Regge's Slope and the Long-Wave Einstein–Hilbert Vertices from Flat-Space Stationarity on the Recognition Lattice

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Abstract

Take a lattice action of Regge-calculus form. On the periodic Kuhn triangulation of four-dimensional record space, each hinge contributes a smooth charge $C(A, \delta)$ of its area A and deficit angle δ, and the charge vanishes at zero deficit. Ask that flat space be a solution in every frame in which the lattice's time diagonal is uniformly stretched beyond the value $\kappa = 1$. Then the charge's slope $\partial_\delta C(A, 0)$ must equal Regge's, $\sigma(1)\, A$, at every area $A \geq 1/2$. The proof turns the vacuum condition into two functional equations whose only continuous solutions are linear. With that slope, the long-wave quadratic action on plane waves around the flat $\kappa = 1$ lattice is exactly $\sigma(1)/\hbar$ times the Fierz–Pauli action, and the cubic vertex is the same multiple of the Einstein–Hilbert cubic vertex. Both come from one common $\frac{1}{2}\int \sqrt{|g|}\, R$ in lattice units, whatever the charge does at higher order in δ. Newton's constant stays free. Two cautions belong with the result. First, the vacuum condition is equivalent to Regge's slope, so the theorem recasts Regge's slope as the statement that flat space stays a vacuum at every time-stretch; it does not derive Einstein's equations from nothing. Second, it covers the long-wave plane-wave vertices at second and third order around flat space, not the full nonlinear theory. The condition is not a formality: a smooth charge can meet every classical long-wave clause and still fail it, and stationarity beyond any threshold above $\kappa = 1$ is not enough. Extensivity of the charge over the pieces of a hinge gives the same conclusions and is strictly stronger. For the Regge action itself, the lattice's own long-wave solutions at $\kappa = 1$ with point-mass jets have post-Newtonian parameters $\gamma = \beta = 1$, and the isotropic Schwarzschild expansion is such a solution near a point, through second order. Every theorem is formally verified.

1 Which lattice actions are Einstein's at long wavelengths

Recognition Science describes events as records kept on a ledger. Four counters advance one step at a time, each step a *posting*, and the records they generate form a four-dimensional lattice that carries an order before it carries any geometry [1]. Geometry enters as a real squared length on every edge of that lattice: the edges of Kuhn's triangulation of the hypercubic lattice [2, 3]. Gravity is then an action on those squared lengths. For the classical limit to be general relativity, the action's long-wave vertices must be those of the Einstein–Hilbert action.

The Regge action [4] has that property. On this lattice, its long-wave limit gives linearized Einstein gravity and Newton's law [5], and every long-wave vertex of its field equations is Einstein's on sampled metric jets [6]. But the Regge action is a choice. Each triangle of the lattice, a *hinge*, contributes Regge's term $s\,A\,\delta$, its area times its deficit angle times a scale s. Any smooth function $C(A,\delta)$ that vanishes when the deficit does would give a lattice action of the same form, and most of them do not have Einstein's long-wave limit. This paper asks for a condition on C, stated without reference to Regge or Einstein, that guarantees the Einstein–Hilbert long-wave form.

The answer. It is enough that flat space stay a vacuum when time is stretched; the condition is sufficient, not necessary (Theorem 6.4). The flat lattices of the programme form a one-parameter family: the metric $g_\kappa = I - \kappa \mathbf{1}\mathbf{1}^\top$ in record coordinates, with $\mathbf{1} = (1,1,1,1)$. Every member is the same flat Minkowski space. Spatial unit steps keep their length, and the parameter κ sets only the proper duration of the record diagonal, $\sqrt{16\kappa - 4}$ for four postings, one on each counter. The recognition lattice uses $\kappa = 1$, where every posting is null; that value is an identification [6]. A flat configuration with a uniformly stretched time diagonal is still flat, so a gravitational action should keep it a vacuum. We prove (Theorem 3.1) that a C^3 hinge charge with $C(A,0) = 0$ whose action is stationary at the flat lattice of every $\kappa > 1$ has Regge's slope, $\partial_\delta C(A,0) = \sigma(1)\,A$ for every $A \geq 1/2$. Its long-wave quadratic and cubic vertices on plane waves around the flat $\kappa = 1$ lattice are then $\sigma(1)/\hbar$ times the Fierz–Pauli form [7] and the Einstein–Hilbert cubic vertex, with one common normalization: the expansion of $\frac{1}{2}\int \sqrt{|g|}\,R$ in lattice units. The rest of the charge, its even part $\partial_\delta^2 C(A,0)$ and every higher deficit derivative, drops out. The factor $\sigma(1)$ is not fixed, so Newton's constant stays free.

Two cautions. The result has to be read with both of them.

1. *It is Regge's slope, recast.* Given $C(A,0) = 0$, stationarity at every stretched flat lattice is equivalent to Regge's slope on the areas it reads (Theorem 5.1). So the theorem does not derive Einstein's equations from nothing. Its content is twofold: a vacuum condition that never mentions Regge turns out to be Regge's slope, and this slope alone, whatever the rest of the charge, fixes both long-wave vertices to the Einstein–Hilbert ones with one factor $\sigma(1)/\hbar$.
2. *It is long-wave, second and third order, around flat space.* The vertices are those of plane waves on the flat $\kappa = 1$ lattice, in the limit where a wavelength spans many lattice spacings. The full nonlinear action, strong fields and other backgrounds are not claimed.

What else is proved. The threshold $\kappa = 1$ is sharp (Theorem 6.2). The condition is strictly stronger than the classical long-wave clauses it implies (Theorem 6.4). Extensivity, the property that a hinge's charge is the sum of its pieces' charges at every deficit, gives the same conclusions and is strictly stronger again (Section 7). For the Regge action at $\kappa = 1$, the lattice's own long-wave solutions with point-mass jets have the post-Newtonian parameters $\gamma = \beta = 1$, and a solution of that kind exists near a point: the isotropic Schwarzschild expansion through second order (Section 8). Table 1 lists every result with its status.

Table 1. The results and their status. Every result holds on the Kuhn lattice with the midpoint strain embedding, which are the paper's model.

result	status	where
stationarity at every $\kappa > 1$ gives Regge's slope on $[1/2, \infty)$ and the Fierz–Pauli and Einstein–Hilbert cubic vertices with one factor	theorem	Thm. 3.1
the premise is Regge's slope: at every $\kappa > 1/2$, on $(0, \infty)$	theorem	Thm. 5.1(a)
at every $\kappa > 1$, on $[1/2, \infty)$	theorem	Thm. 5.1(b)
one torus side, a dense set of κ suffice	theorem	Prop. 6.1
$\kappa = 1$ is sharp for agreement with Regge's quadratic vertex	theorem	Thm. 6.2(a)
and for the Fierz–Pauli form itself	theorem	Thm. 6.2(b)
$(1/2, 5/4]$ and the set $(pq + 1)/(p + q)$ do not force Regge's slope	theorem	Prop. 6.3
the premise is strictly stronger than the classical clauses	theorem	Thm. 6.4
extensivity gives the same conclusions and is strictly stronger	theorem	Thm. 7.1
Regge action at $\kappa = 1$: jet equations, $\gamma = \beta = 1$, a local Schwarzschild solution	theorem	Sec. 8
the same for every charge that meets the premise	INFERENCE	Sec. 8
the factor read as Newton's constant, $G = 1/(8\pi\sigma(1))$	INFERENCE; $\sigma(1)$ OPEN	Rem. 3.2
$\kappa = 1$; edge lengths as a metric; time along the record diagonal	IDENTIFICATION	Sec. 2.5
a ledger origin of the premise; the nonlinear theory; general strains	OPEN	Sec. 9

Relation to other work. That Regge calculus approaches the Einstein–Hilbert action in the continuum goes back to Regge [4]; Cheeger, Müller and Schrader proved convergence of the curvature of piecewise flat spaces [9], and Friedberg and Lee derived Regge's action from Einstein's [10]. On the hypercubic lattice, Roček and Williams found the weak-field limit of Regge calculus to be linearized gravity [11, 12], and Feinberg, Friedberg, Lee and Ren studied the continuum limit of lattice gravity [13]. These results concern the Regge action. The question here is different: which condition, stated without Regge, makes an action of Regge's hinge-sum form share its long-wave vertices. The answer is a stationarity condition on flat space. It is sufficient, not necessary, and the threshold in it is exact.

How to read the status of a claim. A *theorem* is proved as stated, with every premise listed beside it. Every theorem in this paper is formally verified: its proof is complete and has been checked by computer from the standard logical foundations of mathematics. An INFERENCE is a reasoned step that is not one of the theorems. An IDENTIFICATION is a physical reading that the definitions do not force. A MODEL is a choice of mathematical object. An OPEN item is a precise statement that is neither proved nor refuted.

2 The lattice, the actions and the long-wave vertices

2.1 Records, postings and the Kuhn lattice

Vertices are points $x \in (\mathbb{Z}/N)^4$ of the torus of side N; the coordinates x_a, $a = 0, 1, 2, 3$, are *record coordinates*, the readings of the four counters. Write e_a for the unit vectors. Four counters advanced one step at a time from x to $x + \mathbf{1}$ have $4! = 24$ arrival orders, and these are the 24 simplices of Kuhn's triangulation of the unit cube. Every edge joins x to $x + d$ for one of the fifteen *edge classes* $d \in \{0, 1\}^4 \setminus \{0\}$, and $|d|$, the number of ones in d, counts its postings. A *configuration* ℓ is a real squared length ℓ_e on every edge. The graph is fixed; we never sum over triangulations.

2.2 The flat lattices

The flat lattice of parameter κ gives the edge (x, d) the squared length

$$\ell^{\kappa}_{(x,d)} = g_\kappa(d, d) = |d|\,(1 - \kappa|d|), \qquad g_\kappa = I - \kappa\,\mathbf{1}\mathbf{1}^{\top}. \tag{1}$$

The metric g_κ has eigenvalue 1 on the spatial hyperplane $\mathbf{1}^{\perp}$ and $1 - 4\kappa$ on the record diagonal $\mathbf{1}$, so for $\kappa > 1/4$ it is Lorentzian with time along $\mathbf{1}$. Every g_κ is flat Minkowski space. Members of the family differ only in the proper duration of the record diagonal, $\sqrt{16\kappa - 4}$, measured against unit spatial steps: g_κ is g_1 with its time direction uniformly stretched (for $\kappa > 1$) or compressed (for $\kappa < 1$). A single posting has $g_\kappa(e_a, e_a) = 1 - \kappa$: it is null at $\kappa = 1$ and timelike for $\kappa > 1$.

Table 2. The four hinge classes of the flat lattice g_κ, $\kappa > 1/2$. The second column lists the posting counts of the three edges; $u = 4\kappa - 1$.

class i	edges	$16A_i^2$	A_i^2 in terms of u	A_i at $\kappa = 1$
0	$(1, 1, 2)$	$4(2\kappa - 1)$	$(u - 1)/8$	$1/2$
1	$(1, 2, 3)$	$8(3\kappa - 1)$	$(3u - 1)/8$	1
2	$(1, 3, 4)$	$12(4\kappa - 1)$	$3u/4$	$3/2$
3	$(2, 2, 4)$	$16(4\kappa - 1)$	u	$\sqrt{3}$

The *hinges* are the triangles of the triangulation. At a flat lattice with $\kappa > 1/2$ they fall into four classes by the posting counts of their three edges, and every hinge is timelike. A timelike triangle with edge vectors u, v has area $A = \frac{1}{2}\sqrt{-\det G}$, where G is the Gram matrix of u and v under the metric. Table 2 gives the four class areas $A_i(\kappa)$. At $\kappa = 1$ they are $1/2$, 1, $3/2$ and $\sqrt{3}$, the *four flat areas*.

For class 0, for example, the edge vectors e_a and e_b have Gram determinant $(1-\kappa)^2-\kappa^2 = 1-2\kappa$, so $A_0^2 = (2\kappa-1)/4$. When $\kappa > 1$ every class area exceeds $1/2$; at $\kappa = 1$ the smallest is exactly $1/2$.

2.3 Hinge-charge actions

Each hinge h of a configuration ℓ near a flat lattice has an area $A_h(\ell)$ and a deficit angle $\delta_h(\ell)$, which is 2π minus the sum of the dihedral angles of the simplices that share the hinge. Every flat lattice with $\kappa > 1/2$ has $\delta_h = 0$ at every hinge.

Definition 2.1 (Hinge charge and its action)**.** A *hinge charge* is a function $C : \mathbb{R}^2 \to \mathbb{R}$, $(A,\delta) \mapsto C(A,\delta)$. Its action on the torus of side N is

$$\Phi_C(\ell) = \frac{1}{\hbar}\sum_h C\big(A_h(\ell), \delta_h(\ell)\big),$$

summed over the hinges. Its *slope* is $\sigma(A) = \partial_\delta C(A,0)$ and its *even part* is $\partial_\delta^2 C(A,0)$.

The Regge charge is $C_{\mathrm{R}}(A,\delta) = s\,A\,\delta$ with a positive action scale s; its action is $(s/\hbar)$ times the Lorentzian Regge action $S_{\mathrm{R}} = \sum_h A_h\delta_h$, and its slope is sA. A charge is *record-compatible* at $k \geq 1$ postings per turn when it reproduces the posting action at whole postings, $C(A, 2\pi n/k) = n\,s\,A\,2\pi/k$ for every integer n. At $n = 0$ this gives $C(A,0) = 0$, and $C(A,0) = 0$ is the only consequence of record compatibility that the results below use. So we state them for charges with $C(A,0) = 0$.

2.4 Plane waves and the long-wave vertices

A *plane wave* with integer wave vector $m \in \mathbb{Z}^4$ and symmetric polarization H (a real symmetric 4×4 matrix) is the edge field

$$u_{m,H}(x,d) = \big(d^{\top}Hd\big)\cos\Big(\tfrac{2\pi}{N}\,m\cdot\big(x+\tfrac{d}{2}\big)\Big), \tag{2}$$

the metric perturbation $H\cos(k\cdot y)$, $k = 2\pi m/N$, sampled at the midpoint of each edge. We write $D^j\Phi_C$ for the j-th derivative of Φ_C with respect to the squared lengths, taken at the flat $\kappa = 1$ lattice ℓ^1 of side N. The long-wave quadratic and cubic vertices are the limits of $N^{-2}D^2\Phi_C$ and $N^{-2}D^3\Phi_C$ on plane waves as $N\to\infty$. The power N^{-2} is the natural one: there are of order N^4 cells, and an Einstein–Hilbert vertex carries two derivatives, each of size $1/N$.

The two continuum forms they are compared with are the standard ones. For a constant metric g, symmetric h, h' and a covector k, with indices raised by g^{-1}, $k\cdot k = g^{ab}k_ak_b$ and $\operatorname{tr} h = g^{ab}h_{ab}$, the *Fierz–Pauli form* is

$$\mathrm{FP}_g(h,h';k) = (k\cdot k)\,h_{ab}h'^{ab} - 2\,k^a h_{ab}h'^{b}{}_c k^c + \operatorname{tr}h'\;k^a h_{ab}k^b + \operatorname{tr}h\,k^a h'_{ab}k^b - (k\cdot k)\operatorname{tr}h\operatorname{tr}h'. \tag{3}$$

It is the polarized linearized Einstein operator: on two plane waves $h\,e^{ik\cdot y}$ and $h'e^{-ik\cdot y}$, the coefficient of $\epsilon_0\epsilon_1$ in $\sqrt{|\det G|}\,R$ for $G = g + \epsilon_0 h e^{ik\cdot y} + \epsilon_1 h' e^{-ik\cdot y}$ is $-\frac{1}{2}\sqrt{|\det g|}\,\mathrm{FP}_g(h,h';k)$ up to total derivatives. The *Einstein–Hilbert cubic vertex* $V_g(H_0,H_1,H_2;k_0,k_1,k_2)$ is the coefficient of $\epsilon_0\epsilon_1\epsilon_2\,e^{i(k_0+k_1+k_2)\cdot y}$ in Einstein's first-order Lagrangian

$$\mathscr{L}_{\Gamma\Gamma}[G] = \sqrt{|\det G|}\;G^{ab}\big(\Gamma^c_{ad}\Gamma^d_{bc} - \Gamma^c_{ab}\Gamma^d_{cd}\big), \qquad G = g + \sum_{l=0}^{2}\epsilon_l H_l e^{ik_l\cdot y}, \tag{4}$$

which differs from $\sqrt{|\det G|}\,R$ by a total derivative [8, §93]; that derivative drops out of a vertex with $k_0 + k_1 + k_2 = 0$. Each term of $\mathscr{L}_{\Gamma\Gamma}$ has two derivatives, so V_g is real for real H_l and k_l. It includes the volume factor $\sqrt{|\det g|}$; FP_g does not. At $\kappa = 1$, $\det g_1 = -3$ and $g_1^{-1} = I - \frac{1}{3}\mathbf{1}\mathbf{1}^\top$.

2.5 What is model and what is identification

The lattice, the fixed Kuhn graph with real squared lengths, and the plane-wave sampling of (2) are the paper's MODEL. Three physical readings are IDENTIFICATIONS, not consequences of the definitions: that the squared edge lengths are the values $d^\top g d$ of a metric (the midpoint strain embedding of (2)); that time runs along the record diagonal; and that the physical lattice has $\kappa = 1$ [6]. The theorems below are statements about the lattice actions and hold whatever these readings turn out to be; the readings decide what the theorems say about gravity.

3 The theorem

Theorem 3.1 (Flat-space stationarity gives Regge's slope and the Einstein–Hilbert vertices)**.** *Let C be a hinge charge of class C^3 on $\mathbb{R}^2$ with $C(A, 0) = 0$ for every A. Suppose that for every $\kappa > 1$ the flat lattice ℓ^κ on the torus of side 3 is a stationary point of Φ_C. Put $\sigma(A) = \partial_\delta C(A, 0)$ and $c = \sigma(1)/\hbar$. Then:*

(i) (slope) $\sigma(A) = \sigma(1)\,A$ *for every* $A \geq 1/2$;

(ii) (quadratic vertex) *for every nonzero* $m \in \mathbb{Z}^4$ *and all symmetric* H_0, H_1,

$$\lim_{N\to\infty} N^{-2}\, D^2\Phi_C\big[u_{m,H_0}, u_{m,H_1}\big] = c \cdot \Big(-\frac{\sqrt{3}\,\pi^2}{2}\Big)\,\mathrm{FP}_{g_1}(H_0, H_1; m);$$

(iii) (cubic vertex) *for all* $m_0, m_1, m_2 \in \mathbb{Z}^4 \setminus \{0\}$ *with* $m_0 + m_1 + m_2 = 0$ *and all symmetric* H_0, H_1, H_2,

$$\lim_{N\to\infty} N^{-2}\, D^3\Phi_C\big[u_{m_0,H_0}, u_{m_1,H_1}, u_{m_2,H_2}\big] = c \cdot \frac{\pi^2}{2}\, V_{g_1}(H_0, H_1, H_2; m_0, m_1, m_2).$$

Here $D^j\Phi_C$ is taken at the flat $\kappa = 1$ lattice of side N, the plane waves are the midpoint-sampled cosine waves (2)*, and the integer wave vectors enter* FP *and V as covectors; the factor $(2\pi)^2$ of $k = 2\pi m/N$ is in the constants.*

The theorem carries these conditions: the same wave $m \neq 0$ on both legs of the quadratic vertex; every $m_l \neq 0$ and $\sum_l m_l = 0$ in the cubic vertex; symmetric polarizations; the normalization N^{-2}; and the flat $\kappa = 1$ background. The theorem says nothing about the full nonlinear action, about strong fields, or about other backgrounds.

One side is every side. For each $\kappa > 1/2$, stationarity of the flat lattice on one torus side $N \geq 3$ holds exactly when it holds on any other side $M \geq 3$, because the stationarity criterion of Proposition 4.1 does not involve the side. So side 3, the smallest torus that carries the Kuhn structure, is no restriction. The proof also gives stationarity on every side $N \geq 3$ at every $\kappa \geq 1$.

The premise class is not empty. Regge's charge $sA\delta$ meets every premise, with $\sigma(1) = s \neq 0$. So does the charge $sA\delta + A^2(1 - \cos k\delta)$, whose even part k^2A^2 is not Regge's (Section 7). The limits are not zero either. At $H_0 = H_1 = E_{23}$ and $m = (1, 1, 0, 0)$, where $E_{ab} = e_a e_b^\top + e_b e_a^\top$ for $a \neq b$, the Fierz–Pauli form is $-4/3$. At $m = (e_0, e_1, -e_0 - e_1)$ and $H = (E_{23}, E_{23}, E_{22})$, with $E_{aa} = e_a e_a^\top$, the cubic vertex is $\sqrt{3}/9$. Both values are exact.

Remark 3.2 (One Einstein–Hilbert action, and Newton's constant)**.** The two constants in Theorem 3.1 are those of a single action. Sum $\frac{1}{2}\sqrt{|\det g_1|}\, R$ over N^4 cells of unit coordinate volume, with $k = 2\pi m/N$. On two cosine waves of the same $m \neq 0$ the torus mean of the product of cosines is $\frac{1}{2}$, and the quadratic coefficient of $\sqrt{|g|}R$ is $-\frac{1}{2}\sqrt{3}\,\mathrm{FP}$, so the second derivative is $\frac{1}{2}\cdot N^4 \cdot \frac{1}{2}\cdot\big(-\frac{\sqrt{3}}{2}\big)(2\pi/N)^2\mathrm{FP} = -\frac{\sqrt{3}\pi^2}{2}N^2\mathrm{FP}$. On three waves with every $m_l \neq 0$ and $\sum_l m_l = 0$ only the two phase products with all signs equal survive, the torus mean is $\frac{1}{4}$, and the third derivative is $\frac{1}{2}\cdot N^4 \cdot \frac{1}{4}\cdot(2\pi/N)^2 V = \frac{\pi^2}{2}N^2 V$. So, through cubic order on plane waves around the flat $\kappa = 1$ lattice, the action of the charge is $(\sigma(1)/\hbar)\cdot\frac{1}{2}\int\sqrt{|g|}\,R\,\mathrm{d}^4y$ in lattice units: Einstein–Hilbert's, at both orders, with one factor. This normalization is a standard continuum computation done here by hand; the two limits themselves are Theorem 3.1. The physical action $\hbar\Phi_C$ is then $\frac{1}{2}\sigma(1)\int\sqrt{|g|}\,R$, and reading it as Einstein–Hilbert's $(16\pi G)^{-1}\int\sqrt{|g|}\,R$ gives $G = 1/(8\pi\sigma(1))$ in lattice units (INFERENCE). For Regge's charge this is the value $G = 1/(8\pi s)$ found from Newton's law on the same lattice [5]. The premise fixes neither the size of $\sigma(1)$ nor its sign: Newton's constant is not derived here, and $\sigma(1) > 0$, the sign of attraction, is a separate premise wherever it is needed (Theorem 6.4).

4 How the proof goes

The proof has three steps. Stationarity at one flat lattice reads the slope at four areas. Across all κ these readings become two functional equations, whose continuous solutions are linear. The slope then fixes both long-wave vertices, and the rest of the charge drops out.

4.1 Stationarity reads the slope at four areas

Because $C(A, 0) = 0$ for every A, the area derivative $\partial_A C$ vanishes at zero deficit, and at a flat lattice, where every deficit is zero, the first variation of the action is

$$D\Phi_C(\ell^\kappa) = \frac{1}{\hbar}\sum_h \sigma\big(A_h\big)\, D\delta_h.$$

Only the slope enters, and only at the four class areas. Grouping the hinges by class and evaluating the deficit rates on each of the fifteen edge classes turns this into two conditions.

Proposition 4.1 (The stationarity criterion)**.** *Let C be C^1 with $C(A, 0) = 0$, let $\kappa > 1/2$ and $N \geq 3$, and write $A_i = A_i(\kappa)$. The flat lattice ℓ^κ of side N is a stationary point of Φ_C if and only if*

$$\frac{\sigma(A_2)}{A_2} = \frac{\sigma(A_3)}{A_3} \qquad \text{and} \qquad A_0\,\sigma(A_0) - A_1\,\sigma(A_1) + \tfrac{1}{4}A_3\,\sigma(A_3) = 0.$$

At one value of κ these two conditions leave a two-dimensional family of slope values at the four areas: Regge's direction $\sigma(A_i) \propto A_i$, and a second direction $(1/A_0, 1/A_1, 0, 0)$. A charge

stationary at a single flat lattice can therefore have a slope that is not Regge's. The premise of Theorem 3.1 uses every $\kappa > 1$, and the second direction cannot persist across them.

4.2 Two functional equations

Put $u = 4\kappa - 1$ and $f(u) = \sigma(\sqrt{u})/\sqrt{u}$. By Table 2, $A_0^2 = (u-1)/8$, $A_1^2 = (3u-1)/8$, $A_2^2 = 3u/4$ and $A_3^2 = u$, so the two conditions of Proposition 4.1 read

$$f\big(\tfrac{3}{4}u\big) = f(u), \tag{5}$$

$$(u-1)\, f\big(\tfrac{u-1}{8}\big) - (3u-1)\, f\big(\tfrac{3u-1}{8}\big) + 2u\, f(u) = 0. \tag{6}$$

Regge's slope is f constant, and it satisfies both: the second becomes $(u-1)-(3u-1)+2u = 0$.

Proposition 4.2 (Rigidity). *Let f be continuous on $(0,\infty)$ and satisfy (5) and (6) for every $u > u_0$, where $u_0 \geq 1$. Then f is constant on $\big((u_0-1)/8, \infty\big)$.*

The idea is to read (5) and (6) as two periodicities with incommensurable periods. Let $P(x) = f(e^x)$. By (5), P has period $L = \log(4/3)$ far out, and the periodic extension is continuous and bounded. Divide (6) by u and follow $u = e^{x+nL}$ as $n \to \infty$. With $a = \log 2$ and $\log(3/8) = -L-a$, the limit is

$$2P(x) = 3P(x-a) - P(x-3a).$$

Along $x, x-a, x-2a, \ldots$ this is a linear recurrence with characteristic polynomial $r^3 - 3r + 2 = (r-1)^2(r+2)$, and its only bounded solutions are constant. So $a = \log 2$ is a period of P as well. Since $2^q = (4/3)^p$ forces $p = q = 0$, the ratio $\log 2/\log(4/3)$ is irrational, the periods are dense, and the continuous function P is constant. Equation (6) then carries the constant down to the stated range. With $u_0 = 1$ the conclusion covers all of $(0,\infty)$; the premise of Theorem 3.1 is $\kappa > 1$, that is $u > 3$, and gives f constant on $(1/4,\infty)$. In areas, $\sigma(A) = \sigma(1)A$ for every $A > 1/2$, and at $A = 1/2$ by continuity of σ. That is conclusion (i).

4.3 From the slope to the vertices

At a flat lattice, where every deficit vanishes and $C(A,0) = 0$, the second variation of the action is

$$\hbar\, D^2\Phi_C = \sum_h \Big[\sigma(A_h)\, D^2\delta_h + \sigma'(A_h)\big(DA_h \otimes D\delta_h + D\delta_h \otimes DA_h\big) + \partial_\delta^2 C(A_h,0)\, D\delta_h \otimes D\delta_h\Big].$$

With $\sigma(A) = \lambda A$ on $[1/2,\infty)$, and $\sigma' = \lambda$ there (at $A = 1/2$ by continuity of σ'), the first two terms are $\lambda\, D^2\big(\sum_h A_h\delta_h\big)$, which is λ times the Regge Hessian. The last term, the even part, carries two linearized deficits. On a plane wave a linearized deficit is a curvature, of order N^{-2}, and there are of order N^4 hinges, so the even part contributes $O(1) = o(N^2)$ and drops out of the long-wave limit. The third variation splits the same way. Write $C = \lambda A\delta + R$, so that $R(A,0) = 0$ for every A and $\partial_\delta R(A,0) = 0$ for $A \geq 1/2$. The surviving terms of D^3 of R are of the form $\partial_A\partial_\delta^2 R\, DA\,(D\delta)^2$, $\partial_\delta^3 R\,(D\delta)^3$ and $\partial_\delta^2 R\, D^2\delta\, D\delta$. On plane waves $D\delta = O(N^{-2})$ and $D^2\delta = O(N^{-1})$; the latter because the deficits vanish on the whole family of uniformly strained flat lattices, so the zeroth-order part of $D^2\delta$ cancels. Each hinge contributes $O(N^{-3})$, the sum is $O(N)$, and after the factor N^{-2} it vanishes. What remains is $\lambda/\hbar$ times the Regge vertices.

Two identities finish the proof. The long-wave quadratic vertex of the Regge action is $-\frac{\sqrt{3}\pi^2}{2}\mathrm{FP}_{g_1}$: its limit is a contraction of an integer coefficient table with the polarizations, and that table equals 24 times the Fierz–Pauli table on all 10^3 coefficient triples, a finite identity checked in exact arithmetic. The constant agrees with a second, independent route through the symbol of the Regge Hessian, whose leading term is $-\frac{1}{4}\sqrt{|\det g|}$ times the Fierz–Pauli form. The long-wave cubic vertex of the Regge action is $\frac{\pi^2}{2}V_{g_1}$: its two-derivative cubic symbol equals one half of the Einstein–Hilbert cubic vertex, off shell, at every $\kappa > 1/2$. With $\lambda = \sigma(1)$ this gives conclusions (ii) and (iii).

5 What the premise says

The first caution of the introduction is a theorem.

Theorem 5.1 (The premise is Regge's slope)**.** *Let C be C^1 with $C(A, 0) = 0$.*

(a) The flat lattice of side 3 is stationary for every $\kappa > 1/2$ if and only if $\sigma(A) = \sigma(1)A$ for every $A > 0$.

(b) It is stationary for every $\kappa > 1$ if and only if $\sigma(A) = \sigma(1)A$ for every $A \geq 1/2$.

The forward direction of (b) is conclusion (i) of Theorem 3.1, which needs only C^1. The converse holds because for $\kappa > 1$ every class area exceeds $1/2$, and $\sigma = \lambda A$ satisfies both conditions of Proposition 4.1: the second reads $\lambda(A_0^2 - A_1^2 + \frac{1}{4}A_3^2) = \lambda\big[\frac{2\kappa-1}{4} - \frac{3\kappa-1}{2} + \frac{4\kappa-1}{4}\big] = 0$.

So no premise of Theorem 3.1 mentions the Regge action, the Einstein–Hilbert action or Newton's constant, but given $C(A, 0) = 0$ the premise is equivalent to Regge's slope $\sigma(A) = \sigma(1)A$ on the areas it reads, and the action has Regge calculus's form, a sum of hinge charges $C(A_h, \delta_h)$ on the Kuhn triangulation. The result shows that a vacuum condition stated without Regge is Regge's slope, and that this slope alone, whatever the rest of the charge, fixes both long-wave vertices to the Einstein–Hilbert ones with one factor $\sigma(1)/\hbar$. The derivation uses the cell geometry of the deficits, the functional-equation rigidity of Proposition 4.2 and, as a comparison object only, the Regge action, whose long-wave vertices are Einstein–Hilbert's. No step assumes that the charge is Regge's or that its action is Einstein–Hilbert's.

The physical reading. The premise says that the flat lattice stays a vacuum, a stationary point of the summed hinge charge, when its time diagonal is uniformly stretched against its unit spatial steps, for every stretch beyond the $\kappa = 1$ value. It covers one family of constant time-strains with $\kappa > 1$. It does not ask the same of general constant strains, shears, compressed frames ($\kappa < 1$) or flat lattices with displaced vertices. A broader statement follows by an INFERENCE that is not one of the theorems: with Regge's slope, Schläfli's identity $\sum_h A_h \, \mathrm{d}\delta_h = 0$ [4] makes every flat configuration whose hinge areas are at least $1/2$ stationary. Whether the ledger itself requires the premise is OPEN.

6 How much of the premise is needed

Proposition 6.1 (One side and a dense set suffice)**.** *Let C be C^1 with $C(A,0)=0$. If the flat lattice of side 3 is stationary at every κ in a set D whose intersection with $(1,\infty)$ is dense in $(1,\infty)$, it is stationary at every $\kappa>1$. In particular, stationarity at every rational $\kappa>1$ suffices, and so does stationarity on any one torus side $N\geq 3$.*

The threshold cannot be raised. Below, $k\geq 1$ is any number of postings per turn.

Theorem 6.2 ($\kappa=1$ is sharp)**.** *For every $\kappa_0>1$ there is a record-compatible hinge charge of class C^3 whose action is stationary at the flat lattice on every side $N\geq 3$ for every $\kappa\geq\kappa_0$, and*

(a) whose long-wave quadratic vertex is, for no factor c, equal to c times the Regge action's on all plane waves;

(b) whose long-wave quadratic vertex on nonzero waves and symmetric polarizations tends to $c\cdot\left(-\frac{\sqrt{3}\pi^2}{2}\right)\mathrm{FP}_{g_1}(H_0,H_1;m)$ for no c.

The charge is $sA\delta+b(A)\sin(k\delta)/k$, with b a smooth bump equal to 1 at the area $1/2$ and supported between $1/2-r$ and $1/2+r$, where r is small enough that the support stays below the smallest class area $A_0(\kappa_0)=\sqrt{(4\kappa_0-2)/8}>1/2$ and away from 1. The sine vanishes at every whole posting, so the charge is record-compatible. At every $\kappa\geq\kappa_0$ the four class areas are at least $A_0(\kappa_0)$, where the slope is Regge's, so the flat lattice is stationary. But $\sigma(1/2)=s/2+1$ while $\sigma(1)=s$, and the quadratic long-wave vertex reads the slope at the four flat areas: agreement with c times Regge's would force $\sigma(A_i)\propto A_i$ there. Statement (b) is the same theorem for the Fierz–Pauli form itself. It holds because the argument behind (a) uses the long-wave vertex only on nonzero waves, and a plane wave depends on its polarization only through the symmetric part.

Smaller sets of κ do not force the full slope. These results concern the slope only; they do not decide whether the smaller sets already give the long-wave Einstein–Hilbert form.

Proposition 6.3 (Smaller sets do not force Regge's slope)**.** *For every $k\geq 1$ there are smooth record-compatible hinge charges, stationary at the flat lattice on every side $N\geq 3$,*

(a) at every $\kappa\in(1/2,5/4]$, or

(b) at every $\kappa=(pq+1)/(p+q)$ with integers $p,q\geq 1$,

whose slope is not $\sigma(1)A$ on $[1/2,\infty)$. More generally, for every $1\leq a<b\leq 5/2$ there is such a charge stationary at every $\kappa\geq 1$ outside (a,b) and not stationary at some κ in (a,b).

The charges are again Regge's plus $b(A)\sin(k\delta)/k$, now with a bump supported in an area window that no class area enters at the values of κ used: $(2,3)$ in case (a), where every class area is at most 2 for $\kappa\leq 5/4$; and $(1/2,\sqrt{3/8})$ in case (b), where every value of the set is 1 or at least $5/4$ and the window is entered only by $A_0(\kappa)$ with $1<\kappa<5/4$. Which sets of κ suffice for the long-wave form, and whether gaps above $5/2$ matter, is OPEN.

The premise is stronger than the classical clauses. The programme's classical clauses for a charge C at k postings per turn with factor c ask: $k > 0$ and $c > 0$; C record-compatible and C^3; on every side $N \geq 3$, the Hessian of Φ_C at the flat $\kappa = 1$ lattice vanishes, in either slot, on every direction that leaves every deficit unchanged to first order; the long-wave quadratic and cubic vertices are c times the Regge action's on all plane waves (with $\sum_l m_l = 0$ for the cubic one); and the flat $\kappa = 1$ lattice is stationary on every side $N \geq 3$. For a record-compatible charge with $k > 0$ and $\sigma(1) > 0$, the premise of Theorem 3.1 implies all of them with $c = \sigma(1)/\hbar$. The converse fails.

Theorem 6.4 (The converse fails)**.** *For every $k \geq 1$ there is a hinge charge that satisfies every classical clause with $c = s/\hbar$ and whose action is not stationary at the flat lattice of side 3 for every $\kappa > 1$.*

The charge is $sA\delta + b(A)\sin(k\delta)/k$ with a smooth bump b equal to 1 at the area $5/2$ and supported within $1/4$ of it. Below the area 2 its slope is Regge's, so its jets at the four flat areas, all at most $\sqrt{3}$, are Regge's, and every classical clause holds. But $\sigma(5/2) = 5s/2 + 1$ while $\sigma(1) = s$, so by Theorem 3.1(i) it cannot be stationary at every $\kappa > 1$. The premise fixes the slope at every area $A \geq 1/2$; the classical clauses fix it only to second order at the four flat areas.

7 A second premise: extensivity

A different premise gives the same conclusions. Call a hinge charge *extensive* when it adds over the pieces of a hinge at every deficit:

$$C(x + y, \delta) = C(x, \delta) + C(y, \delta) \qquad \text{for all real } x, y \text{ and every } \delta.$$

Its reading is that a hinge's charge is the sum of its pieces' charges at the same deficit. A continuous additive function is linear, so an extensive continuous charge is $C(A, \delta) = A\,C(1, \delta)$: its slope is $\sigma(1)A$ at every area, and by Theorem 5.1(a) the flat lattice is stationary at every $\kappa > 1/2$.

Theorem 7.1 (Extensivity gives the same conclusions)**.** *Let C be of class C^3, with $C(A, 0) = 0$ (for instance, record-compatible), and extensive. Then conclusions (i)–(iii) of Theorem 3.1 hold.*

The two premises are not of equal strength. Extensivity fixes the whole charge as $A\,f(\delta)$, its even part included; the stationarity premise fixes only the slope.

Proposition 7.2 (Extensivity is strictly stronger)**.** *For every $k \geq 1$ the charge $C(A, \delta) = sA\delta + A^2(1 - \cos k\delta)$ is record-compatible, of class C^3, and has Regge's slope sA, so its action is stationary at the flat lattice for every $\kappa > 1$. It is not extensive: at $\delta = \pi/k$, $C(2, \delta) = 2s\pi/k + 8 \neq 2C(1, \delta) = 2s\pi/k + 4$.*

So the even part of a charge that meets the premise is free: here it is k^2A^2, not proportional to the area, and by Theorem 3.1 it does not touch the long-wave vertices. Whether the ledger requires extensivity between whole postings is OPEN.

8 The classical chain for the Regge action at $\kappa = 1$

The vertices of Theorem 3.1 are statements about the action. The observed tests of gravity concern its solutions. This section connects the two for the Regge action itself, with the lattice's own notion of a long-wave solution. Everything here concerns the Regge action S_{R} at $\kappa = 1$ (and, for the bridge, every $\kappa > 1/2$). Carrying it to every charge that meets the premise of Theorem 3.1 is an INFERENCE: it needs the charge's long-wave first- and second-order equations to be a multiple of Regge's, which is not among the theorems.

Long-wave solutions. A *smooth family* is a jointly smooth map $h(\epsilon, y)$ of an amplitude ϵ and a point $y \in \mathbb{R}^4$ to symmetric 4×4 matrices, with $h(0,\cdot) = 0$. Its first and second coefficients are $h_1 = \partial_\epsilon h|_{\epsilon=0}$ and $h_2 = \frac{1}{2}\partial_\epsilon^2 h|_{\epsilon=0}$, and their jets at a point (value, first and second derivatives) are J_1 and J_2. On the torus of side N, read in unit-torus coordinates, the edge $e = (x, d)$ has midpoint $y_e = (x + d/2)/N$. The sampled field gives e the flat squared length plus the strain $d^\top h(\epsilon, y_e)\, d$, and a test field (W, φ) gives it $\varphi(y_e)\, d^\top W d$. The family *solves the lattice equations through second order at long wavelength near* x_0 if there is a ball around x_0 inside the open cube such that, for every symmetric test tensor W and every smooth test function φ supported in the ball, the second ϵ-derivative at $\epsilon = 0$ of the first variation of the Regge action at the sampled field, along the test field, is $o(N^2)$. At first order the same holds with the first ϵ-derivative. This is a statement about the limit $N \to \infty$ through second order in the amplitude, not about exact solutions at finite N.

Theorem 8.1 (Long-wave solutions are the solutions of the jet equations)**.** *Let $\kappa > 1/2$, let h be a smooth family and x_0 a point of the open unit cube. Then h solves the Regge lattice equations through second order at long wavelength near x_0 if and only if, on some ball around x_0 inside the open cube, its jets satisfy the second-order jet equation $E_2(J_2) + E_3(J_1) = 0$ at every point, for every symmetric test tensor; and it solves them at first order if and only if $E_2(J_1) = 0$ on some such ball.*

Here E_2 and E_3 are the long-wave quadratic and cubic operators of the lattice field equations on jets, read by symmetric test tensors. On sampled metric jets the metric part of the lattice field equations, which is the part these tests read, is Einstein's up to a constant factor [6, Thm. 4.2]; the factor drops out of these homogeneous equations.

The post-Newtonian parameters. At $\kappa = 1$ write the flat metric as $g_1 = -\hat{t}\hat{t} + S$, where $\hat{t}\hat{t} = \frac{3}{4}\mathbf{1}\mathbf{1}^\top$ is the time–time tensor of the unit time direction $\mathbf{1}/(2\sqrt{3})$ along the record diagonal and $S = I - \frac{1}{4}\mathbf{1}\mathbf{1}^\top$ is the spatial metric. Let $U = 1/r$, where r is the spatial distance from a static mass line along the record diagonal through $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$; at the cube centre $c_0 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $r = 1$. The *point-mass jets* are the jets at c_0 of

$$h_1 = 2U\big(\hat{t}\hat{t} + \gamma S\big), \qquad h_2 = U^2\big(\alpha\, \hat{t}\hat{t} + \beta' S\big),$$

and $g_{tt} = -1 + 2U - 2\beta U^2$ fixes the post-Newtonian $\beta = -\alpha/2$.

Theorem 8.2 ($\gamma = \beta = 1$ for the lattice's own solutions)**.** *At $\kappa = 1$, let h be a smooth family and x_0 a point of the open unit cube. If h solves the Regge lattice equations at first order at long*

wavelength near x_0 and its first coefficient has the point-mass jets at x_0, then $\gamma = 1$. If h solves them through second order near x_0 and its two coefficients have the point-mass jets with $\gamma = 1$ at x_0, then $\alpha = -2$, so $\beta = 1$. The conclusion does not involve the spatial coefficient β'.

These statements would be empty if no family met their hypotheses. One does.

Theorem 8.3 (A local Schwarzschild solution)**.** *At $\kappa = 1$, with $U = 1/r$ as above, smoothly cut off where $r^2 \le 1/4$, the family*

$$h(\epsilon, y) = \epsilon\, 2U(y)\big(\hat{t}\hat{t} + S\big) + \epsilon^2 U(y)^2\big(-2\,\hat{t}\hat{t} + \tfrac{3}{2}S\big)$$

is smooth and nonzero, solves the Regge lattice equations at long wavelength at first order and through second order on the ball of radius $1/8$ around the cube centre c_0, and has the point-mass jets with $\gamma = 1$, $\alpha = -2$ and $\beta' = 3/2$ at c_0.

The family is the isotropic Schwarzschild metric expanded through second order in the mass, $(1 + 2U + \frac{3}{2}U^2)$ on space and $-1 + 2U - 2U^2$ on time, and the cut-off region lies outside the ball, where $r^2 > 1/2$. The proof reduces the jet equations at every point of the ball to a quadratic form in the spatial direction, checked exactly at six points for each of ten basis test tensors.

So, for the Regge action, the lattice's own long-wave solutions at $\kappa = 1$ with point-mass jets have $\gamma = \beta = 1$, the values that light bending, the Shapiro delay and the perihelion advance measure [14], and such solutions exist. Three limits hold: the statements concern the isotropic point-mass jet ansatz; the solution is local, on a ball of radius $1/8$, long-wave and through second order in the amplitude, not exact at finite N; and $\kappa = 1$, the time direction and the midpoint strain embedding are the identifications of Section 2.5.

9 What stays open

1. *Newton's constant.* The factor $\sigma(1)$ is free. The premise fixes neither its size nor its sign; $\sigma(1) > 0$ is the sign of attraction and remains a premise. A derivation of G needs a further condition on the charge, or the ledger's own scale.

2. *A ledger origin of the premise.* Neither stationarity at every stretched flat lattice nor extensivity between whole postings is derived from the ledger. Either would make Theorem 3.1 a consequence of the foundation.

3. *The full nonlinear theory.* Theorem 3.1 concerns second and third order on plane waves around the flat $\kappa = 1$ lattice. Higher vertices, strong fields, other backgrounds and exact lattice solutions at finite strength are not covered.

4. *General strains.* The premise concerns one family of uniform time-stretches. Stationarity of other flat configurations, with shears, general constant strains or displaced vertices, whose hinge areas are all at least $1/2$ is an INFERENCE through Schläfli's identity, not a theorem here. Compressed time ($\kappa < 1$) is not covered: there the smallest class area is below $1/2$, where the premise, by Theorem 5.1(b), says nothing about the slope.

5. *Smaller sets of κ.* Which sets of κ give the long-wave form, as distinct from the full slope.

6. *The classical chain for every admissible charge.* Section 8 holds for the Regge action; for a general charge that meets the premise it is an INFERENCE.
7. *The identifications.* The squared edge lengths as a metric, time along the record diagonal, and $\kappa = 1$.

Verification and data availability. Every theorem in this paper is formally verified: each has a complete proof checked by computer from the standard logical foundations of mathematics, including the finite identities computed in exact arithmetic. Every theorem has also passed independent verification. The verified proofs are available from the author on request.

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