

Record-Time Root Stability of the Recognition Path Sum: Gaussian Evolution and the Continuum Limit of Lattice Quantum Gravity

Jonathan Washburn

Recognition Physics Institute, Austin, Texas

jon@recognitionphysics.org

Abstract

What evolution does a Lorentzian sum over lattice geometries define? On the recognition ledger the record count of four posting counters supplies a time before geometry is assigned. At Gaussian order, a positive norm preserved by a record tick requires every physical exponential solution to have a unimodular record factor. The bare Regge equation fails: at the identified value $\kappa = 1$ a spatially uniform physical solution grows by $\sqrt{2}$ per tick at every spatial period, and no nonzero rescaling of the action removes it. Its corner doublers also fail the weaker continuum test: an exact physical family decays at a nonzero rate per unit physical time as the spacing shrinks. For the equation with the ledger's derived curvature weight, we prove continuum-limit record-time root stability: $\log|w_n|/b_n \to 0$ for every non-gauge Gaussian mode family with spacing $b_n \to 0^+$ and $|w_n| \to 1$, at fixed nonzero weight strength. The proof combines decoupling away from the zero wave with a cubic bound on record growth near it. For specified holomorphic, gauge-invariant two-point observables at fixed momentum, contour contributions separated from the unit circle vanish faster than every power of the spacing at nonzero physical time. The native phase coefficient gives a fixed weight strength, independent of spacing. The proofs of the correlator suppression, the bare doubler family and the coupling relation await independent verification. The physical reading retains $\kappa = 1$ and two weight identifications as premises. These are spectral statements about a Gaussian equation; no positive physical Hilbert space or interacting unitary evolution is constructed. The proposed record-time Wick rotation ends at an indefinite quadratic form.

1 Does the path sum evolve unitarily in record time?

An evolution needs a time. On the recognition ledger the time comes before geometry: four counters advance one step at a time, each step a posting, and the record count $t(x) = \sum_a x_a$ orders the records [3]. The quantum gravity construction [1] assigns a real-time path sum to this lattice: the Lorentzian Regge action [4] on the Kuhn triangulation, at the identified value $\kappa = 1$ of the flat metric $I - \kappa \mathbf{1}\mathbf{1}^\top$, with a curvature weight derived under two physical identifications. That weight gaps the four corner doublers. Here we ask what the Gaussian field equation permits in record time.

An exponential solution $z^x u$ is multiplied by z^τ under a record tick, a translation τ that advances the record count by one. A positive norm preserved by the tick therefore requires $|z^\tau| = 1$ for every physical exponential solution. We call that necessary spectral condition record-time stability, and reserve record-time unitarity for the existence of a positive invariant Hermitian form on the physical solutions (Section 2). The distinction matters: stability of roots alone constructs neither a Hilbert space of physical states nor a unitary transfer operator.

What this paper establishes.

1. *A bare-action obstruction.* At $\kappa = 1$ a spatially uniform physical solution grows by $\sqrt{2}$ per record tick, so the bare linearized Regge equation has no record norm at any spatial period. The obstruction survives restriction to posting-null solutions and every nonzero rescaling of the action. At $\kappa = 3/2$ the growth factor is $(1 + \sqrt{10})/3$ (Section 3). An exact family near a bare corner doubler also fails continuum-limit stability (Theorem 5.4).
2. *The effect of a real weight.* On the unit torus a Hermitian action symbol and a positive semidefinite weight have only common null vectors. A gauge-definite weight therefore rules out lattice stability whenever a physical exponential solution away from the zero wave exists (Section 4). That existence hypothesis is essential.
3. *A weaker continuum criterion.* We follow non-gauge mode families with record factors approaching the unit circle and ask whether their growth rate per unit physical time vanishes. A model with fourth-order damping passes; its second-order counterpart fails (Section 5).
4. *The derived-weight result.* At every fixed nonzero strength of the derived weight, all such families approach the zero wave. Near it the absolute record growth per tick is bounded by a constant times the cube of the spatial momentum. Hence the Gaussian equation is stable in the continuum limit. Separate contour estimates suppress the far contribution to specified holomorphic, gauge-invariant two-point functions at fixed momentum faster than any power of the spacing. The native coupling is fixed in this limit (Section 6).
5. *Two related diagnostics.* The proposed record-time Wick rotation ends at an indefinite rational Regge form. Jensen's formula separates the frequency average of a log-determinant into a leading-coefficient term and an outside-root term; interpreting that split as a statement about a normalized quantum transfer kernel needs an additional identification (Section 7).

The scope of the result. The value $\kappa = 1$ is an identification; the foundation as stated does not force it [1]. Reading the weight as the ledger's requires two further identifications: (I1) the cost of a hinge record reads its area as the action does, and (I2) the Born weight of a geometry is the Gibbs weight of its cost. The continuum estimates are at fixed coupling and fixed physical momentum. They neither prove a nonempty physical graviton branch nor justify integrating the estimates over all momenta. They do not construct a positive physical Hilbert norm, resolve the zero-spatial-momentum correlator prescription, or establish an interacting transfer evolution. The posting-null obstruction, off-circle correlator estimates, exact bare doubler family and native coupling relation have complete proofs whose independent verification is pending; each is marked at its statement. Section 8 gives the remaining targets, and Table 1 records the conditions of the results.

Relation to other work. Lorentzian simplicial path integrals raise related questions about causal structure and quantum evolution [7, 8, 9]. Causal dynamical triangulations use a time-ordered triangulation and a Wick rotation [12, 13]; here the graph is fixed, the time is the record count and the proposed rotation acts on the metric parameter. Euclidean approaches ask for reflection positivity [10] and confront the indefiniteness of the gravitational action [11]. Our result concerns the roots of a specified Lorentzian Gaussian equation: the bare Regge equation fails both lattice and continuum stability, while the equation with the derived weight passes the continuum criterion under its stated identifications.

2 Record time and what unitarity requires

Status: definitions

Defined. Record time, record ticks and spatially periodic solutions of a linear lattice equation; record-time stability (every physical exponential solution changes by a unimodular factor per tick), record-time unitarity (a positive Hermitian form on the physical solutions that the tick preserves) and the weakest record norm; stability at every spatial period $N \geq 1$ as the target.

Proved. Unitarity implies stability, and in addition excludes Jordan growth at eigenvalue 1. The definitions are not empty: the zero phase fails them, and a propagating lattice wave equation is stable at every spatial period $N \geq 1$ (Proposition 2.2).

Not claimed. This invariant form does not choose an energy sign or establish the physical inner product induced by a Feynman propagator.

2.1 The lattice and its linearized equation

Vertices are points $x \in \mathbb{Z}^4$, the record coordinates of the programme: x_a is the reading of the a-th of four counters [3]. Every edge joins x to $x+d$ for one of the fifteen classes $d \in \{0,1\}^4 \setminus \{0\}$; the $4! = 24$ arrival orders of one step on each counter are the simplices of Kuhn's triangulation of the unit cube [5, 6]. An edge with $|d| = 1$ is a *posting*. The flat lattice of parameter κ has the metric $g_\kappa = I - \kappa \mathbf{1}\mathbf{1}^\top$ in record coordinates, Lorentzian for $\kappa > 1/4$; at $\kappa = 1$ every posting is null. That value is an identification: it is the value at which the lattice interval reads the ledger's posting order, and the foundation as stated does not force it [1]. Every statement below that is made at $\kappa = 1$ carries this identification.

A field h assigns a complex number $h(x,d)$ to every edge, the change of its squared length. The Lorentzian Regge action [4] of the flat lattice has a quadratic form at the flat configuration, and its linearized field equation $E_\kappa h = 0$ is a finite-range linear equation on $\mathbb{Z}^4$ with integer translations as symmetries. It is the operator of the path sum's own quadratic form: on every torus of side $N \geq 3$ and for $\kappa > 1/2$, for every real edge field h, the flat Hessian evaluated on (h,h) is the real part of the pairing of its periodic complexification with E_κ applied to that lift. A *vertex move* is the change of squared lengths produced by a displacement $\zeta : \mathbb{Z}^4 \to \mathbb{C}^4$ of the vertices, of any shape, growing ones included; vertex moves solve the Regge and derived-weight equations and represent gauge directions.

An *exponential field* has the form $h(x,d) = z^x z^{D(d)} u_d$ with phases $z \in (\mathbb{C}^\times)^4$, $z^x = \prod_a z_a^{x_a}$, a fixed monomial convention $z^{D(d)}$ for the edge class, and a polarization $u \in \mathbb{C}^{15}$. On such fields the equation acts through a Laurent polynomial matrix: $E_\kappa h = 0$ exactly when $B_\kappa(z^{-1})\, u = 0$, where B_κ is the Bloch matrix of the flat Hessian of [2, Sec. 4.2], now evaluated at complex phases. On the unit torus $|z_a| = 1$ this is the usual Bloch analysis; off it, an exponential solution grows or decays along the lattice.

2.2 Record time and ticks

The posting order supplies the natural time, the *record count* $t(x) = \sum_a x_a$. Its gradient $c = \mathbf{1}$ is timelike for every $\kappa > 1/4$, since $g_\kappa^{-1}(c,c) = 4/(1-4\kappa) < 0$, so the slices of constant record count are spacelike and the continuum limit of a good theory has real frequencies in record time; a failure in record time is not an artefact of a bad slicing. A *record tick* is a lattice translation τ with $t(\tau) = 1$. A field is *spatially periodic of period N* if it is invariant under the translations $N(e_a - e_b)$ for all a, b; these have record count zero, so record time stays open. For a linear equation E write $V_N(E)$ for its spatially periodic solutions of period N. A solution is *physical* if it is not a vertex move.

An exponential field of phases z is multiplied by z^τ under the tick τ. A positive norm that the tick preserves therefore needs $|z^\tau| = 1$ for every physical exponential solution: a physical mode with $|z^\tau| \neq 1$ would change its own norm.

2.3 Stability, an invariant form and a record norm

Definition 2.1 (Record-time stability and unitarity)**.** Let E be a linear lattice equation, N a spatial period and τ a record tick.

(i) E is *record-time stable* at (N,τ) if every physical exponential solution in $V_N(E)$ has $|z^\tau| = 1$.

(ii) E is *record-time unitary* at (N,τ) if there is a Hermitian sesquilinear form on fields that vanishes against every vertex move in $V_N(E)$, has positive real part $\mathrm{Re}\langle h,h\rangle > 0$ on every physical $h \in V_N(E)$, and is invariant under the tick on $V_N(E)$.

(iii) A *record norm* at (N,τ) is a real-valued function n on fields with $n(ch) = |c|^2 n(h)$ for complex c, strictly positive on every physical solution in $V_N(E)$, and unchanged by τ on $V_N(E)$.

E is *stable at every period* if it is record-time stable at every (N, τ) with $N \geq 1$ and $t(\tau) = 1$, and *unitary at every period* in the same sense.

A record-time unitary equation has a record norm, $n(h) = \mathrm{Re}\langle h, h\rangle$, and is record-time stable: if a physical mode is multiplied by c under the tick, invariance gives $|c|^2 = 1$. Unitarity says more. It also excludes Jordan growth at eigenvalue 1, solutions that grow linearly in record time while their phases stay unimodular; stability allows them. For that reason the target below is stability, and the strong form is kept as a stronger test.

The positive form in this definition concerns the solution space modulo vertex moves. It does not choose an energy sign or identify the form induced by a Feynman propagator. The norm obstructions below rule out this defined evolution on the stated physical solution space.

Period $N = 0$, no spatial periodicity at all, is excluded on purpose. It admits exponentials that grow along a spatial direction, and a tick that moves along that direction then multiplies a perfectly propagating mode by a factor off the unit circle: the propagating equation of Proposition 2.2 is record-time stable at every $N \geq 1$ and fails at $N = 0$, its physical mode $i^{x_0+x_1}2^{x_3}$ being doubled by the tick e_3.

Proposition 2.2 (The definitions are not empty)**.** *(i) The zero phase, $E = 0$, is neither record-time unitary nor stable at every period: the field equal to $2^{t(x)}$ on the class e_2 and zero elsewhere is a physical solution, and every tick doubles it.*

(ii) The equation $E = \mathrm{id}$ is unitary at every period; its only solution is zero.

(iii) The record-time wave equation

$$h(x + e_0) + h(x - e_0) - \tfrac{1}{4}\big(h(x + e_1 - e_2) + h(x - e_1 + e_2)\big) = 0$$

on each edge class, with dispersion $w + w^{-1} = \frac{1}{4}(z_1/z_2 + z_2/z_1)$, has physical solutions and is record-time stable at every spatial period $N \geq 1$ and every record tick.

Statements (i) and (iii) need the exclusion of vertex moves of any shape at complex phases. It comes from the pattern of the solution on three edge classes: an exponential field that vanishes on the classes e_0 and $e_0 + e_2$ and is nonzero on e_2 is not a vertex move whenever $(z_0 - 1)(z_0 z_2 - 1) \neq 0$ (Appendix A.1).

Two weaker notions are useful because they allow cutoff-scale behaviour that a continuum theory may never see.

Definition 2.3 (Gapped stability)**.** E is *gapped stable* if there is $\delta > 0$ such that every physical exponential solution of every period $N \geq 1$ whose phases have a common modulus r has either $r = 1$ or $|r - 1| \geq \delta$.

Stability at every period implies gapped stability; the zero phase is not gapped stable, and $E = \mathrm{id}$ is. The third notion, stability in the continuum limit, is the subject of Section 5.

3 The bare path sum has no unitary record-time evolution

Status: the bare Regge path sum

Proved. At $\kappa = 1$ the linearized Regge equation has a spatially uniform physical solution that every record tick multiplies by $\sqrt{2}$, so at every spatial period and every tick it is neither record-time stable nor record-time unitary and has no record norm (Theorem 3.2). At $\kappa = 3/2$ the factor is $(1 + \sqrt{10})/3$ (Theorem 3.4). Stability and unitarity depend only on the solution space, so every nonzero multiple of the Regge equation fails in the same way (Theorem 3.5).

Premises. $\kappa = 1$ (IDENTIFICATION) where stated; Gaussian order, the linearized equation in open record time.

Proof complete, independent verification pending. Ordering the path sum by record time does not remove the four corner doublers (Theorem 3.1). The corner modes and their causal cones, including their removal at $\kappa = 3/2$, have this status. The posting-null solutions carry the same norm obstruction (Proposition 3.3). The doublers are a gapless obstruction: the bare equation is not even gapped stable (Theorem 3.6), because they form a continuum species that is not stable in the continuum limit (Theorem 5.4).

Not claimed. Anything about the weighted equation (Sections 4 and 6), a nonlinear constrained evolution, or histories of finite record length.

3.1 Record time keeps the doublers

At $\kappa = 1$ the gauge-fixed operator of the flat lattice has a one-dimensional kernel at each of the four axis corners πe_c of the Brillouin zone, spanned by a vector that is not a vertex move: four massless doublers, each with its own Lorentzian cone, whose cones share no time direction with the graviton's [2, Prop. 6.1, Thm. 6.15]. Ordering the sum by record time does not remove them.[1]

Theorem 3.1 (Record time keeps the doublers [1, Thm. 3.2])**.** *Let $\kappa = 1$, with the Feynman regulator. At each axis corner πe_c and for every time direction of the long-wave graviton's cone, some spatial direction carries two real record-time frequencies of the gauge-fixed operator, both shrinking linearly with the distance from the corner, while along other directions there is no real record-time frequency: the doubler propagates in record time, and the corner spectrum is mixed. At πe_0 the directions with two real frequencies form an open set. For record time itself the two real frequencies are zeros of the native block on the eleven non-posting channels, not gauge artefacts. Summing the regulated path sum slice by slice in record time changes nothing at any $\varepsilon > 0$.*[2]

The forms of record-time ordering examined in [1] are the order of summation, a sign rule for the record frequency, a record-time vacuum, the first-order record-time Wick rotation, a rescaled or conjugated corner regulator, the projection onto real record frequencies and the record-extension operator; none removes the doublers. The present paper asks the next question: whether the record-time evolution, doublers included, can be unitary.

[1] Proof complete, independent verification pending.
[2] Proof complete, independent verification pending.

3.2 A growing mode at the identified value

Theorem 3.2 (Record-time growing mode). *Let $\kappa = 1$ and $r = \sqrt{2}$. There is a polarization $u \in \mathbb{Q}(\sqrt{2})^{15}$ such that the field*

$$h_d(x) = r^{t(x)+|d|}\, u_d$$

solves the linearized Regge equation on $\mathbb{Z}^4$, is spatially uniform, is not a vertex move of any shape, and is multiplied by $\sqrt{2}$ by every record tick. Consequently, at every spatial period N and every record tick τ, the linearized Regge equation is neither record-time stable nor record-time unitary, and it has no record norm.

Proof. The polarization is the null vector of the Bloch matrix at the phases $(r^{-1}, r^{-1}, r^{-1}, r^{-1})$ of [2, Prop. 6.16]; its entries and the exclusion of vertex moves are in Appendix A.2. The field has record count as its only dependence, so it is invariant under $N(e_a - e_b)$ for every N, and $t(x+\tau) = t(x) + 1$ gives $h(\cdot + \tau) = rh$. If n were a record norm, then

$$0 < n(h) = n\big(h(\cdot + \tau)\big) = n(rh) = r^2\, n(h) = 2\, n(h),$$

a contradiction. A record-time unitary form would give the record norm $\mathrm{Re}\langle h, h\rangle$, and stability would need $|r| = 1$. □

The theorem is about physical solutions, not about every evolution one could write down. What fails is a positive norm on the physical solutions that the tick preserves. A unitary operator on slice wavefunctions would not by itself establish that norm; the physical reduction and its equality with the same gravitational functional would also have to be supplied. Removing the growing mode would take a boundary condition or a constraint that follows from the ledger; discarding it by definition is not one.

The native integral varies eleven non-posting channels. The full bare equation has a posting-null growing representative, as follows. Relating the constrained equation of that integral to the full equation is a separate issue; this proposition concerns full-equation solutions with zero posting entries.

Proposition 3.3 (The obstruction survives the posting-null restriction). *Let $\kappa = 1$, $N \geq 1$ and τ a record tick. Among the solutions of Theorem 3.2, keep those whose posting components vanish, $h(x, e_a) = 0$ for every vertex x and every axis a. This smaller space still contains a solution that is not a vertex move and is multiplied by $\sqrt{2}$ by the tick, so it carries no record norm either.*[3]

Proof. Let h be the field of Theorem 3.2 and $p(a)$ the class of the posting e_a; the posting components of h are $h(x, e_a) = r^{t(x)+1} u_{p(a)}$. The vertex move generated by the lowered displacement $\zeta_a(x) = r^{t(x)} c_a$ with $c_a = r u_{p(a)} / \big(2(r-1)\big)$ has posting components $2\big(\zeta_a(x + e_a) - \zeta_a(x)\big) = 2r^{t(x)}(r-1)c_a = h(x, e_a)$, so $\hat{h} = h -$ (that move) has all posting components zero. The move solves the linearized equation at these exponential phases, since the gauge identity on the unit torus is an identity of Laurent polynomials and so holds at every nonzero complex phase; hence $\hat{h}$ is a solution. It has every spatial period and is multiplied by r by the tick, as h and the move are. It is not a vertex move, since otherwise h would be one. The argument of Theorem 3.2 applies to $\hat{h}$. □

[3] Proof complete, independent verification pending.

3.3 Other values and other phases

Record-time growth does not single out the identified value.

Theorem 3.4 (The value $\kappa = 3/2$)**.** *At $\kappa = 3/2$ the linearized Regge equation has a spatially uniform physical solution that every record tick multiplies by $x_0 = (1 + \sqrt{10})/3$. It is neither stable nor unitary at every period; already its spatially uniform modes fail, at $\kappa = 1$ and at $\kappa = 3/2$.*

The kernel vectors at both values are certified by exact arithmetic [2, Prop. 6.16]. At $\kappa = 3/2$ the four corner doublers are gone [1].[4] Thus the growing mode is not a doubler effect; it is a property of the Regge equation in record time.

Theorem 3.5 (Only the solution space matters)**.** *Record-time stability and record-time unitarity at (N, τ) depend on E only through its solution space $V_N(E)$. In particular, for every $c \neq 0$ the equation cE_1 is neither stable nor unitary at every period.*

At $\kappa = 1$ on a periodic lattice of side $N \geq 3$, a phase $cS + \sum_h \delta_h^3 \Gamma_h$, with $c \neq 0$ and every Γ_h C^3 at the flat field, has Hessian c times Regge's [1, Thm. 6.7]. Once its Gaussian field equation on $\mathbb{Z}^4$ is identified with the operator of that Hessian, Theorem 3.5 gives the same obstruction (INFERENCE: the operator of a general phase on $\mathbb{Z}^4$ is not constructed here). A phase that passes must change the quadratic action at the lattice scale.

3.4 The doublers as a gapless obstruction

The growing modes of Theorems 3.2 and 3.4 are isolated: their record factor stays a fixed distance from the unit circle, so they do not violate gapped stability. The doublers are a different matter. The exact construction in Theorem 5.4 gives physical record factors near πe_0 that approach the unit circle from off it.

Theorem 3.6 (The bare doublers are gapless)**.** *The $\kappa = 1$ linearized Regge equation is not gapped stable.*[5]

The statement is not trivially true: gauge modes are excluded, and the uniform growing mode does not decide it. It follows from its continuum form, Theorem 5.4, which exhibits an explicit family of exact solutions near the corner πe_0 whose record factors approach the unit circle from off it.

[4] Proof complete, independent verification pending.
[5] Proof complete, independent verification pending.

4 A weight is a damping

Status: a real weight in record time

Proved. For a Hermitian action symbol and a positive semidefinite weight symbol, the null vectors of the weighted symbol are the common null vectors (Theorem 4.1). In a record-time stable weighted equation of Fourier form every physical mode lies on the unit torus and the weight vanishes on it (Theorem 4.3). A weight that sees every physical polarization away from the zero wave therefore leaves no record-time stable equation as soon as one physical mode away from the zero wave exists (Corollary 4.4).

Not claimed. A weight that vanishes on every physical mode, the zero weight for instance, is not excluded; the statement "a real weight cannot give a unitary evolution" needs the premise of Corollary 4.4.

A measure weight e^{-W} with $W \geq 0$ multiplies the phase e^{iaS} of the path sum, where $a > 0$ is the phase coefficient that couples the action, not a lattice spacing. Absorb this fixed coefficient and the quadratic normalization into the action symbol M and the weight symbol $\mathscr{D}$. At Gaussian order the integrand e^{iaS-W} gives the equation $(M + i\mathscr{D})u = 0$, and with $M' = -M$ this is $(M' - i\mathscr{D})u = 0$, so nothing is lost by stating the results for $M - i\mathscr{D}$.

Theorem 4.1 (A weight damps)**.** *Let M be Hermitian and $\mathscr{D}$ positive semidefinite. If $(M-i\mathscr{D})v = 0$ then $\mathscr{D}v = 0$ and $Mv = 0$.*

Proof. $0 = v^\dagger(M - i\mathscr{D})v = v^\dagger Mv - i\, v^\dagger \mathscr{D} v$, where $v^\dagger Mv$ is real and $v^\dagger \mathscr{D} v \geq 0$. So $v^\dagger \mathscr{D} v = 0$, which for a positive semidefinite $\mathscr{D}$ gives $\mathscr{D}v = 0$, and then $Mv = 0$. □

To use this in record time, the weighted equation must act on exponential fields through symbols.

Definition 4.2 (Fourier form)**.** A linear lattice equation E has *Fourier form* with action symbol $M(z)$ and weight symbol $\mathscr{D}(z)$ if, at every phase $z \in (\mathbb{C}^\times)^4$, the exponential field of phases z and polarization u solves E exactly when $(M(z) - i\mathscr{D}(z))u = 0$, and on the unit torus $M(z)$ is Hermitian and $\mathscr{D}(z)$ positive semidefinite.

Theorem 4.3 (In a stable weighted equation the weight vanishes on the physical modes)**.** *Let E have Fourier form, $N \geq 1$ and τ a record tick. If E is record-time stable at (N, τ), every physical exponential solution in $V_N(E)$ lies on the unit torus, $|z_a| = 1$ for all a, and its polarization satisfies $\mathscr{D}(z)u = 0$. The same holds if E is record-time unitary at (N, τ).*

Proof. Spatial periodicity gives $(z_a/z_b)^N = 1$, so all four phases have one modulus ρ; stability gives $|z^\tau| = \rho^{t(\tau)} = \rho = 1$. On the unit torus Theorem 4.1 applies. □

Call a weight symbol *gauge-definite* if on the unit torus, at every phase $z \neq \mathbf{1}$, every null vector of $\mathscr{D}(z)$ is the polarization of a vertex move.

Corollary 4.4 (No stable equation with a gauge-definite weight)**.** *Let E have Fourier form with a gauge-definite weight symbol, $N \geq 1$ and τ a record tick. If $V_N(E)$ contains a physical exponential solution with phases $z \neq \mathbf{1}$, then E is neither record-time stable nor record-time unitary at (N, τ).*

Proof. By Theorem 4.3 such a solution would lie on the unit torus with $\mathscr{D}(z)u = 0$, so u would be a vertex-move polarization and the solution a vertex move. □

This is the exact content of the statement that a real weight cannot give a unitary evolution: it can, if it vanishes on the physical modes, and it cannot if it sees them. The weight derived from the ledger's cost of curvature sees them. It is fourth order in the wave number on long waves, positive on the doublers, and away from the zero wave its kernel consists of vertex moves [1]; its operator on $\mathbb{Z}^4$ has Fourier form (Theorem 6.1). Consequently any physical exponential solution away from the zero wave obstructs lattice stability (Theorem 6.9(iii)).

A cubic bound on absolute logarithmic growth per tick would give a bound of order $b^2|k|^3$ per unit physical time, at spacing b and fixed momentum k. It would therefore vanish in the continuum limit. We prove that bound for the derived equation in Section 6; fourth order of a general weight alone would not suffice. Continuum root stability remains a possible target even when physical modes obstruct lattice stability.

5 Stability in the continuum limit

Status: the continuum notion

Defined. A continuum family: physical exponential solutions at lattice spacings $b_n \to 0^+$ whose record factor tends to the unit circle; stability in the continuum limit: every such family grows by a vanishing amount per unit physical time (Definition 5.1).

Proved. Stability at every period implies gapped stability, which implies stability in the continuum limit; the zero phase fails, a propagating wave equation passes (Proposition 5.2). On a model wave equation a real damping of fourth order in the wave number, the order of the derived weight, is stable in the continuum limit for every nonzero strength but not gapped; a damping of second order, the order of the action, is not stable in the continuum limit (Theorem 5.3).

Proof complete, independent verification pending. The bare $\kappa = 1$ doublers are a continuum species that is not stable in the continuum limit: an explicit family of exact, spatially periodic physical solutions near a corner is multiplied by at most $e^{-3/4}$ per unit physical time (Theorem 5.4).

Not constrained by the definition. Modes that stay a fixed distance from the unit circle. Their contribution requires a separate estimate. For specified fixed-momentum observables the derived weight supplies one (Theorem 6.10).

Stability on the lattice may ask for more than physics needs, since a damping has to vanish only at long wavelengths. The weaker notion follows mode families whose record factor approaches the unit circle as the spacing shrinks.

Definition 5.1 (Stability in the continuum limit)**.** A *continuum family* of a linear lattice equation E consists of a unimodular lattice momentum $\zeta \in (S^1)^4$, a physical momentum $p \in \mathbb{R}^4$, spacings $b_n \to 0^+$, record factors $w_n \neq 0$ with $|w_n| \to 1$, and for each n a physical solution of some period $N_n \geq 1$ with phases $z_{n,a} = w_n \zeta_a e^{ib_n p_a}$. E is *stable in the continuum limit* if every continuum family has $\log|w_n|/b_n \to 0$.

Stability in the continuum limit is a property of the record-time roots of the Gaussian equation. It is weaker than stability on the lattice, and it does not construct a norm-preserving evolution on a space of states; an equation that has it may still have no record norm (Definition 2.1).

A record tick multiplies the n-th mode by w_n times a unimodular factor, and one tick is a time b_n, so $\log|w_n|/b_n$ is the growth per unit physical time. The definition says nothing about modes that stay a fixed distance from the circle: an equation in which every physical exponential mode has its amplitude doubled per tick has no continuum family and satisfies it vacuously. Lattice-scale branches of this kind are cut off at the lattice scale, and whether they leave the continuum theory alone is a statement about the regulated propagator that must be proved for each theory and observable. The fixed-momentum result for the derived weight is Theorem 6.10.

Proposition 5.2 (The hierarchy)**.** *Stability at every period implies gapped stability, and gapped stability implies stability in the continuum limit. The propagating equation of Proposition 2.2(iii) has all three properties. The zero phase has none: its family $w_n = e^{b_n}$ grows at rate exactly* 1 *per unit physical time.*

The second implication holds because a family with $|w_n| \to 1$ is eventually within the gap of the unit circle and hence on it. Both implications are strict. An equation whose physical exponential modes all have their amplitude doubled per tick is gapped stable and stable in the continuum limit but not stable; the damped model below with damping of fourth order is stable in the continuum limit but not gapped stable.

5.1 A damped model

The model is a record-time wave equation with a spatial Laplacian ℓ along $e_1 - e_2$ and a real damping of order ℓ^m,

$$h(x + e_0) + h(x - e_0) - 2h(x) + \ell h(x) - ic\,\ell^m h(x) = 0,$$

acting on each of the fifteen edge classes. It is a stand-in for a weighted equation, not the derived operator. A physical mode with $|z_1| = |z_2|$ has the factor $w = z_0$ for the tick e_0, with

$$w + w^{-1} = 2 - l + i\,c\,l^m, \qquad l \in [0, 4],$$

where l is the value of the Laplacian on the mode. Write $R = |w|^2$ and $g = c\,l^m$. Eliminating the argument of w gives the exact identity

$$g^2 R\,(R+1)^2 = (R-1)^2\big((R-1)^2 + (4 - (2-l)^2)R\big),$$

which controls how far the root sits from the circle.

Theorem 5.3 (Damping in the continuum limit)**.** *Let $c \neq 0$.*

(i) With damping of fourth order in the wave number, $m = 2$, the model is stable in the continuum limit: along a family with $\zeta_1 = \zeta_2$, eventually $(R-1)^2 \le 2R\,c^2(b_n q)^6$ with q fixed by the physical momentum, so $\log|w_n|/b_n = O(b_n^2)$; at a lattice momentum with $\zeta_1 \neq \zeta_2$ the damping keeps the roots away from the circle and no family exists. It is not gapped stable: its roots approach the circle from off it.

(ii) With damping of second order, $m = 1$, the model is not stable in the continuum limit: it has a continuum family of periods $n + 2$, with $b_n = \theta_n = 2\pi/(n+2)$ and $\big(\log|w_n|/\theta_n\big)^2 \ge \min(1, c^2/8)/(4\pi^2)$ eventually, so its growth per unit physical time has magnitude bounded away from zero (Appendix A.3).

The physical reading is the dispersion relation. With $\omega^2 \approx \ell - ic\ell^2$ the imaginary part of the frequency is about $cb^3k^3/2$ per tick, a growth of order b^2k^3 per unit time; with $\omega^2 \approx \ell(1-ic)$ it is of order bk per tick and the growth rate per unit time has magnitude bounded away from zero. So a damping of the order of the derived weight is invisible in the continuum limit, and one of the order of the action is not.

5.2 The bare doublers in the continuum

At $\mu = 0$, without a weight, the question is whether the doublers of the bare equation form a continuum species that is not stable.

Theorem 5.4 (The bare doublers are not stable in the continuum limit)**.** *The $\kappa = 1$ linearized Regge equation is not stable in the continuum limit. Near the corner πe_0 it has a continuum family of exact, spatially periodic physical solutions with physical momentum $p = (0, 2, -1, -1)$ and $\log|w_n|/b_n \le -3/4$ for every n.*[6]

The theorem implies Theorem 3.6. The spatially uniform growing modes of Theorem 3.2 stay off the circle and form no continuum family, so they do not settle this question. The exact doubler construction instead uses the null polarization at $c = (-1, 1, 1, 1)$ and the fixed momentum $p = (0, 2, -1, -1)$. After eliminating the complementary directions, the leading scalar equation along $\varphi = b\big(s(1,1,1,1) + p\big)$ is $(57s^2 + 99)/4 = 0$. Its roots are $s = \pm i\sqrt{99/57}$: this particular two-dimensional restriction has no real frequency. A bordered-matrix reduction and the minimum-modulus principle continue the root near $i\sqrt{99/57}$ to an exact root for every sufficiently small b. A sequence of spacings realizes exact spatial periods $2(n + N_0 + 1)$, and a plane functional that annihilates every vertex move proves that the resulting fields are physical. The bound $-3/4$ holds for the chosen decaying branch; it gives a nonvanishing rate even as $b \to 0$.

6 The derived weight: Gaussian roots in the continuum limit

Status: the derived-weight Gaussian equation

Proved. The finite-range weight has symbol twice the derived Bloch symbol. Both action and weight annihilate vertex moves at nonzero complex phases. The established kernel identity removes continuum families away from the zero wave. Every physical root near that wave obeys a cubic bound on absolute record growth, giving continuum root stability at each fixed $\mu \neq 0$ (Theorems 6.1, 6.5, 6.8 and 6.9 and Proposition 6.4). A physical off-zero exponential solution, when present, obstructs lattice stability. At $\mu = 0$ a static corner family does not decouple (Proposition 6.6).

Proof complete, independent verification pending. The native Gaussian coupling is $\mu = \pi/\log 2$, independent of spacing (Theorem 6.2). For holomorphic, gauge-invariant observable symbols at fixed physical momentum, far contour coefficients are exponentially small in record separation (Theorem 6.10).

Premises. $\kappa = 1$ (IDENTIFICATION); dual-cycle channel counts; Gaussian order. The physical

[6] Proof complete, independent verification pending.

reading of the weight retains (I1), the hinge-area cost identification, and (I2), the geometry amplitude/Born-weight identification.

Not claimed. Existence of a physical graviton branch; momentum-integrated convergence; a physical prescription at zero spatial momentum; a positive physical Hilbert norm or interacting transfer evolution.

The quantum measure of [1] has amplitude modulus $e^{-C/2}$ for ledger cost C, giving Born/Gibbs probability e^{-C}. At Gaussian order, under two identifications, (I1) that the cost of a hinge record reads its area as the action does and (I2) that the Born weight of a geometry is the Gibbs weight of its cost, the weight acts through the positive semidefinite form

$$\mathscr{D} = \tfrac{1}{2} \sum_h \frac{A_h}{n_h} \delta'_h \delta'^{\top}_h,$$

with A_h the flat area of the hinge h, δ'_h the derivative of its deficit at the flat lattice and n_h the channel count of the hinge, here the dual-cycle count [1, Thm. 5.4]. It vanishes on constant strains and vertex moves, is of fourth order in the wave number on long waves, and is positive on the four doublers [1, Thm. 5.5]. This section follows that weight into record time.

6.1 The weight as an operator on the lattice

Two symbol conventions will be used explicitly. In the exponential field of Section 2.1, the actual edge amplitude is $v = T(z)u$, with $T(z) = \mathrm{diag}(z^{D(d)})$. Let $W_{\mathrm{bp}}(z)$ denote the weight symbol acting on this base-point amplitude. In the monomial polarization u it is $\widehat{W}(z) = T(z)^{-1} W_{\mathrm{bp}}(z) T(z)$. The base-point Regge and weighted symbols are

$$H_1(z) = T(z) B_1(z^{-1}) T(z)^{-1}, \qquad \mathscr{S}_\mu(z) = H_1(z) + i\mu W_{\mathrm{bp}}(z).$$

The conjugation follows from the exact cell-symbol identity. On the unit torus $T(z)$ is unitary, so the two conventions preserve Hermiticity, positivity and the physical kernel.

Theorem 6.1 (The derived weight on $\mathbb{Z}^4$)**.** *There is a finite-range linear operator W on fields on $\mathbb{Z}^4$, built from the linearized deficits of the hinges, whose Gram form, a sum over the slots of the hinges with weights A/n^2, equals $\sum_h (A_h/n_h)\, \delta'_h(g)\, \delta'_h(h)$, twice the derived Hessian. For real μ let $E_\mu = E_1 + i\mu W$.*

(i) On the base-point amplitudes of exponential fields, W acts through a Laurent symbol $W_{\mathrm{bp}}(z)$, a sum of products of a column and a row per hinge class, at every phase $z \in (\mathbb{C}^\times)^4$.

(ii) At $z = e^{ik}$ the base-point symbol is twice the derived Bloch symbol $\mathscr{D}(k)$ of [1], and it is positive semidefinite on the whole unit torus.

(iii) E_μ has Fourier form (Definition 4.2): for $\mu \geq 0$ with action symbol $-B_1(z^{-1})$ and weight symbol $\mu\widehat{W}(z)$, for $\mu \leq 0$ with action symbol $B_1(z^{-1})$ and weight symbol $-\mu\widehat{W}(z)$.

6.2 The phase coefficient and the native weight strength

The normalization can be fixed exactly. Write $a = s/\hbar \neq 0$ for the phase coefficient, $\mathscr{G}(k) = \Gamma(e^{ik})\Gamma(e^{ik})^\dagger$ for the canonical gauge-fixing term, where $\Gamma(z)_{d,a} = 2D(d)_a(z^{D(d)} - 1)$, and $\mathcal{S}_\mu(z)$ for the weighted symbol in the same Bloch convention as the Gaussian precision.

Theorem 6.2 (The Gaussian coupling)**.** *For real k and regulator ε, the gauge-fixed precision is*

$$\mathscr{P}_\varepsilon(k) = \varepsilon I - ia\big(\mathcal{S}_{1/(2a)}(e^{ik}) + \mathscr{G}(k)\big).$$

At $\varepsilon = 0$ its kernel is therefore the kernel of the bracketed matrix. The native phase coefficient $a = \log 2/(2\pi)$ gives $\mu = 1/(2a) = \pi/\log 2 \neq 0$, so the native weighted equation has the continuum root stability of Theorem 6.9.[7]

The factor 2 is fixed by $W_{\mathrm{bp}}(e^{ik}) = 2\mathscr{D}(k)$; multiplying by $-ia$ gives the Gaussian precision's real weight $\mathscr{D}(k)$ and imaginary action part. The phase coefficient a is never a lattice spacing. In the native geometric scaling $\ell = t^{-2}(\ell_{\mathrm{flat}} + tx)$, the action and cost weight carry the same factor t^{-2}. At quadratic order both factors cancel against the two powers of the fluctuation tx, leaving this spacing-independent precision [1]. Thus the fixed-μ theorem covers the native coefficient. A separately chosen family with varying phase coefficient would require additional uniform estimates; no such uniformity is asserted here.

6.3 Vertex moves at complex phases

The gauge invariance of the action and of the weight is proved on the torus, at real waves. Record time needs it at complex phases, where the modes grow or decay. A polynomial identity carries it there.

Lemma 6.3 (Laurent extension)**.** *A Laurent polynomial in four variables with exponents in $[-L, L]^4$ that vanishes at all points $(\omega^{m_0}, \dots, \omega^{m_3})$, $\omega = e^{2\pi i/N}$, for some $N > 2L$, vanishes at every point of $(\mathbb{C}^\times)^4$.*

Proof. By orthogonality of the N-th roots of unity, $\sum_m z_m^y = N^4[y = 0]$ for $|y_a| < N$, where $z_m = (\omega^{m_0}, \dots, \omega^{m_3})$. So every coefficient is an average of the values at the points z_m and vanishes. □

Proposition 6.4 (Vertex moves at every complex phase)**.** *The linearized Regge equation E_1 and the derived weight W annihilate every exponential vertex move at every phase $z \in (\mathbb{C}^\times)^4$. Hence so does E_μ, for every μ.*

For the action the input is the Bianchi identity of the flat Bloch matrix on the torus, a Laurent polynomial with exponents in $[-2, 2]^4$ checked on the torus of side 5; for the weight it is the vanishing of the linearized deficits on vertex moves, with exponents in $[-3, 3]^4$ on the torus of side 7 [1].

[7] Proof complete, independent verification pending.

6.4 The lattice species decouple

The kernel identity of the derived form says that at every real wave $k \notin 2\pi\mathbb{Z}^4$ the kernel of $\mathscr{D}(k)$ consists of vertex moves [1, Thm. 5.6]. With Theorem 4.1 it removes every species away from the zero wave from the continuum limit.

Theorem 6.5 (The damped species decouple)**.** *Let $\kappa = 1$ and $\mu \neq 0$. Every continuum family of E_μ has phases tending to the zero wave $z = \mathbf{1}$: no doubler and no other lattice species away from $k = 0$ approaches the unit circle.*

Proof. Suppose a continuum family has a subsequence of phases converging to a point $z^* \neq \mathbf{1}$; since $|w_n| \to 1$, the point lies on the unit torus. Split each polarization into its vertex-move part and the part orthogonal to the vertex-move columns, and normalize the second, which is nonzero because the solutions are physical. A further subsequence converges to a unit vector v orthogonal to the vertex moves at z^* with $(H_1 + i\mu W_{\mathrm{bp}})(z^*)v = 0$. By Theorem 4.1 the weight annihilates it, by the kernel identity it is a vertex-move polarization, and being orthogonal to the vertex moves it is zero, a contradiction. □

The weight is what decouples them.

Proposition 6.6 (Without the weight the doubler stays)**.** *Let $z_c = (-1, 1, 1, 1)$ and let v be the corner null polarization of the Bloch matrix at πe_0 in the monomial convention of Section 2.1. The static field $h_d(x) = (-1)^{x_0 + D(d)_0} v_d$ is physical and has spatial period 2. With $\zeta = z_c$, $p = 0$, $w_n = 1$, $b_n = 1/(n+1)$ and $N_n = 2$, it is a continuum family of the bare equation whose phases stay at z_c.*

It is a solution because v is a null vector of the Bloch matrix itself, the gauge-fixing term dropping out by the Bianchi identity, and it is physical because the plane functional of the (e_0, e_1) plane, which annihilates every vertex move, is nonzero on it. The doubler sits on the unit circle, so this is a failure of decoupling, not a growing mode.

Decoupling has exactly the meaning of Theorem 6.5: no continuum family accumulates away from the zero wave. Off-circle record roots of a weighted equation need not decay: the damped model of Section 5 has reciprocal roots, growing ones included. For the derived-weight equation, the far contour contributions to the specified fixed-momentum correlators are bounded at the end of this section (Theorem 6.10).

6.5 The graviton's branch

Any physical branch that remains must approach the zero wave. Its record roots must approach the circle fast enough: a growth per tick of order $b|p|$ would survive the continuum limit, one of order $(b|p|)^3$ does not. Four exact facts decide it.

Proposition 6.7 (Ingredients of the branch bound)**.** *Let $\kappa = 1$, let $\mathrm{EF}(k)$ be the linearized Einstein (Fierz–Pauli) form of the flat metric extended to complex covectors k, and let $\tau = \mathbf{1}$.*

(i) In a frame centred on the edge midpoints the Regge symbol at complex covectors is even in k, and the strain block of its second-order part is the lattice scale times $\mathrm{EF}(k)$.

(ii) *The derived weight's hinge rows annihilate long-wave strains to second order at complex covectors:* $|\mathrm{row}(k)(Vh)| \le 180\, R_A |k|^2 |h|$ *for* $|k| \le 1/3$, *where* V *maps a strain to its midpoint field and* R_A *is a constant of the rows. Here* $|k| = \sum_a |k_a|$ *and* $|h|$ *is the maximum absolute strain coordinate.*

(iii) *For each fixed* μ *there are* $C, \delta > 0$ *such that every physical null vector of the weighted symbol at* $|k| \le \delta$ *yields a unit strain* h *orthogonal to the corresponding midpoint vertex-move strains, with* $|\mathrm{EF}(k)h| \le C|k|^4$. *After rescaling, the midpoint vertex-move strains tend to their continuum counterparts.*

(iv) *At a real null covector* $q \ne 0$ *and a nonzero null vector* h *of* $\mathrm{EF}(q)$ *orthogonal to the vertex moves, the Krein form* $h^\dagger B(q,\tau)h = -\frac{1}{3}\big(\sum_a q_a\big)\Phi(h,h)$ *is nonzero, where* Φ *is the squared norm of the spatial part in temporal gauge and* B *is the polarization of the quadratic form* EF, $\mathrm{EF}(q + t\tau) = \mathrm{EF}(q) + 2tB(q,\tau) + t^2\mathrm{EF}(\tau)$; *and for real* q *and* ρ,

$$\mathrm{Im}\big(h^\dagger\, \mathrm{EF}(q - i\rho\tau)\, h\big) = -2\rho\, \mathrm{Re}\big(h^\dagger B(q,\tau)\, h\big).$$

Statement (ii) uses only the gauge invariance of the rows: their Taylor coefficients vanish on vertex-move strains, and a lattice Christoffel identity removes the first two orders. Statement (iii) has no term of order $|k|^3$ because of (i) and (ii): the weight is of fourth order on the strains and the action's correction beyond the Einstein form is even. A null vector of EF is Ricci-flat and, off the null cone, a vertex move, and the continuum cone has no complex record frequency: $\mathrm{EF}(q - i\rho\tau)$ singular on a physical strain with q, ρ real forces $\rho = 0$.

Theorem 6.8 (The graviton-branch bound)**.** *Let* $\kappa = 1$ *and* μ *real. There are* $C, \delta > 0$ *such that every physical null vector of the weighted symbol of* E_μ *at phases* z *of one modulus within* δ *of the zero wave satisfies*

$$\big|\log|z_0|\big| \le C\Big(\sum_{\alpha,\beta} |z_\alpha/z_\beta - 1|\Big)^3.$$

Proof. Suppose no such constants exist. Choose physical roots $z_n \to \mathbf{1}$ for which $|\rho_n| > (n+1)r_n^3$, where $\rho_n = \log|z_{n,0}|$ and $r_n = \sum_{\alpha,\beta} |z_{n,\alpha}/z_{n,\beta} - 1|$. In a logarithmic chart write $z_n = e^{ik_n}$ with $k_n = q_n - i\rho_n\tau$, q_n real and $\tau = \mathbf{1}$; equality of the phase moduli gives this common imaginary part. Put $\lambda_n = \sum_a |k_{n,a}| > 0$ and $\hat{k}_n = k_n/\lambda_n$. By Proposition 6.7(iii) there are unit strains h_n with $\|\mathrm{EF}(k_n)h_n\| \le C\lambda_n^4$, orthogonal to the midpoint vertex-move strains.

Pass to a subsequence on which $(\hat{k}_n, h_n) \to (k_*, h_*)$. Both limiting norms are one. Quadratic homogeneity of EF gives $\mathrm{EF}(k_*)h_* = 0$, and the rescaled midpoint gauge strains tend to the continuum gauge strains, so h_* is orthogonal to the latter. A nonzero physical Einstein kernel vector requires a null covector. A complex null covector whose imaginary part is parallel to the timelike τ is real. Thus $k_* = q_*$ is real, nonzero and null. The Krein form of Proposition 6.7(iv) is nonzero at (q_*, h_*).

Write $\hat{q}_n = q_n/\lambda_n$. By continuity, the absolute value of $\mathrm{Re}(h_n^\dagger B(\hat{q}_n, \tau)h_n)$ is bounded below by a positive constant along this subsequence. The exact imaginary-part identity and the reduced-root estimate therefore give

$$2|\rho_n|\lambda_n \big|\mathrm{Re}(h_n^\dagger B(\hat{q}_n, \tau)h_n)\big| \le C'\lambda_n^4, \qquad |\rho_n| \le C''\lambda_n^3.$$

A nonzero real null covector cannot be parallel to the timelike τ, so $\sum_{\alpha,\beta}|q_{*,\alpha}-q_{*,\beta}|>0$. For small real s, $|e^{is}-1|$ bounds $|s|$ below by a fixed positive multiple. Since $z_{n,\alpha}/z_{n,\beta}=e^{i(q_{n,\alpha}-q_{n,\beta})}$, it follows that $r_n\geq c\lambda_n$ eventually for some $c>0$. Hence $|\rho_n|\leq C'''r_n^3$, contradicting the chosen sequence. □

The ratios z_α/z_β remove the record factor and measure the spatial momentum. The theorem bounds record growth per tick by a constant times the cube of that momentum, without constructing a physical branch.

6.6 The derived-weight path sum in record time

Theorem 6.9 (The derived-weight path sum in record time)**.** *Let $\kappa=1$ and $\mu\neq 0$.*

(i) Every continuum family of E_μ has phases tending to the zero wave.

(ii) E_μ is stable in the continuum limit: along every continuum family, eventually $|\log|w_n||/b_n\leq CS^3b_n^2\to 0$, with S fixed by the family's physical momentum.

(iii) E_μ is not record-time stable at any spatial period $N\geq 1$ at which it has a physical exponential solution away from the zero wave.

Proof. (i) is Theorem 6.5. Along a family that tends to $z=\mathbf{1}$ the lattice momentum is uniform, $\zeta_\alpha=\zeta_\beta$, so the phase ratios are $e^{ib_n(p_\alpha-p_\beta)}$ and $\sum_{\alpha,\beta}|z_\alpha/z_\beta-1|\leq b_nS$; Theorem 6.8 gives (ii). (iii) is Corollary 4.4, since the kernel identity makes every nonzero multiple of the weight symbol gauge-definite. □

6.7 Far contour coefficients at fixed momentum

Root stability constrains families approaching the unit circle. A separate analytic estimate is needed for the remaining contribution. We state that estimate for linear observable symbols $O(z),O'(z)\in\mathbb{C}^{15}$ that are holomorphic on $(\mathbb{C}^\times)^4$ and annihilate every vertex-move polarization there; linearized hinge deficits are examples.

For the symbol $\mathcal{S}_\mu(z)$, define the bilinear correlator $F(z)=O(z)\cdot u$ by solving $\mathcal{S}_\mu(z)u=O'(z^{-1})$. Where a transverse gauge fixing is invertible this pairing is independent of the chosen solution and of that fixing. At a point where no solution exists we set the pairing to zero; that algebraic convention supplies no physical singularity prescription. For fixed p,ζ set $F_b(w)=F(w\zeta e^{ibp})$ and define

$$C_b(n;\rho)=\frac{1}{2\pi i}\oint_{|w|=\rho}w^{-n-1}F_b(w)\,\mathrm{d}w.$$

When the unit contour is regular, set $G_b(n)=C_b(n;1)$. These are coefficients of the specified Gaussian symbol, with the overall covariance scalar kept separate. At $\mu=1/(2a)$, $a\neq 0$ and $\varepsilon=0$ in a valid invertible fixing, Theorem 6.2 gives that scalar as i/a; it does not change the decay estimates.

Theorem 6.10 (Off-circle contour suppression)**.** Proof complete, independent verification pending. *Fix $\kappa=1$, $\mu\neq 0$, the two observable symbols just described and $p\in\mathbb{R}^4$.*

(i) *Let $|\zeta_a| = 1$ for all a, with $\zeta_a \neq \zeta_c$ for some pair a, c. There exist $R > 1$, M and $b_0 > 0$, independent of $0 < b \leq b_0$ and $n \in \mathbb{Z}$, such that $|G_b(n)| \leq MR^{-|n|}$.*

(ii) *In the zero-wave sector $\zeta = \mathbf{1}$, there exist $R > 1$, $C > 0$, M and $b_0 > 0$ such that F_b is holomorphic on both closed annuli $e^{Cb^3} \leq |w| \leq R$ and $R^{-1} \leq |w| \leq e^{-Cb^3}$ for $0 < b \leq b_0$. Define*

$$G_b^{\mathrm{far}}(n) = \begin{cases} C_b(n; R), & n \geq 0, \\ C_b(n; R^{-1}), & n < 0. \end{cases}$$

Then $|G_b^{\mathrm{far}}(n)| \leq MR^{-|n|}$, and either contour may be moved within its stated annulus without changing the coefficient. Whenever the unit contour is regular, the remaining part is the contour difference $G_b(n) - G_b^{\mathrm{far}}(n)$.

(iii) *For $T > 0$, $n = \pm\lceil T/b \rceil$ and every integer $k \geq 0$, $b^{-k} G_b(n) \to 0$ in (i), and $b^{-k} G_b^{\mathrm{far}}(n) \to 0$ in (ii), as $b \to 0^+$.*

The proof uses the kernel identity to obtain an invertible transverse fixing near each spatial off-diagonal phase. Compactness in the record phase supplies an annulus with a bound uniform in small b. In the zero-wave sector the cubic branch bound gives the inner edges $e^{\pm Cb^3}$, and the kernel identity supplies the outer circles. Cauchy's theorem moves the contours; their lengths and the factor w^{-n-1} give $MR^{-|n|}$. Exponential decay at $|n| \geq T/b$ dominates every fixed power of b^{-1}.

Here a transverse fixing must obey the gauge injectivity and surjectivity conditions as well as invertibility. For the zero-wave sector with some $p_a \neq p_c$, the unit-circle coefficient is regular for sufficiently small b. If all p_a agree, the contour encounters the zero-wave singularity; a physical regulator-removal prescription remains open. The contour difference above is not yet identified with a sum of residues. The constants may depend on p, μ, ζ, O, O'; no interchange with an integral over momentum or a coincident-time limit follows.

The two conclusions therefore have distinct scopes: Theorem 6.9 bounds the roots of every admitted Gaussian mode family, and Theorem 6.10 bounds particular fixed-momentum contour coefficients. Neither constructs a norm-preserving physical evolution.

6.8 Numerical evidence and the existence question

Exploratory finite scans of the axial-gauge weighted symbol found candidate roots near the zero wave consistent with cubic logarithmic growth (NUMERICAL). They motivate the search for a nonempty physical branch but do not prove branch existence, a limiting rate, an asymptotically sharp bound, two physical polarizations or an exact pairing of growth rates. We therefore retain branch existence as Target 8.2; the universal bound of Theorem 6.8 is independent of this numerical evidence.

7 Wick rotation and the one-loop remainder

Status: two consequences

Proved. At the endpoint of the record-time Wick rotation the rational Regge cell form is indefinite on every torus (Theorem 7.1). For a polynomial in the record-time frequency, the log-modulus averaged over frequencies splits exactly into a normalization part and a growth part, and the growth part vanishes exactly when every root lies in the closed unit disk (Proposition 7.2).

Inferences. That the endpoint of the rotation is $\kappa = -1/2$; that a unitary record-time evolution with the normalization of a unitary kernel has no imaginary part in the zero-momentum remainder.

7.1 The record-time Wick rotation

A Euclidean formulation would make the Gaussian measure a real weight and allow reflection positivity [10] to be posed. In record time the natural rotation turns the record-time part of the interval: rotating it by an angle φ is a complex value $\kappa = \frac{1}{4} + \frac{3}{4}e^{i\varphi}$ of the metric parameter [1], whose endpoint $\varphi = \pi$ is $\kappa(\pi) = -1/2$. That the continuation lands there is a reading of the rotation, not a contour theorem (INFERENCE). For rational $\kappa > 1/2$ the Lorentzian Hessian is a rational cell form divided by $\sqrt{4\kappa - 1}$, so the rotated form is the rational cell form at $\kappa = -1/2$.

Theorem 7.1 (The rotated form is indefinite)**.** *At $\kappa = -1/2$ the rational Regge cell form on the torus of side $N \geq 1$ takes the value $-\frac{3}{2}N^4$ on the field equal to one on every edge of class $(1, 1, 1, 1)$ and zero elsewhere, and the value $\frac{15}{4}N^4$ on the field equal to one on every edge of class e_0 and zero elsewhere.*

Indefiniteness makes the real Gaussian integral of the exponential of any nonzero real multiple of this form diverge on the full field space (INFERENCE, the standard Gaussian consequence). A real positive Gaussian measure on that space is therefore unavailable without an additional prescription. No reflection-positivity construction is supplied here. The endpoint metric $g_{-1/2} = I + \frac{1}{2}\mathbf{1}\mathbf{1}^\top$ is positive definite, so the proposed continuation has a Euclidean endpoint metric but an indefinite quadratic action, as in the familiar Euclidean gravitational problem [11]. This does not identify the particular negative lattice modes with the continuum conformal factor.

7.2 Unitarity and the zero-momentum remainder

The reported computation of the one-loop remainder in [1, Sec. 6] gives a mostly imaginary response to its transverse-traceless probe in the native eleven non-posting channels (NUMERICAL). This is evidence from that diagram model, rather than an established weighted continuum limit. An exact polynomial identity provides a related record-time diagnostic.

Proposition 7.2 (Jensen split)**.** *For a nonzero polynomial $P(w) = c\prod_j (w - w_j)$ in the record-time variable, with $c \neq 0$ and roots counted with multiplicity,*

$$\frac{1}{2\pi}\int_0^{2\pi} \log|P(e^{is})|\,\mathrm{d}s = \log|c| + \sum_{|w_j|>1} \log|w_j|,$$

a normalization part and a growth part; the growth part vanishes exactly when every root lies in the closed unit disk. In particular, if every root of a unitary slicing lies on the unit circle and the polynomial carries the normalization of the unitary kernel, the average is the normalization alone.

Jensen's formula [14] gives the identity by adding the circle averages of the logarithms of the linear factors. Each root contributes $\max(0, \log|w_j|)$, a nonnegative number, giving the stated zero criterion.

Applying this identity to a physical one-loop remainder requires further bridges. One must identify the native determinant with the record-frequency polynomial, identify its leading coefficient with the normalization of a physical transfer kernel, and establish the required root condition for all nearby strains before differentiating the averaged expression. Under those conditions the outside-root term gives no imaginary strain response and the remaining contribution is the normalization term (INFERENCE). The continuum root-growth bound proved here does not supply those conditions or prove that the imaginary part of the native zero-momentum remainder vanishes.

8 What remains

At Gaussian order the bare Regge equation has a physical growing mode and no positive invariant record norm. Its corner doubler also supplies a continuum family with a nonzero logarithmic rate. For the derived-weight equation the established result is different: at fixed $\mu \neq 0$, every admitted continuum family approaches the zero wave, where absolute record growth per tick has a cubic upper bound. The specified fixed-momentum contour coefficients obey a separate exponential suppression estimate. A weighted lattice-instability conclusion additionally requires a physical off-zero solution at the period in question.

The native phase coefficient gives fixed $\mu = \pi/\log 2$, so no varying-coupling estimate is needed for that Gaussian equation. The following propositions remain open; they identify what would strengthen the result. None is supplied by the universal bound on an already admitted family.

Target 8.1 (The physical weight identifications)**.** Derive (I1), the hinge-area cost identification with its channel counts, and (I2), the geometry amplitude identification giving amplitude modulus $e^{-C/2}$ and Born weight e^{-C} for cost C, from the ledger's physical records and operations.

Target 8.2 (A nonempty long-wave physical branch)**.** At the fixed native coupling, construct a continuum family of the weighted equation with phases tending to **1** and nonzero spatial physical momentum, meaning $p_a \neq p_c$ for some pair. Each member must have an admissible spatial period and be outside the vertex-move space. A two-polarization graviton and its dispersion require further information beyond this nonvacuity target.

Target 8.3 (Uniform control before integrating momentum)**.** For each compact set of physical momenta and fixed spatial off-diagonal phase, obtain a common radius $R > 1$, constant M and spacing bound $b_0 > 0$ for the coefficient estimate in Theorem 6.10(i), uniformly over that compact set. A full momentum-integrated limit also requires control of the remaining sectors and tails; compact uniformity alone does not establish it.

Target 8.4 (Positive realization of supplied correlation data)**.** For a supplied correlation kernel K, record translations R_t and a nonempty set of physical labels, construct a complete complex Hilbert space, vectors with dense linear span reproducing K as their inner products, and a strongly continuous unitary group implementing R_t. The vectors of physical labels must be nonzero.

The full physical objective also requires deriving those data from a continuum limit of the same weighted gravitational functional. The limiting physical kernel, label space and record translations have not yet been constructed here. Realizing supplied data would leave their physical correspondence to be established; it would not by itself finish that objective.

The zero-spatial-momentum singularity needs a physical regulator prescription before it can enter such a kernel. Normalization and the Feynman-vacuum identification also remain to be established. The present analysis constructs neither those data nor the full interacting amplitude and its constrained record-time transfer. Its completed result is the Gaussian spectral and contour statement at the conditions listed below. The zero equation fails the stability tests, whereas the propagating model satisfies lattice, gapped and continuum stability, so these conditions distinguish concrete cases without asserting a physical state-space construction.

Premises and status. Table 1 records the conditions of the results. The Feynman regulator $+i\varepsilon$, where used, is a model. "Pending" means proof complete, independent verification pending. A mathematical statement about the chosen weight retains (I1) and (I2) when it is read as a statement about the ledger's physical weight.

Table 1. Conditions and status of the results.

result	κ	other conditions
nontriviality (Proposition 2.2)	any	stated model equations; periods $N \geq 1$
record time keeps the doublers (Theorem 3.1), pending	1	Feynman regulator; stated corner and time directions
growing mode (Theorem 3.2)	1	bare linearized equation; open record time
posting-null obstruction (Proposition 3.3), pending	1	full bare equation with posting entries zero; every period and tick
other parameter and rescalings (Theorems 3.4 and 3.5)	3/2; any	same solution space for the rescaling statement
bare gapless/continuum doubler (Theorems 3.6 and 5.4), pending	1	exact physical family; Gaussian order
common kernel and weight-free stable modes (Theorems 4.1 and 4.3)	any	Hermitian action, positive semidefinite weight; Fourier form
weighted lattice obstruction (Corollary 4.4)	any	gauge-definite weight and an existing physical off-zero mode at that period
hierarchy and damped models (Proposition 5.2 and Theorem 5.3)	any	specified equations; damping strength $c \neq 0$
real-space weight and complex gauge identity (Theorem 6.1 and Proposition 6.4)	1	dual-cycle counts; nonzero coordinate phases

Table 1, continued

result	κ	other conditions
native coupling (Theorem 6.2), pending	1	fixed phase coefficient $a \neq 0$; same Gaussian and Bloch conventions
weighted-family decoupling (Theorem 6.5)	1	fixed $\mu \neq 0$; established real-wave kernel identity
static unweighted corner (Proposition 6.6)	1	$\mu = 0$; period 2; unit record factor
ingredients and cubic root bound (Proposition 6.7 and Theorem 6.8)	1	each fixed real μ; non-gauge roots near zero with a common phase modulus
continuum root stability (Theorem 6.9)	1	fixed $\mu \neq 0$; admitted physical Gaussian families; existence not asserted
off-circle coefficients (Theorem 6.10), pending	1	fixed $\mu \neq 0$, p; holomorphic gauge-invariant linear symbols; specified contour sectors; $T > 0$
branch evidence (Section 6.8)	1	numerical evidence only; no existence theorem
rotated form (Theorem 7.1)	$-1/2$	proposed endpoint of the rotation is an inference
Jensen split (Proposition 7.2)	any	nonzero polynomial; physical determinant and normalization reading remains an inference

A Proofs and data

A.1 A pattern that no vertex move has

Statements about physical solutions need a way to exclude vertex moves of every shape, growing ones included, at complex phases. One pattern on three edge classes suffices.

Lemma A.1 (The pattern lemma)**.** *Let $h(x,d) = z^x c_d$ be an exponential field with phases $z \in (\mathbb{C}^\times)^4$ and constants c_d, with $c_{e_0} = c_{e_0+e_2} = 0$ and $c_{e_2} \neq 0$. If $(z_0-1)(z_0z_2-1) \neq 0$, then h is not a vertex move.*

Proof. A displacement of the vertices changes the squared length of the edge from x to $x+d$, to first order, by $2\sum_a d_a\big(\zeta_a(x+d)-\zeta_a(x)\big)$, where ζ is the displacement with its index lowered by the flat metric. Suppose that h has this form. Its e_0 component vanishes, so ζ_0 is invariant under e_0. Its e_0+e_2 component vanishes, so $\zeta_0(x+e_2)-\zeta_0(x) = -\big(P(x+e_0+e_2)-P(x)\big)$ with $P = \zeta_2$; the left side is invariant under e_0, hence so is the right,

$$P(x+2e_0+e_2) - P(x+e_0) = P(x+e_0+e_2) - P(x) \qquad \text{for every } x.$$

Its e_2 component gives $P(y+e_2)-P(y) = \frac{1}{2}c_{e_2}z^y$ for every y. Subtract the displayed identity at x from the same identity at $x+e_2$ and write each of the four resulting differences as an e_2 difference, at $x+2e_0+e_2$, $x+e_0$, $x+e_0+e_2$ and x. The result is

$$\tfrac{1}{2}c_{e_2}\,z^x\big(z_0^2z_2 - z_0 - z_0z_2 + 1\big) = \tfrac{1}{2}c_{e_2}\,z^x(z_0-1)(z_0z_2-1) = 0,$$

which the hypotheses exclude. □

Proof of Proposition 2.2. (i) The field equal to $2^{t(x)}$ on the class e_2 and zero elsewhere has phases $z = (2, 2, 2, 2)$ and solves $E = 0$. It has every spatial period, every tick multiplies it by 2, and $(z_0 - 1)(z_0 z_2 - 1) = 3$, so it is physical by Lemma A.1. (ii) Only the zero field solves $E = \mathrm{id}$, so the conditions hold vacuously. (iii) A nonzero exponential field of phases z solves the wave equation exactly when $z_0 + z_0^{-1} = \frac{1}{4}(z_1/z_2 + z_2/z_1)$. Spatial periodicity of period $N \geq 1$ gives $(z_a/z_b)^N = 1$, so all four phases have one modulus and $z_1/z_2 = e^{i\theta}$ with θ real. The right side is then $\frac{1}{2}\cos\theta \in [-\frac{1}{2}, \frac{1}{2}]$, both roots of $w^2 - \frac{1}{2}\cos\theta\, w + 1$ are unimodular, and the common modulus is 1: every tick multiplies every admitted exponential mode by a unimodular factor. The field $i^{x_0+x_1}$ on the class e_2, with phases $(i, i, 1, 1)$, solves the equation, has spatial period 4, and has $(z_0 - 1)(z_0 z_2 - 1) = (i - 1)^2 \neq 0$, so physical solutions exist. At period $N = 0$ the field $i^{x_0+x_1} 2^{x_3}$ on the class e_2 solves the same equation and is doubled by the tick e_3.

A.2 The growing mode

Index the fifteen edge classes by the binary digits of $d + 1$ for $0 \leq d < 15$, so that the classes 0, 3 and 4 are e_0, e_2 and $e_0 + e_2$. Put $r = \sqrt{2}$. A null polarization of the $\kappa = 1$ Bloch matrix at the phases $(r^{-1}, r^{-1}, r^{-1}, r^{-1})$ is

$$u_3 = 2 + 2\sqrt{2}, \qquad u_6 = -1, \qquad u_7 = -(2 + 2\sqrt{2}), \qquad u_{10} = 1, \qquad u_d = 0 \text{ otherwise},$$

checked by exact arithmetic in $\mathbb{Q}(\sqrt{2})$. With the phase convention of [2, Prop. 6.16], under which the field equation acts on $z^x z^{D(d)} u$ through the Bloch matrix at z^{-1}, the field $h_d(x) = r^{t(x)+|d|} u_d$ of Theorem 3.2 solves the linearized Regge equation. Its phases are $z = (r, r, r, r)$, its components on the classes e_0 and $e_0 + e_2$ vanish and its component on e_2 does not, and $(z_0 - 1)(z_0 z_2 - 1) = (r - 1)(r^2 - 1) \neq 0$; by Lemma A.1 it is not a vertex move of any shape. At $\kappa = 3/2$ the corresponding null vector, with the record factor $(1 + \sqrt{10})/3$, is certified in the same way [2, Prop. 6.16].

A.3 The damped model

Write $R = |w|^2$ for a root of $w + w^{-1} = 2 - l + ig$ with $g = c\, l^m$. With $w = \sqrt{R}\, e^{i\varphi}$ the real and imaginary parts give $(R + 1)\cos\varphi = (2 - l)\sqrt{R}$ and $(R - 1)\sin\varphi = g\sqrt{R}$, and eliminating φ gives the identity of Section 5,

$$g^2 R\,(R + 1)^2 = (R - 1)^2\big((R - 1)^2 + l(4 - l)R\big),$$

since $4 - (2 - l)^2 = l(4 - l)$. The roots come in pairs w, w^{-1}; the identity holds for either choice. A spatially periodic mode has phases of one modulus. For $c \neq 0$ and $l > 0$, the phases $w(1, 1, e^{-i\theta}, 1)$ with $e^{i\theta}$ an N-th root of unity give a solution of period N whenever w is a root at $l = 2 - 2\cos\theta$, and its component on one edge class can be chosen so that Lemma A.1 applies.

Proof of Theorem 5.3(i). Put $D = (R - 1)^2$. The root identity implies $Dl(4 - l) \leq 4Rg^2$ and $g^2 R \leq D$. For $m = 2$ and $0 < l \leq 1$, the first inequality gives $D \leq 2Rc^2 l^3$; for $l = 0$ the record root is unimodular. Along a continuum family with $\zeta_1 = \zeta_2$, $l_n = 2 - 2\cos(b_n q) \leq (b_n q)^2$ and $l_n \to 0$, so the claimed bound holds eventually. Since $R_n \to 1$, eventually $|\log R_n| \leq 2|R_n - 1|$, proving $\log|w_n|/b_n \to 0$. If $\zeta_1 \neq \zeta_2$, then $l_n \to l^* > 0$, and $c^2 l_n^4 R_n \leq (R_n - 1)^2$ contradicts $R_n \to 1$. The model is not gapped stable: arbitrarily small positive l give physical periodic roots off the circle but tending to it.

Proof of Theorem 5.3(ii). Take $\theta_n = 2\pi/(n+2)$, $l_n = 2 - 2\cos\theta_n$, and a nonzero root w_n of $w + w^{-1} = 2 - l_n + icl_n$. Use phases $w_n e^{i\theta_n}$ on coordinate 1 and w_n on the other coordinates, spacing $b_n = \theta_n$, momentum e_1, and a polarization supported on class e_2. These phases have spatial period $n+2$; the pattern lemma excludes vertex moves. The root identity gives $R_n \to 1$. Eventually $1/2 \le R_n \le 2$ and $l_n \le 1$. Put $\kappa_0 = \min(1, c^2/8) > 0$ and $D_n = (R_n - 1)^2$. The identity gives $D_n \ge \kappa_0 l_n$: if $D_n \ge l_n$ this is immediate; otherwise the identity and those bounds give $(9/8)c^2 l_n \le 9D_n$. Also $|\log R_n| \ge |R_n - 1|/2$ and $l_n \ge 4\theta_n^2/\pi^2$. Therefore

$$\left(\frac{\log|w_n|}{\theta_n}\right)^2 \ge \frac{\kappa_0}{4\pi^2} > 0$$

eventually. This excludes continuum stability, without choosing a sign for the logarithmic rate.

References

[1] J. Washburn, Quantum gravity from the recognition ledger: weighted one-loop graviton responses with finite real parts, Einstein gravity at long wavelengths, and the massless-graviton problem, Zenodo (2026), doi:10.5281/zenodo.23806703.

[2] J. Washburn, Linearized Einstein gravity from Lorentzian Regge calculus on the Kuhn lattice, and the divergence of its one-loop graviton limit, Zenodo (2026), doi:10.5281/zenodo.23764491; Pith preprint doi:10.67347/pith.preprint.2026.000017.

[3] J. Washburn, The record cell of the four-dimensional lattice: time, Lorentzian signature and the Regge action near flatness, in *The Formation of the Universe from Recognition: The Integrated Manuscript and Its Focused Papers*, Zenodo (2026), doi:10.5281/zenodo.23738724.

[4] T. Regge, General relativity without coordinates, *Nuovo Cimento* **19**, 558–571 (1961).

[5] H. Freudenthal, Simplizialzerlegungen von beschränkter Flachheit, *Ann. of Math.* **43**, 580–582 (1942).

[6] H. W. Kuhn, Some combinatorial lemmas in topology, *IBM J. Res. Dev.* **4**, 518–524 (1960).

[7] S. K. Asante, B. Dittrich and H. M. Haggard, Discrete gravity dynamics from effective spin foams, *Class. Quantum Grav.* **38**, 145023 (2021).

[8] B. Dittrich, S. Gielen and S. Schander, Lorentzian quantum cosmology goes simplicial, *Class. Quantum Grav.* **39**, 035012 (2022).

[9] D. Jia, Complex, Lorentzian, and Euclidean simplicial quantum gravity: numerical methods and physical prospects, *Class. Quantum Grav.* **39**, 065002 (2022).

[10] K. Osterwalder and R. Schrader, Axioms for Euclidean Green's functions, *Commun. Math. Phys.* **31**, 83–112 (1973).

[11] G. W. Gibbons, S. W. Hawking and M. J. Perry, Path integrals and the indefiniteness of the gravitational action, *Nucl. Phys. B* **138**, 141–150 (1978).

[12] J. Ambjørn, J. Jurkiewicz and R. Loll, Nonperturbative Lorentzian path integral for gravity, *Phys. Rev. Lett.* **85**, 924–927 (2000).

[13] R. Loll, Quantum gravity from causal dynamical triangulations: a review, *Class. Quantum Grav.* **37**, 013002 (2020).

[14] J. L. W. V. Jensen, Sur un nouvel et important théorème de la théorie des fonctions, *Acta Math.* **22**, 359–364 (1899).