the process quotient is reached from  $[0]$ , because  $\bar{F}^n[0] = [n]$ . An onto successor on a finite set is a permutation. Since one orbit reaches every state, this permutation consists of a single cycle of length  $|\mathcal{Q}|$ . Returning after that many steps and applying (9.9) makes  $|\mathcal{Q}|$  a global observation period. Theorem 9.9 now gives

$$N \mid |\mathcal{Q}|.$$

Finiteness and an onto successor are explicit here; the latter excludes transient quotient states. Equation (9.10) separately characterizes when this multiple is exactly one.

## 9.11 Changes and elapsed intervals

A clock may also count intervals during which the pattern holds. Suppose each interval either leaves the pattern unchanged or changes one coordinate.

Over the  $L$  transitions starting at  $n$ , let  $P(n, L)$  count changes and  $H(n, L)$  count holds. Each transition belongs to exactly one class, so

$$P(n, L) + H(n, L) = L. \quad (9.12)$$

These transitions include  $w(n + L - 1) \rightarrow w(n + L)$ . The complete window itself contains only the  $L$  presentations through  $w(n + L - 1)$ . Keeping this convention explicit prevents an endpoint from being counted twice.

If the window covers  $N$  patterns, at least  $N - 1$  changes are needed: each pattern other than  $w(n)$  has a first arrival, and that arrival must be a change. Distinct new patterns have distinct arrivals. If  $d > 0$  and  $w(n + L) = w(n)$  as well, at least  $N$  changes are needed. The start pattern must then be re-entered after a different pattern has been visited, giving one more arrival. When every transition is a finest change,  $P(n, L) = L$  directly.

**Proposition 9.10** (Holds leave elapsed length free). *For every integer  $k \geq 0$ , there is a completing process in dimension three that holds or makes a finest change at each interval, whose first complete window has length  $8 + k$  and whose  $8 + k$  transitions contain exactly eight changes.*

*Proof.* Let  $u$  be the repeated Gray cycle (9.2) and define

$$w(n) = u(\max(n - k, 0)).$$

The first  $k$  transitions hold the initial pattern. Thereafter every transition is one of the finest changes of  $u$ . The process completes from every starting index, since it eventually follows the complete cycle.

The eight distinct patterns in its first cycle appear at indices  $k, k + 1, \dots, k + 7$ , with the initial pattern also present earlier. The last of these patterns has no earlier occurrence. The shortest complete window starting at zero therefore ends at index  $k + 7$  and has length  $k + 8$ . Among its  $k + 8$  transitions, precisely those starting at  $k, \dots, k + 7$  are changes. There are eight; the last closes the Gray cycle.  $\square$

The construction gives an exact family of different elapsed lengths under the same completion and change-count conditions. To convert the count to a physical duration, a realization must supply how the counted intervals relate to a physical clock. That relation is additional to the binary count.

For example, a specified fixed posting rate  $0 < r \leq 1$  can supply a waiting law in which the next change arrives after  $j + 1$  intervals with weight

$$r(1 - r)^j, \quad j = 0, 1, 2, \dots$$

The weights sum to one by the geometric series. With  $z = 1-r$ , the convergent series identities  $\sum_{j \geq 0} z^j = 1/(1-z)$  and  $\sum_{j \geq 0} jz^j = z/(1-z)^2$  give

$$\sum_{j \geq 0} (j+1)r(1-r)^j = r \left( \frac{z}{(1-z)^2} + \frac{1}{1-z} \right) = \frac{1}{r}.$$

Under this law, assigning that mean wait to each of  $N$  changes gives total mean wait  $N/r$ . At  $d = 3$  this is  $8/r$ ; at  $r = 1$  it is eight. The rate is an input to this waiting law. The coverage theorem has not selected it.

The clock argument therefore has a definite endpoint. The specified binary spatial item has  $2^d$  positions. Coverage forces the minimum complete count, and the reflected construction attains it with finest changes. Completion together with an autonomous pattern successor forces a genuine period of that length. Future observation constructs the state space on which the successor is well-defined and exposes any additional phases. A physical duration then requires the clock realization that assigns time to those changes. In three dimensions, eight is the forced minimum count at this floor, with the conditions for its use as a period now explicit.

# Weights, readings and physical events

A completed comparison can leave several possible records. The structure of those records and the weight assigned to each are different questions. Knowing how many registers there are does not say how much a particular register reads. Knowing the growth of a hierarchy does not yet say how often a physical process visits its levels.

We can now make both distinctions precise. The hierarchy of Chapter 6 supplies a relative weight: the base reading divided by the reading at a later level. The comparison apparatus of Chapter 4 supplies another quantity: the energy deposited by each input, divided by one common reference. This chapter derives the laws of both quantities and then states the additional correspondence required to read a hierarchy weight as an event probability.

## 10.1 The hierarchy supplies its own weight

Keep the actual framework, successor and additive reading of Chapter 6. Under its preserved-comparison and initial costly-join conditions, the positive orbit readings satisfy

$$s_n = s_0 \varphi^n, \quad s_0 > 0, \quad \varphi^2 = \varphi + 1, \quad \varphi > 1. \quad (10.1)$$

Define the relative weight of level  $n$  by

$$w_H(n) = \frac{s_0}{s_n}. \quad (10.2)$$

This is a ratio of the readings already constructed. It introduces no new sequence of physical states and no new adjustable multiplier.

**Theorem 10.1** (Weight of the realized hierarchy). *The weight (10.2) is positive and satisfies*

$$w_H(n) = \varphi^{-n}, \quad (10.3)$$

$$w_H(m+n) = w_H(m)w_H(n), \quad (10.4)$$

$$w_H(1) = \frac{1}{1 + w_H(1)}. \quad (10.5)$$

*Proof.* Substitution of (10.1), followed by cancellation of the positive base reading, gives

$$w_H(n) = \frac{s_0}{s_0 \varphi^n} = \varphi^{-n} > 0.$$

The exponent addition law gives (10.4). The golden equation gives  $\varphi^{-1} = \varphi - 1$ , hence

$$\frac{1}{1 + w_H(1)} = \frac{1}{1 + \varphi^{-1}} = \frac{1}{\varphi} = w_H(1).$$

□

The balance equation is therefore a consequence for this particular weight. Its supplier is the realized hierarchy, whose levels came from the recognizing join. Calling a different quantity a weight does not identify it with  $s_0/s_n$ . In particular, a visit frequency or a survival fraction needs its own correspondence with these ratios.

## 10.2 Uniqueness among factorizing weights

We can also start with an arbitrary function  $w : \mathbb{N} \rightarrow \mathbb{R}$ . Require

$$\begin{aligned} w(n) &> 0 & (n \in \mathbb{N}), \\ w(m+n) &= w(m)w(n) & (m, n \in \mathbb{N}), \\ w(1) &= \frac{1}{1 + w(1)}. \end{aligned} \quad (10.6)$$

The first condition excludes signed weights. The second concerns composition of depths. The third fixes the single-step balance. The preceding theorem has supplied all three for the hierarchy's own ratio.

**Theorem 10.2** (Unique lattice weight). *Every function satisfying (10.6) equals  $w_H$  at every depth. Thus these three conditions characterize the hierarchy's relative weight exactly.*

*Proof.* Factorization at zero gives  $w(0) = w(0)^2$ . Positivity allows us to cancel  $w(0)$ , so  $w(0) = 1$ . Write  $r = w(1) > 0$ . Induction, using  $w(n+1) = w(n)w(1)$ , gives

$$w(n) = r^n.$$

Multiplying the balance equation by its positive denominator yields  $r^2 + r - 1 = 0$ . The factorization

$$r^2 + r - 1 = (r - \varphi^{-1})(r + \varphi)$$

shows that its roots are  $\varphi^{-1}$  and  $-\varphi$ . Only the first is positive. Hence  $w(n) = \varphi^{-n} = w_H(n)$ . Conversely, the preceding theorem shows that  $w_H$  satisfies every condition.  $\square$

Each part of this argument has a separate role. Factorization makes the sequence a power of its first step. The balance equation selects two algebraic candidates. Positivity selects the one appropriate to a positive weight. Indeed, the signed sequence  $(-\varphi)^n$  factorizes and satisfies the step equation, but is negative at odd depths.

There is also a direct way to recognize the balance equation in the hierarchy. For any positive factorizing  $w$ ,

$$w(1) = \frac{1}{1 + w(1)} \iff \frac{1}{w(2)} = \frac{1}{w(1)} + \frac{1}{w(0)}. \quad (10.7)$$

To prove this, use  $w(0) = 1$ ,  $w(2) = r^2$  and multiply the right-hand equation by  $r^2 > 0$ . Both sides of the equivalence become  $1 = r + r^2$ . For  $w_H$ , the reciprocal weights are  $s_n/s_0$ , so the right-hand side is precisely the first additive posting  $s_2 = s_1 + s_0$ , divided by the base reading.

The weight equation and the hierarchy equation thus concern the same levels through the explicit reciprocal map. Their equality is stronger than the appearance of the same numerical constant in two formulas.

### 10.3 Normalization and the mean depth

Put

$$\rho = \varphi^{-1}, \quad 0 < \rho < 1.$$

The total relative weight and its normalized mass are

$$Z = \sum_{n=0}^{\infty} \rho^n, \quad P(n) = \frac{\rho^n}{Z}. \quad (10.8)$$

These sums concern the complete nonnegative ladder. A finite truncation has a different normalization.

**Theorem 10.3** (Normalized hierarchy law).

$$\begin{aligned}
 Z &= \varphi^2, \\
 P(n) &= (1 - \rho)\rho^n > 0, \\
 \sum_{n=0}^{\infty} P(n) &= 1, \\
 P(0) &= \varphi^{-2}, \\
 \sum_{n=0}^{\infty} nP(n) &= \varphi.
 \end{aligned} \tag{10.9}$$

*Proof.* For every finite  $N$ , cancellation of adjacent terms gives

$$(1 - \rho) \sum_{n=0}^N \rho^n = 1 - \rho^{N+1}.$$

Since  $0 < \rho < 1$ , the final power tends to zero. Therefore  $Z = (1 - \rho)^{-1}$ . The golden equation implies

$$1 - \rho = 1 - \varphi^{-1} = \varphi^{-2},$$

which proves the first four assertions.

For the mean, all terms below are nonnegative. Counting the term  $\rho^n$  once for each integer  $k$  from 1 to  $n$  gives the geometric-moment identity

$$\sum_{n=0}^{\infty} n\rho^n = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \rho^n = \sum_{k=1}^{\infty} \frac{\rho^k}{1 - \rho} = \frac{\rho}{(1 - \rho)^2}.$$

Multiplication by  $1 - \rho$  yields

$$\sum_{n=0}^{\infty} nP(n) = \frac{\rho}{1 - \rho} = \frac{\varphi^{-1}}{\varphi^{-2}} = \varphi.$$

□

This is an exact probability mass function on the mathematical depth index. Its normalization does not, by itself, construct a physical experiment whose recorded depths have this distribution. We return to that correspondence after deriving the register reading.

## 10.4 Extending the weight to a real variable

Suppose the additive variable can take every nonnegative real value. Let  $f : [0, \infty) \rightarrow \mathbb{R}$  satisfy

$$\begin{aligned} f(a+b) &= f(a)f(b) \quad (a, b \geq 0), \\ f(a) &\geq f(b) \quad (0 \leq a \leq b), \\ f(1) &= \rho. \end{aligned} \tag{10.10}$$

The variable is expressed in units whose unit step has weight  $\rho$ . The conditions supply neither a power-law ansatz nor continuity of  $f$ .

**Theorem 10.4** (Continuum weight). *The only function satisfying (10.10) is*

$$f(t) = \rho^t = \exp(-t \log \rho) \quad (t \geq 0). \tag{10.11}$$

*Proof.* At zero, factorization gives  $f(0) = f(0)^2$ . Monotonicity gives  $f(0) \geq f(1) = \rho > 0$ , so  $f(0) = 1$ . For any  $x \geq 0$ ,

$$f(x) = f(x/2)^2 \geq 0.$$

Repeated factorization proves  $f(kx) = f(x)^k$  for every natural  $k$ . In particular,  $f(k) = \rho^k$ .

Let  $x = p/q$  with  $p, q \in \mathbb{N}$  and  $q > 0$ . Then

$$f(p/q)^q = f(p) = \rho^p = (\rho^{p/q})^q.$$

Both bases are nonnegative. The  $q$ th-power map is strictly increasing on nonnegative reals, so  $f(p/q) = \rho^{p/q}$ .

Fix  $t > 0$  and put  $L = f(t)$ . Every rational  $q \geq t$  gives the lower bound  $\rho^q \leq L$ ; every rational  $0 \leq q \leq t$  gives the upper bound  $L \leq \rho^q$ . The first bound also shows  $L > 0$ . We can make the squeeze explicit without assuming continuity of  $f$ .

If  $L < \rho^t$ , then, because  $\log \rho < 0$ ,

$$t < \frac{\log L}{\log \rho}.$$

Choose a rational strictly between these two numbers. It satisfies  $q > t$  and  $\rho^q > L$ , contradicting the lower bound. If  $L > \rho^t$ , then

$$\frac{\log L}{\log \rho} < t.$$

Choose a rational  $q$  strictly between  $\max\{0, \log L / \log \rho\}$  and  $t$ . It satisfies  $0 \leq q < t$  and  $\rho^q < L$ , contradicting the upper bound. Thus  $L = \rho^t$ . The case  $t = 0$  was established at the start.

Finally,  $\rho = \varphi^{-1}$  gives  $\rho^t = \exp(-t \log \varphi)$ . This function factorizes, is decreasing, and takes the required value at one. It supplies a witness for all the conditions.  $\square$

The exponential form follows from composition and order. Its numerical rate follows from the unit-step calibration. The variable may be called a cost only after it has been identified with the additive physical quantity being counted. The proof does not identify every energy, duration or history length with the same variable.

## 10.5 Identical registers on normalized states

Return to the eight-register complex comparison carrier. A vector and its load are written

$$\psi = (\psi_0, \dots, \psi_7), \quad \|\psi\|^2 = \sum_{k=0}^7 |\psi_k|^2.$$

The conserved quadratic reading supplies this load. A normalized state has load one. Let  $\mu_\psi(S)$  initially be an arbitrary real reading of a subset  $S$  of the eight registers.

The conditions below concern normalized states only.

1. Identical local moduli give identical singleton readings, even across different states and different registers:

$$|\psi_k| = |\chi_\ell| \implies \mu_\psi(\{k\}) = \mu_\chi(\{\ell\}). \quad (10.12)$$

2. Disjoint subsets have additive readings, and the empty subset reads zero.
3. The whole register set reads one.
4. Each singleton reading is nonnegative.
5. A register with zero amplitude reads zero.

The first condition is a statement about the allowed family of states. It excludes a change in one register's response caused solely by redistributing load among the other registers. Relabeling the registers within one fixed state is a weaker condition.

**Theorem 10.5** (Squared-amplitude reading). *Under these five conditions,*

$$\mu_\psi(S) = \sum_{k \in S} |\psi_k|^2 \quad (10.13)$$

*for every normalized  $\psi$  and every register subset  $S$ .*

*Proof.* For  $0 \leq x \leq 1$ , use the normalized probe

$$p_x = (\sqrt{x}, 0, \sqrt{1-x}, 0, \dots, 0), \quad g(x) = \mu_{p_x}(\{0\}).$$

The complementary load remains in register two. Every state consulted in the proof has total load one.

Each component of a normalized state has modulus at most one. Applying (10.12) to  $\psi$  and  $p_{|\psi_k|^2}$  gives

$$\mu_\psi(\{k\}) = g(|\psi_k|^2).$$

Finite disjoint additivity then gives

$$\mu_\psi(S) = \sum_{k \in S} g(|\psi_k|^2). \quad (10.14)$$

Silence at zero gives  $g(0) = 0$ . Total reading on  $p_1$  gives  $g(1) = 1$ , since the other seven registers have zero amplitude.

For  $x, y \geq 0$  with  $x + y \leq 1$ , compare the normalized states

$$(\sqrt{x}, \sqrt{y}, \sqrt{1-x-y}, 0, \dots, 0) \quad \text{and} \quad p_{x+y}.$$

Their total readings, expressed by (10.14), are

$$\begin{aligned} g(x) + g(y) + g(1-x-y) &= 1, \\ g(x+y) + g(0) + g(1-x-y) &= 1. \end{aligned}$$

Subtracting gives

$$g(x+y) = g(x) + g(y) \quad (x, y \geq 0, x+y \leq 1). \quad (10.15)$$

Nonnegativity of singleton readings gives  $g(x) \geq 0$ . Consequently, when  $0 \leq x \leq y \leq 1$ ,

$$g(y) - g(x) = g(y - x) \geq 0.$$

Thus  $g$  is increasing on the unit interval.

For an integer  $q > 0$ , repeated restricted additivity gives  $1 = g(1) = qg(1/q)$ . Hence  $g(1/q) = 1/q$ . For  $0 \leq p \leq q$ , the same argument gives  $g(p/q) = p/q$ . These equalities determine  $g$  on all rational points of the interval.

Suppose  $g(x) > x$ . Since  $g(x) \leq g(1) = 1$ , a rational  $r$  strictly between  $x$  and  $g(x)$  lies in  $[0, 1]$ . Monotonicity would give  $g(x) \leq g(r) = r < g(x)$ , a contradiction. If  $g(x) < x$ , nonnegativity permits a rational  $r$  strictly between  $g(x)$  and  $x$ . Then  $g(x) < r = g(r) \leq g(x)$ , another contradiction. Therefore  $g(x) = x$  throughout  $[0, 1]$ . Substituting into (10.14) proves the result.  $\square$

No continuity of the reading, trigonometric calibration or proposed response exponent occurs in the premises. The proof uses normalized load, the ability to compare the displayed probes, and the five register conditions. The cost function  $J$  does not appear in this uniqueness argument; its physical connection enters through the conserved comparison apparatus below.

For a normalized state supported on registers zero and one, put  $\theta = \arcsin |\psi_1|$ . Then  $\sin^2 \theta = |\psi_1|^2$  and  $\cos^2 \theta = 1 - |\psi_1|^2 = |\psi_0|^2$ . The theorem gives the corresponding  $\cos^2 \theta$  and  $\sin^2 \theta$  readings. The angle is extracted from the supplied state. Choosing its physical preparation is a separate task.

## 10.6 A physical supplier for the register conditions

The five conditions have just constrained an arbitrary read. We now construct a read that satisfies them using the comparison energy already derived. This establishes that the conditions can hold together on that apparatus, and gives each number a specific physical reading.

Let  $P$  denote the comparison's orthogonal projection and put  $Q = I - P$ . The comparison sends an input  $f$  to its retained and complementary parts  $Pf$  and  $Qf$ . Both belong in the energy account. Introduce two ports and the exchange

$$X(f, g) = (Pf + Qg, Qf + Pg). \quad (10.16)$$

On a ready second port,

$$X(f, 0) = (Pf, Qf). \quad (10.17)$$

Thus the component absent from the first port is present in the second.

**Proposition 10.6** (Conserving reversible record). *The exchange (10.16) is its own inverse and preserves the sum of the two port energies for the conserved quadratic energy law  $E(f) = c\|f\|^2$ ,  $c > 0$ .*

*Proof.* The projection identities are  $P^2 = P$ ,  $Q^2 = Q$ ,  $PQ = QP = 0$ , and  $P + Q = I$ . Applying  $X$  twice and using them gives

$$X^2(f, g) = (P^2 f + Q^2 f, Q^2 g + P^2 g) = (f, g).$$

The ranges of  $P$  and  $Q$  are orthogonal. Hence

$$\begin{aligned} \|Pf + Qg\|^2 + \|Qf + Pg\|^2 &= \|Pf\|^2 + \|Qg\|^2 + \|Qf\|^2 + \|Pg\|^2 \\ &= \|f\|^2 + \|g\|^2. \end{aligned}$$

Multiplication by  $c$  proves energy conservation. □

Use the same normalized comparison pulse as in Chapter 4, now with one complex input  $z$  and the second operand absent. Write it as  $C(z, 0)$ . Its normalization and discarded component satisfy

$$\|C(z, 0)\|^2 = |z|^2, \quad \|QC(z, 0)\|^2 = \frac{|z|^2}{2}. \quad (10.18)$$

These are evaluated properties of that pulse and projection. The factor one-half comes from the balanced comparison construction, rather than from a response law fitted to the input.

Encode each coordinate  $\psi_k$  in its own ready pair  $(C(\psi_k, 0), 0)$ . Before the exchange, the total input load of the bank is  $\sum_k |\psi_k|^2$ , by (10.18). After the exchange, the energy deposited in the second port of coordinate  $k$  is

$$D(\psi_k) = E(QC(\psi_k, 0)) = \frac{c}{2} |\psi_k|^2. \quad (10.19)$$

The same device supplied with a unit pulse gives the common positive reference  $D(1) = c/2$ .

Pool the selected deposited-energy records and divide once by this reference:

$$\mu_{\psi}^E(S) = \frac{\sum_{k \in S} D(\psi_k)}{D(1)}. \quad (10.20)$$

The calibrated meter of Chapter 4 gives precisely such a ratio. Its common energy unit cancels, leaving

$$\mu_{\psi}^E(S) = \sum_{k \in S} |\psi_k|^2. \quad (10.21)$$

This calculation holds for any finite set of encoded coordinates. The eight-register theorem is its specialization to the comparison carrier.

Every register condition is now a consequence. Equal local moduli have equal deposited energies and use the same reference. Disjoint sums add; the empty sum vanishes. A normalized input gives total read one. Deposited energies are nonnegative, and a zero pulse deposits zero. The arbitrary-read uniqueness theorem and the physical calculation therefore arrive at the same reading by their respective premises.

Common calibration is essential to this construction. Dividing each occupied channel's deposited energy by its own input load gives

$$\frac{D(z)}{\|C(z, 0)\|^2} = \frac{c}{2} \quad (z \neq 0).$$

That operation removes the very load the bank is intended to compare. It cannot replace (10.20).

The construction specifies the encoding, ready ports, exchange and meter. It does not assert that every environment prepares this bank. Nor does coherent storage alone select a single outcome in one trial. Reusing the same exchange twice returns  $(C(z, 0), 0)$ , so the second port becomes empty again. A persistent history requires retaining the old outputs while supplying the next recording operation with its required resources. These distinctions keep the physical energy account intact.

## 10.7 Why weaker measurement conditions do not suffice

It is useful to see exactly what the register condition rules out. Consider the following alternative reading on normalized states:

$$\mu_{\psi}^{\text{ctx}}(S) = \begin{cases} \sum_{k \in S} |\psi_k|^2, & \#\{k : \psi_k \neq 0\} \leq 2, \\ \frac{\sum_{k \in S} |\psi_k|^4}{\sum_{j=0}^7 |\psi_j|^4}, & \#\{k : \psi_k \neq 0\} > 2. \end{cases} \quad (10.22)$$

The denominator in the second case is positive. Each case has total reading one, nonnegative singleton readings and silence at zero. Disjoint subsets have additive readings. Changing any component's phase leaves the reading unchanged. Every two-branch calibration also agrees with the squared-amplitude law, because it uses the first case.

Nevertheless, for

$$\psi_* = (\sqrt{1/2}, 1/2, 1/2, 0, \dots, 0)$$

the squared load of register zero is  $1/2$ , whereas

$$\mu_{\psi_*}^{\text{ctx}}(\{0\}) = \frac{1/4}{1/4 + 1/16 + 1/16} = \frac{2}{3}.$$

The normalized probe  $p_{1/2}$  has the same modulus in register zero and only two occupied registers, so its reading there is  $1/2$ . The same local input has acquired two different responses solely because the rest of the state is different. This violates (10.12).

Normalization, phase invariance, additivity and correct two-branch calibration therefore do not determine the general reading. This alternative can coexist with the canonical reciprocal cost and the complex carrier; neither fact removes its dependence on the other registers. The constructed energy bank excludes it for a concrete reason: its deposited energy depends on the local pulse and its common reference, with no state-dependent renormalization across the bank.

## 10.8 From a reading to a law of recorded events

The hierarchy weight and the register reading now have explicit suppliers. To connect them, one must identify the depth index with a recorded event and prove that its load is the hierarchy mass. An equality between two numerical formulas is insufficient when their indices refer to different physical objects.

In a retained history, keep the whole source state together with its successive record labels. An event selects histories; its load includes all retained source coordinates, not only the visible label. Consistency under extension means that the load of a previously recorded event is the sum of the loads of its one-step extensions. This is the property needed for a first-stop record.

Choose a fixed rule on the complete current label that distinguishes “continue” from “stop.” Let  $S_k$  be the load of histories that continue through their first  $k$  steps, and let  $D_k$  be the load that first stops at the next step. The surviving histories partition into those two disjoint possibilities. Consistency gives

$$S_{k+1} + D_k = S_k. \tag{10.23}$$

For a unit initial state,  $S_0 = 1$ . No independence assumption is used in this partition identity.

**Proposition 10.7** (The exact continuation condition). *If the same history process also satisfies*

$$S_{k+1} = \rho S_k \quad (k \in \mathbb{N}), \quad (10.24)$$

*then its first-stop loads are*

$$D_k = (1 - \rho)\rho^k = P(k).$$

*Proof.* Starting from  $S_0 = 1$ , induction in (10.24) gives  $S_k = \rho^k$ . Subtraction in (10.23) then gives  $D_k = \rho^k - \rho^{k+1} = (1 - \rho)\rho^k$ .  $\square$

Equation (10.24) is an additional statement about that history process. The earlier derivation of  $s_0/s_k$  does not establish it for arbitrary physical dynamics. Neither does the rule for reading the load of a supplied state select the state's preparation or its repeated evolution.

A fixed finite record cannot carry this exact law at every natural depth. If a finite set of displayed records is assigned depths in  $\mathbb{N}$ , its image has an upper bound. Every depth above that bound selects an empty event and has load zero. In contrast,  $P(k) > 0$  for every  $k$ . Thus an exact realization on all depths requires a record family beyond a fixed finite display. This is a restriction on that display, not a proof that a proposed unbounded history realizes the continuation equation.

There is a stronger limitation for a particular repeated recorder. In the fixed current recorder, retaining the complete source and using one unchanged continuation selector gives a finite rational transfer on the source covariance. This recorder cannot satisfy (10.24) at all depths, even when its initial amplitudes are arbitrary complex numbers. The restriction is on that unchanged operation and selector. It is not an exclusion of every possible recording process.

The algebra behind this exclusion is short enough to expose here. Write  $A$  for that recorder's selected covariance transfer,  $R_0$  for its initial surviving covariance, and  $A^*$  for the dual transfer on readings. The actual covariance reduction gives

$$S_k = \text{Re tr}(A^k R_0).$$

In matrix coordinates,  $A$  has rational entries, and the dual has the same characteristic polynomial  $p \in \mathbb{Q}[X]$ . Completeness of the recorder makes the dual positive and gives  $0 \leq A^*(I) \leq I$ . These are properties of the fixed

recorder; the preceding register uniqueness theorem alone does not supply them.

Positivity and the last inequality imply  $0 \leq (A^*)^k(I) \leq I$  for every  $k$ . For a positive matrix  $B$ , the order bound  $0 \leq B \leq \|B\|I$  therefore gives  $\|(A^*)^k B\| \leq \|B\|$ . A complex matrix is the sum of four positive parts, with coefficients  $1, -1, i, -i$ , each of norm at most its original norm: take the positive and negative spectral parts of its real and imaginary Hermitian parts. Thus

$$\|(A^*)^k B\| \leq 4\|B\| \quad \text{for every complex matrix } B.$$

If  $A^*B = \lambda B$  and  $B \neq 0$ , this gives  $|\lambda|^k \leq 4$  for all  $k$ . Consequently every eigenvalue of  $A^*$  has modulus at most one.

Now suppose that  $S_k = \rho^k$  for every  $k$ . The Cayley–Hamilton identity  $p(A) = 0$ , followed by the real trace against  $R_0$ , yields

$$0 = \operatorname{Re} \operatorname{tr}(p(A)R_0) = p(\rho).$$

The minimal polynomial of  $\rho$  over the rationals is  $X^2 + X - 1$ : it vanishes at  $\rho$ , and its irrational roots make this quadratic irreducible. It divides  $p$ , so the other root,  $-\varphi$ , is also a root of  $p$ . It would therefore be an eigenvalue of  $A^*$ , contradicting  $|\rho| = \varphi > 1$ . Initial-state amplitudes were never required to be rational.

This exclusion concerns exact equality at every depth. It does not settle a finite approximation or an operation that changes with the retained history. Nor does it supply such a replacement. A physical realization of the hierarchy mass must state its recording process and derive the continuation equation for that process.

The realized hierarchy fixes its reciprocal level weight. The conserved comparison bank supplies the squared-amplitude register read. The partition identity then identifies the precise extra condition that would connect recorded stopping depths to the hierarchy mass. That physical identification remains to be supplied by a suitable process; the unchanged fixed recorder just examined is excluded. None of these three statements can substitute for either of the others.



PART VI

# Scale and the connected foundation



# The Scale of a Primitive Event

A hierarchy supplies ratios. An event also has a duration and an energy. The question is what connects them. A number obtained by multiplying five golden factors becomes a physical action only when those factors count something in the event and its energy and duration read the same construction.

This chapter makes that connection explicit. The recognizing hierarchy supplies the weight of each depth. A finite construction supplies five distinguishable classes of primitive operation. Two physical identifications then attach duration and energy to the construction. Under those identifications, every realized primitive posting has the same native action, and a positive rescaling that preserves both identifications must be trivial. Reporting this action in joule seconds is a further, dimensional step.

## 11.1 What the hierarchy already supplies

Let  $s_n$  be the levels of the recognizing hierarchy, with  $s_0 > 0$ . The preserved comparisons and initial costly binary join established earlier give

$$s_n = s_0 \phi^n, \quad \phi = \frac{1 + \sqrt{5}}{2}.$$

The relative weight of depth  $n$  is therefore

$$W(n) = \frac{s_0}{s_n} = \phi^{-n}. \quad (11.1)$$

Its normalization is relative to the reference level. In particular,  $W(0) = 1$ , all weights are positive, and successive depths compose:

$$W(m+n) = W(m)W(n).$$

The one-step value also obeys

$$W(1) = \frac{1}{1 + W(1)}.$$

These properties are consequences of the hierarchy. They need not be postulated again when the very same weights are used below.

It is useful to prove the converse because it identifies exactly what could count as an alternative weighting.

**Proposition 11.1** (Unique positive block weight). *Fix a realized event  $e$ . Suppose its block weights  $B_e(n)$ , for  $n \in \mathbb{N}$ , satisfy*

$$\begin{aligned} B_e(n) &> 0, \\ B_e(m+n) &= B_e(m)B_e(n), \\ B_e(1) &= \frac{1}{1+B_e(1)}. \end{aligned}$$

*Then  $B_e(n) = \phi^{-n}$  for every  $n$ .*

*Proof.* At zero, composition gives  $B_e(0) = B_e(0)^2$ . Positivity permits cancellation, so  $B_e(0) = 1$ . Put  $b = B_e(1)$ . Multiplying the last condition by  $1+b > 0$  gives  $b^2 + b = 1$ . Its two roots are  $(-1 + \sqrt{5})/2$  and  $(-1 - \sqrt{5})/2$ ; positivity selects the first, which equals  $\phi^{-1}$ . Finally,

$$B_e(n+1) = B_e(n)B_e(1)$$

and induction give  $B_e(n) = b^n = \phi^{-n}$ . □

Thus the hierarchy supplies a block weight for every event by setting  $B_e(n) = W(n)$ , and every block weight satisfying these conditions agrees with it. The event index allows a physical attachment; the conditions remove event-dependent numerical freedom in the resulting weight.

This result concerns multiplicative weights. A fixed positive multiple of all weights generally fails the composition condition, including at zero. It does not follow that a physical energy is already identified with those weights. That is a separate statement about the event, made below.

## 11.2 Five classes of constructed operation

The exponent must come from a carrier whose elements can be distinguished. For this purpose consider the finite spatial realization on a three-dimensional periodic lattice with three sites in each coordinate direction. A primitive spatial event joins nearest neighbors along one coordinate axis, in either orientation. The event construction supplies a minimum-cost posting witness, a committed successor, and a conserved elementary current.

There are three families of operation data associated with such an event. A spatial operation retains its realized edge. A commit operation retains its initial and successor count. A balance operation retains its elementary current. The families are distinct: a particular event can supply an operation of each family. The spatial family is further distinguished by the axis of its edge.

Observe these operations with the following five tests:

1. Does the spatial operation use the first axis?
2. Does the spatial operation use the second axis?
3. Does the spatial operation use the third axis?
4. Does the commit operation make one successor step?
5. Does the balance operation carry the elementary conserved current?

Each answer is 0 or 1. Tests from a different operation family return 0. The tests inspect the edge, the committed count, and the current; the axis is recovered from the edge rather than attached as an arbitrary label. Two operations belong to the same effect class when all five tests give the same answers.

**Proposition 11.2** (The constructed effect carrier). *The effect classes of these operations are in bijection with the five coordinate profiles*

$$(1, 0, 0, 0, 0), \quad (0, 1, 0, 0, 0), \quad (0, 0, 1, 0, 0), \\ (0, 0, 0, 1, 0), \quad (0, 0, 0, 0, 1).$$

*In particular, the effect carrier has exactly five elements.*

*Proof.* Every realized spatial event lies on an axis. It cannot lie on two distinct axes: a step on the first candidate changes that coordinate by one unit modulo three, whereas a step on a different axis leaves that coordinate unchanged. The two endpoints would then both differ and agree in the same coordinate. Forward and backward orientations do not affect this argument. Thus every spatial operation has exactly one of the first three profiles.

Every commit operation has the actual successor witness, so its fourth answer is 1. Every balance operation has the elementary-current witness, so its fifth answer is 1. The cross-family answers vanish by the observation rule. This proves that no other profile occurs.

Conversely, start at the zero site and take a forward nearest-neighbor step on each axis. Each is a realized minimum-cost event and supplies the corresponding spatial profile. Any one of these events also supplies a commit operation and a balance operation, giving the last two profiles. Thus all five occur. Equal profiles are exactly the equivalence relation used to form the effect classes, so the profile map is both injective and surjective on those classes.  $\square$

The construction proves more than the availability of five names. Each class has an operation witness; distinct classes can be separated by one of the tests; and no constructed operation is silent to all the tests. Duplicating an operation's presentation does not create an extra effect class.

The scope of the count matters. It is the cardinality of this operational carrier under these observations. One spatial operation has one axis profile, not three simultaneous spatial effects. Nor does the construction classify every conceivable physical operation. A claim about a larger carrier must show that its complete relevant observations are represented here. In the constructed carrier that completeness is a bijection, with the witnesses just given.

We shall write its cardinality as  $d_e$ . For the constructed carrier,

$$d_e = D + 2 = 5, \quad D = 3. \quad (11.2)$$

Here the extra two count the commit and balance classes. They are not two additional spatial directions.

### 11.3 The duration attached to a posting

A primitive account posting increments one selected debit or credit count by one. For account states  $A$  and  $B$ , its mutation count is

$$M(A, B) = \sum_i |B_i^{\text{debit}} - A_i^{\text{debit}}| + \sum_i |B_i^{\text{credit}} - A_i^{\text{credit}}|. \quad (11.3)$$

The counts are integers, so  $M(A, B)$  is a natural number. In a primitive posting exactly one summand is one and all others vanish. Hence  $M(A, B) = 1$ . The elementary spatial current separately has zero total divergence; it supplies the conserved-current witness used in the event construction. These two statements describe different parts of the construction and should not be conflated.

Let  $T(e) > 0$  be the native duration assigned to a realized primitive event. The duration law used here is the following physical identification:

For each realized event there is a committed minimum-cost account posting, with the event's conserved-current witness, whose mutation count equals the event's native duration.

The law requires a posting witness and equality to its count. It does not assert uniqueness of that witness. Since every primitive posting witness has count one, the conclusion is independent of which qualifying witness is supplied.

**Proposition 11.3** (One mutation gives one native tick). *Under the duration law, every realized primitive event has  $T(e) = 1$ . Its duration equals its committed successor-count interval.*

*Proof.* Choose the witness  $A, B$  required by the law. The primitive posting property gives  $M(A, B) = 1$ , and the law gives  $T(e) = M(A, B)$ . The event's committed count ends at the successor of its starting count, whose difference is also one.  $\square$

Write  $\tau_0 = 1$  for this native tick. No value in seconds has entered. Counting a successor and identifying a physical duration with that count are distinct steps. The proposition uses the stated identification. In particular, a duration assignment of two native ticks to each of the same primitive events fails the law: it cannot equal the count of any primitive posting witness.

## 11.4 Energy at the depth of the carrier

Let  $E(e) > 0$  be the native energy assigned to the same event. The energy law connects that assignment to the hierarchy and the effect carrier:

The energy of a realized posting is the hierarchy weight at the cardinality of its complete classified effect carrier.

In notation, this is

$$E(e) = W(d_e). \quad (11.4)$$

The hierarchy determines  $W$ ; the carrier construction determines  $d_e$ . Equation (11.4) identifies their composition with physical energy. Neither a cardinality nor a relative weight alone makes that identification.

**Theorem 11.4** (Native energy and action of a primitive event). *Use the recognizing hierarchy, the complete constructed effect carrier, and positive energy and duration assignments to realized primitive events. If the duration and energy laws above hold, then every such event satisfies*

$$T(e) = \tau_0 = 1, \quad E(e) = E_{\text{coh}} = \phi^{-5},$$

*and its energy–duration action is*

$$A(e) = E(e)T(e) = E_{\text{coh}}\tau_0 = \phi^{-5}. \quad (11.5)$$

*Proof.* Proposition 11.3 gives the duration. Proposition 11.2 gives  $d_e = 5$ . Substitution into the energy law and (11.1) gives  $E(e) = \phi^{-5}$ . Multiplying by  $T(e) = 1$  proves the action formula.  $\square$

Equivalently, the product of one inverse hierarchy step for each effect class is

$$\prod_{q \text{ an effect class}} \phi^{-1} = \phi^{-5}.$$

The product is an exact count of five factors. Interpreting it as the posting's energy uses the energy law. We denote the native action value by  $\hbar_{\text{native}}$ ; identifying a laboratory quantum of action with this event action requires the same physical identification, followed by an external calibration.

The assumptions have a simultaneous mathematical realization. The canonical complete channel carrier has the five required classified elements, and canonical event assignments have duration  $\tau_0$  and energy  $E_{\text{coh}}$ . The event constructor provides the required posting and conservation witnesses. The hierarchy supplies the block weights, which agree with the canonical weights by Proposition 11.1. Thus the duration and energy laws can hold together; the theorem is not an implication with an empty premise class. This witness establishes consistency of the attachments. It does not substitute for identifying an independently given physical energy and duration with them.

## 11.5 Which alternatives are excluded

The two laws restrict energy and duration separately. Preserving only their product would leave a reciprocal freedom between them. The stronger result below uses both laws on both assignments.

**Theorem 11.5** (No nontrivial native rescaling). *Suppose  $T, E$  satisfy both event laws on a complete carrier. For positive  $a, b$ , suppose the assignments*

$$T'(e) = aT(e), \quad E'(e) = bE(e)$$

*also satisfy both laws on a complete carrier for the same event system. Then  $a = b = 1$ .*

*Proof.* The event construction supplies at least one realized posting. Apply Theorem 11.4 to that event in both assignments. It gives  $T(e) = T'(e) = 1$ , hence  $a = 1$ . It also gives  $E(e) = E'(e) = \phi^{-5} > 0$ . Cancelling this positive value gives  $b = 1$ .  $\square$

The result permits different complete carrier presentations on the two sides. What matters is that each really is complete for the specified classified effects and each energy assignment reads its depth by the same weight law. A relabeling cannot restore a changed energy scale.

There is also a sharper arithmetic obstruction to doubling. It rules out repairing a doubled energy merely by choosing another integer depth.

**Lemma 11.6** (Two is not an integer golden power). *For every  $k \in \mathbb{Z}$ ,  $\phi^k \neq 2$ .*

*Proof.* If  $k \leq 1$ , monotonicity of integer powers of  $\phi > 1$  gives  $\phi^k \leq \phi < 2$ . Otherwise  $k \geq 2$ , so  $\phi^k \geq \phi^2 = \phi + 1 > 2$ . Both cases exclude equality.  $\square$

Consequently  $2\phi^{-5}$  cannot equal  $\phi^{-n}$  at any natural-number depth  $n$ : such an equality would give  $2 = \phi^{5-n}$ . A doubled energy fails the block-energy law even if one changes the finite carrier's cardinality. A different integer rung also cannot retain the original action, because  $k \mapsto \phi^k$  is strictly increasing. In particular, two inverse steps per effect class give  $\phi^{-10} \neq \phi^{-5}$ .

These exclusions depend on keeping the physical laws under comparison. The event geometry, conserved current and dimensionless hierarchy by themselves admit weaker assignments with a stretched duration. The extra duration law rejects them. This is why the law must appear in the theorem's statement: the contradiction is to that requirement, not to the existence of a positive duration assignment.

## 11.6 A hierarchy weight is not a comparison cost

The comparison law and hierarchy use related numbers for different purposes. Their distinction can be checked exactly. The comparison cost is

$$J(x) = \frac{x + x^{-1}}{2} - 1.$$

Since  $\phi^{-1} = \phi - 1$ ,

$$J(\phi) = \phi - \frac{3}{2}, \quad \phi^{-1} - J(\phi) = \frac{1}{2}.$$

Thus the weight  $\phi^{-1}$  of one inverse hierarchy step is not its comparison cost  $J(\phi)$ . Replacing one by the other changes the construction. The native action argument counts multiplicative hierarchy weights and attaches their product to energy. The earlier comparison argument reads a residual through a calibrated apparatus. Their shared origin does not identify every quantity they produce.

## 11.7 Native coordinates and unit counts

An energy written in the original native coordinate and a count of coherence quanta are two ways of reporting the same amount. Define one coherence quantum to have raw native value  $E_{\text{coh}} = \phi^{-5}$ . If  $u$  is a count of such quanta, its raw value is

$$E_{\text{raw}}(u) = uE_{\text{coh}}. \quad (11.6)$$

Similarly, one action quantum has raw value  $\hbar_{\text{native}} = E_{\text{coh}}\tau_0$ . An action count  $v$  has raw value

$$A_{\text{raw}}(v) = v\hbar_{\text{native}}. \quad (11.7)$$

One coherence quantum therefore has count 1 and raw value  $\phi^{-5}$ . One action quantum likewise has count 1 and raw value  $\phi^{-5}$  when  $\tau_0 = 1$ . These are compatible statements in different coordinates.

An external conversion must specify which coordinate it consumes. A factor of “joules per coherence quantum” multiplies the count  $u$ , not the raw value  $u\phi^{-5}$ . Using the latter would apply the native quantum factor twice. This bookkeeping distinction becomes essential when comparing a dimensionless derivation with a laboratory report.

## 11.8 One dimensional anchor for external reporting

Let  $t_* > 0$  be the measured duration of one native tick in seconds. Write  $\ell_*$  for meters per native spatial unit and  $j_*$  for joules per coherence quantum. An external calibration makes these quantities positive and makes its reported speed consistent with the chosen speed convention. That condition alone does not fix  $j_*$ .

To obtain the single-anchor reporting construction, identify one native spatial unit per tick with the vacuum light speed and one native action quantum with the reduced quantum of action. With the exact SI conventions [3], write

$$c_{\text{SI}} = 299\,792\,458 \text{ m s}^{-1},$$

$$h_{\text{SI}} = 6.62607015 \times 10^{-34} \text{ J s}, \quad \hbar_{\text{SI}} = \frac{h_{\text{SI}}}{2\pi}.$$

The calibration is then constructed by

$$\ell_* = c_{\text{SI}} t_*, \quad j_* = \frac{\hbar_{\text{SI}}}{t_*}. \quad (11.8)$$

All factors are positive. Direct cancellation gives

$$\frac{\ell_*}{t_*} = c_{\text{SI}}, \quad j_* t_* = \hbar_{\text{SI}}.$$

Thus one native action quantum reports as  $\hbar_{\text{SI}}$ , and the native unit speed reports as  $c_{\text{SI}}$ . In the count coordinates of the previous section, the conversions are

$$u \text{ coherence quanta} \mapsto u j_* \text{ joules},$$

$$v \text{ action quanta} \mapsto v j_* t_* \text{ joule seconds}.$$

The cancellation is exact. It is a consistency result for this reporting construction, not an independent prediction of the numerical SI value used to construct it. The physical tick still needs a stable laboratory identification and a measured value  $t_*$ . Exact unit conventions do not supply that observation.

Likewise, an arbitrary positive external energy conversion is not removed by Theorem 11.5. That theorem forbids changing the native event assignments while retaining their laws. External reporting changes the chart used for an unchanged assignment. The single-anchor construction ties the external energy conversion to the time conversion through the stated action identification.

## 11.9 The precise foundational endpoint

The scale argument has three linked achievements. The recognizing hierarchy forces its inverse-step weights. The constructed operation carrier has five distinguishable effect classes. The duration and energy laws then force  $E_{\text{coh}} = \phi^{-5}$  and  $\hbar_{\text{native}} = E_{\text{coh}}\tau_0$ , excluding both nontrivial native rescaling directions. The laws also have a simultaneous mathematical witness.

The remaining physical question is precise: for the physical event being described, does its actual energy read the hierarchy weight at this complete effect carrier, and does its actual duration read the committed mutation count? A derivation of that bridge must preserve the same event, the same energy observable and the same duration. Agreement with an unrelated energy budget or an independently chosen clock would not establish the required equalities. The present action theorem retains those two identifications as its premises.

External reporting has its own stated bridge: a stable tick in seconds, the speed identification, and the action identification. Once these are supplied, the single-anchor conversion follows by the displayed algebra. The foundational conclusion is therefore exact in native structure and explicit about its physical and dimensional attachments. These are the conditions to carry into the cumulative statement of the book.

# The connected foundation

The argument has produced a comparison law, a hierarchy and a spatial restriction, together with constructions on which those results agree. It has also identified what must be added to obtain a clock, a physical read and the scale of a primitive event. We can now state the foundation without hiding these connections behind a sequence of chapter titles.

The distinction floor supplies an operational beginning. A comparison has represented operands; their information must be available; examining an earlier record is a later occurrence. Record arithmetic supplies counts, differences and ratios of those records. These conclusions make the later questions precise. They do not, by themselves, supply a signal space, a spatial complement or a physical energy assignment. The constructions and requirements that introduce those objects are part of the argument.

## 12.1 One cumulative statement

Here is the combined result for the comparison, hierarchy and spatial objects already defined. The cost is normalized to the unit-two reference; Chapter 5 gives its general-gauge form. The statement collects the preceding results at their common scope.

**Theorem 12.1** (Comparison, hierarchy and space). *Take the normalized native comparison conditions of Chapter 5. Let  $F$  be their reported cost on positive ratios. Let a closed observable framework have the additive reading (6.1), preserve all positive signed comparisons under its additive update, and have the initial costly join of Chapter 6 on its positive orbit. Write  $s_n$  for the readings of that orbit.*

*In addition, let  $f : S^1 \rightarrow S^D$  be an embedding. On its whole complement, suppose the path observation is additive under concatenation and invariant un-*

der endpoint homotopy, and reads some closed path nontrivially, as in Chapter 7. Then

$$\begin{aligned}
 F(x) &= J(x) = \frac{(x-1)^2}{2x} & (x > 0), \\
 s_n &= s_0 \phi^n & (n \in \mathbb{N}), \\
 \frac{s_0}{s_n} &= \phi^{-n} & (n \in \mathbb{N}), \\
 D &= 3, \quad 2^D = 8,
 \end{aligned}$$

where  $\phi = (1 + \sqrt{5})/2$  is the unique positive solution of  $\phi^2 = \phi + 1$ .

*Proof.* The native comparison conditions give the cost conclusion by Chapter 5. For a general reference gauge the conclusion is  $(\kappa/2)F = J$ ; the stated normalization is  $\kappa = 2$ . Preservation of positive additive comparisons makes the successor reading a uniform positive multiple of the original reading. The initial costly join excludes duplication of one earlier level and gives  $s_2 = s_1 + s_0$ . Chapter 6 therefore forces the multiplier to be  $\phi$  and constructs the hierarchy from this very orbit. Its positive base cancels in  $s_0/s_n$ , giving  $\phi^{-n}$ .

The path observation descends to a homomorphism on first homology and retains the nonzero reading of the supplied closed path. The complement therefore has nonzero first homology. The circle-complement argument in Chapter 7 forces  $D = 3$ , after which  $2^D = 8$  is the binary state count. These steps retain the given embedding and observation; no independent dimension parameter is fitted in the conclusion.  $\square$

The last equality counts states. It has not yet asserted a least period or a duration. Likewise, the hierarchy weight is the ratio of two readings of the stated hierarchy. It has not yet asserted the frequency of every physical event. Those further conclusions have their own suppliers below.

## 12.2 How the constructions meet

The general theorem is accompanied by explicit simultaneous witnesses. The census update acts on the original pairs of counts, and its additive reading reproduces the actual successor. At the selected price its positive comparisons persist and its first costly join supplies the hierarchy. The geometric observation reads the credit trace of that same hierarchy in the complement

of the embedded debit circle. The cost, hierarchy and spatial conditions can therefore hold together.

Chapter 8 gives a more detailed construction on the original eight-edge polygon. Its signal code retains both census coordinates. The two atom images and primitive-response conditions select the formation response; the reference is evaluated from the same operation used for the comparison. The relative residual agrees on every native comparison input. The ambient homeomorphism preserves the entire complement, including its point at infinity, so the geometric transport does not discard a troublesome part of the space.

These constructions have distinct jobs. The generic hierarchy argument selects its ratio without first installing a golden matrix. The explicit apparatus then realizes the selected structure and proves agreement of its observable operations. The dimension argument admits arbitrary  $D$  in the stated sphere class; the three-dimensional witness establishes inhabitation of its requirements. An eight-coordinate witness does not itself prove that every carrier has eight coordinates.

The statements also preserve the distinction between retaining a state and preserving a comparison value. An invertible or fully recorded update can keep the original information while changing a particular readout. The hierarchy needs the displayed preservation law for its original reading. The counterexamples in Chapters 6 and 8 show where the weaker requirement stops.

## **12.3 The additional conclusions and their suppliers**

### **12.3.1 From the binary floor to a clock**

The integer spatial carrier has a binary occupation pattern. At dimension three there are eight such patterns. Completion from every index ensures that none remains permanently unvisited. If the next pattern depends only on the present pattern, the pattern successor is a single permutation cycle. Chapter 9 proves that its least period is then eight. Even-translation covariance of the stated spatial update supplies this successor condition, and the explicit lift realizes it.

The implication uses completion and state-local succession together. Completion alone allows holds, repeated visits and a larger period. Autonomy of an underlying state also does not ensure autonomy of its visible pattern

if that observation has omitted relevant state. The sixteen-phase example isolates that difference. Physical duration enters only when the event clock and its duration law have been specified.

### 12.3.2 From the hierarchy to weights and readings

The weight  $s_0/s_n$  belongs to the hierarchy just constructed. Its positivity, multiplicative composition and one-step balance are proved from that hierarchy. Chapter 10 establishes the corresponding unique lattice weight and its normalized distribution, and gives the continuous extension under its stated regularity conditions.

A physical register needs a read map as well. The normalized-probe argument starts with local dependence on squared amplitudes, a common response for the registers, disjoint additivity, nonnegativity, silence and unit total. The energy-bank construction supplies those conditions by actual deposits and a common reference. The resulting registered weights are the squared amplitudes. The proof does not assume continuity of an otherwise arbitrary response function.

A history of irreversible outcomes needs a further connection. Conserved retained weight separates into survival and new deposits. A fixed continuation ratio would then give the geometric event law. Conservation alone does not select that ratio. The exact obstruction in Chapter 10 excludes the specified fixed rational contractive recorder as a supplier of the golden law at every depth, even with arbitrary complex initial amplitudes. It does not exclude all physical realizations of the weight. Thus a normalized read and an enacted event distribution remain different achievements.

### 12.3.3 From operations to native action

Chapter 11 constructs the finite carrier of distinguishable operation effects. All five classes have witnesses, and the listed observations distinguish them. The hierarchy supplies the block weight. The two physical reading laws then identify duration with the committed mutation count and energy with the hierarchy weight at the complete effect depth. On a primitive event these laws give

$$T(e) = 1, \quad E(e) = \phi^{-5}, \quad E(e)T(e) = \phi^{-5}.$$

They also exclude a positive rescaling of either native assignment while preserving both laws on the same event. Canonical assignments witness that the laws are mutually consistent.

This is a derived native action at the stated physical identification. To read it as the action of a complete dynamical preparation, the energy, duration, preparation and clock must describe that same event with its resources retained. The construction of an energy transfer does not alone identify its transfer with the posting price; equality of a total energy budget does not select its source contribution. The precise remaining connection is this common physical realization, rather than another algebraic evaluation of  $\phi^{-5}$ .

## **12.4 What the foundation excludes**

The proofs do more than produce a set of compatible formulas. Each has an alternative against which its requirements can be checked.

The literal comparator fixes the report on population ratios. Continuity then excludes changes at the remaining positive real arguments. The selected reference and balanced residual remove the unwanted operand self-terms. Positive additive comparison preservation excludes census prices that merge a previously positive distinction. Nontrivial additive linking excludes sphere dimensions other than three. Complete state-local succession excludes longer pattern cycles. A normalized local additive register excludes contextual replacements of its squared-amplitude read. The specified native energy and duration laws exclude further positive rescaling on their common event.

Each exclusion has the same logical form: preserve the stated operations and observations, and the alternative fails. It is neither a comparison of numerical fits nor a count of how many equations a framework contains. The force of the argument lies in the requirements and in the proofs that carry them to a unique result.

## **12.5 The endpoint at T10**

The labels T–2 through T10 name the intended progression from an operational distinction floor to foundational physical scale. The chapter structure has exposed the mathematical content behind those labels. The cumulative theorem establishes the comparison law, golden hierarchy, reciprocal weights and three-dimensional restriction at their displayed premises. Separate constructed suppliers extend the account to the eight-state clock, normalized physical readings and the conditional native action of a primitive event.

Two physical identifications remain explicit at this endpoint. The hierarchy's normalized weight needs a suitable retained-event dynamics to become an event distribution at every depth. The priced primitive posting needs a complete dynamical realization that identifies its energy and duration with the same preparation and clock. The examples and obstructions have narrowed both questions; neither is discharged by renaming a mathematical weight or a resource budget.

External units enter after those distinctions. A measured duration anchor and the defining constants of the SI give the conversion rules of Chapter 11. Cancellation of that anchor in the converted action is a consistency result for the reporting scheme. It is not a second measurement or an independent prediction of the constant used to define the conversion.

The established foundation is the connected set of necessities and constructions proved here. Its physical claims retain the operations, observations and identifications that make them claims about reality. Closing the remaining endpoint requires the same objects to meet the remaining conditions in one physical construction.

# The topology of retained linking

This appendix proves the complement result used in Chapter 7. All homology groups have integer coefficients. The embeddings are topological; no smoothness or local straightening is assumed. The proof uses singular chains, homotopy invariance, the homology of spheres and the Mayer–Vietoris exact sequence. These standard ingredients are developed in Hatcher’s *Algebraic Topology*, Chapter 2; the complement argument is the arc and circle case of Proposition 2B.1.

The work specific to the present argument is to retain the path reading through the homological passage and to show exactly how this complement theorem restricts its possible ambient dimensions. We give the topological proof in enough detail to expose every use of compactness and exactness.

## A.1 Chains, support and inclusion

A singular  $n$ -chain is a finite sum

$$w = \sum_{j=1}^m n_j \sigma_j, \quad \sigma_j : \Delta^n \rightarrow X, \quad n_j \in \mathbb{Z}.$$

Its support is the union of the images of the finitely many simplices having nonzero coefficient. Each image is compact because  $\Delta^n$  is compact. In a Hausdorff ambient space the support is therefore compact and closed.

A continuous map sends each simplex to its composite with that map and thus induces a homomorphism on chains. This homomorphism commutes with the boundary operator. In particular, inclusion of one complement in another transports cycles and bounding chains without changing their ambient geometric maps.

We will use the following elementary support fact. Suppose a chain in  $S^D \setminus A$  has support disjoint from another set  $B$ . Every one of its finitely many simplices can then be regarded as a map into  $S^D \setminus B$ . Keeping the same coefficients gives a chain there with the same ambient pushforward. This is

a statement about every point in each simplex image, not merely about its vertices.

Inclusion into the ambient space is injective on chain groups: distinct simplex maps into a subspace remain distinct maps after inclusion. Therefore, if the ambient boundary of the reinterpreted chain equals an included cycle, their equality already holds in the complement where both lie. This is the last step of the compact-support argument below.

## A.2 The exactness step for two pieces

Let  $U$  and  $V$  be open subsets covering a space  $X$ . The required part of the Mayer–Vietoris sequence is

$$H_2(X) \xrightarrow{\delta} H_1(U \cap V) \xrightarrow{(i_*, -j_*)} H_1(U) \oplus H_1(V). \quad (\text{A.1})$$

The maps  $i$  and  $j$  are the two inclusions. Exactness at  $H_1(U \cap V)$  identifies the kernel of the inclusion pair with the image of  $\delta$ .

**Lemma A.1** (Bounding in both pieces). *Assume  $H_2(X) = 0$ . If a one-cycle  $z$  in  $U \cap V$  bounds after inclusion into  $U$  and after inclusion into  $V$ , it already bounds in  $U \cap V$ .*

*Proof.* The two bounding assumptions say  $i_*[z] = 0$  and  $j_*[z] = 0$ . Thus  $[z]$  is in the kernel of the second arrow of (A.1). Exactness supplies a class  $a \in H_2(X)$  with  $\delta a = [z]$ . Since  $H_2(X) = 0$ , this class is zero and therefore  $[z] = 0$ . By the definition of homology, a zero cycle class is a boundary.  $\square$

One immediate consequence will be used later. If  $H_2(X) = 0$  and both  $H_1(U)$  and  $H_1(V)$  vanish, then  $H_1(U \cap V) = 0$ . Every cycle class in the intersection maps to zero in both pieces, so the lemma applies.

## A.3 Why an embedded arc has acyclic complement in degree one

**Theorem A.2** (Arc-complement first homology). *For every topological embedding  $a : [0, 1] \rightarrow S^D$ ,*

$$H_1(S^D \setminus a([0, 1])) = 0.$$

*Proof.* Suppose instead that there is a one-cycle  $z$  in the arc complement whose class is nonzero. It does not bound there. For a closed interval  $[u, v] \subseteq [0, 1]$ , write

$$K_{u,v} = a([u, v]).$$

The complement of the full arc includes into the complement of  $K_{u,v}$ . Let  $z_{u,v}$  be the image of  $z$  under this inclusion. Call  $[u, v]$  a bad interval if  $z_{u,v}$  does not bound in  $S^D \setminus K_{u,v}$ . The starting interval  $[0, 1]$  is bad.

We first show that a bad interval has a bad half. Put  $m = (u + v)/2$ . Injectivity of  $a$  gives

$$\begin{aligned} K_{u,m} \cup K_{m,v} &= K_{u,v}, \\ K_{u,m} \cap K_{m,v} &= \{a(m)\}. \end{aligned}$$

Both subarc images are compact, hence closed. Their complements are open and cover  $S^D \setminus \{a(m)\}$ ; their intersection is  $S^D \setminus K_{u,v}$ .

A sphere with one point removed is homeomorphic to Euclidean space by stereographic projection. It is contractible, so its second homology vanishes. If  $z$  bounded in the complements of both half-arcs, Lemma A.1 would make it bound in the complement of their union. That would contradict the badness of  $[u, v]$ . At least one half is therefore bad.

Choose a bad half repeatedly. This gives nested closed intervals

$$I_k = [u_k, v_k], \quad I_0 = [0, 1], \quad I_{k+1} \subseteq I_k, \quad v_k - u_k = 2^{-k}. \quad (\text{A.2})$$

The left endpoints increase and are bounded above. Define  $t_* = \sup_k u_k$ . Nesting gives  $u_k \leq t_* \leq v_k$  for every  $k$ : every left endpoint lies below every right endpoint. Thus  $t_*$  belongs to each interval, and their lengths tend to zero. Write  $p = a(t_*)$ .

The original cycle lies in  $S^D \setminus \{p\}$ , because it avoids the entire arc. This punctured sphere is contractible. Consequently its image bounds a finite singular two-chain  $w$  there. Let  $K$  be the support of  $w$ , regarded as a subset of  $S^D$ . The set  $K$  is compact and closed, and  $p \notin K$ .

Continuity of  $a$  now supplies an  $\varepsilon > 0$  such that

$$|t - t_*| < \varepsilon, \quad t \in [0, 1] \implies a(t) \notin K. \quad (\text{A.3})$$

Choose  $k$  with  $2^{-k} < \varepsilon$ . Every point of  $I_k$  is at distance at most its length  $2^{-k}$  from  $t_*$ . Hence  $a(I_k) \cap K = \emptyset$ .

Every simplex of  $w$  therefore lies in  $S^D \setminus a(I_k)$ . Regard the same finite sum as a two-chain  $w_k$  in that complement. Its ambient boundary is the original cycle. The inclusion into  $S^D$  is injective on chains and commutes with boundary, so

$$\partial w_k = z_{u_k, v_k} \quad \text{in } S^D \setminus a(I_k).$$

This contradicts the badness of  $I_k$ . No nonzero one-cycle class in the full arc complement exists.  $\square$

The argument requires neither a tubular neighborhood nor a smooth arc. Finite chain support does the work that a straightening picture might otherwise hide. The potentially complicated arc is reduced to one point only after a nested sequence of nonbounding conditions has been retained. The final bounding chain then avoids an entire sufficiently small subarc, giving the contradiction.

## A.4 The complement of two points

Let  $p \neq q$  be points of  $S^D$ , with  $D \geq 1$ . Stereographic projection from  $p$  identifies  $S^D \setminus \{p\}$  with  $\mathbb{R}^D$ . The second point becomes an ordinary point of  $\mathbb{R}^D$ . Translation moves that point to the origin, leaving  $\mathbb{R}^D \setminus \{0\}$ . Polar coordinates give

$$\mathbb{R}^D \setminus \{0\} \cong S^{D-1} \times (0, \infty).$$

The positive radial factor is contractible. These maps yield a homotopy equivalence

$$S^D \setminus \{p, q\} \simeq S^{D-1}. \quad (\text{A.4})$$

The positive-degree homology of a sphere vanishes outside its dimension, and  $H_2(S^2; \mathbb{Z}) \cong \mathbb{Z}$ . Therefore

$$D \neq 3 \implies H_2(S^D \setminus \{p, q\}; \mathbb{Z}) = 0. \quad (\text{A.5})$$

Only this vanishing statement is needed for the exclusion proof. It makes no assumption about the shape of an embedded circle between  $p$  and  $q$ .

## A.5 Cutting the circle and excluding dimensions

Let  $f : S^1 \rightarrow S^D$  be a topological embedding with  $D \geq 2$ , and suppose  $D \neq 3$ . The images of its two semicircles are embedded arcs  $K_+$  and  $K_-$ , whose

union is  $f(S^1)$ . They meet exactly at two distinct points  $p$  and  $q$ , because  $f$  is injective.

Apply Mayer–Vietoris with

$$X = S^D \setminus \{p, q\}, \quad U = S^D \setminus K_+, \quad V = S^D \setminus K_-.$$

The arc theorem gives  $H_1(U) = H_1(V) = 0$ . Equation (A.5) gives  $H_2(X) = 0$ . The exactness consequence following Lemma A.1 now yields

$$H_1(U \cap V) = H_1(S^D \setminus f(S^1)) = 0.$$

This proves the exclusion in every dimension  $D \geq 2$  other than three.

For  $D = 0$ , the ambient sphere has two points. Every subspace is totally disconnected and has zero singular homology in degree one. It cannot support the required class.

For  $D = 1$ , we show that every continuous injection  $f : S^1 \rightarrow S^1$  is onto. If it omitted a point, stereographic projection from that point would produce a continuous injection  $g : S^1 \rightarrow \mathbb{R}$ . Choose three distinct points on the circle. Their three distinct real images have a strictly middle value; call its preimage  $x_0$ , and call the other two preimages  $x_-$  and  $x_+$ , ordered so that

$$g(x_-) < g(x_0) < g(x_+).$$

The punctured circle  $S^1 \setminus \{x_0\}$  is connected, again by stereographic projection. Its continuous image under  $g$  is a connected subset of  $\mathbb{R}$  containing  $g(x_-)$  and  $g(x_+)$ . It must contain every value between them, including  $g(x_0)$ . Injectivity says that value has only the removed preimage  $x_0$ , a contradiction. Thus  $f$  is surjective and its complement is empty. Its first homology is zero.

All excluded dimensions have now been covered. The explicit three-dimensional construction in Chapter 7 supplies a closed complement path whose additive angular observation is nonzero. The path-to-homology result makes its class nonzero. This completes both the necessity and existence parts of Theorem 7.3.



# Straightening the source polygon

The family in the apparatus chapter is defined on the whole complement of the original Gray polygon. This appendix constructs the ambient map used there. The construction proceeds through a bent rectangle, a planar rectangle and a round circle, then extends across the point at infinity.

## B.1 The bent rectangle and an ambient inverse

Let  $R = [0, 3] \times [0, 1]$ . Its spanning strip is

$$\Sigma(s, y) = \begin{cases} (1 - s, y, 0), & 0 \leq s \leq 1, \\ (0, y, s - 1), & 1 \leq s \leq 2, \\ (s - 2, y, 1), & 2 \leq s \leq 3. \end{cases}$$

The formulas agree at the two joins. The inverse on this strip is

$$(x, y, z) \mapsto (1 + z + (2z - 1)x, y).$$

Substitution in each of the three cases returns  $(s, y)$ , proving injectivity. The strip is compact and the ambient space is Hausdorff, so it is embedded. Its boundary runs along the eight distinct cube edges of the original Gray cycle; the three edges along  $y = 0$ , three along  $y = 1$ , and two end edges account for the whole boundary.

To straighten the ambient space, define the continuous polygonal function

$$\begin{aligned} g(w) = & -w - 1 + \frac{2}{3} \max(0, w + 1) \\ & - \frac{2}{3} \max(0, w - 2) + 2 \max(0, w - 3). \end{aligned}$$

For an arbitrary point  $(x, y, z) \in \mathbb{R}^3$ , set

$$u = x - 2z, \quad w = 4z - |u + 1|,$$

and define

$$H(x, y, z) = (w, y, u - g(w)). \quad (\text{B.1})$$

Its inverse is explicit. Given  $(w, y, h)$ , put

$$u = h + g(w), \quad z = \frac{w + |u + 1|}{4}, \quad x = u + 2z.$$

Substituting these values into (B.1) returns  $(w, y, h)$ . Conversely, starting from  $(x, y, z)$ , the same relations recover  $u$ , then  $z$ , then  $x$ . All functions used are continuous on the entire ambient space. Thus  $H$  is an ambient homeomorphism with the displayed inverse.

On the strip, the third output coordinate vanishes. The first coordinate is

$$w(s) = \begin{cases} s - 2, & 0 \leq s \leq 1, \\ 6s - 7, & 1 \leq s \leq 3/2, \\ 2s - 1, & 3/2 \leq s \leq 2, \\ s + 1, & 2 \leq s \leq 3. \end{cases}$$

Each formula follows by resolving the absolute value and the maxima on its interval. Their endpoint values are  $-2, -1, 2, 3, 4$ , in order. They therefore cover  $[-2, 4]$  continuously and strictly increasingly. The second coordinate remains  $y$ . It follows that  $H$  maps the whole strip onto  $[-2, 4] \times [0, 1] \times \{0\}$ , and its boundary onto the boundary of that rectangle.

## B.2 Rounding and compactification

A compact convex planar set with nonempty interior has an ambient homeomorphism taking its interior, closure and boundary to the open unit disk, closed unit disk and unit circle respectively. This is the convex gauge-rescaling theorem: after translating an interior point to the origin, the gauge supplies the radial normalization. Its hypotheses hold for the planar rectangle. Apply such a plane homeomorphism and extend it by the identity in the third coordinate. Composing with  $H$  gives an ambient homeomorphism  $H_0 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  taking the original polygon to the unit circle in the first two coordinates.

Every ambient homeomorphism of Euclidean space extends to its one-point compactification, fixing infinity. Choose the sphere chart so the north pole is  $(0, 0, 0, 1)$ , and map the Euclidean coordinates first to  $(2x, 2y, 2z, 0)$  in the orthogonal hyperplane. The factor two matches the radius-two stereographic convention. Under its inverse stereographic map, a unit circle point  $(x, y, 0)$

becomes exactly  $(x, y, 0, 0)$ . Hence the image is the round unknot used in the spatial chapter, with the same coordinate normalization.

Conjugate the compactified  $H_0$  by this fixed sphere chart. The resulting sphere homeomorphism sends the embedded original polygon onto the round unknot. A bijection sends the complement of a set onto the complement of its image. Restriction therefore gives the required homeomorphism of whole complements. Infinity lies in the source complement and is retained: the polygon consists only of finite chart points. No additional point has been removed to make the map work.



# References

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