Symplectic Symmetry from a Source with Memory

Jonathan Washburn

Recognition Science Institute

jon@recognitionphysics.org

Abstract

Which changes of frame preserve a source together with the records it can carry? For the native eight-state cell of Recognition Science, we classify the unitary transformations preserving its six-dimensional contact space and admitting covariance of the full populated-record source. The source forces commutation with the literal half-cycle. In the contact coordinates this is the conformal symplectic equation $B^T J B = wJ$. The represented group is $(\mathrm{Sp}(2) \times \mathrm{U}(1))/\mathbb{Z}_2$, with an eleven-dimensional Lie algebra. We derive the source commutant, the faithful representation and its observable kernel, and show how one shared control transports both retained references while leaving the readers unchanged in operator closure. The mechanism is explicit: record mixing identifies three source components, and the half-cycle turns that identification into an alternating form on the contact carrier.

1 A symmetry must survive the record

A source can look symmetric when it is tested only on a fresh record. That test does not determine how a proposed transformation acts when information is already present. The distinction matters for a physical change of frame: the same transformation must remain compatible with the source on every allowed input, including a populated record.

We study this question for the native eight-state cell of Recognition Science (RS). The cell carries a complex signal, six directed contact operators, a two-port recording source, and a separate retained clock. Its contact space alone has a seventeen-dimensional unitary normalizer, parametrized by $\mathrm{U}(4) \times \mathrm{U}(1)$. Requiring a covariance extension to the full source reduces this freedom to eleven dimensions. We identify both the reduction and the physical operation that causes it.

The decisive step is small enough to see directly. Split the main signal and the decoded record into their odd and even half-cycle components. The source transfers amplitude between two components; its fixed decoder mixes a third into that transfer. A transformation commuting with both operations must act by the same matrix on all three. The remaining component has its own matrix. Consequently the main transformation commutes with the half-cycle.

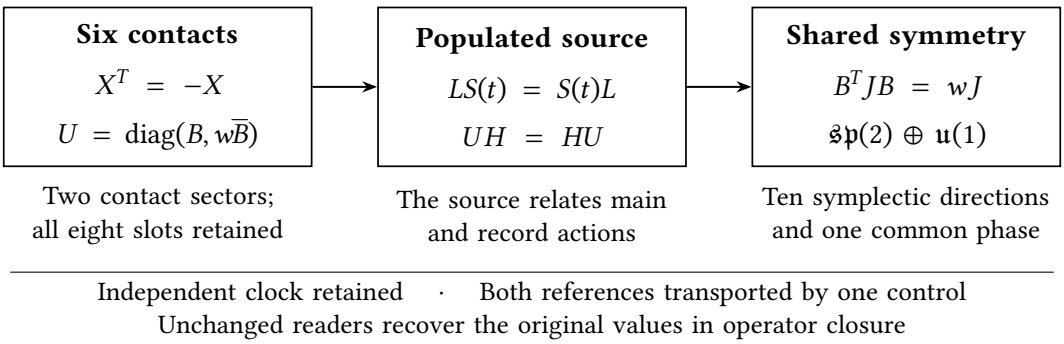


Figure 1. The selection mechanism. Contact preservation first determines two conjugate four-state actions. Full source covariance then requires commutation with the half-cycle H. In the contact chart this becomes preservation, up to phase, of the alternating form J. The clock and read protocol admit every transformation surviving these two constraints.

When written in the independently defined contact coordinates, this condition says that a nondegenerate alternating form is preserved up to phase. Figure 1 follows the complete implication.

The compact symplectic group and its relation to unitary transformations are classical [1]. Our contribution is the selection mechanism on this concrete apparatus: the full source commutant, its intersection with the contact normalizer, and the simultaneous treatment of the record, clock, references and read protocol. The resulting symmetry is a covariance group of specified finite operations. It supplies a derived internal symmetry for this RS source, with a definite carrier and representation.

The operational distinction between a frame and a fixed reference is also established in quantum information [2]. Here both reference states are transported by one available control. We prove exact read identities at the level of the operator closure of finite control words. This is the scope at which the admission theorem is stated throughout.

2 The signal, contacts and recording source

2.1 Eight slots and two complete coordinate systems

Let $\mathscr{V} = \mathbb{C}^8$, with coordinates numbered $0, \dots, 7$ and its standard Hermitian norm. The literal half-cycle is the involution

$$(Hf)_j = f_{j+4 \bmod 8}. \tag{1}$$

For $j = 0, \dots, 3$, put

$$a_j = \frac{f_j - f_{j+4}}{2}, \qquad b_j = \frac{f_j + f_{j+4}}{2}, \qquad f = \mathrm{join}(a, b) = (a + b, -a + b). \tag{2}$$

Thus $H(a, b) = (-a, b)$ in these coordinates and $\|f\|^2 = 2(\|a\|^2 + \|b\|^2)$. This is a change of coordinates on the entire cell.

The contacts use a different decomposition, $\mathscr{V} = \mathscr{V}_+ \oplus \mathscr{V}_-$. Its orthonormal column frames $Z, W : \mathbb{C}^4 \to \mathbb{C}^8$ are the matrices in Appendix A divided by $\sqrt{2}$. They satisfy

$$Z^\dagger Z = W^\dagger W = I_4, \quad Z^\dagger W = 0, \quad ZZ^\dagger + WW^\dagger = I_8. \tag{3}$$

The subscripts + and − name the two contact sectors; they are not the half-cycle eigenspaces. Keeping these decompositions distinct is essential.

Write E_{jk} for a matrix unit on $\mathbb{C}^4$ and $F_{jk} = E_{jk} - E_{kj}$. The six original contacts are $C_\alpha = 2ZK_\alpha W^\dagger$, where

$$\begin{aligned} K_0 &= -F_{03} - F_{12}, & K_3 &= iF_{03} - iF_{12}, \\ K_1 &= -iF_{02} - iF_{13}, & K_4 &= -F_{02} + F_{13}, \\ K_2 &= iF_{01} - iF_{23}, & K_5 &= F_{01} + F_{23}. \end{aligned} \tag{4}$$

These pairs recover each F_{jk} by complex linear combinations. Hence their complex span is exactly

$$\mathcal{N} = \{ZXW^\dagger \,:\, X^T = -X\}. \tag{5}$$

The classification below preserves this whole space under conjugation. Individual contact labels may mix.

2.2 A record that can already contain a signal

The source carrier is $\mathcal{V} \oplus \mathcal{V}$: a main signal f and an incoming record r. Its fixed decoder D acts on opposite slots by

$$D\begin{pmatrix} f_j \\ f_{j+4} \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} f_j \\ f_{j+4} \end{pmatrix}, \qquad c, s \in \mathbb{R}, \quad c^2 + s^2 = 1, \quad s \neq 0. \tag{6}$$

For the native census decoder,

$$c = \frac{1}{\phi\sqrt{\phi}\gamma}, \qquad s = \frac{\phi}{\phi\sqrt{\phi}\gamma}, \qquad \gamma = \sqrt{1 - \phi^{-4}}, \quad \phi = \frac{1+\sqrt{5}}{2}. \tag{7}$$

The established golden hierarchy fixes these coefficients. The classification itself uses only the conditions in (6), in particular nonzero mixing.

Use the full source chart

$$\Phi(f, r) = (a, b, p, q), \qquad (a, b) = \mathrm{split}(f), \quad (p, q) = \mathrm{split}(D^{-1}r). \tag{8}$$

Each of its four entries is in $\mathbb{C}^4$. In this chart the continuous source family has the form

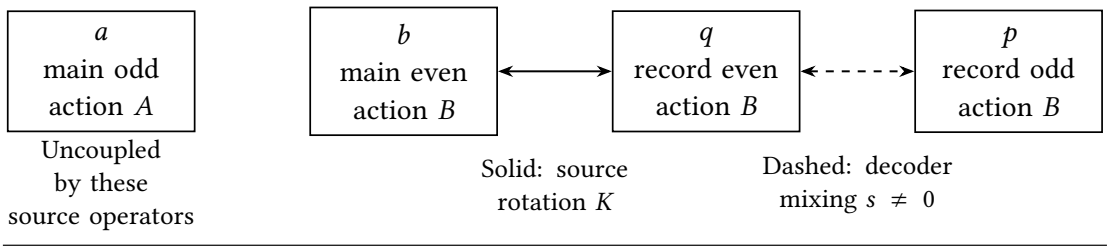
$$\mathcal{S}(t) = e^{tK}S_0, \qquad K(a, b, p, q) = (0, -q, 0, b), \tag{9}$$

where

$$S_0(a, b, p, q) = (a, b, cp + sq, -isp + icq). \tag{10}$$

Here t is the source rotation angle; the native source parameter u gives $t = \theta u$ with fixed $\theta \neq 0$. Both S_0 and e^{tK} are unitary. Although $\mathcal{S}(0)$ is the identity on a fresh input $(f, 0)$, it acts nontrivially on a populated record. Thus replacing $\mathcal{S}(t)$ by its rotation factor would change the problem.

Equations (4)–(10) specify the finite apparatus used in the theorem. In RS the recording source, decoder and contacts have distinct operational roles. Their compatibility is an additional constraint on a frame; it does not follow from the dimension of the carrier alone.



$$L = \mathrm{diag}(A, B, B, B)$$

Equal actions on independent components; no identification of their states

Figure 2. Why the record constrains the main signal. Each box denotes four complex amplitudes in the invertible chart Φ. Solid and dashed connections refer to different source operators, not to spatial links. Solving their common commutant forces the same B on b, p, q and eliminates all cross-block mixing. The free block a carries A. This directly yields $UH = HU$ on the main signal.

3 The populated source forces the half-cycle

Definition 3.1. A complex linear map L on the two source ports is a source covariance if $L\mathcal{S}(t) = \mathcal{S}(t)L$ for every real t, expressed in the chart (8). It extends a main transformation U if the main output of $L(f, 0)$ is Uf for every f.

No separation of the two ports is imposed on L. The following lemma derives that separation.

Lemma 3.2 (Full source commutant)**.** *For $s \neq 0$, the complete source commutant is*

$$\Phi L \Phi^{-1} = \mathrm{diag}(A, B, B, B), \qquad A, B \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^4). \tag{11}$$

Proof. Since S_0 is invertible, commuting with every $e^{tK}S_0$ is equivalent to commuting with S_0 and K. Indeed, set $t = 0$, cancel S_0, and differentiate at zero; the converse follows from the exponential series.

Write L as a four-by-four block matrix, with each block in $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^4)$. Solving $LK = KL$ gives

$$L = \begin{pmatrix} A & 0 & C & 0 \\ 0 & B & 0 & E \\ F & 0 & G & 0 \\ 0 & -E & 0 & B \end{pmatrix}. \tag{12}$$

Now impose $LS_0 = S_0L$. Its block entries $(2,1)$, $(0,3)$ and $(3,0)$ give respectively $sE = 0$, $sC = 0$ and $isF = 0$. The $(2,3)$ entry gives $sG = sB$. Since $s \neq 0$, these are $E = C = F = 0$ and $G = B$. This proves necessity, and direct substitution proves sufficiency. □

The source transfer couples b to q; the decoder couples p to q. The same B must act on all three. Figure 2 distinguishes this equality of actions from an equality of states: b, p, q remain independent inputs.

Theorem 3.3 (Main-port extension criterion). *A main matrix $U \in M_8(\mathbb{C})$ admits a source covariance extension if and only if*

$$UH = HU. \tag{13}$$

The extension is unique. If U is unitary, it preserves the norm of each port on every populated input.

Proof. Lemma 3.2 makes the main action $(a, b) \mapsto (Aa, Bb)$. These are exactly the matrices commuting with $H = \mathrm{diag}(-I_4, I_4)$ in the half-cycle chart. Conversely, such a main action has the extension

$$L(f, r) = (\mathrm{join}(Aa, Bb),\ D\,\mathrm{join}(Bp, Bq)), \tag{14}$$

with $(a, b, p, q) = \Phi(f, r)$. The main matrix determines both A and B, so the extension is unique. When U is unitary, the norm identity after (2) makes A and B unitary. The same identity and unitarity of D show separately that $\|L(f, r)_1\| = \|f\|$ and $\|L(f, r)_2\| = \|r\|$. □

In particular, the extension preserves every source energy law of the form $\kappa(\|f\|^2 + \|r\|^2)$ with $\kappa > 0$. The native positive conserved size laws reduce to this form. Energy preservation therefore adds no further restriction at this step.

The nonzero mixing in (6) is the exact physical premise doing the work. The numerical value of the golden coefficient is not used to tune the final symmetry. The same commutant holds for every lossless decoder with $s \neq 0$.

4 The contact normalizer meets the source

Lemma 4.1 (Contact normalizer). *For unitary U on $\mathscr{V}$, $U\mathcal{N}U^\dagger = \mathcal{N}$ if and only if*

$$U = ZBZ^\dagger + wW\overline{B}W^\dagger, \qquad B \in \mathrm{U}(4), \quad w \in \mathrm{U}(1), \tag{15}$$

Equivalently, in the contact chart, $U = \mathrm{diag}(B, w\overline{B})$.

Proof. The common kernel of $\mathcal{N}$ is $\mathscr{V}_+$, and its combined range is $\mathscr{V}_+$. For the kernel claim, every nonzero vector of $\mathbb{C}^4$ is moved by some F_{jk}; the range claim follows from the ranges of these matrix units. Thus a unitary normalizer preserves $\mathscr{V}_+$ and its orthogonal complement $\mathscr{V}_-$. Write its blocks as $B, D \in \mathrm{U}(4)$.

The transformed coefficient $BXD^\dagger$ must be skew for every skew X. Transposing and cancelling unitary factors gives

$$QX = XQ^T \quad \text{for every } X^T = -X, \qquad Q = B^\dagger\overline{D}. \tag{16}$$

Take $X = F_{jk}$. For $\ell \notin \{j, k\}$, the (ℓ, k) entry forces $Q_{\ell j} = 0$. Every off-diagonal entry is of this form using a third index. The (j, k) entry then gives $Q_{jj} = Q_{kk}$. Hence Q is scalar, and $D = w\overline{B}$ with $|w| = 1$.

Conversely these blocks send X to $\overline{w}BXB^T$, which is skew. The inverse has the same property, so the contact space is preserved exactly. □

The formula is most simply written

$$U(B, w) = ZBZ^\dagger + wW\overline{B}W^\dagger. \tag{17}$$

The parameters are unique, and their multiplication is $(B, w)(B', w') = (BB', ww')$. This yields the stated $\mathrm{U}(4) \times \mathrm{U}(1)$ normalizer.

The half-cycle exchanges the two contact frames through the real matrix

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad HZ = WJ, \quad HW = -ZJ, \quad J^2 = -I_4. \tag{18}$$

All these identities follow directly from the slot matrices in Appendix A. In particular J is supplied by the literal half-cycle acting between the original contact sectors.

Theorem 4.2 (Exact source-compatible contact group)**.** *A unitary U preserves the contact space and admits a source covariance extension if and only if*

$$U = U(B, w), \qquad B^T JB = wJ. \tag{19}$$

Define

$$\mathrm{Sp}(2) = \{A \in \mathrm{U}(4) \,:\, A^T JA = J\}. \tag{20}$$

The exact representation of this group on the original eight slots is

$$\rho(A, z) = z\big(ZAZ^\dagger + W\overline{A}W^\dagger\big), \quad (A, z) \in \mathrm{Sp}(2) \times \mathrm{U}(1), \tag{21}$$

with kernel $\{(I, 1), (-I, -1)\}$. Consequently the represented group is

$$\mathscr{G} \cong (\mathrm{Sp}(2) \times \mathrm{U}(1))/\langle(-I, -1)\rangle. \tag{22}$$

Proof. Use Lemma 4.1 and Theorem 3.3. Applying $UH = HU$ to Z and W gives $w\overline{B}J = JB$ and $BJ = wJ\overline{B}$. The first equation, multiplied on the left by B^T, is $B^T JB = wJ$. Conversely, that equation gives the first identity by unitarity. Complex conjugation, $|w| = 1$ and $J^2 = -I$ give the second.

Choose $z \in \mathrm{U}(1)$ with $z^2 = w$ and put $A = z^{-1}B$. Then $A^T JA = J$ and (17) becomes (21). Every (A, z) in (21) satisfies (19). The formula is a homomorphism because the two contact frames are orthogonal. If $\rho(A, z) = I$, its blocks give $zA = I$ and $z\overline{A} = I$. Thus $A = z^{-1}I$, $z^2 = 1$, and the kernel is exactly the two stated elements. □

The alternating form $\omega(x, y) = x^T Jy$ has therefore been derived as a compatibility condition between two parts of the same apparatus. The contact space supplies the conjugate pair of four-state actions; the populated source supplies the half-cycle constraint. Neither requirement alone gives (19).

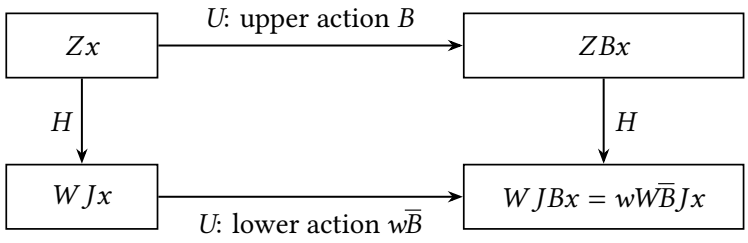


Figure 3. The alternating form is the half-cycle in contact coordinates. The relation $HZ = WJ$ is computed from the original slot matrices. Commuting the frame change with that physical half-cycle makes the square commute. Unitarity converts the resulting intertwining equation into the conformal symplectic equation.

5 Representation and infinitesimal symmetry

The two contact blocks of (21) are zA and $z\overline{A}$. For $A \in \mathrm{Sp}(2)$, unitarity and the symplectic equation imply

$$J\overline{A} = AJ. \tag{23}$$

Hence the fixed change of lower coordinate $y \mapsto Jy$ makes both blocks equal to zA. The two four-state components carry equivalent symplectic actions and the same central phase. Their opposite contact-sector labels do not turn that common phase into opposite phase charges.

Proposition 5.1 (Observable kernel)**.** *The conjugation representation on all of $M_8(\mathbb{C})$ has kernel*

$$\ker(\mathrm{Ad}\,\rho) = \{I, -I\} \times \mathrm{U}(1). \tag{24}$$

Its image is thus the quotient by this kernel, whereas the vector representation has the two-element kernel in Theorem 4.2.

Proof. A matrix commuting with every matrix unit is scalar. If $\rho(A, z)$ is scalar, the upper block makes $A = \lambda I$. The symplectic equation gives $\lambda^2 = 1$. Conversely $A = \pm I$ makes $\rho(A, z)$ scalar for every $z \in \mathrm{U}(1)$. □

Theorem 5.2 (Infinitesimal classification and integration)**.** *The infinitesimal contact-block matrices are exactly*

$$\mathscr{L} = \{Y \,:\, Y^\dagger = -Y,\ Y^T J + JY = 2i\vartheta J \text{ for some } \vartheta \in \mathbb{R}\}. \tag{25}$$

Every $Y \in \mathscr{L}$ has a unique decomposition

$$Y = X + i\vartheta I, \qquad X \in \mathfrak{sp}(2) = \{X \,:\, X^\dagger = -X,\ X^T J + JX = 0\}. \tag{26}$$

This is a Lie algebra direct sum $\mathfrak{sp}(2) \oplus \mathfrak{u}(1)$, of real dimension $10 + 1 = 11$. Moreover (25) is equivalent to

$$e^{tY} \in \mathrm{U}(4), \qquad (e^{tY})^T J e^{tY} = e^{2i\vartheta t} J \quad \text{for every } t \in \mathbb{R}. \tag{27}$$

Proof. Differentiate $B(t)^\dagger B(t) = I$ and $B(t)^T J B(t) = w(t) J$ at $B(0) = I, w(0) = 1$. The tangent to U(1) has $w'(0) = 2i\vartheta$, giving (25). Subtracting $i\vartheta I$ gives (26). If two such decompositions existed, a scalar $i\eta I$ would lie in $\mathfrak{sp}(2)$, so $2i\eta J = 0$ and $\eta = 0$. Scalar matrices commute with everything, so the decomposition preserves brackets.

For completeness the ten real coordinates of X are displayed in Appendix B. Their independent entry values prove the dimension without a dimension-only identification of the algebra. If (25) holds, set $B(t) = e^{tY}$. Skew-adjointness gives unitarity. Differentiating $B(t)^T J B(t)$ gives $2i\vartheta B(t)^T J B(t)$, whose initial value is J. This proves (27). Conversely, differentiation of (27) at zero gives (25). □

The Lie algebra is therefore identified by necessary and sufficient equations and by their integrated curves. The number eleven counts the allowed directions after the source constraint; it is not used to select a familiar algebra.

6 The same clock, references and readers

6.1 The retained clock

The clock is a separate copy of $\mathscr{V}$. The full carrier is $(\mathscr{V} \oplus \mathscr{V}) \oplus \mathscr{V}$, with states written $((f, r), k)$. For the extension L in (14), define

$$\hat{L}((f, r), k) = (L(f, r), k). \tag{28}$$

The direct flow has source factor e^{tK} and the fixed clock flow $T(t)$. Because L commutes with K, $\hat{L}$ commutes with this direct flow. It also commutes with the grounded flow obtained by multiplying the entire direct flow by its scalar phase. The clock state, its phase reader and its quadratic form are unchanged. The independent clock therefore imposes no additional reduction on (22).

6.2 One control transports both references

Let $\mathscr{K}$ be the operator-norm closure of the main-port matrices of finite original control words, augmented by the returned-record source cycle. The augmentation stores the complementary component in the record, applies one original tick to the main port, then returns the stored component. Appendix C gives the resulting connected control graph and the proof that

$$\mathrm{SU}(8) \subseteq \mathscr{K}. \tag{29}$$

This is an availability statement about the control inventory, distinct from commuting with the source family.

For any $U \in \mathrm{U}(8)$, choose $\zeta \in \mathrm{U}(1)$ with $\zeta^8 = \det U$ and put $V = \zeta^{-1} U$. Then $V, V^\dagger \in \mathscr{K}$. For any two retained references r_1, r_2, a sequence of the same finite words $V_n \to V$ gives

$$(V_n r_1, V_n r_2) \longrightarrow (V r_1, V r_2). \tag{30}$$

The two references are carried jointly; independent controls for their separate marginals are unnecessary.

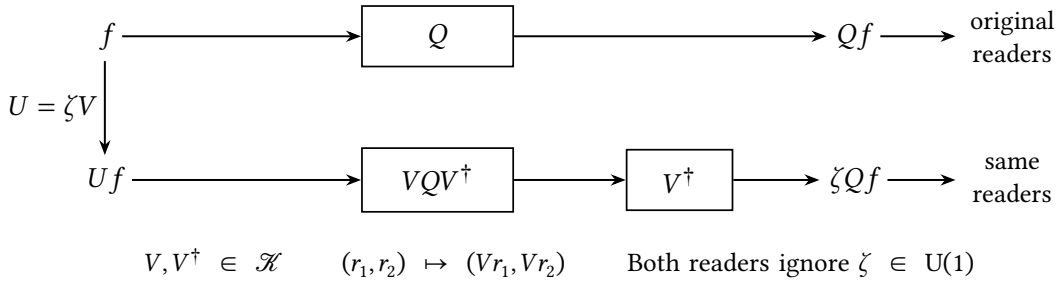


Figure 4. An enacted change of read frame. One control V carries both retained references and conjugates every finite word Q. Its adjoint is applied before the original readers. The output differs from Qf only by a common phase, so both original readings agree exactly at the operator-closure level. The clock remains on its separate, unchanged port.

Write Q for any finite original or source-assisted word. Its transported operation is $VQV^\dagger \in \mathscr{K}$, and

$$V^\dagger(VQV^\dagger)Uf = \zeta Qf. \tag{31}$$

The existing readers are invariant under a common unit phase. The gate-cost reader is the squared norm of the complementary projection after the original paired-slot phase gate. The phase reader is the ratio of Fourier coefficients at modes one and zero, defined as zero when the denominator vanishes. Equation (31) therefore reproduces both original readings for every input. It does so by applying $V^\dagger$ before the unchanged reader.

Corollary 6.1 (Simultaneous operational admission)**.** *For any two retained reference states, a unitary on the original cell preserves the contact space, has an energy-preserving full source extension, retains the independent clock, and admits the shared control/read protocol above if and only if it lies in $\mathscr{G}$ of* (22)*.*

Proof. Necessity follows from contact preservation and the source extension alone. Theorem 4.2 gives their exact intersection. Theorem 3.3, equation (28), and equations (29)–(31) supply all remaining witnesses for every element of that intersection. □

The read identities are exact for the limiting controls. Finite words approximate these controls in operator norm. A phase ratio near a zero denominator need not have uniform numerical stability; operator approximation and uniform readout accuracy are different requirements. Likewise, the source extension in (14) is an exact covariance map on populated ports. The admission theorem does not require a finite-word synthesis of every such two-port map.

7 What this fixes in the RS architecture

The result ties a symmetry to the operations that give the native cell its physical meaning. The full source preserves a distinction between the half-cycle components of the main signal, while its populated record ties three source components to a common action. The contact frame reads that same half-cycle as the alternating matrix J. This yields a precise route from a recording operation to compact symplectic covariance.

The distinction between the contact sectors remains useful for describing a contact, but it cannot be used to remove part of the apparatus. In fact the contacts and their adjoints generate

all of $M_8(\mathbb{C})$. To see this in the contact chart, take $N_{jk} = \left(\begin{smallmatrix} 0 & F_{jk} \\ 0 & 0 \end{smallmatrix}\right)$. For three distinct indices p, q, r, the upper block of $N_{pr}N_{qr}^\dagger$ is E_{pq}. Multiplying matrix units gives every diagonal unit as well. The adjoint products give the entire lower diagonal block; multiplying these block units by the N_{jk} and their adjoints gives every cross-block unit. Thus a projector commuting with all these operations is either 0 or I.

There are consequently three different mathematical objects with three different jobs. The generated operation algebra is the full matrix algebra. The contact normalizer preserves the specified space of primitive directed operations. The source-compatible group further preserves their relation to recording. The smaller group is a symmetry of this operational structure within the full algebra.

The two uses of the decoder should be distinguished. Nonzero mixing alone forces the source commutant and the resulting symplectic constraint. The fixed golden coefficients do more in the control argument: their phase has infinite order, providing continuous rotations in the closure of the original discrete controls. Thus the algebraic selection is stable under changing the nonzero mixing, while the enacted read protocol uses the actual native coefficients.

Within RS, this is a concrete internal symmetry result on a retained physical carrier. Its downstream use is correspondingly definite: any proposed matter or interaction interpretation built on this apparatus must respect (19), the common phase in (21), and the observable kernel (24). Identifying local fields requires a further construction specifying how neighboring cells, their source ports and their frames are related. The present classification fixes the local carrier data that such a construction must carry forward.

A The contact frames in the original slots

The unnormalized frames are

$$\sqrt{2}Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -i \end{pmatrix}, \qquad \sqrt{2}W = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -i & 0 \\ 0 & -1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \tag{32}$$

Each column has squared norm two before normalization; different columns are orthogonal. Applying (1) exchanges the upper and lower groups of four rows and gives (18) directly. The contact coefficients in (4) are the original coefficients divided by four in the unnormalized chart. Thus $C_\alpha = \frac{1}{4}(\sqrt{2}Z)(4K_\alpha)(\sqrt{2}W)^\dagger = 2ZK_\alpha W^\dagger$. No state or contact is removed by this normalization.

B Ten real coordinates for the compact algebra

Solving $X^\dagger = -X$ and $X^T J + JX = 0$ entry by entry gives exactly

$$X = \begin{pmatrix} ia & b+ic & g+ih & k+i\ell \\ -b+ic & -ia & -k+i\ell & g-ih \\ -g+ih & k+i\ell & id & e+if \\ -k+i\ell & -g-ih & -e+if & -id \end{pmatrix}, \tag{33}$$

where $a, b, c, d, e, f, g, h, k, \ell$ are arbitrary real numbers. Conversely, $a = \operatorname{Im} X_{00}$, $d = \operatorname{Im} X_{22}$ and the four complex entries $X_{01}, X_{23}, X_{02}, X_{03}$ recover all ten real parameters. Substitution verifies both defining equations, proving a real linear bijection with $\mathbb{R}^{10}$.

C Why the read controls reach every special unitary

We record the control argument because source covariance and control availability play different roles in Corollary 6.1. Use the complete signal coordinates $(a_0, \dots, a_3, b_0, \dots, b_3)$ from (2). We first explain why the original discrete controls provide continuous local curves. For the decoder coefficients c, s in (6), put $q = c + is$. Their exact values give $c^2 + s^2 = 1$ and $5c^2s^2 = 1$, hence

$$q^4 + \bar{q}^4 = -\frac{6}{5}. \tag{34}$$

If q were a root of unity, both summands would be algebraic integers. Their rational sum would then be an integer, a contradiction. The positive powers of q are therefore dense in the unit circle: an infinite cyclic subgroup has arbitrarily small nonzero angles, whose successive multiples approximate any angle. Taking powers of the fixed decoder consequently supplies, in closure, every simultaneous real rotation $R(z)$ of the four opposite-slot pairs.

Write T for the original cyclic slot shift and P for the original paired gate, which multiplies slots 4 and 5 by i and leaves the others fixed. Direct multiplication gives

$$R(z)TR(z^{-1})T^{-1} = R_{3,7}(z^2), \tag{35}$$

where the right side rotates only the pair $(3, 7)$. All other pairs cancel; at the wraparound pair the shift reverses the orientation. Every circle element has a square root, and conjugating by powers of T moves this local rotation to any opposite pair. Conjugation of the $(0, 4)$ rotation by P gives a local matrix with imaginary off-diagonal entries. In the odd/even coordinates it is an opposite-phase action; translating it gives that action on every pair. Thus the original controls provide in closure the curves

$$e^{tR_{p,p+4}}, \qquad e^{tD_{p,p+4}}, \qquad p = 0, 1, 2, 3, \tag{36}$$

where $R_{jk} = E_{jk} - E_{kj}$ and $D_{jk} = i(E_{jj} - E_{kk})$ now act on $\mathbb{C}^8$. These are the real rotation and opposite-phase actions on each opposite-slot pair, in the source coordinates.

The source supplies an additional returned-record operation. At rotation angle $\pi/2$, it stores b in the record while leaving a in the main signal. Applying the original tick to the main port and reversing the source rotation returns the record to its incoming fresh value and acts on the signal as

$$T_{\mathrm{ret}}(a_0, a_1, a_2, a_3, b) = (a_1, a_2, a_3, -a_0, b). \tag{37}$$

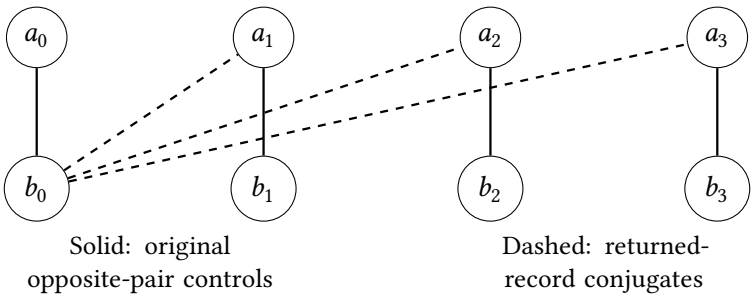


Figure 5. The record connects the control graph. Original local controls connect each a_p only to its partner b_p. The store–tick–return operation cycles the four a coordinates while leaving b fixed. Conjugating one original edge adds the three dashed edges. The resulting connected graph generates every special unitary through products and commutator limits. The edges denote available couplings, not spatial adjacency.

The intermediate record is populated and retained in this construction. Conjugating the 0–4 real rotation by successive powers of T_{ret} supplies the edges 1–4, 2–4 and 3–4. Together with the original edges p–$(p+4)$ they form a connected graph on all eight coordinates (Figure 5).

Here is the remaining Lie argument explicitly. For distinct j, k, ℓ, the commutator of R_{jk} and $R_{k\ell}$ is $R_{j\ell}$. Paths in the connected graph therefore generate every real skew matrix unit. The commutator of a diagonal difference D_{jk} with R_{jk} supplies $i(E_{jk} + E_{kj})$, up to a nonzero real factor. Conjugating a supplied diagonal difference by the available real rotations moves it to any pair of coordinates. These matrices span all traceless skew-Hermitian matrices.

Products and commutator product limits put their exponentials in the closed control set. Every special unitary has a traceless skew-Hermitian logarithm: diagonalize it unitarily, choose real eigenphases whose sum is an integer multiple of 2π, and alter one phase by that multiple to make the sum zero. Thus (29) follows. The use of Lie closure here is the standard operator-controllability argument [3]; the source-specific step is the returned-record operation (37) that connects the graph.

References

[1] J. C. Baez, *Symplectic, Quaternionic, Fermionic*, 2014. `https://math.ucr.edu/home/baez/symplectic.html`.

[2] S. D. Bartlett, T. Rudolph and R. W. Spekkens, Reference frames, superselection rules, and quantum information, *Rev. Mod. Phys.* **79** (2007), 555–609. doi:10.1103/RevModPhys.79.555.

[3] F. Albertini and D. D'Alessandro, Notions of controllability for bilinear multilevel quantum systems, *IEEE Trans. Autom. Control* **48** (2003), 1399–1403. doi:10.1109/TAC.2003.815027.