

Recognition Boundary Theory

How One Ledger Holds Many Perspectives

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Abstract

Recognition Science describes one ledger of recognition events, yet experience arrives from inside separate perspectives. We ask what in the ledger's own dynamics makes a perspective, for one explicit operation: the native recognition step, in which two eight-state clock channels exchange occupation, post a three-component current and move an integer loop flux.

Boundaries. For the algebra of native events, every closed invariant subspace of histories is a join of atoms, the atoms are exact superselection sectors with no environment involved, and the atoms an actual history occupies are fixed at its beginning. Neither a symmetric rule nor an element of that algebra selects among them, although a larger read algebra crosses these walls.

Participants. Inside a boundary the postings select. Exactly a four-dimensional subspace of the 64-dimensional cell never acquires neutral weight under any record: two persistent loop currents and an occupation oscillation. Three postings expose every other state, and the selected states are exactly those that pass the eight-tick window on the loop edge with certainty. On every network of contacts in its round order the kept states obey a parity law, the kept projection commutes with every round word, so composition creates no kept weight, and participation on a connected network is joint. Other timings act through one token mechanism, which decides births, fair and gated participation, random timing and what a participant's own record reads of its world.

Records. A participant's own record decides its status; the conditional present purifies; its long-run motion is a classical chain that is reversible exactly on the participant classes and carries a stationary circulation elsewhere. Every record reads each site only within the same twelve directions. When two members obey the product law, a composite that reads at each site no more than they do together adds to their two-site reads at most the crossed products of their single-site reads, and exactly these when it obeys the product law with their combined single-site reads.

Among the stated premises are the step itself, with the routing of its exchange current through the loop edge; the account and tick on which the eight-tick window is read; the genesis state; the locality of contacts, their orientation and their posting order; the gate and the independence of attempts; the relay's Coulomb difference, momentum phase and recoil, and the logarithm branches and common rate of its native terms; the least-cost booking that decides whether a padding post is a physical write; and the capture apparatus. The photon dictionary, the reading of fractional postings as dynamics, the reading of the conditional state as the present and the experiential reading of a sure subject are identifications. On the ledger's torus, locality, the symmetry of the contact rule and the existence of a contact place the contacts on nearest-neighbour pairs; an attempt law computed from the native data alone gives every attempt a positive chance; and the native posting is the product of eighth-turns of the relay's photon-free clock and reciprocal terms. We locate each premise, give the case for the experiential reading, and state every open problem exactly.

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Chapter 1

One ledger, many perspectives

Recognition Science starts from the fact that occurrences can be told apart, that comparing them needs information about both, and that examining a record is itself a further occurrence [1]. Followed through, these facts give one ledger: every recognition event is a posting, and every posting leaves a record. Experience, though, does not arrive as one ledger. It arrives from inside separate perspectives, each with its own boundary, its own continuing presence and its own past. If the ledger is the whole of what happens, the perspectives have to be structure inside it. Recognition Boundary Theory asks what that structure is and where it comes from.

We answer for one explicit finite operation, the native recognition step of recognition dynamics [3]. Two eight-state clock channels exchange occupation and advance. Each step writes a three-component current, one component on each edge of a periodic axis, and moves an integer loop flux by the current on the loop edge. The step is unitary, its records follow the Born law of their effects, and the state of the cell given its record is a well-defined conditional state. Everything in the book is a theorem about this step, its records and networks of its contacts, stated with its hypotheses, or a named premise, identification or open problem. This book is the complete version of the preprint [2]: it keeps its notation and proves its results in full, with the further results and open problems named below.

1.1 Three layers

Three layers of structure come out of the step.

Boundaries. Call a closed subspace of histories a boundary when no native event can carry a state out of it. Every boundary is a join of indivisible atoms, and each atom collects the coordinates that read back to one root. The atoms an actual history can occupy are the lineage atoms, one for each occupation and loop at which a history can begin; the others, rooted at a label no input could write, are never populated. The atoms are exact superselection sectors: an operator commutes with every native event exactly when it acts as one scalar on each atom. Nothing is averaged and no environment is traced out. Which atoms a history occupies is fixed at its beginning. Evolution never changes it, and neither a symmetric rule nor any operation in the norm-closed algebra of native events picks one atom out. A larger read algebra, containing the retained read of recognition dynamics, crosses these walls.

Participants. Inside a boundary the postings themselves select. A state can carry weight in a neutral sector, where its writing of the loop edge halves with every step. Exactly a four-dimensional subspace of the 64-dimensional cell never acquires such weight under any record: two persistent loop currents of opposite sense and an oscillation of occupation across the loop edge. Three postings already cut this subspace out, and every other state reaches

the neutral sector within three postings. The selected states are exactly the *sure subjects*, the states that pass the eight-tick window on the loop edge with certainty. On every network of contacts posting in its round order the kept states obey a parity law, composition creates no kept weight, and on a connected network participation is joint: one participant's own test keeps exactly what the whole keeps.

Records. A participant's status is decided by its own record: for some states one posting decides, but no fixed finite record event decides for every state, and every infinite record decides. Given the record, the present cell purifies, and in the long run it moves as a classical Markov chain over five conserved classes. That chain is reversible exactly on the three classes that hold participants. On the other two it carries a stationary circulation that no potential explains, set by the orientation of the native step, with no bath and no imposed boundary condition: an arrow inside the ledger.

We use the following words throughout. A state of the kept class, the states that no record of the postings ever leaves with neutral weight on one cell or on a network, is a *participant*; the name is introduced with the class in Chapter 7. A *sure subject* is a state that passes every field-current window with probability one. The theorems of Chapters 7 and 13 say that the two coincide. The class a word names is *kept*, and its weight is the *kept weight*. On a network, a configuration of dressed clocks and bits is *awake* when every posting of the round order meets a mixed pair. A *perspective* is a participant inside a boundary together with its record (Chapter 22).

1.2 A map of the book

The step and its verdicts. Chapter 2 derives the native step from its two clock channels, with its records, their laws and the present cell. Chapter 3 turns the eight-tick criterion into the window effects of the record law and their verdicts.

Boundaries. Chapter 4 proves that boundaries are joins of atoms. Chapter 5 shows that the genesis state, and no symmetry or admitted operation, decides which boundary a history occupies. Chapter 6 goes beyond the event algebra to the retained read and the pointer.

Participants on one cell. Chapter 7 proves the selection theorem and reads the four modes as motions under the photon dictionary. Chapter 8 shows that the record decides.

Records and the present. Chapter 9 determines what the record law reads and what it cannot see. Chapter 10 follows the present into its pointer chain, and Chapter 11 proves the arrow of that chain and what the record remembers of genesis.

Networks. Chapter 12 sets up networks of contacts and proves the parity law. Chapter 13 proves selection, commutation and joint participation on every network. Chapter 14 treats every other timing through one token mechanism, and Chapter 15 asks what a participant's own record reads of its world.

Pooled records. Chapter 16 treats what a record reads at each site, which writes keep insides private and what composites add. Chapter 17 decides when a pool of records is as good as the full record.

Protected insides. Chapters 18 and 19 study what survives when the cell is coupled to the relay,

a supplied interaction with the photon field.

Activity and attachment. Chapter 20 proves that writing a record is never a passive flow and develops capture and control at the scope where they hold. Chapter 21 attaches postings to events.

Perspective and open problems. Chapter 22 assembles the three layers into a definition of a perspective and states the experiential reading with its evidence and its open identity. Chapter 23 collects the premises and open problems. The appendix holds the tables and finite certificates.

1.3 What is supplied

Some inputs are not derived in the book. Each is named where it enters; here is the list, with what is known about removing it.

- *The step.* We take the native operation from recognition dynamics [3]. Each channel has 2^3 states, the configurations of three binary coordinates (Chapter 2), and single unit postings are the cheapest ledger extensions (Proposition 2.1). Part of the step is the routing of the exchange's transfer through the loop edge: the occupations fix the current only up to a circulation, and every result that reads the loop edge rests on this routing (Proposition 2.8). The retained read and the grammar of placed reads of Chapters 5 and 6 are also taken from recognition dynamics. Which physical interactions enact the exchange and the clock is not derived here.
- *The window.* The length eight of the window and its condition that no tick is silent come from the eight-tick criterion. Reading it on the field-current account, one tick per posting, is a choice; Chapter 3 shows what other accounts and clocks give.
- *The beginning.* Which lineage atoms a history occupies is fixed by its genesis state, an initial condition that no symmetric rule or event selects (Chapter 5).
- *Where contacts sit.* On the bare network carrier the native postings cannot place contacts: they are unchanged by every relabelling of sites, so every rule computed from them places all contacts or none. On the ledger's torus, whose sites we identify with the network's sites and where one post moves one unit one step along one axis, three premises place them: a contact joins only sites one post can join (locality), the contact rule is computed from the native data alone and so carries their symmetries, and at least one contact exists. Then every nearest-neighbour pair is a contact, and we take one contact per pair (Proposition 12.4). Its orientation is not forced; it changes the written labels but not what is kept.
- *When contacts post.* No round order, schedule or prefix is forced: a timing rule computed from the native data inherits their symmetries and therefore picks out no fair schedule. A supplied local gate can replace the schedule; it restricts the native instrument by silencing the posting on a balanced pair. Under random timing only the positivity of the attempt chances matters. If the attempt law is computed from the native data alone (a premise), it gives every one of m contacts the chance $1/m$, and positivity follows. The independence of the attempts is supplied.
- *Event attachment.* Whether a padding post on the ledger counts as a physical write decides both event attachment and whether counts of executed posts agree with net-current

counts. The ledger facts established so far cannot decide it. One step-level premise does: if each native step is booked at least cost, padding is not a write and every count agrees (Chapter 21).

- *The relay.* The protected-inside results of Chapters 18 and 19 use the relay, a supplied operator coupling the cell to the photon field. The native posting is the product of eighth-turns of the relay's photon-free clock and reciprocal terms, for the branches of the logarithm fixed in Proposition 18.1, and alternating these terms at a common rate gives the photon-free evolution as the limit of fractional native postings. The branches, the common rate, the Coulomb difference, the momentum phase, the recoil of the reciprocal term and the reading of fractional postings as dynamics are not derived.
- *Capture and control.* The carriers, blanks, donor slots, schedules, strengths and matchings of Chapter 20 are supplied apparatus, and reading a non-passive write as the activity of a registering boundary is an identification.
- *Identifications.* Reading the conditional state as the physical present, reading the axis as a plaquette of the photon field (the photon dictionary of Section 7.7), and reading a sure subject as a perspective that experiences are identifications, not theorems. That the record law gives the frequencies with which records occur in repeated trials is an apparatus statement, separate from its construction (Section 2.7). The physical status of orphan coordinates, which no beginning populates, is open.

1.4 How the results are established

Most results are proved on the page. Where a proof is an exact finite computation, we say what is computed, give the mechanism and print the table in the appendix; such a claim is marked as a direct exact computation and can be repeated from the definitions. Refutations of posed laws are stated as results, each with its counterexample. One statement of Chapter 15, that two members sharing one clock tell apart the two wholes of every exceptional pair, is recorded without proof and is marked so where it appears. Where a question is only partly settled, the proved part is a theorem and the rest is a named open problem with its exact content; Chapter 23 collects them. Notation is fixed in Chapter 2 and used unchanged throughout.

Chapter 2

The native recognition step

Every result in this book concerns one explicit finite operation and the records it writes. This chapter constructs that operation from recognition dynamics [3], a real orthogonal step on a 64-dimensional cell whose fifteen current labels form a quantum instrument, and derives the record law and the objects the later chapters use.

Every matrix identity below can be checked by hand from Tables 2.1 and 2.2 and the explicit matrices of Appendix A.1.

2.1 From two eight-state channels to sixteen step blocks

The native step is the native operation of recognition dynamics, not an operation designed for this book, written in coordinates that separate an occupation bit from a four-state internal clock. This section carries out that change of coordinates. The result is the block formula (2.7); a reader willing to take it as given can begin at Section 2.2.

2.1.1 The posting rule

Recognition dynamics starts from a ledger. A ledger on finitely many accounts holds a debit count d_k and a credit count c_k for each account k; its *potential* on account k is $d_k - c_k$. An extension $(d, c) \to (d', c')$ of a ledger is *monotone* if no count decreases. Under the canonical recognition cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$, so that $J(e^t) = \cosh t - 1$ [1], applied to each count separately, the extension costs

$$\mathscr{C} = \sum_k \big[\cosh(d'_k - d_k) - 1\big] + \sum_k \big[\cosh(c'_k - c_k) - 1\big]. \tag{2.1}$$

A *unit posting* raises one count on one account by one: a single debit or a single credit.

Proposition 2.1 (Unit postings are the cheapest extensions)**.** *Every monotone extension $(d, c) \to (d', c')$ other than the identity costs at least* $\cosh 1 - 1$*. Equality holds exactly when (d', c') is obtained from (d, c) by one unit posting. A unit posting changes the potential by ± 1 on one account and leaves every other account unchanged.*

Proof. The increments $d'_k - d_k$ and $c'_k - c_k$ are nonnegative integers, and $\cosh t - 1$ vanishes at $t = 0$ and increases for $t \geq 0$. A nontrivial extension has a positive increment, which contributes at least $\cosh 1 - 1$; every other term is nonnegative. Equality forces that increment to be 1 and every other increment to be 0. A unit posting attains the bound, and it moves $d_k - c_k$ by $+1$ (debit) or -1 (credit) on its own account only. □

Every record in this book is written by such postings. In this chapter the cost enters only here: no cost function appears in the cell, its step blocks, its slices or its record law.

2.1.2 One eight-state channel and its occupation chart

A *channel* is the clock $\mathbb{C}^8$ with basis $|t\rangle$, $t \in \mathbb{Z}/8\mathbb{Z}$. Eight is 2^3: the label t is three binary coordinates, and the clock runs through all eight of their configurations [1]. Three operators act on a channel:

$$S_8 = \mathrm{diag}(1,-1,-1,1,-1,1,1,-1), \qquad F_8|t\rangle = |t \oplus 1\rangle, \qquad \mathbb{K}|t\rangle = |t-1\rangle. \tag{2.2}$$

S_8 is the parity of the three bits of t, F_8 flips the lowest bit, and $\mathbb{K}$ advances the clock. Flipping one bit reverses the parity, so $S_8F_8 = -F_8S_8$, and the signed flip $R_8 = \mathrm{i}S_8F_8$ is a Hermitian involution. The projection $N = \frac{1}{2}(I - R_8)$ is the *occupation* of the channel.

The occupation chart separates the occupation from the rest of the clock. Let $D = \mathrm{diag}(1,-1,-1,1)$ on $\mathbb{C}^4$, so that D_{ii} is the parity of the two bits of $i \in \mathbb{Z}/4$, and let L be the cyclic shift $Le_i = e_{i-1}$ on $\mathbb{C}^4$. For a bit b write $s(b) = 1 - 2b$. For $a \in \{0,1\}$ and $i \in \mathbb{Z}/4$ put

$$|a,i\rangle = \tfrac{1}{\sqrt{2}}\big(|2i\rangle - \mathrm{i}\, s(a)\, D_{ii}\, |2i+1\rangle\big). \tag{2.3}$$

The two slots $2i$ and $2i+1$ differ only in the lowest bit; the internal index i is the two higher bits, and the occupation a fixes the relative phase of the two slots. Define the *clock maps* on $\mathbb{C}^4$,

$$K(b',b) = \tfrac{1}{2}\, D\big(s(b)\, I - s(b')\, L\big), \qquad b,b' \in \{0,1\}. \tag{2.4}$$

Proposition 2.2 (The occupation chart)**.** *The eight vectors $|a,i\rangle$ form an orthonormal basis of $\mathbb{C}^8$. In this basis*

$$N|a,i\rangle = a\,|a,i\rangle, \qquad S_8|a,i\rangle = D_{ii}\,|1-a,i\rangle, \qquad F_8|a,i\rangle = -\mathrm{i}\, s(a)\, D_{ii}\,|1-a,i\rangle,$$

and the clock has the matrix elements

$$\langle a',i'|\,\mathbb{K}\,|a,i\rangle = -\mathrm{i}\, K(a',a)_{i'i}. \tag{2.5}$$

Proof. For fixed i the two vectors $|0,i\rangle, |1,i\rangle$ lie in the span of $|2i\rangle, |2i+1\rangle$, and $\langle 0,i|1,i\rangle = \frac{1}{2}(1 - D_{ii}^2) = 0$; vectors with different i have disjoint support. Inverting (2.3),

$$|2i\rangle = \tfrac{1}{\sqrt{2}}\big(|0,i\rangle + |1,i\rangle\big), \qquad |2i+1\rangle = \tfrac{\mathrm{i}\,D_{ii}}{\sqrt{2}}\big(|0,i\rangle - |1,i\rangle\big).$$

The parity of $2i$ is D_{ii} and that of $2i+1$ is $-D_{ii}$, so $S_8|a,i\rangle = \frac{D_{ii}}{\sqrt{2}}(|2i\rangle + \mathrm{i}\, s(a) D_{ii}|2i+1\rangle) = D_{ii}|1-a,i\rangle$. Similarly $F_8|a,i\rangle = \frac{1}{\sqrt{2}}(|2i+1\rangle - \mathrm{i}\, s(a) D_{ii}|2i\rangle) = -\mathrm{i}\, s(a) D_{ii}|1-a,i\rangle$. Then $R_8|a,i\rangle = \mathrm{i}S_8F_8|a,i\rangle = s(a)|a,i\rangle$, which gives the formula for N. For the clock, $\mathbb{K}|a,i\rangle = \frac{1}{\sqrt{2}}(|2(i-1)+1\rangle - \mathrm{i}\, s(a) D_{ii}|2i\rangle)$, and substituting the two inverse formulas gives

$$\mathbb{K}|a,i\rangle = -\tfrac{\mathrm{i}}{2}\Big[s(a) D_{ii}\big(|0,i\rangle + |1,i\rangle\big) - D_{i-1,i-1} \sum_{a'} s(a')\,|a',i-1\rangle\Big].$$

The coefficient of $|a',i'\rangle$ is $-\frac{\mathrm{i}}{2}\big(s(a) D_{ii}\,\delta_{i'i} - s(a') D_{i'i'}\,\delta_{i',i-1}\big)$, and $L_{i'i} = \delta_{i',i-1}$, so this is $-\mathrm{i}\, K(a',a)_{i'i}$. □

So the clock maps are the eight-state clock itself, seen in the occupation chart, up to the phase $-\mathrm{i}$. A clock map $K(b',b)$ takes an occupation b to an occupation b'; it *stays* when $b' = b$ and *flips* otherwise.

2.1.3 Two channels and the reciprocal exchange

Two channels $\mathbb{C}^8 \otimes \mathbb{C}^8$ are coupled by

$$X = S_8 \otimes F_8, \qquad Y = F_8 \otimes S_8, \qquad U = \tfrac{1}{2}\big(I + R_8 \otimes R_8 + \mathrm{i}(X - Y)\big), \tag{2.6}$$

and the native operation of recognition dynamics is $M_{\mathrm{nat}} = (\mathbb{K} \otimes \mathbb{K})\,U$ [3]. The operators X and Y are commuting Hermitian involutions with $XY = R_8 \otimes R_8$, so $U = e^{\mathrm{i}\pi X/4} e^{-\mathrm{i}\pi Y/4}$: a quarter turn generated by each channel's parity acting with the other's flip, in opposite senses. We call U the *reciprocal exchange*. Write $|a, i; b, i'\rangle = |a, i\rangle \otimes |b, i'\rangle$ for the product chart.

Proposition 2.3 (The exchange swaps mixed occupations)**.** *In the product chart,*

$$\begin{aligned} U|00; i, i'\rangle &= |00; i, i'\rangle, & U|11; i, i'\rangle &= |11; i, i'\rangle, \\ U|10; i, i'\rangle &= D_{ii} D_{i'i'}\, |01; i, i'\rangle, & U|01; i, i'\rangle &= -D_{ii} D_{i'i'}\, |10; i, i'\rangle. \end{aligned}$$

Proof. By Proposition 2.2, $R_8 \otimes R_8$ multiplies $|a, i; b, i'\rangle$ by $s(a)s(b)$, while X and Y send it to $-\mathrm{i}\, s(b) D_{ii} D_{i'i'}$ and $-\mathrm{i}\, s(a) D_{ii} D_{i'i'}$ times $|1 - a, i; 1 - b, i'\rangle$. Hence

$$U|a, i; b, i'\rangle = \tfrac{1}{2}\big(1 + s(a)s(b)\big)|a, i; b, i'\rangle + \tfrac{1}{2}\big(s(b) - s(a)\big) D_{ii} D_{i'i'}\, |1 - a, i; 1 - b, i'\rangle.$$

For $a = b$ the first coefficient is 1 and the second 0; for $(a, b) = (1, 0)$ they are 0 and 1; for $(a, b) = (0, 1)$ they are 0 and -1. □

Theorem 2.4 (The native step in blocks)**.** *Order the product chart as an occupation pair $x = (x_1, x_2) \in \{0, 1\}^2$ followed by the two internal indices. Then the block of M_{nat} from input occupation x to output occupation y is*

$$M_{yx} = \sigma(x)\,\big(K(y_1, x_2)\, C_x\big) \otimes \big(K(y_2, x_1)\, C_x\big), \tag{2.7}$$

where $C_x = I$ if $x_1 = x_2$, $C_x = D$ if $x_1 \neq x_2$, $\sigma(0, 1) = +1$, and $\sigma(x) = -1$ for the other three occupations.

Proof. By Proposition 2.3, U carries occupation x to its swap (x_2, x_1) with internal factor I when x is balanced, $D \otimes D$ when $x = (1, 0)$, and $-D \otimes D$ when $x = (0, 1)$. By (2.5), $\mathbb{K} \otimes \mathbb{K}$ then carries occupation (x_2, x_1) to y with internal block $(-\mathrm{i})^2 K(y_1, x_2) \otimes K(y_2, x_1)$. Multiplying, the block is $-K(y_1, x_2) \otimes K(y_2, x_1)$ for balanced x, $-\big(K(y_1, 0)D\big) \otimes \big(K(y_2, 1)D\big)$ for $x = (1, 0)$, and $+\big(K(y_1, 1)D\big) \otimes \big(K(y_2, 0)D\big)$ for $x = (0, 1)$. These are the three cases of (2.7). □

The chart is unitary, so the change of coordinates loses nothing: all 64 dimensions of the two channels are kept. From here on we work with the blocks (2.7) and call the operation they define the *native step*. Clock cell 1 reads the occupation bit x_2 and writes y_1; clock cell 2 reads x_1 and writes y_2. The crossing is the exchange: by the time the clocks act, the two occupations have traded places. The step is taken from recognition dynamics as given. Which physical interactions enact it is a separate question that this book does not address.

2.2 One clock cell

The internal part of a channel is the *clock cell* $\mathscr{H}_{\mathrm{c}} = \mathbb{C}^4$ of Section 2.1, with basis $e_0, \dots, e_3$ and indices modulo 4; in coordinates $(Lv)_i = v_{i+1}$. The four clock maps (2.4) are

$$K(0,0) = \text{stay} = \tfrac{1}{2}D(I-L), \quad K(1,0) = \text{flip} = \tfrac{1}{2}D(I+L), \quad K(0,1) = -\text{flip}, \quad K(1,1) = -\text{stay}.$$

The *pole charge* is $c = \frac{1}{2}(L + L^{\top})$ and the *dressed pole charge* is $c' = DcD$. Both are diagonal in the orthonormal *pole basis*

$$\begin{aligned} p &= \tfrac{1}{2}(1,1,1,1), & a &= \tfrac{1}{2}(1,-1,1,-1), \\ P &= Dp = \tfrac{1}{2}(1,-1,-1,1), & A &= Da = \tfrac{1}{2}(1,1,-1,-1). \end{aligned} \tag{2.8}$$

The vector p is fixed by the shift and a is reversed by it; P and A are their images under the dressing. We write $\mathscr{U} = \operatorname{span}\{p, a\}$ and $D\mathscr{U} = \operatorname{span}\{P, A\}$: by Table 2.1, $D\mathscr{U}$ is the zero plane of c and $\mathscr{U}$ is the zero plane of c'.

	p	a	P	A
c (pole charge)	$+1$	-1	0	0
$c' = DcD$ (dressed pole charge)	0	0	$+1$	-1

Table 2.1. The two pole charges on the pole basis. Each is ± 1 on two lines and vanishes on the other two; their product is $cc' = 0$.

	p	a	P	A
stay	0	A	$\frac{1}{2}(p+a)$	$\frac{1}{2}(a-p)$
flip	P	0	$\frac{1}{2}(p-a)$	$\frac{1}{2}(p+a)$
D	P	A	p	a

Table 2.2. The clock maps and the dressing on the pole basis. For example, $Lp = p$ gives stay $p = 0$ and flip $p = Dp = P$.

Lemma 2.5 (Clock Grams)**.** *The clock maps satisfy*

$$\text{stay}^{\top}\text{stay} = \tfrac{1}{2}(I - c), \qquad \text{flip}^{\top}\text{flip} = \tfrac{1}{2}(I + c), \qquad \sum_{b'} K(b', b)^{\top} K(b', b'') = \delta_{bb''}\, I.$$

Proof. Since $D^{\top}D = I$ and $L^{\top}L = I$, $K(b',b)^{\top}K(b',b'') = \frac{1}{4}\big(s(b)s(b'')I - s(b)s(b')L - s(b'')s(b')L^{\top} + I\big)$. The cases $b = b''$ with $b' = 0, 1$ give the two Grams. Summing over b' cancels the terms linear in $s(b')$ and leaves $\frac{1}{2}\big(s(b)s(b'') + 1\big)I$, which is I for $b = b''$ and 0 otherwise. □

The Grams say how a cell moves. A cell on a line where its pole charge is ± 1 moves surely: p always flips and a always stays. A cell on the plane where its pole charge vanishes stays or flips with weight $\frac{1}{2}$ each, and lands in $\mathscr{U}$. When the cell is dressed by D first, as happens on a mixed input, the same statements hold with c' in place of c: then P always flips (to P) and A always stays (to A). A basis vector e_i always splits evenly, because both pole charges have zero diagonal.

2.3 The native cell and its step

An *occupation* is a pair $x = (x_1, x_2) \in \mathcal{O} = \{0,1\}^2$. It is *balanced* when $x_1 = x_2$ and *mixed* otherwise. The *native cell* is

$$\mathcal{H} = \mathbb{C}^{\mathcal{O}} \otimes \mathcal{H}_{\mathrm{c}} \otimes \mathcal{H}_{\mathrm{c}} \cong \mathbb{C}^{64} \ :$$

an occupation pair and two internal clock cells. The step blocks M_{yx} are those of Theorem 2.4, and the *native step* is

$$U_{\mathrm{nat}} = \sum_{x,y\in\mathcal{O}} |y\rangle\langle x| \otimes M_{yx}.$$

A worked block shows how the formula reads. For $x = y = (1,0)$, $\sigma(x) = -1$, $K(y_1, x_2) = K(1,0) = \mathrm{flip}$ and $K(y_2, x_1) = K(0,1) = -\mathrm{flip}$, so $M_{(1,0),(1,0)} = (\mathrm{flip}\, D) \otimes (\mathrm{flip}\, D)$. Since $\mathrm{flip}\, D\, P = \mathrm{flip}\, p = P$, this block fixes $P \otimes P$. Appendix A.1 lists all sixteen blocks in this factored form.

Proposition 2.6 (The step is orthogonal)**.** *U_{nat} is a real orthogonal matrix on $\mathcal{H}$.*

Proof. All entries are rational. The (x, z) block of $U_{\mathrm{nat}}^{\top} U_{\mathrm{nat}}$ is

$$\sigma(x)\sigma(z) \Big[C_x^{\top} \sum_{y_1} K(y_1, x_2)^{\top} K(y_1, z_2)\, C_z \Big] \otimes \Big[C_x^{\top} \sum_{y_2} K(y_2, x_1)^{\top} K(y_2, z_1)\, C_z \Big].$$

If $x \neq z$, then $x_1 \neq z_1$ or $x_2 \neq z_2$, and Lemma 2.5 makes one factor vanish. If $x = z$, both brackets equal $C_x^{\top} C_x = I$. □

2.4 Currents, the loop field and the ledger

Each step writes a current on a three-edge periodic axis, and the ledger books it in unit postings.

2.4.1 Current labels

A transition $x \to y$ writes the *current label*

$$j(y, x) = \big(y_1 - x_2,\ x_1 - x_2,\ y_2 - x_1\big) \in \{-1, 0, 1\}^3. \tag{2.9}$$

Table 2.3 lists the sixteen transitions, with $\bar{1} = -1$.

input x \ output y	00	10	01	11
00	$(0,0,0)$	$(1,0,0)$	$(0,0,1)$	$(1,0,1)$
10	$(0,1,\bar{1})$	$(1,1,\bar{1})$	$(0,1,0)$	$(1,1,0)$
01	$(\bar{1},\bar{1},0)$	$(0,\bar{1},0)$	$(\bar{1},\bar{1},1)$	$(0,\bar{1},1)$
11	$(\bar{1},0,\bar{1})$	$(0,0,\bar{1})$	$(\bar{1},0,0)$	$(0,0,0)$

Table 2.3. The current label $j(y, x)$ of each of the sixteen transitions. The middle component is the input's occupation imbalance; only the zero label occurs twice.

Proposition 2.7 (Fifteen labels, each reading back its input)**.** *The sixteen transitions write exactly fifteen distinct labels. The only coincidence is the zero label, written by* $(0,0) \to (0,0)$ *and by* $(1,1) \to (1,1)$*. Given the output y and the label j, the input is recovered as* $x_2 = y_1 - j_1$*,* $x_1 = y_2 - j_3$*. The middle component* j_2 *vanishes exactly when the input is balanced.*

Proof. The recovery formulas invert the first and third components of (2.9). So two transitions with the same label and the same output have the same input; a coincidence of labels needs different outputs. Equal labels with outputs $y \neq y'$ and inputs x, x' need $y_1 - x_2 = y_1' - x_2'$, $y_2 - x_1 = y_2' - x_1'$ and $x_1 - x_2 = x_1' - x_2'$; reading Table 2.3 row by row, the only solution is the zero label. The last claim is $j_2 = x_1 - x_2$. □

Write $\mathcal{J}$ for the set of fifteen labels.

2.4.2 The complete current on a periodic axis

The three components of a label are currents on the three edges of the periodic axis $\mathbb{Z}/3\mathbb{Z}$ [3]. Its vertices are a reference vertex b_0 and two sites s, t; the edges are $e_1 : b_0 \to s$ and $e_2 : s \to t$ in the positive direction, and the periodic edge $e_3 : b_0 \to t$, which runs in the negative periodic direction. (These edges are not the basis vectors of $\mathcal{H}_c$.) The edges e_1, e_3 form a tree; e_2 closes the cycle and is the *loop edge*. For a current $f = (f_1, f_2, f_3)$ on (e_1, e_2, e_3) the net inflow at the vertices (b_0, s, t) is

$$\partial f = \big(-f_1 - f_3,\ f_1 - f_2,\ f_2 + f_3\big).$$

Proposition 2.8 (The label moves the occupations)**.** *For every transition,*

$$\partial j(y, x) = \big(-(y_1 + y_2 - x_1 - x_2),\ y_1 - x_1,\ y_2 - x_2\big).$$

The inflow at s and at t is the change of the occupation of channel 1 and of channel 2, and the reference vertex supplies the balance.

Proof. $j_1 - j_2 = (y_1 - x_2) - (x_1 - x_2) = y_1 - x_1$ and $j_2 + j_3 = (x_1 - x_2) + (y_2 - x_1) = y_2 - x_2$; the first component is minus their sum. □

The proof reads as a sequence of transfers. The exchange moves the imbalance $x_1 - x_2$ from s to t along the loop edge: channel 1 gives up x_1 and receives x_2. Then each clock writes its own bit change along the tree edge from the reference vertex: $y_1 - x_2$ into s and $y_2 - x_1$ into t. The divergence fixes a current on the axis only up to an integer multiple of the circulation $(1, 1, -1)$: the tree-routed current $(y_1 - x_1, 0, y_2 - x_2)$ moves the occupations equally well and never uses the loop edge. The label is the representative that routes the exchange's transfer $x_1 - x_2$ through the loop edge and each clock's bit change through its tree edge. This routing is part of the step taken from recognition dynamics [3]; every result that reads j_2, the field-current window first among them, depends on it.

2.4.3 The loop field

The occupations are charges, and the currents need a field. Fix background charges $\bar{c}, \bar{d} \in \{0, 1\}$ at the sites s and t and a *loop flux* $n \in \mathbb{Z}$, the flux on the loop edge. The *edge flux* of a cell at

occupation x is

$$m(n, x; \bar{c}, \bar{d}) = \big(n + x_1 - \bar{c},\ n,\ -n + x_2 - \bar{d}\big). \tag{2.10}$$

Proposition 2.9 (Gauss incidence)**.** *The edge flux has divergence $\partial m(n, x; \bar{c}, \bar{d}) = (\bar{c} + \bar{d} - x_1 - x_2,\ x_1 - \bar{c},\ x_2 - \bar{d})$: its inflow at each site is the occupation charge there, measured from the background charge. For every transition $x \to y$,*

$$m(n, x; \bar{c}, \bar{d}) + j(y, x) = m\big(n + j_2(y, x),\ y; \bar{c}, \bar{d}\big).$$

Proof. Both statements are direct substitution: for instance the first component of the identity is $(n + x_1 - \bar{c}) + (y_1 - x_2) = (n + x_1 - x_2) + y_1 - \bar{c}$. □

So a posting with label j moves the loop flux by j_2, the current on the loop edge, and the tree fluxes follow from the occupations. The whole integer field is carried by the single integer n. No record law depends on the background charges, and we suppress them; the field energy of Chapter 7 is computed with both set to 0.

2.4.4 Ledger accounts and physical writes

The step is recorded on a ledger with three *observed accounts* $1, 2, 3$, one for each edge; account 2, the account of the loop edge, is the *field-current account*. A *padding account* $\circ$ is used by the padded encoding below. (Chapter 21 adds two accounts that neither encoding touches.)

Definition 2.10 (Physical writes and the padded encoding)**.** The *physical writes* of a step with label j are one unit posting on observed account i for each i with $j_i \neq 0$: a debit when $j_i = +1$ and a credit when $j_i = -1$. The *padded encoding* makes the same posts and, in addition, one debit on the padding account for each i with $j_i = 0$, so that every step makes exactly three posts.

Proposition 2.11 (The ledger carries the current)**.**

(a) Under either encoding, a step with label j changes the potential of observed account i by exactly j_i, from every ledger state. Each post is a unit posting in the sense of Proposition 2.1.

(b) Hence, if the potentials of the observed accounts equal the edge flux $m(n, x; \bar{c}, \bar{d})$ before a transition $x \to y$, they equal $m(n + j_2, y; \bar{c}, \bar{d})$ after it.

(c) The zero label has no physical writes, and the padded encoding gives it three padding posts. The zero-label step is nevertheless a nonzero operation on the cell: its blocks are $M_{(0,0),(0,0)} = -\mathrm{stay} \otimes \mathrm{stay}$ and $M_{(1,1),(1,1)} = -\mathrm{stay} \otimes \mathrm{stay}$.

(d) The two encodings post on the field-current account at exactly the same steps: those with $j_2 \neq 0$.

Proof. (a) A debit on account i adds 1 to its potential and a credit subtracts 1; padding posts touch account $\circ$ only. (b) follows from (a) and Proposition 2.9. (c) The zero label has no nonzero component; its two blocks are $\sigma(x)K(x_1, x_2) \otimes K(x_2, x_1)$ with x balanced, and $K(1, 1) \otimes K(1, 1) = \mathrm{stay} \otimes \mathrm{stay}$. (d) Account 2 receives a post exactly when $j_2 \neq 0$ in both encodings, since a vanishing component is padded on account $\circ$. □

Taking the physical writes to be the events of a step is an identification, and Chapter 21 examines it. The padded encoding contains every physical write, but its padding posts write nothing the current does; whether a padding post counts as a physical write is settled there by one premise, least-cost booking of each step.

2.5 The posting instrument

A step does more than move the cell: it leaves its label. The operators that do both are the slices.

For each label $j \in \mathcal{J}$ the *slice* is

$$V_j = \sum_{(y,x)\,:\,j(y,x)=j} |y\rangle\langle x| \otimes M_{yx}. \tag{2.11}$$

Every slice has one block except $V_{(0,0,0)}$, which keeps the two balanced self-transitions together, coherently; and $\sum_j V_j = U_{\mathrm{nat}}$. For a chronological word $w = (j^{(1)}, \dots, j^{(k)})$ put $V_w = V_{j^{(k)}} \cdots V_{j^{(1)}}$, with the first label applied first, and call $E_w = V_w^* V_w$ its *effect*. The empty word has $V_\emptyset = E_\emptyset = I$. The *free pull*, the Heisenberg form of one posting with its record summed, is

$$\Phi(Y) = \sum_{j\in\mathcal{J}} V_j^*\, Y V_j, \tag{2.12}$$

so that $\Phi^k(Y) = \sum_{w\in\mathcal{J}^k} V_w^* Y V_w$, and in particular $\Phi^k(I)$ is the sum of the effects of all words of length k.

Proposition 2.12 (The postings form an instrument)**.** *$\sum_{j\in\mathcal{J}} V_j^* V_j = I$. Hence $\rho \mapsto (V_j \rho V_j^*)_{j\in\mathcal{J}}$ is a quantum instrument [15] with fifteen outcomes, Φ is unital, and $\sum_{w\in\mathcal{J}^k} E_w = I$ for every k.*

Proof. In $V_j^* V_j$ a cross term between two transitions with the same label needs the same output, and then, by Proposition 2.7, the same input. So $V_j^* V_j = \sum_{j(y,x)=j} |x\rangle\langle x| \otimes M_{yx}^\top M_{yx}$. Summing over j gives $\sum_x |x\rangle\langle x| \otimes \sum_y M_{yx}^\top M_{yx} = I$ by Proposition 2.6. The last claim follows by induction on k. □

The label is what makes this work. Attaching records to arbitrary pieces of a unitary matrix can destroy the cancellations between its columns and spoil normalization; here the label and the output recover the input, and that supplies the needed orthogonality.

The instrument is not sharp. By the proof, the one-step effect of a label is a sum of terms $|x\rangle\langle x| \otimes M_{yx}^\top M_{yx}$, and by Lemma 2.5 each $M_{yx}^\top M_{yx}$ is a product of two Grams $\frac{1}{2}(I \pm c)$ (balanced input) or $\frac{1}{2}(I \pm c')$ (mixed input). So every one-step effect is diagonal in the product pole basis, with entries in $\{0, \frac{1}{4}, \frac{1}{2}, 1\}$. Longer effects need not be diagonal; Chapter 9 determines exactly which part of the cell they read.

The instrument has an exact symmetry that no single internal coordinate shares. Let $H_1 = I \otimes L^2 \otimes I$, which moves the first clock cell half way round its cycle ($e_i \mapsto e_{i+2}$), and $H_2 = I \otimes I \otimes L^2$ likewise for the second.

Proposition 2.13 (Half-cycle symmetry of the records)**.** *For each $H \in \{H_1, H_2\}$ and each label j, $HV_jH = -V_j$ if $j_2 = 0$ and $HV_jH = V_j$ if $j_2 \neq 0$. Hence $HE_wH = E_w$ for every word w, and every polynomial in the effects commutes with H_1 and H_2. In particular the projection onto a single basis vector $|x\rangle \otimes e_i \otimes e_{i'}$ is not a polynomial in the effects.*

Proof. L^2 commutes with L, and $L^2DL^2 = -D$, because shifting the diagonal $(1, -1, -1, 1)$ by two places reverses its sign. So $L^2K(b', b)L^2 = -K(b', b)$ and $L^2K(b', b)DL^2 = K(b', b)D$. A balanced block has $C_x = I$, so conjugating either cell factor by L^2 reverses its sign; a mixed block has $C_x = D$, and the conjugation leaves it unchanged. By Proposition 2.7, $j_2 = 0$ exactly for balanced inputs. Then $HE_wH = (HV_wH)^*(HV_wH) = E_w$, since the signs square away. The projection onto $|x\rangle \otimes e_i \otimes e_{i'}$ does not commute with H_1, which moves e_i to e_{i+2}. □

2.6 Records and the history space

A *record of length k* is a word $h = (j^{(1)}, \dots, j^{(k)}) \in \mathcal{J}^k$, oldest label first. Its *accumulated loop current* is $\mathrm{acc}(h) = \sum_t j_2^{(t)}$. For $k \geq 0$ let

$$\mathcal{H}_k = \ell^2(\mathbb{Z} \times \mathcal{J}^k) \otimes \mathcal{H},$$

the square-summable functions $\xi(n, h) \in \mathcal{H}$ of a loop flux n and a record h of length k. A *genesis state* is a unit vector $v \in \mathcal{H}_0 = \ell^2(\mathbb{Z}) \otimes \mathcal{H}$, with loop slices $v(n) \in \mathcal{H}$. A genesis state is an initial condition: no result of this chapter selects or prepares one, and the ready vector of Example 2.18 is an example, not a derived state. The *posting* $T : \mathcal{H}_k \to \mathcal{H}_{k+1}$ is

$$(T\xi)(n,\, h \cdot j) = V_j\, \xi(n - j_2,\, h) \ : \tag{2.13}$$

it applies the slice, moves the loop flux by the loop-edge current, and appends the label to the record.

Theorem 2.14 (Posting is an isometry that keeps its records)**.**

(a) T is an isometry $\mathcal{H}_k \to \mathcal{H}_{k+1}$ for every k.
(b) For a genesis state v and a record h of length k, $(T^k v)(n, h) = V_h\, v\big(n - \mathrm{acc}(h)\big)$.
(c) The coefficients of $T\xi$ on the records extending h depend on ξ only through $\xi(\cdot, h)$: a posting never alters an earlier record.

Proof. (a) $\|T\xi\|^2 = \sum_{h,j} \sum_n \|V_j \xi(n - j_2, h)\|^2 = \sum_{h,m} \langle \xi(m, h), \sum_j V_j^* V_j\, \xi(m, h) \rangle = \|\xi\|^2$, reindexing $m = n - j_2$ and using Proposition 2.12. (b) Induction on k: $(T^{k+1}v)(n, h \cdot j) = V_j (T^k v)(n - j_2, h) = V_j V_h\, v(n - j_2 - \mathrm{acc}(h))$, and $\mathrm{acc}(h \cdot j) = \mathrm{acc}(h) + j_2$. (c) is read off (2.13). □

All sums over labels are finite, so the theorem holds for every square-summable field state and for any entanglement of the field with the cell.

The stages assemble into one space. The *complete history space* is the Hilbert inductive limit $\mathcal{H}_\infty$ of the stages along T.

Proposition 2.15 (The complete history space)**.** *There are a Hilbert space $\mathcal{H}_\infty$ and isometries $\iota_k : \mathcal{H}_k \to \mathcal{H}_\infty$ with $\iota_{k+1} T = \iota_k$ and $\bigcup_k \iota_k(\mathcal{H}_k)$ dense. If $\mathcal{K}$ is another Hilbert space with isometries*

$\jmath_k$ having the same two properties, there is exactly one unitary $W : \mathscr{H}_\infty \to \mathscr{K}$ with $W\iota_k = \jmath_k$ for every k. For a bounded operator B on $\mathscr{H}_k$ and genesis states u, v,

$$\langle \iota_0 u,\ \iota_k B \iota_k^*\, \iota_0 v\rangle = \langle T^k u,\ B T^k v\rangle.$$

Proof. Take $\mathscr{H}_\infty$ to be the completion of the union of the stages, with $\xi \in \mathscr{H}_k$ identified with $T\xi \in \mathscr{H}_{k+1}$; the identification preserves inner products by Theorem 2.14(a). Uniqueness: $W(\iota_k\xi) = \jmath_k\xi$ is forced, is well defined and isometric on the dense union, and extends to a unitary because the range of $\jmath$ is dense. The matrix element follows from $\iota_0 = \iota_k T^k$ and $\iota_k^*\iota_k = I$. □

Since $\iota_k T^k v = \iota_0 v$, a genesis state and its whole posted history are one vector: a vector of $\mathscr{H}_\infty$ is a history whose records are all kept. Every finite-time quantity of the native dynamics is a matrix element in this one space. Chapter 4 studies its events.

Each posting is also one half of a reversible operation. On $\mathscr{H}_k \oplus \mathscr{H}_{k+1}$ put

$$\mathbb{U} = \begin{pmatrix} 0 & T^* \\ T & I - TT^* \end{pmatrix}. \tag{2.14}$$

Proposition 2.16 (A reversible completion of each posting)**.** *$\mathbb{U}$ is unitary and self-adjoint, $\mathbb{U}^2 = I$, and $\mathbb{U}(\xi, 0) = (0, T\xi)$ for every $\xi \in \mathscr{H}_k$.*

Proof. Since $T^*T = I$, the operator TT^* is a projection and $T^*(I - TT^*) = 0$. Squaring (2.14) gives diagonal blocks $T^*T = I$ and $TT^* + (I - TT^*)^2 = I$ and zero off-diagonal blocks. Self-adjointness is visible in (2.14). □

The completion places the input in a ready slot and returns its posted image in an output slot. It shows that nothing in the posting forbids reversible dynamics on the enlarged space. It does not supply the duration of a step, the preparation of the output slot, or the interaction that would enact it; those remain outside the construction.

2.7 The Born law and the law of infinite records

The history keeps its coherence: T is an isometry on the whole space. What a record *has*, at each length, is a weight: the squared norm of the history at that record. These weights form the record law.

For a genesis state v let $\rho_v = \sum_n v(n)v(n)^*$, the *cell density*: the density matrix of the cell with the loop field traced out. It has trace $\|v\|^2 = 1$.

Theorem 2.17 (The record law)**.** *Let v be a genesis state. For a record h of length k define its* load

$$\mathbb{P}_v(h) = \sum_{n\in\mathbb{Z}} \big\|(T^k v)(n, h)\big\|^2.$$

(a) (Born form) $\mathbb{P}_v(h) = \sum_n \big\|V_h\, v(n - \mathrm{acc}(h))\big\|^2 = \mathrm{tr}(\rho_v E_h)$.
(b) $\mathbb{P}_v(h) \geq 0$ and $\sum_{h\in\mathscr{J}^k} \mathbb{P}_v(h) = 1$.
(c) (Consistency) $\sum_{j\in\mathscr{J}} \mathbb{P}_v(h\cdot j) = \mathbb{P}_v(h)$ for every record h.

(d) (Infinite records) There is exactly one probability measure μ_v on the space $\mathcal{J}^{\mathbb{N}}$ of infinite records, with its product σ-algebra, such that for every k and every $h \in \mathcal{J}^k$ the set of infinite records beginning with h has measure $\mathbb{P}_v(h)$.

(e) $\mathbb{P}_v$ depends on v only through ρ_v. In particular it is unchanged when the loop flux is translated, $v(n) \mapsto v(n - n_0)$ for a fixed $n_0 \in \mathbb{Z}$.

Proof. (a) The first equality is Theorem 2.14(b). For fixed h, $n \mapsto n - \mathrm{acc}(h)$ is a bijection of $\mathbb{Z}$, so the sum is $\sum_n \langle v(n), E_h v(n)\rangle = \mathrm{tr}(\rho_v E_h)$. (b) Each load is a sum of squared norms, and the loads at length k add up to $\|T^k v\|^2 = 1$. (c) $E_{h\cdot j} = V_h^* V_j^* V_j V_h$; summing over j with Proposition 2.12 gives E_h. (d) By (b) and (c) the laws $\mathbb{P}_v$ on $\mathcal{J}^k$, $k \geq 0$, are a consistent family of probability laws on finite products of the finite set $\mathcal{J}$. Kolmogorov's extension theorem [27] gives a probability measure on $\mathcal{J}^{\mathbb{N}}$ with these cylinder masses. It is unique because the sets of infinite records beginning with a given word, together with the empty set, are closed under intersection and generate the product σ-algebra. (e) The trace formula in (a) involves v only through ρ_v, and translation leaves ρ_v unchanged. □

The theorem constructs a probability law from the dynamics: the load is the conserved norm of the history at a record, and the laws of all lengths fit together. That this law gives the frequencies with which records occur in repeated trials is an apparatus statement, separate from the construction. We call the load the probability of the record throughout and name the apparatus link once, here.

No probability rule is needed to single out the load. Let κ assign a number to each vector of the complete history space $\mathscr{H}_\infty$, nonnegative, equal to one on unit vectors, and additive over pairs u, w with $\mathrm{Re}\langle u, w\rangle = 0$ whose sum has norm at most one. Restricted to any real subspace of dimension at least three, these conditions force $\kappa(u) = \|u\|^2$ for $\|u\| \leq 1$, and every vector lies in such a subspace. Applied to the part of a genesis history $\iota_0 v$ whose records of length k lie in a set S, every such κ therefore returns its squared norm, $\sum_{h \in S} \mathbb{P}_v(h)$. So every ratio of record probabilities, the present verdict of Chapter 3 among them, is a ratio of κ-counts of record events, for every κ.

Example 2.18 (The ready vector)**.** Take the genesis state $r = |00\rangle \otimes p \otimes a$ at loop flux 0, the *ready vector*. First posting: the input is balanced. Cell 1 reads $x_2 = 0$; since stay $p = 0$ and flip $p = P$, it must flip, $y_1 = 1$. Cell 2 reads $x_1 = 0$; since stay $a = A$ and flip $a = 0$, it must stay, $y_2 = 0$. So the label $(1, 0, 0)$ is written with probability 1 and the cell becomes $|10\rangle \otimes (-P \otimes A)$. Second posting: the input $(1, 0)$ is mixed and both cells are dressed, $DP = p$ and $DA = a$. Cell 1 reads $x_2 = 0$ and must flip; cell 2 reads $x_1 = 1$, and $K(0,1)a = -\mathrm{flip}\, a = 0$ while $K(1,1)a = -\mathrm{stay}\, a = -A$, so $y_2 = 1$. The label $(1, 1, 0)$ is written with probability 1, and the cell becomes $|11\rangle \otimes (-P \otimes A)$. Third posting: the input $(1, 1)$ is balanced and P, A lie on the zero plane of c, so each cell stays or flips with weight $\frac{1}{2}$. The four labels $(-1, 0, -1)$, $(0, 0, -1)$, $(-1, 0, 0)$ and $(0, 0, 0)$ each have probability $\frac{1}{4}$. So

$$\mathbb{P}_r\big((1,0,0),(1,1,0),j\big) = \tfrac{1}{4} \qquad \text{for each of these four } j,$$

and every other record of length three has probability 0. The first two labels are certain; the third splits evenly.

2.8 The present cell

Given a record, the future of the record law depends on the past only through one density matrix.

Definition 2.19 (Present cell)**.** For a genesis state ν and a record h with $\mathbb{P}_\nu(h) > 0$, the *present cell* is the conditional state

$$\rho_{\nu,h} = \frac{V_h \, \rho_\nu \, V_h^*}{\mathbb{P}_\nu(h)}.$$

The present cell is the cell density of an actual state. The unit genesis state $\nu_h(n) = V_h \nu(n - \mathrm{acc}(h))/\sqrt{\mathbb{P}_\nu(h)}$, the history's coefficients at the record h renormalized and with the record itself set aside, has cell density $\rho_{\nu_h} = \rho_{\nu,h}$. In particular $\rho_{\nu,h}$ has trace 1.

Proposition 2.20 (The record law factors through the present)**.** *Let* $\mathbb{P}_\nu(h) > 0$.

(a) For every continuation w, $\mathbb{P}_\nu(h \cdot w) = \mathbb{P}_\nu(h)\ \mathrm{tr}(\rho_{\nu,h} E_w) = \mathbb{P}_\nu(h)\, \mathbb{P}_{\nu_h}(w)$.
(b) If also $\mathbb{P}_\nu(h \cdot j) > 0$, *then* $\rho_{\nu,h\cdot j} = V_j \rho_{\nu,h} V_j^* / \mathrm{tr}(\rho_{\nu,h} E_j)$.

Proof. (a) $E_{h\cdot w} = V_h^* E_w V_h$, so $\mathrm{tr}(\rho_\nu E_{h\cdot w}) = \mathrm{tr}(V_h \rho_\nu V_h^* E_w)$; the second equality is Theorem 2.17(a) for ν_h. (b) $V_{h\cdot j} = V_j V_h$, and by (a) with $w = j$, $\mathbb{P}_\nu(h \cdot j) = \mathbb{P}_\nu(h)\,\mathrm{tr}(\rho_{\nu,h} E_j)$. □

Example 2.21 (A present cell of the ready vector)**.** In Example 2.18, take the record h whose third label is $(0,0,0)$, so that $h = \big((1,0,0),\,(1,1,0),\,(0,0,0)\big)$. Its third block is $-\mathrm{stay} \otimes \mathrm{stay}$, and stay $P = \frac{1}{2}(p + a)$, stay $A = \frac{1}{2}(a - p)$. The present cell is the pure state

$$\begin{aligned} \rho_{r,h} &= |\phi\rangle\langle\phi|, \\ \phi &= |11\rangle \otimes \tfrac{1}{\sqrt{2}}(p + a) \otimes \tfrac{1}{\sqrt{2}}(a - p) = -|11\rangle \otimes \tfrac{1}{\sqrt{2}}(e_0 + e_2) \otimes \tfrac{1}{\sqrt{2}}(e_1 + e_3). \end{aligned}$$

Both factors are fixed by L^2, as p and a are, and each has pole charge expectation 0; the occupation is balanced, so by Lemma 2.5 the next posting again splits each cell evenly.

The present cell is a mathematical object: the state of the cell given its written record, with the record set aside. It is a conditional-state identity of the record law; it adds no collapse rule to the dynamics, which remains the isometry T on the whole history. Reading the present cell as the physical present of the cell is an identification, and we name it as one wherever it is used.

2.9 Sectors, four modes and the field-current window

The rest of the book turns on a small part of the cell. This section names it.

2.9.1 Sectors of the pole product

On a mixed occupation, let $\Theta = c' \otimes c'$ act on the two clock cells: the *pole product*. By Table 2.1 its spectrum is $\{1, -1, 0\}$ and $\Theta^3 = \Theta$. Define the *aligned*, *opposed* and *neutral* sectors

$$\begin{gathered} Q_{\mathrm{al}} = \sum_{x \text{ mixed}} |x\rangle\langle x| \otimes \tfrac{1}{2}(\Theta^2 + \Theta), \qquad Q_{\mathrm{op}} = \sum_{x \text{ mixed}} |x\rangle\langle x| \otimes \tfrac{1}{2}(\Theta^2 - \Theta), \\ Q_{\mathrm{ne}} = \sum_{x \text{ mixed}} |x\rangle\langle x| \otimes (I - \Theta^2). \end{gathered} \tag{2.15}$$

At each mixed occupation the aligned sector is span$\{P\otimes P, A\otimes A\}$, where the two dressed pole charges agree; the opposed sector is span$\{P\otimes A, A\otimes P\}$; and the neutral sector, of dimension 12, holds the states with at least one cell where the dressed pole charge vanishes. "Neutral" refers to this pole product, not to electric charge. The three sectors are orthogonal projections that sum to the projection onto the mixed occupations, with ranks 4, 4 and 24; all three vanish on the balanced occupations, which carry the remaining 32 dimensions. For a cell density ρ write $w_{\mathrm{al}}(\rho) = \mathrm{tr}(\rho Q_{\mathrm{al}})$ and $w_{\mathrm{ne}}(\rho) = \mathrm{tr}(\rho Q_{\mathrm{ne}})$, the *aligned* and *neutral weights*.

2.9.2 Four modes

Four vectors of the aligned sector carry the rest of the book:

$$\psi_+ = |10\rangle\otimes P\otimes P, \quad \psi_- = |01\rangle\otimes P\otimes P, \quad \psi_\circ = |10\rangle\otimes A\otimes A, \quad \psi_\circ' = |01\rangle\otimes A\otimes A. \tag{2.16}$$

They are orthonormal and span the aligned sector: $Q_{\mathrm{al}} = \sum |\psi\rangle\langle\psi|$ over the four.

Proposition 2.22 (What the four modes do)**.** *Under the native step,*

$$V_{(1,1,-1)}\psi_+ = \psi_+, \qquad V_{(-1,-1,1)}\psi_- = -\psi_-, \qquad V_{(0,1,0)}\psi_\circ = \psi_\circ', \qquad V_{(0,-1,0)}\psi_\circ' = -\psi_\circ,$$

and every other slice annihilates the mode in question.

Proof. Take ψ_+, with $x = (1,0)$ mixed, so both cells are dressed: $DP = p$. Cell 1 applies $K(y_1, 0)$ to p, which is 0 for $y_1 = 0$ and P for $y_1 = 1$. Cell 2 applies $K(y_2, 1)$ to p, which is $-P$ for $y_2 = 0$ and 0 for $y_2 = 1$. So only $y = (1,0)$ survives, with block value $\sigma(1,0)\, P\otimes(-P) = P\otimes P$ and label $(1,1,-1)$. The other three cases use the same table: for ψ_-, only $y = (0,1)$ survives, with $\sigma(0,1)(-P)\otimes P$; for $\psi_\circ$, $DA = a$ gives $y = (0,1)$ with $\sigma(1,0)\, A\otimes(-A)$; for $\psi_\circ'$, $y = (1,0)$ with $\sigma(0,1)(-A)\otimes A$. □

mode	occupation	cells	one posting	label
ψ_+	$(1,0)$	$P\otimes P$	ψ_+	$(1,1,-1)$
ψ_-	$(0,1)$	$P\otimes P$	$-\psi_-$	$(-1,-1,1)$
$\psi_\circ$	$(1,0)$	$A\otimes A$	$\psi_\circ'$	$(0,1,0)$
$\psi_\circ'$	$(0,1)$	$A\otimes A$	$-\psi_\circ$	$(0,-1,0)$

Table 2.4. The four modes and their postings. Each mode writes one label with certainty. The loop flux moves by the middle component of the label.

So ψ_+ and ψ_- are *persistent loop currents*: each writes the same label at every step, and its loop flux moves by $+1$ or -1 per step without bound. The pair $\psi_\circ, \psi_\circ'$ is an *occupation oscillation*: it alternates between the two mixed occupations, writes $(0,1,0)$ and $(0,-1,0)$ in turn, keeps the loop flux within one unit of its start, and returns to itself with phase $+1$ after four steps. The names describe the records; that the accumulated current of a persistent mode is a field winding up without bound is the Gauss reading of Proposition 2.9.

2.9.3 The field-current window

Earlier work on consciousness in this framework uses an eight-tick completeness criterion: a pattern passes when each of eight consecutive ticks writes [4]. We read the criterion on the field-current account, one tick per posting; this choice of account and clock is supplied, and Chapter 3 shows what other choices give. The *field-current window* starting after posting k *passes* when each of the eight postings $k+1, \dots, k+8$ writes the loop edge, that is, when each of their labels has $j_2 \neq 0$. Equivalently, each of those postings starts from a mixed occupation, and each makes a physical write on the field-current account. By Proposition 2.11(d) this reading is the same under the padded encoding. Its length matches the eight-state period of each channel, but nothing in this chapter derives the window from the step. Let G_k be the summed effect of the words of length k that write the loop edge at every step,

$$G_k = \sum_{w \in \mathcal{J}^k,\ j_2^{(t)} \neq 0 \text{ for all } t} E_w .$$

Calling a state that passes every window with certainty a *sure subject* is the framework's identification of the criterion with a perspective that is present at every tick. Chapter 7 proves that the sure subjects are exactly the states of a class defined from the postings alone, without reference to the window.

The next question is what the framework's eight-tick criterion reads on this carrier: which windows pass, with what probability, and what a single window can and cannot decide about the state that writes it. Chapter 3 answers it.

Chapter 3

The eight-tick window and its verdicts

Chapter 2 read the framework's eight-tick completeness criterion [4] on the native cell: a window of eight postings passes when each posting writes the loop edge. This chapter asks what that verdict is, as an object of the step's own record law, and what it is a property of.

The answer is explicit. On the first window the verdict is the read $Q_{\mathrm{al}} + \frac{1}{128} Q_{\mathrm{ne}}$: weight one on the aligned sector, $\frac{1}{128}$ on the neutral sector and nothing elsewhere. It commutes with everything the record effects generate, so no coherence of a state, inside a sector or between sectors, changes the probability of passing; among all sets of accounts on which a window can pass, only those whose sole observed account is the loop edge have this property. Referred back to genesis, the property belongs to the first three windows only; read on the state the written record leaves, it holds at every window. From the fourth window on, the loop-edge verdict carries a coherence that shrinks but never vanishes, and centrality returns only in the limit, as an explicit central read that every state passes with probability at least $\frac{1}{448}$.

3.1 Cuts, clocks and window effects

A window verdict needs three choices: which accounts count, what counts as a tick, and how many ticks make a window. We fix each and then turn the verdict into an operator on the cell.

A posting with label $j = (j_1, j_2, j_3)$ is booked on the ledger of Chapter 2, whose observed accounts $1, 2, 3$ carry the three current components; account 2, the field-current account, carries j_2. We list the physical writes of j (Definition 2.10) as signed unit posts,

$$\mathrm{phys}(j) = \big[(i, \operatorname{sgn} j_i) \,:\, j_i \neq 0\big], \qquad i = 1, 2, 3 \text{ in this order},$$

so the zero label writes nothing. The ledger has further accounts, among them the padding account $\circ$ of the padded encoding studied in Chapter 21; physical writes never touch them. A *cut* is any set C of accounts.

On the *native-step clock* there is one tick per posting: posting t is a *tick of* C when $\mathrm{phys}(j^{(t)})$ has an entry on C, that is, when $j_i^{(t)} \neq 0$ for some observed account $i \in C$. The *window of* C *starting after posting* k passes when the eight postings $k+1, \dots, k+8$ are all ticks of C. The field-current window of Chapter 2 is the window of the cut $\{2\}$.

Definition 3.1 (Window effects)**.** For a cut C and $t \geq 0$ let $G_t(C) = \sum E_h$, summed over the records $h \in \mathscr{J}^t$ all of whose postings are ticks of C, with $G_0(C) = I$. For $k \geq 0$ the *window effect* is

$$W_k(C) = \sum_{h \in \mathscr{J}^{k+8} :\ \text{window } k \text{ of } C \text{ passes on } h} E_h .$$

We write $W_k = W_k(\{2\})$ for the field-current window. Since a tick of $\{2\}$ is a posting with

$j_2 \neq 0$, $G_t(\{2\}) = G_t$ is the summed effect of the t-letter words that write the loop edge at every step (Chapter 2).

Proposition 3.2 (Window effects). *For every cut C, every $k \geq 0$ and every genesis state v:*

(a) $\mathbb{P}_v(\text{window } k \text{ of } C \text{ passes}) = \operatorname{tr}\big(\rho_v W_k(C)\big)$;
(b) $G_{t+1}(C) = \sum_j V_j^* G_t(C) V_j$, *summed over the labels j that are ticks of C, and* $W_0(C) = G_8(C)$;
(c) $W_{k+1}(C) = \Phi\big(W_k(C)\big)$, *so* $W_k(C) = \Phi^k\big(G_8(C)\big)$;
(d) $W_k(C) = W_k\big(C \cap \{1,2,3\}\big)$, *and* $W_k(C) = 0$ *when C holds no observed account.*

Proof. (a) is the record law $\mathbb{P}_v(h) = \operatorname{tr}(\rho_v E_h)$ summed over the passing records. For (b) and (c), split a record as $h = j \cdot h'$ with first label j, so that $E_{j\cdot h'} = V_j^* E_{h'} V_j$. All postings of $j \cdot h'$ are ticks exactly when j is a tick and all postings of h' are; and window $k+1$ passes on $j \cdot h'$ exactly when window k passes on h', whatever j is. Summing gives the two recursions. (d) Physical writes land only on observed accounts. □

Part (c) says what a later window is: the first window pulled back through k postings whose record is summed. A later verdict is still a verdict about the genesis state, not about a state conditioned on what was written; the conditioned verdict appears in Section 3.6.

The cut and the clock are supplied; Section 3.8 shows that the clock matters. The factor $\frac{1}{2}$ per tick found below does not depend on the window length eight; the number $\frac{1}{128} = 2^{-7}$ does. For a cut without the padding account the verdict is the same under every attachment between the physical writes and the padded encoding (Theorem 21.10(c)).

3.2 The field-current verdict reads three sectors

The first field-current window has a closed form. It reads the aligned sector with weight one, the neutral sector with weight $\frac{1}{128}$, and nothing else.

Write Π_{mix} for the projection onto the mixed occupations. The sectors of Chapter 2 project onto $\Theta = 1, -1$ and 0 there, so $Q_{\text{al}} + Q_{\text{op}} + Q_{\text{ne}} = \Pi_{\text{mix}}$.

Lemma 3.3 (Awake effects). $G_1 = \Pi_{\text{mix}}$, *and for every $k \geq 1$*

$$G_{k+1} = \sum_{x \text{ mixed}} |x\rangle\langle x| \otimes \big(\tfrac{1}{2}(I + \Theta)\big)^k = Q_{\text{al}} + 2^{-k} Q_{\text{ne}}.$$

Proof. A label has $j_2 \neq 0$ exactly when its input is mixed, and for a fixed input x the effects of its four transitions sum to $|x\rangle\langle x| \otimes \sum_y M_{yx}^\top M_{yx} = |x\rangle\langle x| \otimes I$. This gives G_1.

Two facts about a mixed input x carry the induction. First, its transitions to the two mixed outputs have Gram sum $\frac{1}{2}(I + \Theta)$. For $x = (1,0)$ the output $(1,0)$ applies flip D to both cells, up to sign, and the output $(0,1)$ applies stay D to both. Conjugating the Grams of Chapter 2 by D gives $(\text{flip } D)^\top(\text{flip } D) = \frac{1}{2}(I + c')$ and $(\text{stay } D)^\top(\text{stay } D) = \frac{1}{2}(I - c')$, and

$$\tfrac{1}{4}(I + c') \otimes (I + c') + \tfrac{1}{4}(I - c') \otimes (I - c') = \tfrac{1}{2}(I + c' \otimes c').$$

The input $(0,1)$ is the same with the two outputs exchanged. Second, every such block commutes with Θ: $c = \frac{1}{2}(L + L^{-1})$ commutes with L, so stay D and flip D commute with $c' = DcD$.

Suppose $G_k = \sum_{x \text{ mixed}} |x\rangle\langle x| \otimes g_k(\Theta)$ for a polynomial g_k. Then $G_{k+1} = \sum_{j_2 \neq 0} V_j^* G_k V_j$. A transition into a balanced output meets the zero block of G_k. The transitions into the two mixed outputs contribute $g_k(\Theta)\,\frac{1}{2}(I+\Theta)$ by the two facts. So $g_{k+1} = \left(\frac{1}{2}(1+\Theta)\right)^k$. The operator $\frac{1}{2}(I+\Theta)$ is 1 on the aligned sector, 0 on the opposed sector and $\frac{1}{2}$ on the neutral sector. □

Theorem 3.4 (The first window)**.** *Let v be a genesis state.*

(a) $W_0 = G_8 = Q_{\text{al}} + \frac{1}{128} Q_{\text{ne}}$*, and the first field-current window passes with probability* $w_{\text{al}}(\rho_v) + \frac{1}{128} w_{\text{ne}}(\rho_v)$*.*

(b) The aligned sector certifies every window length and the opposed sector fails at the second tick: $Q_{\text{al}} G_{k+1} = Q_{\text{al}}$ *and* $Q_{\text{op}} G_{k+2} = 0$ *for every* $k \geq 0$*.*

(c) The sectors are combinations of window effects: $Q_{\text{ne}} = 4(G_2 - G_3)$*,* $Q_{\text{al}} = 2G_3 - G_2$ *and* $Q_{\text{op}} = G_1 - Q_{\text{al}} - Q_{\text{ne}}$*.*

(d) The verdict is not sharp: G_2 *and* G_8 *are not projections.*

Proof. (a), (b) and (c) follow from Lemma 3.3, Proposition 3.2(a) and the orthogonality of the three sectors. (d) $G_8^2 = Q_{\text{al}} + 2^{-14} Q_{\text{ne}} \neq G_8$ because $Q_{\text{ne}} \neq 0$, and likewise $G_2^2 \neq G_2$. □

So the verdict is a classical mixture over three sectors. An aligned state writes the loop edge at every posting. An opposed state writes it once and is silent at the second posting. A neutral state writes it with probability one half at each posting after the first.

Example 3.5 (Three verdicts)**.** The persistent mode $\psi_+ = |10\rangle \otimes P \otimes P$ writes $(1, 1, -1)$ at every posting, so it passes every window of every cut holding an observed account with probability one. The balanced state $\alpha = |00\rangle \otimes a \otimes a$ writes the zero label at its first posting with probability one, so it fails the first window of every cut surely. The *chance state*

$$\chi = |10\rangle \otimes \left(\tfrac{3}{5}\, P \otimes P + \tfrac{4}{5}\, A \otimes P\right)$$

has $w_{\text{al}} = \frac{9}{25}$, $w_{\text{op}} = \frac{16}{25}$ and $w_{\text{ne}} = 0$, since $A \otimes P$ has dressed pole product -1. By Theorem 3.4 it passes the first field-current window with probability exactly $\frac{9}{25}$. The mechanism is visible in the record. The aligned part writes $(1, 1, -1)$ forever. The opposed part moves surely to the balanced occupation $(0, 0)$ with label $(0, 1, -1)$, after which the next posting is silent on the loop edge. The two parts write different first labels, so the passing records are exactly $(1, 1, -1)^8$, of probability $(\frac{3}{5})^2$.

A genesis state therefore fixes the probability of the verdict, not the verdict.

3.3 The verdict is central

The window verdict commutes with every record effect, and with everything the effects generate. That is why no coherence can move its probability.

Let $\mathscr{A}$ be the algebra generated by all effects E_w, the *current algebra.* An element of $\mathscr{A}$ is *central* when it commutes with every element of $\mathscr{A}$. At occupation x write $c_x = C_x c C_x$ for the pole charge of a clock cell (c on balanced and c' on mixed occupations) and $\Pi_+^x, \Pi_-^x, \Pi_0^x$ for its spectral projections onto $1, -1, 0$. The 36 *sector projections* are $\Pi(x, s, t) = |x\rangle\langle x| \otimes \Pi_s^x \otimes \Pi_t^x$.

We use the planes $\mathcal{U} = \mathrm{span}\{p, a\}$ and $D\mathcal{U} = \mathrm{span}\{P, A\}$ of Chapter 2, the zero planes of c' and of c.

Lemma 3.6 (Effects respect occupation and poles)**.** *Every effect E_w is block diagonal in the occupation, and on the block of x it commutes with $c_x \otimes I$ and with $I \otimes c_x$. Hence so does every element of $\mathscr{A}$.*

Proof. By the table of clock maps in Chapter 2, a cell map $f = K(b', b)C_x$ annihilates one of the two pole lines of c_x, sends the other into $D\mathcal{U}$, and sends the zero plane of c_x into $\mathcal{U}$. For every output y, both $\mathcal{U}$ and $D\mathcal{U}$ are sums of spectral subspaces of c_y. Let Y commute with c_y, so Y preserves $\mathcal{U}$ and $D\mathcal{U}$. If u lies in the zero plane of c_x and e on a pole line, then $fu \in \mathcal{U}$ and $fe \in D\mathcal{U}$, and Y preserves both planes, so $\langle f^\top Y f u, e\rangle = \langle Y f u, fe\rangle = 0$ and $\langle f^\top Y f e, u\rangle = \langle Y f e, fu\rangle = 0$. For two vectors e, e' on the two pole lines, one of fe, fe' vanishes, so $\langle f^\top Y f e, e'\rangle = 0$. So $f^\top Y f$ preserves the three spectral subspaces of c_x and commutes with c_x. The same holds factor by factor for $f_1 \otimes f_2$ and the two cells.

Two transitions with the same label and different inputs have different outputs, because label and output recover the input. So for Y block diagonal in the occupation,

$$V_j^* Y V_j = \sum_{j(y,x)=j} |x\rangle\langle x| \otimes M_{yx}^\top Y_{yy} M_{yx}.$$

Induction from $E_\varnothing = I$ with $E_{j\cdot w} = V_j^* E_w V_j$ proves the claim for effects. Both properties survive sums and products. □

Theorem 3.7 (The window verdict is central)**.** *For every $k \geq 0$, G_k is a central element of $\mathscr{A}$. So are Π_{mix}, Q_{al}, Q_{op} and Q_{ne}, and in particular the first field-current window effect $W_0 = G_8$.*

Proof. G_k is a sum of effects, so it lies in $\mathscr{A}$. By Lemma 3.3, G_{k+1} is block diagonal with block $(\frac{1}{2}(I + c_x \otimes c_x))^k$ on each mixed x and 0 on each balanced x: on every block a polynomial in $c_x \otimes I$ and $I \otimes c_x$. By Lemma 3.6 every element of $\mathscr{A}$ is block diagonal and commutes with these on each block, so it commutes with G_{k+1}. The sectors are combinations of G_1, G_2, G_3 by Theorem 3.4(c). □

Because the first window effect commutes with every record effect, its probability also does not depend on the order in which reads are made. Centrality is relative to $\mathscr{A}$; Chapter 9 computes its centre.

3.4 Outer accounts give contextual verdicts

Read on an outer account, the window verdict depends on coherence inside a sector. Two states with the same occupation and the same pole sectors can be a sure failure and a chance pass.

Account 1 is written exactly when $y_1 \neq x_2$, that is, when the first clock cell flips. Account 3 is written exactly when the second cell flips. One tick of an outer account is still a sector read.

Proposition 3.8 (One outer tick is central)**.** $G_1(\{1\}) = \frac{1}{2}\big(I + \sum_x |x\rangle\langle x| \otimes c_x \otimes I\big)$ *and* $G_1(\{3\}) = \frac{1}{2}\big(I + \sum_x |x\rangle\langle x| \otimes I \otimes c_x\big)$*. Both are central in $\mathscr{A}$.*

Proof. For input x, the transitions that flip the first cell are the two outputs with $y_1 \neq x_2$. They apply flip C_x to the first cell, up to sign, and the two possible maps to the second, whose Grams sum to I. By the Grams of Chapter 2 conjugated by C_x, their effects sum to $|x\rangle\langle x| \otimes \frac{1}{2}(I + c_x) \otimes I$. Centrality follows as in Theorem 3.7. Account 3 is the same with the cells exchanged. □

From the second tick on, the verdict becomes contextual. The witness is a short native loop. Let

$$w^* = \big((0,0,1),(0,-1,1),(-1,0,-1)\big), \qquad w^{*\prime} = \big((1,0,0),(1,1,0),(-1,0,-1)\big),$$

the records of the occupation loops $(0,0) \to (0,1) \to (1,1) \to (0,0)$ and $(0,0) \to (1,0) \to (1,1) \to (0,0)$. On the block of $(0,0)$, direct exact computation gives

$$E_{w^*}\big|_{(0,0)} = \big(\tfrac{1}{2}|a\rangle\langle a| + \tfrac{1}{4}|A\rangle\langle A|\big) \otimes \big(\tfrac{1}{2}|p\rangle\langle p| + \tfrac{1}{4}|A\rangle\langle A|\big). \tag{3.1}$$

This effect weighs A and not P, although both lie in the zero plane of the pole charge c. That is the whole mechanism: a native record event can tell apart two directions that every pole polynomial treats alike.

Consider the unit states

$$s_A = |00\rangle \otimes \tfrac{1}{\sqrt{2}}(P + A) \otimes p, \qquad s_B = |00\rangle \otimes \tfrac{1}{\sqrt{2}}(A - P) \otimes p.$$

Both sit at the occupation $(0,0)$ with the first cell on the zero plane of c and the second on its $+1$ line: they lie in the same sector projection $\Pi\big((0,0),0,+\big)$, so every combination of sector projections gives them the same weight.

Lemma 3.9 (A commutation test)**.** *Let $\phi_A = |00\rangle \otimes A \otimes p$ and $\phi_P = |00\rangle \otimes P \otimes p$, so $s_A = (\phi_A + \phi_P)/\sqrt{2}$ and $s_B = (\phi_A - \phi_P)/\sqrt{2}$. If a symmetric matrix K commutes with E_{w^*}, then $\langle s_A, Ks_A\rangle = \langle s_B, Ks_B\rangle$.*

Proof. By (3.1), $E_{w^*}\phi_A = \frac{1}{8}\phi_A$ and $E_{w^*}\phi_P = 0$. Then $E_{w^*}K\phi_P = KE_{w^*}\phi_P = 0$, so $K\phi_P$ lies in the kernel of the symmetric matrix E_{w^*}, which is orthogonal to its eigenvector ϕ_A of eigenvalue $\frac{1}{8}$. Hence $\langle s_A, Ks_A\rangle - \langle s_B, Ks_B\rangle = 2\langle \phi_A, K\phi_P\rangle = 0$. □

Theorem 3.10 (Outer verdicts are not central)**.** *Let w^*, $w^{*\prime}$, s_A and s_B be as above.*

(a) For every cut C with $1 \in C$ and $3 \notin C$ and every $t \geq 2$, E_{w^*} does not commute with $G_t(C)$. For every cut with $3 \in C$ and $1 \notin C$ the same holds with $w^{*\prime}$.
(b) For $C = \{1,3\}$ and $C = \{1,2,3\}$, E_{w^*} commutes neither with $G_2(C)$ nor with the eight-tick effect $G_8(C) = W_0(C)$.
(c) s_B fails the first window of $\{1\}$ with probability one, while s_A passes it with positive probability.

Proof. By Lemma 3.9 it suffices to show that $G_t(C)$ gives s_A and s_B different weights.

(a) and (c). Let $1 \in C$, $3 \notin C$. At a balanced input no posting writes account 2, so a tick of C there must flip the first cell. Start from s_B. The first cell must flip: $\mathrm{flip}(A - P) = a$. The second cell sits on p, which stay annihilates, so it flips too, to P, and the occupation becomes $(1,1)$. There the first cell must flip again, and flip $a = 0$. So s_B has weight zero on $G_t(C)$ for every $t \geq 2$. Start from s_A instead: $\mathrm{flip}(P + A) = p$, which flips surely at the next posting. One path

keeps a positive amplitude at every tick: $(0,0) \to (1,1) \to (0,1)$, followed by the self-loop at $(0,1)$, whose label $(-1,-1,1)$ writes account 1. On that loop the first cell lies on the $+1$ line of its dressed pole charge and is fixed, while the second turns by 45 degrees inside the zero plane with amplitude $\frac{1}{\sqrt{2}}$, so the amplitude never vanishes. Hence s_A has positive weight on every $G_t(C)$, and on $W_0(\{1\}) = G_8(\{1\})$ in particular. The mirror case is the same with the two cells exchanged: E_{w^*} on the block of $(0,0)$ is (3.1) with its two factors exchanged (direct exact computation), and the states are $|00\rangle \otimes p \otimes \frac{1}{\sqrt{2}}(P + A)$ and $|00\rangle \otimes p \otimes \frac{1}{\sqrt{2}}(A - P)$.

(b) Direct exact computation: on $G_2(C)$ the states s_A and s_B have weights 1 and $\frac{3}{4}$, for both cuts. On $G_8(C)$ their weights differ by $\frac{957}{16384}$ for $\{1,3\}$ and by $\frac{1023}{16384}$ for $\{1,2,3\}$. □

The verdict reads the direction inside the zero plane of the first cell, because a flip turns that direction into the sign of the next pole, and the sign decides whether the cell can flip again.

The field-current account is the only one that avoids this on the first window.

Theorem 3.11 (The first window selects the loop edge)**.** *For every set C of accounts, $W_0(C)$ is central in $\mathscr{A}$ if and only if C contains neither account* 1 *nor account* 3*. A cut can pass the window on some record written by an occupation path, on the native-step clock, if and only if it holds an observed account. Consequently a cut both can pass and has a central first-window verdict if and only if*

$$2 \in C, \qquad 1 \notin C, \qquad 3 \notin C.$$

The same holds with passability on the one-post clock of Section 3.8.

Proof. If C holds no outer account, then $W_0(C)$ is W_0 or 0 by Proposition 3.2(d), and both are central. If C holds exactly one outer account, Theorem 3.10(a) applies at $t = 8$; if it holds both, $W_0(C) = W_0(C \cap \{1,2,3\})$ and part (b) applies. Passability: the persistent mode's record $(1,1,-1)^n$ ticks on every observed account, and a cut with no observed account never ticks. For the one-post clock, Theorem 3.20 gives the passable cuts, and every passable cut with no outer account holds account 2. □

This is a selection only under a criterion we supply: that a verdict meant to mark presence at every tick should be a sector fact, a central read, rather than a record event that depends on coherence. The theorem says exactly what that criterion selects. Whether nature imposes the criterion is not decided here.

3.5 Later windows: centrality from genesis belongs to the first three

The loop edge's advantage is a property of the first windows, not of the edge. A window that starts after three or more free postings has a verdict that depends on coherence, on the field-current cut too.

Theorem 3.12 (Which field-current windows are central)**.** *W_k is central in $\mathscr{A}$ if and only if $k \le 2$. For every $k \ge 3$, E_{w^*} does not commute with W_k. The separating vectors are*

$$\phi_{PP} = |00\rangle \otimes P \otimes P, \qquad \phi_{AA} = |00\rangle \otimes A \otimes A,$$

both in the sector of $(0,0)$ *where both pole charges vanish, and for every* $k \geq 1$

$$\langle \phi_{AA}, W_k \phi_{PP} \rangle = 2^{-(k+7)}\, q_{k-1}, \tag{3.2}$$

where q_n *is the integer sequence of Table A.4, with* $q_0 = q_1 = 0$ *and* q_n *odd for every* $n \geq 2$.

Proof. By (3.1), $E_{w^*}\phi_{AA} = \frac{1}{16}\phi_{AA}$ and $E_{w^*}\phi_{PP} = 0$, so the argument of Lemma 3.9 shows that every symmetric matrix commuting with E_{w^*} has $\langle \phi_{AA}, K\phi_{PP} \rangle = 0$. Equation (3.2) with q_{k-1} odd gives the failure for $k \geq 3$. For $k \leq 2$, W_k is a combination of the pole tables described below, with no coherence weight, and every such combination is central by the argument of Theorem 3.7. Equation (3.2) and the parity of q_n come from the mode recursion below. □

The mechanism is a one-cell rule. At occupation x, a clock cell carries five reads: the pole polynomials c_x^2, c_x and $\Pi_0^x = I - c_x^2$, and two coherence reads inside the zero plane of c_x,

$$Z_x = C_x\big(PA^\top + AP^\top\big)C_x, \qquad X_x = C_x\big(AA^\top - PP^\top\big)C_x.$$

Pulling one of these reads back through one cell map gives a combination of the five reads at the input, with coefficients that depend only on whether the cell stays or flips and whether the output is balanced (Table 9.1 of Chapter 9, with $c_x^2 = \Pi_+^x + \Pi_-^x$ and $c_x = \Pi_+^x - \Pi_-^x$; direct exact computation). On a mixed output, pole polynomials pull back to pole polynomials, and Z, X rotate into each other. On a balanced output the pole sign does not: c_y pulls back to $-\frac{1}{2}Z_x$ through stay and to $+\frac{1}{2}Z_x$ through flip. The sign of the output pole is a coherence inside the zero plane of the input. This is the same effect that defeats the outer accounts, now reached by free postings before the window.

The verdict's tables close on eight modes. For $m = 1, \ldots, 8$ let $M_m = \sum_x |x\rangle\langle x| \otimes B_m(x)$ with the blocks listed in Table A.3: the balanced and mixed weights $c^2 \otimes c^2$ and $c \otimes c$, the both-zero read $\Pi_0 \otimes \Pi_0$, the one-zero read $c^2 \otimes \Pi_0 + \Pi_0 \otimes c^2$, and the coherence reads $Z \otimes Z$ and $X \otimes X$ (with sign reversed on mixed occupations).

Proposition 3.13 (Mode form of the later windows)**.** *For every* $k \geq 1$, $W_k = \sum_{m=1}^{8} \theta_m(k) M_m$, *where*

$$\theta(1) = \big(0, 0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{256}, \tfrac{1}{256}, 0, 0\big), \qquad \theta(k+1) = S\theta(k),$$

with the 8×8 *matrix* S *printed in Section A.2. In integer form, let* $v(n) = (z_n, g_n, d_n, u_n, q_n, e_n)$ *with* $v(0) = (1, 0, 0, 0, 0, 0)$ *and*

$$v(n+1) = \big(z_n + g_n + d_n,\ z_n + e_n,\ u_n,\ 2z_n - 2e_n,\ e_n + g_n - d_n,\ q_n\big)$$

(Table A.4). Then, with $\epsilon = 2^{-(n+8)}$,

$$\theta(n+1) = \big(\epsilon(g_n + d_n),\ \epsilon(g_n - d_n),\ \tfrac{1}{2} + \tfrac{\epsilon}{2}u_n,\ \tfrac{1}{2} - \tfrac{\epsilon}{2}u_n,\ \epsilon z_n,\ \tfrac{1}{256},\ \epsilon q_n,\ \epsilon e_n\big).$$

Three quantities are conserved: $\theta_6 = \frac{1}{256}$, $\theta_3 + \theta_4 = 1$, *and* $\frac{1}{2}\theta_1 + \frac{1}{4}\theta_3 - \frac{1}{4}\theta_4 + \theta_5 = \frac{1}{256}$.

Proof. Every W_k is block diagonal in the occupation, and on each block it is a combination of products $R \otimes R'$ of the five one-cell reads; Φ acts on such combinations cell by cell through Table 9.1. Applying the table to $W_0 = G_8 = Q_{\mathrm{al}} + \frac{1}{128}Q_{\mathrm{ne}}$ (Theorem 3.4) gives $W_1 = \Phi(W_0) =$

$\sum_m \theta_m(1)M_m$, a direct exact computation. Applying it to each mode gives $\Phi\big(\sum_m \theta_m M_m\big) = \sum_m (S\theta)_m M_m$ (Section A.2), so $W_{k+1} = \Phi(W_k)$ (Proposition 3.2(c)) and induction on k give the mode form. For the integer form, at $n = 0$ the formula gives $\epsilon = \frac{1}{256}$ and returns $\theta(1)$. If $\theta(n+1)$ has the stated form, multiplying by S halves ϵ and replaces $\nu(n)$ by $\nu(n+1)$ row by row: for instance the first row gives $\frac{1}{2}(\theta_3 - \theta_4 + \theta_5 + \theta_8) = \frac{\epsilon}{2}(u_n + z_n + e_n) = \frac{\epsilon}{2}(g_{n+1} + d_{n+1})$, and the last gives $\frac{1}{2}\theta_7 = \frac{\epsilon}{2}q_n = \frac{\epsilon}{2}e_{n+1}$. The conserved quantities are left eigenvectors of S for the eigenvalue 1: the sixth row of S is the sixth unit row, the third and fourth rows sum to $(0,0,1,1,0,0,0,0)$, and $\eta = (\frac{1}{2}, 0, \frac{1}{4}, -\frac{1}{4}, 1, 0, 0, 0)$ satisfies $\eta S = \eta$. Their values at $\theta(1)$ are $\frac{1}{256}$, 1 and $\frac{1}{256}$. □

The coherence weight of $Z \otimes Z$ is $\theta_7(k) = 2^{-(k+7)}q_{k-1}$, and only this mode couples ϕ_{PP} to ϕ_{AA}, which gives (3.2). It first appears at $k = 3$: the balanced pole-sign weight θ_2 is created at the second pull-back, with value $\frac{1}{512}$, and the third pull-back turns it into $Z \otimes Z$ coherence through the balanced-output rule. From then on the recursion keeps q_n odd, so the coherence never cancels. It does shrink, and Section 3.6 computes where the windows go.

Referred back to genesis, the field-current verdict is central exactly on the windows $k \le 2$; read on the present cell (Proposition 3.16) the first-window selection applies at every window.

3.6 The long-run verdict

Far from genesis the field-current windows converge to one central read. Every state passes it with probability at least $\frac{1}{448}$, the states that pass it surely are exactly the sure subjects, and along the record its value is a fair game.

Let $Q_{\mathrm{oz}} = \sum_x |x\rangle\langle x| \otimes \big(\Pi_0^x \otimes (I - \Pi_0^x) + (I - \Pi_0^x) \otimes \Pi_0^x\big)$ project onto the states in which exactly one cell has pole charge zero, on any occupation.

Theorem 3.14 (The long-run window). *Let* $\theta^\infty = \big(\frac{1}{448}, 0, \frac{1}{2} + \frac{1}{896}, \frac{1}{2} - \frac{1}{896}, \frac{1}{448}, \frac{1}{256}, 0, 0\big)$.

(a) Every mode weight converges to θ^∞ *at an explicit rate:*

$$\big|\theta_m(n+1) - \theta_m^\infty\big| \le \tfrac{3}{448}\, 4^{-\lfloor n/8 \rfloor} \qquad (m = 1, \dots, 8,\ n \ge 0).$$

(b) So $W_k \to W_\infty = \sum_m \theta_m^\infty M_m$, *and*

$$W_\infty = \tfrac{1}{448}\, I + \tfrac{447}{448}\, Q_{\mathrm{al}} + \tfrac{3}{1792}\, Q_{\mathrm{oz}}.$$

W_∞ is central in $\mathscr{A}$, it is fixed by one free posting, $\Phi(W_\infty) = W_\infty$, and so are Q_{al} and Q_{oz} separately. $W_\infty \ne W_0$.

(c) For every genesis state ν, $\mathbb{P}_\nu$(window k passes) $\to \mathrm{tr}(\rho_\nu W_\infty)$, and $\frac{1}{448} \le \mathrm{tr}(\rho_\nu W_\infty) \le 1$. The lower bound is attained.

(d) $\mathrm{tr}(\rho_\nu W_\infty) = 1$ if and only if $w_{\mathrm{al}}(\rho_\nu) = 1$, if and only if ν passes every field-current window with probability one.

Proof outline. (a) The three conserved quantities of Proposition 3.13 are left fixed vectors of S, and θ^∞ is the fixed point of S with their values and no coherence weight. Put $\delta(n) = 7\nu(n) - 2^{n+1}(2,1,1,2,0,0)$. The conserved row gives $\delta_u = -4\delta_z - 2\delta_g - 2\delta_d$, and the other

five coordinates obey $\delta(n+1) = \Delta\delta(n)$ with an integer 5×5 matrix Δ whose eighth power has absolute row sums at most 64. Since $\max_j |\delta_j(r)| \le 3 \cdot 2^r$ for $r < 8$, induction gives $|\delta_j(n)| \le 3 \cdot 2^n 4^{-\lfloor n/8 \rfloor}$. Finally $\theta_m(n+1) - \theta_m^\infty = 2^{-(n+8)}\beta_m(\delta(n))/7$ with linear forms β_m of absolute coefficient sum at most 4, which gives the bound $\frac{4\cdot 3}{7\cdot 256} 4^{-\lfloor n/8 \rfloor} = \frac{3}{448} 4^{-\lfloor n/8 \rfloor}$.

(b) Sum the modes with the weights θ^∞; the three-term form is a direct exact computation on the 36 sector projections (Table A.5). $\mathscr{A}$ is finite-dimensional, hence closed, so $W_\infty \in \mathscr{A}$, and it is central because $\theta_7^\infty = \theta_8^\infty = 0$. $\Phi(W_\infty) = W_\infty$ follows from $W_{k+1} = \Phi(W_k)$ by continuity of Φ. (c) Continuity of the trace, and the sector values of Table A.5, all at least $\frac{1}{448}$; the ready vector below attains the floor. (d) The only sector value equal to 1 is the aligned one, and every other value is at most $\frac{1}{256}$, so $\mathrm{tr}(\rho_v W_\infty) = 1$ exactly when $w_{\mathrm{al}}(\rho_v) = 1$. A state with $w_{\mathrm{al}} = 1$ passes every window surely (Chapter 7); conversely, a state that passes the first window surely has $w_{\mathrm{al}} + \frac{1}{128} w_{\mathrm{ne}} = 1$ by Theorem 3.4, hence $w_{\mathrm{al}} = 1$. □

The characteristic polynomial of S is $\frac{1}{8}(z-1)^3(8z^5 + 4z^4 - 2z^2 - 2z - 1)$; the quintic's roots lie in the unit disc, and the root of largest modulus is real, 0.7792 … (Chapters 7 and 11), so $4^{-1/8}$ per posting is a safe rate, not the sharp one.

Remark 3.15 (Where the floor comes from). The floor is the long-run neutral leak seen through the first-window formula. Since $W_k = \Phi^k(G_8)$ and $\mathbb{P}_v(h)\rho_{v,h} = V_h \rho_v V_h^*$, the window k passes with probability $\sum_{h \in \mathscr{J}^k} \mathbb{P}_v(h)\big[w_{\mathrm{al}}(\rho_{v,h}) + \frac{1}{128} w_{\mathrm{ne}}(\rho_{v,h})\big]$. The first term is constant in k, because $\Phi(Q_{\mathrm{al}}) = Q_{\mathrm{al}}$. The record-averaged neutral weight converges to $\frac{1}{2} w_{\mathrm{oz}} + \frac{2}{7}(1 - w_{\mathrm{oz}} - w_{\mathrm{al}})$, with $w_{\mathrm{oz}} = \mathrm{tr}(\rho_v Q_{\mathrm{oz}})$ (Chapter 7). Together,

$$W_\infty = Q_{\mathrm{al}} + \tfrac{1}{128}\Big(\tfrac{1}{2} Q_{\mathrm{oz}} + \tfrac{2}{7}\big(I - Q_{\mathrm{oz}} - Q_{\mathrm{al}}\big)\Big),$$

which is the three-term form of Theorem 3.14(b). The ready vector has $w_{\mathrm{al}} = w_{\mathrm{oz}} = 0$; its long-run neutral weight $\frac{2}{7}$, read at $\frac{1}{128}$, is the floor $\frac{1}{448}$.

The verdict can also be read after a written record, on the present cell $\rho_{v,h}$ of Chapter 2. Reading it as the state the record leaves is the identification named there; the probabilities below do not depend on it.

Proposition 3.16 (The present verdict is a ratio of record probabilities)**.** *Let v be a genesis state and h a record of length k with $\mathbb{P}_v(h) > 0$. For every cut C,*

$$\mathrm{tr}\big(\rho_{v,h} W_0(C)\big) = \frac{\mathbb{P}_v\big(\textit{the record begins with } h \textit{ and window } k \textit{ of } C \textit{ passes}\big)}{\mathbb{P}_v(h)},$$

and this is also the conditional probability of the same event given the prefix h under the law on infinite records. For the field-current cut it equals $w_{\mathrm{al}}(\rho_{v,h}) + \frac{1}{128} w_{\mathrm{ne}}(\rho_{v,h})$, a fixed linear form in the 36 sector weights $\mathrm{tr}\big(\rho_{v,h}\Pi(x,s,t)\big)$ of the present cell: coefficient 1 on the two aligned sectors of each mixed occupation, $\frac{1}{128}$ on the mixed sectors with a zero pole, and 0 elsewhere.

Proof. By the factorization of the record law through the present cell, $\mathbb{P}_v(h \cdot w) = \mathbb{P}_v(h)\,\mathrm{tr}(\rho_{v,h} E_w)$; sum over the eight-letter continuations w that pass. The infinite law has these finite laws as its cylinder masses. The sector form is Theorem 3.4(a) applied to $\rho_{v,h}$. □

Define the *long-run verdict* of a record h of positive probability by $\lambda_v(h) = \operatorname{tr}(\rho_{v,h} W_\infty)$, the long-run pass probability of the present cell.

Theorem 3.17 (The long-run verdict is a fair game)**.** *For every genesis state v:*

(a) $\sum_{j \in \mathcal{J}} \mathbb{P}_v(h \cdot j)\, \lambda_v(h \cdot j) = \mathbb{P}_v(h)\, \lambda_v(h)$ *for every record h, with $\mathbb{P}_v(h)\lambda_v(h)$ read as $\operatorname{tr}(V_h \rho_v V_h^* W_\infty)$ when $\mathbb{P}_v(h) = 0$;*

(b) $\sum_{h \in \mathcal{J}^n} \mathbb{P}_v(h)\, \lambda_v(h) = \operatorname{tr}(\rho_v W_\infty)$ *for every n;*

(c) if $\lambda_v(h) = 1$, then $\lambda_v(h \cdot w) = 1$ for every continuation w with $\mathbb{P}_v(h \cdot w) > 0$; the same holds with the value $\frac{1}{448}$;

(d) λ_v evaluated on the prefixes of an infinite record converges almost surely under the law on infinite records.

The aligned weight and the one-zero weight of the present cell are martingales in the same sense.

Proof. $\mathbb{P}_v(h)\rho_{v,h} = V_h \rho_v V_h^*$ for every record, so the left side of (a) is $\operatorname{tr}(V_h \rho_v V_h^* \Phi(W_\infty))$, and $\Phi(W_\infty) = W_\infty$. Iterating gives (b). For (c), a weighted average of values at most 1 equals 1 only if every term of positive weight does; the floor is the same argument with values at least $\frac{1}{448}$. (d) is the martingale convergence theorem for the bounded martingale of (a). The last sentence uses $\Phi(Q_{\mathrm{al}}) = Q_{\mathrm{al}}$ and $\Phi(Q_{\mathrm{oz}}) = Q_{\mathrm{oz}}$. □

The value 1 and the floor $\frac{1}{448}$ both absorb, and the two worked states below sit at these two values after one posting. Chapter 8 finds the almost sure limits of the aligned weight, in $\{0, 1\}$, and of the long-run verdict.

3.7 Two states across all windows

Table 3.1 lists the exact pass probabilities of the chance state and the ready vector.

window k	0	1	2	3	4	5	6	7	$k \to \infty$
chance state χ	$\frac{9}{25}$	$\frac{9}{25}$	$\frac{29}{80}$	$\frac{289}{800}$	$\frac{289}{800}$	$\frac{289}{800}$	$\frac{1157}{3200}$	$\frac{2313}{6400}$	$\frac{9}{25} + \frac{1}{700}$
ready vector r	0	0	0	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{512}$	$\frac{1}{512}$	$\frac{5}{2048}$	$\frac{1}{448}$

Table 3.1. Probabilities that the field-current window starting after posting k passes, for the chance state χ of Example 3.5 and the ready vector $r = |00\rangle \otimes p \otimes a$. For $k \geq 1$ they are $\frac{9}{25} + \frac{16}{25} 2^{-(k+7)} u_{k-1}$ and $2^{-(k+6)} d_{k-1}$ with the integers of Table A.4.

Proposition 3.18 (The chance state)**.** *Let χ be the chance state of Example 3.5.*

(a) χ passes the windows $k = 0$ and $k = 1$ with probability exactly $\frac{9}{25}$ and the window $k = 2$ with probability $\frac{9}{25} + \frac{1}{400} = \frac{29}{80}$.

(b) For every $k \geq 2$ it passes with probability at least $\frac{9}{25} + \frac{1}{800}$, with equality for $k = 3, 4, 5$, so $\frac{1}{800}$ is the exact infimum of the excess over $\frac{9}{25}$ on these windows.

(c) Its pass probabilities converge to $\frac{9}{25} + \frac{1}{700}$. In particular the long-run pass probability is not the persistent weight $\frac{9}{25}$.

(d) Its first label settles its long-run verdict: with probability $\frac{9}{25}$ *it is* $(1,1,-1)$ *and* $\lambda_\chi = 1$ *on every later record of positive probability; with probability* $\frac{16}{25}$ *it is* $(0,1,-1)$ *and* $\lambda_\chi = \frac{1}{448}$ *on every later record of positive probability. No other first label has positive probability.*
(e) For every $k \geq 1$, $\mathbb{P}_\chi(\text{window } k \text{ passes}) = \frac{9}{25} + \frac{16}{25}\,\mathbb{P}_r(\text{window } k+1 \text{ passes})$.

Proof. The window $k = 0$ is Example 3.5. For $k \geq 1$, (a), (b) and (e) follow from the closed forms of Table 3.1: $u_0 = 0$, $u_1 = 2$, and $u_n \geq 2^{n-1}$ for $n \geq 1$ with equality at $n = 2, 3, 4$ (for $n < 24$ this is a finite check of the recursion; for $n \geq 24$, $7u_n = 2^{n+2} + \delta_u(n)$ and $|\delta_u(n)| \leq 8 \cdot 3 \cdot 2^n 4^{-3} < 2^{n-1}$ by the proof of Theorem 3.14(a), and equality fails); part (e) is the recursion $d_{n+1} = u_n$. (c) is Theorem 3.14(c): ψ_+ lies in an aligned sector, of value 1, and $|10\rangle \otimes A \otimes P$ in an opposed sector, of value $\frac{1}{448}$, so $\mathrm{tr}(\rho_\chi W_\infty) = \frac{9}{25} \cdot 1 + \frac{16}{25} \cdot \frac{1}{448}$. (d) After $(1,1,-1)$ the present cell is ψ_+, of value 1. After $(0,1,-1)$ it is $|00\rangle \otimes A \otimes P$, which lies in the balanced sector with both poles zero, of value $\frac{1}{448}$. Absorption (Theorem 3.17(c)) carries both values forward. □

So the chance state is decided by its first posting, although its later windows never show it. Its decayed branch, of weight $\frac{16}{25}$, sits at the long-run floor and keeps re-entering passing windows by chance.

Proposition 3.19 (The ready vector)**.** *For* $r = |00\rangle \otimes p \otimes a$*, the window starting after posting* k *passes with positive probability if and only if* $k \geq 3$*. It passes the window* $k = 3$ *with probability* $\frac{1}{256}$*, every later window with probability at least* $\frac{1}{512}$*, and in the long run with probability* $\frac{1}{448}$*. Every record of positive probability leaves it at* $\lambda_r = \frac{1}{448}$ *with aligned weight zero, so no record makes it a sure subject. The reverse ready vector* $|00\rangle \otimes a \otimes p$ *has the same pass probability on every window.*

Proof. For $k = 0$, r sits on a balanced occupation, where $W_0 = G_8$ vanishes. For $k \geq 1$ the pass probability is $2^{-(k+6)} d_{k-1}$ (Table 3.1), and the integers have $d_0 = d_1 = 0$, $d_2 = 2$ and $d_n = u_{n-1} \geq 2^{n-2}$ for $n \geq 2$ (proof of Proposition 3.18), so $2^{-(k+6)} d_{k-1} \geq 2^{-9}$ for every $k \geq 3$. The limit is Theorem 3.14(c): r lies in the balanced sector with poles of opposite sign, of value $\frac{1}{448}$. Since this is the floor, Theorem 3.17(c) keeps every record there, and the value $\frac{1}{448}$ forces aligned weight zero by Theorem 3.14(b). □

So the ready vector is not always silent on the loop edge. Its silence on the first three windows is forced by its first moves (Example 2.18): its first and third postings are written from balanced occupations. Every pass it makes afterwards is a chance pass.

3.8 What the verdict is a property of

The window verdict is defined on records. Three results say how much of it survives when we ask about something coarser: a different clock, a pointer label, or the boundary a history lies in.

The clock. On the *one-post clock* the record is serialized into its physical posts, in the account order $1, 2, 3$ within each posting, and each post is one tick, passing C when it lands on C. Call a cut *post-passable* when some record written by an occupation path from genesis

has at least eight posts and its last eight posts land on C, and *step-passable* when some such record of length at least eight passes its last window of C on the native-step clock.

Theorem 3.20 (The clock decides which cuts can pass)**.** *Consider the records written by occupation paths from genesis.*

(a) On such a record, no three consecutive physical posts land on the same outer account. Two can: the path $(1,0) \to (0,0) \to (0,1)$ *posts on the accounts* $2, 3, 3$.
(b) C *is post-passable if and only if* $2 \in C$*, or* $1 \in C$ *and* $3 \in C$*. It is step-passable if and only if it holds an observed account.*
(c) The two clocks disagree exactly on the cuts that hold one outer account and not account 2*; every post-passable cut is step-passable.*
(d) The classification in (b) is the same for all six orders of the accounts within a posting; under every order a run on an outer account has length at most three, and the order $3, 1, 2$ *attains three.*
(e) The record $(1,0,0)^8$*, which no occupation path writes, posts eight times on account* 1.

Proof outline. The postings that write only account 1 are $(0,0) \to (1,0)$ and $(1,1) \to (0,1)$; those that write only account 3 are $(0,0) \to (0,1)$ and $(1,1) \to (1,0)$. Each ends on a mixed occupation, and the next posting starts from it and writes account 2. The only silent postings are the balanced self-loops, which do not change the occupation. So a run on one outer account consists of the tail of one posting, at most one single-account posting, and the head of the next (silent postings in between post nothing). This gives runs of at most three under every order. Under the order $1, 2, 3$ a posting whose last post is on account 1 writes account 1 only, and a posting from a mixed occupation posts on account 2 before account 3, so a run has at most two posts. Induction along the record with a bound on the current run, checked over the sixteen transitions, gives (a) and its analogue for every order in (d). (b) follows from the run bound and two witnesses: the oscillation $(1,0) \leftrightarrow (0,1)$ posts only on account 2, and the balanced oscillation $(0,0) \leftrightarrow (1,1)$ posts $1, 3, 1, 3, \ldots$. Step-passability uses the persistent record $(1,1,-1)^n$. (e) The label $(1,0,0)$ is written only by $(0,0) \to (1,0)$, which cannot follow itself. □

The field-current cut passes on both. Record (e) belongs to an orphan coordinate in the sense of Chapter 4, which no genesis state populates; the constraint that a record be written by an occupation path does real work at the level of verdicts.

The pointer. Chapter 6 studies a pointer with two sectors, labelled 0 and 1. When the genesis state lies in one pointer sector, the sector of every later slice of the history follows a simple rule (Lemma 6.9 and Proposition 6.10): the label flips at every posting that is silent on the loop edge and is kept at every other. Starting from a label b, the label after a record h is

$$b \cdot h = b + \#\{t \,:\, j_2^{(t)} = 0\} \pmod 2.$$

Proposition 3.21 (The pointer reads silence parity)**.** *For every record* h *of length eight and every* b*:*

(a) $b \cdot h \equiv b + \mathrm{acc}(h) \pmod 2$*: the label reads the parity of the loop-flux change over the window;*
(b) if the field-current window passes on h*, the label returns,* $b \cdot h = b$*;*

(c) if the label is flipped, $b \cdot h \neq b$, the window fails;
(d) no rule on the initial and final labels decides whether the window passes.

Proof. Each $j_2^{(t)}$ is 0 or ± 1, so the number of awake postings is congruent to $\mathrm{acc}(h) = \sum_t j_2^{(t)}$ modulo 2, and the number of silent ones is 8 minus that number. This gives (a). A passing window has no silent posting, which gives (b) and, by contraposition, (c). For (d), the record $(0,0,0)^8$ of the balanced self-loop at $(0,0)$ and the record $(1,1,-1)^8$ of the self-loop at $(1,0)$ both return the label; only the second passes. □

A flipped pointer certifies a failed window; a returned pointer certifies nothing.

The branch. Chapter 4 shows that the history space is built from indivisible atoms, and that the atoms a history occupies are fixed at its beginning. The window verdict is not a property of the atom. Give a coordinate of stage $k \geq 8$ the verdict of the window formed by the last eight labels of its record.

Theorem 3.22 (The verdict belongs to the branch)**.** *For every coordinate a there are coordinates a' and a'' in the atom of a such that the last window of a' passes every cut holding an observed account and the last window of a'' passes no cut. Consequently, for a cut C, the verdict of C is constant on every atom if and only if C holds no observed account, and then it never passes. No function of the atom agrees with the field-current verdict.*

Proof outline. Every native step joins a coordinate to each of its four children with a nonzero overlap, because the diagonal of every occupation block is nonzero, so a coordinate and all its descendants lie in one atom. From any coordinate, the descendant that first moves to $(1,0)$ and then stays there eight times ends with eight labels $(1,1,-1)$, which pass every cut holding an observed account; the descendant that first moves to $(1,1)$ and then stays ends with eight zero labels, which pass no cut. □

Selecting a boundary therefore does not decide the verdict. Which atoms a history occupies, and why every boundary it cannot leave is a join of atoms, is the subject of the next two chapters.

Chapter 4

Boundaries are joins of atoms

The complete history of the native step divides into pieces that no native event connects. This chapter finds those pieces and shows that the division is exact. In the complete history space of Chapter 2 we ask which closed subspaces the coordinate events cannot leave. Each such subspace, a *boundary*, turns out to be a closed join of irreducible ones, the *atoms*. Reading a record backwards names its atom: every atom consists of the recorded continuations of one root event, and the atoms that hold histories from genesis are the *lineages*, one for each original occupation and original loop.

The algebra generated by the events treats the atoms as classical alternatives, and the record law is blinder still. Nothing here uses an environment or a limit of large systems: everything follows from the pattern of nonzero posting amplitudes, which the step blocks of Chapter 2 fix.

4.1 One space for every posted history

A history is a genesis state together with every record the posting writes from it. Chapter 2 placed all of them in one Hilbert space: the stages $\mathscr{H}_k = \ell^2(\mathbb{Z} \times \mathscr{J}^k) \otimes \mathscr{H}$, the isometric posting $T : \mathscr{H}_k \to \mathscr{H}_{k+1}$, $(T\xi)(n, h \cdot j) = V_j \xi(n - j_2, h)$, and the complete history space $\mathscr{H}_\infty$, with isometries $\iota_k : \mathscr{H}_k \to \mathscr{H}_\infty$, $\iota_{k+1} T = \iota_k$, whose ranges have dense union (Proposition 2.15). Since $\iota_k T^k v = \iota_0 v$, a genesis state v and its posted history after any number of steps are one vector; identifying the stages is bookkeeping, not a second time evolution. Boundaries and atom weights do not depend on normalization, so in this chapter and the next a *genesis state* is any $v \in \mathscr{H}_0$; where a probability is meant (the record law, the genesis distribution) we say that v is a unit vector.

4.1.1 Coordinates and events

Label the basis of the native cell by $z = (x, i)$, with $x \in \mathcal{O}$ an occupation and $i = (i_1, i_2) \in (\mathbb{Z}/4)^2$ the pair of clock indices, so that $|z\rangle = |x\rangle \otimes e_{i_1} \otimes e_{i_2}$. A *coordinate* is $a = (k, n, z, h)$ with $k \geq 0$, $n \in \mathbb{Z}$ and $h \in \mathscr{J}^k$. Its vector and its *native event* are

$$\chi_a = \iota_k(\delta_n \otimes \delta_h \otimes |z\rangle), \qquad F_a = |\chi_a\rangle\langle\chi_a|.$$

F_a is a rank-one projection: it asks whether the history is at this coordinate. Let $\mathscr{E}$ be the unital algebra the events generate and $\overline{\mathscr{E}}$ its closure in operator norm, the *norm-closed event algebra*. Write $a \sim b$, *connected*, for the equivalence relation generated by $\langle \chi_a, \chi_b \rangle \neq 0$. A vector ξ *responds* to a when $\langle \chi_a, \xi \rangle \neq 0$. The *original loop* of a is $\ell(a) = n - \mathrm{acc}(h)$, the flux the history had before its record was written.

Every boundary, atom and superselection result of this chapter is relative to $\overline{\mathscr{E}}$. That this

algebra is the set of operations available inside the ledger is an identification, not a theorem, and we name it once here. Section 5.7 and Chapter 6 exhibit reads from recognition dynamics that lie outside $\overline{\mathscr{E}}$ and cross the walls found below.

Lemma 4.1 (Coordinates). *(i) Coordinates of one stage are orthonormal. For $k' \geq k$, $\langle \chi_{(k',n',z',h')}, \chi_{(k,n,z,h)} \rangle$ is the coefficient of $T^{k'-k}(\delta_n \otimes \delta_h \otimes |z\rangle)$ at (n', h', z'). It vanishes unless h is an initial segment of h'. For one step, $h' = h \cdot j$, $z = (x, i)$ and $z' = (y, i')$, it equals $[n' = n + j_2]\,[j = j(y, x)]\,(M_{yx})_{i'i}$.*

(ii) If $\xi \in \mathscr{H}_\infty$ responds to no coordinate, then $\xi = 0$.

(iii) The original loop is constant on connected classes, and coordinates with different original loops are orthogonal.

Proof. (i) $\iota_k = \iota_{k'} T^{k'-k}$ and $\iota_{k'}$ is isometric, so the inner product is computed in $\mathscr{H}_{k'}$. Posting keeps the old record and appends, so $T^{k'-k}$ of a vector with record h lives on records extending h. The one-step formula is the definition of T together with $\langle y, i' | V_j | x, i \rangle = [j(y, x) = j](M_{yx})_{i'i}$. (ii) The stage-$k$ coordinate vectors are the images of an orthonormal basis of $\mathscr{H}_k$, so $\iota_k^* \xi = 0$ for every k; then ξ is orthogonal to a dense set. (iii) $T^m(\delta_n \otimes \delta_h \otimes |z\rangle)$ lives on pairs $(n + \mathrm{acc}(w), h \cdot w)$, whose original loop is again $n - \mathrm{acc}(h)$. □

Which overlaps are nonzero is decided by one fact about the step blocks.

Lemma 4.2 (Support of the step blocks). *For all occupations x, y and clock indices $i, i' \in (\mathbb{Z}/4)^2$,*

$$(M_{yx})_{i'i} = \pm\tfrac{1}{4} \ \text{ if } i - i' \in \{0, 1\}^2 \text{ componentwise mod } 4, \qquad (M_{yx})_{i'i} = 0 \ \text{ otherwise.}$$

So every row and every column of every block has exactly four nonzero entries, the diagonal entries are nonzero, and every column has squared norm $\frac{1}{4}$.

Proof. $Le_i = e_{i-1}$ puts the entries of L at positions $(r, r+1)$. Hence $K(b', b) = \frac{1}{2} D\big(s(b) I - s(b') L\big)$ has entries $\pm\frac{1}{2}$ exactly at (r, r) and $(r, r+1)$, and right multiplication by $C_x \in \{I, D\}$ changes only signs. An entry of the Kronecker product $\sigma(x)(KC_x) \otimes (KC_x)$ is the product of one entry from each factor. □

So one posting carries the clock index i to i', with amplitude $\pm\frac{1}{4}$, exactly when each clock index stays or steps back by one, and a basis cell passes to each of the four output occupations with probability exactly $\frac{1}{4}$, the squared norm of a column.

The events generate an algebra with a simple description.

Lemma 4.3 (The event algebra is a path algebra). $\mathscr{E} = \mathbb{C}1 + \mathrm{span}\{|\chi_a\rangle\langle\chi_b| : a \sim b\}$. *In particular $\mathscr{E}$ is closed under adjoints and $\overline{\mathscr{E}}$ is a C^*-algebra.*

Proof. A product of two transitions is $|\chi_a\rangle\langle\chi_b| \cdot |\chi_c\rangle\langle\chi_d| = \langle\chi_b, \chi_c\rangle\, |\chi_a\rangle\langle\chi_d|$, which is zero unless b and c overlap, and then $a \sim d$. So the right side is an algebra; it contains 1 and every F_a. Conversely, along a chain of overlaps $a = a_0, a_1, \ldots, a_r = b$ the product $F_{a_0} F_{a_1} \cdots F_{a_r}$ equals $\prod_s \langle\chi_{a_s}, \chi_{a_{s+1}}\rangle\, |\chi_a\rangle\langle\chi_b|$ with a nonzero coefficient, so every transition along a path lies in $\mathscr{E}$. The adjoint of $|\chi_a\rangle\langle\chi_b|$ is $|\chi_b\rangle\langle\chi_a|$. □

4.2 Every boundary is a closed join of atoms

Definition 4.4 (Boundaries and atoms)**.** A *boundary* is a closed subspace $S \subseteq \mathscr{H}_\infty$ with $F_a S \subseteq S$ for every coordinate a; equivalently, S is closed and invariant under $\overline{\mathscr{E}}$. A boundary is *irreducible* if it is nonzero and contains no boundary other than 0 and itself. The *atom* of a coordinate a is

$$B_a = \overline{\operatorname{span}}\{\chi_b : b \sim a\}.$$

The *boundary generated* by ξ is $\mathscr{C}(\xi) = \overline{\overline{\mathscr{E}}\xi}$, the least boundary containing ξ. We write Q_S for the orthogonal projection onto a closed subspace S.

Invariance under the events passes to sums, products and norm limits, which is why the two forms of the definition agree. Since each F_a is self-adjoint, $S^\perp$ is a boundary whenever S is, and then Q_S commutes with every element of $\overline{\mathscr{E}}$.

Everything in this section rests on one observation. If S is a boundary and $\xi \in S$ responds to a, then

$$\chi_a = \langle \chi_a, \xi \rangle^{-1} F_a \xi \in S. \tag{4.1}$$

A boundary that contains a vector contains every coordinate that vector responds to.

Theorem 4.5 (Boundaries are joins of atoms)**.** *For the norm-closed event algebra $\overline{\mathscr{E}}$:*

(i) B_a is the least boundary containing χ_a. $\chi_b \in B_a$ if and only if $a \sim b$, and $B_a = B_b$ if and only if $a \sim b$. Two atoms are equal or orthogonal.

(ii) Every boundary S is the closed join of the atoms B_a with $\chi_a \in S$.

(iii) A boundary is irreducible if and only if it is an atom.

(iv) $\mathscr{C}(\xi)$ is the closed join of the atoms of the coordinates ξ responds to. For $\xi \neq 0$ it is an atom if and only if all those coordinates are connected. Every boundary is $\mathscr{C}(\xi)$ for some ξ; in particular a single vector generates $\mathscr{H}_\infty$.

(v) $\mathscr{H}_\infty$ is the orthogonal Hilbert sum of its atoms: $\|\xi\|^2 = \sum_B \|Q_B \xi\|^2$, and $Q_B \xi \neq 0$ exactly when ξ responds to some coordinate whose vector lies in B.

(vi) The exterior $B_a^\perp$ of an atom is a boundary and is not irreducible. $\mathscr{H}_\infty$ is a boundary and is not irreducible; no atom equals $\mathscr{H}_\infty$.

Proof. (i) For a coordinate c and $b \sim a$, $F_c \chi_b$ is 0 or a nonzero multiple of χ_c with $c \sim b$, so F_c maps the span of $\{\chi_b : b \sim a\}$ into itself; by continuity B_a is a boundary. Any boundary containing χ_a contains, by (4.1), every χ_b along a chain of overlaps, so it contains B_a. If $a \nsim b$, then $\langle \chi_c, \chi_b \rangle = 0$ for every $c \sim a$, so $\chi_b \perp B_a$ and, being a unit vector, $\chi_b \notin B_a$. The remaining claims follow.

(ii) Let S_0 be the closed span of the coordinate vectors in S, and write $\xi \in S$ as $\eta + \zeta$ with $\eta \in S_0$ and $\zeta \perp S_0$. Then $\zeta \in S$. If ζ responded to some a, (4.1) would put χ_a in S, hence in S_0, and $\langle \chi_a, \zeta \rangle = 0$. So ζ responds to nothing and vanishes by Lemma 4.1(ii). Thus $S = S_0$, and S_0 is the closed join of the atoms B_a with $\chi_a \in S$, each of which lies in S by (i).

(iii) A nonzero boundary contains a nonzero vector, which responds to some a; by (4.1) and (i) it contains B_a. An irreducible boundary therefore equals an atom. Conversely, a nonzero boundary inside B_a contains some χ_b with $b \sim a$, hence $B_b = B_a$.

(iv) $\mathscr{C}(\xi)$ contains the atom of each coordinate ξ responds to, by (4.1). If ξ responds to no coordinate of B_a, then $\xi \perp B_a$, so ξ lies in the boundary $B_a^{\perp}$ and so does $\mathscr{C}(\xi)$. Apply (ii). If all responding coordinates are connected they share one atom, which is then $\mathscr{C}(\xi)$; if $\mathscr{C}(\xi) = B_c$, every responding coordinate lies in B_c and is connected to c. For the last claim, the coordinates are countable, hence so are the atoms of a boundary S; choose one coordinate a_r in each and put $\xi = \sum_r 2^{-r}\chi_{a_r}$. The atoms are orthogonal, so $\langle \chi_{a_r}, \xi\rangle = 2^{-r}$, and every coordinate ξ responds to lies in S by (4.1). So $\mathscr{C}(\xi) = S$.

(v) The atoms are pairwise orthogonal by (i), and their sum contains every coordinate vector, which is total by Lemma 4.1(ii). Since B is the closed span of its coordinate vectors, $Q_B\xi = 0$ exactly when ξ is orthogonal to all of them.

(vi) The genesis coordinates $(0, \ell(a)+1, z, \varnothing)$ and $(0, \ell(a)+2, z, \varnothing)$ have original loops different from $\ell(a)$ and from each other, so by Lemma 4.1(iii) they are connected neither to a nor to each other. Both lie in $B_a^{\perp}$, which therefore contains two distinct atoms and is not an atom, hence not irreducible by (iii). The same two coordinates show that $\mathscr{H}_\infty$ is not an atom. □

Part (ii) gives the whole lattice of boundaries at once.

Corollary 4.6 (Boundaries are sets of atoms)**.** *The map $S \mapsto \{B : B \subseteq S\}$ is an order isomorphism from the boundaries, ordered by inclusion, onto the sets of atoms. Its inverse sends a set of atoms to its closed join. In particular the closed join of any family of boundaries is a boundary, the least one containing every member; the closed join of all atoms is $\mathscr{H}_\infty$; and the boundaries containing a coordinate vector χ_a are exactly the boundaries containing B_a.*

Proof. For a set $\mathscr{A}$ of atoms with closed join S, an atom $B \notin \mathscr{A}$ is orthogonal to every member of $\mathscr{A}$, hence to S, so the atoms inside S are exactly $\mathscr{A}$. Theorem 4.5(ii) gives the other composition. The closed join of boundaries is invariant because each event is linear and continuous. □

Membership is therefore nested, never exclusive.

4.3 Atoms are rooted lineages

The record names the atom. To read it we run the record backwards. The output occupation and the label recover the input, so each step back is forced, until it reaches genesis or meets a label that no input could have written.

Definition 4.7 (Read-back and roots)**.** Let $a = (k, n, (x, i), h)$ with $h = (j^{(1)}, \dots, j^{(k)})$. Put $y_k = x$ and $n_k = n$. For $t = k, k-1, \dots, 1$: if some input y_{t-1} has $j(y_t, y_{t-1}) = j^{(t)}$ (it is then unique), set $n_{t-1} = n_t - j_2^{(t)}$ and continue; otherwise stop. If the read-back stops at stage t, the *root* of a is

$$\operatorname{root}(a) = \big(t,\ n_t,\ y_t,\ (j^{(1)}, \dots, j^{(t)})\big).$$

If $t = 0$, a is *genesis-compatible*, with *original occupation* $o(a) = y_0$; then $n_0 = \ell(a)$. Otherwise a is an *orphan*. A *root point* is (t, m, y, g) with $g \in \mathscr{J}^t$ such that $t = 0$ or the last label of g is written into y by no input. The *lineage atom* of $(x, n) \in \mathcal{O} \times \mathbb{Z}$ is $\Lambda(x, n) = B_{(0,n,(x,(0,0)),\varnothing)}$.

Two kinds of overlap drive the classification.

Lemma 4.8 (Parent edges and index connectivity). *(i) If $a = (k+1, n, (y, i), h \cdot j)$ and $j = j(y, x)$, then $\langle \chi_a, \chi_{(k,\, n-j_2,\, (x,i),\, h)} \rangle = (M_{yx})_{ii} \neq 0$.*
(ii) For every stage t, loop m, occupation x and record $g \in \mathcal{J}^t$, all sixteen coordinates $(t, m, (x, i), g)$, $i \in (\mathbb{Z}/4)^2$, are connected.

Proof. (i) is Lemma 4.1(i) with Lemma 4.2. (ii) Let $e \in \{(1,0), (0,1)\}$ and $j = j(00, x)$. The child $c = (t+1, m+j_2, (00, i), g \cdot j)$ overlaps $(t, m, (x, i), g)$ with amplitude $(M_{00,x})_{ii}$ and $(t, m, (x, i+e), g)$ with amplitude $(M_{00,x})_{i,i+e}$, both nonzero by Lemma 4.2. So $i \sim i + e$ through c, and the two steps $(1,0)$, $(0,1)$ generate $(\mathbb{Z}/4)^2$. □

Theorem 4.9 (Atoms are rooted lineages). *(i) For all coordinates a, b: $a \sim b$ if and only if $\mathrm{root}(a) = \mathrm{root}(b)$. The atoms correspond one to one with root points; the atom of (t, m, y, g) contains exactly the coordinates whose record reads back to it.*
(ii) A genesis-compatible a has $\mathrm{root}(a) = (0, \ell(a), o(a), \emptyset)$ and lies in $\Lambda(o(a), \ell(a))$. For genesis-compatible a, $a \sim b$ if and only if b is genesis-compatible with $o(b) = o(a)$ and $\ell(b) = \ell(a)$.
(iii) $\Lambda(x, n) = \Lambda(x', n')$ if and only if $x = x'$ and $n = n'$. There are four lineage atoms at each loop value, and infinitely many atoms.
(iv) The clock indices carry no atom information: coordinates that differ only in i are connected.

Proof. (i) Suppose $\xi \in \mathcal{H}_k$ is supported on coordinates with root ρ. The component of $T\xi$ at $(n, h \cdot j)$ and output (y, i') is $\big(M_{yx}\, \xi(n - j_2, h)_x\big)_{i'}$ for the unique input x with $j(y, x) = j$, or zero if there is none. When it is nonzero the read-back from $(k+1, n, y, h \cdot j)$ first steps to $(k, n - j_2, x, h)$, which has root ρ. So T keeps root supports, and so does every power of T. A basis vector at a is supported on $\{a\}$, so by Lemma 4.1(i) a later coordinate overlaps a only if it has the same root; overlaps are symmetric, so this holds whichever is later. Hence $a \sim b$ implies equal roots. Conversely, Lemma 4.8(i) connects every coordinate, one parent at a time, to a coordinate at its root point, and Lemma 4.8(ii) connects all clock indices there. So equal roots imply $a \sim b$. A root point (t, m, y, g) is the root of the coordinate $(t, m, (y, i), g)$, because the read-back stops at once; distinct root points therefore give distinct atoms.

(ii) For genesis-compatible a the read-back completes, $y_0 = o(a)$ and $n_0 = n - \mathrm{acc}(h) = \ell(a)$. The genesis coordinate defining $\Lambda(o(a), \ell(a))$ has the same root. Apply (i).

(iii) The root of $(0, n, (x, (0,0)), \emptyset)$ is $(0, n, x, \emptyset)$; apply (i). Different loop values give different atoms.

(iv) is Lemma 4.8(ii). □

The original loop alone does not classify atoms: $\Lambda(00, n)$ and $\Lambda(11, n)$ share their loop and are different atoms, although the all-zero record 0^k reads back to genesis from both. The pair (occupation, loop) is exact on genesis-compatible coordinates, and the root is exact everywhere.

4.3.1 Orphans

Most records do not read back to genesis. The labels written into an occupation y are the column y of Table 2.3; a label outside that column cannot have been written there.

Proposition 4.10 (Orphans exist and are never populated)**.** *(i) For every n and i, the coordinate $(1, n, (00, i), ((1, 0, 0)))$ is an orphan: only the transition $00 \to 10$ writes $(1, 0, 0)$.*
(ii) Every record of positive length is carried by an orphan: for every $k \geq 1$, n, i and $h \in \mathcal{J}^k$ there is an occupation y such that $(k, n, (y, i), h)$ is an orphan and its own root.
(iii) For every genesis state $v \in \mathscr{H}_0$ and every orphan a, $\langle \chi_a, \iota_0 v \rangle = 0$.
(iv) Orphan atoms are orthogonal to every lineage atom.

Proof. (i) Read off column 00 of Table 2.3. (ii) The last label of h is written into at most two occupations, so at least two occupations do not receive it, and the read-back stops at once at either. (iii) The inner product is the coefficient of $T^k v$ at a. At stage 0 there are no orphans. A stage-$(k+1)$ orphan either ends in a label that no input writes into its occupation, where every vector of the form $T\xi$ has zero coefficient, or its read-back steps to a stage-k orphan, whose coefficient in $T^k v$ vanishes by induction; the coefficient at the child is a block applied to that coefficient. (iv) Different roots; Theorem 4.9(i). □

So the lineage atoms, one for each original occupation and original loop, hold every history posted from genesis. The orphan atoms are plentiful and empty. Each is rooted at a record that no earlier state could have written, after which posting continues normally. Whether such coordinates have any physical standing is open; Proposition 4.13 below shows that no boundary generated by a history from genesis contains one.

4.3.2 Counting root records and their symmetry types

Fix a loop value. Atoms rooted at stage t correspond to *root records* (y, g): an occupation and a record of length t whose last label is not written into y (for $t = 0$, an occupation alone).

Proposition 4.11 (Root records)**.** *At each loop value there are 4 root records of stage 0 and $44 \cdot 15^{t-1}$ of stage $t \geq 1$.*

Proof. Each output receives exactly four labels, one from each input and distinct because the output and label recover the input. So 16 of the $4 \cdot 15 = 60$ pairs (occupation, label) are written and 44 are not. The earlier $t - 1$ labels are free. □

Two relabellings act on these records. The *complement* g_{c} flips both bits, $x \mapsto (1-x_1, 1-x_2)$, and negates labels, $j \mapsto -j$. The *swap* g_{s} exchanges the channels, $x \mapsto (x_2, x_1)$ and $j \mapsto (j_3, -j_2, j_1)$. Both satisfy $j(gy, gx) = g\, j(y, x)$, as one checks from $j(y, x) = (y_1 - x_2,\ x_1 - x_2,\ y_2 - x_1)$. So they send written pairs to written pairs and act letter by letter on root records of every stage. Together with their product they form a Klein four-group $\mathbb{V} = \{1, g_{\mathrm{c}}, g_{\mathrm{s}}, g_{\mathrm{c}} g_{\mathrm{s}}\}$. Chapter 5 shows that the orbits of $\mathbb{V}$ are exactly what a symmetric rule can read.

Proposition 4.12 (Orbit types of root records)**.** *The group $\mathbb{V}$ has 2 orbits on root records of stage 0 and*

$$11 \cdot 15^{t-1} + \tfrac{1}{2}\big(3^{t-1} + 3 \cdot 5^{t-1}\big)$$

orbits on root records of stage $t \geq 1$.

Proof. By Burnside's lemma the number of orbits is the average number of fixed points. A root record of stage $t \geq 1$ is fixed by g exactly when g fixes its last pair (occupation, unwritten label) and each of its $t-1$ earlier labels, so the fixed count is the product shown in Table 4.1. The complement fixes only the zero label and no occupation. The swap fixes the occupations 00, 11 and the labels with $j_1 = j_3$, $j_2 = 0$, namely $(0,0,0)$, $(1,0,1)$, $(\bar{1},0,\bar{1})$; of the resulting pairs, only $(00,(1,0,1))$ and $(11,(\bar{1},0,\bar{1}))$ are unwritten. The product fixes 10, 01 and the five labels with $j_3 = -j_1$; three of them are unwritten at 10 and three at 01. Hence

$$4N_t = 44 \cdot 15^{t-1} + 0 + 2 \cdot 3^{t-1} + 6 \cdot 5^{t-1}.$$

At stage 0 the fixed occupations number $4, 0, 2, 2$, giving $8/4 = 2$ orbits. □

	1	g_c	g_s	$g_c g_s$
fixed occupations	4	0	2	2
fixed labels	15	1	3	5
fixed unwritten pairs (occupation, label)	44	0	2	6
fixed root records of stage $t \geq 1$	$44 \cdot 15^{t-1}$	0	$2 \cdot 3^{t-1}$	$6 \cdot 5^{t-1}$

root stage t	0	1	2	3	4
root records per loop value	4	44	660	9900	148500
orbit types	2	13	174	2517	37326

Table 4.1. Fixed points of the four occupation relabellings, and the resulting counts of root records and of their orbit types.

4.3.3 The genesis whole

The histories from genesis together occupy a definite boundary. Its exact description for every genesis state is the subject of Chapter 5; here we record what the atoms already imply.

Proposition 4.13 (The genesis whole)**.** *(i) Some $v \in \mathscr{H}_0$ has nonzero amplitude at every genesis coordinate, for example $v(n) = 2^{-|n|} \sum_z |z\rangle$.*

(ii) For such v, $\mathscr{C}(\iota_0 v)$ contains every lineage atom, contains χ_a if and only if a is genesis-compatible, and is not an atom: it properly contains the nonzero boundary $\Lambda(00,0)$.

(iii) For every $v \in \mathscr{H}_0$, $\mathscr{C}(\iota_0 v)$ is orthogonal to every orphan atom. In particular no genesis state generates $\mathscr{H}_\infty$, although some vector of $\mathscr{H}_\infty$ does.

Proof. (i) The amplitudes are square-summable. (iii) By Proposition 4.10(iii), $\iota_0 v$ is orthogonal to every coordinate vector of an orphan atom B, hence lies in the boundary $B^\perp$, and so does $\mathscr{C}(\iota_0 v)$. Theorem 4.5(iv) supplies the generating vector of $\mathscr{H}_\infty$. (ii) $\langle \chi_{(0,n,z,\varnothing)}, \iota_0 v \rangle = v(n)_z \neq 0$, so by Theorem 4.5(iv) every lineage atom lies in $\mathscr{C}(\iota_0 v)$. A genesis-compatible coordinate lies in a lineage atom, and an orphan coordinate is excluded by (iii). The boundary holds the distinct atoms $\Lambda(00,0)$ and $\Lambda(00,1)$. □

4.4 Exact superselection without an environment

The atoms are the superselection sectors of the event algebra.

Theorem 4.14 (The commutant of the events)**.** *For a bounded operator X on $\mathscr{H}_\infty$ the following are equivalent.*

(a) $XF_a = F_aX$ *for every coordinate a.*
(b) There is a function c on coordinates, constant on connected classes, with $X\chi_a = c(a)\chi_a$ for every a.
(c) X acts on each atom B as one scalar c_B.

Such operators commute with each other and with every element of $\overline{\mathscr{E}}$, and they satisfy $\|Q_BX\xi\| = |c_B|\,\|Q_B\xi\|$. Every atom projection Q_B is one of them, and none is a scalar.

Proof. (a)⇒(b). $X\chi_a = XF_a\chi_a = F_aX\chi_a$ is a multiple $c(a)\chi_a$. Applying $XF_a = F_aX$ to ξ and pairing with χ_a gives $\langle\chi_a, X\xi\rangle = c(a)\langle\chi_a, \xi\rangle$. With $\xi = \chi_b$ this reads $c(b)\langle\chi_a, \chi_b\rangle = c(a)\langle\chi_a, \chi_b\rangle$, so c is constant across every nonzero overlap and hence on connected classes. (b)⇒(c). X is continuous and equals $c(a)$ on the coordinate vectors spanning B_a. (c)⇒(a). By Theorem 4.5(v), X preserves each atom and the closed join of the others, so it commutes with each Q_B. Then $F_aX\xi = \langle\chi_a, Q_{B_a}X\xi\rangle\chi_a = c_{B_a}\langle\chi_a, \xi\rangle\chi_a = XF_a\xi$. Two such operators agree in either order on each atom, and the atoms span a dense subspace. Commuting with the generators F_a passes to $\mathscr{E}$ and to norm limits. Finally $Q_BX\xi = XQ_B\xi = c_BQ_B\xi$; Q_B is 1 on B and 0 on every other atom, and there are at least two atoms. □

Theorem 4.15 (What the event algebra observes)**.** *Let $A \in \overline{\mathscr{E}}$.*

(i) A maps every boundary into itself; in particular $AQ_B = Q_BA$ for every atom B.
(ii) If $\xi_1, \dots, \xi_r$ lie in pairwise distinct atoms, then $\big\langle \sum_s \xi_s, A\sum_s \xi_s\big\rangle = \sum_s \langle\xi_s, A\xi_s\rangle$. So if $\xi = \sum_B Q_B\xi$ with finitely many nonzero terms, $\langle\xi, A\xi\rangle = \sum_B \|Q_B\xi\|^2\langle\hat{\xi}_B, A\hat{\xi}_B\rangle$ with unit parts $\hat{\xi}_B = Q_B\xi/\|Q_B\xi\|$: the expectation of a superposition over atoms is that of the mixture of its parts.
(iii) Let $\xi_1 \in B$ and $\xi_2 \in B'$ be nonzero vectors of two atoms. Every element of $\overline{\mathscr{E}}$ has a θ-independent expectation in $\xi_1 + e^{i\theta}\xi_2$ if and only if $B \neq B'$. When $B = B'$, a self-adjoint element of $\mathscr{E}$ sees the phase.

Proof. (i) The events preserve boundaries, and so do sums, products and norm limits of them. (ii) For $s \neq t$, $A\xi_t$ lies in the atom of ξ_t, which is orthogonal to that of ξ_s; every cross term vanishes. (iii) If $B \neq B'$, apply (ii). If $B = B'$, the nonzero vectors ξ_1, ξ_2 respond to coordinates c, d of B, and $c \sim d$, so $N = |\chi_c\rangle\langle\chi_d|$ lies in $\mathscr{E}$ by Lemma 4.3, with $\langle\xi_1, N\xi_2\rangle = \langle\xi_1, \chi_c\rangle\langle\chi_d, \xi_2\rangle \neq 0$. For self-adjoint A the expectation is $\langle\xi_1, A\xi_1\rangle + \langle\xi_2, A\xi_2\rangle + 2\,\mathrm{Re}\big(e^{i\theta}\langle\xi_1, A\xi_2\rangle\big)$. If both $A = N + N^*$ and $A = i(N - N^*)$ had zero cross amplitude, then $\langle\xi_1, N\xi_2\rangle = 0$; so one of them has a nonzero cross amplitude, and its expectation varies with θ. □

Inside one atom an event can see a relative phase that leaves the norm unchanged.

Example 4.16 (Interference inside one lineage)**.** Take the orthogonal genesis coordinates $g_0 = (0, 0, (00, (0, 0)), \varnothing)$ and $g_1 = (0, 0, (00, (1, 0)), \varnothing)$, both in $\Lambda(00, 0)$, and the stage-one

coordinate $c = (1, 0, (00, (0, 0)), ((0, 0, 0)))$. Since $M_{00,00} = -\text{stay} \otimes \text{stay}$ with $\text{stay}_{00} = \frac{1}{2}$ and $\text{stay}_{01} = -\frac{1}{2}$,

$$\langle \chi_c, \chi_{g_0} \rangle = (M_{00,00})_{(0,0),(0,0)} = -\frac{1}{4}, \qquad \langle \chi_c, \chi_{g_1} \rangle = (M_{00,00})_{(0,0),(1,0)} = \frac{1}{4}.$$

For $\xi_\theta = \chi_{g_0} + e^{i\theta}\chi_{g_1}$ the norm is $\sqrt{2}$ for every θ, while the event F_c reads

$$\langle \xi_\theta, F_c\, \xi_\theta \rangle = \left| -\tfrac{1}{4} + \tfrac{1}{4} e^{i\theta} \right|^2 = \tfrac{1}{8}(1 - \cos\theta).$$

Two clock indices of one genesis occupation post coherently into one child, and the child's event depends on their relative phase.

Theorem 4.17 (Atom weights are conserved)**.** *Let $\xi \in \mathscr{H}_\infty$.*

(i) For every $A \in \overline{\mathscr{E}}$ and atom B: if $Q_B\xi = 0$ then $Q_B A\xi = 0$. No operation of the algebra puts weight on an atom the state does not load.
(ii) Every isometry $U \in \overline{\mathscr{E}}$ conserves every atom weight: $\|Q_B U\xi\| = \|Q_B\xi\|$.
(iii) If ξ has nonzero weight on two distinct atoms, no isometry in $\overline{\mathscr{E}}$ maps ξ into a single atom.
(iv) If $A \in \overline{\mathscr{E}}$ maps ξ into an atom B, then $AQ_{B'}\xi = 0$ for every atom $B' \neq B$. If ξ responds to b, then $F_b\xi$ is a nonzero vector of B_b.

The isometries in (ii) are not only the identity: for every coordinate c, $1 - 2F_c$ is a self-adjoint unitary in $\mathscr{E}$ that sends χ_c to $-\chi_c$.

Proof. (i) and (ii) follow from $AQ_B = Q_B A$ (Theorem 4.15(i)). (iii) If $U\xi \in B''$, then $Q_B U\xi = 0$ for every atom $B \neq B''$, so $Q_B\xi = 0$ by (ii), leaving at most one loaded atom. (iv) $AQ_{B'}\xi = Q_{B'}A\xi = 0$ because $A\xi \in B \perp B'$. And $F_b\xi = \langle \chi_b, \xi \rangle \chi_b$. □

Concentration onto one atom is an outcome step, not a transformation the events can perform.

This is superselection in the sense of Wick, Wightman and Wigner [43]: no observable has a matrix element between two sectors, so a superposition across sectors has the statistics of a mixture [6]. Here the sectors are not imposed through a symmetry or a conserved charge. They come from the native step and its coordinate events. Decoherence suppresses interference between pointer states approximately, through an environment [46]; between atoms the suppression is exact, because no native event connects two atoms and every cross term is identically zero. The statement is relative to $\overline{\mathscr{E}}$, and Section 5.7 shows how placed reads of recognition dynamics behave.

4.5 The record law cannot see the lineage

The events at least see which atom a state loads. The record law of Theorem 2.17 does less. For every $\nu \in \mathscr{H}_0$ we write $\mathbb{P}_\nu(h) = \sum_n \|V_h\, \nu(n - \mathrm{acc}(h))\|^2 = \mathrm{tr}(\rho_\nu E_h)$ for the load of a record h; for a unit ν it is the record law. Let $(\mathrm{tr}_c \nu)(n) = \nu(n - c)$ translate the loop, on genesis states and likewise on every stage. Write $P_x = |x\rangle\langle x| \otimes I$ and $\rho^{\mathrm{occ}} = \sum_x P_x \rho P_x$ for the occupation-diagonal part of a cell density.

Theorem 4.18 (The record law is blind to lineage)**.** *Let $\nu, \nu' \in \mathscr{H}_0$.*

(i) $T^k \mathrm{tr}_c = \mathrm{tr}_c T^k$, *and* $\mathbb{P}_{\mathrm{tr}_c v}(h) = \mathbb{P}_v(h)$ *for every record* h *of every length; for a unit* v *the law on infinite records is unchanged by loop translation.*

(ii) *If* $\rho_v^{\mathrm{occ}} = \rho_{v'}^{\mathrm{occ}}$, *then* $\mathbb{P}_v = \mathbb{P}_{v'}$ *on records of every length.*

(iii) *If* v *and* v' *populate no common lineage, that is, for every* (x, n) *one of* $P_x v(n)$ *and* $P_x v'(n)$ *vanishes, then for all* $\alpha, \beta \in \mathbb{C}$ *and every record* h,

$$\sum_n \left\| V_h(\alpha v + \beta v')(n - \mathrm{acc}(h)) \right\|^2 = |\alpha|^2 \, \mathbb{P}_v(h) + |\beta|^2 \, \mathbb{P}_{v'}(h).$$

For unit v, v' *and* $|\alpha|^2 + |\beta|^2 = 1$, *the superposition has the record law of the mixture, whatever the relative phase.*

(iv) *The record determines the change of the loop since genesis,* $n - \ell = \mathrm{acc}(h)$, *and nothing about the original loop* ℓ.

(v) *Populations of occupations are read. For unit genesis states at a basis cell* $(10, i)$ *and at* $(01, i)$, *at any loop, the first label is* $(1, 1, -1)$ *with probability* $\frac{1}{4}$ *and* 0 *respectively.*

Proof. (i) $(T^k \mathrm{tr}_c v)(n, h) = V_h v(n - c - \mathrm{acc}(h)) = (\mathrm{tr}_c T^k v)(n, h)$, and the load sums over all n. The infinite law is determined by its finite marginals. (ii) The output occupation and the record recover the input occupation step by step, so $\langle y, i' | V_h | x, i \rangle \neq 0$ only if the read-back of h from y reaches x. Each y reaches at most one x, so for $x \neq x'$ the vectors $V_h P_x u$ and $V_h P_{x'} u'$ live on disjoint output occupations and are orthogonal. Hence $P_x E_h P_{x'} = 0$ for $x \neq x'$ and $\mathrm{tr}(\rho E_h) = \mathrm{tr}(\rho^{\mathrm{occ}} E_h)$. (iii) The hypothesis makes every cross term $P_x v(n)\, v'(n)^* P_x$ vanish, so $\rho^{\mathrm{occ}}_{\alpha v + \beta v'} = |\alpha|^2 \rho_v^{\mathrm{occ}} + |\beta|^2 \rho_{v'}^{\mathrm{occ}}$; apply the computation in (ii). (iv) $\ell = n - \mathrm{acc}(h)$ by definition, and (i) shows that the law of h does not depend on ℓ. (v) By column 10 of Table 2.3 only the transition $10 \to 10$ writes $(1, 1, -1)$, so $V_{(1,1,-1)} = |10\rangle\langle 10| \otimes M_{10,10}$. Its value on the basis cell has squared norm $\frac{1}{4}$ by Lemma 4.2, and it annihilates occupation 01. □

Example 4.19 (Two atoms, one record law)**.** For a unit $e \in \mathscr{H}_c \otimes \mathscr{H}_c$ and an occupation x, the genesis states $\delta_0 \otimes |x\rangle \otimes e$ and $\delta_1 \otimes |x\rangle \otimes e$ lie in the distinct atoms $\Lambda(x, 0)$ and $\Lambda(x, 1)$. They have the same record law at every length, and so does every unit superposition of them, with any weights and any relative phase. The projection onto original loop 0, which multiplies each coordinate vector by $[\ell(a) = 0]$, separates them. It is one scalar on each atom, so it lies in the commutant of Theorem 4.14; it reads the loop, which the record never does.

The record law is in fact coarser than ρ_v^{occ}: Chapter 9 shows that it does not even determine the occupation-diagonal blocks, and identifies exactly what it does read.

4.6 Which atom holds a history: finite reads and the location wall

Which atom holds a state is a question the events answer at every finite stage and cannot answer as one operation. The finite answer comes first.

Definition 4.20 (Stage reads of an atom)**.** For an atom B and a stage k, the *stage read* is $F_{B,k} = \sum F_b$, summed over the stage-k coordinates b with $\chi_b \in B$.

Theorem 4.21 (Finite reads and the location wall)**.** *Let* B *be an atom.*

(i) B has finitely many stage-k coordinates, at most $64 \cdot 15^k$. $F_{B,k}$ is a self-adjoint projection in $\mathscr{E}$, and $F_{B,k} = Q_B$ on $\iota_k(\mathscr{H}_k)$.
(ii) For every genesis state ν and every k, $F_{B,k}\iota_0\nu = F_{B,0}\iota_0\nu = Q_B\iota_0\nu$, and

$$\|Q_B\iota_0\nu\|^2 = \sum |\nu(n)_z|^2,$$

summed over the genesis coordinates $(0, n, z, \varnothing)$ of B. For $B = \Lambda(x, n)$ this is $\sum_i |\nu(n)_{(x,i)}|^2$, sixteen terms; for an orphan atom it is 0. For a unit ν the lineage weights sum to 1.
(iii) $F_{B,k}\xi \to Q_B\xi$ for every $\xi \in \mathscr{H}_\infty$, but $F_{B,k}$ does not converge to Q_B in operator norm, and $Q_B \notin \overline{\mathscr{E}}$.

For a unit ν the weights in (ii) form the *genesis distribution* over lineages. Part (iii) is the *location wall.* The proof of (iii) needs a supply of orthonormal coordinates inside every atom.

Lemma 4.22 (Every atom holds an infinite comb)**.** *Let $a = (k, n, (x, i), h)$ and $\bar{x} = (1 - x_1, x_2)$. For $m \geq 0$ let c_m be the coordinate at stage $k+m+1$ with cell $(\bar{x}, i)$ and record $h \cdot j(x, x)^m \cdot j(\bar{x}, x)$, at the loop $n + (m+1)(x_1 - x_2)$. Then every c_m is connected to a, and the vectors χ_{c_m} are orthonormal.*

Proof. The chain from a stays m times at occupation x and then flips the first bit; each step overlaps its parent through a diagonal entry of M_{xx} or $M_{\bar{x}x}$, nonzero by Lemma 4.2. For $m' < m$, the record of $c_{m'}$ has $j(\bar{x}, x)$ at the position where the record of c_m has $j(x, x)$, and these labels differ in their first component, $\bar{x}_1 - x_2 \neq x_1 - x_2$. So neither record is an initial segment of the other, and Lemma 4.1(i) gives orthogonality. □

In particular every atom, lineage or orphan, is infinite-dimensional: a branching tree of records with infinitely many orthogonal branches.

Proof of Theorem 4.21. (i) By Lemma 4.1(iii) every coordinate of B has the original loop $\ell(B)$, so a stage-k coordinate of B has loop $\ell(B) + \mathrm{acc}(h)$, fixed by its record; it is determined by its cell and record. Same-stage coordinates are orthonormal, so $F_{B,k}$ is a projection, and it lies in $\mathscr{E}$. The stage-k coordinate vectors form an orthonormal basis of $\iota_k(\mathscr{H}_k)$; Q_B keeps those in B and kills the others, which lie in other atoms. So does $F_{B,k}$.

(ii) $\iota_0\nu = \iota_k T^k \nu \in \iota_k(\mathscr{H}_k)$ for every k, so (i) applies at every stage. At stage 0, Parseval in the basis of genesis coordinates gives the sum. By Theorem 4.9(ii) the genesis coordinates of $\Lambda(x, n)$ are the sixteen $(0, n, (x, i), \varnothing)$, and an orphan atom has none. By Theorem 4.5(v) the weights over all atoms sum to $\|\iota_0\nu\|^2 = \|\nu\|^2$.

(iii) If $\xi \in \iota_m(\mathscr{H}_m)$ then $\xi \in \iota_k(\mathscr{H}_k)$ for all $k \geq m$, so $F_{B,k}\xi = Q_B\xi$ from $k = m$ on. These vectors are dense and $\|F_{B,k}\| \leq 1$, so the convergence holds for every ξ. By Lemma 4.3 every element of $\mathscr{E}$ is a scalar plus an operator of finite rank, so $\overline{\mathscr{E}} \subseteq \mathbb{C}1 + \mathscr{K}$, with $\mathscr{K}$ the compact operators. If $Q_B = \lambda 1 + Z$ with Z compact, compressing to B and to another atom B' (Theorem 4.5(vi)), both infinite-dimensional by Lemma 4.22, makes $(1 - \lambda)1$ compact on B and $\lambda 1$ compact on B', which is impossible. So $Q_B \notin \overline{\mathscr{E}}$. Since $\overline{\mathscr{E}}$ is closed and contains every $F_{B,k}$, they cannot converge to Q_B in norm. □

Remark 4.23 (Where the atom projection lives). Q_B commutes with every event (Theorem 4.14) and is the strong limit of the stage reads $F_{B,k} \in \mathscr{E}$. It therefore lies in the strong closure of $\mathscr{E}$,

the von Neumann algebra generated by the events [41], and outside its norm-closed part. For a history from genesis the wall is escaped at every finite stage: the stage read already equals Q_B on the posted history.

Chapter 5

Which boundary is actual: genesis and the walls

Chapter 4 cut the complete history into atoms and showed that no native event connects two of them. This chapter asks which atoms an actual history occupies, and whether anything in the native structure can choose among them. The answer has three parts, and each is exact.

The genesis state decides: an actual history occupies exactly the lineages, pairs of an original occupation and an original loop, on which its genesis state has amplitude, and posting never changes that set. Structure cannot decide: the relabellings of the step that no posting can detect move every atom, and the only property of a lineage they all respect is a single bit, whether its first posting moves the loop flux. No admitted operation can even ask: the norm-closed event algebra contains no projection onto any boundary other than $\{0\}$ and $\mathscr{H}_\infty$, although finite reads at a growing stage answer the question. These walls belong to the coordinate events; the placed reads of recognition dynamics reach every coordinate of the complete history from the actual range, orphans included.

5.1 The boundary of an actual history is its set of lineages

We use the coordinates, events, boundaries, atoms and roots of Chapter 4.

Definition 5.1 (Actual history, genesis support)**.** As in Chapter 4, a genesis state is any $v \in \mathscr{H}_0$; its posted history at stage k is $T^k v$, and $\iota_k T^k v = \iota_0 v$. An *actual history* is $\iota_0 v$ for a nonzero genesis state. The *genesis support* of v is

$$\operatorname{supp} v = \{(x,n) \in \mathcal{O} \times \mathbb{Z} \,:\, (|x\rangle\langle x| \otimes I)\, v(n) \neq 0\},$$

the lineages on which v has amplitude. For $\mathscr{L} \subseteq \mathcal{O} \times \mathbb{Z}$, $\Lambda(\mathscr{L})$ is the closed join of the lineage atoms $\Lambda(x,n)$, $(x,n) \in \mathscr{L}$, with $\Lambda(\varnothing) = \{0\}$. The *genesis whole* is $\Lambda(\mathcal{O} \times \mathbb{Z})$.

Example 5.2. Take $v = \delta_0 \otimes |00\rangle \otimes e_0 \otimes e_0 + \lambda\, \delta_3 \otimes |10\rangle \otimes e_1 \otimes e_2$ with $\lambda \neq 0$. Its genesis support is $\{(00,0),(10,3)\}$, and Theorem 5.4 below gives $\mathscr{C}(\iota_0 v) = \Lambda(00,0) \oplus \Lambda(10,3)$, the sum of two orthogonal atoms. Nothing else about v matters. Any other nonzero λ, or amplitude moved to other clock indices of the same two occupations at the same two loops, gives the same boundary; so does the posted history after any number of postings, since $\iota_k T^k v$ is the same vector. Setting $\lambda = 0$ removes $\Lambda(10,3)$. The amplitudes enter one level down: by Theorem 4.21(ii), the two atoms carry the weights 1 and $|\lambda|^2$ (before normalization) at every stage. The boundary records which lineages are populated; the weights record how much.

The proof rests on one observation about stage vectors.

Lemma 5.3 (Support)**.** *Let $S \subseteq \mathscr{H}_\infty$ be a closed subspace and $w \in \mathscr{H}_k$. If $\chi_{(k,n,z,h)} \in S$ whenever the coefficient $w(n,h)_z$ is nonzero, then $\iota_k w \in S$.*

Proof. Stage-k coordinate vectors are orthonormal and ι_k is an isometry, so the expansion

$$\iota_k w = \sum_{n,h,z} w(n,h)_z \, \chi_{(k,n,z,h)}$$

converges in norm. Every nonzero term lies in S, and S is closed. □

Theorem 5.4 (The boundaries of actual histories)**.** *For the norm-closed event algebra $\overline{\mathscr{E}}$ and every genesis state v:*

(i) $\mathscr{C}(\iota_0 v) = \Lambda(\operatorname{supp} v)$, and the same holds for the posted history $\iota_k T^k v$ at every stage k.
(ii) For every lineage, $\Lambda(x,n) \subseteq \mathscr{C}(\iota_0 v)$ iff $(x,n) \in \operatorname{supp} v$. The atoms of the coordinates $\iota_0 v$ responds to are exactly the $\Lambda(x,n)$ with $(x,n) \in \operatorname{supp} v$.
(iii) A boundary S is $\mathscr{C}(\iota_0 w)$ for some nonzero genesis state w iff $S \neq \{0\}$ and S contains no orphan coordinate vector, iff $S = \Lambda(\mathscr{L})$ for a nonempty $\mathscr{L}$. Every nonempty $\mathscr{L}$ is realized by a genesis state whose amplitudes sit only on the lineages of $\mathscr{L}$.
(iv) $\Lambda(x,n) \subseteq \Lambda(\mathscr{L})$ iff $(x,n) \in \mathscr{L}$; a coordinate vector χ_b lies in $\Lambda(\mathscr{L})$ iff b is genesis-compatible with $(o(b), \ell(b)) \in \mathscr{L}$. In particular $\mathscr{L} \mapsto \Lambda(\mathscr{L})$ is injective.
(v) Every genesis state that responds to every genesis coordinate has the same boundary, the genesis whole $\Lambda(\mathcal{O} \times \mathbb{Z})$ of Proposition 4.13.

Proof. (i) and (ii). Split v by lineage: $v = \sum_{(x,n)} v_{x,n}$ with $v_{x,n} = \delta_n \otimes (|x\rangle\langle x| \otimes I) v(n)$. The coefficients of $v_{x,n}$ sit on genesis coordinates $(0, n, (x,i))$, whose vectors lie in $\Lambda(x,n)$; by Lemma 5.3, $\iota_0 v_{x,n} \in \Lambda(x,n)$. Genesis coordinates are orthonormal, so $\langle \chi_{(0,n,(x,i))}, \iota_0 v \rangle = v(n)_{(x,i)}$, and $\iota_0 v$ responds to a genesis coordinate of $\Lambda(x,n)$ exactly when $(x,n) \in \operatorname{supp} v$. Conversely, let $\iota_0 v$ respond to some coordinate b. Then b is not an orphan (Proposition 4.10(iii)), so $\chi_b \in \Lambda(x,n)$ for $(x,n) = (o(b), \ell(b))$. If $(x,n) \notin \operatorname{supp} v$, every part $\iota_0 v_{x',n'}$ lies in an atom other than $\Lambda(x,n)$, hence is orthogonal to it, and $\iota_0 v$ could not respond to b. So the atoms responded to are exactly the $\Lambda(x,n)$ with $(x,n) \in \operatorname{supp} v$, and (i) follows from Theorem 4.5(iv); the first statement of (ii) then follows from (iv). The posted history is the same vector.

(iv) If $(x,n) \notin \mathscr{L}$, then $\Lambda(x,n)$ is orthogonal to every atom of $\Lambda(\mathscr{L})$, hence to $\Lambda(\mathscr{L})$, and being nonzero it is not contained in it. A coordinate vector of $\Lambda(\mathscr{L})$ lies in one of the atoms joined, since its own atom is orthogonal to every other atom (Theorem 4.5(i)); and every coordinate of $\Lambda(x,n)$ is genesis-compatible with $(o, \ell) = (x,n)$.

(iii) If $S = \mathscr{C}(\iota_0 w)$ with $w \neq 0$, then $S \ni \iota_0 w \neq 0$, and $S = \Lambda(\operatorname{supp} w)$ by (i), which contains no orphan vector by (iv). Conversely, a nonzero boundary without orphan vectors is the closed join of the atoms whose coordinates it contains, all of them lineage atoms; with $\mathscr{L} = \{(x,n) : \Lambda(x,n) \subseteq S\}$, nonempty, $S = \Lambda(\mathscr{L})$. For realization, given nonempty $\mathscr{L}$ let w have amplitude $2^{-|n|}$ on the cell $(x,(0,0))$ at loop n for each $(x,n) \in \mathscr{L}$ and zero elsewhere. It is square-summable and nonzero with $\operatorname{supp} w = \mathscr{L}$, so $\mathscr{C}(\iota_0 w) = \Lambda(\mathscr{L})$ by (i).

(v) Such a state has $\operatorname{supp} v = \mathcal{O} \times \mathbb{Z}$, so its boundary is $\Lambda(\mathcal{O} \times \mathbb{Z})$ by (i). That it is neither an atom nor $\mathscr{H}_\infty$ is Proposition 4.13(ii) and (iii). □

Corollary 5.5 (The exact range)**.** *The boundaries an actual history can occupy correspond one to one with the nonempty sets of lineages, by $\mathscr{L} \mapsto \Lambda(\mathscr{L})$, and $\Lambda(\mathscr{L}) \subseteq \Lambda(\mathscr{L}')$ iff $\mathscr{L} \subseteq \mathscr{L}'$. The*

boundary of an actual history is fixed at genesis by its genesis support and does not change under posting.

Proof. Theorem 5.4(iii) and (iv); the order statement follows from (iv) and the definition of $\Lambda(\mathscr{L})$ as a closed join. □

Every boundary is $\mathscr{C}(\xi)$ for some ξ (Theorem 4.5(iv)), but a boundary containing an orphan coordinate vector is the boundary of no actual history.

The theorem lists every boundary an actual history can occupy and chooses none. The genesis state is an input to every statement in this chapter. Calling $\iota_0 v$ an actual history is a definition; taking the physical history of the ledger to be a posted genesis state, and the genesis lineages to be its physical initial conditions, are identifications, and the theorems use neither.

5.2 Posting symmetries

A rule that picks a boundary without being handed a genesis state has only the step to work with, so it must respect the step's symmetries. These are relabellings of occupations, clock indices, labels and loop flux that reproduce every posting up to a sign. They act on the step blocks of Theorem 2.4,

$$M_{yx} = \sigma(x)\,\big(K(y_1, x_2)\,C_x\big) \otimes \big(K(y_2, x_1)\,C_x\big), \tag{5.1}$$

Below, σ without an argument denotes a posting symmetry, not the leading sign $\sigma(x)$.

Definition 5.6 (Posting symmetry)**.** A *posting symmetry* $\sigma = (\pi, \tau, \kappa, f, \epsilon)$ consists of bijections π of $\mathscr{O}$, τ of $(\mathbb{Z}/4)^2$, κ of $\mathscr{J}$ and f of $\mathbb{Z}$, and signs $\epsilon : \mathscr{O} \to \{\pm 1\}$, such that for all $x, y \in \mathscr{O}$, $j \in \mathscr{J}$, $n \in \mathbb{Z}$ and clock indices i', i:

$$j(\pi y, \pi x) = \kappa\big(j(y, x)\big), \qquad f(n - j_2) = f(n) - \kappa(j)_2, \qquad (M_{\pi y, \pi x})_{\tau i', \tau i} = \epsilon(x)\,(M_{yx})_{i'i}. \tag{5.2}$$

It acts on coordinates by $\sigma(k, n, (x, i), h) = (k, f(n), (\pi x, \tau i), \kappa(h))$, with κ applied to each label of the record. The *record sign* $\eta_\sigma(a) \in \{\pm 1\}$ of $a = (k, n, (x, i), h)$ is the product of $\epsilon(y)$ over the inputs y recovered when the record is read back from the output x, stopping where read-back stops; it is 1 at stage 0.

The three conditions say that σ preserves which transition writes which label, moves the loop flux consistently with the relabelled current, and reproduces every step block up to a sign fixed by the input.

Proposition 5.7 (Symmetries act on the complete history)**.** *Let σ be a posting symmetry.*

(i) There is a linear isometry U_σ of $\mathscr{H}_\infty$ with $U_\sigma \chi_a = \eta_\sigma(a)\,\chi_{\sigma a}$ for every coordinate a.
(ii) $\langle \chi_{\sigma a}, \chi_{\sigma b}\rangle = \eta_\sigma(a)\eta_\sigma(b)\langle \chi_a, \chi_b\rangle$. Hence $a \sim b$ iff $\sigma a \sim \sigma b$, and U_σ maps the atom B_a onto $B_{\sigma a}$.
(iii) $o(\sigma a) = \pi(o(a))$ for genesis-compatible a, σa is an orphan iff a is, and $\ell(\sigma a) = f(\ell(a))$. So $U_\sigma \Lambda(x, n) = \Lambda(\pi x, f(n))$.

Proof. On $\mathscr{H}_k$ let W_k relabel and sign: $(W_k w)(f(n), \kappa(h))_{(\pi x, \tau i)} = \eta_\sigma(k, n, (x, i), h)\, w(n, h)_{(x,i)}$. It is an isometry because it permutes an orthonormal basis up to signs. We check $W_{k+1} T = T W_k$

coefficientwise. The coefficient of Tw at loop n, record $h \cdot j$ and cell (y, i') is $\sum_i (M_{yx})_{i'i}\, w(n - j_2, h)_{(x,i)}$, where x is the unique input with $j(y, x) = j$. Under W_{k+1} it is multiplied by $\eta_\sigma = \epsilon(x)\,\eta_\sigma(k, n - j_2, (x, i), h)$ and moved to $(f(n), \kappa(h)\kappa(j), (\pi y, \tau i'))$. On the other side, TW_k at that coordinate reads the input πx (first condition), at loop $f(n) - \kappa(j)_2 = f(n - j_2)$ (second condition), with block entries $(M_{\pi y,\pi x})_{\tau i',\tau i} = \epsilon(x)(M_{yx})_{i'i}$ (third condition). The two agree. So the isometries $\iota_k W_k$ satisfy $\iota_{k+1} W_{k+1} T = \iota_k W_k$, and they extend to one isometry U_σ on the closure of $\bigcup_k \iota_k(\mathscr{H}_k)$, which is $\mathscr{H}_\infty$. Evaluating on basis vectors gives (i), and (ii) follows because U_σ preserves inner products, σ is a bijection of coordinates, and an isometry maps a closed span onto the closed span of the images. For (iii), the first condition says that read-back at $(\pi y, \kappa(j))$ recovers πx exactly when read-back at (y, j) recovers x; iterating the second condition gives $f(n) - \mathrm{acc}(\kappa(h)) = f(n - \mathrm{acc}(h))$. □

Example 5.8 (Four kinds of posting symmetry)**.** Each of the following satisfies (5.2); composing posting symmetries gives a posting symmetry, with the signs multiplied.

(a) *Loop translation* σ_c^{tr}: $f(n) = n + c$, all other maps the identity, $\epsilon \equiv 1$.
(b) *Complement* σ^{co}: π flips both bits, $\tau = \mathrm{id}$, $\kappa(j) = -j$, $f(n) = -n$, and $\epsilon = +1$ on balanced, -1 on mixed inputs. Since $s(1 - b) = -s(b)$, $K(1 - b', 1 - b) = -K(b', b)$; the two factor signs cancel, and the leading sign of (5.1) changes exactly when the input is mixed. The labels satisfy $j(\pi y, \pi x) = -j(y, x)$.
(c) *Swap* σ^{sw}: $\pi(x_1, x_2) = (x_2, x_1)$, $\tau(i_1, i_2) = (i_2, i_1)$, $\kappa(j) = (j_3, -j_2, j_1)$, $f(n) = -n$, and ϵ as for the complement. The swap exchanges the two clock cells, hence the two tensor factors of (5.1), and changes the leading sign on mixed inputs; $j(\pi y, \pi x) = (y_2 - x_1,\ x_2 - x_1,\ y_1 - x_2)$ is $\kappa(j(y, x))$.
(d) *Half-turns* $\sigma_1^{\mathrm{ht}}, \sigma_2^{\mathrm{ht}}$: π, κ, f the identity, $\tau^{(1)}(i) = i + (2, 0)$, $\tau^{(2)}(i) = i + (0, 2)$, and $\epsilon = -1$ on balanced, $+1$ on mixed inputs. By the proof of Proposition 2.13, relabelling one clock cell by $i \mapsto i + 2$ changes the sign of every balanced block and fixes every mixed block, which is dressed by D.

Write $\tau^{(12)} = \tau^{(1)}\tau^{(2)}$ for the joint half-turn.

5.3 Translation moves every atom

The posting does not care where the loop flux starts. A posting with label j shifts the flux by j_2 whatever its value, so translating the flux is a symmetry, and it moves every atom.

Theorem 5.9 (The translation wall)**.** *Write $a + c = (k, n + c, z, h)$ for $a = (k, n, z, h)$, and let $c \neq 0$.*

(i) $\langle \chi_{a+c}, \chi_{b+c} \rangle = \langle \chi_a, \chi_b \rangle$ for all a, b; hence $a \sim b$ iff $a + c \sim b + c$.
(ii) $B_{a+c} \neq B_a$ for every coordinate a, lineage or orphan.
(iii) If $\xi \in \mathscr{H}_\infty$ satisfies $\langle \chi_{a+c}, \xi \rangle = \langle \chi_a, \xi \rangle$ for every a, then $\xi = 0$. In particular $U_{\sigma_c^{\mathrm{tr}}}$ fixes no nonzero vector.
(iv) Let $\mathscr{A}$ be a set of closed subspaces with $B_{b+c'} \in \mathscr{A}$ whenever $B_b \in \mathscr{A}$, for all b and c'. If $\mathscr{A}$ contains one atom, it is infinite. In particular $\mathscr{A} \neq \{B_a\}$ for every a.
(v) If v is a genesis state supported on the single lineage (x, n), then $\iota_0 v \in \Lambda(x, n)$ and $U_{\sigma_c^{\mathrm{tr}}} \iota_0 v \in \Lambda(x, n + c)$.

Proof. (i) The translation has all signs $+1$, so $\eta \equiv 1$ and Proposition 5.7(ii) applies. (ii) $\ell(a +$

$c) = \ell(a) + c$, and the original loop is constant on atoms. (iii) Fix k, z, h. The sequence $n \mapsto \langle \chi_{(k,n,z,h)}, \xi \rangle$ is the coefficient sequence of $\iota_k^* \xi \in \mathscr{H}_k$, hence square-summable, and it is c-periodic. A periodic square-summable sequence takes each of its values infinitely often, so it vanishes. Thus ξ responds to no coordinate and $\xi = 0$. If $U\xi = \xi$ with $U = U_{\sigma_c^{\mathrm{tr}}}$, then $\langle \chi_{a+c}, \xi \rangle = \langle U\chi_a, U\xi \rangle = \langle \chi_a, \xi \rangle$. (iv) If $B_a \in \mathscr{A}$, the atoms B_{a+m}, $m \in \mathbb{Z}$, all lie in $\mathscr{A}$ and are pairwise distinct by (ii). (v) Lemma 5.3 and Proposition 5.7(iii). □

A normalizable history cannot be translation invariant, so every actual history breaks the translation, and the atom it occupies is set by its genesis data, which the translation moves. The record law adds nothing here: it is unchanged by loop translation (Theorem 4.18(i)), so the loop half of the genesis datum is invisible in the record statistics as well.

5.4 Structure reads one bit of a lineage

Translation acts on the loop. The other symmetries act on the occupation, and there they can do very little: every posting symmetry keeps the balance of every occupation, so it cannot carry a lineage that starts balanced to one that starts mixed.

By Proposition 2.7, a lineage is balanced exactly when its first posting leaves the loop flux still.

Theorem 5.10 (What a posting symmetry does to lineages)**.** *Let $\sigma = (\pi, \tau, \kappa, f, \epsilon)$ be a posting symmetry.*

(i) πx is balanced iff x is balanced.
(ii) π is the identity, the swap $(x_1, x_2) \mapsto (x_2, x_1)$, the complement, or the complement composed with the swap.
(iii) Put $\omega_\sigma = (\pi(10))_1 - (\pi(10))_2 \in \{\pm 1\}$, the orientation*. Then $f(n) = \omega_\sigma n + f(0)$ for all n, and $\omega_\sigma = -1$ iff π exchanges* 10 *and* 01*.*
(iv) The pairs (π, f) that occur are exactly $(\mathrm{id}, n \mapsto n+c)$, $(\mathrm{swap}, n \mapsto -n+c)$, $(\mathrm{complement}, n \mapsto -n+c)$ and $(\mathrm{complement} \circ \mathrm{swap}, n \mapsto n+c)$, and each occurs for every $c \in \mathbb{Z}$.
(v) For genesis-compatible a, the atom of σa is $\Lambda(\pi o(a), f(\ell(a)))$.

Proof. (i) The second condition of (5.2) at $n = 0$ reads $f(-j_2) = f(0) - \kappa(j)_2$, and f is injective, so $\kappa(j)_2 = 0$ iff $j_2 = 0$. By the first condition and (2.9), $\kappa(j(y,x))_2 = j(\pi y, \pi x)_2 = (\pi x)_1 - (\pi x)_2$. Hence πx is balanced iff $j(y,x)_2 = 0$, iff x is balanced. (ii) A bijection of $\mathcal{O}$ preserving balance permutes $\{00, 11\}$ and $\{10, 01\}$ separately, which leaves four maps: the identity, the swap (which fixes 00 and 11 and exchanges 10 and 01), the complement (which exchanges both pairs), and their product (which exchanges 00 and 11 only). (iii) Take $j = j(y, 10)$, so $j_2 = 1$ and $\kappa(j)_2 = (\pi(10))_1 - (\pi(10))_2 = \omega_\sigma$, which is ± 1 because $\pi(10)$ is mixed. The second condition gives $f(n-1) = f(n) - \omega_\sigma$ for every n, so f is affine with slope ω_σ; $\omega_\sigma = -1$ iff $\pi(10) = 01$, and then $\pi(01) = 10$. (iv) By (ii) and (iii) the slope is -1 exactly for the swap and the complement. The pairs are realized by σ_c^{tr} composed with the identity, σ^{sw}, σ^{co} and $\sigma^{\mathrm{co}}\sigma^{\mathrm{sw}}$. (v) Proposition 5.7(iii). □

Theorem 5.11 (Symmetric rules read one bit)**.** *Let the posting symmetries act on lineages and atoms as in Theorem 5.10.*

(i) A set $\mathscr{L} \subseteq \mathcal{O} \times \mathbb{Z}$ with $(\pi x, f(n)) \in \mathscr{L}$ for every $(x, n) \in \mathscr{L}$ and every posting symmetry is one of four sets: $\emptyset$, all balanced lineages $\{(x, n) : x_1 = x_2\}$, all mixed lineages, or $\mathcal{O} \times \mathbb{Z}$. Invariance under loop translations and the complement alone already forces this.

(ii) Let $\mathscr{A}$ be a set of closed subspaces closed under the symmetries: $B_{\sigma a} \in \mathscr{A}$ whenever $B_a \in \mathscr{A}$. Then the lineages it selects, $\{(x, n) : \Lambda(x, n) \in \mathscr{A}\}$, form one of the four sets of (i).

(iii) Every posting symmetry keeps the type of every coordinate: balanced origin, mixed origin, or orphan. Under posting symmetries and connection, the genesis-compatible coordinates form exactly two classes, those of balanced origin and those of mixed origin.

Proof. (i) The four sets are invariant by Theorem 5.10(i). Conversely, if $(x, n) \in \mathscr{L}$, translations put every (x, m) in $\mathscr{L}$, and the complement adds $(\bar{x}, m)$ for the complement $\bar{x}$ of x, which is the other occupation of the same balance class. So $\mathscr{L}$ is a union of balance classes times $\mathbb{Z}$. (ii) If $\Lambda(x, n) = B_a \in \mathscr{A}$ for a genesis coordinate a, then $\Lambda(\pi x, f(n)) = B_{\sigma a} \in \mathscr{A}$ by Theorem 5.10(v); apply (i). (iii) Type is kept by Theorem 5.10(i) and Proposition 5.7(iii). Every genesis-compatible coordinate is connected to a genesis coordinate of its lineage, with any clock index, and translations and the complement carry that genesis coordinate to $(0, 0, (00, (0, 0)), \emptyset)$ or to $(0, 0, (10, (0, 0)), \emptyset)$ according to its balance. The two classes are distinct by (i). □

The swap is not needed for these orbits: translations and the complement already produce both classes. On orphan atoms a symmetric rule can read a little more, but still only types. The posting symmetries act on atoms through roots, and two atoms lie in one orbit of the posting symmetries together with $\sim$ if and only if they have the same root stage and their root records lie in one orbit of the four occupation maps, the Klein group $\mathbb{V}$ of Chapter 4; the loop value is free. So a set of atoms closed under every posting symmetry contains an orphan atom exactly when it contains every orphan atom of the same root stage and the same orbit type, and every union of orbit types is closed. Proposition 4.12 counts the types: two at root stage 0 (the balanced and mixed lineages) and $11 \cdot 15^m + \frac{1}{2}(3^m + 3 \cdot 5^m)$ at root stage $m + 1$, thirteen at root stage 1 and 174 at root stage 2, each holding infinitely many atoms.

Selecting a lineage needs data the structure does not supply, and in this setting the genesis state supplies it. An operator that reads the original loop, one scalar $g(\ell)$ on each atom, breaks translation and would rank lineages by their loop; such operators exist on $\mathscr{H}_\infty$, but for nonconstant g none lies in $\overline{\mathscr{E}}$ (Theorem 5.16), and none is derived from the step here.

5.5 The internal symmetries are the clock half-turns

A symmetry that fixes every occupation can still relabel the clock. There are exactly four ways to do it, and the field-moving postings alone decide this: the step from a mixed input contains the dressed one-step advance of each clock, and only half-turns commute with those advances.

Theorem 5.12 (Internal posting symmetries)**.** *Let $\sigma = (\mathrm{id}, \tau, \kappa, f, \epsilon)$ be a posting symmetry that fixes every occupation.*

(i) $\kappa = \mathrm{id}$ and $f(n) = n + c$ for some $c \in \mathbb{Z}$.

(ii) $\epsilon(x) = +1$ for both mixed inputs.

(iii) τ *is one of* $\mathrm{id}, \tau^{(1)}, \tau^{(2)}, \tau^{(12)}$.
(iv) ϵ *is determined by* τ*:* $\epsilon(00) = \epsilon(11)$*, and* $\epsilon(00) = +1$ *iff* $\tau \in \{\mathrm{id}, \tau^{(12)}\}$.
(v) So σ *is a loop translation composed with the identity,* σ_1^{ht}, σ_2^{ht} *or* $\sigma_1^{\mathrm{ht}}\sigma_2^{\mathrm{ht}}$*, and all of these occur.*

Proof. (i) With $\pi = \mathrm{id}$ the first condition reads $\kappa(j(y, x)) = j(y, x)$, and every label is written by some transition. The second condition then reads $f(n - j_2) = f(n) - j_2$; with $j_2 = 1$ it gives $f(n) = f(0) + n$.

(ii) Take a mixed input x and output 00. Summing the third condition over the diagonal, and using that τ is a bijection, gives $\operatorname{tr} M_{00,x} = \epsilon(x) \operatorname{tr} M_{00,x}$. Now flip $D = \frac{1}{2}(I + DLD)$ and stay $D = \frac{1}{2}(I - DLD)$, and DLD has zero diagonal, so both have trace 2. Since $M_{00,10} = (\text{stay } D) \otimes (\text{flip } D)$ and $M_{00,01} = -(\text{flip } D) \otimes (\text{stay } D)$, the traces are 4 and -4, so $\epsilon(x) = 1$.

(iii) By (ii), τ preserves every entry of the four blocks $M_{y,10}$, hence of the signed sums

$$S_1 = M_{10,10} + M_{11,10} - M_{00,10} - M_{01,10}, \qquad S_2 = M_{00,10} + M_{10,10} - M_{01,10} - M_{11,10}.$$

From (5.1), $M_{y,10} = (K(y_1, 0)D) \otimes (K'(y_2)D)$ with $K'(0) = \text{flip}$ and $K'(1) = \text{stay}$, and flip D + stay $D = I$, flip D − stay $D = DLD$. Hence

$$S_1 = DLD \otimes I, \qquad S_2 = I \otimes DLD.$$

The matrix DLD has one nonzero entry in row r, at column $r + 1$, equal to $d_r d_{r+1}$, which is -1 for even r and $+1$ for odd r. So row i of S_1 has its single nonzero entry at column $i + (1, 0)$, with the sign fixed by the parity of i_1. A bijection preserving S_1 entrywise must therefore satisfy $\tau(i + (1, 0)) = \tau(i) + (1, 0)$ and keep the parity of the first index; likewise for S_2 and the second index. Hence $\tau(i) = \tau(0) + i$ with both components of $\tau(0)$ even: $\tau(0) \in \{0, 2\}^2$.

(iv) Every block M_{yx} is nonzero, so the third condition fixes $\epsilon(x)$ once τ is known. The four maps of (iii) are realized by the identity and the half-turns of Example 5.8(d), whose signs on balanced inputs are $+1$, -1, -1 and $(-1)^2 = +1$. (v) Compose with σ_c^{tr}, which changes only f. □

Corollary 5.13 (Sixteen cell relabellings)**.** *The cell relabellings* (π, τ) *of all posting symmetries are exactly sixteen distinct maps:* π *is one of the four occupation maps of Theorem 5.10(ii), and* $\tau = \tau_\pi \tau_0$*, where* $\tau_0 \in \{\mathrm{id}, \tau^{(1)}, \tau^{(2)}, \tau^{(12)}\}$ *and* τ_π *exchanges the two clock indices when* π *is the swap or the complement composed with the swap, and is the identity otherwise.*

Proof. Let σ have occupation map π and let σ_π be the posting symmetry among id, σ^{sw}, σ^{co}, $\sigma^{\mathrm{co}}\sigma^{\mathrm{sw}}$ with the same occupation map. Each of these maps is an involution on occupations and on clock indices, so $\sigma_\pi \sigma$ fixes every occupation and has clock map $\tau_\pi \tau$. Theorem 5.12(iii) gives $\tau_\pi \tau = \tau_0$. All sixteen occur by composition, and they differ on cells. □

The half-turns are not arbitrary relabellings. The two spectral projections $\frac{1}{2}(1 \mp \tau^{(1)})$ of the half-turn of the first clock cell, and likewise for the second, are the pointer projections studied in Chapter 6. Theorem 5.12 shows that this marker is forced: an internal posting symmetry with $\epsilon(00) = -1$ has the clock map of one of the two half-turns (it is that half-turn composed with a loop translation), and no relabelling of the clock alone, other than the four of the theorem, is compatible with the posting. Half-turns fix π and f, so they fix every lineage atom, but they act on lineage histories with a sign, and those signs are as constrained as the maps.

Proposition 5.14 (Record signs). *Let σ be a posting symmetry with orientation ω_σ, and let $a = (k, n, z, h)$ be genesis-compatible. Write* $\mathrm{sil}(h)$ *for the number of labels of h with $j_2 = 0$, the* silent *postings.*

(i) $\epsilon(x) = \epsilon(00)$ *for balanced x and* $\epsilon(x) = \omega_\sigma$ *for mixed x.*
(ii) $\eta_\sigma(a) = \epsilon(00)^{\mathrm{sil}(h)}\, \omega_\sigma^{k-\mathrm{sil}(h)}$.
(iii) Hence $\eta_\sigma(a)$ *is* $1, (-1)^{\mathrm{sil}(h)}, (-1)^k(-1)^{\mathrm{sil}(h)}$ *or* $(-1)^k$ *according as* $(\omega_\sigma, \epsilon(00))$ *is* $(+,+), (+,-), (-,+)$ *or* $(-,-)$. *All four occur: for a loop translation, a half-turn, the swap or complement, and the swap composed with a half-turn.*
(iv) For the swap and the complement, $\eta_\sigma(a) = (-1)^{\mathrm{acc}(h)} = (-1)^{n-\ell(a)}$, *the parity of the loop displacement since genesis.*
(v) Reflecting the loop does not by itself give the swap's sign law: $\sigma^{\mathrm{sw}}\sigma_1^{\mathrm{ht}}$ *has* $\omega = -1$ *and* $\eta(a) = (-1)^k$, *which differs from* $(-1)^k(-1)^{\mathrm{sil}(h)}$ *whenever* $\mathrm{sil}(h)$ *is odd. The law in (iii) for* $(\omega, \epsilon(00)) = (-,+)$ *needs both conditions.*

Proof. (i) Mixed inputs: summing the third condition of (5.2) over the diagonal gives $\operatorname{tr} M_{\pi 00,\pi x} = \epsilon(x) \operatorname{tr} M_{00,x}$. With the traces $\operatorname{tr} M_{y,10} = 4$ and $\operatorname{tr} M_{y,01} = -4$ for balanced y (the proof of Theorem 5.12(ii), and $M_{11,10} = (\mathrm{flip}\, D) \otimes (\mathrm{stay}\, D)$, $M_{11,01} = -(\mathrm{stay}\, D) \otimes (\mathrm{flip}\, D)$), $\epsilon(x) = 1$ when π fixes the mixed occupations and -1 when it exchanges them, which is ω_σ. Balanced inputs: $K(b', 1) = -K(1 - b', 0)$ gives $M_{y,11} = M_{\bar{y},00}$, with $\bar{y}$ the complement. The four occupation maps commute with the complement, so the third condition at $x = 11$ and at $x = 00$ compares the same nonzero entries, and $\epsilon(11) = \epsilon(00)$. (ii) For genesis-compatible a all k inputs are recovered, and by Proposition 2.7 the input of step t is balanced iff $j_2^{(t)} = 0$. (iii) Read off from (ii); the signs of the four examples are in Example 5.8. (iv) The swap and complement have $(\omega, \epsilon(00)) = (-1, +1)$, and $k - \mathrm{sil}(h)$ counts the labels with $j_2 = \pm 1$, which has the parity of $\mathrm{acc}(h)$. (v) The swap composed with a half-turn has signs $(+1)(-1)$ on balanced and $(-1)(+1)$ on mixed inputs, so $\epsilon \equiv -1$; apply (ii). □

Remark 5.15. The half-turn acts on a lineage history by $(-1)^{\mathrm{sil}(h)}$: its sign flips once per silent posting. This is the rule by which the pointer sector of Chapter 6 is carried along a record, so that transport rule is the record-sign law of an internal posting symmetry. On genesis-compatible coordinates every record sign of every posting symmetry lies in the four-element group generated by the stage parity $(-1)^k$ and the silence parity $(-1)^{\mathrm{sil}(h)}$.

5.6 No admitted operation reads which boundary

Which boundary holds a history has an exact answer at every finite stage and no answer in the admitted algebra. Chapter 4 proved this for one atom. Here it holds for every boundary, and we identify the operators that the finite answers converge to.

We use the stage reads $F_{B,k}$ of Definition 4.20: finite sums of events, projections in $\mathscr{E}$, equal to Q_B on $\iota_k(\mathscr{H}_k)$ (Theorem 4.21(i)). Let $\mathcal{S}_k$ be the linear span of the stage reads $F_{B,k}$ over all atoms B. Its elements are finite sums of events.

Theorem 5.16 (The location wall for every boundary). *For the norm-closed event algebra $\overline{\mathscr{E}}$:*

(i) For a boundary S, the projection Q_S lies in $\overline{\mathscr{E}}$ iff $S = \{0\}$ or $S = \mathscr{H}_\infty$.
(ii) An element of $\overline{\mathscr{E}}$ commutes with every native event iff it is a scalar multiple of 1.

Proof. By the proof of Theorem 4.21(iii), $\overline{\mathscr{E}} \subseteq \mathbb{C}1 + \mathscr{K}$, with $\mathscr{K}$ the compact operators; and every atom is infinite-dimensional (Lemma 4.22). (i) Let $0 \neq S \neq \mathscr{H}_\infty$. Then S contains an atom. Its orthogonal complement is a boundary, because events are self-adjoint, and is nonzero, so it contains an atom too. If $Q_S = \lambda 1 + Z$ with Z compact, restriction to the first atom makes $(1-\lambda)1$ compact on an infinite-dimensional space, so $\lambda = 1$; restriction to the second makes $\lambda 1$ compact, so $\lambda = 0$. This is impossible. The projections 0 and 1 are admitted. (ii) An operator commuting with every event is one scalar c_B on each atom B (Theorem 4.14). If it is also $\lambda 1 + Z$, then $(c_B - \lambda)1_B$ is compact on the infinite-dimensional B, so $c_B = \lambda$ for every atom, and the atoms span a dense subspace. □

So no nonconstant function of which boundary, and in particular no readout of the original loop, is one admitted operation. The finite reads still determine the answer. The next theorem says exactly what they determine.

Theorem 5.17 (Boundary-level operators are limits of stage reads)**.** *For a bounded operator X on $\mathscr{H}_\infty$ the following are equivalent:*

(a) X commutes with every native event;
(b) there are $S_k \in \mathscr{S}_k$ with $\|S_k\| \leq \|X\|$ and $S_k\xi \to X\xi$ for every $\xi \in \mathscr{H}_\infty$;
(c) there are $S_k \in \mathscr{S}_k$ with $S_k\chi_b \to X\chi_b$ for every coordinate b.

In particular, for every boundary S there are projections $R_k \in \mathscr{S}_k$, each a finite sum of events, with $R_k\xi \to Q_S\xi$ for every ξ; if $0 \neq S \neq \mathscr{H}_\infty$, the limit Q_S is not in $\overline{\mathscr{E}}$.

Proof. (a)⇒(b). The atoms are countable; enumerate them as $B_1, B_2, \ldots$ and put $P_k = \sum_{r \leq k} F_{B_r,k}$. Distinct atoms have disjoint sets of stage-k coordinates and these coordinates are orthonormal, so P_k is a projection. For $\xi \in \iota_m(\mathscr{H}_m)$ and $k \geq m$, $P_k\xi = \sum_{r \leq k} Q_{B_r}\xi$, because $\iota_m(\mathscr{H}_m) \subseteq \iota_k(\mathscr{H}_k)$; this tends to ξ since $\mathscr{H}_\infty$ is the orthogonal sum of its atoms. With $\|P_k\| \leq 1$ and the density of $\bigcup_m \iota_m(\mathscr{H}_m)$, $P_k \to 1$ strongly. Now X is a scalar c_B on each atom and commutes with Q_B, while $F_{B,k} = F_{B,k}Q_B$; so $F_{B,k}X = F_{B,k}Q_BX = F_{B,k}XQ_B = c_BF_{B,k}$. Hence $S_k = P_kX = \sum_{r \leq k} c_{B_r}F_{B_r,k}$ lies in $\mathscr{S}_k$, $\|S_k\| \leq \|X\|$, and $S_k\xi \to X\xi$. (b)⇒(c) is immediate. (c)⇒(a). If the stage of b is at most k, then $\chi_b \in \iota_k(\mathscr{H}_k)$, so $F_{B,k}\chi_b = Q_B\chi_b$, which is χ_b if $\chi_b \in B$ and 0 otherwise. Thus each S_k acts on χ_b, once k reaches the stage of b, as a scalar that depends only on the atom of b. In the limit $X\chi_b = c(b)\chi_b$ with c constant on atoms, and an operator of this form commutes with every event (Theorem 4.14). For the last statement take $X = Q_S$, which commutes with every event because S is a boundary. Then $c_B \in \{0, 1\}$ and $R_k = P_kQ_S$ is the sum of the stage-k reads of the atoms $B_r \subseteq S$ with $r \leq k$. Theorem 5.16(i) excludes Q_S from $\overline{\mathscr{E}}$. □

The stage index has to grow. A read at one fixed stage is never boundary-level unless it is zero.

Proposition 5.18 (The stage must grow)**.** *Let B be an atom and $k \geq 0$.*

(i) $F_{B,k}$ commutes with every native event iff B has no coordinate of stage k, that is, iff $F_{B,k} = 0$.
(ii) A lineage atom has coordinates at every stage, so none of its stage reads commutes with the events. An orphan atom has no coordinate below the stage of its root; its stage-0 read is 0 and commutes with everything.

Proof. (i) If b is a stage-k coordinate of B, then $F_{B,k}\chi_b = \chi_b$. If $F_{B,k}$ commuted with every event it would be one scalar on B, necessarily 1; but $F_{B,k}$ has finite rank and B is infinite-dimensional. (ii) For a genesis coordinate g of $\Lambda(x,n)$, $T^k\delta_g \neq 0$ because T is an isometry, so $\iota_0\delta_g = \iota_k T^k\delta_g$ responds to some stage-k coordinate, which then lies in $\Lambda(x,n)$. Every coordinate of an orphan atom reads back to the common root, so its stage is at least the root's. For example, the atom of the orphan $(1, n, (00, i), ((1,0,0)))$ of Chapter 4 has root stage 1. □

5.7 The walls belong to the events: reads reach every record

The three walls hold for the coordinate events. Recognition dynamics supplies further reads [3]: the retained read and the grammar of placed reads below are taken from it, not derived from postings. These reads reach every coordinate from the actual range; Chapter 6 adds the archive reads.

Definition 5.19 (Retained read, live reads, placed reads)**.** Group the cell into two channels, bit x_c with clock cell c, and let β flip the bit. The *retained read* is the rank-16 projection $\mathrm{R} = r \otimes r$ with $r = \frac{1}{2}(I - \beta \otimes L^2)$; its entries are

$$\mathrm{R}_{(y,i'),(x,i)} = \tfrac{1}{4} \prod_{c=1,2} \big([y_c = x_c,\ i'_c = i_c] - [y_c = x_c + 1,\ i'_c = i_c + 2] \big), \tag{5.3}$$

bits modulo 2 and indices modulo 4. The *live read algebra* $\mathscr{A}_{\mathrm{R}}$ is the algebra of 64×64 matrices generated by the effects $E_w = V_w^* V_w$ and the *retained effects* $E_w^{\mathrm{R}} = V_w^* \mathrm{R} V_w$ over all words w, the empty word included. A cell matrix A *placed* at stage k and record $h \in \mathscr{J}^k$ is $A[k,h] = \iota_k(1 \otimes |h\rangle\langle h| \otimes A)\iota_k^*$. The *placed-read algebra* $\overline{\mathscr{G}}_{\mathrm{pl}}$ is the norm closure of the unital $*$-algebra generated by the placed reads $E_w[k,h]$ and $E_w^{\mathrm{R}}[k,h]$.

As $\iota_k^*\iota_k = 1$, $A \mapsto A[k,h]$ is linear and multiplicative, so $A[k,h]$ lies in that $*$-algebra for every $A \in \mathscr{A}_{\mathrm{R}}$. One fact does the work: $\mathscr{A}_{\mathrm{R}}$ consists exactly of the cell matrices commuting with both half-turn relabellings $\tau^{(1)}$, $\tau^{(2)}$ of the clock index, occupation fixed (Corollary 6.14).

The construction moves one cell at a time. For an occupation y and a cell z, let $U_{z,y}$ be the cell matrix with $(U_{z,y})_{u,v} = 1$ when $v = (y, e)$ and $u = (z_{\mathrm{occ}}, z_{\mathrm{idx}} + e)$ for some $e \in \{(0,0), (2,0), (0,2), (2,2)\}$, and 0 otherwise. Shifting u and v by the same half-turn keeps this pattern, so $U_{z,y} \in \mathscr{A}_{\mathrm{R}}$. It carries the half-turn orbit of the cell $(y, (0,0))$ onto that of z.

Lemma 5.20 (One transfer per posting)**.** *Let $x, y \in \mathcal{O}$, $j = j(y,x)$, h a record of length k, $n \in \mathbb{Z}$ and z a cell. Then*

$$U_{z,y}[k+1, h\cdot j]\ \chi_{(k,\, n-j_2,\, (x,(0,0)),\, h)} = (M_{yx})_{(0,0),(0,0)}\ \chi_{(k+1,\, n,\, z,\, h\cdot j)}, \qquad (M_{yx})_{(0,0),(0,0)} = \pm\tfrac{1}{4}.$$

Proof. The coordinate vector is ι_{k+1} of the posting of $\delta_{n-j_2} \otimes \delta_h \otimes |x,(0,0)\rangle$. The placed read keeps only the record $h \cdot j$, where the posting is $\delta_n \otimes \delta_{h\cdot j} \otimes |y\rangle \otimes M_{yx} e_{(0,0)}$; the input and the label fix the output y. Each clock map $K(b', b)$ has nonzero entries only at (r, r) and $(r, r+1)$, so the column $(0,0)$ of M_{yx} is supported on the clock indices $\{0,3\}^2$. Of these, only $(0,0)$ lies in the half-turn orbit $\{(0,0), (2,0), (0,2), (2,2)\}$, so $U_{z,y}$ keeps exactly the entry $(M_{yx})_{(0,0),(0,0)}$ and moves it to the cell z. That entry is the leading sign times $K(y_1, x_2)_{00} K(y_2, x_1)_{00} = \frac{1}{4} s(x_2) s(x_1)$, because $D_{00} = 1$. □

Theorem 5.21 (Reads reach every coordinate). *For the placed reads and the algebra $\overline{\mathscr{G}}_{\mathrm{pl}}$:*

(i) For every coordinate $a = (k, n, z, h)$ there are a genesis coordinate state $\nu = \delta_m \otimes |z_0\rangle$ (with $z_0 = z$ if $k = 0$, and $z_0 = (x, (0,0))$ for some occupation x if $k \geq 1$) and a product $A = A_k \cdots A_1$ of placed reads, A_t placed at stage t and at the first t labels of h, each the placement of an element of $\mathscr{A}_{\mathrm{R}}$, with $A\,\iota_0\nu = \chi_a$. This holds for orphan coordinates as well.
(ii) The closed span of the vectors $A\xi$, $A \in \overline{\mathscr{G}}_{\mathrm{pl}}$, ξ in the genesis whole, is $\mathscr{H}_\infty$.
(iii) The only closed subspace that contains the genesis whole and is invariant under $\overline{\mathscr{G}}_{\mathrm{pl}}$ is $\mathscr{H}_\infty$.

Proof. (i) Induction on k; at $k = 0$ take $A = 1$ and $\nu = \delta_n \otimes |z\rangle$; the step below uses only targets with clock index $(0,0)$. For $a = (k+1, n, z, h \cdot j)$ choose (y, x) with $j(y, x) = j$, which exists for every label, and by induction A' and ν with $A'\iota_0\nu = \chi_{(k, n-j_2, (x,(0,0)), h)}$. By Lemma 5.20, $A = (M_{yx})^{-1}_{(0,0),(0,0)}\, U_{z,y}[k+1, h\cdot j]\, A'$ satisfies $A\iota_0\nu = \chi_a$. Nothing in the step uses the genesis compatibility of a. (ii) Each coordinate vector lies in the span by (i), and coordinate vectors are total. (iii) Such a subspace contains every $A\iota_0\nu$, hence by (ii) everything. □

The genesis coordinate used in (i) sits at the original loop $m = n - \mathrm{acc}(h)$ of the target; its occupation, the first input chosen along the record, need not be compatible with the record at all, and that is how orphans are reached.

Example 5.22 (An orphan of root stage seven). Let $j = (1,0,0) = j(10,00)$ and $\nu = \delta_0 \otimes |00\rangle \otimes e_0 \otimes e_0$. The coordinate $o = (8, 0, (10, (0,0)), j^8)$, eight postings of j ending in occupation 10, is an orphan: its last label reads back to the input 00, and no input writes $(1,0,0)$ into 00: the labels written into 00 are $(0,0,0)$, $(0,1,-1)$, $(-1,-1,0)$ and $(-1,0,-1)$. Read-back stops at stage 7, and $\langle \chi_o, \iota_0 w\rangle = 0$ for every genesis state w.

Seven transfers $U_{(00,(0,0)),10}[m, j^m]$, $m = 1, \ldots, 7$, each return the occupation from 10 to 00 and carry $\iota_0\nu$ to $(-\frac{1}{4})^7 \chi_{(7,0,(00,(0,0)),j^7)}$, since $(M_{10,00})_{(0,0),(0,0)} = -\frac{1}{4}$. The overlap of that vector with χ_o is one more posting coefficient, $-\frac{1}{4}$, so the product A of the seven reads gives

$$\langle \chi_o, A\,\iota_0\nu\rangle = 4^{-8} = \frac{1}{65536}.$$

The retained read alone does the same. Place R itself at the records j^m, $m = 1, \ldots, 7$. Only the component on occupation 00 can post j again, $M_{10,00} = -(\mathrm{flip} \otimes \mathrm{stay})$, and the entry formula gives the block of R from 10 to 00 as $-\frac{1}{4}(L^2 \otimes I)$. The amplitude on o is therefore the $((0,0),(0,0))$ entry of

$$M_{10,00}\big(-\tfrac{1}{4}(L^2 \otimes I)\, M_{10,00}\big)^7 = -\big(\tfrac{1}{4}\big)^7 \big(\mathrm{flip}(L^2\,\mathrm{flip})^7\big) \otimes \mathrm{stay}^8.$$

By direct exact computation $\big(\mathrm{flip}(L^2\mathrm{flip})^7\big)_{00} = -\frac{1}{32}$ and $(\mathrm{stay}^8)_{00} = \frac{1}{32}$; the second also follows from $\mathrm{stay}^4 = \frac{1}{2}\mathrm{stay}^2$ and $(\mathrm{stay}^2)_{00} = \frac{1}{4}$. The amplitude is

$$\big\langle \chi_o,\ \mathrm{R}[7, j^7] \cdots \mathrm{R}[1, j]\, \iota_0\nu\big\rangle = \big(\tfrac{1}{4}\big)^7 \cdot \frac{1}{1024} = 2^{-24}.$$

A single read can already leave the actual range, and we can say exactly which generators do. Say that the occupation y is *genesis-compatible at* the record h when the coordinates $(k, n, (y, i), h)$ are genesis-compatible.

Proposition 5.23 (Which single reads keep the genesis whole)**.** *Let h be a record of length k, w a word, and A either $V_w^* V_w$ or $V_w^* \mathrm{R} V_w$. The placed read $A[k,h]$ maps the genesis whole into itself iff one of the following holds:*

(a) $A = V_w^* V_w$*;*
(b) $k = 0$*;*
(c) w *contains a nonzero label;*
(d) w *is empty, so* $A = \mathrm{R}$*, and no occupation is genesis-compatible at* h*;*
(e) w *is a nonempty word of zero labels, and the occupations* 00 *and* 11 *are both genesis-compatible at* h *or both not.*

Proof. A vector $\iota_k w'$ with $w' \in \mathscr{H}_k$ lies in the genesis whole iff w' has no orphan coefficients (Lemma 5.3 one way, orthogonality to orphan vectors the other), and ι_k^* maps the genesis whole onto the stage-k vectors without orphan coefficients. So $A[k,h]$ keeps the genesis whole iff $A_{u,v} = 0$ whenever the occupation of v is genesis-compatible at h and that of u is not. (a) $V_w^* V_w$ is block diagonal in the occupation, because a label and an output fix the input. (b) A read placed at stage 0, whatever its matrix, maps $\mathscr{H}_\infty$ into $\iota_0(\mathscr{H}_0)$, which lies in the genesis whole. (c) Every nonzero label is written by one transition only, so V_w has a single input occupation x and $V_w^* \mathrm{R} V_w$ is supported on x. (d) R connects every occupation to every other, and every record of positive length has an occupation that is not genesis-compatible, since each label is written into at most two occupations. (e) V_{0^m} acts on 00 and 11 only, by $(-1)^m \,\mathrm{stay}^m \otimes \mathrm{stay}^m$ on each, and R maps the 00 block to the 11 block by $\frac{1}{4} L^2 \otimes L^2$. So the cross block of $V_{0^m}^* \mathrm{R} V_{0^m}$ is $\frac{1}{4}\, C_m \otimes C_m$ with $C_m = (\mathrm{stay}^m)^\top L^2 \,\mathrm{stay}^m$, and likewise in the other direction. Its entry $(C_m)_{0,2}$ is $-\frac{1}{2}, \frac{3}{8}, -\frac{3}{16}$ for $m = 1, 2, 3$, and $C_{m+4} = \frac{1}{4} C_{m+2}$ from $\mathrm{stay}^4 = \frac{1}{2}\mathrm{stay}^2$, so it never vanishes. The read therefore keeps the genesis whole exactly when 00 and 11 are compatible together or not at all. □

These are statements about operator images. Reading $A\iota_0 v / \|A\iota_0 v\|$ as the state after a read with outcome A is the Lüders update, which the read algebra does not supply; on that reading, seven successive retained outcomes would place a posted history, with amplitude 2^{-24} in the unnormalized image, on a record that no posted history populates. Without it, Theorem 5.21 says that the distinction between actual and orphan coordinates, and the boundaries of Section 5.1, belong to unread posted evolution and to the coordinate events, not to $\overline{\mathscr{G}}_{\mathrm{pl}}$.

Sections 5.1 to 5.6 take the admitted operations to be the norm-closed algebra of the coordinate events; which algebra describes the operations available inside the ledger is open, and the native reads cross these walls. Chapter 6 takes up the algebra of those reads, together with the pointer whose projections are the spectral projections of the half-turns found in Section 5.5.

Chapter 6

Beyond the events: retained reads and the pointer

The coordinate events of Chapter 4 ask where a history is, and nothing finer. Recognition dynamics supplies more reads than the events. A current word can be read at a stored record; the stored record can itself be read; and the native operation carries a read of its own, the retained read R [3]. This chapter asks what these reads see.

Two answers organize it. Every read that consults only the record keeps each atom in place, and the retained read does not. Inside every atom each clock cell carries a binary label, its half-turn sector, which no event-compatible read sees and every native read respects; we call it the *pointer*.

Besides the retained read and the placed-read grammar, supplied by recognition dynamics (Chapter 5), one item enters as given, and we name it here once. A read placed at stage k acts on the stage-k part of a history and is zero on the directions born later; we call this the *corner convention*. Every statement below concerns operators; the Lüders reading of an image (Chapter 5) enters only where we say so.

6.1 Reads placed on the record

We use the complete history, coordinates, events and atoms of Chapter 4, and the retained read R with its entries (5.3), the effects E_w and retained effects E_w^{R}, the live read algebra $\mathscr{A}_{\mathrm{R}}$ and the placed reads $A[k,h]$ of Definition 5.19; the current algebra $\mathscr{A}$, generated by the E_w, is that of Chapter 3. Records are written oldest label first, and 0^k is the record of k zero labels.

Definition 6.1 (Native reads)**.** For $m \geq 0$ and $g : \mathscr{J}^m \to \mathbb{C}$ the *archive read* $\mathrm{Ar}_m(g)$ is the bounded operator with

$$\mathrm{Ar}_m(g)\,\iota_k\xi = \iota_k\xi', \qquad \xi'(n,h) = g(h_{\leq m})\,\xi(n,h) \qquad (k \geq m),$$

where $h_{\leq m}$ is the oldest m labels of h. The *native read algebra* $\mathscr{G}$ is the unital $*$-algebra generated by the archive reads and the placed reads $E_w[k,h]$, $E_w^{\mathrm{R}}[k,h]$; $\overline{\mathscr{G}}$ is its operator-norm closure. For $n \geq 0$ the *late reads* $\overline{\mathscr{G}}_{\geq n}$ form the norm-closed $*$-algebra generated by the archive reads and the placed reads $E_w[m,h]$, $E_w^{\mathrm{R}}[m,h]$ with $m \geq n$; thus $\overline{\mathscr{G}}_{\geq 0} = \overline{\mathscr{G}}$ and $\overline{\mathscr{G}}_{\geq n+1} \subseteq \overline{\mathscr{G}}_{\geq n}$.

The archive read is well defined because posting keeps every old label, so multiplying by $g(h_{\leq m})$ commutes with T, and the stages $k \geq m$ have dense union. A placed read is zero on $\iota_k(\mathscr{H}_k)^{\perp}$: its certain event is the stage projection $Q_k = \iota_k\iota_k^*$, not the identity. The algebra $\overline{\mathscr{G}}$ contains the placed-read algebra $\overline{\mathscr{G}}_{\mathrm{pl}}$ of Chapter 5, which has the same placed generators and no archive reads. Since $A \mapsto A[k,h]$ is multiplicative and $*$-preserving, every $A[k,h]$ with

$A \in \mathscr{A}_{\mathrm{R}}$ lies in $\mathscr{G}$, and in $\overline{\mathscr{G}}_{\geq n}$ when $k \geq n$. By (5.3) R couples occupations, while every E_w is block diagonal in the occupation, because output and label fix the input.

Definition 6.2. A bounded operator B on $\mathscr{H}_\infty$ *keeps atoms* if $B\,B_a \subseteq B_a$ for every coordinate a.

Proposition 6.3 (Atom-keeping reads see mixtures)**.** *(i) B keeps atoms if and only if $B\chi_a \in B_a$ for every coordinate a. (ii) If B keeps atoms, $\xi \in B_a$, $\xi' \in B_b$ and $a \nsim b$, then for every θ*

$$\langle \xi + e^{i\theta}\xi',\ B(\xi + e^{i\theta}\xi')\rangle = \langle \xi, B\xi\rangle + \langle \xi', B\xi'\rangle.$$

Proof. (i) An atom is the closed span of its coordinate vectors and B is continuous. (ii) $B\xi' \in B_b \perp B_a \ni \xi$ and $B\xi \in B_a \perp \xi'$, so both cross terms vanish. □

Theorem 6.4 (Record reads keep atoms)**.** *Each of the following keeps every atom, orphan atoms included: (i) every element of the norm-closed event algebra $\overline{\mathscr{E}}$, and every bounded operator commuting with all coordinate events, in particular every loop readout (multiplication of each χ_a by $f(\ell(a))$ for a bounded f on $\mathbb{Z}$); (ii) every archive read, and every stage read $\iota_k(1 \otimes \mathrm{diag}(d) \otimes I)\iota_k^*$ that multiplies each stored record h by a number $d(h)$; (iii) every placed read $A[k,h]$ whose matrix A is block diagonal in the occupation, in particular every $E_w[k,h]$; (iv) every operator-norm limit of operators that keep atoms.*

Proof. For a stage k and any coordinate a, $\iota_k^*\chi_a = \sum_b \langle \chi_b, \chi_a\rangle \delta_b$ over stage-k basis vectors δ_b with $\iota_k\delta_b = \chi_b$, and only $b \sim a$ contribute. (i) Atoms are boundaries, and an operator commuting with all events is one scalar on each atom (Chapter 4). (ii) On a coordinate born at stage $k \geq m$ the archive read multiplies by a number; a coordinate born earlier is $\chi_a = \iota_m T^{m-k}\delta_a$, a combination of stage-m coordinates overlapping a, each multiplied by a number. The stage read sends χ_a to $\sum_b \langle \chi_b, \chi_a\rangle d(h_b)\chi_b$. (iii) A sends $\delta_{(k,n,(x,i),h)}$ into the span of the $\delta_{(k,n,(x,i'),h)}$, connected to it by Lemma 4.8(ii). (iv) Atoms are closed. □

By Proposition 6.3, events, loop readouts, archive reads, record masks and placed current-word reads all see the genesis whole as the mixture of its lineage atoms, with weights $\|Q_{\Lambda(x,n)}\iota_0\nu\|^2$.

6.2 The retained read sees the genesis whole

The retained read is not block diagonal in the occupation, and Theorem 6.4 does not apply to it. One pair of coordinates shows what it sees. The zero label $(0,0,0)$ is written only by the balanced self-transitions $00 \to 00$ and $11 \to 11$, so the all-stay record 0^k reads back to 00 from occupation 00 and to 11 from occupation 11.

Theorem 6.5 (The retained read sees the phase between lineages)**.** *Let $k \geq 0$ and $n \in \mathbb{Z}$, and put $\chi_{\mathrm{lo}} = \chi_{(k,n,(00,(0,0)),0^k)}$ and $\chi_{\mathrm{hi}} = \chi_{(k,n,(11,(2,2)),0^k)}$. (i) $\chi_{\mathrm{lo}} \in \Lambda(00,n)$ and $\chi_{\mathrm{hi}} \in \Lambda(11,n)$, two distinct atoms. (ii) For $\xi_\theta = \chi_{\mathrm{lo}} + e^{i\theta}\chi_{\mathrm{hi}}$, $\langle \xi_\theta, \mathrm{R}[k,0^k]\,\xi_\theta\rangle = \frac{1}{2}(1+\cos\theta)$. So $\mathrm{R}[k,0^k] \in \mathscr{G}$ does not keep atoms, at every stage. (iii) Let $k \geq 1$. If a placed read $A[k,h]$ has a nonzero matrix element between two genesis-compatible coordinates of stage k with original occupations o and o', then $o_1 - o_2 = o_1' - o_2'$, that is, $o = o'$ or $\{o,o'\} = \{00,11\}$; and 00, 11 do share every all-stay record.*

Proof. (i) Both coordinates are genesis-compatible with the original occupations stated, and atoms of distinct original occupations differ (Chapter 4). (ii) By (5.3) the four entries of R among the cells $(00,(0,0))$ and $(11,(2,2))$ equal $\frac{1}{4}$: on the diagonal nothing moves, and off it each channel moves its bit by 1 and its index by 2, contributing $(-1)^2$. The two coordinates are orthonormal, and $\mathrm{R}[k,0^k]$ acts on the record-0^k slice by R, so the value is $\frac{1}{4}(1+1+e^{i\theta}+e^{-i\theta})$. (iii) A placed read acts at one record h. Reading h back to genesis, its oldest label is $j(y,o)$ for an output y, with middle entry o_1-o_2. Two lineages sharing h therefore have the same middle current, which for $o\neq o'$ forces $\{o,o'\}=\{00,11\}$. □

Remark 6.6 (Two further phase-sensitive reads). A read built from the native positive clock also sees inter-lineage phase. Let the cell dwell under the clock generator H_E, the native clock with any real diagonal field energy E added (zero allowed), for a time $t_*>0$ fixed by the rate bound $1+\|H_E\|$; then post once and read the event that the loop-edge current is -1, that is, that the input occupation was 01. On the unit genesis states $(\chi_L\pm i\chi_R)/\sqrt{2}$, with χ_L, χ_R the stage-0 coordinates of the cells $(01,(0,0))$ and $(00,(0,1))$ at one loop, in the distinct atoms $\Lambda(01,n)$ and $\Lambda(00,n)$, its expectation on the $-$ state exceeds that on the $+$ state by at least t_*. A third read applies a supplied rotation of the two mixed occupations, not built from native words, before reading occupation 01; it reads $1-\sin\theta$ on $\chi_{(0,n,(10,(0,0)),\varnothing)}+e^{i\theta}\chi_{(0,n,(01,(0,0)),\varnothing)}$. After the first posting this read is blind to the phase between any two genesis lineage atoms, because no current is written from both mixed occupations: from a genesis-compatible coordinate of the cell $(10,(0,0))$ it moves amplitude $-i/2$ onto an orphan coordinate instead. The retained read, a real symmetric read, sees only $\cos\theta$; the dwell read and the rotated read separate phases that differ in $\sin\theta$.

6.3 Reads do not keep actual histories actual

An actual history is $\iota_0 v$ for a nonzero genesis state v, and its boundary is the join $\Lambda(\operatorname{supp} v)$ of the lineage atoms it populates (Chapter 5). Orphan atoms hold no actual history. Does a read applied to an actual history leave a vector whose boundary is again that of some genesis state?

Theorem 6.7 (The retained read writes orphan records)**.** *(i) Every operator B that keeps atoms maps each lineage join $\Lambda(L)$ into itself. Hence for every genesis state v, $B\iota_0 v$ is zero or has the boundary of a nonzero genesis state. This applies to every read of Theorem 6.4. (ii) Every read placed at stage* 0*, whatever its matrix, maps $\mathscr{H}_\infty$ into the genesis whole $\Lambda(\mathcal{O}\times\mathbb{Z})$ (proof of Proposition 5.23(b)). The retained read $\mathrm{R}[k,0^k]$ still moves weight between lineage atoms there and at every later stage: it sends $\chi_{\mathrm{lo}}\in\Lambda(00,n)$ out of $\Lambda(\{00\}\times\mathbb{Z})$. (iii) Let $k\geq 1$, $v=\delta_n\otimes|00\rangle\otimes e_0\otimes e_0$ and $o_k=(k,n,(10,(2,0)),0^k)$. Then o_k is an orphan, its own root, and*

$$\langle\chi_{o_k},\,\mathrm{R}[k,0^k]\,\iota_0 v\rangle=-\tfrac{1}{4}(-1)^k\big((\mathrm{stay}^k)_{00}\big)^2\neq 0,$$

which is $\frac{1}{16}$ at $k=1$. The boundary of $\mathrm{R}[k,0^k]\iota_0 v$ is that of no genesis state. So $\mathrm{R}[k,0^k]$ keeps every actual history actual if and only if $k=0$. (iv) Let $k\geq 1$ and y balanced. From $\chi_{(k,n,(y,i),0^k)}$ the read $\mathrm{R}[k,0^k]$ reaches, with nonzero amplitude, an orphan coordinate at occupation y' if and only if y' is mixed.

Proof. (i) $\Lambda(L)$ is a closed join of atoms; an image that is a nonzero join of lineage atoms is the boundary of a genesis state (Chapter 5). (ii) For the second claim: by (5.3), R sends the cell $(00,(0,0))$ onto $(11,(2,2))$ with amplitude $\frac{1}{4}$. (iii) The record-0^k part of $T^k v$ is $|00\rangle \otimes M^k_{00,00}(e_0 \otimes e_0)$ with $M_{00,00} = -\text{stay} \otimes \text{stay}$; only the 00 cells carry amplitude at that record, since the zero label is written only by $00 \to 00$ and $11 \to 11$. Row $(10,(2,0))$ of R meets the 00 cells only at $(0,0)$, with $-\frac{1}{4}$. The clock map stay is singular (it kills the uniform vector p) but satisfies $\text{stay}^4 = \frac{1}{2}\text{stay}^2$, so $(\text{stay}^{k+2})_{00} = \frac{1}{2}(\text{stay}^k)_{00}$ for $k \geq 2$; with $(\text{stay}^k)_{00} = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ for $k = 1, 2, 3$, the entries are never zero. The coordinate o_k is an orphan because its newest label, the zero label, enters no mixed occupation; its read-back stops at once. The image responds to o_k, so its boundary contains the orphan atom B_{o_k} and is no lineage join. At $k = 0$, part (ii) applies. (iv) This is the proof of Proposition 5.23(d) at the record 0^k. There R sends the cell (y, i) to the cell (y', i'), $i'_c = i_c + 2[y'_c \neq y_c]$, with amplitude $\pm\frac{1}{4}$ for every y', and the occupations that are not genesis-compatible at 0^k are exactly the mixed ones: a mixed occupation never receives the zero label, and a balanced y' receives it from itself and reads back along 0^k to y'. □

Under the Lüders reading, a retained read performed on an actual history produces a state outside every actual boundary; without it, the theorem says that $\mathrm{R}[k, 0^k]$ does not commute with the projection onto the genesis whole.

6.4 The pointer: a half-turn label carried by the posting

The pointer starts from one fact about a single clock cell. The half-turn L^2, which sends e_i to e_{i+2}, anticommutes with every clock map $K(b', b)$ (proof of Proposition 2.13): one free tick moves a cell from one eigenspace of L^2 to the other. By the pole table, $L^2 p = p$, $L^2 a = a$, $L^2 P = -P$ and $L^2 A = -A$, so the eigenspaces are the two planes $\mathcal{U} = \text{span}\{p, a\}$ ($L^2 = +1$) and $D\mathcal{U} = \text{span}\{P, A\}$ ($L^2 = -1$).

Definition 6.8 (Pointer sectors on the cell)**.** For a clock cell $c \in \{1, 2\}$ let L^2_c be the half-turn acting on that cell of $\mathscr{H}$, and put

$$\Pi^{(c)}_1 = \tfrac{1}{2}(1 + L^2_c), \qquad \Pi^{(c)}_0 = \tfrac{1}{2}(1 - L^2_c),$$

the projections onto $\mathcal{U}$ and onto $D\mathcal{U}$ in cell c, every occupation and the other cell retained. For $s = (s_1, s_2)$ the *joint sector* is $\Pi_s = \Pi^{(1)}_{s_1}\Pi^{(2)}_{s_2}$, of rank 16. Write $\varsigma_b = 2b - 1$, so that $\Pi^{(c)}_b = \frac{1}{2}(1 + \varsigma_b L^2_c)$.

These are not the pole projections $\Pi^x_{\pm,0}$ of Chapter 3, nor the pointer lines of Chapter 10; Remark 6.20 relates the first two. A label j is *silent* when its middle current vanishes, $j_2 = 0$, that is, when it was written from a balanced occupation; $\mathrm{sil}(h)$ counts the silent labels of a record h.

Lemma 6.9 (The step and the half-turn)**.** *For all occupations x, y and $c \in \{1, 2\}$,*

$$L^2_c\, M_{yx}\, L^2_c = \epsilon(x)\, M_{yx}, \qquad \epsilon(x) = \begin{cases} -1, & x \text{ balanced}, \\ +1, & x \text{ mixed}. \end{cases}$$

Consequently $\Pi^{(c)}_{b'} M_{yx} = M_{yx} \Pi^{(c)}_b$ with $b' = 1 - b$ when $j(y, x)$ is silent and $b' = b$ otherwise.

Proof. The slice V_j carries the block M_{yx} at (y,x) for each transition with $j(y,x)=j$, and $j_2(y,x)=x_1-x_2$ vanishes exactly for balanced x; so the identity is Proposition 2.13 read block by block. Then $M\Pi_b^{(c)}=\frac{1}{2}(M+\varsigma_b ML_c^2)=\frac{1}{2}(M+\varsigma_b\epsilon(x)L_c^2M)=\Pi_{b'}^{(c)}M$ with $\varsigma_{b'}=\epsilon(x)\varsigma_b$. □

So a posting flips each cell's pointer when its label is silent and keeps it otherwise. Along a record the label is carried as a parity: starting from $b\in\{0,1\}$, the label after a record h is $b\cdot h=b+\mathrm{sil}(h)\bmod 2$, as in Chapter 3. Chapter 3 reads $\mathrm{sil}(h)$ as the silence parity of the field-current account; a single record thus fixes the change of the pointer, never its initial value (the record law does determine the sector weights, Theorem 6.21).

Proposition 6.10 (The completed pointer)**.** *Let $c\in\{1,2\}$ and $b\in\{0,1\}$. (i) The stage operator $\Pi_{k,b}^{(c)}$ on $\mathscr{H}_k$, $(\Pi_{k,b}^{(c)}\xi)(n,h)=\Pi_{b\cdot h,b}^{(c)}\xi(n,h)$, is an orthogonal projection and $T\Pi_{k,b}^{(c)}=\Pi_{k+1,b}^{(c)}T$. (ii) There is a unique bounded operator $\widehat{\Pi}_b^{(c)}$ on $\mathscr{H}_\infty$ with $\widehat{\Pi}_b^{(c)}\iota_k=\iota_k\Pi_{k,b}^{(c)}$ for all k. It is an orthogonal projection, $\widehat{\Pi}_0^{(c)}+\widehat{\Pi}_1^{(c)}=1$, and on coordinates*

$$\widehat{\Pi}_b^{(c)}\chi_a=\tfrac{1}{2}\big(\chi_a+\varsigma_{b\cdot h}\chi_{a^{\sharp c}}\big),\qquad a=(k,n,(x,i),h),\quad a^{\sharp c}=(k,n,(x,i^{\sharp c}),h),$$

where $i^{\sharp 1}=(i_1+2,i_2)$ and $i^{\sharp 2}=(i_1,i_2+2)$. The pointer charge *$\zeta_c=\widehat{\Pi}_1^{(c)}-\widehat{\Pi}_0^{(c)}$ acts by $\zeta_c\chi_a=(-1)^{\mathrm{sil}(h)}\chi_{a^{\sharp c}}$, and $\widehat{\Pi}_b^{(c)}=\frac{1}{2}(1+\varsigma_b\zeta_c)$. (iii) If a genesis state ν lies in sector b of cell c, $\Pi_b^{(c)}\nu(n)=\nu(n)$ for all n, then $\Pi_{k,b}^{(c)}T^k\nu=T^k\nu$ for every k, and $\widehat{\Pi}_b^{(c)}\iota_0\nu=\iota_0\nu$.*

Proof. (i) On each slice (n,h) the operator is an orthogonal projection. The component of $T\xi$ at $(n,h\cdot j)$ and output y is $\sum_x[j(y,x)=j]M_{yx}\xi(n-j_2,h)_x$, and $b\cdot(h\cdot j)=1-b\cdot h$ exactly when j is silent, so Lemma 6.9 gives the intertwining slice by slice. (ii) The family is compatible with the embeddings and bounded by 1, so it extends by continuity from the dense union $\bigcup_k\iota_k(\mathscr{H}_k)$; idempotence and self-adjointness pass to the extension. The coordinate formula is the entry formula of $\Pi_{b\cdot h}$; subtracting the two sectors gives $\frac{1}{2}(\varsigma_{1+\mathrm{sil}(h)}-\varsigma_{\mathrm{sil}(h)})=(-1)^{\mathrm{sil}(h)}$. (iii) Induction on k with (i). □

Part (iii) maintains a sector the genesis state already occupies; it does not choose one.

6.5 The pointer splits every atom, unseen by the events

The completed pointer acts at fixed stage, loop, occupation and record and moves only the internal index. By Lemma 4.8(ii) that keeps it inside every atom. The coordinate events, on the other hand, resolve the internal index, and the pointer mixes it.

Theorem 6.11 (The pointer inside the atoms)**.** *Let $c\in\{1,2\}$ and $b\in\{0,1\}$. (i) $\widehat{\Pi}_b^{(c)}B_a\subseteq B_a$ for every coordinate a, genesis-compatible or orphan; every atom is the orthogonal sum of its two sector parts $\widehat{\Pi}_0^{(c)}B_a$ and $\widehat{\Pi}_1^{(c)}B_a$. (ii) Both parts of every atom are nonzero: $\widehat{\Pi}_b^{(c)}\chi_a\neq 0$. (iii) $\widehat{\Pi}_b^{(c)}$ commutes with no coordinate event. (iv) The range of $\widehat{\Pi}_b^{(c)}$ contains no nonzero boundary. In particular neither the range nor its intersection with any atom is a boundary. (v) $\widehat{\Pi}_b^{(c)}$ commutes with every bounded operator that commutes with all coordinate events.*

Proof. (i) By Proposition 6.10(ii), $\widehat{\Pi}_b^{(c)}\chi_a$ is a combination of χ_a and $\chi_{a^{\sharp c}}$, connected by Lemma 4.8(ii); use Proposition 6.3(i) and $\widehat{\Pi}_0 + \widehat{\Pi}_1 = 1$. (ii) $\langle \chi_a, \widehat{\Pi}_b^{(c)}\chi_a\rangle = \frac{1}{2}$, since $a \neq a^{\sharp c}$ are orthonormal. (iii) If F_a commuted with it, $\widehat{\Pi}_b^{(c)}\chi_a = \widehat{\Pi}_b^{(c)}F_a\chi_a$ would lie in $\mathbb{C}\chi_a$; its component along $\chi_{a^{\sharp c}}$ is $\pm\frac{1}{2}$. (iv) A nonzero boundary contains a coordinate vector χ_a (Chapter 4), and $\widehat{\Pi}_b^{(c)}\chi_a \neq \chi_a$. The range and its atom parts are nonzero by (ii). (v) Such an operator is one scalar on each atom (Chapter 4), and by (i) the pointer maps each atom, and the closed join of all the others, into itself. □

By (v) the pointer lies in the double commutant of the coordinate events, so by von Neumann's bicommutant theorem it is a strong limit of elements of the event algebra. It is not in the norm-closed event algebra $\overline{\mathscr{E}} \subseteq \mathbb{C}1 + \mathscr{K}$ (proof of Theorem 4.21(iii)): by (i) and (ii) both $\widehat{\Pi}_b^{(c)}$ and $1 - \widehat{\Pi}_b^{(c)} = \widehat{\Pi}_{1-b}^{(c)}$ have nonzero range in each of the infinitely many orthogonal atoms, so both ranges are infinite-dimensional, and $\widehat{\Pi}_b^{(c)} = \lambda 1 + Z$ with Z compact would make $(1-\lambda)1$ and $\lambda 1$ compact on them. The events generate the pointer only through their noncommuting structure inside atoms: a pointer sector is not a boundary, and the pointer is not a reading of which atom is occupied.

Theorem 6.12 (Event-compatible reads cannot see the pointer)**.** *Let B be a bounded operator commuting with every coordinate event, and let $c \in \{1,2\}$. (i) For each atom there is a number λ with $\langle \xi, B\xi\rangle = \lambda\|\xi\|^2$ for every ξ in the atom. So B has one expectation on all unit vectors of an atom, whichever pointer sector they occupy, and $\langle \widehat{\Pi}_0^{(c)}\xi, B\widehat{\Pi}_1^{(c)}\xi\rangle = 0$ for ξ in an atom. (ii) An operator with two different expectations on unit vectors of one atom fails to commute with some coordinate event. In every atom there are unit vectors u_0, u_1 with $\langle u_0, \widehat{\Pi}_0^{(c)}u_0\rangle = 1$ and $\langle u_1, \widehat{\Pi}_0^{(c)}u_1\rangle = 0$. (iii) The pointer $\widehat{\Pi}_b^{(c)}$ commutes with every stage read $\iota_k(1 \otimes \mathrm{diag}(d) \otimes I)\iota_k^*$ of the record, in particular with the record events $C_{k,S} = \iota_k(1 \otimes 1_S \otimes I)\iota_k^*$, $S \subseteq \mathscr{J}^k$; with every loop readout; and with every index-blind event $\sum_i F_{(k,n,(x,i),h)}$. (iv) For a genesis state ν, $k \geq 0$ and $S \subseteq \mathscr{J}^k$, the product $\widehat{\Pi}_b^{(c)}C_{k,S}$ is an orthogonal projection, and*

$$\|\widehat{\Pi}_b^{(c)}C_{k,S}\iota_0\nu\|^2 = \sum_{n\in\mathbb{Z}}\sum_{h\in S}\|\Pi_{b\cdot h}^{(c)}(T^k\nu)(n,h)\|^2.$$

Summed over b it is the record-branch weight $\|C_{k,S}\iota_0\nu\|^2$; summed over S and its complement it is the sector weight $\|\widehat{\Pi}_b^{(c)}\iota_0\nu\|^2$.

Proof. (i) B is a scalar λ on the atom, and $\widehat{\Pi}_1^{(c)}\xi$ stays in the atom by Theorem 6.11(i), so $B\widehat{\Pi}_1^{(c)}\xi = \lambda\widehat{\Pi}_1^{(c)}\xi \perp \widehat{\Pi}_0^{(c)}\xi$. (ii) The first claim is (i) read backwards; take $u_b = \widehat{\Pi}_b^{(c)}\chi_a/\|\widehat{\Pi}_b^{(c)}\chi_a\|$. (iii) Since the pointer is self-adjoint, $\iota_k^*\widehat{\Pi}_b^{(c)} = \Pi_{k,b}^{(c)}\iota_k^*$, so $\widehat{\Pi}_b^{(c)}$ commutes with $\iota_k X\iota_k^*$ whenever $\Pi_{k,b}^{(c)}$ commutes with X. The stage pointer acts slice by slice, so it commutes with every multiplication by a function of (n,h) and with $|x\rangle\langle x| \otimes I$ on each slice; this covers record reads and index-blind events, whose stage form is $\delta_n\delta_n^* \otimes |h\rangle\langle h| \otimes |x\rangle\langle x| \otimes I$. Loop readouts commute with all events; apply Theorem 6.11(v). (iv) $\iota_0\nu = \iota_k T^k\nu$, $C_{k,S}\iota_k = \iota_k 1_S$, $\widehat{\Pi}_b^{(c)}\iota_k = \iota_k\Pi_{k,b}^{(c)}$, and ι_k is isometric; the two projections commute by (iii). □

Reading a pointer sector disturbs the coordinate resolution only in its internal-index part, and a single record shows the change of the sector, never the sector itself.

6.6 The four joint sectors are what native reads never superpose

The atoms are superselection sectors of the coordinate events, and Section 6.2 shows that the native reads cross them. The native reads have superselection labels of their own, and they are the pointer's.

Theorem 6.13 (The native superselection labels)**.** *(i) A* 64×64 *matrix commutes with every element of the live read algebra* $\mathscr{A}_{\mathrm{R}}$ *if and only if it is* $\sum_s c_s \Pi_s$ *for numbers* c_s, $s \in \{0,1\}^2$*. The four joint sectors* Π_s *lie in* $\mathscr{A}_{\mathrm{R}}$ *and span its centre. (ii) Every completed pointer* $\widehat{\Pi}_b^{(c)}$*, and so every joint sector* $\widehat{\Pi}_s = \widehat{\Pi}_{s_1}^{(1)} \widehat{\Pi}_{s_2}^{(2)}$*, commutes with every element of* $\overline{\mathscr{G}}$*. (iii) The isometry* $U_{\sigma_m^{\mathrm{tr}}}$ *of every loop translation* $n \mapsto n+m$ *commutes with every element of* $\overline{\mathscr{G}}$*, so* $\langle U_{\sigma_m^{\mathrm{tr}}}\xi, A\, U_{\sigma_m^{\mathrm{tr}}}\xi\rangle = \langle \xi, A\xi\rangle$ *for all* $A \in \overline{\mathscr{G}}$*: the original loop is pure multiplicity for the native reads. (iv) With* $\chi_{\mathrm{lo}}, \chi_{\mathrm{hi}}$ *as in Theorem 6.5,* $\langle \chi_{\mathrm{lo}}, \widehat{\Pi}_s \mathrm{R}[k, 0^k] \widehat{\Pi}_s \chi_{\mathrm{hi}}\rangle = \frac{1}{16}$ *for each of the four s, and* $\widehat{\Pi}_s \mathrm{R}[k, 0^k] \widehat{\Pi}_t = 0$ *for* $s \neq t$*. (v) A bounded operator B on* $\mathscr{H}_\infty$ *commutes with every element of* $\overline{\mathscr{G}}$ *if and only if (a) it commutes with every stage projection* Q_k*; (b) it commutes with the four completed pointers; and (c) for every stage k and loops* n', n *there are numbers* $c(h,s)$ *with*

$$\big\langle \iota_k(\delta_{n'} \otimes \delta_{h'} \otimes |z'\rangle),\ B\, \iota_k(\delta_n \otimes \delta_h \otimes |z\rangle)\big\rangle = [h' = h]\Big(\sum_s c(h,s)\Pi_s\Big)_{z'z}.$$

Proof. (i) Each slice satisfies $L_c^2 V_j L_c^2 = \pm V_j$ by Proposition 2.13, the sign being fixed by j_2; so every E_w commutes with L_1^2 and L_2^2, and so does R, because r commutes with $I \otimes L^2$ on its channel. Hence $\mathscr{A}_{\mathrm{R}}$ is block diagonal in a basis adapted to L_1^2, L_2^2, with four blocks of size 16, and each Π_s commutes with it. The converse, that a matrix commuting with the generators is block diagonal and scalar on each block, is a direct exact computation in that basis. Membership of the Π_s follows from Theorem 6.21 below. (ii) A placed read commutes with the pointer when its matrix commutes with the stage pointer at its record (proof of Theorem 6.12(iii)), which (i) gives; archive reads multiply records by numbers at every later stage. The commutant of the self-adjoint pointer is a norm-closed $*$-algebra. (iii) The loop translation commutes with T, with every placed read and with every archive read, none of which touches the loop. (iv) The joint sectors commute with R and resolve the identity, and their sum of couplings is the coupling $\frac{1}{4}$ of Theorem 6.5; the split into four equal parts is a direct exact computation with (5.3). (v) The stage projections lie in $\overline{\mathscr{G}}$, and the pointers are strong limits of elements of $\mathscr{G}$ (Theorem 6.15(i)); a commutant is strongly closed, which gives (a) and (b), and compressing to stage k and applying (i) record by record gives (c). Conversely, by (a) B and B^* map each $\iota_k(\mathscr{H}_k)$ into itself, so $B\iota_k = \iota_k B_k$ and $\iota_k^* B = B_k \iota_k^*$ with $B_k = \iota_k^* B \iota_k$. By (c), B_k is record-diagonal with a combination of the Π_s at each record, so it commutes with $1 \otimes |h\rangle\langle h| \otimes A$ for $A \in \mathscr{A}_{\mathrm{R}}$ by (i), and with every record multiplication; hence B commutes with every placed read and every archive read, and so with $\overline{\mathscr{G}}$. □

Corollary 6.14. $\mathscr{A}_{\mathrm{R}} = \bigoplus_s \Pi_s \mathscr{A}_{\mathrm{R}} \Pi_s$*, and each summand is the full algebra of operators on the* 16*-dimensional range of* Π_s*.*

Proof. $\mathscr{A}_{\mathrm{R}}$ is generated by self-adjoint matrices and contains 1, so it equals its double commutant. By Theorem 6.13(i) its commutant is spanned by the four orthogonal projections Π_s, whose commutant is $\bigoplus_s \mathrm{End}(\Pi_s \mathscr{H})$. □

Inside one record and one joint sector, then, the native reads are the full matrix algebra: they see every relative phase there, including phases between distinct lineage atoms. Across joint sectors they see none. The kept pair of Theorem 6.5 illustrates both halves: its two coordinates have weight $\frac{1}{4}$ in every joint sector, and the retained read couples them inside each sector and never across.

Theorem 6.15 (Depth of the pointer and of the record). *Let $c \in \{1,2\}$ and $b \in \{0,1\}$. (i) For every k the stage corner $Q_k \widehat{\Pi}_b^{(c)} Q_k = \sum_{h \in \mathcal{J}^k} \Pi_{b\cdot h}^{(c)}[k,h]$ lies in $\mathcal{G}$, and these corners converge strongly to $\widehat{\Pi}_b^{(c)}$ as $k \to \infty$. (ii) Every $X \in \mathcal{G}$ satisfies $\|\widehat{\Pi}_0^{(c)} - X\| \geq \frac{1}{2}$. Hence $\widehat{\Pi}_b^{(c)} \notin \overline{\mathcal{G}}$. (iii) For $n \leq m$, every archive read $\mathrm{Ar}_n(g)$ commutes with every placed read $A[m,h]$ and with every archive read. So the record of length n is central in the late reads $\overline{\mathcal{G}}_{\geq n}$. (iv) Let $\chi_{\mathrm{s}} = \chi_{(1,n,(00,(0,0)),((0,0,0)))}$ and $\chi_{\mathrm{f}} = \chi_{(1,n,(10,(1,0)),((1,0,0)))}$, two stage-1 coordinates with different records, both written from occupation 00. Then $\langle \chi_{\mathrm{s}}, Q_0 \chi_{\mathrm{f}} \rangle = \frac{1}{8}$. The stage projection $Q_0 \in \overline{\mathcal{G}}$ does not commute with the archive read of the first label, so the stored record is not central in $\overline{\mathcal{G}}$.*

Proof. (i) $Q_k \widehat{\Pi}_b^{(c)} Q_k = \iota_k \Pi_{k,b}^{(c)} \iota_k^*$, which splits over records into placed reads of the $\Pi_{b\cdot h}^{(c)} \in \mathscr{A}_{\mathrm{R}}$; and $Q_k \to 1$ strongly. (ii) An element $X \in \mathcal{G}$ involves finitely many placed reads and archive reads; let N exceed all their stages and depths. Let $h = 0^N \cdot (1,0,0)$ and $\psi = \chi_{(N+1,0,(00,(0,0)),h)}$. No input writes $(1,0,0)$ into 00, so the posting puts no amplitude on any coordinate of stage $N+1$, occupation 00 and record h; these coordinates are orthogonal to $\iota_N(\mathscr{H}_N)$. On them every placed read of X vanishes and every archive read multiplies by one number, so by induction over the algebra X acts on their span as a scalar λ. That span contains the unit vectors $u_b = \widehat{\Pi}_b^{(c)}\psi / \|\widehat{\Pi}_b^{(c)}\psi\|$, on which $\widehat{\Pi}_0^{(c)} - X$ acts by $1-\lambda$ and by $-\lambda$; one of these has modulus at least $\frac{1}{2}$. Since $\widehat{\Pi}_1^{(c)} = 1 - \widehat{\Pi}_0^{(c)}$, neither pointer is a norm limit of $\mathcal{G}$. (iii) At every stage $m' \geq m \geq n$ an archive read of depth n multiplies each record by a number, and a placed read acts inside one record. (iv) $\langle \chi_{\mathrm{s}}, Q_0 \chi_{\mathrm{f}} \rangle = \sum_i (M_{00,00})_{(0,0),i} (M_{10,00})_{(1,0),i}$, the overlap of two posting rows from the input 00, which is $\frac{1}{8}$ by direct exact computation. With g the indicator of the zero first label, $\mathrm{Ar}_1(g)\chi_{\mathrm{s}} = \chi_{\mathrm{s}}$ and $\mathrm{Ar}_1(g)\chi_{\mathrm{f}} = 0$, so $\langle \chi_{\mathrm{s}}, \mathrm{Ar}_1(g) Q_0 \chi_{\mathrm{f}} \rangle = \frac{1}{8} \neq 0 = \langle \chi_{\mathrm{s}}, Q_0 \, \mathrm{Ar}_1(g) \chi_{\mathrm{f}} \rangle$. □

So the pointer lies in the commutant of the native reads and in their strong closure, but not in their norm closure: a label maintained at every depth, never one finite read. A stored record is a classical fact for every read made at or after the stage it was written, while a read placed before it, such as Q_0, still sees two records of one ancestor coherently. Whether such an earlier read can act on a history that already carries the newer record is the corner-convention question of Section 6.9; if it cannot, the operations available after stage n are exactly the late reads $\overline{\mathcal{G}}_{\geq n}$, and a record is classical for all of them.

6.7 On actual histories the pointer is a symmetry charge

The half-turn of one clock cell is a symmetry of the posting: with π, κ and f the identities, clock map $\tau^{(c)}(i) = i^{\#c}$ and signs ϵ (-1 on balanced inputs, $+1$ on mixed), Lemma 6.9 is exactly the defining identity of a posting symmetry. This is the half-turn σ_c^{ht} of Chapter 5. Write $U_c^{\#} = U_{\sigma_c^{\mathrm{ht}}}$ for its isometry and $r_c(a) = \eta_{\sigma_c^{\mathrm{ht}}}(a)$ for its record sign, so that $U_c^{\#} \chi_a = r_c(a) \chi_{a^{\#c}}$, with $r_c(a)$ the product of ϵ over the inputs recovered by reading the record of a back, stopping at the

root. Each recovered input is balanced exactly when its label is silent. On a genesis-compatible coordinate every label is recovered, so $r_c(a) = (-1)^{\mathrm{sil}(h)}$, the sign of ζ_c. On an orphan the read-back stops early. Write $\mathrm{root}(a) = (t, m, y, g)$ with root record $g \in \mathscr{J}^t$ (Chapter 4).

Theorem 6.16 (The pointer and the half-turn symmetry)**.** *Let $c \in \{1, 2\}$. (i) $(U_c^\#)^2 = 1$, $U_c^\#$ is self-adjoint, and $U_1^\# U_2^\# = U_2^\# U_1^\#$. (ii) For every coordinate a, $U_c^\# \chi_a = (-1)^{\mathrm{sil}(g)} \zeta_c \chi_a$, with g the root record of a. So $U_c^\# = \zeta_c$ on the closed span of the genesis-compatible coordinates, which is the genesis whole $\Lambda(\mathcal{O} \times \mathbb{Z})$, and there $\widehat{\Pi}_b^{(c)} = \frac{1}{2}(1 + \varsigma_b U_c^\#)$. (iii) At the orphan $o = (1, n, (00, (0, 0)), ((1, 0, 0)))$, whose label is written only by $00 \to 10$ and is silent, $U_c^\# \chi_o = -\zeta_c \chi_o$. Neither $\widehat{\Pi}_b^{(c)} = \frac{1}{2}(1 + \varsigma_b U_c^\#)$ nor $\widehat{\Pi}_b^{(c)} = \frac{1}{2}(1 - \varsigma_b U_c^\#)$ holds on $\mathscr{H}_\infty$. (iv) $U_c^\#$ does not commute with $\overline{\mathscr{G}}$: the retained read placed at the record $((1, 0, 0))$ has amplitude $-\frac{1}{4}$ from the genesis-compatible coordinate $(1, n, (10, (0, 0)), ((1, 0, 0)))$, record sign -1, to the orphan coordinate $(1, n, (00, (2, 0)), ((1, 0, 0)))$, record sign $+1$. Nevertheless $\langle \varphi, U_c^\# A \psi \rangle = \langle \varphi, A U_c^\# \psi \rangle$ for all $A \in \overline{\mathscr{G}}$ and all φ, ψ in the genesis whole. Every $X \in \mathscr{G}$ satisfies $\|U_c^\# - X\| \geq 1$, so $U_c^\# \notin \overline{\mathscr{G}}$. (v) $U_1^\# U_2^\# = \zeta_1 \zeta_2$ on all of $\mathscr{H}_\infty$, and it commutes with $\overline{\mathscr{G}}$. (vi) The isometry of a posting symmetry commutes with every element of $\overline{\mathscr{G}}$ if and only if the symmetry fixes every occupation and has sign $+1$ on 00, if and only if it is a loop translation, possibly composed with the simultaneous half-turn of both clock cells. No posting symmetry has isometry ζ_c.*

Proof. (i) $U_c^\#$ maps an orthonormal stage basis to itself up to signs, and the half-turn is an involution with $r_c(a^{\#c}) = r_c(a)$. (ii) $r_c(a)$ counts the silent labels among the recovered ones, ζ_c counts all silent labels, and the unrecovered labels are those of the root record. The genesis-compatible coordinates are those with root at stage 0; by the root theorem of Chapter 4 they span exactly the lineage atoms. (iii) At o the read-back stops at once, so $r_c(o) = 1$, while $\mathrm{sil} = 1$. The first identity fails at o; the second fails at every genesis-compatible coordinate. (iv) The amplitude is the entry (5.3) with channel 1 moved and channel 2 kept, $\frac{1}{4}(-1)(+1)$. The record sign at $(1, n, (10, (0, 0)), ((1, 0, 0)))$ is $\epsilon(00) = -1$, and the orphan has $+1$; a read coupling coordinates of different record signs at one record cannot commute with $U_c^\#$. Between vectors of the genesis whole, $U_c^\# = \zeta_c$ by (ii), and ζ_c commutes with $\overline{\mathscr{G}}$ by Theorem 6.13(ii). For the bound, take N and the fresh coordinate ψ of the proof of Theorem 6.15(ii): $X\psi = \lambda\psi$, while $U_c^\# \psi$ is a unit vector orthogonal to ψ. (v) The record signs of the two half-turns agree and cancel, $r_1(a) r_2(a) = 1$, while $\zeta_1 \zeta_2 \chi_a = \chi_{(k,n,(x,i+(2,2)),h)}$. Centrality follows from Theorem 6.13(ii). (vi) A symmetry moving an occupation y fails to commute with the occupation read $(|y\rangle\langle y| \otimes I)[0, \varnothing]$, which lies in $\mathscr{G}$ because $|y\rangle\langle y| \otimes I$ commutes with the four Π_s and so lies in $\mathscr{A}_{\mathrm{R}}$ by Corollary 6.14. A symmetry fixing occupations with sign -1 on 00 is detected by the retained read of (iv). By Theorem 5.12(iii),(iv) a symmetry fixing every occupation has $\tau \in \{\mathrm{id}, \tau^{(1)}, \tau^{(2)}, \tau^{(12)}\}$ and signs fixed by τ; sign $+1$ on 00 leaves id and $\tau^{(12)}$, whose record signs are trivial. The remaining symmetries have trivial record signs and act as $U_{\sigma_m^{\mathrm{tr}}}$ or $U_{\sigma_m^{\mathrm{tr}}} \zeta_1 \zeta_2$, which commute with $\overline{\mathscr{G}}$ by Theorem 6.13. Finally, ζ_c commutes with $\overline{\mathscr{G}}$ and moves the internal index of cell c alone, while $U_{\sigma_m^{\mathrm{tr}}}$ and $U_{\sigma_m^{\mathrm{tr}}} \zeta_1 \zeta_2$ do not act that way, so by the classification no posting symmetry has isometry ζ_c. □

On the whole carrier the charge and the symmetry differ: ζ_c reads the silence parity of the record itself, the half-turn reads it only through the occupations the record determines, and the discrepancy is the silence parity of the root record, the part that no history from genesis

explains. Native reads detect the single half-turn only through amplitude they move off the actual histories. The labels that posting symmetries protect from every native read are the original loop and the relative pointer parity $\zeta_1\zeta_2$; each single pointer label remains a superselection label (Theorem 6.13) with no symmetry of the posting behind it on the whole carrier.

6.8 How the posting writes, carries and shows the pointer

The pointer's mechanics are local: one posting acting on one cell. We state them for one recorded step with an arbitrary reference system attached, because that is the generality in which they hold. Let E be any complex Hilbert space, possibly infinite-dimensional, and $(U_j)_{j\in\mathcal{J}}$ any isometries of E, one for each label. A cell state correlated with E is $v \in \mathscr{H} \otimes E$, with occupation components $v_x \in (\mathscr{H}_c \otimes \mathscr{H}_c) \otimes E$. The *recorded step with reference* is

$$(Wv)(y,j) = (1 \otimes U_j) \sum_x [j(y,x) = j]\,(M_{yx} \otimes 1)\, v_x,$$

an isometry because output and label fix the input. One posting T is of this form, with $E = \ell^2(\mathbb{Z}\times\mathcal{J}^k)$ and U_j the shift of the loop by j_2. Every identity below holds for all E and all (U_j); no reference vector is traced out. The *ready space* is $\mathscr{H}_{\text{ready}} = \text{span}\{|a,0\rangle : a \in \{0,1\}\} \otimes \mathscr{U} \otimes \mathscr{U}$: sender bit $x_1 = a$, receiver bit $x_2 = 0$, and both cells in the plane $\mathscr{U}$ where the half-turn is $+1$.

Proposition 6.17 (The posting copies the pointer and does not select it)**.** *Let $c \in \{1,2\}$. (i) For $a \in \{0,1\}$ and every output y, $\Pi_b^{(c)} M_{y,(a,0)} \Pi_1^{(1)}\Pi_1^{(2)} = [b = a]\, M_{y,(a,0)}\Pi_1^{(1)}\Pi_1^{(2)}$: from the ready input with sender bit a, every output lies in sector a of both clock cells. (ii) Let $v_0, v_1 \in (\mathscr{H}_c\otimes\mathscr{H}_c)\otimes E$ and $\psi = \sum_a |a,0\rangle \otimes (\Pi_1^{(1)}\Pi_1^{(2)} \otimes 1)v_a$. Then $(\Pi_b^{(c)} \otimes 1)W\psi = W\big(|b,0\rangle \otimes (\Pi_1^{(1)}\Pi_1^{(2)} \otimes 1)v_b\big)$, and its squared norm is the ready load $\|(\Pi_1^{(1)}\Pi_1^{(2)} \otimes 1)v_b\|^2$, the same for both cells. (iii) $W\psi$ lies in sector b of cell c if and only if the ready load of v_{1-b} vanishes. If both ready loads are nonzero, $W\psi$ lies in neither sector of either cell.*

Proof. (i) Lemma 6.9 with $b = 1$: the input $(a,0)$ is balanced exactly when $a = 0$, so sector 1 goes to 0 when $a = 0$ and stays 1 when $a = 1$. (ii) W acts branch by branch and U_j acts after the cell step, so by (i) the projection keeps the branch b and removes the other; W is isometric. (iii) By (ii), $W\psi$ is in sector b exactly when W of the other branch vanishes, that is, when that branch does. □

The posting copies the sender bit into both pointers and does not choose it: a coherent pair of senders yields a coherent pair of pointer sectors with the input loads. Free ticks then carry the label. The free clock of one channel is the 8×8 matrix C with blocks $C_{b'b} = -iK(b',b)$ on $\mathbb{C}^2 \otimes \mathscr{H}_c$ (bit and cell), the eight-state clock of Chapter 2 in its occupation chart.

Proposition 6.18 (Free ticks alternate the pointer)**.** *With Π_b the pointer on the cell factor, $\Pi_b C = C\,\Pi_{1-b}$, and $\Pi_{b+n}C^n = C^n\Pi_b$ for $n \geq 0$, indices modulo 2. The same holds for $C \otimes C$ on two channels with either cell's pointer, with any reference system attached: a state in sector b of a cell is in sector $b + n$ after n free ticks.*

Proof. Each block $K(b',b)$ anticommutes with L^2; induct on n. □

The pointer also shows in the statistics of the currents. For an input occupation x the Born effect of output y on the internal space of x is $G_{yx} = M_{yx}^{\top} M_{yx}$, and $\sum_y G_{yx} = I$. Clock cell 1 writes the tree current $\hat{\jmath}_1 = j_1 = y_1 - x_2$ and cell 2 writes $\hat{\jmath}_2 = j_3 = y_2 - x_1$. Put $m_c(x) = \sum_y \hat{\jmath}_c(y,x) G_{yx}$ and $N_c(x) = \sum_y \hat{\jmath}_c(y,x)^2 G_{yx} - m_c(x)^2$, the connected second moment of that current, and let $\nu(x) = 0$ for balanced x and $\nu(x) = 1$ for mixed x.

Theorem 6.19 (Current noise reveals the pointer)**.** *For every occupation x and $c \in \{1,2\}$: (i) $4N_c(x) = \Pi^{(c)}_{\nu(x)}$ on $\mathscr{H}_c \otimes \mathscr{H}_c$. The connected second moment of the middle current, and the connected covariances between distinct currents, vanish. (ii) For every $w \in (\mathscr{H}_c \otimes \mathscr{H}_c) \otimes E$, $(N_c(x) \otimes 1)w = 0$ if and only if $(\Pi^{(c)}_{1-\nu(x)} \otimes 1)w = w$: the quiet sector $1 - \nu(x)$ is exactly the kernel of the current noise. (iii) If an input v has $(N_c(x) \otimes 1)v_x = 0$ for every x, then $(\Pi^{(c)}_0 \otimes 1)Wv = Wv$.*

Proof. (i) Take $c = 1$. G_{yx} factors into a cell-1 effect of writing y_1 and a cell-2 effect of writing y_2, and the cell-2 effects sum to I. So $\hat{\jmath}_1$ is a binary variable shifted by $-x_2$, with cell-1 effect $e = C_x K(1,x_2)^{\top} K(1,x_2) C_x$ for the value $1 - x_2$, and $N_1(x) = e - e^2$. By the Gram lemma of Chapter 2, $e = \frac{1}{2}(I \pm c_x)$ with $c_x = C_x c C_x$, so $e - e^2 = \frac{1}{4}(I - c_x^2)$, a quarter of the projection onto the zero plane of c_x. That plane is $D\mathscr{U}$ for balanced x and $\mathscr{U}$ for mixed x, which is $\Pi^{(1)}_{\nu(x)}$. The middle current $x_1 - x_2$ is fixed by the input, and the two tree currents have product effects, so the remaining connected moments vanish. (ii) $4N_c(x) + \Pi^{(c)}_{1-\nu(x)} = 1$ with $N_c(x) \geq 0$. (iii) Lemma 6.9: the quiet sector is 1 on balanced inputs, which go to 0, and 0 on mixed inputs, which keep it. □

Remark 6.20 (Quiet means sharp). The proof identifies the noisy sector at occupation x with the zero plane of the cell's pole charge c_x, where the cell stays or flips with weight $\frac{1}{2}$ each and its tree current has variance $\frac{1}{4}$; the quiet sector is the pair of pole lines, where the cell moves surely and its current is sharp. Equivalently, each pointer projection is a sum of the 36 sector projections that span the centre of the current algebra (Corollary 9.9): the pointer is central in $\mathscr{A}$ but far from a minimal central projection.

The current algebra contains the pointer itself, not only its occupation-dressed form.

Theorem 6.21 (The pointer is a central projection of the current algebra)**.** *Let $c \in \{1,2\}$ and write $E_{(j)} = V_j^* V_j$ for one-letter effects. (i) $M_{\mathrm{bal}} = \sum_j [j_2 = 0]\, E_{(j)}$ is the projection $\sum_{x\ bal} |x\rangle\langle x| \otimes I$. (ii) With $m_c = \sum_j \hat{\jmath}_c E_{(j)}$ and $s_c = \sum_j \hat{\jmath}_c^2 E_{(j)}$, $N = 4(s_c - m_c^2) = \sum_x |x\rangle\langle x| \otimes \Pi^{(c)}_{\nu(x)}$. (iii) $\Pi^{(c)}_0 = M_{\mathrm{bal}} N + (1 - M_{\mathrm{bal}})(1 - N)$ and $\Pi^{(c)}_1 = 1 - \Pi^{(c)}_0$; both lie in $\mathscr{A}$. (iv) Every element of $\mathscr{A}$ commutes with $\Pi^{(c)}_0$ and $\Pi^{(c)}_1$: they are central projections of $\mathscr{A}$.*

Proof. (i) $j_2(y,x) = x_1 - x_2$ and $\sum_y G_{yx} = I$. (ii) $E_{(j)} = \sum_x |x\rangle\langle x| \otimes \sum_{y\,:\,j(y,x)=j} G_{yx}$ is block diagonal, so weighted sums and products act block by block; apply Theorem 6.19(i). (iii) On a balanced block the expression is $N = \Pi^{(c)}_0$; on a mixed block it is $1 - \Pi^{(c)}_1 = \Pi^{(c)}_0$. (iv) Every element of $\mathscr{A}$ commutes with L_c^2 (Proposition 2.13). □

Theorem 6.21 is algebraic: it constructs the pointer from current effects with signed coefficients and products. A finite experiment that records currents and decides from them is a different thing, and it fails. Let V_0 be the zero-label slice, which keeps $00 \to 00$ and $11 \to 11$ coherently: $V_0 = M_{\mathrm{bal}}(1 \otimes (-\mathrm{stay} \otimes \mathrm{stay}))$.

Theorem 6.22 (No finite current history decodes the pointer). *Let $c \in \{1,2\}$, $b \in \{0,1\}$ and $n \geq 0$, and let 0^n be the word of n zero labels. (i) $V_{0^n}\Pi_b^{(c)} \neq 0$. Witnesses: $w_0 = (1,1,-1,-1) \in D\mathcal{U}$ and $w_1 = (0,-1,0,-1) \in \mathcal{U}$ satisfy $\mathrm{stay}^2 w = \frac{1}{2}w$, and $|00\rangle \otimes w_b \otimes w_0$ (for $c = 1$; $|00\rangle \otimes w_0 \otimes w_b$ for $c = 2$) lies in sector b of cell c and is multiplied by $\frac{1}{4}$ under V_0^2. (ii) No function d from words of length n to $\{0,1\}$ has $V_w\Pi_{1-d(w)}^{(c)} = 0$ for every word w of length n. (iii) If a finite set S of words has $\sum_{w\in S} E_w = \Pi_b^{(c)}$, then $V_w\Pi_{1-b}^{(c)} = 0$ for every $w \in S$. Hence no such S contains 0^n, and the words of length n cannot be split into two sets whose summed effects are $\Pi_0^{(c)}$ and $\Pi_1^{(c)}$.*

Proof. (i) On the 00 block V_0^2 acts as $\mathrm{stay}^2 \otimes \mathrm{stay}^2$, and $\mathrm{stay}^2 w_b = \frac{1}{2}w_b$ by direct computation; $L^2 w_0 = -w_0$ and $L^2 w_1 = w_1$. So V_0^n is nonzero on the witness for every n. (ii) Apply (i) to $w = 0^n$ and $b = 1 - d(0^n)$. (iii) $\sum_{w\in S} \Pi_{1-b}^{(c)} E_w \Pi_{1-b}^{(c)} = \Pi_{1-b}^{(c)}\Pi_b^{(c)}\Pi_{1-b}^{(c)} = 0$ is a sum of positive matrices, so each term $(V_w\Pi_{1-b}^{(c)})^* V_w \Pi_{1-b}^{(c)}$ vanishes. One of the two sets of a split contains 0^n. □

The scope is a fixed finite number of unchanged contacts followed by a classical decision on the recorded currents. The theorem does not exclude added controls, inputs prepared in a subspace, approximate reads or limiting protocols, and it is consistent with Theorem 6.15: the pointer is a strong limit of native reads, never a finite one. One supplied encoding makes the currents read the pointer exactly.

Proposition 6.23 (On the ready encoding the middle current reads the pointer). *Let ψ be as in Proposition 6.17, and let K_b keep the components of $W\psi$ whose label has middle current $j_2 = b$ and remove the others. Then $K_b W\psi = (\Pi_b^{(c)} \otimes 1)W\psi$ for both c, its squared norm is the ready load of ν_b, and $K_0 W\psi + K_1 W\psi = W\psi$.*

Proof. From the input $(a,0)$ every label has $j_2 = a$, so K_b keeps exactly the branch b; apply Proposition 6.17(ii). □

The ready encoding is supplied: receiver bit 0 and both cells in $\mathcal{U}$. The proposition identifies a current event with the pointer event on that encoding; it does not implement a pointer read on arbitrary inputs. Can the native words themselves supply the encoding? Only from the right internal parity.

Proposition 6.24 (Ready preparation needs positive internal parity). *(i) The total internal parity $L_1^2 L_2^2$ commutes with every slice V_j, hence with every word V_w. (ii) Let P_{ready} project onto $\mathscr{H}_{\mathrm{ready}}$ and $P_- = \frac{1}{2}(1 - L_1^2 L_2^2)$. Then $P_{\mathrm{ready}} V_w P_- = 0$ for every word w. If $P_- v = v$ and $V_w v$ lies in $\mathscr{H}_{\mathrm{ready}}$, then $V_w v = 0$. (iii) The six-posting herald word along the occupations $00 \to 10 \to 00 \to x \to 11 \to y \to 00$, with $(x,y) = (01,10)$ or $(10,01)$, maps every input to a multiple of $|00\rangle \otimes p \otimes a$ or of $|00\rangle \otimes a \otimes p$ respectively; both lie in $\mathscr{H}_{\mathrm{ready}}$, and the herald annihilates the negative-parity sector.*

Proof. (i) By Lemma 6.9 the two half-turns each contribute $\epsilon(x)$, and $\epsilon(x)^2 = 1$. (ii) $\mathscr{H}_{\mathrm{ready}}$ lies in the range of $\frac{1}{2}(1 + L_1^2 L_2^2)$, and $V_w P_- = P_- V_w$. (iii) Only inputs at occupation 00 follow the route; on that block the six blocks multiply to the rank-one matrix $\frac{1}{16}|r\rangle\langle r'|$ with $r = p \otimes a$

or $a \otimes p$, by direct exact computation. Then (ii) forces r' to vanish on the negative-parity sector. □

With these contacts alone, a ready pointer can be prepared only from the positive total parity of the two internal cells. Other cells, transferred resources, changed interactions and other ready encodings lie outside the statement.

6.9 What remains open

Nothing here enacts a read, prepares or selects a pointer sector, or selects an atom. Three questions stay open. Which algebra describes the operations available inside the ledger: the coordinate events, under which atoms are superselection sectors, or the native reads, under which only the joint pointer sectors are? Can a read placed at an earlier stage act on a history that already carries newer records, which decides whether a stored record is classical for every available read? And which physical process, if any, enacts the retained read or the dwell read on the live cell? Reading a pointer sector, a lineage or the genesis whole as a participant or a perspective is an identification that this chapter does not make.

Chapter 7 asks which states of the native cell the ledger keeps, from the postings alone: words of three postings cut out a four-dimensional subspace of the cell that never turns neutral, and every other state leaks.

Chapter 7

The postings select four modes

Boundaries tell us where a history lives. They do not tell us which states inside a boundary persist as themselves while the ledger keeps posting. This chapter answers that question for one native cell. The answer is a four-dimensional subspace of the 64-dimensional cell, spanned by the four modes of Chapter 2, and the ledger's own postings select it: it is exactly the part of the cell that no record, of any length, can leave with neutral weight. We then prove that the selected states are exactly the sure subjects of the eight-tick window on the loop edge. Each selected state writes one of four words. Under the photon dictionary these are two circulations around a lattice plaquette and a unit charge hopping across its loop edge, and, among the selected one-loop states, bounded field energy picks out the hop.

7.1 Neutral weight is the part that falls silent

Of the three sectors, only the neutral one loses the loop edge gradually. By Lemma 3.3, $G_1 = \Pi_{\mathrm{mix}}$ and $G_{k+1} = Q_{\mathrm{al}} + 2^{-k} Q_{\mathrm{ne}}$ for $k \geq 1$: an aligned state writes the loop edge at every posting, an opposed state is surely silent at the second, and neutral weight falls silent at the rate one half per posting.

The simplest test of this kind looks at one label. The four modes write the labels $(1, 1, -1)$, $(-1, -1, 1)$, $(0, 1, 0)$ and $(0, -1, 0)$ (Table 2.4); call them the *mode labels*.

Proposition 7.1 (The one-posting mode test)**.** *The mode labels are exactly the labels of the four transitions from a mixed occupation to a mixed occupation, and their summed effect is*

$$E_{(1,1,-1)} + E_{(-1,-1,1)} + E_{(0,1,0)} + E_{(0,-1,0)} = G_2 = Q_{\mathrm{al}} + \tfrac{1}{2} Q_{\mathrm{ne}}.$$

Hence, for every unit genesis state ν, the first label is a mode label with probability $w_{\mathrm{al}}(\rho_\nu) + \frac{1}{2} w_{\mathrm{ne}}(\rho_\nu)$.

Proof. By Table 2.3, the transitions $(1,0) \to (1,0)$, $(1,0) \to (0,1)$, $(0,1) \to (0,1)$ and $(0,1) \to (1,0)$ write the four mode labels, and no other transition writes them. A two-letter word writes the loop edge twice exactly when its first transition starts and ends on mixed occupations, so $G_2 = \sum_j V_j^* G_1 V_j$ over labels with $j_2 \neq 0$ keeps exactly these four transitions, and $G_2 = \sum E_j$ over the mode labels. Lemma 3.3 with $k = 1$ gives the formula, and the record law gives the probability. □

A state with no neutral weight passes this test with probability exactly its aligned weight. That makes the question of this chapter precise. Is having no neutral weight a stable property of a state, or can the postings write neutral weight into a state that had none?

7.2 Where the neutral sector lives

The neutral sector is carried by product states in which a clock cell sits on its pole-zero plane, and one per-cell rule says how the postings move that plane. We use the planes $\mathcal{U} = \mathrm{span}\{p, a\}$ and $D\mathcal{U} = \mathrm{span}\{P, A\}$ of Chapter 2 and, from Chapter 3, the pole charge $c_x = C_x c C_x$ of a clock cell at occupation x (so c on balanced and c' on mixed occupations), and its spectral projections $\Pi_+^x, \Pi_-^x, \Pi_0^x$. By Table 2.1, the pole-zero plane $\mathrm{ran}\,\Pi_0^x$ is $D\mathcal{U}$ on a balanced occupation and $\mathcal{U}$ on a mixed one.

Lemma 7.2 (The pole-zero rule). *Let $x \to y$ be a transition and let $f = K(b', b)\, C_x$ be either of the two cell maps in the block M_{yx}. Then*

$$\Pi_0^y f = f \Pi_0^x \quad \text{if } y \text{ is mixed}, \qquad \Pi_0^y f = f\big(I - \Pi_0^x\big) \quad \text{if } y \text{ is balanced}.$$

A posting into a mixed occupation keeps each cell's pole-zero status; a posting into a balanced occupation reverses it. Consequently the pole-product-zero block $Z_x = I - c_x^2 \otimes c_x^2$, *the projection onto the states of the two cells in which at least one cell is on its pole-zero plane, satisfies $Z_y M_{yx} = M_{yx} Z_x$ for every transition into a mixed occupation y.*

Proof. By Table 2.2, stay and flip map $\mathcal{U}$ into $D\mathcal{U}$ and $D\mathcal{U}$ into $\mathcal{U}$, and D exchanges the two planes. So f exchanges the planes when x is balanced ($C_x = I$) and keeps each plane when x is mixed ($C_x = D$). Write $\Pi_{\mathcal{U}}$ and $\Pi_{D\mathcal{U}}$ for the two plane projections. If y is mixed, $\Pi_0^y = \Pi_{\mathcal{U}}$, and $\Pi_{\mathcal{U}} f$ equals $f\Pi_{D\mathcal{U}}$ for balanced x and $f\Pi_{\mathcal{U}}$ for mixed x: in both cases $f\Pi_0^x$. If y is balanced, $\Pi_0^y = \Pi_{D\mathcal{U}}$, and $\Pi_{D\mathcal{U}} f$ equals $f\Pi_{\mathcal{U}}$ for balanced x and $f\Pi_{D\mathcal{U}}$ for mixed x: in both cases $f(I - \Pi_0^x)$. For the last statement, $c_x^2 = I - \Pi_0^x$, so for mixed y each factor satisfies $c_y^2 f = f c_x^2$, hence $(c_y^2 \otimes c_y^2) M_{yx} = M_{yx}(c_x^2 \otimes c_x^2)$. □

On a mixed occupation $\Theta^2 = c'^2 \otimes c'^2$, so the neutral block $I - \Theta^2$ is Z_x. With $c'^2 = I - \Pi_0^x$ it splits into the part where exactly one cell is on its pole-zero plane and the part where both are. The first part is the mixed half of the one-zero sector Q_{oz} of Chapter 3; write

$$Q_{\mathrm{zz}} = \sum_{x \text{ mixed}} |x\rangle\langle x| \otimes \Pi_0^x \otimes \Pi_0^x$$

for the second, the *zero-zero sector*.

Proposition 7.3 (Carriers of the neutral sector). *(a)* $Q_{\mathrm{ne}} = \Pi_{\mathrm{mix}}\, Q_{\mathrm{oz}} + Q_{\mathrm{zz}}$.

(b) At each mixed occupation the 12*-dimensional neutral block has an orthonormal basis of product vectors $e \otimes f$ from the pole basis: eight* one-zero carriers, *with one factor in $\{p, a\}$ and the other in $\{P, A\}$, and four* zero-zero carriers, *with both factors in $\{p, a\}$.*

(c) At the occupation $(1, 0)$, each one-zero carrier writes exactly two first labels, each with probability $\frac{1}{2}$. For $v = |10\rangle \otimes a \otimes P$ they are $(1, 1, -1)$ and $(0, 1, -1)$.

Proof. (a) On a mixed occupation, $I - (I - \Pi_0^x) \otimes (I - \Pi_0^x) = \Pi_0^x \otimes (I - \Pi_0^x) + (I - \Pi_0^x) \otimes \Pi_0^x + \Pi_0^x \otimes \Pi_0^x$; the first two terms are the block of Q_{oz} and the last is the block of Q_{zz}. (b) On a mixed occupation Π_0^x projects onto $\mathcal{U} = \mathrm{span}\{p, a\}$ and $I - \Pi_0^x$ onto $D\mathcal{U} = \mathrm{span}\{P, A\}$, so the one-zero block has the $2 \cdot 2 \cdot 2 = 8$ product vectors listed and the zero-zero block the $2 \cdot 2 = 4$. (c) The input is mixed, so both cells are dressed by D first. The carrier's cell in $\{P, A\}$ is dressed onto a pole

line of c, where by Lemma 2.5 it moves surely: it fixes its output bit. The other cell is dressed into $D\mathcal{U}$, where c vanishes, and stays or flips with weight $\frac{1}{2}$ each; its output bit is free. For v, cell 2 is P, dressed to p; it reads $x_1 = 1$, and $K(y_2, 1)p$ is nonzero only for $y_2 = 0$. Cell 1 is a, dressed to A, so $y_1 \in \{0, 1\}$ with weight $\frac{1}{2}$ each. The outputs are $(0, 0)$ and $(1, 0)$, with labels $(0, 1, -1)$ and $(1, 1, -1)$. □

So the neutral sector is concrete: a live cell, on a mixed occupation, at least one of whose clock cells carries no definite dressed pole.

7.3 Four transitions write neutral weight

Having no neutral weight now does not last. Exactly four of the sixteen transitions write neutral weight into states that had none, at a rate of at most one half per posting. We first say what closure would require.

Proposition 7.4 (Closure criterion)**.** *The following are equivalent.*

(i) For every unit genesis state v with $w_{\mathrm{ne}}(\rho_v) = 0$ and every record h with $\mathbb{P}_v(h) > 0$, the present cell has $w_{\mathrm{ne}}(\rho_{v,h}) = 0$.
(ii) $Q_{\mathrm{ne}} V_j (I - Q_{\mathrm{ne}}) = 0$ for every label $j \in \mathcal{J}$.

Proof. For an orthogonal projection Q, $\mathrm{tr}(\rho_v Q) = \sum_n \|Qv(n)\|^2$, so $w_{\mathrm{ne}}(\rho_v) = 0$ says that every loop slice $v(n)$ lies in $\ker Q_{\mathrm{ne}}$. (ii)⇒(i): by (ii) each V_j maps $\ker Q_{\mathrm{ne}}$ into itself, hence so does every V_h; the present cell $\rho_{v,h}$ is a positive combination of the operators $V_h v(n) v(n)^* V_h^*$, so $Q_{\mathrm{ne}} \rho_{v,h} = 0$. (i)⇒(ii): if $Q_{\mathrm{ne}} V_j (I - Q_{\mathrm{ne}}) \neq 0$, choose a unit vector $u \in \ker Q_{\mathrm{ne}}$ with $Q_{\mathrm{ne}} V_j u \neq 0$ and place it at loop flux 0. It has no neutral weight, its first label is j with probability $\|V_j u\|^2 \geq \|Q_{\mathrm{ne}} V_j u\|^2 > 0$, and the present cell has neutral weight $\|Q_{\mathrm{ne}} V_j u\|^2 / \|V_j u\|^2 > 0$. □

Closure fails, and the failure is located exactly. Let

$$Q_{\mathrm{bz}} = \sum_{x \text{ balanced}} |x\rangle\langle x| \otimes Z_x = \sum_{x \text{ balanced}} |x\rangle\langle x| \otimes \left(I - c^2 \otimes c^2\right)$$

be the *balanced pole-product-zero sector*: a balanced occupation with at least one clock cell on its pole-zero plane $D\mathcal{U}$. It has rank 24. It is not part of the neutral sector, which lives on mixed occupations only.

Theorem 7.5 (Creation of neutral weight)**.** *(a) For every label j, $Q_{\mathrm{ne}} V_j (I - Q_{\mathrm{ne}}) = \Pi_{\mathrm{mix}} V_j Q_{\mathrm{bz}}$.*
(b) $Q_{\mathrm{ne}} V_j (I - Q_{\mathrm{ne}}) \neq 0$ exactly when j is the label of a transition from a balanced to a mixed occupation. These are the four labels

$$(1, 0, 0), \quad (0, 0, 1), \quad (0, 0, -1), \quad (-1, 0, 0)$$

of $(0, 0) \to (1, 0)$, $(0, 0) \to (0, 1)$, $(1, 1) \to (1, 0)$ and $(1, 1) \to (0, 1)$.
(c) The aligned and opposed sectors feed no neutral weight in one posting: $Q_{\mathrm{ne}} V_j Q_{\mathrm{al}} = 0$ and $Q_{\mathrm{ne}} V_j Q_{\mathrm{op}} = 0$ for every j.
(d) The state $\zeta = |00\rangle \otimes P \otimes A$ has no neutral weight. Its first posting writes $(1, 0, 0)$ with probability $\frac{1}{4}$, and the present cell after that record lies wholly in the neutral sector.

In particular the equivalent conditions of Proposition 7.4 fail.

Proof. (a) Every slice is a sum of blocks $|y\rangle\langle x| \otimes M_{yx}$; only the zero label has two, and both of its outputs are balanced. Write N_x for the block of Q_{ne} at x: 0 if x is balanced and Z_x if x is mixed. The (y, x) block of the left side is $N_y M_{yx}(I - N_x)$. If y is balanced, $N_y = 0$, and the right side vanishes there too. If y and x are mixed, Lemma 7.2 gives $Z_y M_{yx}(I - Z_x) = M_{yx} Z_x (I - Z_x) = 0$, and Q_{bz} vanishes at mixed x. If y is mixed and x balanced, $N_x = 0$ and the block is $Z_y M_{yx} = M_{yx} Z_x$, which is the right side.

(b) By (a), only transitions from balanced to mixed occupations can contribute. For each of them $M_{yx} Z_x \neq 0$: on $D\mathcal{U} \otimes D\mathcal{U}$, where both cells lie on the pole-zero plane of c, each cell map from a balanced input has Gram $\frac{1}{2}(I \mp c)$ (Lemma 2.5), which acts as $\frac{1}{2} I$ on $D\mathcal{U}$, so $\|M_{yx} u\|^2 = \frac{1}{4}\|u\|^2$ for every u in $D\mathcal{U} \otimes D\mathcal{U}$. The labels are read from Table 2.3.

(c) $Q_{\mathrm{ne}} Q_{\mathrm{al}} = 0$, so $Q_{\mathrm{ne}} V_j Q_{\mathrm{al}} = Q_{\mathrm{ne}} V_j (I - Q_{\mathrm{ne}}) Q_{\mathrm{al}} = \Pi_{\mathrm{mix}} V_j Q_{\mathrm{bz}} Q_{\mathrm{al}}$ by (a), and $Q_{\mathrm{bz}} Q_{\mathrm{al}} = 0$ because the two live on different occupations. The opposed sector is the same.

(d) ζ sits on a balanced occupation, so $Q_{\mathrm{ne}} \zeta = 0$. Cell 1 applies $K(y_1, 0)$ to P and cell 2 applies $K(y_2, 0)$ to A. The output $(1, 0)$ takes flip on cell 1 and stay on cell 2; by Table 2.2 the block value is $-\frac{1}{2}(p - a) \otimes \frac{1}{2}(a - p)$, of squared norm $\frac{1}{4}$. Both factors lie in $\mathcal{U}$, the pole-zero plane at the mixed occupation $(1, 0)$, so the present cell lies in the zero-zero sector. □

The mechanism is the pole-zero rule: a clock cell on a balanced occupation can sit on $D\mathcal{U}$, its pole-zero plane, and a posting into a mixed occupation keeps that status. The opposed sector reaches the neutral sector only through a balanced occupation, in two postings; the aligned sector never reaches it (Lemma 7.12 below). The amount created in one posting has a closed form.

Theorem 7.6 (How much one posting creates). *(a)* $\Phi(Q_{\mathrm{ne}}) = \frac{1}{2}\big(Q_{\mathrm{ne}} + Q_{\mathrm{bz}}\big)$*: the free pull of the neutral sector is one half of the pole-product-zero sector on all four occupations.*

(b) For every unit genesis state v,

$$\sum_{j \in \mathcal{J}} \mathbb{P}_v(j)\, w_{\mathrm{ne}}(\rho_{v,j}) = \tfrac{1}{2}\, w_{\mathrm{ne}}(\rho_v) + \tfrac{1}{2}\, \mathrm{tr}\big(\rho_v Q_{\mathrm{bz}}\big),$$

where a record of probability zero contributes zero.

(c) $(I - Q_{\mathrm{ne}})\, \Phi(Q_{\mathrm{ne}})\, (I - Q_{\mathrm{ne}}) = \frac{1}{2}\, Q_{\mathrm{bz}}$*. Over unit genesis states with no neutral weight, the largest value of the left side of (b) is* $\frac{1}{2}$*, attained by* ζ*.*

Proof. (a) Φ maps operators that are block diagonal in the occupation to block diagonal operators, with block $\sum_y M_{yx}^{\top} Y_y M_{yx}$ at the input x: a cross term would need two transitions with the same label and the same output, and those determine the input. For $Y = Q_{\mathrm{ne}}$ only mixed outputs contribute, and Lemma 7.2 gives the block $\big(\sum_{y\ \mathrm{mixed}} M_{yx}^{\top} M_{yx}\big) Z_x$. For mixed x the Gram sum is $\frac{1}{2}(I + \Theta)$, as in the proof of Lemma 3.3, and $\frac{1}{2}(I + \Theta)(I - \Theta^2) = \frac{1}{2}(I - \Theta^2)$ because $\Theta^3 = \Theta$. For balanced x the two mixed outputs apply flip $\otimes$ stay and stay $\otimes$ flip up to sign, so by Lemma 2.5 the Gram sum is $\frac{1}{4}(I + c) \otimes (I - c) + \frac{1}{4}(I - c) \otimes (I + c) = \frac{1}{2}(I - c \otimes c)$, and $\frac{1}{2}(I - c \otimes c)(I - c^2 \otimes c^2) = \frac{1}{2}(I - c^2 \otimes c^2)$ because $c^3 = c$. Every block of $\Phi(Q_{\mathrm{ne}})$ is therefore $\frac{1}{2} Z_x$.

(b) $\mathbb{P}_v(j)\, w_{\mathrm{ne}}(\rho_{v,j}) = \mathrm{tr}(V_j \rho_v V_j^* Q_{\mathrm{ne}}) = \mathrm{tr}(\rho_v V_j^* Q_{\mathrm{ne}} V_j)$; sum over j and use (a).

(c) $Q_{\mathrm{ne}}Q_{\mathrm{bz}} = 0$, so $(I - Q_{\mathrm{ne}})Q_{\mathrm{bz}}(I - Q_{\mathrm{ne}}) = Q_{\mathrm{bz}}$, while $(I - Q_{\mathrm{ne}})Q_{\mathrm{ne}}(I - Q_{\mathrm{ne}}) = 0$. For a state with no neutral weight the left side of (b) is $\frac{1}{2}\operatorname{tr}(\rho_v Q_{\mathrm{bz}}) \le \frac{1}{2}$, with equality for ζ, which lies in $\operatorname{ran} Q_{\mathrm{bz}}$ because $P, A \in D\mathcal{U}$. □

The factor $\frac{1}{2}$ is the Gram of a pole-zero cell under either move. Two states met earlier show the effect along actual records.

Example 7.7 (Two states that start with no neutral weight)**.** *The ready vector* $r = |00\rangle \otimes p \otimes a$ of Example 2.18 writes $(1, 0, 0)$ and then $(1, 1, 0)$ with certainty and becomes $|11\rangle \otimes (-P \otimes A)$. Both cells now lie on $D\mathcal{U}$, the pole-zero plane at the balanced occupation $(1, 1)$. The third posting goes to $(1, 0)$ with label $(0, 0, -1)$ with probability $\frac{1}{4}$, and by the pole-zero rule both cells arrive on $\mathcal{U}$. So

$$\mathbb{P}_r\big((1,0,0),(1,1,0),(0,0,-1)\big) = \tfrac{1}{4},$$

and the present cell after this record lies wholly in the neutral sector, although r has none.

The chance state $\chi = |10\rangle \otimes (\frac{3}{5}P \otimes P + \frac{4}{5}A \otimes P)$ of Example 3.5 has aligned weight $\frac{9}{25}$, opposed weight $\frac{16}{25}$ and no neutral weight. Its opposed part moves surely to $(0, 0)$ with label $(0, 1, -1)$ and becomes $\frac{4}{5}|00\rangle \otimes A \otimes P$, both cells on the pole-zero plane of c. The next posting goes to $(1, 0)$ with label $(1, 0, 0)$ with probability $\frac{1}{4}$, leaving both cells on $\mathcal{U}$. So $\mathbb{P}_\chi\big((0,1,-1),(1,0,0)\big) = \frac{16}{25}\cdot\frac{1}{4} = \frac{4}{25}$, and the present cell after this record is wholly neutral.

The created weight matters for every finite test made later along the record. A state with no neutral weight passes the one-posting mode test with probability exactly its aligned weight (Proposition 7.1), but present cells do not inherit that exactness: after the record $(1, 0, 0)$ the present cell of ζ lies in the neutral sector (Theorem 7.5(d)), so it has aligned weight 0 and passes the test with probability $\frac{1}{2}$.

Which records guarantee a present with no neutral weight, whatever the genesis state? For short records the answer depends only on the last occupation.

Proposition 7.8 (Record certificates)**.** *(a) For every transition* $x \to y$, $Q_{\mathrm{ne}}V_{j(y,x)} = 0$ *if and only if* y *is balanced.*

(b) For every occupation path $x \to y_1 \to y_2$ *and its record* $h = \big(j(y_1, x),\ j(y_2, y_1)\big)$, $Q_{\mathrm{ne}}V_h = 0$ *if and only if* y_2 *is balanced.*

Proof. If the last output is balanced, every block of V_h ends on a balanced occupation (a nonzero label is written by exactly one transition, and both transitions of the zero label end on balanced occupations), and Q_{ne} vanishes there. Conversely, let the last output be mixed and take $e_0 \otimes e_0$ at x, where $e_0 = \frac{1}{2}(p + a + P + A)$ is the first basis vector of a clock cell. Since $De_0 = e_0$, the dressing does not matter, and by Table 2.2

$$\operatorname{stay} e_0 = \tfrac{1}{2}(a + A), \qquad \operatorname{flip} e_0 = \tfrac{1}{2}(p + P).$$

Both are fixed by D, and one more move gives $\operatorname{stay}(a+A) = A + \frac{1}{2}(a-p)$, $\operatorname{flip}(a+A) = \frac{1}{2}(p+a)$, $\operatorname{stay}(p+P) = \frac{1}{2}(p+a)$, $\operatorname{flip}(p+P) = P + \frac{1}{2}(p-a)$. So after one or two cell maps, up to sign, each cell has a nonzero component in $\mathcal{U}$. At the mixed last occupation $\mathcal{U} \otimes \mathcal{U}$ lies in the neutral sector, so $Q_{\mathrm{ne}}V_h(|x\rangle \otimes e_0 \otimes e_0) \neq 0$. □

For longer records the witness $e_0 \otimes e_0$ need not survive, and which records with a mixed last occupation still certify a present off the neutral sector is not settled here.

Remark 7.9 (Invariant subspaces of an instrument). Proposition 7.4 is the standard invariance criterion for a family of Kraus operators: a subspace with projection Q is kept by every outcome exactly when $(I-Q)K_jQ = 0$ for every j. It underlies decoherence-free subspaces [28] and noiseless subsystems [25]. Here the Kraus operators are the fifteen slices. In those terms, $\ker Q_{\mathrm{ne}}$ is not invariant, four slices break it, and $\frac{1}{2}Q_{\mathrm{bz}}$ is the exact one-posting leakage operator. The content specific to the native step is which subspace fails and at what rate.

7.4 The never-neutral subspace

The states that no record can ever leave with neutral weight form the aligned sector, and three postings already cut it out.

Definition 7.10 (Never-neutral subspace)**.** The *never-neutral subspace* is

$$\mathcal{N} = \bigcap_{k\geq 0} \bigcap_{w\in\mathcal{J}^k} \ker\bigl(Q_{\mathrm{ne}}V_w\bigr) \subseteq \mathscr{H},$$

the empty word included. For $L \geq 0$, $\mathcal{N}_L$ is the same intersection over words of length at most L.

A vector of $\mathcal{N}$ is a cell state that no record, of any length, can leave with neutral weight. Equivalently, $\mathcal{N}$ is the largest subspace of $\ker Q_{\mathrm{ne}}$ that every slice maps into itself: it is invariant, since $Q_{\mathrm{ne}}V_wV_ju = Q_{\mathrm{ne}}V_{j\cdot w}u$, and every invariant subspace of $\ker Q_{\mathrm{ne}}$ lies in it.

Theorem 7.11 (The postings select four modes)**.** *$\mathcal{N}$ is the aligned sector:*

$$\mathcal{N} = \operatorname{ran} Q_{\mathrm{al}} = \operatorname{span}\{\psi_+, \psi_-, \psi_\circ, \psi_\circ'\}, \qquad \dim \mathcal{N} = 4.$$

Words of length three cut it out: $\mathcal{N}_L = \mathcal{N}$ for every $L \geq 3$. Three is the least such length: the ready vector $r = |00\rangle \otimes p \otimes a$ lies in $\mathcal{N}_2$ but not in $\mathcal{N}$.

The proof has an easy half and a hard half. The easy half says the aligned sector never leaks.

Lemma 7.12 (The aligned sector is invariant)**.** *Every slice commutes with Q_{al}: $V_jQ_{\mathrm{al}} = Q_{\mathrm{al}}V_j$ for all $j \in \mathcal{J}$. Consequently $\Phi(Q_{\mathrm{al}}) = Q_{\mathrm{al}}$.*

Proof. By Proposition 2.22, each slice sends each mode to $\pm$ a mode or to 0, so V_j maps $\operatorname{ran} Q_{\mathrm{al}}$ into itself. The same holds for $V_j^\top$. Indeed $U_{\mathrm{nat}} = \sum_j V_j$ permutes the four modes up to sign, so its inverse $U_{\mathrm{nat}}^\top$ does too. A mode $\psi = |y\rangle \otimes \phi$ sits on one occupation, and $U_{\mathrm{nat}}^\top\psi = \sum_x |x\rangle \otimes M_{yx}^\top\phi$ is $\pm$ a mode, which also sits on one occupation x^*. So $M_{yx}^\top\phi = 0$ for $x \neq x^*$, and $V_j^\top\psi$ equals $U_{\mathrm{nat}}^\top\psi$ when $j = j(y, x^*)$ and 0 otherwise. A matrix that maps a subspace into itself, and whose transpose does too, commutes with the orthogonal projection onto it. Finally $\Phi(Q_{\mathrm{al}}) = \sum_j V_j^\top V_jQ_{\mathrm{al}} = Q_{\mathrm{al}}$ by Proposition 2.12. □

The hard half needs a way to add up all the words of one length at once. The free pull does that.

Lemma 7.13 (Word Grams). *For every $u \in \mathscr{H}$ and $k \geq 0$, $\sum_{w \in \mathcal{J}^k} \|Q_{\mathrm{ne}} V_w u\|^2 = \langle u, \Phi^k(Q_{\mathrm{ne}})u\rangle$. Hence $u \in \mathcal{N}_L$ if and only if $\langle u, \Phi^k(Q_{\mathrm{ne}})u\rangle = 0$ for $k = 0, \dots, L$, that is, $\mathcal{N}_L = \ker \sum_{k \leq L} \Phi^k(Q_{\mathrm{ne}})$.*

Proof. Q_{ne} is an orthogonal projection, so $\|Q_{\mathrm{ne}} V_w u\|^2 = \langle u, V_w^\top Q_{\mathrm{ne}} V_w u\rangle$. Summing over the last letter of w and inducting on k gives $\Phi^k(Q_{\mathrm{ne}})$. Each summand is nonnegative, so the sum vanishes exactly when every term does, and the kernel of a sum of positive semidefinite matrices is the intersection of their kernels. □

Lemma 7.14 (A three-step certificate). *As an operator inequality on $\mathscr{H}$,*

$$\sum_{k=0}^{3} \Phi^k(Q_{\mathrm{ne}}) \;\geq\; \tfrac{1}{2}\big(I - Q_{\mathrm{al}}\big).$$

Proof. Block reduction. By the first step in the proof of Theorem 7.6, Φ maps block diagonal operators to block diagonal operators. The blocks of $\Phi(Y)$ depend only on whether x is balanced or mixed, provided those of Y do, because, up to sign, the pairs of cell maps leading from x to balanced outputs, and those leading to mixed outputs, depend only on the kind of x. So every $\Phi^k(Q_{\mathrm{ne}})$ is described by two 16×16 blocks, one for each kind of occupation.

The tables. Work in the product pole basis $\{e \otimes f \,:\, e, f \in \{p, a, P, A\}\}$. Pulling a rank-one projection $|e \otimes f\rangle\langle e \otimes f|$ through a block M_{yx} gives the rank-one projection onto $M_{yx}^\top(e \otimes f)$, and Table A.6 gives the transposed clock maps on the pole basis. Starting from Q_{ne}, whose mixed block is diagonal with entry 1 exactly when at least one factor lies in $\{p, a\}$, three pulls give the blocks of Table A.7 (direct exact computation). They are diagonal in the product pole basis except at $k = 3$. There the mixed block has coherence $-\frac{1}{8}$ between $p \otimes p$ and $a \otimes a$ and between $p \otimes a$ and $a \otimes p$; the balanced block has coherence $+\frac{1}{8}$ between $P \otimes P$ and $A \otimes A$ and between $P \otimes A$ and $A \otimes P$.

Positivity. Subtract $\frac{1}{2}(I - Q_{\mathrm{al}})$. Table A.8 shows the diagonal of the difference in both kinds of block: every entry is nonnegative. The only off-diagonal entries are the four coherences of size $\frac{1}{8}$, each joining two diagonal entries equal to $\frac{3}{2}$ (mixed) or $\frac{1}{2}$ (balanced). Each such 2×2 block $\left(\begin{smallmatrix} d & \pm 1/8 \\ \pm 1/8 & d \end{smallmatrix}\right)$ with $d \geq \frac{1}{2}$ is positive definite. So the difference is positive semidefinite. □

Proof of Theorem 7.11. If $u = Q_{\mathrm{al}} u$, Lemma 7.12 gives $Q_{\mathrm{ne}} V_w u = Q_{\mathrm{ne}} Q_{\mathrm{al}} V_w u = 0$ for every word, since the sectors are orthogonal. So $\operatorname{ran} Q_{\mathrm{al}} \subseteq \mathcal{N} \subseteq \mathcal{N}_3$. Conversely, if $u \in \mathcal{N}_3$, Lemma 7.13 makes the left side of Lemma 7.14 vanish on u, so $\frac{1}{2}\|(I - Q_{\mathrm{al}})u\|^2 \leq 0$ and $u \in \operatorname{ran} Q_{\mathrm{al}}$. Hence $\mathcal{N} = \mathcal{N}_L = \operatorname{ran} Q_{\mathrm{al}}$ for $L \geq 3$, of dimension $\operatorname{tr} Q_{\mathrm{al}} = 4$, spanned by the four modes (2.16).

For $r = |00\rangle \otimes p \otimes a$, the balanced blocks of Table A.7 have diagonal entry at $p \otimes a$ equal to $0, 0, 0, \frac{1}{2}$ for $k = 0, 1, 2, 3$, and no coherence touches it. So $r \in \mathcal{N}_2$ by Lemma 7.13. But $Q_{\mathrm{al}} r = 0$, since r sits on a balanced occupation, so $r \notin \mathcal{N}$, and $\mathcal{N}_2 \neq \mathcal{N}$. Since $\mathcal{N}_L$ decreases in L, no $L \leq 2$ cuts out $\mathcal{N}$. □

The tables give more than the endpoint. For $L \leq 2$ the summed Gram $\sum_{k \leq L} \Phi^k(Q_{\mathrm{ne}})$ is diagonal in the product pole basis, so $\mathcal{N}_L$ is spanned by the pole products at which its diagonal vanishes.

Corollary 7.15 (How fast the exposure proceeds). $\dim \mathcal{N}_0 = 40$, $\dim \mathcal{N}_1 = 16$, $\dim \mathcal{N}_2 = 8$ *and* $\dim \mathcal{N}_L = 4$ *for $L \geq 3$. The postings expose the cell in the order of Table 7.1.*

Proof. $\mathcal{N}_0 = \ker Q_{\mathrm{ne}}$ has dimension $64 - 24 = 40$. By Table A.7, the diagonal of $\Phi^1(Q_{\mathrm{ne}})$ vanishes on a balanced block exactly at the four products in $\mathscr{U} \otimes \mathscr{U}$, which removes $2 \cdot 12 = 24$ dimensions; adding $\Phi^2(Q_{\mathrm{ne}})$ removes $p \otimes p$, $a \otimes a$ on balanced blocks and $P \otimes A$, $A \otimes P$ on mixed blocks, another 8. Theorem 7.11 gives the rest. □

occupation	clock cells	first postings at which neutral weight can appear
mixed	at least one cell in $\mathscr{U}$	already neutral ($k = 0$)
mixed	$P \otimes A$, $A \otimes P$ (opposed)	$k = 2$
mixed	$P \otimes P$, $A \otimes A$ (aligned)	never
balanced	at least one cell in $D\mathscr{U}$	$k = 1$
balanced	$p \otimes p$, $a \otimes a$	$k = 2$
balanced	$p \otimes a$, $a \otimes p$	$k = 3$

Table 7.1. When each part of the cell is first exposed: the least k for which the summed neutral weight after k postings, $\langle u, \Phi^k(Q_{\mathrm{ne}})u\rangle$, is positive on the product pole vectors u of each kind (read from Table A.7). Only the aligned directions are never exposed, and only $p \otimes a$ and $a \otimes p$ on balanced occupations need three postings.

The ready vector is the extreme case: its first two postings are deterministic and leave both cells on the pole-zero plane at $(1, 1)$ (Example 7.7), so the leak arrives only at the third posting, with probability one half.

7.5 Never neutral is sure on the loop edge

The theorem is about vectors of one cell. A genesis state spreads over the loop flux, and its present cells follow its records, so we now state the consequence for states. The class selected by the postings turns out to be the class selected by the window.

Theorem 7.16 (Equivalent descriptions of the selected states)**.** *For a unit genesis state ν the following are equivalent:*

(a) $w_{\mathrm{al}}(\rho_\nu) = 1$*;*
(b) every loop slice $\nu(n)$ lies in $\mathcal{N}$;
(c) for every record h with $\mathbb{P}_\nu(h) > 0$, the present cell has no neutral weight: $w_{\mathrm{ne}}(\rho_{\nu,h}) = 0$;
(d) for every record h with $\mathbb{P}_\nu(h) > 0$, the present cell passes the one-posting mode test with probability exactly its aligned weight;
(e) ν is a sure subject: every field-current window passes with probability one.

Moreover, for every unit genesis state,

$$\sum_{k=0}^{3} \sum_{h \in \mathscr{J}^k} \mathbb{P}_\nu(h)\, w_{\mathrm{ne}}(\rho_{\nu,h}) \;\geq\; \tfrac{1}{2}\big(1 - w_{\mathrm{al}}(\rho_\nu)\big), \tag{7.1}$$

and $\frac{1}{2}$ is the largest constant for which this holds.

Proof. (a)⇔(b): $1 - w_{\mathrm{al}}(\rho_\nu) = \sum_n \|(I - Q_{\mathrm{al}})\nu(n)\|^2$, and $\mathcal{N} = \operatorname{ran} Q_{\mathrm{al}}$ by Theorem 7.11.

For (7.1), $\mathbb{P}_\nu(h)\, w_{\mathrm{ne}}(\rho_{\nu,h}) = \mathrm{tr}(V_h \rho_\nu V_h^* Q_{\mathrm{ne}})$, so by Lemma 7.13 the inner sum is $\mathrm{tr}\big(\rho_\nu \Phi^k(Q_{\mathrm{ne}})\big)$, and Lemma 7.14 gives the bound. For the ready vector at one loop value the four inner sums are $0, 0, 0, \frac{1}{2}$ and $w_{\mathrm{al}} = 0$, so no larger constant works.

(b)⇒(c): $\rho_{\nu,h}$ is a positive combination of the operators $V_h \nu(n) \nu(n)^* V_h^*$, and $Q_{\mathrm{ne}} V_h \nu(n) = 0$ for $\nu(n) \in \mathcal{N}$. (c)⇒(a): if $w_{\mathrm{al}}(\rho_\nu) < 1$, then (7.1) makes some term with $k \le 3$ positive.

(c)⇔(d): by Proposition 7.1 applied to the present cell, which is the cell density of a unit genesis state (Chapter 2), the test passes with probability $w_{\mathrm{al}}(\rho_{\nu,h}) + \frac{1}{2} w_{\mathrm{ne}}(\rho_{\nu,h})$. This equals $w_{\mathrm{al}}(\rho_{\nu,h})$ exactly when $w_{\mathrm{ne}}(\rho_{\nu,h}) = 0$.

(a)⇒(e): each mode writes a label with $j_2 \neq 0$ at every step and is annihilated by every other slice (Proposition 2.22), so $V_w Q_{\mathrm{al}} = 0$ for every word w with some $j_2 = 0$. With $\rho_\nu = Q_{\mathrm{al}} \rho_\nu Q_{\mathrm{al}}$, every such word has probability zero, at every position in the record. So every window passes surely. (e)⇒(a): by Theorem 3.4, the first window passes with probability $w_{\mathrm{al}} + \frac{1}{128} w_{\mathrm{ne}}$. If this is 1, then $w_{\mathrm{al}} + w_{\mathrm{ne}} \le 1$ forces $w_{\mathrm{ne}} = 0$ and $w_{\mathrm{al}} = 1$. □

Corollary 7.17 (Every other state leaks within three postings)**.** *If $w_{\mathrm{al}}(\rho_\nu) < 1$, then some record h of length at most 3 has $\mathbb{P}_\nu(h) > 0$ and $w_{\mathrm{ne}}(\rho_{\nu,h}) > 0$.*

Proof. The right side of (7.1) is positive, so some term on the left is. □

Items (a) to (c) use only the postings and the sectors; item (e) is where the window, and with it the word "subject", enters. The set of selected genesis states is $\ell^2(\mathbb{Z}) \otimes \mathrm{ran}\, Q_{\mathrm{al}}$: the dimension four is per loop slice, and the loop flux may carry any distribution. Items (c), (d) and (7.1) concern present cells, conditional states of the record law; no proof uses the reading of a present cell as the physical present (Chapter 2).

We call the states of this class *participants*: the states that remain present to the ledger at every tick of its own clock. That name is our reading of the theorem, not part of it.

7.6 Four motions and what each cut sees

The selected class is not a sector label alone. It is spanned by four definite motions, and each writes a single infinite record. Four words of each length $n \ge 1$ carry them. The *persistent words* are the constant words

$$u_n^+ = \big((1,1,-1), \dots, (1,1,-1)\big), \qquad u_n^- = \big((-1,-1,1), \dots, (-1,-1,1)\big),$$

and the *oscillation words* alternate between the two hop labels:

$$o_n = \big((0,1,0), (0,-1,0), (0,1,0), \dots\big), \qquad o_n' = \big((0,-1,0), (0,1,0), (0,-1,0), \dots\big),$$

each of length n, o_n starting with $(0,1,0)$ and o_n' with $(0,-1,0)$. Write $x_n = (1,0)$ for even n and $x_n = (0,1)$ for odd n; this is the occupation the oscillation started at $(1,0)$ occupies after n postings. For a unit cell vector ψ write $w_\psi(\rho) = \langle \psi, \rho\, \psi \rangle$, the Born weight of the rank-one read "the cell is in ψ"; for a genesis state $w_\psi(\rho_\nu) = \sum_{n \in \mathbb{Z}} |\langle \psi, \nu(n) \rangle|^2$. For the four modes these are the *mode weights* $w_m = w_{\psi_m}$, $m \in \{+, -, \circ, \circ'\}$, and $w_{\mathrm{al}} = w_+ + w_- + w_\circ + w_\circ'$ because the four modes are an orthonormal basis of $\mathrm{ran}\, Q_{\mathrm{al}}$. When the state is clear we write w_m for $w_m(\rho_\nu)$.

Proposition 7.18 (Each mode runs on one word)**.** *Let $n \geq 1$ and $h \in \mathscr{J}^n$.*

(a) $V_h\psi_+ = [h = u_n^+]\,\psi_+$ *and* $V_h\psi_- = [h = u_n^-]\,(-1)^n\psi_-$.

(b) $V_h\psi_\circ = 0$ *unless* $h = o_n$, *and* $V_{o_n}\psi_\circ$ *is* $\pm\psi_\circ$ *for even* n *and* $\pm\psi_\circ'$ *for odd* n; *likewise* $V_h\psi_\circ' = 0$ *unless* $h = o_n'$, *and* $V_{o_n'}\psi_\circ'$ *is* $\pm\psi_\circ'$ *for even* n *and* $\pm\psi_\circ$ *for odd* n. *The phase of the oscillation returns to* $+1$ *after four postings. The four words* u_n^+, u_n^-, o_n, o_n' *differ in their first letter.*

(c) $\mathrm{acc}(u_n^+) = n$, $\mathrm{acc}(u_n^-) = -n$, $\mathrm{acc}(o_n) = n \bmod 2$ *and* $\mathrm{acc}(o_n') = -(n \bmod 2)$; *every prefix of* o_n *has accumulated current* 0 *or* 1, *and every prefix of* o_n' *has* 0 *or* -1.

(d) Let $\nu = \delta_q \otimes f$ *be a one-loop genesis state at loop* $q \in \mathbb{Z}$ *with unit cell vector* f *in the aligned sector. The stage-*n *history* $T^n\nu$ *vanishes off the four words. It carries weight* w_+ *on* u_n^+, *at loop* $q + n$ *and occupation* $(1, 0)$; *weight* w_- *on* u_n^-, *at loop* $q - n$ *and occupation* $(0, 1)$; *weight* $w_\circ$ *on* o_n, *at loop* $q + (n \bmod 2)$ *and occupation* x_n; *and weight* $w_\circ'$ *on* o_n', *at loop* $q - (n \bmod 2)$ *and occupation* x_{n+1}. *In each case the loop-edge flux of the cell is the loop.*

Proof. (a), (b) Induction on n with Proposition 2.22: each mode is carried to plus or minus a mode by exactly one slice, and every other slice annihilates it. $V_{(1,1,-1)}\psi_+ = \psi_+$, and ψ_- is the same with sign -1 per step. Along the oscillation the chain is $\psi_\circ \mapsto \psi_\circ' \mapsto -\psi_\circ \mapsto -\psi_\circ' \mapsto \psi_\circ$ under the labels $(0, 1, 0), (0, -1, 0), (0, 1, 0), (0, -1, 0)$, so the phase returns to $+1$ after four hops. The first letters are $(1, 1, -1)$, $(-1, -1, 1)$, $(0, 1, 0)$ and $(0, -1, 0)$. (c) The middle components of the four words are $1, -1$, alternately $1, -1$ and alternately $-1, 1$. (d) By Theorem 2.14(b), $(T^n\nu)(n', h) = [n' = q + \mathrm{acc}(h)]\,V_h f$. By (a) and (b), $V_h f = \langle\psi, f\rangle\,V_h\psi$ for the one mode ψ whose word is h, a vector of squared norm $|\langle\psi, f\rangle|^2 = w_\psi$ at the occupation stated, and $V_h f = 0$ if h is none of the four words; (c) gives the loops. The loop-edge flux $m_2(n', x; c, d) = n'$ is read from (2.10). □

Theorem 7.19 (The record law of a sure subject)**.** *Let ν be a unit genesis state with $w_{\mathrm{al}}(\rho_\nu) = 1$, with any loop distribution and any coherence between modes and loops. For every $n \geq 1$ and every record $h \in \mathscr{J}^n$,*

$$\mathbb{P}_\nu(h) = w_+\,[h = u_n^+] + w_-\,[h = u_n^-] + w_\circ\,[h = o_n] + w_\circ'\,[h = o_n'].$$

Proof. Since $\sum_{n'} \|(1 - Q_{\mathrm{al}})\nu(n')\|^2 = \mathrm{tr}(\rho_\nu(1 - Q_{\mathrm{al}})) = 0$, every loop slice lies in the aligned sector: $\nu(n') = \sum_\psi \langle\psi, \nu(n')\rangle\,\psi$ over the four modes. Fix h. By Proposition 7.18 at most one mode ψ_h survives V_h, because the four words are distinct, and $V_h\psi_h$ is a unit vector. So $V_h\nu(n') = \langle\psi_h, \nu(n')\rangle\,V_h\psi_h$, and by Theorem 2.17(a)

$$\mathbb{P}_\nu(h) = \sum_{n'} \big\|V_h\,\nu(n' - \mathrm{acc}(h))\big\|^2 = \sum_{n'} \big|\langle\psi_h, \nu(n')\rangle\big|^2 = w_{\psi_h}(\rho_\nu).$$

If no mode survives, $V_h\nu(n') = 0$ for every n' and $\mathbb{P}_\nu(h) = 0$. □

The cross terms are absent for a structural reason: two modes never write the same word, so the record never asks how they interfere. A coherent superposition of the modes and the mixture with the same weights have the same record law at every length, and each mode, placed at one loop value, writes its own word with certainty.

We now ask what each motion shows a cut of accounts, in the sense of Chapter 3: a cut C ticks at a posting when the label has a nonzero component on an observed account in C. Account 2, the field-current account, books the loop edge; accounts 1 and 3 are the *outer accounts*.

Theorem 7.20 (What each cut sees of the selected states)**.** *Let v be a unit genesis state, C a cut and $k \geq 0$ a window start.*

(a) If $w_\circ(\rho_v) + w'_\circ(\rho_v) = 1$, window k of C passes with probability 1 if $2 \in C$ and 0 otherwise.
(b) If $w_+(\rho_v) + w_-(\rho_v) = 1$, window k of C passes with probability 1 if C holds an observed account and 0 otherwise.
(c) If $w_{\mathrm{al}}(\rho_v) = 1$, window k of C passes with probability

$$(w_+ + w_-)\big[C \text{ holds an observed account}\big] + (w_\circ + w'_\circ)\big[2 \in C\big].$$

In particular v passes every field-current window with probability 1, and every window of a cut that holds an outer account but not account 2 with probability $w_+ + w_-$.
(d) Let C hold account 1 or account 3 but not account 2. Every field-current window passes with probability one and the first window of C fails with probability one, both together, if and only if $w_\circ(\rho_v) + w'_\circ(\rho_v) = 1$.

Proof. (c) By Proposition 3.2(a) the pass probability is the total record probability, at length $k + 8$, of the records on which the window passes; by Theorem 7.19 only the four words carry probability, $w_+, w_-, w_\circ, w'_\circ$. The persistent words write all three observed accounts at every posting. The oscillation words write account 2 at every posting and no outer account at any posting. (a) and (b) are the cases $w_+ + w_- = 0$ and $w_\circ + w'_\circ = 0$ of (c), since each hypothesis gives $w_{\mathrm{al}} = 1$. (d) If $w_\circ + w'_\circ = 1$, then $w_{\mathrm{al}} = 1$ and (c) gives both statements. Conversely, sure passage of every field-current window gives $w_{\mathrm{al}} = 1$ by Theorem 7.16, and then the first window of C passes with probability $w_+ + w_-$ by (c); sure failure forces $w_+ + w_- = 0$, and $w_\circ + w'_\circ = w_{\mathrm{al}} - w_+ - w_- = 1$. □

So on a selected state every window verdict is a function of the four mode weights, although the verdict of a cut holding an outer account is in general not central (Theorem 3.10).

The postings also never carry a state into the selected class from outside. Every V_h commutes with Q_{al} (Lemma 7.12), so if $Q_{\mathrm{al}}v(n) = 0$ for every n, then $Q_{\mathrm{al}}V_h v(n) = 0$, and every present cell of a genesis state with no aligned weight has no aligned weight. The ready vector, a natural balanced source, shows the same thing on the window.

Proposition 7.21 (The ready vectors start late)**.** *At every loop value, the ready vector $r = |00\rangle \otimes p \otimes a$ and the reversed ready vector $r' = |00\rangle \otimes a \otimes p$ fail with certainty every field-current window that starts after $k = 0$, 1 or 2 postings.*

Proof. Each moves surely from $(0, 0)$ to a mixed occupation and then to $(1, 1)$; for r this is Example 2.18, and for r' the cells exchange roles, giving $(0, 0) \to (0, 1) \to (1, 1)$. Every transition out of $(1, 1)$ has $j_2 = 0$ (Table 2.3), so the third posting is surely silent on the loop edge. Every window starting at $k \leq 2$ contains the third posting. □

Preparing a selected state from a source outside the aligned sector, such as a ready vector, needs something beyond the postings; what that is lies outside this chapter.

7.7 The modes on a plaquette: currents and field energy

Under the framework's photon dictionary the three-edge axis of Chapter 2 is embedded in one square plaquette of the cubic lattice, its periodic edge becoming a two-edge path, and every label is a lattice current on it. The persistent modes then circulate around the plaquette, while the oscillation moves one unit of charge across one edge, the edge the field-current account books. We price the Gauss field these currents leave behind by the convex dual of twice the recognition cost, and read the price as electric energy (an identification, named in Section 7.7.3). Along a persistent mode that price grows without bound; along the oscillation it takes two values. Among one-loop states of aligned weight one, bounded field energy (for a positive scale) therefore picks out exactly the states that cut exclusivity, Theorem 7.20(d), singles out.

We use the axis, edges e_1, e_2, e_3 and edge flux $m(n, x; \bar{c}, \bar{d})$ of Chapter 2; account i books j_i on e_i.

7.7.1 The photon dictionary

The cubic lattice $\mathbb{Z}^3$ has oriented edges $[\mathbf{r} \to \mathbf{r} + \hat{a}]$, with $\hat{a}$ one of the three unit vectors $\hat{x}, \hat{y}, \hat{z}$. A *lattice current* is a finitely supported real function F on the oriented edges; its *inflow* ∂F at a site is the current entering the site minus the current leaving it. Place the reference vertex b_0 at the origin, the site s at $\hat{x}$ and the site t at $\hat{x} + \hat{y}$, and add the auxiliary site $u = \hat{y}$. The three paths

$$\pi_1 = [b_0 \to s], \qquad \pi_2 = [s \to t], \qquad \pi_3 = [b_0 \to u] + [u \to t]$$

run along the axis edges e_1, e_2, e_3; the third path has two lattice edges. The *photon dictionary* assigns to each label j the lattice current

$$\mathbf{F}(j) = j_1\, \pi_1 + j_2\, \pi_2 + j_3\, \pi_3, \tag{7.2}$$

and to each posting with label j a coherent displacement of the free photon field: the Weyl operator whose one-photon profile is the transverse part of the Fourier transform of $\mathbf{F}(j)$, weighted by a fixed radial factor with two supplied constants. The zero label posts the identity on the field. The *posted current* of a record h is $\mathbf{F}(h) = \sum_t \mathbf{F}(j^{(t)})$.

The dictionary is an identification, and we name it once, here. It joins two operators, a posting of the native step and a displacement of the photon field sourced by the posting's current. It does not show that a posting emits radiation, and nothing below uses the field displacement: Proposition 7.22 and Theorem 7.23 use only the lattice currents (7.2).

Proposition 7.22 (The accounts are edges of the plaquette)**.**

(i) For every label j, the current $\mathbf{F}(j)$ is j_1 on $[b_0 \to s]$, j_2 on $[s \to t]$, j_3 on $[b_0 \to u]$ and on $[u \to t]$, and 0 on every other edge. The map $j \mapsto \mathbf{F}(j)$ is injective.

(ii) The field-current window starting after posting k passes on a record h exactly when each of the eight posted currents $\mathbf{F}(j^{(k+1)}), \dots, \mathbf{F}(j^{(k+8)})$ is nonzero on the edge $[s \to t]$. For a general cut C the window passes exactly when each of the eight is nonzero on the account edge *of some observed account in C, the account edges of accounts $1, 2, 3$ being $[b_0 \to s]$, $[s \to t]$ and $[b_0 \to u]$.*

(iii) On the edge $[s \to t]$ the posted current of every record h is $\mathrm{acc}(h)$.

(iv) Let $\mathbf{q}(x) = x_1(\delta_s - \delta_{b_0}) + x_2(\delta_t - \delta_{b_0})$ *be the* occupation charge. *Then* $\partial\mathbf{F}(j(y,x)) = \mathbf{q}(y) - \mathbf{q}(x)$ *for every transition, and for every occupation path* $x^{(0)}, \dots, x^{(n)}$ *the record it writes has posted current with inflow* $\mathbf{q}(x^{(n)}) - \mathbf{q}(x^{(0)})$.

Proof. (i) The four lattice edges are distinct; π_1, π_2 are single edges and π_3 is the other two. Reading off j_1, j_2, j_3 on the first three edges inverts the map. (ii) A posting is a tick of C when $j_i \neq 0$ for an observed account $i \in C$, and by (i) j_i is the value of $\mathbf{F}(j)$ on the account edge of i. (iii) is (i) summed over the record. (iv) $\partial\pi_1 = \delta_s - \delta_{b_0}$, $\partial\pi_2 = \delta_t - \delta_s$ and $\partial\pi_3 = \delta_t - \delta_{b_0}$, the current passing through u. Hence

$$\partial\mathbf{F}(j) = (j_1 - j_2)\,\delta_s + (j_2 + j_3)\,\delta_t - (j_1 + j_3)\,\delta_{b_0},$$

and Proposition 2.8 gives $j_1 - j_2 = y_1 - x_1$ and $j_2 + j_3 = y_2 - x_2$. Along a path the inflows telescope. □

7.7.2 What each mode posts

Write $\gamma = \pi_1 + \pi_2 - \pi_3$ for the boundary of the plaquette, traversed $b_0 \to s \to t \to u \to b_0$; by Proposition 7.22(iv), $\partial\gamma = 0$.

Theorem 7.23 (Circulations and a hopping charge)**.**

(a) Every stay $x \to x$ *posts* $(x_1 - x_2)\,\gamma$. *The label* $(1,1,-1)$ *posts* γ *and* $(-1,-1,1)$ *posts* $-\gamma$*; the persistent words post* $\mathbf{F}(u_n^+) = n\gamma$ *and* $\mathbf{F}(u_n^-) = -n\gamma$*, with inflow* 0*: no persistent posting moves charge.*

(b) The label $(0,1,0)$ *posts* π_2 *and carries one unit of charge from* s *to* t*; the label* $(0,-1,0)$ *posts* $-\pi_2$ *and carries it back. The oscillation words post* $\mathbf{F}(o_n) = (n \bmod 2)\,\pi_2$ *and* $\mathbf{F}(o_n') = -(n \bmod 2)\,\pi_2$*; every prefix of* o_n *posts* 0 *or* π_2*, and* $\partial\mathbf{F}(o_n) = \mathbf{q}(x_n) - \mathbf{q}(1,0)$.

(c) $\mathbf{F}(j)$ *is a multiple of* π_2 *if and only if* $j_1 = j_3 = 0$*, if and only if* $j \in \{(0,0,0),(0,1,0),(0,-1,0)\}$. *The transitions that write these labels are exactly those with* $y = (x_2, x_1)$*: the two balanced stays and the two hops between the mixed occupations. The labels with all three components nonzero are exactly* $(1,1,-1)$ *and* $(-1,-1,1)$.

(d) Let ν *be a unit genesis state with* $w_{\mathrm{al}}(\rho_\nu) = 1$ *and* $n \geq 1$. *Every record of positive probability posts nonzero current on* $[s \to t]$ *at every posting. If* $w_\circ + w_\circ' = 1$*, every such record is* o_n *or* o_n'*, posts* $\mathrm{acc}(h)\,\pi_2$*, and each of its postings carries one unit of charge across* $[s \to t]$. *If* $w_+ + w_- = 1$*, every such record is* u_n^+ *or* u_n^-*, posts* $\mathrm{acc}(h)\,\gamma$*, and moves no charge.*

Proof. (a) By (2.9), $j(x,x) = (x_1 - x_2,\ x_1 - x_2,\ x_2 - x_1)$, so $\mathbf{F}(j(x,x)) = (x_1 - x_2)(\pi_1 + \pi_2 - \pi_3)$; the labels of $(1,0) \to (1,0)$ and $(0,1) \to (0,1)$ are $(1,1,-1)$ and $(-1,-1,1)$. Sums follow, and $\partial\gamma = 0$. (b) $\partial\pi_2 = \delta_t - \delta_s$. The sums use Proposition 7.18(c), and the inflow is Proposition 7.22(iv) along the path $(1,0),(0,1),(1,0),\dots$. (c) The first equivalence is Proposition 7.22(i). By (2.9), $j_1 = y_1 - x_2$ and $j_3 = y_2 - x_1$ vanish together exactly when $y = (x_2, x_1)$, and Table 2.3 lists the labels of these four transitions and the labels with three nonzero components. (d) By Theorem 7.19 the records of positive probability are among the four words, all of which have $j_2 \neq 0$ at every posting; with $w_\circ + w_\circ' = 1$ or $w_+ + w_- = 1$ only two of them carry probability. Then apply (a), (b) and Proposition 7.22(iii). □

Measured from the reference vertex, ψ_+ holds a unit charge at s and ψ_- at t, and their circulations pass through without moving it; the oscillation carries one unit of charge across the very edge the field-current window reads.

Remark 7.24 (Orientation). The sense of the circulation is fixed by the dictionary's orientation of π_3. With $\hat{x}$ and $\hat{y}$ drawn as the usual axes, ψ_+ circulates counterclockwise and ψ_- clockwise, along $[s \to b_0] + [b_0 \to u] + [u \to t] + [t \to s]$. A second chart of the framework, built for a negatively charged source, books the same labels with the opposite sign, and at the occupation of ψ_+ it posts the clockwise loop. The two constructions displace the field in opposite senses at that occupation. Which sign an actual field displacement carries is not derived here.

7.7.3 The Gauss field has a price

The loop flux is the flux of the Gauss field through the loop edge (Proposition 2.9). We price each edge flux by the convex dual of twice the recognition cost and compute the expected price along a sure subject's record. The answer separates the modes: bounded for the oscillation, unbounded for any positive persistent weight.

Definition 7.25 (Electric density and bare electric energy)**.** The *electric density* is

$$\varrho(z) = z\,\mathrm{arsinh}(z/2) - \sqrt{z^2+4} + 2, \qquad z \in \mathbb{R}.$$

For $\eta \in \mathbb{R}$, a loop flux $n \in \mathbb{Z}$ and an occupation x, the *bare electric energy* is

$$\mathscr{E}_\eta(n,x) = \eta \sum_{i=1}^{3} \varrho\big(m_i(n,x)\big), \qquad m(n,x) = (n + x_1,\ n,\ -n + x_2),$$

the edge flux (2.10) with both background charges set to 0. Unlike the record law, the energy depends on the background charges; the values below and Remark 7.30 are for this choice.

Proposition 7.26 (The density is the dual of the cost)**.**

(i) $\varrho(z) = \max_{r\in\mathbb{R}}\big(zr - 2J(e^r)\big) = \max_{r\in\mathbb{R}}\big(zr - 2(\cosh r - 1)\big)$*, the maximum attained at* $r = \mathrm{arsinh}(z/2)$*.*

(ii) ϱ *is even,* $\varrho(0) = 0$ *and* $\varrho(z) > 0$ *for* $z \neq 0$*. Its derivative is* $\varrho'(z) = \mathrm{arsinh}(z/2)$*, so* ϱ *increases strictly with* $|z|$*. At unit flux,* $\varrho(1) = 2 - \sqrt{5} + \ln\varphi$*, with* φ *the golden ratio.*

(iii) $\varrho(z) \geq |z|\big(\mathrm{arsinh}(|z|/2) - 1\big)$ *for every* z*, and this is at least* $|z|$ *when* $|z| \geq 2\sinh 2$*.*

(iv) For $\eta \geq 0$*,* $\mathscr{E}_\eta(n,x) \geq \eta\,\varrho(n) \geq 0$*.*

Proof. (i) For fixed z the function $r \mapsto zr - 2(\cosh r - 1)$ is strictly concave, with derivative $z - 2\sinh r$, which vanishes at $r_* = \mathrm{arsinh}(z/2)$. Since $\cosh r_* = \sqrt{1 + z^2/4}$, the maximum is $z\,\mathrm{arsinh}(z/2) - \sqrt{z^2+4} + 2$. (ii) arsinh is odd, so ϱ is even. Differentiating, $\varrho'(z) = \mathrm{arsinh}(z/2) + z/\sqrt{z^2+4} - z/\sqrt{z^2+4} = \mathrm{arsinh}(z/2)$, which has the sign of z; with $\varrho(0) = 0$ this gives positivity and strict increase in $|z|$. Finally $\mathrm{arsinh}(1/2) = \ln\big(\frac{1}{2} + \frac{\sqrt{5}}{2}\big) = \ln\varphi$. (iii) $\sqrt{z^2+4} \leq |z| + 2$, so $\varrho(z) = \varrho(|z|) \geq |z|\,\mathrm{arsinh}(|z|/2) - |z|$; and $\mathrm{arsinh}(|z|/2) \geq 2$ exactly when $|z| \geq 2\sinh 2$. (iv) The middle component of $m(n,x)$ is n, and every term is nonnegative by (ii). □

Part (i) is why we call ϱ the price of the field: it is the convex dual of twice the recognition cost $J(e^r) = \cosh r - 1$, the same cost that makes unit postings the cheapest ledger extensions

(Proposition 2.1). Reading $\mathscr{E}_\eta$ as the electric energy of the Gauss field is an identification, and the scale $\eta > 0$ is supplied. $\mathscr{E}_\eta$ is the energy of the bare field. It omits the energy of whatever interaction drives the step and supplies its writes, and it uses no cutoff on the flux. Even one posting of a persistent mode raises it: $\mathscr{E}_\eta(1,(1,0)) - \mathscr{E}_\eta(0,(1,0)) = \eta\big(\varrho(2) + \varrho(1)\big) > 0$, although the cell returns to the same occupation and the same internal state.

Definition 7.27 (Expected field energy along the record)**.** For $q \in \mathbb{Z}$ and a unit cell vector f, let $v = \delta_q \otimes f$ be the one-loop genesis state at loop q. Its *expected field energy at stage n* is

$$\varepsilon_n(q,f) = \sum_{n' \in \mathbb{Z}} \sum_{h \in \mathscr{J}^n} \sum_{x \in \mathscr{O}} \big\| (|x\rangle\langle x| \otimes I)\,(T^n v)(n',h) \big\|^2 \, \mathscr{E}_\eta(n',x).$$

This is the Born expectation, in the stage-n history, of the bare electric energy at each coordinate: loop flux n', record h, occupation x. By Theorem 2.14(b) the history at record h sits at the single loop $q + \mathrm{acc}(h)$, so the sum over n' has one nonzero term for each record and $\varepsilon_n(q,f)$ is finite.

Theorem 7.28 (Field energy along a sure subject)**.** *Let $\eta \in \mathbb{R}$, $q \in \mathbb{Z}$, f a unit cell vector in the aligned sector, $w_\psi = |\langle \psi, f\rangle|^2$, and $n \geq 1$. Then*

$$\begin{aligned}\varepsilon_n(q,f) = {} & w_+ \, \mathscr{E}_\eta(q+n,(1,0)) + w_- \, \mathscr{E}_\eta(q-n,(0,1)) \\ & + w_\circ \, \mathscr{E}_\eta\big(q + (n \bmod 2), x_n\big) + w'_\circ \, \mathscr{E}_\eta\big(q - (n \bmod 2), x_{n+1}\big).\end{aligned}$$

For the four modes themselves and every $n \geq 0$: $\varepsilon_n(q,\psi_+) = \mathscr{E}_\eta(q+n,(1,0))$ and $\varepsilon_n(q,\psi_-) = \mathscr{E}_\eta(q-n,(0,1))$; $\varepsilon_n(q,\psi_\circ)$ is $\mathscr{E}_\eta(q,(1,0))$ for even n and $\mathscr{E}_\eta(q+1,(0,1))$ for odd n; and $\varepsilon_n(q,\psi'_\circ)$ is $\mathscr{E}_\eta(q,(0,1))$ for even n and $\mathscr{E}_\eta(q-1,(1,0))$ for odd n.

Proof. By Proposition 7.18(d) the stage-n history is a vector of squared norm w_ψ at a single loop and a single occupation on each of the four words, and vanishes on every other record; those loops and occupations are the arguments of $\mathscr{E}_\eta$ above. A vector at loop n' and one occupation x contributes its squared norm times $\mathscr{E}_\eta(n',x)$. For a single mode the same computation holds at $n = 0$, where the history is the mode itself. □

Theorem 7.29 (Bounded field energy picks out the oscillation)**.** *Let $\eta > 0$, $q \in \mathbb{Z}$, f a unit cell vector in the aligned sector and $v = \delta_q \otimes f$. The following are equivalent:*

(i) the expected field energy $\varepsilon_n(q,f)$ is bounded in n;
(ii) $w_\circ + w'_\circ = 1$;
(iii) $w_+ + w_- = 0$;
(iv) v passes every field-current window with probability 1, and fails with probability 1 the first window of some (equivalently, of every) cut that holds an outer account but not the field-current account.

Moreover $\varepsilon_n(q,\psi_\pm) \to \infty$, with $\varepsilon_n(q,\psi_\pm) \geq \eta(n - |q|)$ for $n \geq |q| + 2\sinh 2$, while $\varepsilon_n(q,\psi_\circ)$ and $\varepsilon_n(q,\psi'_\circ)$ each alternate between two values.

Proof. (ii) and (iii) are equivalent because the four weights sum to 1. (iii) and (iv) are equivalent by Theorem 7.20(c): the field-current windows pass surely because v has aligned weight one,

and the first window of such a cut passes with probability $w_+ + w_-$. (ii) implies (i): by Theorem 7.28, for $n \geq 1$ the energy is at most $w_\circ\big(\mathscr{E}_\eta(q,(1,0)) + \mathscr{E}_\eta(q+1,(0,1))\big) + w'_\circ\big(\mathscr{E}_\eta(q,(0,1)) + \mathscr{E}_\eta(q-1,(1,0))\big)$. (i) implies (iii): suppose $w_+ > 0$. Every term of Theorem 7.28 is nonnegative by Proposition 7.26(iv), so $\varepsilon_n(q,f) \geq w_+\,\eta\,\varrho(q+n)$. For $n \geq |q| + 2\sinh 2$ we have $|q+n| \geq n - |q| \geq 2\sinh 2$, and Proposition 7.26(iii) gives $\varrho(q+n) \geq n - |q|$, which is unbounded. The case $w_- > 0$ is the same with $q - n$. The last sentence is the case of a single mode in Theorem 7.28, with the same bound. □

Theorem 7.29 compares two properties of states; it does not say that nature selects the state of bounded energy, and no cut is selected by cost here.

Remark 7.30 (Where the oscillation is cheapest). Let $\eta > 0$. The two values of $\varepsilon_n(q,\psi_\circ)$ sum to $3\eta\big(\varrho(q)+\varrho(q+1)\big)$, and those of $\varepsilon_n(q,\psi'_\circ)$ to $3\eta\big(\varrho(q-1)+\varrho(q)\big)$: in each case 3η times the density summed over the two loop-edge fluxes the oscillation visits. Indeed $m(q,(1,0)) = (q+1,q,-q)$ and $m(q+1,(0,1)) = (q+1,q+1,-q)$, and similarly for $\psi'_\circ$. Since ϱ is even and increases strictly with $|z|$ (Proposition 7.26(ii)), the sum is least exactly when one visited flux is 0, that is, for $q \in \{-1,0\}$ along $\psi_\circ$ and $q \in \{0,1\}$ along $\psi'_\circ$; there the energy alternates between $\eta\varrho(1)$ and $2\eta\varrho(1)$. Otherwise both visited fluxes are nonzero and one has absolute value at least 2, so the two values sum to at least $3\eta\big(\varrho(1)+\varrho(2)\big)$. Nothing in the native step moves a state toward that loop: the record law does not depend on the loop at all (Theorem 2.17(e)).

Theorems 7.28 and 7.29 are stated for one-loop genesis states. A state spread over many loops needs a summability condition on its loop distribution before its expected energy is finite.

Three questions about the field are open. Does the field displacement posted by the oscillation, whose current is a single open edge, have a nonzero transverse profile, and what is its coherent energy beside that of the circulating loop? Which sign does an actual displacement carry (Remark 7.24)? And does the split of Theorem 7.29 survive when the interaction and control energies of the step are added, and for a general loop distribution? One composition question is open in the same terms. On two native cells, are the joint sure subjects of both field-current windows with bounded joint field energy exactly the products of two oscillations, or do entangled pairs of oscillations also qualify? It is not decided here. Chapter 12 develops the networks on which that question lives.

7.8 The leak never stops

The selection does not delete the other states. A state outside the aligned sector does not leak once and fall silent: its present cells keep returning to the neutral sector, and the load-weighted neutral weight has an exact limit. Write

$$\ell_k(\nu) = \sum_{h \in \mathscr{J}^k} \mathbb{P}_\nu(h)\, w_{\mathrm{ne}}(\rho_{\nu,h}) = \mathrm{tr}\big(\rho_\nu \Phi^k(Q_{\mathrm{ne}})\big)$$

for the expected neutral weight of the present after k postings, and $w_{\mathrm{oz}}(\rho) = \mathrm{tr}(\rho Q_{\mathrm{oz}})$ for the one-zero weight.

Theorem 7.31 (The long-run neutral weight)**.** *For every unit genesis state ν,*

$$\lim_{k\to\infty} \ell_k(\nu) = \ell(\nu) := \tfrac{1}{2}\, w_{\mathrm{oz}}(\rho_\nu) + \tfrac{2}{7}\big(1 - w_{\mathrm{oz}}(\rho_\nu) - w_{\mathrm{al}}(\rho_\nu)\big),$$

with $|\ell_{m+1}(v) - \ell(v)| \le 4^{-\lfloor m/8 \rfloor}\big(1 + |\ell(v)|\big)$ *for every* $m \ge 0$. *In particular the ready vector has* $\ell_k \to \frac{2}{7}$.

Proof. Let $\Lambda = \frac{1}{2}Q_{\mathrm{oz}} + \frac{2}{7}(I - Q_{\mathrm{oz}} - Q_{\mathrm{al}})$. The pulls of the neutral sector satisfy

$$8\,\Phi^6(Q_{\mathrm{ne}}) + 4\,\Phi^5(Q_{\mathrm{ne}}) - 2\,\Phi^3(Q_{\mathrm{ne}}) - 2\,\Phi^2(Q_{\mathrm{ne}}) - \Phi(Q_{\mathrm{ne}}) = 7\Lambda, \tag{7.3}$$

a direct exact computation from Table A.7, entry by entry. For example, at $p \otimes p$ on a mixed block the left side is $8\cdot\frac{18}{64} + 4\cdot\frac{20}{64} - 2\cdot\frac{1}{4} - 2\cdot\frac{1}{4} - \frac{1}{2} = 2$, and 7Λ there is $7\cdot\frac{2}{7}$, both cells being on their pole-zero plane. The free pull fixes Λ: $\Phi(Q_{\mathrm{al}}) = Q_{\mathrm{al}}$ by Lemma 7.12, $\Phi(I) = I$ by Proposition 2.12, and $\Phi(Q_{\mathrm{oz}}) = Q_{\mathrm{oz}}$ (Theorem 3.14). The last identity is also a two-line consequence of the pole-zero rule: through a block into a mixed output each cell keeps its status, and through a block into a balanced output both cells reverse it, which exchanges the two terms of the one-zero block; in either case $M_{yx}^{\top}(Q_{\mathrm{oz}})_y M_{yx} = M_{yx}^{\top} M_{yx}(Q_{\mathrm{oz}})_x$, and $\sum_y M_{yx}^{\top} M_{yx} = I$. Applying Φ^m to (7.3) and taking the trace against ρ_v gives $8\ell_{m+6} + 4\ell_{m+5} - 2\ell_{m+3} - 2\ell_{m+2} - \ell_{m+1} = 7\ell(v)$, since $\mathrm{tr}(\rho_v\Lambda) = \ell(v)$. The coefficients on the left add up to 7, so the deviations $d_m = \ell_{m+1} - \ell(v)$ satisfy

$$8\,d_{m+5} = d_m + 2\,d_{m+1} + 2\,d_{m+2} - 4\,d_{m+4},$$

whose characteristic polynomial is $8x^5 + 4x^4 - 2x^2 - 2x - 1$. Let S be its 5×5 companion matrix acting on $(d_m, \dots, d_{m+4})$. Every row of S^8 has absolute sum at most $\frac{15}{64} \le \frac{1}{4}$ (direct exact computation). Since $0 \le \ell_k(v) \le 1$, every $|d_i| \le 1 + |\ell(v)|$; applying S^8 repeatedly to the vector starting at $i = m \bmod 8$ gives the bound, and the limit follows. For the ready vector, $w_{\mathrm{oz}} = w_{\mathrm{al}} = 0$, because both cells sit on pole lines of c. □

The polynomial $8x^5 + 4x^4 - 2x^2 - 2x - 1$ is the quintic that governs the long-run window of Chapter 3. Every root has modulus below one, and the dominant root is real, near 0.7792; the bound $4^{-1/8} \approx 0.841$ per posting is a safe rate, not the sharp one. The limit separates two classes. A state of one-zero weight one, such as ν of Proposition 7.3, has $\ell = \frac{1}{2}$. A state with no one-zero and no aligned weight, such as r or ζ, has $\ell = \frac{2}{7}$. The chance state, with $w_{\mathrm{oz}} = 0$ and $w_{\mathrm{al}} = \frac{9}{25}$, has $\ell = \frac{16}{25}\cdot\frac{2}{7} = \frac{32}{175}$: its aligned part never leaks, and its opposed part leaks like the ready vector.

Nothing in this chapter selects or prepares a genesis state, or selects the oscillation, or attaches experience to a mode. A state with $0 < w_{\mathrm{al}} < 1$ is neither selected nor excluded at genesis; Chapter 8 shows that its own record drives its present, almost surely, to aligned weight 0 or 1.

Chapter 8

The record decides

A genesis state with aligned weight strictly between 0 and 1 is neither selected nor excluded by the postings of Chapter 7. Its own record settles the matter. Along every infinite record, without exception, the aligned weight of the present cell converges to 0 or to 1, and under the law on infinite records it converges to 1 with probability equal to the genesis aligned weight.

Throughout, v is a genesis state, a unit vector of $\ell^2(\mathbb{Z}) \otimes \mathscr{H}$, with cell density ρ_v and record law $\mathbb{P}_v$ (Chapter 2). For a record h write

$$a_h = w_{\mathrm{al}}(\rho_{v,h}) \quad \text{if } \mathbb{P}_v(h) > 0, \qquad a_h = 0 \quad \text{if } \mathbb{P}_v(h) = 0,$$

the *present aligned weight*; reading it as the state's present membership in the selected class reads the present cell as the physical present (Chapter 2). The convention at probability zero never affects a probability, since every sum below weights a_h by $\mathbb{P}_v(h)$. The four modes are indexed by $m \in \{+, -, \circ, \circ'\}$, with mode weights $w_m(\rho) = \langle \psi_m, \rho\, \psi_m \rangle$ summing to w_{al} (Chapter 7). For a set S of records of one length, $E(S) = \sum_{w \in S} E_w$ and $\mathbb{P}_v(S) = \mathrm{tr}(\rho_v E(S))$.

8.1 Writing never mixes the aligned sector with the rest

The record can decide only if writing it keeps the aligned part of a state apart from the rest. It does, and the reason is a general property of projections that the free pull leaves fixed.

Lemma 8.1 (Fixed projections commute with every slice)**.** *Let E be an orthogonal projection on $\mathscr{H}$ with $\Phi(E) = E$. Then $V_j E = E V_j$ for every label j, and $V_h E = E V_h$ for every record h.*

Proof. Since Φ is unital (Proposition 2.12), $\Phi(I - E) = I - E$. Put $X_j = E V_j (I - E)$. Then

$$\sum_j X_j^* X_j = (I - E)\, \Phi(E)\, (I - E) = (I - E) E (I - E) = 0,$$

and a sum of positive operators vanishes only if each does, so $X_j = 0$, that is $E V_j = E V_j E$. The same argument with E and $I - E$ exchanged gives $(I - E) V_j E = 0$, that is $V_j E = E V_j E$. Hence $V_j E = E V_j$, and the record operators V_h, products of slices, commute with E too. □

The aligned sector is fixed by the free pull (Lemma 7.12), and so is the one-zero sector Q_{oz} of Chapter 3 (Theorem 3.14(b)); so both commute with every slice.

Split the genesis state along the sector: $v = Q_{\mathrm{al}} v + (I - Q_{\mathrm{al}}) v$, slice by slice in the loop flux. For a record h put

$$\alpha_h = \mathrm{tr}\big(V_h Q_{\mathrm{al}} \rho_v Q_{\mathrm{al}} V_h^*\big), \qquad \beta_h = \mathrm{tr}\big(V_h (I - Q_{\mathrm{al}}) \rho_v (I - Q_{\mathrm{al}}) V_h^*\big),$$

the probability of h written by the aligned part and by the rest. Both are nonnegative, being sums of squared norms.

Proposition 8.2 (The record law is a mixture)**.** *For every genesis state ν and every record h:*

(a) $\mathbb{P}_\nu(h) = \alpha_h + \beta_h$*: the record law of ν is the sum of the record laws of its aligned part and of its rest, with no cross term;*
(b) $\mathbb{P}_\nu(h)\, a_h = \alpha_h$ *and* $\mathbb{P}_\nu(h)(1 - a_h) = \beta_h$*;*
(c) $\sum_{j \in \mathcal{J}} \mathbb{P}_\nu(h \cdot j)\, a_{h\cdot j} = \mathbb{P}_\nu(h)\, a_h$*, and* $\sum_{h \in \mathcal{J}^n} \mathbb{P}_\nu(h)\, a_h = w_{\mathrm{al}}(\rho_\nu)$ *for every n.*

Proof. (a) By Lemma 8.1, $V_h Q_{\mathrm{al}} = Q_{\mathrm{al}} V_h$. The cross term $\operatorname{tr}(V_h Q_{\mathrm{al}} \rho_\nu (I - Q_{\mathrm{al}}) V_h^*)$ equals $\operatorname{tr}(Q_{\mathrm{al}} V_h \rho_\nu V_h^* (I - Q_{\mathrm{al}})) = \operatorname{tr}(V_h \rho_\nu V_h^* (I - Q_{\mathrm{al}}) Q_{\mathrm{al}}) = 0$, and likewise the other one. (b) $\mathbb{P}_\nu(h)\rho_{\nu,h} = V_h \rho_\nu V_h^*$ when $\mathbb{P}_\nu(h) > 0$, so $\mathbb{P}_\nu(h) a_h = \operatorname{tr}(V_h \rho_\nu V_h^* Q_{\mathrm{al}}) = \operatorname{tr}(Q_{\mathrm{al}} V_h \rho_\nu V_h^* Q_{\mathrm{al}}) = \alpha_h$ by commutation. When $\mathbb{P}_\nu(h) = 0$, both α_h and β_h vanish by (a), and so do both sides. The second identity follows from (a). (c) is the martingale property of the present aligned weight in Theorem 3.17, written with (b). □

Part (a) is the point: coherence between the aligned sector and its complement never reaches a record, so whatever the record decides, it decides by telling two laws apart.

8.2 A record that decides slowly

One state shows the whole mechanism: the one-zero carrier

$$\nu = |10\rangle \otimes a \otimes P$$

of Proposition 7.3. Its first cell is dressed to $Da = A$ and its second to $DP = p$ (Table 2.2). The second cell reads $x_1 = 1$; since $-\text{stay}\, p = 0$, it must write $y_2 = 0$, landing on P up to sign, exactly as it does in the persistent mode ψ_+. The first cell sits on the zero plane of the dressed pole charge, $c'a = 0$, so it stays or flips with weight $\frac{1}{2}$ each (Lemma 2.5). Flipping keeps the occupation at $(1, 0)$, writes the persistent label $(1, 1, -1)$, and leaves the first cell at $\frac{1}{\sqrt{2}}(p + a)$, again on the zero plane of c'. Repeating the argument,

$$\mathbb{P}_\nu\big((1, 1, -1)^n\big) = 2^{-n} \qquad (n \geq 0),$$

and after that record the present cell lies wholly in the neutral sector. The state ν has $w_{\mathrm{al}} = 0$ and $w_{\mathrm{ne}} = 1$. It imitates the persistent mode for n postings with probability 2^{-n}.

Example 8.3 (An even mixture)**.** Take $\nu = \frac{1}{\sqrt{2}}(\psi_+ + \nu)$ at one loop flux, so $w_{\mathrm{al}}(\rho_\nu) = w_{\mathrm{ne}}(\rho_\nu) = \frac{1}{2}$. The aligned part writes $(1, 1, -1)^n$ with certainty. By Proposition 8.2, for a record h of length n,

$$\mathbb{P}_\nu(h) = \tfrac{1}{2}\,[h = (1, 1, -1)^n] + \tfrac{1}{2}\,\mathbb{P}_\nu(h), \qquad \alpha_h = \tfrac{1}{2}\,[h = (1, 1, -1)^n],$$

where $[\,\cdot\,]$ is 1 when the condition holds and 0 otherwise. Every record other than $(1, 1, -1)^n$ therefore leaves $a_h = 0$: the first posting that departs from the persistent label refutes the aligned part for good. The record $(1, 1, -1)^n$ itself has probability $\frac{1}{2}(1 + 2^{-n})$ and leaves

$$a_{(1,1,-1)^n} = \frac{1}{1 + 2^{-n}} = \frac{2^n}{2^n + 1},$$

which tends to 1 but reaches it at no finite length. The load-weighted overlap is

$$\sum_{h \in \mathcal{J}^n} \mathbb{P}_\nu(h)\, a_h (1 - a_h) = \tfrac{1}{2}(1 + 2^{-n}) \cdot \frac{2^n}{2^n + 1} \cdot \frac{1}{2^n + 1} = \frac{1}{2(2^n + 1)}.$$

It falls by a factor that tends to one half at each posting.

The opposite extreme is the chance state $\chi = |10\rangle \otimes \big(\frac{3}{5}P \otimes P + \frac{4}{5}A \otimes P\big)$ of Example 3.5, whose rest lies in the opposed sector. Its first label is $(1, 1, -1)$ with probability $\frac{9}{25}$, leaving $a = 1$, or $(0, 1, -1)$ with probability $\frac{16}{25}$, leaving $a = 0$. One posting decides. The difference between the two states is the neutral sector: an opposed state is silent on the loop edge at its second posting at the latest, while a neutral one can keep writing it.

8.3 The record decides in mean, at one half per posting

The even mixture is typical. Summed over records, the undecided part of the present aligned weight falls by one half per posting, for every genesis state. Write $\mathcal{J}_* = \{j \in \mathcal{J} : j_2 \neq 0\}$ for the labels that write the loop edge, so that $\mathcal{J}_*^n$ is the set of records that write it at every posting and $G_n = E(\mathcal{J}_*^n)$.

Theorem 8.4 (The record decides in mean)**.** *Let ν be a genesis state.*

(a) The aligned part writes the loop edge at every posting: $\sum_{h \notin \mathcal{J}_*^n} \alpha_h = 0$ for every n. The rest writes it at every one of $n \geq 2$ postings with probability exactly

$$\sum_{h \in \mathcal{J}_*^n} \beta_h = 2^{-(n-1)}\, w_{\mathrm{ne}}(\rho_\nu).$$

(b) For every $n \geq 2$,

$$\sum_{h \in \mathcal{J}^n} \mathbb{P}_\nu(h)\, a_h(1 - a_h) \leq 2^{-(n-1)}\, w_{\mathrm{ne}}(\rho_\nu).$$

In particular, after $8r$ postings with $r \geq 1$ the left side is at most $2^{-(8r-1)} w_{\mathrm{ne}}(\rho_\nu) \leq 128^{-r}\big(1 - w_{\mathrm{al}}(\rho_\nu)\big)$.

(c) The left side of (b) is nonincreasing in n and tends to 0.

Proof. (a) By Lemma 3.3, $G_n = Q_{\mathrm{al}} + 2^{-(n-1)} Q_{\mathrm{ne}}$ for $n \geq 2$, and $G_1 = \Pi_{\mathrm{mix}}$. In every case $Q_{\mathrm{al}} G_n Q_{\mathrm{al}} = Q_{\mathrm{al}}$, so $\sum_{h \in \mathcal{J}_*^n} \alpha_h = \mathrm{tr}(\rho_\nu Q_{\mathrm{al}} G_n Q_{\mathrm{al}}) = w_{\mathrm{al}}(\rho_\nu) = \sum_{h \in \mathcal{J}^n} \alpha_h$ by Proposition 8.2(c), and the remaining records carry no aligned probability. For $n \geq 2$, $(I - Q_{\mathrm{al}}) G_n (I - Q_{\mathrm{al}}) = 2^{-(n-1)} Q_{\mathrm{ne}}$, because the three sectors are orthogonal; tracing against ρ_ν gives the second statement.

(b) Since $0 \leq a_h \leq 1$, Proposition 8.2(b) gives $\mathbb{P}_\nu(h) a_h(1 - a_h) \leq \min(\alpha_h, \beta_h)$. Bound the terms with $h \in \mathcal{J}_*^n$ by β_h and the others by α_h, and apply (a). For $n = 8r$, $2^{-(8r-1)} \leq 2^{-7r}$ when $r \geq 1$, and $w_{\mathrm{ne}} \leq 1 - w_{\mathrm{al}}$.

(c) Fix a record h and write $P_j = \mathbb{P}_\nu(h \cdot j)$, $x_j = a_{h \cdot j}$. Then $\sum_j P_j = \mathbb{P}_\nu(h)$ and $\sum_j P_j x_j = \mathbb{P}_\nu(h) a_h$ by Proposition 8.2(c), so expanding $\sum_j P_j (x_j - a_h)^2 \geq 0$ gives $\sum_j P_j x_j^2 \geq \mathbb{P}_\nu(h) a_h^2$, and

$$\sum_j P_j x_j (1 - x_j) = \mathbb{P}_\nu(h) a_h - \sum_j P_j x_j^2 \leq \mathbb{P}_\nu(h)\, a_h(1 - a_h).$$

Summing over $h \in \mathcal{J}^n$ shows that the overlap at length $n + 1$ is at most the overlap at length n. It tends to 0 by (b). □

The rate in (b) cannot be improved: for the even mixture of Example 8.3 the bound is 2^{-n} and the overlap is $1/(2(2^n+1))$, so their ratio tends to $\frac{1}{2}$. The factor $\frac{1}{2}$ per posting is the neutral sector's own, not the window's: a cell on the zero plane of the dressed pole charge repeats a loop-edge posting with probability one half. The bound 128^{-r} is the weaker window form, one factor $\frac{1}{128}$ per window of eight postings.

A single record can stay undecided for a long time, as $(1,1,-1)^n$ does in Example 8.3. Section 8.5 shows that every single infinite record decides nonetheless.

8.4 No finite record event decides

A natural hope is that some event of finitely many postings has probability exactly $w_{\mathrm{al}}(\rho_v)$, for every genesis state. There is none. The obstruction is the state v: an event that every selected state writes surely is written by v, which has no aligned weight, with probability at least 2^{-n}.

The four modes write four words. For $n \geq 1$ let $\mathscr{M}_n = \{u_n^+, u_n^-, o_n, o_n'\}$ be the set of the mode words of length n (Proposition 7.18): four different words, since their first labels differ. Let $\mathscr{M}_0$ be the empty record alone. A set $S \subseteq \mathscr{J}^n$ is *surely passed* if $\mathbb{P}_v(S) = 1$ for every genesis state with $w_{\mathrm{al}}(\rho_v) = 1$; these are the states that Chapter 7 selects. Let Q_{zz} be the zero-zero sector of Chapter 7, the rank-8 part of the neutral sector in which both cells lie on the zero plane of c'.

Theorem 8.5 (No finite record event decides)**.**

(a) For every $n \geq 0$, every surely passed set $S \subseteq \mathscr{J}^n$ contains the mode words of length n, and $\mathbb{P}_v(S) \geq 2^{-n}$ while $w_{\mathrm{al}}(\rho_v) = 0$. Consequently, for no n and no $S \subseteq \mathscr{J}^n$ does $\mathbb{P}_v(S) = w_{\mathrm{al}}(\rho_v)$ hold for every genesis state v.

(b) For $n \geq 1$ and $\varepsilon = 2^{-n}$, the mode-word event has the effect

$$E(\mathscr{M}_n) = Q_{\mathrm{al}} + \varepsilon\, Q_{\mathrm{ne}} + (2\varepsilon^2 - \varepsilon)\, Q_{\mathrm{zz}}.$$

Hence $w_{\mathrm{al}} + 2\varepsilon^2 w_{\mathrm{ne}} \leq \mathbb{P}_v(\mathscr{M}_n) \leq w_{\mathrm{al}} + \varepsilon\, w_{\mathrm{ne}}$ for every genesis state, with the weights taken at ρ_v. If $w_{\mathrm{ne}}(\rho_v) = 0$, then $\mathbb{P}_v(\mathscr{M}_n) = w_{\mathrm{al}}(\rho_v)$ exactly.

(c) For $n \geq 1$, $\mathscr{M}_n$ is surely passed, and among surely passed sets it is the best test: the worst false pass $\sup\{\mathbb{P}_v(S) : w_{\mathrm{al}}(\rho_v) = 0\}$ is at least 2^{-n} for every surely passed $S \subseteq \mathscr{J}^n$, and equals 2^{-n} for $S = \mathscr{M}_n$. For $n \geq 2$ the loop-edge event $\mathscr{J}_*^n$ is surely passed and has worst false pass $2^{-(n-1)}$.

Proof. (b) The word $(1,1,-1)^n$ is written only by the transition $(1,0) \to (1,0)$, whose block is $F \otimes F$ with $F = \mathrm{flip}\, D$ (Chapter 2). Conjugating Lemma 2.5 by D gives $F^\top F = \frac{1}{2}(I + c')$, and F commutes with $c' = DcD$ because L commutes with c. So

$$(F^n)^\top F^n = \left(\tfrac{1}{2}(I + c')\right)^n = \Pi_+' + \varepsilon\, \Pi_0',$$

where Π_+', Π_-', Π_0' are the spectral projections of c', and $E_{(1,1,-1)^n} = |10\rangle\langle 10| \otimes (\Pi_+' + \varepsilon\Pi_0') \otimes (\Pi_+' + \varepsilon\Pi_0')$. The word $(-1,-1,1)^n$ gives the same at $|01\rangle$, with block $(-F) \otimes F$. The oscillation words alternate the blocks $M_{01,10} = S \otimes S$ and $M_{10,01} = (-S) \otimes S$ with $S = \mathrm{stay}\, D$, for which $S^\top S = \frac{1}{2}(I - c')$ and S commutes with c'; so the word starting at the mixed occupation x has effect $|x\rangle\langle x| \otimes (\Pi_-' + \varepsilon\Pi_0') \otimes (\Pi_-' + \varepsilon\Pi_0')$. Adding the four effects, on each mixed occupation

$$\textstyle\sum_\pm (\Pi_\pm' + \varepsilon\Pi_0')^{\otimes 2} = \big(\Pi_+' \otimes \Pi_+' + \Pi_-' \otimes \Pi_-'\big) + \varepsilon\big(\sum_\pm (\Pi_\pm' \otimes \Pi_0' + \Pi_0' \otimes \Pi_\pm')\big) + 2\varepsilon^2\, \Pi_0' \otimes \Pi_0'.$$

The first bracket is the aligned block. The neutral block is the second bracket plus $\Pi_0' \otimes \Pi_0'$. This is the formula. For the bounds, $0 \le Q_{zz} \le Q_{ne}$ and $2\varepsilon^2 - \varepsilon \le 0$ for $n \ge 1$.

(a) Each mode, placed at one loop flux, is a genesis state with $w_{al} = 1$ whose record law is a point mass on its own word (Proposition 2.22). So a surely passed S contains $\mathscr{M}_n$. Then $\mathbb{P}_\nu(S) \ge \mathbb{P}_\nu((1,1,-1)^n) = 2^{-n}$ by Section 8.2. If $\mathbb{P}_\nu(S) = w_{al}(\rho_\nu)$ for every ν, then S is surely passed, and ν gives $2^{-n} \le 0$, which is false.

(c) A genesis state with $w_{al} = 1$ has $\mathbb{P}_\nu(\mathscr{M}_n) = 1$ by (b), since then $w_{ne} = 0$. By (b), every ν with $w_{al} = 0$ has $\mathbb{P}_\nu(\mathscr{M}_n) \le \varepsilon w_{ne} \le \varepsilon$, and ν, with $w_{ne} = 1$ and no weight on Q_{zz}, attains ε. A surely passed S contains $\mathscr{M}_n$ by (a), so its worst false pass is at least that of $\mathscr{M}_n$. For the loop-edge event, $\mathbb{P}_\nu(\mathscr{J}_*^n) = w_{al} + 2^{-(n-1)} w_{ne}$ by Lemma 3.3; this is 1 when $w_{al} = 1$, at most $2^{-(n-1)}$ when $w_{al} = 0$, and equal to it at ν. □

So Q_{al} is the norm limit of the unsharp effects $E(\mathscr{M}_n)$ and G_n (Lemma 3.3) and a signed combination of two record effects (Theorem 3.4(c)), but the effect of no single finite record event.

8.5 Every infinite record decides

What no finite stretch of the record can do, the infinite record does. The event is the obvious one, and the decision holds along every record, not only almost every one.

Let $\mathscr{J}^{\mathbb{N}}$ be the space of infinite records, with the law μ_ν of Theorem 2.17(d), and write $\omega_{\le n}$ for the first n labels of ω. The four *infinite mode words* $\omega_+, \omega_-, \omega_\circ, \omega_\circ'$ are the infinite records whose prefixes of length n are u_n^+, u_n^-, o_n and o_n'. Let

$$\mathscr{M} = \{\omega \in \mathscr{J}^{\mathbb{N}} : \ \omega_{\le n} \in \mathscr{M}_n \text{ for every } n \ge 1\},$$

the *mode-record event*, and $a_n(\omega) = a_{\omega_{\le n}}$, the present aligned weight along ω.

Theorem 8.6 (Every infinite record decides)**.** *Let ν be a genesis state with mode weights $w_m = w_m(\rho_\nu)$.*

(a) $\mathscr{M} = \{\omega_+, \omega_-, \omega_\circ, \omega_\circ'\}$. A prefix of a mode word is a mode word, so once a record leaves the mode words it never returns.

(b) If $\omega_{\le N} \notin \mathscr{M}_N$, then $a_n(\omega) = 0$ for every $n \ge N$. Along the infinite mode word ω_m, for every $n \ge 1$,

$$a_n(\omega_m) = \frac{w_m}{w_m + r_n}, \qquad 0 \le r_n \le 2^{-n}\, w_{ne}(\rho_\nu),$$

when $w_m > 0$, and $a_n(\omega_m) = 0$ when $w_m = 0$.

(c) Hence for every infinite record ω the sequence $a_n(\omega)$ converges to 0 or to 1, and it converges to 1 exactly when $\omega = \omega_m$ for a mode with $w_m > 0$.

(d) $\mu_\nu(\mathscr{M}) = w_{al}(\rho_\nu)$ and $\mu_\nu(\{\omega_m\}) = w_m$ for each mode. For $n \ge 1$,

$$w_{al}(\rho_\nu) \le \mu_\nu\big(\omega_{\le n} \in \mathscr{M}_n\big) \le w_{al}(\rho_\nu) + 2^{-n}\, w_{ne}(\rho_\nu),$$

and the error is attained: $\mu_\nu(\mathscr{M}) = 0$ while $\mu_\nu(\omega_{\le n} \in \mathscr{M}_n) = 2^{-n}$.

(e) μ_ν-almost surely, $a_n \to \mathbf{1}_{\mathscr{M}}$. The records along which $a_n \to 1$ have probability $w_{al}(\rho_\nu)$.

Proof. (a) The four words of each length $n \geq 1$ have different first labels, so an $\omega \in \mathscr{M}$ has a first label that names one mode m, and then $\omega_{\leq n}$ is the word of m for every n, that is $\omega = \omega_m$. The prefix statement is visible in the words.

(b) By Proposition 7.18, $V_h\psi_m \neq 0$ exactly when h is the word of m, and then $V_h\psi_m = \pm\psi_m^{(n)}$, the unit mode the word leads to. If $h \notin \mathscr{M}_n$, then $V_hQ_{\mathrm{al}} = 0$, so $\alpha_h = 0$ and $a_h = 0$ by Proposition 8.2(b); by (a) this holds for all later prefixes too. If h is the word of m, then $V_hQ_{\mathrm{al}} = \pm|\psi_m^{(n)}\rangle\langle\psi_m|$, so $\alpha_h = w_m$ and $a_h = w_m/(w_m + \beta_h)$. Put $r_n = \beta_h$. Since Q_{al} commutes with $E(\mathscr{M}_n)$,

$$r_n \leq \sum_{h' \in \mathscr{M}_n} \beta_{h'} = \mathrm{tr}\big(\rho_\nu(I - Q_{\mathrm{al}})E(\mathscr{M}_n)(I - Q_{\mathrm{al}})\big) = \varepsilon\, w_{\mathrm{ne}} + (2\varepsilon^2 - \varepsilon)\, w_{\mathrm{zz}} \leq \varepsilon\, w_{\mathrm{ne}}$$

by Theorem 8.5(b), with $\varepsilon = 2^{-n}$ and $w_{\mathrm{zz}} = \mathrm{tr}(\rho_\nu Q_{\mathrm{zz}}) \geq 0$.

(c) Off $\mathscr{M}$ the sequence is eventually 0 by (b). On ω_m it tends to 1 if $w_m > 0$ and is 0 from the first posting on if $w_m = 0$.

(d) The events $A_n = \{\omega_{\leq n} \in \mathscr{M}_n\}$ decrease, by (a), and their intersection is $\mathscr{M}$. Their probabilities $\mu_\nu(A_n) = \mathbb{P}_\nu(\mathscr{M}_n)$ lie between w_{al} and $w_{\mathrm{al}} + 2^{-n}w_{\mathrm{ne}}$ by Theorem 8.5(b), so continuity from above gives $\mu_\nu(\mathscr{M}) = w_{\mathrm{al}}$. The same argument applied to the cylinders of ω_m, whose probabilities are $w_m + r_n$ by (b), gives $\mu_\nu(\{\omega_m\}) = w_m$. For ν, Theorem 8.5(b) gives $\mathbb{P}_\nu(\mathscr{M}_n) = 2^{-n}$ exactly.

(e) By (c), $a_n(\omega) \to \mathbf{1}_{\mathscr{M}}(\omega)$ except on the words ω_m with $w_m = 0$, which have probability 0 by (d). The records along which $a_n \to 1$ are the ω_m with $w_m > 0$, of total probability $\sum_{w_m > 0} w_m = w_{\mathrm{al}}$. □

The route through the mode-record event shows, beyond the martingale convergence theorem applied to Proposition 8.2(c), that the convergence holds along every record, and names the event on which the limit is 1.

Read as a test, the theorem describes a sequential procedure: reject the aligned sector at the first posting that leaves the mode words. It never rejects a state with $w_{\mathrm{al}} = 1$. It rejects a state with $w_{\mathrm{al}} = 0$ with probability one, and the probability that it has not yet done so after n postings is at most $2^{-n}w_{\mathrm{ne}}$. On a mixture it splits exactly by the aligned weight. In the terms of sequential analysis it is a test of power one with size zero [14]. The samples are the postings of one record, not independent repetitions, and the test exists because every record operator commutes with Q_{al}.

The infinite record also reads which mode is present. Along ω_m with $w_m > 0$, the present cell's weight on the unit mode $\psi_m^{(n)}$ its word leads to (Proposition 7.18) equals $a_n(\omega_m)$, so it tends to 1; for a genesis state with $w_{\mathrm{al}} = 1$ it equals 1 from the first posting on. The present cell may still be spread over the loop flux; each mode word moves the flux deterministically.

The verdicts of Chapter 3 follow the aligned weight. For a record h of positive probability let $\lambda_\nu(h) = \mathrm{tr}(\rho_{\nu,h}W_\infty)$ be its long-run verdict and $\mathrm{tr}(\rho_{\nu,h}W_0)$ its present first-window verdict.

Corollary 8.7 (The verdicts follow the record)**.** *For every record h of positive probability,*

$$a_h \leq \mathrm{tr}(\rho_{\nu,h}W_0) \leq \tfrac{1}{128} + \tfrac{127}{128}\, a_h, \qquad a_h \leq \lambda_\nu(h) \leq 1.$$

Hence along every infinite record on which $a_n \to 1$, in particular along every ω_m with $w_m > 0$, both verdicts tend to 1.

Proof. $W_0 = Q_{\mathrm{al}} + \frac{1}{128}Q_{\mathrm{ne}}$ (Theorem 3.4) and $w_{\mathrm{ne}} \le 1 - w_{\mathrm{al}}$ give the first pair. $W_\infty - Q_{\mathrm{al}} = \frac{1}{448}(I - Q_{\mathrm{al}}) + \frac{3}{1792}Q_{\mathrm{oz}}$ is positive and $W_\infty \le I$ by Theorem 3.14, which gives the second. □

On the records where $a_n \to 0$ the long-run verdict still converges almost surely (Theorem 3.17(d)), but the aligned weight does not fix its limit. The one-zero sector is the missing piece. It is also fixed by the free pull, so by Lemma 8.1 the record law is a mixture across it as well, and the record decides it in the same sense: the present one-zero weight converges almost surely to 0 or 1, and to 1 with probability $w_{\mathrm{oz}}(\rho_\nu) = \mathrm{tr}(\rho_\nu Q_{\mathrm{oz}})$ (Theorem 11.2(c), where $\Lambda_3 = Q_{\mathrm{oz}}$ and $w_3 = w_{\mathrm{oz}}$).

Proposition 8.8 (The long-run verdict settles at three values)**.** *For every genesis state ν, μ_ν-almost surely the long-run verdict $\lambda_\nu(\omega_{\le n})$ converges to* 1*, to $\frac{1}{256}$ or to $\frac{1}{448}$. These limits have probabilities $w_{\mathrm{al}}(\rho_\nu)$, $w_{\mathrm{oz}}(\rho_\nu)$ and $1 - w_{\mathrm{al}}(\rho_\nu) - w_{\mathrm{oz}}(\rho_\nu)$.*

Proof. By Theorem 3.14(b), $\lambda_\nu(h) = \frac{1}{448} + \frac{447}{448}a_h + \frac{3}{1792}z_h$, with z_h the present one-zero weight. Almost surely $a_n \to \mathbf{1}_{\mathscr{M}}$ (Theorem 8.6) and z_n tends to 0 or 1 (Theorem 11.2(c), where $\Lambda_3 = Q_{\mathrm{oz}}$ and $w_3 = w_{\mathrm{oz}}$). The sectors are orthogonal, so $a_h + z_h \le 1$ and the two limits are not both 1. The three cases give $\frac{1}{448} + \frac{447}{448} = 1$, $\frac{1}{448} + \frac{3}{1792} = \frac{1}{256}$ and $\frac{1}{448}$, with the stated probabilities. □

The selection deletes nothing. On its unselected branch a genesis state keeps writing records and keeps passing windows by chance: by Proposition 8.8 its long-run verdict there settles at $\frac{1}{256}$ or $\frac{1}{448}$, never at 0. Averaged over all records, its neutral weight tends to $\ell(\nu) = \frac{2}{7}(1 - w_{\mathrm{al}}) + \frac{3}{14}w_{\mathrm{oz}}$ (Theorem 7.31), which is positive whenever $w_{\mathrm{al}} < 1$.

8.6 After one posting the modes are classical

The record decides which mode a selected state is in. This section shows that, from the first posting on, the modes are also classical alternatives for every read placed after it, and not only for the record law. Before the first posting they are not.

We use the current algebra $\mathscr{A}$ of Chapter 3, the live read algebra $\mathscr{A}_{\mathrm{R}}$ of Definition 5.19, and the late reads $\overline{\mathscr{G}}_{\ge n}$ of Definition 6.1, the native reads made at or after stage n; by Theorem 6.15(iii) the record of length n is central in them. Take a selected state at one loop flux q,

$$\nu_c = \delta_q \otimes \sum_m c_m \psi_m, \qquad c \in \mathbb{C}^4,$$

and write $\nu_m = \delta_q \otimes \psi_m$ for its four modes. By Theorem 7.19 the record law of a unit ν_c sees the four weights $|c_m|^2$ and nothing else; in particular it cannot tell $\alpha\psi_\circ + \beta\psi'_\circ$ from $\alpha\psi_\circ - \beta\psi'_\circ$. Neither can any read built from currents alone, but a native read can.

Theorem 8.9 (The hop phase)**.** *Let $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$, let $q \in \mathbb{Z}$, and let $\nu_\pm = \delta_q \otimes (\alpha\psi_\circ \pm \beta\psi'_\circ)$.*

(i) For every $B \in \mathscr{A}$, $\mathrm{tr}(\rho_{\nu_+} B) = \mathrm{tr}(\rho_{\nu_-} B)$.

(ii) The four modes lie in the joint sector $\Pi_{(0,0)}$ of Chapter 6. The hop exchange $X_{\mathrm{hop}} = \psi_\circ\psi_\circ'^* + \psi_\circ'\psi_\circ^*$ *is self-adjoint, lies in $\mathscr{A}_{\mathrm{R}}$ and not in $\mathscr{A}$, and*

$$\mathrm{tr}(\rho_{\nu_\pm} X_{\mathrm{hop}}) = \pm 2\,\mathrm{Re}(\bar\alpha\beta).$$

So X_{hop} separates ν_+ from ν_- exactly when $\mathrm{Re}(\bar\alpha\beta) \neq 0$; for $\alpha = \frac{3}{5}$, $\beta = \frac{4}{5}$ it reads $\frac{24}{25}$ and $-\frac{24}{25}$. Its read $X_{\mathrm{hop}}[0,\varnothing]$ placed at genesis lies in $\overline{\mathscr{G}}_{\geq 0}$ and, for every $c \in \mathbb{C}^4$, has expectation $2\,\mathrm{Re}(\bar c_\circ c_{\circ'})$ on $\iota_0\nu_c$.

(iii) The self-adjoint $Y_{\mathrm{hop}} = i(\psi_\circ\psi_\circ'^* - \psi_\circ'\psi_\circ^*)$ also lies in $\mathscr{A}_{\mathrm{R}}$, with $\mathrm{tr}(\rho_{\nu_\pm} Y_{\mathrm{hop}}) = \mp 2\,\mathrm{Im}(\bar\alpha\beta)$. Hence $\mathscr{A}_{\mathrm{R}}$ separates ν_+ from ν_- exactly when $\alpha\beta \neq 0$.

Proof. (i) Output and label fix the input, so every effect E_w is block diagonal in the occupation (proof of Proposition 2.12), and so is every element of $\mathscr{A}$. Since $\psi_\circ$ sits at $(1,0)$ and $\psi_\circ'$ at $(0,1)$, $\langle\psi_\circ, B\psi_\circ'\rangle = 0$ for $B \in \mathscr{A}$, and both states give $|\alpha|^2\langle\psi_\circ, B\psi_\circ\rangle + |\beta|^2\langle\psi_\circ', B\psi_\circ'\rangle$.

(ii) $L^2P = -P$ and $L^2A = -A$, so every mode is odd under both half-turns and lies in $\Pi_0^{(1)}\Pi_0^{(2)} = \Pi_{(0,0)}$. X_{hop} is self-adjoint by its form; it maps the range of $\Pi_{(0,0)}$ into itself and vanishes on its complement, so it lies in $\mathscr{A}_{\mathrm{R}}$, which contains every operator on the range of each joint sector (Theorem 6.13(i), Corollary 6.14). X_{hop} maps occupation $(0,1)$ to $(1,0)$, so it is not block diagonal in the occupation and $X_{\mathrm{hop}} \notin \mathscr{A}$. With $g_\pm = \alpha\psi_\circ \pm \beta\psi_\circ'$, $\langle g_\pm, X_{\mathrm{hop}} g_\pm\rangle = \pm(\bar\alpha\beta + \bar\beta\alpha) = \pm 2\,\mathrm{Re}(\bar\alpha\beta)$, and $2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$. The placed read of an element of $\mathscr{A}_{\mathrm{R}}$ lies in the algebra generated by the placed reads of the effects, so $X_{\mathrm{hop}}[0,\varnothing] \in \overline{\mathscr{G}}_{\geq 0}$; since X_{hop} annihilates ψ_+ and ψ_-, its expectation on $\iota_0\nu_c$ is $\bar c_\circ c_{\circ'} + c_\circ \bar c_{\circ'}$.

(iii) Y_{hop} also maps the range of $\Pi_{(0,0)}$ into itself and vanishes on its complement, so $Y \in \mathscr{A}_{\mathrm{R}}$, and $\langle g_\pm, Y_{\mathrm{hop}} g_\pm\rangle = \pm i(\bar\alpha\beta - \alpha\bar\beta) = \mp 2\,\mathrm{Im}(\bar\alpha\beta)$. If $\alpha\beta \neq 0$, then $\bar\alpha\beta \neq 0$ and X_{hop} or Y_{hop} separates the two states. If $\alpha\beta = 0$, then g_+ and g_- differ by a phase, $\rho_{\nu_+} = \rho_{\nu_-}$, and no read separates them. □

The relative phase is a coherence of the charge's position, at s or at t (Section 7.7). A read built from current effects cannot couple the two positions, since every current effect is diagonal in the occupation; a read that exchanges them sees the phase, and the live read algebra contains such reads. Which finite combination of placed current words and retained reads equals X_{hop} is not determined here; the theorem shows only that X_{hop} lies in the algebra they generate. The phase belongs to genesis: the words o_n and o_n' differ in their first letter, and by the next theorem no read made after the first posting sees it.

Theorem 8.10 (The modes part at the first posting)**.**

(a) For $n \geq 1$ and every $A \in \overline{\mathscr{G}}_{\geq n}$,

$$\langle \iota_0\nu_c,\ A\,\iota_0\nu_c\rangle = \sum_m |c_m|^2\,\langle \iota_0\nu_m,\ A\,\iota_0\nu_m\rangle.$$

(b) For $n \geq 1$, ν_c and $\nu_{c'}$ have the same expectation on every element of $\overline{\mathscr{G}}_{\geq n}$ if and only if $|c_m| = |c_m'|$ for every mode, and, for unit vectors, if and only if their record laws of length n agree. At $n = 0$ the criterion fails: by Theorem 8.9(ii), $c = (0,0,1,1)$ and $c' = (0,0,1,-1)$ give different expectations of $X_{\mathrm{hop}}[0,\varnothing]$.

Proof. (a) For a record h of length n let $\mathrm{Ar}_n(\mathbf{1}_h)$ be the archive read of its indicator (Definition 6.1). These are projections with $\sum_h \mathrm{Ar}_n(\mathbf{1}_h) = 1$, and each commutes with every $A \in \overline{\mathscr{G}}_{\geq n}$. So

$$\langle \iota_0 \nu_m, A\, \iota_0 \nu_{m'} \rangle = \sum_{h \in \mathscr{J}^n} \langle \mathrm{Ar}_n(\mathbf{1}_h) \iota_0 \nu_m,\ A\ \mathrm{Ar}_n(\mathbf{1}_h) \iota_0 \nu_{m'} \rangle.$$

Since $\iota_0 = \iota_n T^n$ and $(T^n \nu_m)(\cdot, h) = \delta_{q+\mathrm{acc}(h)} \otimes V_h \psi_m$, the vector $\mathrm{Ar}_n(\mathbf{1}_h)\iota_0 \nu_m$ vanishes unless h is the word of m (Proposition 7.18). For $m \neq m'$ the two words differ, so every term vanishes. Expanding ν_c leaves only the diagonal terms.

(b) If the weights agree, (a) gives equal expectations. Conversely, the identity read placed at stage n and the word h of m lies in $\overline{\mathscr{G}}_{\geq n}$ and has expectation $\| V_h \sum_{m'} c_{m'} \psi_{m'} \|^2 = |c_m|^2$. For unit vectors the record law of length n is $\mathbb{P}_{\nu_c}(h) = \sum_m |c_m|^2 [h \text{ is the word of } m]$ by Theorem 7.19, and the four words are distinct. □

The same holds for a selected state $\sum_q \delta_q \otimes \sum_m c_{q,m} \psi_m$ spread over finitely many loop fluxes, with the weights $\sum_q |c_{q,m}|^2$ in place of $|c_m|^2$: the late reads couple no two loop fluxes, and the expectation of a late read does not depend on the loop flux at which a mode starts. The first posting writes which mode is present into the record, and from then on the record law is a complete account of what any native read placed after the first posting can learn about a selected state. Summed over the mode words, the posting acts on a selected state as

$$\sum_{h \in \mathscr{M}_n} V_h \Big(\sum_m c_m \psi_m \Big) \Big(\sum_m c_m \psi_m \Big)^* V_h^* = \sum_m |c_m|^2\, |\psi_m^{(n)}\rangle\langle \psi_m^{(n)}|,$$

the mixture of the four modes after n postings: each mode goes to its later mode with certainty, and the coherence between modes is gone.

Two premises enter in this section and nowhere else in the chapter. Placed reads follow the corner convention of Chapter 6. And the operations available to a participant after n postings are taken to be the late reads $\overline{\mathscr{G}}_{\geq n}$; whether a read placed earlier can act on a history that already carries newer records is the open question of Chapter 6. If it can, the stage-zero coherence of Theorem 8.9 remains in reach.

8.7 Every posting is a witness

A selected state writes its mode redundantly. Each posting on its own is a complete record of the mode, so n postings are n disjoint witnesses, and two readers of different postings agree with certainty.

At posting t the four modes write the t-th letters of their words: ψ_+ writes $(1, 1, -1)$, ψ_- writes $(-1, -1, 1)$, $\psi_\circ$ writes $(0, (-1)^{t+1}, 0)$ and $\psi_\circ'$ writes $(0, (-1)^t, 0)$. The persistent labels write both outer accounts and the oscillation labels write neither, and the two oscillation labels at one posting have opposite loop-edge current. So the four labels at each posting are distinct, and the *decoder* d_t that returns the mode whose label at posting t is j, and nothing if there is none, recovers the mode from one label.

Theorem 8.11 (Redundant records of a selected state)**.** *Let $n \geq 1$ and let $\nu_c = \delta_q \otimes \sum_m c_m \psi_m$ be a unit genesis state. Then:*

(a) for each posting $t \le n$, the label at t is the label of mode m with probability $|c_m|^2$;
(b) for postings t, t' and modes $m \neq m'$, the probability that posting t writes the label of m and posting t' writes the label of m' is 0; the decoders of any two postings agree with probability 1;
(c) every record of positive probability is the word of a mode m with $c_m \neq 0$, and on these records the label at any one posting determines the whole record. Each of the n postings determines the mode.

Proof. By Theorem 7.19, $\mathbb{P}_{\nu_c}(h) = \sum_m |c_m|^2\,[h \text{ is the word of } m]$: the record law is the image of the law $(|c_m|^2)_m$ of the modes under the map that sends a mode to its word. All three statements follow because the four words differ at every posting. □

This is a form of the redundancy that the program of quantum Darwinism takes as the mark of an objective fact [47], with one difference that matters: the fragments here are successive postings of one record, ordered in time, not separate parts of an environment. Whether several boundaries holding different parts of the record realize several readers is not addressed.

Redundancy is a property of the selected states, not of every record. The chance state χ has the records of length three

$$h_1 = \big((1,1,-1),(1,1,-1),(1,1,-1)\big), \quad h_2 = \big((0,1,-1),(0,0,0),(0,0,0)\big),$$
$$h_3 = \big((0,1,-1),(0,0,0),(1,0,0)\big), \quad h_4 = \big((0,1,-1),(1,0,1),(0,0,0)\big),$$
$$h_5 = \big((0,1,-1),(1,0,0),(1,1,-1)\big),$$

with $\mathbb{P}_\chi(h_1) = \frac{9}{25}$ and $\mathbb{P}_\chi(h_i) = \frac{1}{25}$ for $i = 2,3,4,5$. After the label $(0,1,-1)$ the present cell is $|00\rangle \otimes A \otimes P$, on a balanced occupation with both cells on the zero plane of c, so the second posting splits each cell evenly; a direct computation shows that the third does too, which gives $\frac{16}{25} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{25}$. The records h_2 and h_3 agree at posting 2 and differ at posting 3, while h_2 and h_4 agree at posting 3 and differ at posting 2: neither of these postings determines the other. The records h_1 and h_5 share their third label, so posting 3 does not determine posting 1, which is the label that settles the chance state's verdict. The decayed branch even re-enters the persistent label. A late posting of a state outside the aligned sector is not a witness of its first record.

The record has now told us which of the four modes a selected state is in, and no read placed after the first posting sees more; what it misses is the hop phase at genesis. That is one instance of a general question: on the whole cell, which part of the state does the record law read, and what does it miss? Chapter 9 answers it exactly.

Chapter 9

What the record law reads

A record is a list of current labels, and its law is the Born law $\mathbb{P}_\nu(h) = \mathrm{tr}(\rho_\nu E_h)$ of Theorem 2.17. Two genesis states therefore have the same record law at every length exactly when their cell densities agree on every effect E_w. What the record law reads is one subspace of 64×64 matrices, the span of the effects, and this chapter determines it. It has dimension 92, and the words of three labels already span it. It contains the 36 sector projections, which are exactly the classical facts of the algebra the effects generate. It misses 52 of that algebra's 144 dimensions: the 40 reads that see the direction in which a clock cell circulates, and at each occupation three products of zero-plane reads, the relative circulation of the two cells among them.

The record law is one read among the native reads of Chapter 6, and those see more. A record, once written, is a classical fact for every later native read, and two components of a genesis state at one loop value remain coherent for those reads exactly while some record carries both of them in one joint pointer sector. A record that contains one nonzero label fixes the whole occupation path. And whoever predicts the records and verdicts of a cell of unknown state must carry all of its record law.

Throughout, a cell vector $f \in \mathscr{H}$ also denotes the genesis state $\delta_0 \otimes f$ at loop flux 0, whose cell density is $|f\rangle\langle f|$. Two genesis states have *one record law* when $\mathbb{P}_\nu(h) = \mathbb{P}_{\nu'}(h)$ for every record h of every length. The retained read R and the placed reads are those of Definition 5.19, the native reads and the corner convention those of Chapter 6. That the reads made after the n-th posting are the late reads $\overline{\mathscr{G}}_{\geq n}$ is a premise; Section 9.6 uses it only in readings, never in proofs.

9.1 Six reads of one clock cell

On each occupation block an effect is a sum of products of one-cell matrices, and every one-cell matrix that occurs commutes with the pole charge of its cell. We therefore start with the commutant of one pole charge and with the rule by which a posting moves it.

We use the pole charge c_x, its spectral projections $\Pi^x_\pm, \Pi^x_0$, the 36 sector projections $\Pi(x,s,t)$ and the coherence reads $Z_x = C_x Z C_x$, $X_x = C_x X C_x$ of Chapter 3, with $Z = PA^\top + AP^\top$ and $X = AA^\top - PP^\top$ on $D\mathscr{U}$; the zero plane of c_x is $D\mathscr{U}$ for balanced and $\mathscr{U}$ for mixed x. Put $\Gamma = PA^\top - AP^\top$ and $\Gamma_x = C_x \Gamma C_x$. For a matrix B on $\mathscr{H}_\mathrm{c} \otimes \mathscr{H}_\mathrm{c}$ write $\mathrm{lift}_x(B) = |x\rangle\langle x| \otimes B$, and put

$$\gamma = \tfrac{1}{2}(1, i, -1, -i) \in \mathscr{H}_\mathrm{c}.$$

Lemma 9.1 (The commutant of one pole charge)**.** *Fix an occupation x.*

(a) The six matrices $\Pi^x_+, \Pi^x_-, \Pi^x_0, Z_x, X_x, \Gamma_x$ form a basis of the 4×4 matrices that commute with c_x. They are trace-orthogonal, and $\mathrm{tr}\, B^2 = 1, 1, 2, 2, 2, -2$ in this order.

(b) *Z_x and X_x are symmetric involutions of the zero plane of c_x, and $Z_xX_x = \Gamma_x = -X_xZ_x$. The first five matrices are symmetric; Γ_x is antisymmetric.*

(c) *$\Gamma = \frac{1}{2}(L - L^\top)$. The vector γ satisfies $L\gamma = i\gamma$, hence $c\gamma = 0$ and $\Gamma\gamma = i\gamma$, $\Gamma\bar\gamma = -i\bar\gamma$; and $\gamma^\top\gamma = 0$, so $\gamma \perp \bar\gamma$.*

Proof. (a) c_x has the eigenvalues 1 and -1 on two lines and 0 on a plane, so its commutant is $M_1 \oplus M_1 \oplus M_2$, of dimension 6. In the basis (P, A) of $D\mathscr{U}$ the matrices Π_0^x, X, Z, Γ (for balanced x) are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

a trace-orthogonal basis of M_2; conjugation by the orthogonal C_x carries the statement to every x. (b) is read off the same matrices. (c) From the coordinates, $L^\top P = A$ and $L^\top A = -P$, so $LP = -A$ and $LA = P$, while L and $L^\top$ act on p and a by the same sign. Hence $\frac{1}{2}(L - L^\top)$ vanishes on $\mathscr{U}$ and sends $P \mapsto -A$, $A \mapsto P$, as Γ does. Since $(L\gamma)_i = \gamma_{i+1}$, $L\gamma = i\gamma$ and $L^\top\gamma = -i\gamma$, which gives $c\gamma = 0$ and $\Gamma\gamma = i\gamma$; conjugation gives the statement for $\bar\gamma$. Finally $\gamma^\top\gamma = \frac{1}{4}(1 - 1 + 1 - 1) = 0$. □

A posting moves each clock cell by stay or flip: on a step $x \to y$, cell 1 stays exactly when $y_1 = x_2$ and cell 2 exactly when $y_2 = x_1$. The label and the output fix the input (Proposition 2.7), so for every label j and occupation y there is at most one transition $x \to y$ with $j(y, x) = j$, and then

$$V_j^* \operatorname{lift}_y(B \otimes B')V_j = \operatorname{lift}_x\big(f_1^\top B_y f_1 \otimes f_2^\top B'_y f_2\big), \qquad f_c = K(b'_c, b_c)\, C_x, \tag{9.1}$$

with $B_y = C_yBC_y$; if there is no such transition the left side is 0. The leading sign $\sigma(x)$ of the step block cancels. So one table describes every pullback.

output read	stay, y balanced	flip, y balanced	stay, y mixed	flip, y mixed
Π_+	$\frac{1}{4}\Pi_0 - \frac{1}{4}Z$	$\frac{1}{4}\Pi_0 + \frac{1}{4}Z$	0	Π_+
Π_-	$\frac{1}{4}\Pi_0 + \frac{1}{4}Z$	$\frac{1}{4}\Pi_0 - \frac{1}{4}Z$	Π_-	0
Π_0	Π_-	Π_+	$\frac{1}{2}\Pi_0$	$\frac{1}{2}\Pi_0$
Z	0	0	$-\frac{1}{2}X$	$\frac{1}{2}X$
X	Π_-	$-\Pi_+$	$\frac{1}{2}Z$	$-\frac{1}{2}Z$
Γ	0	0	$\frac{1}{2}\Gamma$	$\frac{1}{2}\Gamma$

Table 9.1. One-cell pullback $f^\top B_y f$ of the six reads through one clock map $f = K(b', b)C_x$, written in the reads at the input occupation x. It depends only on the move and on whether the output y is balanced.

Lemma 9.2 (One-cell pullbacks)**.** *For every step $x \to y$ and either cell, $f^\top B_y f$ is the combination of the six reads at x given in Table 9.1.*

Proof. Direct exact computation from the table of clock maps, Table 2.2. For example, onto a mixed output stay pulls Z back to $\frac{1}{4}C_x(I - L^\top)Z(I - L)C_x$. Here $(I - L)p = 0$ and $(I - L)a = 2a$,

which Z annihilates, while $(I-L)P = P + A$ and $(I-L)A = A - P$; so the middle matrix kills p and a and sends $P \mapsto \frac{1}{2}P$, $A \mapsto -\frac{1}{2}A$, that is, it is $-\frac{1}{2}X$. □

9.2 Ninety-two dimensions, and three labels read them

Definition 9.3 (Record reads and the pole algebra)**.** The *record reads* are the 92 matrices $\mathrm{lift}_x(B \otimes B')$ with $x \in \mathcal{O}$, $B, B' \in \{\Pi_+^x, \Pi_-^x, \Pi_0^x, Z_x, X_x\}$ and $(B, B') \notin \{(Z_x, X_x), (X_x, Z_x)\}$; $\mathscr{R}$ is their span. The *pole algebra* $\mathscr{A}_{\mathrm{pole}}$ is the set of 64×64 matrices that are block diagonal in the occupation and commute with $c_x \otimes I$ and $I \otimes c_x$ on the block of x.

Recall that the current algebra $\mathscr{A}$ is the algebra generated by the effects, and that $\mathscr{A} \subseteq \mathscr{A}_{\mathrm{pole}}$ by Lemma 3.6.

Theorem 9.4 (What the record law reads)**.** *The record law reads exactly the span of the record reads.*

(a) $\mathrm{span}\{E_w\} = \mathscr{R}$*, of dimension* 92*. The effects of the words of length at most* $0, 1, 2, 3$ *span* $1, 15, 56, 92$ *dimensions, so the words of three labels span* $\mathscr{R}$ *and the words of two do not.*

(b) $\mathscr{A} = \mathscr{A}_{\mathrm{pole}}$*, of dimension* 144*, and the* 144 *products* $\mathrm{lift}_x(B \otimes B')$ *of the six reads of Lemma 9.1 are a trace-orthogonal basis of it. The record law reads a* 92*-dimensional proper part of the current algebra.*

(c) Two genesis states ν, ν' *have one record law if and only if* $\mathrm{tr}(\rho_\nu Y) = \mathrm{tr}(\rho_{\nu'} Y)$ *for every* $Y \in \mathscr{R}$*.*

(d) If two unit genesis states have one law of records of length $k+3$*, they have one record law; and after every record* h *of length* k *with* $\mathbb{P}_\nu(h) > 0$*, their present cells* $\rho_{\nu,h}$ *and* $\rho_{\nu',h}$ *have one record law.*

Proof. (a) $\mathscr{R}$ contains $I = \sum_x \mathrm{lift}_x\big(\sum_{s,t} \Pi_s^x \otimes \Pi_t^x\big)$. It is invariant under every pullback $Y \mapsto V_j^* Y V_j$: by (9.1) a record read pulls back factor by factor, both cells moving onto the same output y. If y is balanced, Table 9.1 gives no X component and sends Z to 0, so both factors land in $\mathrm{span}\{\Pi_\pm, \Pi_0, Z\}$ and the product is a combination of record reads. If y is mixed, poles go to poles and (Z, X) goes to multiples of (X, Z) in both cells at once, so a product $Z \otimes X$ or $X \otimes Z$ at x could only come from $X \otimes Z$ or $Z \otimes X$ at y, which are excluded. Γ never arises. Since $E_\varnothing = I$ and $E_{j \cdot w} = V_j^* E_w V_j$, induction on the length gives $E_w \in \mathscr{R}$ for every word. By Lemma 9.1(a), $\mathrm{tr}\big(\mathrm{lift}_x(B \otimes B')\,\mathrm{lift}_{x'}(\tilde{B} \otimes \tilde{B}')\big) = [x = x']\,\mathrm{tr}(B\tilde{B})\,\mathrm{tr}(B'\tilde{B}')$, so the 92 record reads are trace-orthogonal and nonzero, and $\dim \mathscr{R} = 92$. The dimensions $1, 15, 56, 92$ are a direct exact computation: the ranks of the lists of effects of all words of length at most L, as rational 64×64 matrices. Since the words of length at most three span 92 dimensions inside $\mathscr{R}$, they span $\mathscr{R}$.

(b) The commutant of c_x is spanned by the six reads, so the block of $\mathscr{A}_{\mathrm{pole}}$ at x is spanned by the 36 products $\mathrm{lift}_x(B \otimes B')$, trace-orthogonal by Lemma 9.1(a); hence $\dim \mathscr{A}_{\mathrm{pole}} = 144$. By (a), $\mathscr{A} \supseteq \mathscr{R}$. Lifts at one occupation multiply factor by factor, and Π_0 is the unit of the zero plane, so $\mathscr{A}$ contains

$$Z \otimes X = (Z \otimes \Pi_0)(\Pi_0 \otimes X), \quad \Gamma \otimes B = (Z \otimes B)(X \otimes B)\ (B = \Pi_\pm, \Pi_0), \quad \Gamma \otimes Z = (Z \otimes Z)(X \otimes \Pi_0),$$

$$\Gamma \otimes X = (Z \otimes X)(X \otimes \Pi_0), \qquad \Gamma \otimes \Gamma = (Z \otimes Z)(X \otimes X),$$

all at occupation x and dressed by C_x, and their mirror images. These complete the 144 products, so $\mathscr{A} \supseteq \mathscr{A}_{\text{pole}}$.

(c) By the Born form, $\mathbb{P}_v(h) = \operatorname{tr}(\rho_v E_h)$; apply linearity and (a).

(d) By consistency (Theorem 2.17(c)) the law at length $k+3$ fixes the laws at all lengths up to $k+3$, in particular $\operatorname{tr}(\rho_v E_w)$ for $|w| \le 3$. By (a) these span $\mathscr{R}$, so (c) gives one record law. Then $\mathbb{P}_v(h) = \mathbb{P}_{v'}(h)$, and by Proposition 2.20(a), $\operatorname{tr}(\rho_{v,h} E_w) = \mathbb{P}_v(h \cdot w)/\mathbb{P}_v(h)$ is the same for v and v' for every w. □

The native posting stops at three labels, for every cell at once, where the general bound for observable operator models and hidden Markov sources allows 91 [22, 23]; the reason is the clock geometry of Table 9.1. One label reads the poles of each cell: one-label effects are products of the Grams $\frac{1}{2}(I \pm c_x)$. A second label adds Z, the split of the zero plane along the standing waves $\Omega_\pm = (P \pm A)/\sqrt{2}$, because onto a balanced output the pole sign pulls back to $\mp\frac{1}{4}Z$. Only a third label reads X, the split along P and A: X arises from Z onto a mixed output, and Z itself needed a balanced output first.

Proposition 9.5 (Two labels do not suffice)**.** *The unit cells $f_A = |00\rangle\otimes A\otimes p$ and $f_P = |00\rangle\otimes P\otimes p$ have one law of records of length two. The record $h^* = \big((0,0,1),(0,-1,1),(-1,0,-1)\big)$, written by the transitions $00 \to 01 \to 11 \to 00$, has probability $\frac{1}{8}$ under f_A and 0 under f_P.*

Proof. $\rho_{f_A} - \rho_{f_P} = \operatorname{lift}_{00}(X \otimes \Pi_+)$, since p spans the $+1$ line of c. One-label effects lie in the span of products of the projections $\Pi_\pm, \Pi_0$, and by Table 9.1 a second pullback adds Z but never X. So every effect of a word of length at most two is a combination of $\operatorname{lift}_x(B \otimes B')$ with $B \in \{\Pi_\pm, \Pi_0, Z\}$, and each of these is trace-orthogonal to $\operatorname{lift}_{00}(X \otimes \Pi_+)$. For h^* follow the first cell with Table 2.2; the second cell goes $p \mapsto P \mapsto P \mapsto -\frac{1}{2}(p-a)$ with weights $1, 1, \frac{1}{2}$ in both cases. Under f_A the first cell goes $A \mapsto \operatorname{stay} A = \frac{1}{2}(a-p)$, then through $-\operatorname{stay} D$ to $-\frac{1}{2}\operatorname{stay}(A-P) = \frac{1}{2}p$, then through $-\operatorname{flip}$ to $-\frac{1}{2}P$: squared length $\frac{1}{4}$. Under f_P it goes $P \mapsto \frac{1}{2}(p+a) \mapsto -\frac{1}{2}\operatorname{stay}(P+A) = -\frac{1}{2}a$, and $\operatorname{flip} a = 0$. So the probabilities are $\frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$ and 0. □

9.3 What the record law misses

The record law misses two kinds of structure. Between the 36 sectors it sees no coherence at all. Inside the sectors it misses 52 directions, and every one of them involves the zero plane of a pole charge.

Write $\mathscr{P}(\rho) = \sum_{x,s,t} \Pi(x,s,t)\,\rho\,\Pi(x,s,t)$ for the density pinched onto the 36 sectors.

Proposition 9.6 (The record law sees the pinched density)**.** *For every genesis state v and record h,*

$$\mathbb{P}_v(h) = \operatorname{tr}\big(\mathscr{P}(\rho_v) E_h\big).$$

In particular the chance state and its mirror

$$\chi = |10\rangle \otimes \big(\tfrac{3}{5}P \otimes P + \tfrac{4}{5}A \otimes P\big), \qquad \check{\chi} = |10\rangle \otimes \big(\tfrac{3}{5}P \otimes P - \tfrac{4}{5}A \otimes P\big)$$

lie in one lineage atom $\Lambda((1,0),0)$ *and have one record law, and so does* $|10\rangle\otimes(\frac{3}{5}P\otimes P+\frac{4}{5}e^{i\theta}A\otimes P)$ *for every* θ*. Their occupation blocks differ: the diagonal entry at* $e_0\otimes e_0$ *of the block of* $(1,0)$ *of the density is* $\frac{49}{400}$ *for* χ *and* $\frac{1}{400}$ *for* $\check{\chi}$*.*

Proof. Each effect lies in $\mathscr{A}_{\mathrm{pole}}$ and so commutes with the projections $\Pi(x,s,t)$, which resolve the identity; hence $E_h=\sum\Pi(x,s,t)E_h\Pi(x,s,t)$, and the trace is cyclic. The two terms of χ lie in the sectors $\Pi((1,0),+,+)$ and $\Pi((1,0),-,+)$, since $c'P=P$ and $c'A=-A$, so $\mathscr{P}$removes their cross terms. All coordinates of P and A have modulus $\frac{1}{2}$ and $P_0=A_0=\frac{1}{2}$, so the coefficient of $e_0\otimes e_0$ is $\frac{1}{4}(\frac{3}{5}\pm\frac{4}{5})$, with squares $\frac{49}{400}$ and $\frac{1}{400}$. □

Chapter 4 showed that the record law depends on a state only through its occupation blocks; the chance pair shows that it does not even determine them. Nor does it select a loop lineage (Chapter 5).

Inside one sector the decisive fact is that the native step is real.

Theorem 9.7 (The record law cannot see a circulation)**.** *Write* $\bar{\nu}$ *for the complex conjugate of a genesis state* ν*.*

(a) Every effect is real symmetric. Hence $\mathbb{P}_{\bar{\nu}}=\mathbb{P}_{\nu}$ *for every genesis state* ν*.*

(b) Let $f_\gamma=|00\rangle\otimes p\otimes\gamma$*. Then* f_γ *and* $\bar{f}_\gamma$ *are orthogonal unit vectors in the sector* $\Pi((0,0),+,0)$ *and in one lineage atom, they have one record law, and the projection* $\mathrm{lift}_{00}(|p\otimes\gamma\rangle\langle p\otimes\gamma|)$ *lies in* $\mathscr{A}$ *and reads* 1 *on* f_γ *and* 0 *on* $\bar{f}_\gamma$*. So even inside one atom and one sector, the record law is strictly coarser than the cell density.*

(c) For every occupation x *and* $u,u'\in\mathscr{H}_{\mathrm{c}}$*, the cells* $|x\rangle\otimes u\otimes u'$*,* $|x\rangle\otimes\bar{u}\otimes u'$ *and* $|x\rangle\otimes u\otimes\bar{u}'$ *have one record law. In particular the co-rotating cell* $|00\rangle\otimes\gamma\otimes\gamma$ *and the counter-rotating cell* $|00\rangle\otimes\gamma\otimes\bar{\gamma}$ *have one record law, while the real symmetric element* $\mathrm{lift}_{00}(\Gamma\otimes\Gamma)$ *of* $\mathscr{A}$ *reads* -1 *on the first and* $+1$ *on the second.*

(d) $\mathscr{A}=\mathscr{R}\oplus\mathscr{M}_{\mathrm{even}}\oplus\mathscr{M}_{\mathrm{odd}}$*, trace-orthogonally, where* $\mathscr{M}_{\mathrm{odd}}$ *is spanned by the* 40 *products* $\mathrm{lift}_x(\Gamma_x\otimes B)$*,* $\mathrm{lift}_x(B\otimes\Gamma_x)$*,* $B\in\{\Pi^x_\pm,\Pi^x_0,Z_x,X_x\}$*, and* $\mathscr{M}_{\mathrm{even}}$ *by the* 12 *products* $\mathrm{lift}_x(Z_x\otimes X_x)$*,* $\mathrm{lift}_x(X_x\otimes Z_x)$*,* $\mathrm{lift}_x(\Gamma_x\otimes\Gamma_x)$*. The symmetric elements of* $\mathscr{A}$ *form* $\mathscr{R}\oplus\mathscr{M}_{\mathrm{even}}$*, of dimension* 104*, and the antisymmetric ones form* $\mathscr{M}_{\mathrm{odd}}$*. Two genesis states have one record law if and only if they agree on every symmetric element of* $\mathscr{A}$ *that is trace-orthogonal to* $\mathscr{M}_{\mathrm{even}}$*.*

(e) Let $(s,t)\neq(0,0)$*,* $q\in\mathbb{Z}$*, and let* f,g *lie in the range of* $\Pi(x,s,t)$*. If* $\delta_q\otimes f$ *and* $\delta_q\otimes g$ *have one record law, then* $ff^*+\bar{f}f^\top=gg^*+\bar{g}g^\top$*: in every sector other than the doubly zero one, the record law fixes the symmetric part of the density.*

Proof. (a) The slices have real entries, so $E_w=V_w^\top V_w$. The density of $\bar{\nu}$ is $\rho_\nu^\top$, and $\mathrm{tr}(\rho^\top E)=\mathrm{tr}(\rho E)$ for symmetric E. (b) $cp=p$ and $c\gamma=0$, so $p\otimes\gamma$ is a joint eigenvector of $c\otimes I$ and $I\otimes c$, and its projection, lifted to 00, lies in $\mathscr{A}_{\mathrm{pole}}=\mathscr{A}$. Orthogonality: $\langle\bar{f}_\gamma,f_\gamma\rangle=f_\gamma^\top f_\gamma=(p^\top p)(\gamma^\top\gamma)=0$. One record law is (a). Both states are genesis states at occupation 00 and loop 0. (c) By Theorem 9.4(a), on the block of x every effect is a combination of products $B\otimes B'$ of real symmetric matrices, and $\langle\bar{u},B\bar{u}\rangle=\overline{\langle u,Bu\rangle}=\langle u,Bu\rangle$ for real symmetric B. So each record read has the same value on the three product cells. By Lemma 9.1(c), $\langle\gamma\otimes\gamma,(\Gamma\otimes\Gamma)(\gamma\otimes\gamma)\rangle=i\cdot i=-1$ and $\langle\gamma\otimes\bar{\gamma},(\Gamma\otimes\Gamma)(\gamma\otimes\bar{\gamma})\rangle=i\cdot(-i)=1$; and $(\Gamma\otimes\Gamma)^\top=\Gamma\otimes\Gamma$. (d) By Theorem 9.4(b) the 144 lifted products are a trace-orthogonal basis of $\mathscr{A}$, and $\mathscr{R}$ is spanned by 92 of them. Of the remaining 13 products at each occupation, the 10 with exactly one factor Γ_x are antisymmetric and the 3

listed are symmetric, because transposition acts factor by factor and only Γ_x is antisymmetric. By trace-orthogonality, $\mathscr{R}$ is the set of symmetric elements of $\mathscr{A}$ trace-orthogonal to $\mathscr{M}_{\text{even}}$; apply Theorem 9.4(c). (e) Let $Q = \Pi(x, s, t)$. Each product spanning $\mathscr{M}_{\text{even}}$ has both factors supported on zero planes, so $QMQ = 0$ for $M \in \mathscr{M}_{\text{even}}$ when $(s, t) \neq (0, 0)$. For any 64×64 matrix Y, $Y_Q = \frac{1}{2}Q(Y + Y^\top)Q$ is symmetric, commutes with $c_x \otimes I$ and $I \otimes c_x$ (which act on the range of Q as scalars) and so lies in $\mathscr{A}$, and is trace-orthogonal to $\mathscr{M}_{\text{even}}$. By (d), $Y_Q \in \mathscr{R}$. Since $\rho_f = Q\rho_f Q$, one record law gives $\operatorname{tr}\big((\rho_f + \rho_f^\top)Y\big) = 2\operatorname{tr}(\rho_f Y_Q) = 2\operatorname{tr}(\rho_g Y_Q) = \operatorname{tr}\big((\rho_g + \rho_g^\top)Y\big)$ for every Y. □

The missing directions have a plain meaning. f_Y and $\bar{f}_Y$ put the same weight $\frac{1}{16}$ on each of the sixteen internal positions; they differ only in the direction in which the second clock cell runs round its cycle, as an eigenvector of the clock shift with eigenvalue i or $-i$. That direction is read with certainty by an element of the current algebra, and the record law cannot hold it. Calling complex conjugation time reversal would be an identification, which we do not make. Chapter 11 shows that the same wall stands for every sequence of real instruments and finds what can break it. Part (e) locates the even losses: away from the doubly zero sectors the record law misses only the direction of circulation; inside them it also misses $Z \otimes X$, $X \otimes Z$ and the relative circulation $\Gamma \otimes \Gamma$.

9.4 The classical facts are the thirty-six sector weights

Chapter 3 proved that the first window verdict commutes with every element of $\mathscr{A}$. The next theorem says exactly which reads do; we call a central element of $\mathscr{A}$ a *classical fact*.

Theorem 9.8 (The centre of the current algebra)**.** *The central elements of $\mathscr{A}$ are the sector functions.*

(a) An element of $\mathscr{A}$ commutes with every element of $\mathscr{A}$ if and only if it equals

$$\sum_{x,s,t} g(x, s, t)\,\Pi(x, s, t)$$

for a function g on $\mathcal{O} \times \{+, -, 0\}^2$. The function is unique, and the centre of $\mathscr{A}$ has dimension 36*.*

(b) Every central element lies in $\mathscr{R}$. Two genesis states with one record law have the same weight $\operatorname{tr}(\rho\,\Pi(x, s, t))$ *on each of the* 36 *sectors.*

(c) A window effect $W_k(C)$ of Chapter 3 is central exactly when it is a function of the sector. For the field-current cut and $k = 0, 1, 2$ the function is given in Table A.5.

Proof. (a) By Theorem 9.4(b), $\mathscr{A} = \mathscr{A}_{\text{pole}}$. On the block of x this is the commutant of the two commuting symmetric matrices $c_x \otimes I$ and $I \otimes c_x$: the direct sum, over their nine joint eigenspaces (the ranges of $\Pi_s^x \otimes \Pi_t^x$, of dimensions $1, 1, 2, 1, 1, 2, 2, 2, 4$), of the full matrix algebras on those spaces. The centre of a direct sum of full matrix algebras is spanned by the units of the summands. The 36 projections are nonzero and mutually orthogonal, which gives uniqueness. (b) Each $\Pi(x, s, t) = \operatorname{lift}_x(\Pi_s^x \otimes \Pi_t^x)$ is a record read; apply Theorem 9.4(c). (c) $W_k(C)$ is a sum of effects, so (a) applies. The values are a direct exact computation: $W_0 = Q_{\text{al}} + \frac{1}{128}Q_{\text{ne}}$, and $W_{k+1} = \Phi(W_k)$ is obtained with Table 9.1. □

The pointer of Chapter 6 is one of these facts, and the record law reads it through an explicit formula.

Corollary 9.9 (The record reads the pointer statistically)**.** *Let* $\Pi_0^{(c)} = \frac{1}{2}(1 - L_c^2)$ *be pointer sector* 0 *of cell c (Definition 6.8).*

(i) On the block of a mixed occupation x, $\Pi_0^{(c)}$ *is the projection of cell c onto the two pole lines of* c_x*; on the block of a balanced occupation it is the projection onto the zero plane of* c_x*. So* $\Pi_0^{(c)}$ *is a sum of sector projections and lies in* $\mathscr{R}$*.*

(ii) For cell 1 *and every unit genesis state v,*

$$\operatorname{tr}(\rho_v \Pi_0^{(1)}) = \mathbb{P}_v\big(j_2^{(1)} \neq 0\big) - 4\,\mathbb{P}_v\big(j_2^{(1)} \neq 0,\ j_2^{(2)} \neq 0,\ \mathrm{ch}_1\big) + 4\,\mathbb{P}_v\big(j_2^{(1)} = 0,\ j_2^{(2)} \neq 0,\ \mathrm{ch}_1\big),$$

where ch_1 *is the event that exactly one of* $j_1^{(1)}, j_1^{(2)}$ *vanishes (cell 1 changes its move); for cell 2 replace the first component by the third. Two genesis states with one record law have the same pointer weights.*

Proof. (i) Since $L^4 = I$ and $L^\top = L^{-1}$, $c^2 = \frac{1}{2}(I + L^2)$, so $\Pi_0^x = I - c^2 = \frac{1}{2}(I - L^2)$ for balanced x. For mixed x, $L^2 D = -DL^2$ gives $c'^2 = \frac{1}{2}(I - L^2) = \Pi_+^x + \Pi_-^x$. (ii) The first label has $j_2 \neq 0$ exactly when the input is mixed, and the second exactly when the first output is mixed; a cell changes its move when one of its two outer components vanishes and the other does not. A cell on a pole line moves surely (Lemma 2.5) and, onto a mixed output, lands on a pole line of the same sign, so it repeats its move and contributes nothing to the last two events. By Table 9.1, the summed effect of the two-label words in each of these events is $\frac{1}{4}\,\mathrm{lift}_x(\Pi_0^x \otimes I)$, summed over the mixed, respectively balanced, inputs x, while the first event has effect $\sum_{x\ \mathrm{mixed}} |x\rangle\langle x| \otimes I$. Insert (i). □

The record fixes the pointer's weights, not its value on one run: Theorem 6.22 shows that no finite record decodes the pointer.

9.5 What a written record fixes

The record law concerns weights. A single written record also fixes facts outright, and the fact it fixes is the occupation.

Proposition 9.10 (The record fixes the occupation path)**.** *Consider the records written by the native step.*

(a) (Proposition 2.7) A nonzero label is written by exactly one transition. The zero label is written by exactly $00 \to 00$ *and* $11 \to 11$*.*

(b) If a record h of length k contains a nonzero label, all occupation paths $x^{(0)}, \dots, x^{(k)}$ *with* $j(x^{(t)}, x^{(t-1)}) = h_t$ *for every t coincide. In particular, for every genesis state v, every coordinate of* $T^k v$ *with record h and nonzero amplitude has the same occupation: the output of the newest nonzero label.*

(c) A record of $k \geq 1$ *zero labels is written by exactly the two constant balanced paths, and both carry amplitude for the genesis state* $2^{-1/2}(|00\rangle + |11\rangle) \otimes a \otimes a$*. So for every* $k \geq 1$ *there is no function* $d : \mathscr{J}^k \to \mathscr{O}$ *such that* $d(h)$ *is the occupation of every coordinate of* $T^k v$ *with record h and nonzero amplitude, for every genesis state v.*

(d) The empty record is compatible with all four occupations. The statement that a silent record leaves exactly the balanced bit open therefore holds for every length $k \geq 1$ and fails at $k = 0$.

Proof. (a) is Proposition 2.7. (b) The transition of a nonzero label is pinned. From a pinned input, each later label fixes its output, since for a fixed input the label determines the output; from a pinned output, each earlier label fixes its input, by Proposition 2.7. A coordinate with nonzero amplitude lies on a path compatible with its record, by the step formula. (c) Zero labels keep a balanced occupation, so a silent path is constant and balanced. On a balanced occupation the zero slice acts as $-\mathrm{stay} \otimes \mathrm{stay}$ on both 00 and 11, and $\mathrm{stay}^n a \neq 0$ for every n, because $\mathrm{stay}\, a = A$, $\mathrm{stay}\, A = \frac{1}{2}(a - p)$ and $\mathrm{stay}(a - p) = A$. So both coordinates are populated, and any d would have to return two occupations. (d) A record of length zero imposes no condition. □

Each label is a difference between new and held occupations, so the record is relational; it still fixes the absolute occupation, because the fifteen labels have only one coincidence. What a silent record leaves open is one bit, 00 against 11, and on the same record both pointer sectors survive (Theorem 6.22): a silent record hides both the occupation bit and the pointer.

9.6 A record is classical for every later read

The native reads of Chapter 6 see the coherence between pole sectors that the record law cannot. This section shows that the extra sight ends at the first record: a coherence stays visible to the reads made afterwards exactly while its components share a record and a joint pointer sector.

We use the placed reads $A[k, h]$ and the live read algebra $\mathscr{A}_{\mathrm{R}}$ of Definition 5.19, and from Chapter 6 the archive reads $\mathrm{Ar}_m(g)$, the native read algebra $\overline{\mathscr{G}}$ and the late reads $\overline{\mathscr{G}}_{\geq n}$ of Definition 6.1, the joint sectors Π_s and their completions $\widehat{\Pi}_s$, the stage projections Q_k, the loop readouts and the loop translations $U_{\sigma_m^{\mathrm{tr}}}$ (Chapter 5). Write O_q for the loop readout that projects onto original loop q, and $\varphi_f = \iota_0(\delta_q \otimes f)$ for a one-loop genesis state.

First, a native read does see what the record law misses. Write $\psi_+ = |10\rangle \otimes P \otimes P$ and $d = |10\rangle \otimes A \otimes P$, so that $\chi = \frac{3}{5}\psi_+ + \frac{4}{5}d$ and $\check{\chi} = \frac{3}{5}\psi_+ - \frac{4}{5}d$.

Proposition 9.11 (A native read sees the pole coherence)**.** *Let ψ_+, d, χ and $\check{\chi}$ be as above.*

(a) ψ_+ and d lie in the joint pointer sector $\Pi_{(0,0)}$. The coherence read $K_{\mathrm{coh}} = |\psi_+\rangle\langle d| + |d\rangle\langle\psi_+|$ lies in $\mathscr{A}_{\mathrm{R}}$ and reads $\frac{24}{25}$ on χ and $-\frac{24}{25}$ on $\check{\chi}$, while every element of $\mathscr{A}$ has one expectation on χ and $\check{\chi}$. Hence $\mathscr{A} \subsetneq \mathscr{A}_{\mathrm{R}}$, and the pole charge of cell 1 does not commute with $\mathscr{A}_{\mathrm{R}}$: the dressed pole is not a superselection label of the native reads.

(b) The retained read reads $\frac{1}{4}$ on both states. Every single read E_w or E_w^{R}, and every product of two of them, has one expectation on χ and $\check{\chi}$. The product $\mathrm{R}\, E_{(0,0,0),(0,0,0)}\, \mathrm{R} \in \mathscr{A}_{\mathrm{R}}$ reads $\frac{49}{3200}$ on χ and $\frac{1}{3200}$ on $\check{\chi}$.

Proof. (a) $L^2 P = -P$ and $L^2 A = -A$, so every cell of ψ_+ and d lies in $D\mathscr{U}$, the range of $\Pi_0^{(c)}$. By Corollary 6.14, $\mathscr{A}_{\mathrm{R}}$ contains every operator on the range of $\Pi_{(0,0)}$, in particular K_{coh}. Its expectations are $2 \cdot \frac{3}{5} \cdot (\pm\frac{4}{5})$. An element $Y \in \mathscr{A}$ commutes with the central projections

$\Pi((1,0),+,+)$ and $\Pi((1,0),-,+)$, which contain ψ_+ and d, so $\langle\psi_+, Yd\rangle = 0$ and both states read $\frac{9}{25}\langle\psi_+, Y\psi_+\rangle + \frac{16}{25}\langle d, Yd\rangle$. K_{coh} does not commute with $c' \otimes I$ on the block of $(1,0)$, since it couples its $+1$ and -1 eigenvectors. (b) The block of R at (x,x) is $\frac{1}{4}I$ by (5.3), so $\langle\psi_+, \mathrm{R}d\rangle = 0$. The statements about single reads and pairs are a direct exact computation; the mechanism is that a slice keeps ψ_+ only under $(1,1,-1)$ and moves d only under $(0,1,-1)$, so no slice carries both, and one retained read cannot bring them together. Three factors can: R spreads both modes coherently over all four occupations, on the balanced ones both have pole charge 0, and two zero-label postings send the first cells P and A to the same vector, $\mathrm{stay}^2 P = \mathrm{stay}^2 A = \frac{1}{2}A$. The amplitudes then interfere with weights $(\frac{3}{5} \pm \frac{4}{5})^2/128$. □

These reads, like the projection that reads a circulation in Theorem 9.7(b), are elements of the read algebras; $\mathrm{R}\, E_{(0,0,0),(0,0,0)}\, \mathrm{R}$ applies R before a posting, which the sequential grammar of current outcomes followed by a terminal retained read does not do. Whether a native process performs any of them as one operation is open.

The verdict sectors are thus classical for the record law and not for the native reads. The next theorem shows that one posting changes this.

Theorem 9.12 (The first record ends the pole coherence)**.** *Let K_{coh} be the coherence read of Proposition 9.11.*

(a) The placed read $K_{\mathrm{coh}}[0,\varnothing] \in \overline{\mathscr{G}}$ reads $\frac{24}{25}$ on $\iota_0\chi$ and $-\frac{24}{25}$ on $\iota_0\check{\chi}$.

(b) The first posting writes $(1,1,-1)$ on ψ_+ and $(0,1,-1)$ on d, each with certainty. With g the indicator of the first label $(1,1,-1)$, $\mathrm{Ar}_1(g)$ sends both $\iota_0\chi$ and $\iota_0\check{\chi}$ to $\frac{3}{5}\iota_0\psi_+$, and $\mathrm{Ar}_1(1-g)$ sends them to $\pm\frac{4}{5}\iota_0 d$.

(c) For every $A \in \overline{\mathscr{G}}_{\geq 1}$,

$$\langle \iota_0\chi, A\iota_0\chi\rangle = \langle \iota_0\check{\chi}, A\iota_0\check{\chi}\rangle = \tfrac{9}{25}\langle \iota_0\psi_+, A\iota_0\psi_+\rangle + \tfrac{16}{25}\langle \iota_0 d, A\iota_0 d\rangle.$$

(d) More generally, let $g : \mathscr{J}^n \to \mathbb{R}$ and $\xi, \eta \in \mathscr{H}_\infty$ with $\mathrm{Ar}_n(g)\xi = \xi$ and $\mathrm{Ar}_n(g)\eta = 0$. Then $\langle\xi, A\eta\rangle = 0 = \langle\eta, A\xi\rangle$ for every $A \in \overline{\mathscr{G}}_{\geq n}$, so $\alpha\xi + \beta\eta$ and $\alpha\xi - \beta\eta$ have one expectation on $\overline{\mathscr{G}}_{\geq n}$.

(e) $K_{\mathrm{coh}}[0,\varnothing] \notin \overline{\mathscr{G}}_{\geq 1}$, and all of $\overline{\mathscr{G}}_{\geq n}$ gives χ and $\check{\chi}$ one expectation exactly when $n \geq 1$.

Proof. (a) ι_0 is isometric, so a read placed at stage 0 has expectation $\mathrm{tr}(\rho A)$ on a genesis state at one loop; use Proposition 9.11. (b) $V_{(1,1,-1)}\psi_+ = \psi_+$ and every other slice annihilates ψ_+ (Proposition 2.22). For d, the dressed cells are a and p; cell 1 must stay ($\mathrm{flip}\, a = 0$) and cell 2, through $K(y_2, 1)$, must go to $y_2 = 0$ ($\mathrm{stay}\, p = 0$), so only the transition $10 \to 00$ survives, with label $(0,1,-1)$. The archive read multiplies each record by g of its first label. (d) $\mathrm{Ar}_n(g)$ is self-adjoint and central in $\overline{\mathscr{G}}_{\geq n}$ (Theorem 6.15(iii)), so $\langle\xi, A\eta\rangle = \langle \mathrm{Ar}_n(g)\xi, A\eta\rangle = \langle\xi, A\,\mathrm{Ar}_n(g)\eta\rangle = 0$. (c) is (d) with $n = 1$, $\xi = \iota_0\psi_+$, $\eta = \iota_0 d$. (e) By (a) and (c), $K_{\mathrm{coh}}[0,\varnothing]$ separates the two states and no element of $\overline{\mathscr{G}}_{\geq 1}$ does; the algebras decrease with n. □

So the relative sign between the verdict sectors is visible to a native read only before the first record; after it, every read in $\overline{\mathscr{G}}_{\geq 1}$, the coherence read placed later included, sees the chance state as the mixture $\frac{9}{25} : \frac{16}{25}$ of its two verdict modes. Nothing has been erased: the history is still the pure state $T^k\nu$, and the stage-0 read still sees the coherence. What ends is the access

of later reads, and no environment is involved. Decoherence is usually derived by tracing out an environment [46]; here the record that the system writes for itself does the same work, relative to the reads that come after it.

Different first labels are one way to part. The exact criterion is a shared record and a shared joint sector.

Theorem 9.13 (A shared record keeps coherence visible)**.** *Let $q \in \mathbb{Z}$, $f, f' \in \mathscr{H}$ and $n \geq 0$.*

(a) For every record h of length n and every 64×64 matrix A, $\langle \varphi_f, A[n,h]\, \varphi_{f'} \rangle = \langle V_h f, A V_h f' \rangle$.
(b) Some $A \in \overline{\mathscr{G}}_{\geq n}$ has $\langle \varphi_f, A\varphi_{f'} \rangle \neq 0$ if and only if there are a record $h \in \mathscr{J}^n$ and a joint sector s with $\Pi_s V_h f \neq 0$ and $\Pi_s V_h f' \neq 0$.
(c) No element of $\overline{\mathscr{G}}$ couples $\iota_0(\delta_q \otimes f)$ to $\iota_0(\delta_{q'} \otimes f')$ when $q \neq q'$, or when f and f' lie in different joint sectors.
(d) φ_f and $\varphi_{f'}$ have one expectation on every element of $\overline{\mathscr{G}}_{\geq n}$ if and only if $|\Pi_s V_h f\rangle\langle \Pi_s V_h f| = |\Pi_s V_h f'\rangle\langle \Pi_s V_h f'|$ for every $h \in \mathscr{J}^n$ and every s.
(e) For the two verdict modes, some element of $\overline{\mathscr{G}}_{\geq n}$ couples $\iota_0\psi_+$ to $\iota_0 d$ exactly when $n = 0$.

Proof. (a) The posting carries $\delta_q \otimes f$ to $\sum_h \delta_{q+\mathrm{acc}(h)} \otimes \delta_h \otimes V_h f$ at stage n (Theorem 2.14(b)), $\iota_0 = \iota_n T^n$, and $A[n,h]$ keeps the record h. (b) An element of $\overline{\mathscr{G}}_{\geq n}$ has no matrix element between two stage-n records, since the record functions $\mathrm{Ar}_n(g)$ are central in it; none between two original loops, since loop readouts commute with every native read (Theorem 9.15(a)); and at a record h it acts at every loop by one cell matrix M_h, which commutes with the joint sectors (Theorem 6.13). So $\langle \varphi_f, A\varphi_{f'} \rangle = \sum_h \sum_s \langle \Pi_s V_h f, M_h \Pi_s V_h f' \rangle$, which vanishes if every pair (h,s) has a zero side. Conversely, put $u = \Pi_s V_h f$, $u' = \Pi_s V_h f'$ and $A = (\Pi_s|u\rangle\langle u'|\Pi_s)[n,h]$; its matrix lies in $\mathscr{A}_{\mathrm{R}}$ by Corollary 6.14, and by (a) its cross element is $\|u\|^2\|u'\|^2 \neq 0$. (c) is the expansion of (b) at $n = 0$, with the loops kept apart. (d) The expansion of (b) with $f = f'$ gives each expectation as $\sum_{h,s} \mathrm{tr}(M_h|\Pi_s V_h f\rangle\langle \Pi_s V_h f|)$; choosing A as placed matrix units inside one joint sector reads every entry of each branch density. (e) Both modes lie in $\Pi_{(0,0)}$ at stage 0. From stage 1 on, their records begin with different labels (Theorem 9.12(b)), so no record carries both. □

So the reads made after n postings see a one-loop genesis state exactly as the family of its branches $\Pi_s V_h f$, indexed by record and joint sector, with weights $\|\Pi_s V_h f\|^2$. Within one branch they see every superposition; across branches, none. Coherence survives every record exactly when, at every stage, some record carries both components in one joint sector, and it can only be lost as the stage grows.

Theorem 9.14 (What later reads treat as classical)**.** *Let $n \geq 0$.*

(a) The centre of $\overline{\mathscr{G}}_{\geq n}$ is $\{\mathrm{Ar}_n(g) : g : \mathscr{J}^n \to \mathbb{C}\}$: an element of $\overline{\mathscr{G}}_{\geq n}$ commutes with all of it exactly when it is a function of the stage-n record.
(b) Later records are not classical for these reads. For every $h_0 \in \mathscr{J}^n$, the indicator of the stage-$(n+1)$ record $h_0 \cdot (0,0,0)$ lies in $\overline{\mathscr{G}}_{\geq n}$ and is not central there: $Q_n \in \overline{\mathscr{G}}_{\geq n}$ couples a coordinate of record $h_0 \cdot (0,0,0)$ to one of record $h_0 \cdot (1,0,0)$ with amplitude $\frac{1}{8}$.
(c) An element of $\overline{\mathscr{G}}$ lies in $\overline{\mathscr{G}}_{\geq n}$ if and only if it commutes with every $\mathrm{Ar}_n(g)$.
(d) Each completed pointer $\widehat{\Pi}_b^{(c)}$ commutes with every element of $\overline{\mathscr{G}}_{\geq n}$ and is a strong limit of elements of it, but it does not lie in $\overline{\mathscr{G}}_{\geq n}$.

Proof. (a) The record functions are central by Theorem 6.15(iii). For the converse let Y be central. It commutes with every read placed at a stage $M \geq n$, so, as in Theorem 6.13(v), it acts on the live cell at each record of each stage $M \geq n$ as a combination of the four joint sectors. One posting of the label $(1, 0, 0)$ intertwines the cell matrices of a record and its child exactly, and its slice annihilates no joint sector, so every coefficient at a record reappears at its descendants along chains of $(1, 0, 0)$. Deep along such a chain lie directions born after the stage of any given finite read, and there every finite read of $\overline{\mathscr{G}}_{\geq n}$ acts as one number, as in the proof of Theorem 6.15(ii). Given $\varepsilon > 0$, take a finite read A with $\|Y - A\| < \varepsilon$ and a descendant so deep that A acts on its fresh directions as one number λ. These directions meet every joint sector at every loop, and Y acts on each of them by its coefficient there, which is the coefficient at the ancestor; so all coefficients of Y at one record lie within ε of λ, and two of them differ by less than 2ε. Hence Y is one scalar on each record block, with no dependence on the loop or the sector. Every slice is nonzero, so the scalar passes from each record to its children, and $Y = \mathrm{Ar}_n(g)$. (b) The indicator is an archive read; the coupling is the overlap of two posting rows from the input 00, as in Theorem 6.15(iv), at the record h_0. (c) Commuting with the record functions is necessary by (a). For sufficiency, the pinching $X \mapsto \sum_h p_h X p_h$, $p_h = \mathrm{Ar}_n(1_{\{h\}})$, is continuous, fixes every operator that commutes with the record functions, and maps every finite native read into $\overline{\mathscr{G}}_{\geq n}$: between two stage-$n$ record indicators an earlier placed read becomes a stage-n corner that respects the records, and such a corner is a placed read $A[n, h]$ with $A \in \mathscr{A}_{\mathrm{R}}$ at each record, because it commutes with the record functions, the loop readouts, the loop translations and both completed pointers. (d) The corners $Q_m \widehat{\Pi}_b^{(c)} Q_m$, $m \geq n$, lie in $\overline{\mathscr{G}}_{\geq n}$ and converge strongly (Theorem 6.15(i)); commutation is Theorem 6.13(ii); the distance argument of Theorem 6.15(ii) keeps the pointer out of the norm closure. □

For the reads made after stage n, then, the classical content is the record so far and the joint pointer sector, in two different senses: the record is central, a classical fact that the reads contain; the pointer sector is a superselection label, a limit of reads but no read. By (c), "the reads made after a record" and "the reads that respect the records already written" are the same algebra, so the premise named at the start of the chapter reduces to one physical question: whether the reads performed after a record respect it; a negative answer to the corner-convention question of Chapter 6 (paragraph after Theorem 6.15) would settle it.

Theorem 9.15 (What the native reads cannot superpose)**.** *Let O_q be the loop readouts and $U_{\sigma_m^{\mathrm{tr}}}$ the loop translations.*

(a) Every loop readout commutes with every element of $\overline{\mathscr{G}}$.

(b) For every label j and joint sector $s = (s_1, s_2)$, $V_j \Pi_s = \Pi_{s'} V_j$, where $s'_c = 1 - s_c$ when j is silent and $s'_c = s_c$ otherwise; and $V_j \Pi_s \neq 0$, $\Pi_s V_j \neq 0$.

(c) A bounded operator T commutes with every element of $\overline{\mathscr{G}}$ if and only if, for every finite set of vectors and every $\varepsilon > 0$, some polynomial G in the four completed joint sectors $\widehat{\Pi}_s$, the loop readouts and the loop translations satisfies $\|T\xi - G\xi\| < \varepsilon$ on that set. Between two original loops such an operator is four numbers times the loop shift: $O_p T O_q = \sum_s c_s \widehat{\Pi}_s U_{\sigma_{p-q}^{\mathrm{tr}}} O_q$.

(d) The centre of this commutant is the span of the four $\widehat{\Pi}_s$.

(e) For $\overline{\mathscr{G}}_{\geq n}$ the same holds with the archive reads $\mathrm{Ar}_n(g)$ added to the generators in (c), and the centre of its commutant is $\{\sum_s \mathrm{Ar}_n(g_s) \widehat{\Pi}_s\}$.

(f) For a cell basis vector $|z\rangle$, the closed span of $\{A\, \iota_0(\delta_q \otimes |z\rangle) : A \in \overline{\mathscr{G}}\}$ is the closed span of all coordinates whose original loop is q, orphans included.

Proof. (a) Archive reads and placed reads act inside one stored record and one loop, while a loop readout multiplies each coordinate by a function of its original loop $n - \mathrm{acc}(h)$. (b) The intertwining is Lemma 6.9. For non-vanishing, the column of internal index $(0,0)$ of every step block has its only entry in the half-turn orbit of $(0,0)$ at $(0,0)$, with value $\pm\frac{1}{4}$ (proof of Lemma 5.20); so $\Pi_t V_j$ has the entry $\pm\frac{1}{16}$ there for every t, and $V_j\Pi_s \neq 0$ follows from the intertwining. (c) The generators commute with $\overline{\mathscr{G}}$ by (a) and Theorem 6.13(ii),(iii), and commutants are strongly closed. Conversely, by Theorem 6.13(v) the cell matrices of T at each record are combinations of the four joint sectors; by (b) one posting intertwines the cell matrices of a record and its child and kills no sector, so if T acts at a record h by $\sum_s c_s(h)\Pi_s$, then $V_j\Pi_s = \Pi_{s'}V_j \neq 0$ forces $c_{s'}(h\cdot j) = c_s(h)$: every coefficient is a stage-0 coefficient carried along the record, which gives the form of O_pTO_q. The truncations $O_S = \sum_{q\in S} O_q$ over finite sets S are contractions converging strongly to 1, and O_STO_S is a polynomial in the generators. (d) A central T commutes with each E_q and each loop translation, so its stage-0 matrix is diagonal and invariant in the loop; the propagation of (c) finishes. (e) The same argument with the coefficients started at stage n: they may now depend on the stage-n record, which adds the archive reads $\mathrm{Ar}_n(g)$ (Theorem 9.14). (f) The readout O_q commutes with $\overline{\mathscr{G}}$ and fixes $\iota_0(\delta_q \otimes |z\rangle)$, which gives one inclusion. For the other, a read placed at stage 0 moves the starting cell within its half-turn orbit to any occupation with internal index $(0,0)$, and the transfers of Lemma 5.20, applied once per posting, then reach every coordinate whose original loop is q. □

What the native reads can never superpose is therefore exactly three things: the joint pointer sector, the original loop and, for the reads made after a record, the written record. There is no fourth: no hidden freedom on orphan coordinates and no branch along which a commuting operator stops propagating, because every slice carries every joint sector.

9.7 What a tracker of the record must carry

The present verdict of any cell state is a function of the 36 sector weights of its present cell (Proposition 3.16). Can those weights be kept up to date as labels are written, or does the next verdict need more? Call a subspace $\mathscr{W}$ of 64×64 matrices *pull-closed* if $V_j^*\mathscr{W}V_j \subseteq \mathscr{W}$ for every label j. If $\mathscr{W}$ is pull-closed and contains I, the weights $\mathrm{tr}(\rho Y)$, $Y\in\mathscr{W}$, of the present cell after one more label are fixed by the label and by the present weights on $\mathscr{W}$, because $\mathrm{tr}(\rho_{v,h\cdot j}Y) = \mathrm{tr}(\rho_{v,h}V_j^*YV_j)/\,\mathrm{tr}(\rho_{v,h}V_j^*V_j)$ by Proposition 2.20.

Theorem 9.16 (The least content that updates itself)**.** *Let* $\Omega_\pm = (P \pm A)/\sqrt{2}$.

(a) The sector weights have no closed update. The unit cells

$$f_+ = |00\rangle \otimes \Omega_+ \otimes \Omega_+, \qquad f_- = |00\rangle \otimes \Omega_- \otimes \Omega_-$$

have the same weight on all 36 *sectors (all of it on* $\Pi((0,0),0,0)$*). The fill label* $(1,0,1)$*, written by* $00\to 11$*, has probability* $\frac{1}{4}$ *under both; the present cell after it is* $|11\rangle\otimes p\otimes p$ *in the sector* $(+,+)$ *for* f_+*, and* $|11\rangle\otimes a\otimes a$ *in the sector* $(-,-)$ *for* f_-*, up to sign. So no map of the* 36 *weights and the written label returns the* 36 *weights of the present cell.*

(b) The least pull-closed subspace containing the 36 *sector projections is* $\mathscr{R}$*. So is the least pull-closed subspace containing* I*.*

(c) *The record law updates itself: if v, v' have one record law, then after every record h with $\mathbb{P}_v(h) > 0$ their present cells have one record law.*
(d) *Let $\mathscr{C}_0$ be the span of the sector projections and $\mathscr{C}_{m+1} = \mathscr{C}_m + \sum_j V_j^* \mathscr{C}_m V_j$. Then $\dim \mathscr{C}_0, \dim \mathscr{C}_1, \dim \mathscr{C}_2 = 36, 64, 92$, and $\mathscr{C}_2 = \mathscr{R}$. The space $\mathscr{C}_1$ is spanned by the 64 products $\mathrm{lift}_x(B \otimes B')$ with $B, B' \in \{\Pi_+^x, \Pi_-^x, \Pi_0^x, Z_x\}$.*
(e) *The cells $f_\pm$ have one law of one-label records. The record $\big((1,0,1),(-1,0,-1)\big)$ has probability $\frac{1}{4}$ under f_+ and 0 under f_-; the 36 sector facts themselves are two-label facts, since $\mathscr{C}_0$ lies in the span of the effects of words of length at most two and not of length at most one.*

Proof. (a) $\Omega_\pm = (P \pm A)/\sqrt{2}$ lie in the zero plane of c. By Table 2.2, $\mathrm{flip}\,\Omega_+ = p/\sqrt{2}$ and $\mathrm{flip}\,\Omega_- = -a/\sqrt{2}$, and the block of $00 \to 11$ is $-\mathrm{flip} \otimes \mathrm{flip}$; this gives the loads and the present cells, which lie on the $+1$ poles, respectively the -1 poles, of c at 11. (b) $\mathscr{R}$ is pull-closed by the proof of Theorem 9.4(a) and contains I and every sector projection. A pull-closed subspace containing I contains every effect, since $E_{j\cdot w} = V_j^* E_w V_j$, hence $\mathscr{R}$; and the sector projections sum to I. (c) $\mathrm{tr}(\rho_{v,h} E_w) = \mathbb{P}_v(h \cdot w)/\mathbb{P}_v(h)$ by Proposition 2.20(a). (d) Direct exact computation with Table 9.1: pulling a sector projection back once produces Z (onto balanced outputs) but no X, so $\mathscr{C}_1$ is the stated span; a second pullback turns Z into X onto mixed outputs and reaches every record read. (e) From $p \otimes p$ at 11 both cells must flip (stay $p = 0$), writing $(-1, 0, -1)$; from $a \otimes a$ both must stay ($\mathrm{flip}\, a = 0$), writing the zero label. One-label effects are products of $\Pi_\pm, \Pi_0$, on which f_+ and f_- agree. The two-label statement is a direct exact computation; a witness for one label is the pair $|00\rangle \otimes e_0 \otimes p$ and $|00\rangle \otimes A \otimes p$, which have one law of one-label records and weights $\frac{1}{4}$ and 0 on $\Pi((0,0), +, +)$. □

Inside the single sector "empty occupation, both poles zero", the 36 weights cannot see the direction of the clock state within the zero plane, and the fill step turns that direction into opposite pole sectors of the filled occupation. A tracker that keeps only the classical facts can state the present verdict but not the next one; the least content that contains those facts and keeps itself up to date is the whole record content. In the language of predictive state representations [29], the native posting has no predictive compression below 92 dimensions once it is asked to track its sector facts. The same holds if it only tracks the verdict.

Theorem 9.17 (Tracking the verdict needs the whole record law)**.** *Let $W_0 = Q_{\mathrm{al}} + \frac{1}{128} Q_{\mathrm{ne}}$ be the first field-current window effect.*

(a) *The least pull-closed subspace containing W_0 is $\mathscr{R}$.*
(b) *The pullbacks $V_w^* W_0 V_w$ over words of length at most five span $\mathscr{R}$; over words of length at most four they span at most 88 dimensions. The identity and all 36 sector projections lie in the span of the pullbacks of length at most three, and not in that of length at most two.*
(c) *Every posting symmetry of Chapter 5 fixes W_0 under its cell relabelling, but not the pulled reads: the swap carries $V_j^* W_0 V_j$ for $j = j(01, 00)$ to the pullback for $j(10, 00)$, and the two differ. The elements of $\mathscr{R}$ fixed by every posting symmetry span 28 dimensions.*
(d) *The span of the one-label pullbacks $V_j^* W_0 V_j$ and $V_j^* V_j$, $j \in \mathscr{J}$, has dimension 23.*

Proof. (b) is a direct exact computation in the record-read coordinates. Each of the 36 sector projections is an explicit rational combination of pullbacks of W_0 along words of at most three labels. Every element of the span of length at most four satisfies, at each occupation x, the linear identity $a_x + \epsilon_x b_x = 0$ between its coefficients a_x on $\mathrm{lift}_x(X_x \otimes \Pi_0^x)$ and b_x on $\mathrm{lift}_x(\Pi_0^x \otimes X_x)$, with

$\epsilon_x = -1$ for balanced and $+1$ for mixed x. The four identities are independent and the record read $\mathrm{lift}_{00}(X \otimes \Pi_0)$ violates one of them, so that span has dimension at most 88. Two labels miss I and $\Pi((0,0),+,-)$. Since the sector projections are reached at three labels, Theorem 9.16(d) reaches $\mathscr{R}$ at five. (a) follows: the least pull-closed subspace containing W_0 contains all pullbacks, hence $\mathscr{R}$ by (b); and $\mathscr{R}$ is pull-closed and contains W_0, a sum of effects. (c) A posting symmetry with label map κ and cell relabelling Σ satisfies $V_{\kappa(j)}^{*}(\Sigma Y \Sigma^{\top}) V_{\kappa(j)} = \Sigma\,(V_j^{*} Y V_j)\,\Sigma^{\top}$, so if Σ fixes W_0 it carries the pullback along a word to the pullback along the relabelled word. The half-turns of Theorem 5.12 fix every label and every record read; the swap relabels words. The two pullbacks and the dimension 28 are a direct exact computation. (d) Direct exact computation: the eight nonzero one-label pullbacks of W_0 and the fifteen one-label effects are independent. □

So the verdict's depth is five, against three for the record law, but its content is the same 92-dimensional space. Chapter 10 follows the present cell along a growing record.

Chapter 10

The present

After a long record, the present cell of every genesis state lies, except on records of geometrically small total probability, close to one of 256 pointer lines, and on those lines it moves as a classical Markov chain that its own record drives and reads. The present is where every window's verdict is read: conditioned on what has been written, each window is the first window of the state the record leaves. Nothing in the cell escapes the record: there is no dark subspace, and the present purifies. And the one feature the record law cannot see, the direction in which a clock circulates, is not kept in the present either. Recording turns the present into a real standing wave, and the direction survives only as phases between alternative records.

Throughout, $\mathbb{P}_\nu$ is the record law and $\rho_{\nu,h}$ the present cell of Chapter 2 (Definition 2.19), and the record reads, the planes $\mathscr{U} = \mathrm{span}\{p, a\}$, $D\mathscr{U} = \mathrm{span}\{P, A\}$ and the reads $\Pi^x_\pm, \Pi^x_0, Z_x, X_x$ are those of Chapter 9. For a unit genesis state ν and $k \geq 0$ write

$$\eta_k(\nu) = \sum_{h \in \mathscr{J}^k} \mathbb{P}_\nu(h)\big(1 - \mathrm{tr}\,\rho^2_{\nu,h}\big)$$

for the mean impurity of the present after k postings; records of probability zero contribute nothing to this or to any later record average.

10.1 Every window is the first window of its present

Chapter 3 found that the field-current verdict, referred back to genesis, is central only on the windows that start after 0, 1 or 2 postings. Read on the state that the written prefix leaves, it is central on every window. Given a record h of length k with $\mathbb{P}_\nu(h) > 0$, window k of a cut C passes with conditional probability $\mathrm{tr}\big(\rho_{\nu,h} W_0(C)\big)$, the first-window pass probability of the present cell (Proposition 3.16), and $W_0(C)$ is central in the current algebra $\mathscr{A}$ if and only if C contains neither account 1 nor account 3 (Theorem 3.11). So the selection of Chapter 3 holds at every window, relative to the written record: among the cuts that can pass, those whose verdict on the present is a sector fact are the cuts whose only observed account is the loop edge. The restriction to three windows concerns the verdict referred back to genesis, a prediction made before the prefix is written, which averages the present verdicts over the prefixes.

Theorem 10.1 (The genesis verdict averages present verdicts)**.** *Let C be a set of accounts, ν a unit genesis state and $k \geq 0$. Then $W_k(C) = \sum_{h \in \mathscr{J}^k} V_h^* W_0(C) V_h$, and for every record h of length k, $\mathrm{tr}\big(\rho_\nu V_h^* W_0(C) V_h\big) = \mathbb{P}_\nu(h)\,\mathrm{tr}\big(\rho_{\nu,h} W_0(C)\big)$ if $\mathbb{P}_\nu(h) > 0$, and 0 otherwise.*

Proof. $W_k(C) = \Phi^k(W_0(C))$ by Proposition 3.2(c), and $\Phi^k(Y) = \sum_{|h|=k} V_h^* Y V_h$. Next, $\mathrm{tr}(\rho_\nu V_h^* W V_h) = \mathrm{tr}(V_h \rho_\nu V_h^* W)$; if $\mathbb{P}_\nu(h) > 0$ this is $\mathbb{P}_\nu(h)\,\mathrm{tr}(\rho_{\nu,h} W)$ by Definition 2.19, and if $\mathbb{P}_\nu(h) = 0$ the positive matrix $V_h \rho_\nu V_h^*$ has trace zero and vanishes. □

The individual terms of this average are where the coherence enters, and it enters one posting earlier than in the sum.

Proposition 10.2 (A single carried-back term is contextual)**.** *Let $C = \{2\}$ and $T_h = V_h^* W_0 V_h$ for a record h.*

(a) For every record h of length 0 or 1, T_h is central in $\mathscr{A}$.
(b) Let $h^* = \big((1,0,1),(0,0,-1)\big)$, the record of the occupation path $(0,0) \to (1,1) \to (1,0)$, and let s_A, s_B be the states of Chapter 3, which lie in one sector projection $\Pi\big((0,0),0,+\big)$. Then $\langle s_A, T_{h^*} s_A\rangle = 0$ and $\langle s_B, T_{h^*} s_B\rangle = \frac{1}{512}$. Hence T_{h^*} is not central.
(c) The sum $W_2 = \sum_{|h|=2} T_h$ is central.

Proof. (a) $T_\varnothing = W_0$ is central (Theorem 3.7). For a label j, $T_{(j)} = \sum_{j(y,x)=j} |x\rangle\langle x| \otimes M_{yx}^\top (W_0)_{yy} M_{yx}$, as in the proof of Lemma 3.6. The block $(W_0)_{yy}$ vanishes for balanced y and is a polynomial in $c_y \otimes I$ and $I \otimes c_y$ for mixed y. Onto a mixed output each cell map pulls $\Pi_+^y, \Pi_-^y, \Pi_0^y$ back to multiples of $\Pi_+^x, \Pi_-^x, \Pi_0^x$ (Table 9.1). So $T_{(j)}$ is a combination of sector projections, and these commute with every element of $\mathscr{A}$ by Lemma 3.6.

(b) We follow the two postings with Table 2.2. The first posting goes $(0,0) \to (1,1)$, so both cells flip: flip $\frac{1}{\sqrt{2}}(P + A) = \frac{1}{\sqrt{2}} p$, flip $\frac{1}{\sqrt{2}}(A - P) = \frac{1}{\sqrt{2}} a$, and the second cell goes from p to P. The second posting goes $(1,1) \to (1,0)$: cell 1 reads $x_2 = 1$ and writes $y_1 = 1$, so it stays, and cell 2 flips. Since stay $p = 0$, the state s_A writes h^* with probability 0, and $\langle s_A, T_{h^*} s_A\rangle = 0$ by Theorem 10.1. From s_B, stay $a = A$ with weight 1 and flip $P = \frac{1}{2}(p - a)$ with weight $\frac{1}{2}$, so $\mathbb{P}_{s_B}(h^*) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ and the present is the line of $|10\rangle \otimes A \otimes \frac{1}{\sqrt{2}}(p - a)$. At the mixed occupation $(1,0)$ the second cell lies in $\mathscr{U}$, the zero plane of c', so the present lies in the neutral sector and its first-window verdict is $\frac{1}{128}$ (Theorem 3.4). Hence $\langle s_B, T_{h^*} s_B\rangle = \frac{1}{4} \cdot \frac{1}{128} = \frac{1}{512}$. The matrix T_{h^*} is real symmetric and lies in $\mathscr{A}$, so by Lemma 3.9 it does not commute with E_{w^*}.

(c) is Theorem 3.12 with $k = 2$. □

The prefix reads the direction, not the verdict. The step into the balanced occupation $(1,1)$ turns the direction of cell 1 inside the zero plane of c into the pole p or a, and the next stay keeps a and kills p. This is the same mechanism that makes the later windows of Chapter 3 contextual. At prefix length two the sum over prefixes still cancels it; from three postings on it does not. Read on the present, the verdict is always a sector fact.

The same factorization carries the statistical reading of the half-cycle pointer (Corollary 9.9) to the present: the pointer weights of $\rho_{v,h}$ are given by that formula with each one- and two-label probability replaced by the conditional probability of the continuation after h.

10.2 Nothing in the cell stays unread

Every record probability is the Born weight of a record read, an element of the 92-dimensional record content $\mathscr{R}$ of Chapter 9. A part of the cell that no record ever distinguishes would be a dark subspace. There is none, and the present purifies.

Definition 10.3 (Dark subspace)**.** A *dark subspace* is the range of an orthogonal projection Q on $\mathscr{H}$ of rank at least two such that for every word w there is $\lambda_w \in \mathbb{C}$ with $Q E_w Q = \lambda_w Q$.

Every density supported in a dark subspace has the same record law, $\operatorname{tr}(\rho E_w) = \lambda_w$, so the record could never tell such states apart. A dark subspace is exactly what blocks the purification of conditional states in Maassen and Kümmerer's analysis of quantum trajectories [30].

Definition 10.4 (Record basis)**.** Let $\omega_\pm = \frac{1}{\sqrt{2}}(p \pm a)$ and $\Omega_\pm = \frac{1}{\sqrt{2}}(P \pm A)$, the standing waves of the two planes. At an occupation x the *record basis* of a clock cell consists of the two poles of c_x and the two eigenvectors of Z_x in its zero plane: p, a, Ω_+, Ω_- for balanced x, and P, A, ω_+, ω_- for mixed x. Their projections are Π^x_+, Π^x_-, $\frac{1}{2}(\Pi^x_0 + Z_x)$ and $\frac{1}{2}(\Pi^x_0 - Z_x)$. The 64 products $|x\rangle\langle x| \otimes R_1 \otimes R_2$ of these projections are rank-one, mutually orthogonal, and sum to I.

Theorem 10.5 (No dark subspace)**.**

(a) Each of the 64 *record-basis projections lies in $\mathscr{R}$. So the record law fixes, at every occupation, the weight of each clock cell's two poles and of each standing wave in its zero plane.*
(b) The native cell has no dark subspace.

Proof. (a) Expanding the product gives combinations of the matrices $|x\rangle\langle x| \otimes A \otimes B$ with $A, B \in \{\Pi^x_+, \Pi^x_-, \Pi^x_0, Z_x\}$. None of these pairs is (Z_x, X_x) or (X_x, Z_x), so all are record reads, and $\mathscr{R}$ is the span of the effects (Theorem 9.4(a)).

(b) Let Q be a projection of rank at least two and $B = |b\rangle\langle b|$ a rank-one projection with $QBQ = \lambda Q$. Then $(QBQ)^2 = \langle b, Qb\rangle QBQ$ gives $\lambda^2 Q = \langle b, Qb\rangle \lambda Q$, and taking traces in $QBQ = \lambda Q$ gives $\langle b, Qb\rangle = \lambda \operatorname{tr} Q$. So $\lambda^2(1 - \operatorname{tr} Q) = 0$, hence $\lambda = 0$, so $\|QBQ\| = \|Qb\|^2 = 0$ and $BQ = 0$. If Q compressed every effect to a multiple of itself, it would compress all of $\mathscr{R}$, in particular each of the 64 record-basis projections of (a). Then $BQ = 0$ for all of them, and since they sum to I, $Q = 0$. □

Corollary 10.6 (No dark pair survives three labels)**.** *For orthonormal $u, v \in \mathscr{H}$ there is a word w of at most three labels with $\langle u, E_w u\rangle \neq \langle v, E_w v\rangle$ or $\langle u, E_w v\rangle \neq 0$.*

Proof. Otherwise $Q = |u\rangle\langle u| + |v\rangle\langle v|$ would compress every effect of length at most three to a multiple of itself. Those effects span $\mathscr{R}$ (Theorem 9.4(a)), so Q would span a dark subspace. □

The two alternatives in the corollary are different in kind. A state and its complex conjugate have one record law (Theorem 9.7), so for that pair the first alternative never holds; the pair is separated only by a nonzero cross term $\langle u, E_w \bar{u}\rangle$. Section 10.5 shows what such cross terms do as the record grows.

Purification is the dynamical form of the theorem. Write $\rho_\nu = G^* G$ for some matrix G, and for a record h put $A_h = G E_h G^*$ and $t_h = \operatorname{tr} A_h = \mathbb{P}_\nu(h)$.

Proposition 10.7 (The mean purity never decreases)**.** *For every unit genesis state ν and every k,*

$$\sum_{h \in \mathscr{J}^{k+1}} \mathbb{P}_\nu(h) \operatorname{tr} \rho^2_{\nu,h} - \sum_{h \in \mathscr{J}^k} \mathbb{P}_\nu(h) \operatorname{tr} \rho^2_{\nu,h} = \sum_{h \in \mathscr{J}^k} \sum_{j \,:\, t_{h\cdot j} > 0} t_{h\cdot j} \left\| \frac{A_{h\cdot j}}{t_{h\cdot j}} - \frac{A_h}{t_h} \right\|^2_{\mathrm{HS}} \geq 0,$$

the outer sum running over the records with $t_h > 0$. So $\eta_k(\nu)$ is nonincreasing in k.

Proof. With $B = V_h G^*$, $\mathbb{P}_v(h)^2 \operatorname{tr} \rho_{v,h}^2 = \operatorname{tr}\big((BB^*)^2\big) = \operatorname{tr}\big((B^*B)^2\big) = \operatorname{tr} A_h^2$, so the mean purity at length k is $\sum_h \operatorname{tr} A_h^2 / t_h$. Since $\sum_j E_{h\cdot j} = E_h$, $A_h = \sum_j A_{h\cdot j}$ and $t_h = \sum_j t_{h\cdot j}$, and a positive $A_{h\cdot j}$ with $t_{h\cdot j} = 0$ vanishes. The displayed identity is then the variance identity $\sum_j t_j \|Y_j\|^2 - t\|\bar{Y}\|^2 = \sum_j t_j \|Y_j - \bar{Y}\|^2$ for $Y_j = A_{h\cdot j}/t_{h\cdot j}$ and their weighted mean $\bar{Y} = A_h/t_h$. □

Purification has a plain meaning. The present cell is the cell density of the unit genesis state v_h that the record leaves (Chapter 2), a pure state of the loop field and the cell together, so its impurity is the correlation of the cell with the loop field. Purification says that, given its record, the cell becomes uncorrelated with the field.

10.3 The present settles onto a pointer line

The eight vectors $p, a, P, A, \omega_\pm, \Omega_\pm$ of a clock cell are closed under the cell maps: each stay or flip sends each of them to a multiple of one of them. That makes a finite family of cell states on which the record acts as a classical chain.

Definition 10.8 (Pointer lines)**.** A *pointer line* is $\mathscr{P}_\ell = |x\rangle\langle x| \otimes |\ell_1\rangle\langle \ell_1| \otimes |\ell_2\rangle\langle \ell_2|$ for $\ell = (x; \ell_1, \ell_2)$ with $x \in \mathcal{O}$ and $\ell_1, \ell_2 \in \{p, a, P, A, \omega_+, \omega_-, \Omega_+, \Omega_-\}$. There are 256 pointer lines. At each occupation they contain two orthonormal bases of the cell: the record basis, and the *pole lines*, both cells on $\{p, a, P, A\}$.

	p	a	P	A	ω_+	ω_-	Ω_+	Ω_-
stay, x balanced	0	A, 1	ω_+, $\frac{1}{2}$	ω_-, $\frac{1}{2}$	A, $\frac{1}{2}$	A, $\frac{1}{2}$	a, $\frac{1}{2}$	p, $\frac{1}{2}$
flip, x balanced	P, 1	0	ω_-, $\frac{1}{2}$	ω_+, $\frac{1}{2}$	P, $\frac{1}{2}$	P, $\frac{1}{2}$	p, $\frac{1}{2}$	a, $\frac{1}{2}$
stay, x mixed	ω_+, $\frac{1}{2}$	ω_-, $\frac{1}{2}$	0	A, 1	a, $\frac{1}{2}$	p, $\frac{1}{2}$	A, $\frac{1}{2}$	A, $\frac{1}{2}$
flip, x mixed	ω_-, $\frac{1}{2}$	ω_+, $\frac{1}{2}$	P, 1	0	p, $\frac{1}{2}$	a, $\frac{1}{2}$	P, $\frac{1}{2}$	P, $\frac{1}{2}$

Table 10.1. The one-cell line map: the image line of each cell vector and its weight (squared length). On a mixed input the dressing D acts first. A step $x \to y$ moves cell 1 by stay if $y_1 = x_2$ and cell 2 by stay if $y_2 = x_1$, otherwise by flip. For example stay $\Omega_+ = \frac{1}{\sqrt{2}}\big(\frac{1}{2}(p + a) + \frac{1}{2}(a - p)\big) = a/\sqrt{2}$.

Lemma 10.9 (Lines go to lines)**.** *For every label j and pointer line ℓ there are a weight $w_j(\ell) \in \{0, \frac{1}{4}, \frac{1}{2}, 1\}$ and a line ℓ' with $V_j \mathscr{P}_\ell V_j^* = w_j(\ell)\mathscr{P}_{\ell'}$, and $\sum_j w_j(\ell) = 1$. Along a record h, $V_h \mathscr{P}_\ell V_h^* = w_h(\ell)\mathscr{P}_{\ell_h}$, where $w_h(\ell)$ is the product of the weights met along h and ℓ_h the line reached. On a step $x \to y$ the $+1$ pole of c_x always flips and the -1 pole always stays, each with weight 1; every other cell vector stays or flips with weight $\frac{1}{2}$.*

Proof. Each cell map sends each of the eight vectors to a multiple of one of them (Table 10.1, by linearity from Table 2.2), with squared length $\langle \ell_i, \frac{1}{2}(I \mp c_x)\ell_i\rangle$ for a stay and a flip (Lemma 2.5, with the dressing on mixed inputs). A label and an input fix the output (Proposition 2.7), so $V_j \mathscr{P}_\ell V_j^*$ is a single block. Finally $\sum_j w_j(\ell) = \operatorname{tr}\big(\mathscr{P}_\ell \sum_j V_j^* V_j\big) = 1$ by Proposition 2.12. The statement along records follows by induction. □

Let κ be the stochastic matrix on the 256 lines with

$$\kappa(\ell, \ell') = \sum_{j\colon\ \ell' \text{ is the image of } \ell \text{ under } j} w_j(\ell), \tag{10.1}$$

the *pointer kernel.* On pointer lines the present moves classically: if $\rho_{v,h} = \mathscr{P}_\ell$, then by Proposition 2.20 the next label is j with probability $w_j(\ell)$ and the next present is the image line.

Example 10.10 (The ready vector walks along lines)**.** The ready vector $r = |00\rangle \otimes p \otimes a$ of Example 2.18 is the line $(00; p, a)$. By Table 10.1 it moves with weight 1 to $(10; P, A)$, then with weight 1 to $(11; P, A)$, and then with weight $\frac{1}{4}$ each to $(11; \omega_+, \omega_-)$, $(10; \omega_+, \omega_+)$, $(01; \omega_-, \omega_-)$ and $(00; \omega_-, \omega_+)$, with labels $(0, 0, 0)$, $(0, 0, -1)$, $(-1, 0, 0)$ and $(-1, 0, -1)$. The first of these is the present cell of Example 2.21. A state that starts on a line never leaves the lines.

A general genesis state is not on a line. It is carried onto one.

Theorem 10.11 (The present enters the pointer lines)**.** *Let ν be a unit genesis state. For a record h with $\mathbb{P}_\nu(h) > 0$ let $M_\nu(h) = \max_\ell \operatorname{tr}(\rho_{\nu,h}\mathscr{P}_\ell)$, and let $m_k(\nu) = \sum_{h \in \mathscr{J}^k} \mathbb{P}_\nu(h) M_\nu(h)$. Then for every k*

$$1 - m_k(\nu) \le 4096\,(41/48)^{\lfloor k/2 \rfloor}.$$

Along almost every infinite record, under the law μ_ν of Theorem 2.17(d), $M_\nu \to 1$.

Proof. A record names a line. Fix a record h of length k and a line ℓ^*, and let $Y = (I - \mathscr{P}_{\ell^*})V_h$. Then $\mathbb{P}_\nu(h) - \operatorname{tr}(V_h \rho_\nu V_h^* \mathscr{P}_{\ell^*}) = \operatorname{tr}(\rho_\nu Y^* Y) \le \operatorname{tr}(Y^* Y)$, since ρ_ν is a density. The 64 pole lines q sum to I, so by Lemma 10.9

$$\operatorname{tr}(Y^*Y) = \sum_q \operatorname{tr}\big(V_h \mathscr{P}_q V_h^* (I - \mathscr{P}_{\ell^*})\big) = \sum_q w_h(q)\big(1 - \operatorname{tr}(\mathscr{P}_{q_h}\mathscr{P}_{\ell^*})\big).$$

Take for ℓ^* the line q_h^* reached by a pole line q^* of largest weight $w_h(q^*)$. A term with $q_h = \ell^*$ vanishes, and every other term is at most $w_h(q) \le \sqrt{w_h(q) w_h(q^*)}$. Hence

$$\mathbb{P}_\nu(h)\big(1 - M_\nu(h)\big) \le \sum_{q,q'} \sqrt{w_h(q) w_h(q')}\,[q_h \ne q'_h].$$

Pairs carried apart lose affinity. For pole lines q, q' let $\alpha_k(q, q') = \sum_{|h|=k} \sqrt{w_h(q) w_h(q')}\,[q_h \ne q'_h]$. Two lines that meet stay together, and appending two labels multiplies each term by $\sum_{u \in \mathscr{J}^2} \sqrt{w_u(\ell) w_u(\ell')}\,[\ell_u \ne \ell'_u]$ for the current pair $\ell \ne \ell'$. This two-step affinity is at most $\frac{41}{48}$ for every pair of distinct lines, by direct exact computation (Appendix A.4.2). With $\alpha_0 \le 1$ and $\alpha_1 \le 1$ (Cauchy–Schwarz with $\sum_j w_j = 1$), $\alpha_k(q, q') \le (41/48)^{\lfloor k/2 \rfloor}$. Summing over the records and the $64^2 = 4096$ ordered pairs gives the bound.

Almost surely. The mean of $1 - M_\nu$ over the records of length k is $1 - m_k(\nu)$, which is summable in k. By monotone convergence $\sum_k (1 - M_\nu) < \infty$ along almost every infinite record, so its terms tend to zero. □

The constants are a proved envelope, far from sharp: the bound says something only from about 106 postings on, while the almost-sure convergence holds for every state. In the language of decoherence theory this is einselection [46]: the pointer lines are the cell states that the monitoring keeps predictable. Here the monitoring is the record that the postings write, and the pointer basis is fixed by the native slices, not chosen.

Corollary 10.12 (The present purifies geometrically)**.** *For every unit genesis state ν, $\eta_k(\nu) \le 2\big(1 - m_k(\nu)\big) \le 8192\,(41/48)^{\lfloor k/2 \rfloor}$, and $\operatorname{tr} \rho_{\nu,h}^2 \to 1$ along μ_ν-almost every infinite record.*

Proof. If ρ is a density and u a unit vector with $\langle u, \rho u\rangle = M$, then $\operatorname{tr} \rho^2 \ge \|\rho u\|^2 \ge M^2 \ge 2M - 1$. Apply this to $\rho_{\nu,h}$ and the line achieving $M_\nu(h)$, and use Theorem 10.11. □

Reading the present cell as the physical present of the cell, the identification named in Chapter 2, a long-recorded cell has, up to a geometrically small load, a definite present: an occupation and, in each clock cell, a pole of one of the two planes or a standing wave of fixed phase.

10.4 The present as a classical chain

The pointer kernel has five closed classes. They lie in the five classes of cell states that the native step conserves (Chapter 11), and inside the two large ones the present keeps moving at random forever.

Definition 10.13 (The five classes)**.** Sort the 36 sector projections $\Pi(x, s, t) = |x\rangle\langle x| \otimes \Pi_s^x \otimes \Pi_t^x$ of Chapter 3 into five classes. Class 0 is the sector $((1, 0), +, +)$ and class 1 the sector $((0, 1), +, +)$. Class 2 holds $((1, 0), -, -)$ and $((0, 1), -, -)$. Class 3 holds the 16 sectors in which exactly one of s, t is 0. Class 4 holds the remaining 16: the sectors $(x, 0, 0)$, the sectors with $s, t \neq 0$ on a balanced occupation, and $(x, +, -)$, $(x, -, +)$ on a mixed one. Let Λ_c be the sum of the sector projections of class c. By (2.16), Λ_0, Λ_1 and Λ_2 project onto ψ_+, onto ψ_- and onto $\operatorname{span}\{\psi_\circ, \psi_\circ'\}$, so that $\Lambda_0 + \Lambda_1 + \Lambda_2 = Q_{\mathrm{al}}$; $\Lambda_3 = Q_{\mathrm{oz}}$ of Chapter 3 holds the states with exactly one cell on the zero plane of c_x; and $\Lambda_4 = I - Q_{\mathrm{al}} - Q_{\mathrm{oz}}$. The ranks are $1, 1, 2, 32, 28$. We call classes $0, 1, 2$ the *sure-subject classes*, class 3 the *one-zero class* and class 4 the *rest class*. For a genesis state ν write $w_c(\nu) = \operatorname{tr}(\rho_\nu \Lambda_c)$ for its *class weights*, so that $w_3 = w_{\mathrm{oz}}$ and $w_0 + w_1 + w_2 = w_{\mathrm{al}}$, and $\vartheta(\nu) = \operatorname{tr}(\rho_\nu \Psi_\circ)$, $\Psi_\circ = |\psi_\circ\rangle\langle\psi_\circ| - |\psi_\circ'\rangle\langle\psi_\circ'|$, for its *oscillation phase*; $|\vartheta(\nu)| \le w_2(\nu)$. Chapter 11 shows that the Λ_c span the sector reads fixed by the free pull and commute with every slice (Theorem 11.1).

Theorem 10.14 (The pointer chain)**.**

(a) κ has exactly five closed classes, of $1, 1, 2, 64, 44$ lines (Table 10.2), and every line of class c lies in the range of Λ_c.
(b) Each closed class is irreducible. From any line, after $2n$ steps at most 2^{-n} of the weight remains on the other 144 lines.
(c) Class c carries exactly one stationary probability law π_c, given in Table 10.2.
(d) $\sum_\ell \pi_c(\ell) \mathscr{P}_\ell = \Lambda_c / \operatorname{tr} \Lambda_c$.

Proof. (a) and (c), closure and stationarity: the lines of classes $0, 1, 2$ are the four modes, which move as in Proposition 2.22: $\psi_\pm$ return to themselves with weight 1 and the two oscillation lines go to each other with weight 1. Classes 3 and 4 are checked by hand in Appendix A.4.1, by moving whole types of lines with Table 10.1. A line of class 3 has one cell in $\mathscr{U}$ and the other on a pole of $D\mathscr{U}$; whatever the occupation, exactly one of the two is on the zero plane of c_x. A line of class 4 has, at a balanced occupation, both cells in $\mathscr{U}$ (the span of the poles of c)

class	closed lines $(x;\ell_1,\ell_2)$	lines	$\operatorname{tr}\Lambda_c$	π_c
0	$(10;P,P)$	1	1	1
1	$(01;P,P)$	1	1	1
2	$(10;A,A)$, $(01;A,A)$	2	2	$\frac{1}{2}$
3	one cell on p or a, the other on P or A	32	32	$\frac{1}{96}$
	one cell on $\omega_\pm$, the other on P or A	32		$\frac{1}{48}$
4	both cells on $\{p,a\}$	16	28	$\frac{1}{84}$
	both cells on $\{\omega_+,\omega_-\}$	16		$\frac{1}{42}$
	both on $\{P,A\}$ (x balanced); (P,A), (A,P) (x mixed)	12		$\frac{1}{28}$

Table 10.2. The closed classes of the pointer chain, at any occupation unless stated, with the stationary probability of each line and the rank of the class projection. The other 144 lines are transient.

or both in $D\mathcal{U}$ (its zero plane); at a mixed occupation, both cells in $\mathcal{U}$ (the zero plane of c') or on opposite poles of c'. In each case the line lies in the range of Λ_c. (b) Irreducibility of classes 3 and 4 is witnessed by closed walks of positive weight of lengths 112 and 97 that visit every line of the class; these and the transient bound are direct exact computations with Table 10.1. (c) Uniqueness holds for every irreducible finite chain. (d) The two standing waves of a plane average to the same projection as its two poles: $|\omega_+\rangle\langle\omega_+| + |\omega_-\rangle\langle\omega_-| = |p\rangle\langle p| + |a\rangle\langle a|$, and likewise in $D\mathcal{U}$. With the weights of Table 10.2, at every occupation the class-3 lines average to $(\frac{1}{96}+\frac{1}{48})$ times the one-zero projection, which is $\Lambda_3/32$, and the class-4 lines to $\frac{1}{28}$ times the rest projection, $\Lambda_4/28$. □

On a sure subject the chain is the mode dynamics: $\psi_\pm$ stand still and the oscillation alternates. On the one-zero and rest classes the present never settles on one line. It wanders through 64 or 44 lines with the stationary laws above, while the class itself never changes. Which class a long record ends in, and with what probability, is decided by the class weights $w_c(v)$; Chapters 8 and 11 prove that the record decides it.

The theorem also gives the long-run law of the line itself. Two distinct lines overlap by at most $\frac{1}{2}$, so a present cell puts more than $\frac{9}{10}$ of its weight on at most one line; call that line its *best line*. By Theorem 10.11 and Markov's inequality, the records of length k whose present has no best line have total probability at most $10(1-m_k(v))$.

Theorem 10.15 (The law of the best line)**.** *Let v be a unit genesis state and $\beta_k(v,\ell)$ the probability of the records of length k whose present has best line ℓ. Let $\varepsilon(\ell)=1$ for $\ell=(10;A,A)$, $\varepsilon(\ell)=-1$ for $\ell=(01;A,A)$ and $\varepsilon(\ell)=0$ otherwise. Then*

$$\beta_{2n}(v,\ell)\to\sum_{c=0}^{4}w_c(v)\pi_c(\ell)+\tfrac{1}{2}\vartheta(v)\varepsilon(\ell),\qquad \beta_{2n+1}(v,\ell)\to\sum_{c=0}^{4}w_c(v)\pi_c(\ell)-\tfrac{1}{2}\vartheta(v)\varepsilon(\ell)$$

as $n\to\infty$.

Proof. After $m+L$ postings, the last L labels name a line: carry each of the 64 pole lines along them and take the image of the heaviest. Write $D_L=4096\,(41/48)^{\lfloor L/2\rfloor}$. By the affinity bound

in the proof of Theorem 10.11, the present sits on the named line up to a record-mean load at most D_L, so $\beta_{m+L}(\nu,\cdot)$ is, up to $10D_L$, the law of the line named by the last L labels. The probability that postings $m+1,\dots,m+L$ write a given word converges along even and along odd m to an expression in the class weights and the oscillation phase of ν (Theorem 11.6(d)), whose class term is the probability of the word in the uniform state of the class. By Theorem 10.14(d) the uniform state of class c is $\sum_\ell \pi_c(\ell)\mathscr{P}_\ell$, and a record u carries each line ℓ to ℓ_u with weight $w_u(\ell)$; so by stationarity the line reached after L postings has law π_c. From every starting line, the line named by u differs from the line reached only on records of total weight at most $2D_L$, since two distinct lines overlap by at most $\frac{1}{2}$. So the class terms give $\sum_c w_c(\nu)\pi_c$, and the oscillation, which alternates between its two lines, contributes $\pm\frac{1}{2}\vartheta(\nu)\varepsilon$; the limit law of the named line is within $12D_L$ of the stated limit. Let $L\to\infty$. □

The class is a record of genesis; the line inside a moving class is not. A long record fixes the present up to a classical random walk whose law forgets the start, apart from the five class weights and the phase of the oscillation. The long-run law of the present on pure states is atomic, carried by the pointer lines; the invariant-measure theory of purifying quantum trajectories [8] describes such laws in general, and here the atoms and their weights are explicit.

10.5 Recording erases a circulation

Every effect is real symmetric, so the record law cannot tell a state from its complex conjugate (Theorem 9.7). With $\gamma=\frac{1}{2}(1,i,-1,-i)$, the circulating cell $f_\gamma=|00\rangle\otimes p\otimes\gamma$ and its reverse $\bar f_\gamma$ are orthogonal and have one record law; the clock of cell 2 turns one way in the first and the other way in the second. This section follows the circulation as the record is written. It is not stored in the record, it is not kept in the present, and it survives only as phases between alternative records.

Lemma 10.16 (Conjugate presents)**.** *Let ν be a genesis state and $\bar\nu$ its complex conjugate. For every record h, $\mathbb{P}_{\bar\nu}(h)=\mathbb{P}_\nu(h)$, and if this is positive, $\rho_{\bar\nu,h}=\rho_{\nu,h}^{\top}$. If a genesis state u has $\rho_u=\frac{1}{2}(\rho_\nu+\rho_\nu^{\top})$, then $\mathbb{P}_u=\mathbb{P}_\nu$ and $\rho_{u,h}=\frac{1}{2}(\rho_{\nu,h}+\rho_{\nu,h}^{\top})$ for every record h of positive probability. One such u is $u(2n)=\nu(n)/\sqrt{2}$, $u(2n+1)=\bar\nu(n)/\sqrt{2}$.*

Proof. $\rho_{\bar\nu}=\overline{\rho_\nu}=\rho_\nu^{\top}$ because ρ_ν is self-adjoint. The slices are real, so $V_h\rho^{\top}V_h^{*}=(V_h\rho V_h^{*})^{\top}$ and $\operatorname{tr}(\rho^{\top}E_h)=\operatorname{tr}(\rho E_h)$ for the symmetric E_h. The mixture statement follows by linearity. For the interleaved u, distinct loop values carry the two halves, so $\rho_u=\frac{1}{2}\sum_n \nu(n)\nu(n)^{*}+\frac{1}{2}\sum_n\bar\nu(n)\bar\nu(n)^{*}=\frac{1}{2}(\rho_\nu+\rho_\nu^{\top})$, and $\|u\|=\|\nu\|$. □

Define the *conjugation impurity* $\tau_k(\nu)=\sum_{h\in\mathscr{J}^k}\mathbb{P}_\nu(h)\big(1-\operatorname{tr}(\rho_{\nu,h}\rho_{\nu,h}^{\top})\big)$. It vanishes exactly when the present cells of positive probability are real pure states up to phase.

Theorem 10.17 (Circulation erasure)**.** *Let ν be a unit genesis state.*

(a) For u as in Lemma 10.16, $\eta_k(u)=\frac{1}{2}\eta_k(\nu)+\frac{1}{2}\tau_k(\nu)$. Hence $0\le\tau_k(\nu)\le 16384\,(41/48)^{\lfloor k/2\rfloor}$.

(b) For every matrix E with $E^{\top}=-E$, $\sum_h\mathbb{P}_\nu(h)\big|\operatorname{tr}(\rho_{\nu,h}E)\big|\le\|E\|_{\mathrm{HS}}\sqrt{\tau_k(\nu)/2}$.

(c) *Along μ_v-almost every infinite record,* $\operatorname{tr}(\rho_{v,h}\rho_{v,h}^{\top}) \to 1$, *and* $\operatorname{tr}(\rho_{v,h}E) \to 0$ *for every antisymmetric* E.

(d) *If* $v = \delta_q \otimes f$ *lies at one loop value, every present cell is pure,* $\rho_{v,h} = |\psi_h\rangle\langle\psi_h|$ *with* $\psi_h = V_h f/\|V_h f\|$, *so* $\tau_k(v) = \sum_h \mathbb{P}_v(h)\big(1 - |\psi_h^{\top}\psi_h|^2\big)$; *and there is a unit* $c_h \in \mathbb{C}$ *with* $\|\operatorname{Im}(c_h\psi_h)\|^2 = \frac{1}{2}\big(1 - |\psi_h^{\top}\psi_h|\big)$.

Proof. (a) $\operatorname{tr}\big(\frac{1}{2}(\rho + \rho^{\top})\big)^2 = \frac{1}{2}\operatorname{tr}\rho^2 + \frac{1}{2}\operatorname{tr}(\rho\rho^{\top})$, since $\operatorname{tr}(\rho^{\top})^2 = \operatorname{tr}\rho^2$; average over records with Lemma 10.16. By Cauchy–Schwarz $\operatorname{tr}(\rho\rho^{\top}) \le \operatorname{tr}\rho^2 \le 1$, so both means are nonnegative and $\tau_k(v) \le 2\eta_k(u)$; apply Corollary 10.12 to u. (b) For antisymmetric E, $\operatorname{tr}(\rho^{\top}E) = -\operatorname{tr}(\rho E)$, so $\operatorname{tr}(\rho E) = \frac{1}{2}\operatorname{tr}\big((\rho - \rho^{\top})E\big)$, and $\|\rho - \rho^{\top}\|_{\mathrm{HS}}^2 = 2\big(\operatorname{tr}\rho^2 - \operatorname{tr}\rho\rho^{\top}\big) \le 2\big(1 - \operatorname{tr}\rho\rho^{\top}\big)$. Cauchy–Schwarz in tr and then over the records gives the bound. (c) $\mathbb{P}_u = \mathbb{P}_v$, so $\mu_u = \mu_v$ by the uniqueness in Theorem 2.17(d). Along almost every record $\operatorname{tr}\rho_{u,h}^2 \to 1$ (Corollary 10.12), and $1 - \operatorname{tr}\rho_{u,h}^2 \ge \frac{1}{2}\big(1 - \operatorname{tr}\rho_{v,h}\rho_{v,h}^{\top}\big)$. The reads follow from the pointwise bound in (b). (d) By Theorem 2.14(b), $(T^k v)(n,h)$ is $V_h f$ at the single loop value $n = q + \operatorname{acc}(h)$ and zero elsewhere, so the present is pure and $\operatorname{tr}(\rho\rho^{\top}) = |\psi_h^{\top}\psi_h|^2$. Choose c_h with $c_h^2\psi_h^{\top}\psi_h = |\psi_h^{\top}\psi_h|$ and write $c_h\psi_h = r + is$ with r, s real: then $|r|^2 - |s|^2 = |\psi_h^{\top}\psi_h|$ and $|r|^2 + |s|^2 = 1$. □

For a one-loop genesis the present is pure at every length, so what recording changes is not its purity but its realness. A concrete case shows the mechanism.

Example 10.18 (The circulating cell)**.** Let $v_\gamma = \delta_0 \otimes f_\gamma$. For every $k \ge 1$, $\tau_k(v_\gamma) = 2^{1-k}$. The present is circulating, $\psi_h^{\top}\psi_h = 0$, exactly on the records whose first $k-1$ labels are $(1,0,0), (1,1,-1), \dots, (1,1,-1)$, and it is real up to phase on every other record of positive probability.

The vector $\gamma = \frac{1}{2}\big((1-i)P + (1+i)A\big)$ lies in $D\mathscr{U}$, the zero plane of c, and $\gamma^{\top}\gamma = 0$. At the first posting cell 1, on the $+1$ pole p of c, must flip (to P, writing $y_1 = 1$), and cell 2 stays or flips with weight $\frac{1}{2}$: stay $\gamma = \frac{1}{2}(a - ip)$ with label $(1,0,0)$, flip $\gamma = \frac{1}{2}(p + ia)$ with label $(1,0,1)$. Both images lie in $\mathscr{U}$ and still have zero transposed square. Two facts finish the count.

1. At $(1,1)$ the plane $\mathscr{U}$ is spanned by the poles of c, which move surely (p flips to P, a stays to A), so after one more posting the present is a real vector times a phase; real vectors stay real.
2. At $(1,0)$, with cell 1 on P and cell 2 on a vector $u \in \mathscr{U}$ with $u^{\top}u = 0$: cell 1 is on the $+1$ pole of c' and flips with weight 1, staying on P; cell 2 is on the zero plane of c' and moves to $(1,0)$ or $(1,1)$ with weight $\frac{1}{2}$ each, writing $(1,1,-1)$ or $(1,1,0)$. For $M =$ stay D or flip D, $(Mu)^{\top}(Mu) = u^{\top}\frac{1}{2}(I \mp c')u = \frac{1}{2}u^{\top}u \mp \frac{1}{2}u^{\top}c'u = 0$, since $u^{\top}u = 0$ and $c'u = 0$; and $Mu \in \mathscr{U}$.

So after posting $k \ge 1$ the circulating records are those at $(1,0)$, of total probability 2^{-k}, and those that have just entered $(1,1)$, of probability 2^{-k}: together 2^{1-k}. A direct exact computation of all records up to $k = 8$ agrees. At $k = 2$ the four real records begin with $(1,0,1)$, have probability $\frac{1}{8}$ each, and their presents have $\psi_h^{\top}\psi_h = 1, -1, 1, -1$ for second labels $(-1,0,-1)$, $(-1,0,0)$, $(0,0,-1)$, $(0,0,0)$. The surviving circulation rides on the record of a persistent current, and its weight halves at every posting.

The history evolution is an isometry, so the circulation cannot simply disappear. It moves into the phases between records.

Proposition 10.19 (Where the circulation goes)**.** *Let $v = \delta_q \otimes f$ be a unit one-loop genesis state and $v' = \delta_q \otimes \bar{f}$.*

(a) For every k, $\langle T^k v, T^k v' \rangle = \sum_h \big\langle (T^k v)(\cdot, h), (T^k v')(\cdot, h) \big\rangle = \overline{f^\top f}$. The term of h has modulus $\mathbb{P}_v(h)|\psi_h^\top \psi_h|$, and $1 - \tau_k(v) \le \sum_h \mathbb{P}_v(h)|\psi_h^\top \psi_h| \le 1$.
(b) For any reads E_h chosen according to the record,

$$\Big| \sum_h \mathbb{P}_v(h) \big(\mathrm{tr}(\rho_{v,h} E_h) - \mathrm{tr}(\rho_{v',h} E_h) \big) \Big| \le \max_h \|E_h\|_{\mathrm{HS}} \sqrt{2\tau_k(v)}.$$

For $f = f_\gamma$, $f^\top f = 0$: the two histories stay orthogonal at every length, while by Example 10.18 the per-record overlaps have moduli summing to $1 - 2^{1-k}$ and cancel exactly in their sum.

Proof. (a) T is an isometry (Theorem 2.14), records are orthogonal, and the term of h is $\langle V_h f, V_h \bar{f} \rangle = \overline{(V_h f)^\top (V_h f)} = \mathbb{P}_v(h) \overline{\psi_h^\top \psi_h}$. Its sum is $\langle f, \bar{f} \rangle = \overline{f^\top f}$. For the lower bound use $|z| \ge |z|^2$ for $|z| \le 1$. (b) $\rho_{v',h} = \rho_{v,h}^\top$ by Lemma 10.16; bound each term by $\|\rho - \rho^\top\|_{\mathrm{HS}} \|E_h\|_{\mathrm{HS}}$ and use the estimate in the proof of Theorem 10.17(b). □

No read of the record, of the present, or of the present chosen according to the record tells the circulating cell from its reverse after a long record. Only an operation that combines different records coherently can, such as the projection onto $T^k v_\gamma$ in the history space.

A familiar expectation says that a feature which no monitoring reads is preserved. That holds for charges the instrument fixes; the circulation is not one.

Proposition 10.20 (Fixed charges keep their mean; the circulation does not)**.**

(a) If $\Phi(E) = E$, then $\sum_{|h|=k} \mathbb{P}_v(h) \,\mathrm{tr}(\rho_{v,h} E) = \mathrm{tr}(\rho_v E)$ for every genesis state v and every k.
(b) The circulation read $R_\gamma = |f_\gamma\rangle\langle f_\gamma| - |\bar{f}_\gamma\rangle\langle \bar{f}_\gamma|$ is antisymmetric, $\|R_\gamma\|_{\mathrm{HS}} = \sqrt{2}$, it reads 1 on v_γ and -1 on $\bar{v}_\gamma$, and for $k \ge 1$

$$\sum_h \mathbb{P}_{v_\gamma}(h) \big| \mathrm{tr}(\rho_{v_\gamma,h} R_\gamma) \big| \le 2^{(1-k)/2}.$$

Hence $\Phi(R_\gamma) \ne R_\gamma$, and some slice does not commute with R_γ.

Proof. (a) $\mathbb{P}_v(h) \,\mathrm{tr}(\rho_{v,h} E) = \mathrm{tr}(\rho_v V_h^* E V_h)$ and $\sum_{|h|=k} V_h^* E V_h = \Phi^k(E) = E$. (b) $(|f\rangle\langle f|)^\top = |\bar{f}\rangle\langle \bar{f}|$, and $\langle f_\gamma, \bar{f}_\gamma \rangle = \overline{f_\gamma^\top f_\gamma} = 0$. The bound is Theorem 10.17(b) with Example 10.18. If $\Phi(R_\gamma) = R_\gamma$, (a) would keep the mean at 1 for every k, but it is at most $2^{(1-k)/2} < 1$ for $k \ge 2$. If every slice commuted with R_γ, then $\Phi(R_\gamma) = \sum_j V_j^* V_j R_\gamma = R_\gamma$. □

The native postings therefore never read the direction of a circulation and do not keep it in the present. Chapter 11 shows the converse wall, that no sequence of real postings produces a circulating state from a real one, and computes the stationary currents of the one-zero and rest classes, on which the pointer chain is not reversible.

Chapter 11

The arrow, the conjugation wall and genesis memory

A long record carries the present cell close to a pointer line, on which the present moves as a classical Markov chain on 256 lines (Chapter 10). This chapter finds where that motion has a direction, what the postings can and cannot read of a clock's circulation, and how much of its genesis a long record keeps.

11.1 Five classes that every posting keeps

The free pull fixes more than the aligned sector. Among the thirty-six sector reads it fixes exactly a five-dimensional space, and the five projections that span it commute with every slice: the native postings never move weight from one of them to another.

Recall the sector projections $\Pi(x,s,t) = |x\rangle\langle x| \otimes \Pi_s^x \otimes \Pi_t^x$, $x \in \mathcal{O}$, $s,t \in \{+,-,0\}$, the spectral projections of the pole charge c_x in the two clock cells. They span the centre of the current algebra (Theorem 9.8). A *sector read* is $S_g = \sum g(x,s,t)\,\Pi(x,s,t)$ for a function g on the 36 sectors.

The five classes $\Lambda_0, \dots, \Lambda_4$, the class weights $w_c(v)$ and the oscillation phase $\vartheta(v)$ are those of Definition 10.13: classes 0, 1, 2 are the sectors of the four modes, class 3 the sixteen sectors with exactly one zero pole, class 4 the remaining sixteen.

Theorem 11.1 (Five conserved classes)**.** *(a) $\Phi(S_g) = S_g$ if and only if g is constant on each of the five classes. The sector reads fixed by the free pull form a five-dimensional space with basis $\Lambda_0, \dots, \Lambda_4$. In particular they are not spanned by I, Q_{al} and Q_{oz}.*

(b) Every Λ_c commutes with every slice V_j. Hence $\Phi(\Lambda_c) = \Lambda_c$, and the class weight $\mathrm{tr}(\rho_{v,h}\Lambda_c)$ of the present is a martingale under the record law.

(c) The free pull exchanges the two halves of class 2: $\Phi(\Pi((1,0),-,-)) = \Pi((0,1),-,-)$ and $\Phi(\Pi((0,1),-,-)) = \Pi((1,0),-,-)$.

Proof. A sector read pulls back through a step $x \to y$ factor by factor (Lemma 9.2), cell 1 moving by stay if $y_1 = x_2$ and cell 2 if $y_2 = x_1$, otherwise by flip. Table 9.1 gives two rules. Onto a mixed output a $+1$ pole is pulled back only through a flip, to Π_+, a -1 pole only through a stay, to Π_-, and Π_0 to $\frac{1}{2}\Pi_0$: each cell keeps its status and its sign. Onto a balanced output each cell reverses its status: $\Pi_\pm$ pulls back to half the projection onto a standing wave of the zero plane, $\Pi_+ + \Pi_-$ to $\frac{1}{2}\Pi_0$, and Π_0 to Π_- or Π_+.

For $\Pi((1,0),+,+)$ both cells must flip into $y = (1,0)$, so $x_2 \neq 1$ and $x_1 \neq 0$: only $x = (1,0)$ contributes, through the label $(1,1,-1)$, and the pull is $\Pi((1,0),+,+)$ again. The same holds at $(0,1)$. For $\Pi((1,0),-,-)$ both cells must stay, so $x_2 = 1$ and $x_1 = 0$: the only input is $(0,1)$ and

the pull is $\Pi((0,1),-,-)$, which is (c). The status rules show that a mixed output keeps and a balanced output toggles both cells' zero status together, so $\Phi(Q_{\mathrm{oz}}) = Q_{\mathrm{oz}}$ (as in Theorem 3.14). With $\Phi(I) = I$ this makes all five Λ_c fixed, and every S_g with g constant on classes.

Conversely, let $K(p,q) = \operatorname{tr}\big(\Pi_q \sum_j V_j \Pi_p V_j^*\big)/\operatorname{tr}\Pi_p$ for sectors p, q: the probability that the normalized sector state $\Pi_p/\operatorname{tr}\Pi_p$ posts into sector q. It is a stochastic matrix, since $\sum_q \Pi_q = I$ and $\sum_j V_j^* V_j = I$. If $\Phi(S_g) = S_g$, then $g(p)\operatorname{tr}\Pi_p = \operatorname{tr}(\Pi_p \Phi(S_g)) = \sum_q g(q)\operatorname{tr}\big(\Pi_q \sum_j V_j \Pi_p V_j^*\big)$, so $g = Kg$. Each class is closed for K, because $\sum_{q\in c} K(p,q) = \operatorname{tr}(\Phi(\Lambda_c)\Pi_p)/\operatorname{tr}\Pi_p = 1$ for p in class c. Each class is irreducible: a closed walk along positive entries of K passes through all of its sectors (for classes 0 to 2 this is the fixed point and the swap above; for classes 3 and 4 it is a direct exact computation from the same pulls). A K-harmonic function attains its maximum on a closed irreducible class at some sector, hence at every sector reachable from it, hence on the whole class. So g is constant on each class. The five Λ_c are independent, and Λ_0 separates the two mixed occupations, which I, Q_{al} and Q_{oz} do not.

(b) A projection fixed by Φ commutes with every slice (Lemma 8.1). Then $\mathbb{P}_\nu(hj)\operatorname{tr}(\rho_{\nu,hj}\Lambda_c) = \operatorname{tr}(V_j \rho_h V_j^* \Lambda_c)\,\mathbb{P}_\nu(h)$ with $\rho_h = \rho_{\nu,h}$, and summing over j gives $\operatorname{tr}(\rho_h \Phi(\Lambda_c)) = \operatorname{tr}(\rho_h \Lambda_c)$. □

The three aligned classes are the three motions of Sections 7.6 and 7.7: the persistent positive current, the persistent negative current and the occupation oscillation. The long-run verdict of Theorem 3.14 reads only a coarse-graining of the five classes, $\frac{1}{448}I + \frac{447}{448}Q_{\mathrm{al}} + \frac{3}{1792}Q_{\mathrm{oz}}$; it cannot tell the two persistent currents from the oscillation, or one mixed occupation from the other.

11.1.1 The record decides the one-zero class

Chapter 8 showed that a record decides the aligned class. It also decides the one-zero class, through a written pattern that the one-zero class can never produce.

Call two consecutive labels j (earlier) and j' (next) a *bad pair* if j ends on a mixed occupation, j' ends on a mixed occupation exactly when j started on one, and exactly one of j, j' moves cell 1 by stay. All three conditions are read off the labels: $j = (y_1 - x_2,\ x_1 - x_2,\ y_2 - x_1)$ starts on a mixed occupation iff $j_2 \neq 0$, ends on one iff $j_1 - j_2 - j_3 \neq 0$, and moves cell 1 by stay iff $j_1 = 0$. For a record h of the genesis state ν write $a_h = \operatorname{tr}(\rho_{\nu,h} Q_{\mathrm{al}})$ and $z_h = \operatorname{tr}(\rho_{\nu,h}\Lambda_3)$.

Theorem 11.2 (The record decides the one-zero class)**.** *Let ν be a unit genesis state.*

(a) If j, j' is a bad pair, then $V_{j'} V_j \Lambda_3 = 0$. Hence $V_h \Lambda_3 = 0$, and $z_h = 0$, for every record h that contains a bad pair.

(b) From every state of the rest class, four postings write a bad pair with probability at least $\frac{1}{8}$. For every n,

$$\sum_{h\in\mathcal{J}^n} \mathbb{P}_\nu(h)\, z_h(1-z_h) \;\le\; \sum_{h\in\mathcal{J}^n} \mathbb{P}_\nu(h)\, a_h(1-a_h) + \left(\tfrac{7}{8}\right)^{\lfloor n/4\rfloor},$$

and both sides tend to 0.

(c) μ_ν-almost surely $z_{\omega_{\le n}}$ converges to 0 or to 1; the probability of the limit 1 is $w_3(\nu)$. Along every infinite record on which $z_{\omega_{\le n}} \to 1$, the long-run verdict $\lambda_\nu(\omega_{\le n})$ tends to $\frac{1}{256}$.

Proof. (a) Two facts about one step do the work. First, at a mixed input the dressing sends P to p and A to a, and flip $p = P$, stay $p = 0$, stay $a = A$, flip $a = 0$ (Table 2.2): a cell on the $+1$ pole of c' must flip and stays on it if the output is mixed, and a cell on the -1 pole must stay. At a balanced input a cell on p must flip to P and a cell on a must stay to A. Second, a step between a balanced and a mixed occupation, in either direction, moves exactly one cell by stay, while a step between two mixed occupations moves both cells the same way; for instance, from $x = (s, s)$ to a mixed y, cell 1 stays iff $y_1 = s$ and cell 2 iff $y_2 = s$, and $y_1 \neq y_2$.

Since Λ_3 commutes with V_j, it suffices to follow a component of a one-zero state in which a single cell, say cell c, is off its zero plane. A step into a mixed occupation keeps that status, so at the mixed output y of j cell c sits on P or A, and by the first fact it sits on P exactly when it flipped at j. At j' it must flip if on P and stay if on A. So cell c makes the same move at j and at j'. In a bad pair, j and j' are either both steps between a balanced and a mixed occupation or both steps between mixed occupations, so by the second fact cell 1 repeats its move exactly when cell 2 does. Hence cell 1 makes the same move at j and j', which a bad pair forbids. Every component is annihilated. If h contains a bad pair, $V_h\Lambda_3 = 0$, and $\mathbb{P}_\nu(h)z_h = \operatorname{tr}(V_h\Lambda_3\rho_\nu\Lambda_3V_h^*) = 0$.

(b) With $r_h = 1 - a_h - z_h$ the rest weight, $z(1-z) = za + zr \leq a(1-a) + r[z > 0]$, because $z \leq 1 - a$. By (a), $z_h > 0$ only for records with no bad pair, so the second term is at most the rest load $G_n(\nu) = \sum \operatorname{tr}(V_h\Lambda_4\rho_\nu\Lambda_4V_h^*)$ over the records of length n without a bad pair. The first claim of (b) has a plain mechanism. In the rest class two cells that both lie on zero planes move by stay or flip with weight $\frac{1}{2}$ each, whatever their internal state, and so write a bad pair within two postings with probability $\frac{1}{8}$; every other rest sector reaches such a joint zero sector within two forced postings. The bound itself is a direct exact computation of the four-posting effects on the sixteen rest sectors: the records of length four with no bad pair carry at most $\frac{7}{8}$ of the rest weight of any state. Hence $G_{n+4}(\nu) \leq \frac{7}{8}G_n(\nu)$ and $G_n(\nu) \leq (\frac{7}{8})^{\lfloor n/4 \rfloor}$. The aligned term tends to 0 by Theorem 8.4.

(c) By Theorem 11.1(b), z is a bounded martingale under μ_ν, so it converges almost surely; by (b) and bounded convergence its limit z_∞ satisfies $z_\infty(1 - z_\infty) = 0$ almost surely, and $\int z_\infty \, d\mu_\nu = z_\varnothing = w_3(\nu)$. Along a record with $z \to 1$, $a \leq 1 - z \to 0$, and the long-run verdict $\frac{1}{448} + \frac{447}{448}a + \frac{3}{1792}z$ (Theorem 3.14) tends to $\frac{1}{448} + \frac{3}{1792} = \frac{1}{256}$. □

With Theorem 8.6 this settles the long-run verdict along almost every record at one of three values, 1, $\frac{1}{256}$ or $\frac{1}{448}$, with probabilities w_{al}, w_{oz} and the rest weight (Proposition 8.8). The rate $\frac{7}{8}$ per four postings is a proved bound on a survival probability; whether it is sharp is open.

11.2 Off the sure subjects the present runs one way

On pointer lines the present moves classically, and the motion has a direction exactly on the classes that hold no sure subject. We use the pointer lines $\mathscr{P}_\ell$, $\ell = (x; \ell_1, \ell_2)$, of Definition 10.8, the pointer kernel κ of (10.1), its closed classes $\mathscr{C}_0, \dots, \mathscr{C}_4$ of 1, 1, 2, 64, 44 lines in the ranges of $\Lambda_0, \dots, \Lambda_4$, and their stationary laws π_c (Theorem 10.14 and Table 10.2). The two lines of $\mathscr{C}_2$ are the *oscillation lines* $\ell_\circ = (10; A, A)$ and $\ell'_\circ = (01; A, A)$.

Definition 11.3 (Stationary current)**.** The *stationary current* of class c is $F_c(\ell, \ell') = \pi_c(\ell)\kappa(\ell, \ell') - \pi_c(\ell')\kappa(\ell', \ell)$. Class c is *reversible* if $F_c = 0$ (detailed balance). Its *irreversibility*

$\mathrm{irr}_c = \frac{1}{2}\sum_{\ell,\ell'} |F_c(\ell,\ell')|$ is the total-variation distance between the stationary two-step law $\pi_c(\ell)\kappa(\ell,\ell')$ and its reverse.

Theorem 11.4 (Reversible exactly on the sure subjects). *(a) Class c is reversible if and only if $c \in \{0,1,2\}$. Equivalently, the stationary chain of class c gives every finite path of lines the probability of its reversal if and only if $c \le 2$.*

(b) Classes 3 and 4 have one-way steps:

$$\kappa\big((00;p,P),(10;P,\omega_+)\big) = \tfrac{1}{2}, \qquad \kappa\big((10;P,\omega_+),(00;p,P)\big) = 0;$$
$$\kappa\big((00;a,a),(00;A,A)\big) = 1, \qquad \kappa\big((00;A,A),(00;a,a)\big) = 0.$$

(c) Every current is divergence-free, $\sum_{\ell'} F_c(\ell,\ell') = 0$. Around the closed walk

$$W_3 \;:\; (00;p,P) \to (10;P,\omega_+) \to (11;P,a) \to (11;\omega_+,A) \to (11;A,\omega_-) \to (10;\omega_-,P) \to (00;p,P)$$

the class-3 current sums to $\frac{1}{24}$, and around $W_4 \;:\; (00;a,a) \to (00;A,A) \to (01;\omega_-,\omega_+) \to (00;a,a)$ the class-4 current sums to $\frac{3}{112}$. So on classes 3 and 4 the current is not of the form $V(\ell') - V(\ell)$ for any real function V on lines, and Kolmogorov's criterion [26] fails on W_3 and W_4.

(d) $\mathrm{irr}_c = 0, 0, 0, \frac{23}{24}, \frac{20}{21}$ for $c = 0, \dots, 4$.

(e) Class 3 has 156 and class 4 has 136 pairs of distinct lines joined by a step of positive weight in at least one direction. Over these pairs $2688|F_3|$ takes the value 14 on 120 pairs, 28 on 32 and 0 on 4, and $2688|F_4|$ takes 8, 16, 24, 32, 96 on 36, 60, 28, 8, 4 pairs. By Schnakenberg's decomposition each current is a sum of 93 independent cycle currents ($93 = 156 - 64 + 1 = 136 - 44 + 1$), one per chord of a spanning tree.

Proof. All weights come from Table 2.2: a step $x \to y$ moves cell 1 by stay iff $y_1 = x_2$ and cell 2 iff $y_2 = x_1$, a mixed input is dressed first, and the weight of a line is the product of the squared lengths of the two cell images.

(a), classes 0 to 2. From $(10;P,P)$ the dressed cells sit on p, which survives only a flip, so $y_1 \ne x_2 = 0$ and $y_2 \ne x_1 = 1$: the only successor is $(10;P,P)$, with weight 1. The same holds at $(01;P,P)$. From $\ell_\circ$ the dressed cells sit on a, which survives only a stay, so $y = (0,1)$ and the successor is $\ell'_\circ$ with weight 1; symmetrically back. So $F_0 = F_1 = 0$, and $F_2(\ell_\circ,\ell'_\circ) = \frac{1}{2} - \frac{1}{2} = 0$. For a reversible class the probability of a path and of its reversal agree by telescoping detailed balance; conversely paths of length one are detailed balance.

(b) From $(00;p,P)$ to $y = (1,0)$ cell 1 flips, flip $p = P$ with weight 1, and cell 2 stays, stay $P = \frac{1}{2}(p+a) = \omega_+/\sqrt{2}$ with weight $\frac{1}{2}$. Back from the mixed line $(10;P,\omega_+)$, cell 1 is dressed to p, whose only image is P: it never returns to p. From $(00;a,a)$ both cells survive only a stay, so $y = (0,0)$ and stay $a = A$ with weight 1 in each cell; from $(00;A,A)$ the images of A are $\frac{1}{2}(a-p)$ and $\frac{1}{2}(p+a)$, never a. Since π_3 and π_4 are positive on these lines, F_3 and F_4 are not zero, which completes (a).

(c) Divergence: $\sum_{\ell'} F_c(\ell,\ell') = \pi_c(\ell) - \sum_{\ell'} \pi_c(\ell')\kappa(\ell',\ell) = 0$ by stochasticity and stationarity. Along W_3 the forward weights are $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}$; for instance $(11;\omega_+,A) \to (11;A,\omega_-)$ uses two stays, stay $\omega_+ = A/\sqrt{2}$ and stay $A = -\omega_-/\sqrt{2}$. Every reverse weight is 0, because in each

reversed step some cell would need a line outside its image set. The stationary values along the walk are $\frac{1}{96}, \frac{1}{48}, \frac{1}{96}, \frac{1}{48}, \frac{1}{48}, \frac{1}{48}$, so the circulation is

$$\frac{1}{192} + \frac{1}{96} + \frac{1}{192} + \frac{1}{192} + \frac{1}{192} + \frac{1}{96} = \frac{1}{24}.$$

Along W_4 the forward weights are 1, $\frac{1}{4}$ (from $(00; A, A)$ to $(0,1)$ cell 1 stays, $A \mapsto -\omega_-/\sqrt{2}$, and cell 2 flips, $A \mapsto \omega_+/\sqrt{2}$) and $\frac{1}{4}$ (from the mixed line $(01; \omega_-, \omega_+)$ to $(0,0)$ cell 1 flips, $D\omega_- = \Omega_- \mapsto -a/\sqrt{2}$, and cell 2 stays, $D\omega_+ = \Omega_+ \mapsto a/\sqrt{2}$). The reverses vanish, and $\pi_4 = \frac{1}{84}, \frac{1}{28}, \frac{1}{42}$ give $\frac{1}{84} + \frac{1}{112} + \frac{1}{168} = \frac{3}{112}$. A current of the form $V(\ell') - V(\ell)$ sums to zero around every closed walk, and Kolmogorov's criterion compares the forward and reversed kernel products around a walk, here positive against zero.

(d) Classes 0 to 2 have $F_c = 0$. For classes 3 and 4 the values of $2688|F_c|$ are those of (e), a direct exact computation over the 108 lines; their absolute sums are $2(14 \cdot 120 + 28 \cdot 32) = 5152$ and $2(8 \cdot 36 + 16 \cdot 60 + 24 \cdot 28 + 32 \cdot 8 + 96 \cdot 4) = 5120$, so $\mathrm{irr}_c = \sum |F_c|/2 = 5152/5376 = \frac{23}{24}$ and $5120/5376 = \frac{20}{21}$.

(e) The histograms are a direct exact computation on the same tables. The fundamental cycles of a spanning tree of a connected graph with E edges and V vertices number $E - V + 1$, are independent (each passes its own chord once and no other chord) and span its cycle space, and a divergence-free current is the sum of its chord currents times the cycles [35]. □

Since irr_c is a total-variation distance, $1 - \mathrm{irr}_c$ is the part of the stationary two-step law matched by its reverse: $\frac{1}{24}$ on class 3 and $\frac{1}{21}$ on class 4. The types of line circulate: an undressed pole is recorded into a dressed pole, a dressed pole into an undressed wave, and a wave back into a pole, and in one step a dressed pole never turns back into an undressed one.

Remark 11.5 (Entropy production). On classes 3 and 4 the stationary two-step law gives positive probability to the steps of (b) and its reverse gives them probability zero, so the forward law is not absolutely continuous with respect to the reverse. The relative entropy between a stationary step and its reverse, which measures entropy production in the sense of Schnakenberg [35] and of Kawai, Parrondo and Van den Broeck [24], is therefore infinite on these classes and zero on classes 0, 1, 2.

We read the stationary circulation on classes 3 and 4 as an *arrow of time inside the ledger*: a direction in the present's own classical motion, carried by states that are not sure subjects. This is an identification, and we name it here once. The theorem concerns the stationary current of a Markov chain on pointer lines. That this current is experienced as a direction of time is not claimed, and nothing here shows that any physical process prepares a genesis state in class 3 or 4. The current is produced by the native postings alone: the kernel κ is built from the slices, and no bath, coarse-graining or boundary condition enters.

11.3 A long record keeps six numbers of its genesis

Far from genesis, the probability of every word depends on the genesis state through six numbers, and the record itself holds all six. Everything else decays at a proved rate, with an exact slowest mode.

The effects span the record content $\mathscr{R}$, the 92-dimensional span of the record reads (Theorem 9.4). The free pull maps $\mathscr{R}$ into itself, since $\Phi(E_w) = \sum_j E_{j\cdot w}$, where $j \cdot w$ writes j first. The probability that postings $n+1, \dots, n+|w|$ write the word w is

$$\mathbb{P}_\nu^{(n)}(w) = \operatorname{tr}\big(\rho_\nu \Phi^n(E_w)\big),$$

the sum of $\mathbb{P}_\nu$ over all records of length $n+|w|$ that end in w. For $E \in \mathscr{R}$ write $\|E\|$ for the largest modulus of its coefficients in the 92 record reads.

Theorem 11.6 (What a long record remembers)**.** *(a) The elements of $\mathscr{R}$ fixed by Φ are exactly the span of $\Lambda_0, \dots, \Lambda_4$. Those reversed by it, $\Phi(E) = -E$, are exactly the multiples of $\Psi_\circ$. Every eigenvalue of Φ on $\mathscr{R}$ of modulus one is 1 or -1.*

(b) Every other eigenvalue has modulus less than 0.7793. On the rest class the slowest is the real root $\rho_* \in (0.7792, 0.7793)$ of

$$8x^5 + 4x^4 - 2x^2 - 2x - 1,$$

and it is attained: some nonzero $E \in \mathscr{R}$ built from record reads of the rest class has $\Phi(E) = \rho_* E$. On the one-zero class every eigenvalue other than 1 has $|\mu|^2 \le \frac{1}{2}$, and this is attained: some nonzero $E \in \mathscr{R}$ has $\Phi^2(E) = -\frac{1}{2}E$.

(c) If $E \in \mathscr{R}$ has $\operatorname{tr}(\Lambda_c E) = 0$ for every c and $\operatorname{tr}(\Psi_\circ E) = 0$, then eight postings at least halve it: $\|\Phi^8(E)\| \le \frac{1}{2}\|E\|$.

(d) For every genesis state ν, every word w and every $n \ge 0$,

$$\Big|\mathbb{P}_\nu^{(n)}(w) - \sum_{c=0}^{4} \alpha_c(w)\, w_c(\nu) - (-1)^n \beta(w)\, \vartheta(\nu)\Big| \le C_\nu C_w\, 2^{-\lfloor n/8 \rfloor},$$

where $\alpha_c(w) = \operatorname{tr}(\Lambda_c E_w)/\operatorname{tr}\Lambda_c$ is the probability of w in the uniform state of class c, $\beta(w) = \frac{1}{2}\operatorname{tr}(\Psi_\circ E_w)$, C_ν depends only on ν and C_w only on w (Section A.5). In particular $\mathbb{P}_\nu^{(2k)}(w)$ and $\mathbb{P}_\nu^{(2k+1)}(w)$ converge, to limits that differ only in the sign of the phase term.

Proof. Two identities carry the peripheral part. The record instrument is unital, $\sum_j V_j V_j^* = I$ (Proposition 11.8(a)), and Λ_c commutes with each V_j, so

$$\operatorname{tr}(\Lambda_c \Phi(E)) = \sum_j \operatorname{tr}(V_j \Lambda_c V_j^* E) = \operatorname{tr}\Big(\Lambda_c \sum_j V_j V_j^* E\Big) = \operatorname{tr}(\Lambda_c E)\ :$$

every class average is conserved. By Proposition 2.22, $V_{(0,1,0)}\psi_\circ = \psi_\circ'$, $V_{(0,-1,0)}\psi_\circ' = -\psi_\circ$ and every other slice annihilates both, so $\sum_j V_j \Psi_\circ V_j^* = -\Psi_\circ$ and $\operatorname{tr}(\Psi_\circ \Phi(E)) = -\operatorname{tr}(\Psi_\circ E)$. Also $\Phi(\Lambda_c) = \Lambda_c$ (Theorem 11.1) and $\Phi(\Psi_\circ) = -\Psi_\circ$ by Theorem 11.1(c). So the trace-orthogonal projection Per onto $\operatorname{span}\{\Lambda_0, \dots, \Lambda_4, \Psi_\circ\}$, namely $E \mapsto \sum_c \alpha_c \Lambda_c + \frac{1}{2}\operatorname{tr}(\Psi_\circ E)\Psi_\circ$ with $\alpha_c = \operatorname{tr}(\Lambda_c E)/\operatorname{tr}\Lambda_c$, commutes with Φ, and Φ acts on its range by $+1$ on the Λ_c and -1 on $\Psi_\circ$.

The rest is a direct exact computation with one rational matrix. In the basis of the 92 record reads, Φ acts by a 92×92 matrix M whose entries are products of two one-cell pull coefficients from Table 9.1. A record read belongs to the class of its sector, with Z and X counted as zero-plane factors; so M is block diagonal along the classes, with blocks of $1, 1, 2, 48, 40$ reads. On each block an annihilating polynomial of $Y = 16M$ is listed in Table A.9. Apart from

$Y = \pm 16$ and the quintic factor, every root has $|Y| \le 8\sqrt{2}$, that is $|x| \le \frac{\sqrt{2}}{2}$ for $x = Y/16$; on the one-zero block this gives $|\mu|^2 \le \frac{1}{2}$. Substituting $Y = 16x$ in the quintic factor and dividing by 2^{17} gives $8x^5 + 4x^4 - 2x^2 - 2x - 1$. Three root-squaring steps turn it into a quintic whose roots are the eighth powers of the original ones, and a Cauchy bound for that quintic shows that no root has modulus 0.7793 or more; a sign change on $[0.7792, 0.7793]$ locates the real root. For the slow rest read, an explicit integer vector u of the rest block is annihilated by the quintic factor evaluated at Y, while $u, Yu, \dots, Y^4u$ are independent; so the span of these five vectors carries an eigenvector for every root of the quintic, ρ_* among them. The one-zero read with $\Phi^2(E) = -\frac{1}{2}E$ is explicit. Together with the peripheral identities this gives (a) and (b). For (c), the rows of $(16M)^8 - 16^8$ Per have absolute sums at most $\frac{1}{2} \cdot 16^8$ on every block. For (d), write the coefficient vector of E_w as e_w; then $\Phi^n(E_w)$ differs from $\sum_c \alpha_c(w)\Lambda_c + (-1)^n \beta(w)\Psi_\circ$ by $M^n(e_w - \operatorname{Per} e_w)$, whose entries are at most $2^{-\lfloor n/8 \rfloor} C_w$ by (c), and pairing with ρ_ν costs at most $C_\nu = \sum_q |\operatorname{tr}(\rho_\nu R_q)|$ over the record reads R_q. □

The six numbers are the five class weights, which sum to one, and the oscillation phase, which says on which mixed occupation the oscillation sits at even stages. All six are recorded, since Λ_c and $\Psi_\circ$ lie in $\mathscr{R}$. Everything else, including every coherence of the dressed zero planes and the distribution of weight inside each class, is forgotten: the record forgets toward the uniform state of each class, and its long-run memory is classical.

The slowest forgetting sits in the rest class, at the rate $\rho_* \approx 0.779$ per posting. It is the root of the same quintic that sets the approach of the record-averaged neutral weight to its limit (Theorem 7.31). On the one-zero class the slowest modes decay by $\frac{1}{\sqrt{2}}$ per posting. These rates are spectral: the proved decay of every read is the factor two per eight postings of (c), about 0.917 per posting.

The theorem concerns the free pull, with labels summed: it is the unconditioned law of later postings. Conditioning on the written record is a different question, answered by Theorem 11.2 and Chapter 8: there the record decides the class.

11.4 The record reads the arrow, and a clock is not an arrow

The circulation of Section 11.2 is not hidden. The record names the best line of the present at two consecutive stages, and a fixed read of that pair tells forward from backward with a bias equal to the class weight times irr_c.

Let $b_k(h, \ell)$ be 1 if ℓ is the best line of the length-k record h, the unique line carrying more than $\frac{9}{10}$ of its present (Chapter 10), and 0 otherwise. The *pair law* at stage k is

$$\mathcal{Q}_\nu^{(k)}(\ell, \ell') = \sum_{h \in \mathcal{J}^k} \sum_{j \in \mathcal{J}} \mathbb{P}_\nu(hj)\, b_k(h, \ell)\, b_{k+1}(hj, \ell'),$$

the probability that the present sits on ℓ at stage k and on ℓ' at stage $k+1$. A *read* of two consecutive lines is a function f on pairs of lines with values in $[0, 1]$; its *bias* at stage k is

$$\Delta_k(f) = \sum_{\ell, \ell'} f(\ell, \ell')\big(\mathcal{Q}_\nu^{(k)}(\ell, \ell') - \mathcal{Q}_\nu^{(k)}(\ell', \ell)\big),$$

the probability that f accepts the pair in its written order minus the probability that it accepts the reversed pair.

Theorem 11.7 (The record reads the arrow)**.** *Let ν be a unit genesis state.*

(a) The record names the pair. For $m, s \ge 0$, $\mathcal{Q}_\nu^{(m+s)}(\ell, \ell')$ differs by at most $30 \cdot 4096\,(\frac{41}{48})^{\lfloor s/2 \rfloor}$ from the probability that the labels written at postings $m+1, \dots, m+s+1$ name (ℓ, ℓ'): that the first s of them carry the pole line they weight most onto ℓ, and the last carries ℓ to ℓ'.

(b) With $L_\nu^{(n)}(\ell) = \sum_c w_c(\nu)\pi_c(\ell) + \frac{1}{2}(-1)^n \vartheta(\nu)\big([\ell = \ell_\circ] - [\ell = \ell'_\circ]\big)$,

$$\mathcal{Q}_\nu^{(2k)}(\ell, \ell') \to L_\nu^{(0)}(\ell)\,\kappa(\ell, \ell'), \qquad \mathcal{Q}_\nu^{(2k+1)}(\ell, \ell') \to L_\nu^{(1)}(\ell)\,\kappa(\ell, \ell').$$

(c) For every read f the bias converges at each parity n of k, to

$$\sum_{c=0}^{4} w_c(\nu) \sum_{\ell, \ell'} f(\ell, \ell') F_c(\ell, \ell') + (-1)^n \vartheta(\nu)\big(f(\ell_\circ, \ell'_\circ) - f(\ell'_\circ, \ell_\circ)\big).$$

(d) For $c = 3, 4$ the arrow read *$f_c = [F_c > 0]$ has limit $w_c(\nu)\,\mathrm{irr}_c$ at both parities. If $\vartheta(\nu) = 0$, no read has a limit above $\sum_c w_c(\nu)\,\mathrm{irr}_c$.*

(e) If $w_3(\nu) = w_4(\nu) = 0$, the two parity limits of every read are opposite numbers. If also $w_2(\nu) = 0$, the bias of every read tends to 0.

(f) The cell $\nu = \delta_0 \otimes |10\rangle \otimes a \otimes P$ has $w_3 = 1$, and its arrow read f_3 has bias tending to $\frac{23}{24}$. The cell $\delta_0 \otimes |00\rangle \otimes a \otimes a$ has $w_4 = 1$, and f_4 has bias tending to $\frac{20}{21}$. The oscillating mode $\delta_0 \otimes \psi_\circ$ has $w_2 = 1$ and $\vartheta = 1$, and the read $[\ell = \ell_\circ,\ \ell' = \ell'_\circ]$ has bias tending to 1 along even stages and to -1 along odd ones.

Proof. (a) Write $D_s = 4096(\frac{41}{48})^{\lfloor s/2 \rfloor}$. By the pointer-entry estimate of Chapter 10, at stage $m+s$ the present sits on the line named by the last s labels up to a record-mean defect at most D_s. The newest label moves that line by the line map, and at stage $m+s+1$ the present sits on the moved line up to a record-mean defect at most $2D_s$. Where the present's weight off a line is δ, its best-line indicator differs from the indicator of that line by at most 10δ in record mean (Markov's inequality, as before Theorem 10.15). So the two stages cost at most $10D_s + 10 \cdot 2D_s = 30D_s$. (b) The named pair is a function of $s+1$ consecutive labels written after stage m, so its probability is a finite sum of late word probabilities, which converge at each parity by Theorem 11.6(d). In the limit the sum is the late law of the named line, which is $L_\nu^{(n)}$ up to $12D_s$ (proof of Theorem 10.15), times the kernel. Let $s \to \infty$. (c) The bias is linear in the pair law. Substituting (b), the class terms give $\sum_c w_c(\nu) \sum f F_c$, and the phase term contributes only on the two oscillation lines, which go to each other with weight 1; it gives $(-1)^n \vartheta(\nu)\big(f(\ell_\circ, \ell'_\circ) - f(\ell'_\circ, \ell_\circ)\big)$. (d) F_c is antisymmetric, so $\sum |F_c| = 2 \sum \max(F_c, 0)$ and $\sum f F_c \le \sum \max(F_c, 0) = \mathrm{irr}_c$ for $0 \le f \le 1$, with equality for $f = f_c$. The supports of the currents of different classes are disjoint, and $F_3 = F_4 = 0$ on the oscillation pair, so f_c picks up only its own class. If $\vartheta(\nu) = 0$, sum the bound over c with weights $w_c(\nu) \ge 0$. (e) $F_c = 0$ for $c \le 2$, so only the phase term survives, with sign $(-1)^n$. If $w_2(\nu) = 0$ then $\vartheta(\nu) = 0$. (f) At the mixed occupation $(1, 0)$, a lies on the zero plane of c' and P on its $+1$ pole, so ν lies in a one-zero sector. At $(0, 0)$ both cells of $a \otimes a$ are on the -1 pole of c, a rest sector. Apply (d) and (c) with $\mathrm{irr}_3 = \frac{23}{24}$, $\mathrm{irr}_4 = \frac{20}{21}$ (Theorem 11.4). □

On the sure subjects there is no direction to detect. On the persistent classes every read is time-symmetric in the limit. The oscillation needs a closer look, because there a read can have bias ± 1. Its present alternates between $\ell_\circ$ and $\ell'_\circ$, so at each fixed parity the order of two consecutive lines is certain. But the reversed sequence is the forward one shifted by one stage, and the limits at the two parities cancel. The read tells the stage parity, not a direction: the oscillation is a clock, not an arrow. Its phase is the one genesis fact besides the class weights that a long record keeps (Theorem 11.6).

11.5 Running the postings backwards does not undo the arrow

Every quantum instrument has a natural reversal. For the native postings it is a valid native instrument, and on the sure subjects it reproduces the time reversal of the present's chain. On the arrow classes it does not.

Proposition 11.8 (The reversed postings). *(a) $\sum_j V_j V_j^* = I$. Hence the transposed slices $V_j^\top$ form a complete instrument, $\sum_j (V_j^\top)^* V_j^\top = I$: the reversal, in the sense of Crooks [13], of the record instrument relative to its invariant state $I/64$.*

(b) Each $V_j^\top$ carries every pointer line onto a multiple of a pointer line. The weights define a stochastic kernel κ^{rev} on the 256 lines.

(c) On classes 0, 1, 2, κ^{rev} is the time reversal of the present's chain: $\kappa^{\mathrm{rev}}(\ell, \ell') = \kappa^*(\ell, \ell') := \pi_c(\ell')\kappa(\ell', \ell)/\pi_c(\ell)$ for $\ell \in \mathscr{C}_c$. On classes 3 and 4 it is not:

$$\kappa^{\mathrm{rev}}\big((00; A, A), (00; a, a)\big) = 1, \quad \kappa^* = \tfrac{1}{3}; \qquad \kappa^{\mathrm{rev}}\big((00; p, P), (10; \omega_-, P)\big) = \tfrac{1}{2}, \quad \kappa^* = 1.$$

(d) On the 64 pole lines, those with both cells among p, a, P, A, $\kappa^{\mathrm{rev}}(\ell, \ell') = \kappa(\ell', \ell)$. The reversed kernel carries the class-4 line $(00; p, p)$ onto the transient line $(00; \Omega_-, \Omega_-)$ with weight at least $\frac{1}{4}$, and it does not keep π_4: $\sum_\ell \pi_4(\ell)\kappa^{\mathrm{rev}}(\ell, (00; p, p)) = \frac{1}{28}$, while $\pi_4((00; p, p)) = \frac{1}{84}$.

Proof. (a) Write $U_{\mathrm{nat}} = \sum_{x,y} |y\rangle\langle x| \otimes M_{yx}$. Its unitarity gives $\sum_x M_{yx} M_{yx}^* = I$ for every output y. A label and an output fix the input, so two transitions with one label have different outputs, and $\sum_j V_j V_j^*$ is the block-diagonal part $\sum_y |y\rangle\langle y| \otimes \sum_x M_{yx} M_{yx}^* = I$. The slices are real, so $(V_j^\top)^* V_j^\top = V_j V_j^*$. Since $\sum_j V_j (I/64) V_j^* = I/64$, the reversal $\sigma^{1/2} V_j^* \sigma^{-1/2}$ relative to $\sigma = I/64$ is $V_j^* = V_j^\top$. (b), (c), (d) The transposed clock maps send each of the eight cell vectors to a multiple of one of them (Table A.6), so the line property and the kernel follow as for κ; stochasticity is (a). The comparisons are direct exact computations on the two kernels. For the first pair in (c): from $(00; A, A)$, stay$^\top A = a$ in both cells with weight 1, while $\kappa^* = \pi_4((00; a, a)) \cdot 1/\pi_4((00; A, A)) = \frac{1/84}{1/28} = \frac{1}{3}$. For the second, $\kappa^* = \pi_3((10; \omega_-, P)) \cdot \frac{1}{2} / \pi_3((00; p, P)) = \frac{1/96}{1/96} = 1$, the last step of W_3 having weight $\frac{1}{2}$. □

Whether some other instrument built from the native postings realizes κ^* on classes 3 and 4, for instance the reversal relative to the normalized class states $\Lambda_c / \operatorname{tr} \Lambda_c$, is open.

11.6 The conjugation wall

Every native operation on the record side is a real matrix in the native basis. So the sense in which a clock cell circulates is invisible to all of them, conserved by them in record mean, and never created by them. We check realness for the slices and record operators, the retained read and the clock archive below; we claim no census of every operation, and one with non-real entries would act as a reference (Proposition 11.15).

The circulating vector $\gamma = \frac{1}{2}(1, i, -1, -i)$ satisfies $L\gamma = i\gamma$: the clock cell runs round its four ticks one way, and $\bar{\gamma}$ runs the other way. As in Chapter 9, $f_\gamma = |00\rangle \otimes p \otimes \gamma$ and $v_\gamma = \delta_0 \otimes f_\gamma$. The *pairing* of a vector is $\psi^\top \psi = \sum_i \psi_i^2$, with no complex conjugation; it equals $\langle \bar{\psi}, \psi \rangle$, so it vanishes exactly when $\psi \perp \bar{\psi}$, and a unit vector has $|\psi^\top \psi| = 1$ exactly when it is real up to a phase. Here $\gamma^\top \gamma = f_\gamma^\top f_\gamma = 0$. An operation is *real* if its matrix has real entries in the native basis.

Theorem 11.9 (Sequential native reads are blind to conjugation)**.** *(a) Let W be a finite product, in any order and with any repetition, of slices V_j and the retained read* R*. Then W^*W is real symmetric. Let $\mathcal{S}$ be the span of all such effects. For every $E \in \mathcal{S}$ and every one-loop genesis state $\delta_q \otimes f$, $\mathrm{tr}(\rho_{\delta_q \otimes \bar{f}} E) = \mathrm{tr}(\rho_{\delta_q \otimes f} E)$. In particular v_γ and $\bar{v}_\gamma$ have one expectation on every element of $\mathcal{S}$.*

(b) The projection $\mathrm{lift}_{00}(|p \otimes \gamma\rangle\langle p \otimes \gamma|)$ lies in the live read algebra $\mathscr{A}_{\mathrm{R}}$, reads 1 on v_γ and 0 on $\bar{v}_\gamma$, and does not lie in $\mathcal{S}$. So $\mathcal{S}$ is a proper subspace of $\mathscr{A}_{\mathrm{R}}$.

Proof. (a) The slices are real, and so is R, whose entries are 0 and $\pm\frac{1}{4}$ by (5.3). A product of real matrices is real, so $W^* = W^\top$ and $W^*W = W^\top W$ is symmetric. The density of $\delta_q \otimes \bar{f}$ is $\rho^\top$ with ρ that of $\delta_q \otimes f$, and $\mathrm{tr}(\rho^\top E) = \mathrm{tr}((E^\top \rho)^\top) = \mathrm{tr}(E\rho)$ for symmetric E; this extends to the span. (b) The projection lies in $\mathscr{A} \subseteq \mathscr{A}_{\mathrm{R}}$ and reads 1 and 0 by Theorem 9.7(b). Its transpose is the projection onto $p \otimes \bar{\gamma}$, so it is not symmetric and not in $\mathcal{S}$. □

This wall stands for every finite sequence of native instruments, the posting with its record and the retained read, in any order and with outcomes grouped in any way. The direction of a circulation exists in the read algebra as an element, and in no sequence of native instruments as an effect.

Proposition 11.10 (Real operations conserve the pairing)**.** *(a) If V is real and $V^*V = I$, then $(V\psi)^\top (V\psi) = \psi^\top \psi$.*

(b) If the K_w are real and $\sum_w K_w^* K_w = I$, then $\sum_w (K_w\psi)^\top (K_w\psi) = \psi^\top \psi$. Equivalently, $\sum_w \|K_w\psi\|^2 \, \psi_w^\top \psi_w = \psi^\top \psi$ for the conditional states $\psi_w = K_w\psi / \|K_w\psi\|$, terms with $K_w\psi = 0$ omitted.

(c) For every one-loop genesis state $\delta_q \otimes f$ with f a unit vector and every k, the present vectors $\psi_h = V_h f / \|V_h f\|$ satisfy $\sum_{h \in \mathcal{F}^k} \mathbb{P}(h) \, \psi_h^\top \psi_h = f^\top f$. For v_γ the sum is 0 at every k.

Proof. $(V\psi)^\top (V\psi) = \psi^\top V^\top V \psi$ and $V^\top = V^*$ for real V; sum over w for (b). For (c) take $K_h = V_h$, which are real with $\sum_h V_h^* V_h = I$, and $\mathbb{P}(h) = \|V_h f\|^2$. □

Part (c) is the identity of Proposition 10.19(a), here a case of (b).

Theorem 11.11 (Real operations never make a circulation)**.** *(a) Every density reached from real densities by real operations is real. The operations are: conditioning on a real Kraus operator with any real renormalization, a real instrument with its outcomes summed, real mixtures, and adjoining real ancillas.*
(b) If ρ is a real density and φ a unit vector with $\varphi^{\top}\varphi = 0$, then $\langle \varphi, \rho\varphi\rangle \le \frac{1}{2}$. The bound is attained by the real density $\frac{1}{2}(|\varphi\rangle\langle\varphi| + |\bar{\varphi}\rangle\langle\bar{\varphi}|)$.
(c) For every real cell vector f, loop q and record h of positive probability, the present of $\delta_q \otimes f$ after h, read as a unit genesis state, has fidelity at most $\frac{1}{2}$ with ν_γ.

Proof. (a) $\overline{K\rho K^*} = \bar{K}\bar{\rho}\bar{K}^* = K\rho K^*$ for real K and ρ, and sums, real multiples and Kronecker products of real matrices are real. (b) Write $\varphi = r + is$ with r, s real. Then $\varphi^{\top}\varphi = |r|^2 - |s|^2 + 2i\, r^{\top}s = 0$ gives $|r|^2 = |s|^2 = \frac{1}{2}$ and $r \perp s$. A real density is real symmetric, so $\langle \varphi, \rho\varphi\rangle = r^{\top}\rho r + s^{\top}\rho s = \frac{1}{2}\big(\hat{r}^{\top}\rho\hat{r} + \hat{s}^{\top}\rho\hat{s}\big) \le \frac{1}{2}\operatorname{tr}\rho$ with $\hat{r}, \hat{s}$ orthonormal. The density named is real, and $\bar{\varphi} \perp \varphi$ since $\langle \bar{\varphi}, \varphi\rangle = \varphi^{\top}\varphi = 0$. (c) The present vector of a real genesis is real, because the slices are; apply (b) to its projection with $\varphi = \nu_\gamma$, whose pairing is 0. □

Complex conjugation acts here as a two-element superselection rule. Its reference frames, in the sense of Bartlett, Rudolph and Spekkens [6], are the non-real states, and Section 11.8 finds where they come from.

11.7 An archive left alone

Writing a record into clocks is not the same as keeping it. If the reader does not know how long the archive has idled, a record is read only up to a common shift of its codes after whole beats, and not at all after arbitrary durations. And idling never makes a circulation readable.

Definition 11.12 (The clock archive)**.** The *eight-tick clock* is $\mathbb{C}^8$ with basis $e_0, \dots, e_7$, indices modulo 8, and discrete Fourier modes $d_\mu = 8^{-1/2}\sum_t e^{-2\pi i\mu t/8} e_t$, $\mu \in \mathbb{Z}/8$. Its *generator* G has eigenvalue $\frac{\pi}{4}Q(\mu)$ on d_μ, where the charge is $Q(1) = 1$, $Q(3) = 3$, $Q(5) = -3$, $Q(7) = -1$ and $Q(\mu) = 0$ for even μ. The *free flow* of duration $u \in \mathbb{R}$, in beats, is $\mathscr{F}_u = e^{-iuG}$, and $B = \mathscr{F}_1$ is the *beat*. The *pointer states* $\varpi_n = (e_{-n} - e_{4-n})/\sqrt{2}$, $n \in \mathbb{Z}/8$, satisfy $B\varpi_n = \varpi_{n+1}$ and $\varpi_{n+4} = -\varpi_n$, and $\varpi_0, \dots, \varpi_3$ are orthonormal. The *archive* writes a label j on three clocks, one per account axis, as $\varpi_{j_1+1} \otimes \varpi_{j_2+1} \otimes \varpi_{j_3+1}$: the *code* of j is $(j_1 + 1, j_2 + 1, j_3 + 1) \in \{0, 1, 2\}^3$. A record of length k is written on $3k$ clocks as the product of its labels' triples. The *archived history* of a genesis state v at stage k is the stage-k state with its record replaced by this archive vector. While *idle* for a duration u, every clock evolves by $\mathscr{F}_u$ and the cell and loop do not change. Two records h, h' of length k *merge* if one $t \in \mathbb{Z}/4$ adds to every one of the $3k$ codes of h, modulo 4, to give those of h'. A matrix E on the archive *reads* a set S of records at a family of durations if, for every duration u of the family and every record h, E maps the idle archive $\mathscr{F}_u^{\otimes 3k} a_h$ of h to itself when $h \in S$ and to 0 when $h \notin S$.

The archive is an isometric encoding, since distinct code tuples give orthonormal product vectors. We take the recorder and the idle evolution as given: each clock runs under its own free flow, with no coupling to the other clocks, the cell or a field. A reader who does not know the idle time is modelled as one who must read sharply at every element of a family

of durations; unsharp reads are not treated, and how well one recovers a record after an unknown real duration is open.

Theorem 11.13 (What an unattended archive keeps)**.** *Let $k \geq 0$ and let S be a set of records of length k.*

(a) An effect $0 \leq E \leq I$ reads S at every whole number of beats if and only if S is a union of merge classes, and then a projection does it. A matrix E of any kind that reads S at every whole number of beats never separates two merged records.

(b) The persistent records $(1,1,-1)^k$ and $(-1,-1,1)^k$ merge, with shift 2: for $k \geq 1$ no matrix reads the sign of a persistent current at every beat. For $k \geq 2$ each of the two oscillation records is alone in its merge class and the set of the four mode records is a union of classes: a projection reads the oscillation phase, and a projection reads the sure-subject event, at every beat. For $k = 1$ no matrix reads the sure-subject event at every beat: $(0,1,0)$ merges with $(-1,0,-1)$ and $(0,-1,0)$ with $(1,0,1)$.

(c) If a matrix E reads S at every real duration, then S is empty or contains every record.

(d) One beat carries the blank archive, ϖ_0 on every clock, to the archive of the all-zero record $(0,0,0)^k$; no matrix both annihilates the blank archive and fixes the archive of $(0,0,0)^k$ at every beat. With one blank triple written beside the archive as an extra slot, idling with it, every set of records is read by a projection at every whole beat, and still no set other than the empty one and the whole is read at every real duration.

(e) If each code c is recoded into the even mode d_{2c} on its clock, the recoded archive does not move while idle, and every set of records is read by a projection at every real duration.

Proof. (a) By $B\varpi_n = \varpi_{n+1}$ and $\varpi_{n+4} = -\varpi_n$, n beats carry the archive of a code tuple to $\pm$ the archive of the tuple shifted by n modulo 4. If h' is h shifted by t, the archive of h' is, up to sign, the archive of h after t beats, so a read at every beat treats them alike. Conversely, let E project onto the span of the archives of all shifts of the code tuples of records in S. These pointer products are orthonormal, beats permute them up to sign, and when S is a union of merge classes no shift of a record outside S lands among them. (b) The codes are $(2,2,0)$ for $(1,1,-1)$ and $(0,0,2) = (2,2,0)+2$ for $(-1,-1,1)$; $(1,2,1)$ for $(0,1,0)$ and $(0,1,0) = (1,2,1)+3$ for $(-1,0,-1)$; $(1,0,1)$ for $(0,-1,0)$ and $(2,1,2) = (1,0,1)+1$ for $(1,0,1)$. Codes of records lie in $\{0,1,2\}$, so a shift t can keep $(1,2,1)$ inside only for $t \in \{0,3\}$ and $(1,0,1)$ only for $t \in \{0,1\}$. From $k = 2$ on an oscillation record contains both labels $(0,1,0)$ and $(0,-1,0)$, so only $t = 0$ survives. The persistent records merge only with each other. Apply (a). (c) On the archive, $\mathscr{F}_u^{\otimes 3k}$ multiplies the product Fourier vector of modes $m = (m_1, \dots, m_{3k})$ by $e^{-i\pi Q(m)u/4}$ with total charge $Q(m) \in [-9k, 9k]$, and only the product of d_3 on every clock has $Q = 9k$. Since the charge differences lie in $[-18k, 18k]$, the average

$$\frac{1}{18k+1} \sum_{n=0}^{18k} e^{i\pi \cdot 9k\, u_n/4}\, \mathscr{F}_{u_n}^{\otimes 3k}, \qquad u_n = \frac{8n}{18k+1},$$

is the projection onto that single top line τ. Each pointer state has squared modulus $\frac{1}{4}$ on d_3, so the archive of every record has modulus 2^{-3k} on τ. Apply a matrix E that reads S at every duration to the same average of the idle archives of one record h: the average is a nonzero multiple of τ, so $E\tau = \tau$ if $h \in S$ and $E\tau = 0$ if $h \notin S$. If S contained some records and missed others, both would hold. (d) The blank code 0 moves to 1, the code of the zero label, with sign

+1. A blank triple has code 0 for every record, so a common shift that keeps it a blank triple is $t = 0$, and merge classes become single records; (a) applies. The argument of (c) applies to $k + 1$ slots. (e) Even modes have charge 0, so $\mathscr{F}_u$ fixes them; distinct recoded tuples give orthonormal product vectors. □

Record length is not kept passively either: no postings and k zero postings are the same aged archive one beat apart. Keeping a record through unattended evolution takes an operation beyond writing, such as the recoding of (e), which we do not construct, or a reader synchronized with the archive's age, which is a timing premise. The blank triple of (d) is a partial synchronizer: it fixes the shift at whole beats and not between them.

Theorem 11.14 (Waiting does not unhide a circulation)**.** *For every stage k and every real duration u:*

(a) $\mathscr{F}_u$ is a real matrix. Hence the idle archived history of $\bar{v}_\gamma$ is the complex conjugate of that of v_γ, and every real matrix E has the same expectation on the two. The same holds after any further real operation, and beside any real reference register idling under real dynamics, a blank triple included.

(b) Some effect, the projection onto the idle archived history of v_γ, reads 1 on it and 0 on that of $\bar{v}_\gamma$.

(c) Idling and then reading $A \otimes C$, with A a real effect on the archive and C any read of the cell, is the read $\mathscr{F}_u^* A \mathscr{F}_u \otimes C$ on the freshly written archive, where $\mathscr{F}_u^* A \mathscr{F}_u$ is again a real effect.

Proof. (a) The entries of $\mathscr{F}_u$ are $\sum_\mu e^{-i\pi Q(\mu)u/4} (d_\mu)_t \overline{(d_\mu)_s}$. Conjugation sends d_μ to $d_{-\mu}$, and $Q(-\mu) = -Q(\mu)$, so reindexing $\mu \to -\mu$ shows every entry equals its conjugate. The pointer states and the slices are real, so the idle archived history of $\bar{v}_\gamma$ is the conjugate of that of v_γ, and $\langle \bar{\psi}, E\bar{\psi} \rangle = \overline{\langle \psi, E\psi \rangle}$ for real E has the same real part. Real operations and real registers keep the two conjugate. (b) The archived history is the image of v_γ under real isometries, so its pairing is $f_\gamma^\top f_\gamma = 0$ by Proposition 11.10, and the two idle histories are orthogonal. (c) $\mathscr{F}_u$ is unitary and real. □

The charges $\pm 1, \pm 3$ of the clock give it complex phases, but they come in conjugate pairs on conjugate modes. In the native basis the idle archive is real at every duration, so waiting supplies no phase reference.

11.8 What breaks the wall

A circulation can be read only beside a state that is not real up to a phase. A flow is real exactly when its energy is odd under conjugation; an energy even under conjugation turns real non-stationary states into references, and positive energy excludes every antiunitary symmetry of the flow.

Proposition 11.15 (A non-real state is a reference)**.** *Let ψ and φ be unit vectors with $\psi^\top \psi = \varphi^\top \varphi = 0$. The real projection E onto* span$\{\psi \otimes \varphi, \bar{\psi} \otimes \bar{\varphi}\}$ *reads $\psi \otimes \varphi$ with probability 1 and $\bar{\psi} \otimes \varphi$ with probability 0. In particular, beside a second circulating cell, or beside one eight-tick clock in its Fourier mode d_1, for every stage k and duration u some real projection, depending on k and u, reads the idle archived histories of v_γ and $\bar{v}_\gamma$ perfectly.*

Proof. The two spanning vectors are conjugate, so E is real, and they are orthonormal since $\langle\bar{\psi},\psi\rangle = \psi^{\top}\psi = 0$. The vector $\bar{\psi}\otimes\varphi$ is orthogonal to $\psi\otimes\varphi$ (first factor) and to $\bar{\psi}\otimes\bar{\varphi}$ (second factor). For the examples, $f_{\gamma}^{\top}f_{\gamma} = 0$, $d_1^{\top}d_1 = \frac{1}{8}\sum_t e^{\pm 2\pi i\cdot 2t/8} = 0$, and the idle archived history has pairing 0 (Theorem 11.14). □

Theorem 11.16 (Which flows are real, and which break conjugation)**.** *Let H be a complex $N\times N$ matrix and $U_t = e^{-itH}$.*

(a) U_t is real for every real t if and only if $\bar{H} = -H$.
(b) If $\operatorname{Re}\operatorname{tr} H \neq 0$, no unitary W satisfies $W\bar{U}_tW^{*} = U_t$ for all t. So no antiunitary operator commutes with the flow of a nonzero positive semidefinite H.
(c) The clock generator satisfies $\bar{G} = -G$, and the free flow of the clock is $\mathscr{F}_u = e^{-iuG}$ at every real duration.
(d) If H is real symmetric and r is a real unit vector that is not an eigenvector of H, then $|(U_tr)^{\top}(U_tr)| < 1$ for all small $t > 0$. If e_0, e_1 are real orthonormal eigenvectors of H with energies $E_0 \neq E_1$ and $r = (e_0+e_1)/\sqrt{2}$, then

$$(U_tr)^{\top}(U_tr) = \tfrac{1}{2}\big(e^{-2iE_0t} + e^{-2iE_1t}\big),$$

which vanishes at $t_{*} = \pi/(2(E_1-E_0))$. Beside $U_{t_{*}}r$, for every stage k and duration u some real effect, depending on k and u, reads the idle archived histories of v_{γ} and $\bar{v}_{\gamma}$ perfectly.
(e) On the Fock space of the free photon field of Section 7.7, with positive energy scale, no antilinear map C with $\langle Cx, Cy\rangle = \overline{\langle x,y\rangle}$ commutes with the free unitary group at every time.

Proof. (a) $\overline{U_t} = e^{it\bar{H}}$, which is U_t for all t when $\bar{H} = -H$; conversely, differentiate $\overline{U_t} = U_t$ at $t = 0$. (b) Differentiating $W\bar{U}_tW^{*} = U_t$ at $t = 0$ gives $W\bar{H}W^{*} = -H$, so $\overline{\operatorname{tr} H} = -\operatorname{tr} H$ and $\operatorname{Re}\operatorname{tr} H = 0$. An antiunitary operator is $x\mapsto W\bar{x}$ with W unitary, and a nonzero positive semidefinite H has $\operatorname{tr} H > 0$. (c) In the native basis $G = \sum_{\mu}\frac{\pi}{4}Q(\mu)\,d_{\mu}d_{\mu}^{*}$; conjugation sends d_{μ} to $d_{-\mu}$ and $Q(-\mu) = -Q(\mu)$, so $\bar{G} = -G$. The clock's free flow is defined by these energies, so it is e^{-iuG}. (d) H real symmetric gives $U_t^{\top} = U_t$, so $(U_tr)^{\top}(U_tr) = r^{\top}U_{2t}r = \langle r, e^{-2itH}r\rangle$, an average of the phases e^{-2itE} over the spectral weights of r, which has modulus below 1 for small $t > 0$ once two distinct energies carry weight. For $r = (e_0+e_1)/\sqrt{2}$ the average is the one displayed, and $e^{-2iE_1t_{*}} = -e^{-2iE_0t_{*}}$. Then apply Proposition 11.15. (e) Differentiating $CU_tx = U_tCx$ at $t = 0$ puts Cx in the domain of the energy H whenever x is, and, C being antilinear, gives $CH = -HC$ there. Then $\langle Cx, HCx\rangle = -\langle Cx, CHx\rangle = -\overline{\langle x, Hx\rangle} = -\langle x, Hx\rangle$. The left side is nonnegative and the right side nonpositive, so $\langle x, Hx\rangle = 0$ for every vector in the domain, while a one-photon state has positive energy. □

Two different symmetries are at work, and the theorem keeps them apart. An energy odd under conjugation, $\bar{H} = -H$, has a real flow: conjugation commutes with it, as for the clock. An energy even under conjugation, $\bar{H} = H$, has $\overline{U_t} = U_{-t}$: conjugation reverses its time, which is Wigner's time reversal, and such an energy turns a real non-stationary state into a reference by (d). Calling complex conjugation time reversal is therefore an identification, and it is not the symmetry the clock flow respects. The wall of (b) and the reference of (d) are also different statements. Adding a constant $c > 0$ to the clock generator gives a positive energy fenced by (b), but $e^{-it(G+cI)} = e^{-ict}\mathscr{F}_t$ only multiplies a real state by a global phase, and its pairing keeps modulus 1. References need an energy even under conjugation with two distinct energies in the support of the state.

With real initial states the record physics, whose operations and effects are real in the native basis, is a real quantum theory in the sense of Renou and coauthors, a class whose network predictions differ from those of complex quantum theory [33]. Positive energy excludes every antiunitary symmetry, and on the photon field it does so for the free flow itself; a reference needs, in addition, an energy even under conjugation acting on a superposition of two levels. Whether one real read serves every duration beside such a reference is not settled here. Whether the couplings between a cell and the field carry such a reference into a cell's record law, so that a real genesis acquires a circulation, is open.

All of this concerns one cell. Chapter 12 joins cells by contacts, and asks which states survive the native postings when several cells share them.

Chapter 12

Networks and the parity law

One contact posting on its own pair keeps exactly four states (Chapter 7). When contacts share sites, that answer no longer applies contact by contact: a posting on one pair moves a bit that another pair reads at its next posting. This chapter finds what survives the coupling. For product configurations of dressed clocks the answer is a parity law. Every contact writes the loop edge at every posting exactly when two linear conditions modulo 2 hold, one across every contact and one around every cycle of the network.

12.1 Sites, contacts and networks

A network has sites $z = 0, \dots, n-1$. Each site carries an occupation bit $x_z \in \{0, 1\}$ and a clock cell, so its carrier is

$$\mathscr{H}_n = \bigotimes_{z=0}^{n-1} \big(\mathbb{C}^2 \otimes \mathscr{H}_{\mathrm{c}}\big), \qquad \dim \mathscr{H}_n = 8^n.$$

A *contact* $e = (u, v)$ is an ordered pair of distinct sites. It acts by the native step of Chapter 2 on the bits and clocks of u and v: the occupation of its pair is (x_u, x_v), clock cell 1 is the clock of u and clock cell 2 is the clock of v. Its slice V_j^e is V_j on that pair and the identity on every other site, so $\sum_j V_j^{e*} V_j^e = I$ (Proposition 2.12), and $Q_{\mathrm{al}}^e, Q_{\mathrm{op}}^e, Q_{\mathrm{ne}}^e$ are its three sectors (2.15) on the pair. A *network* is a list $E = (e_1, \dots, e_m)$ of contacts, $e_i = (u_i, v_i)$; a pair of sites may occur more than once, as a *doubled* contact. A *posting sequence*, or *schedule*, $g = (g_1, g_2, \dots)$ names the contact of E that posts at each time. The *round order* posts $e_1, \dots, e_m$ and repeats: $g_t = e_i$ for $t \equiv i \pmod m$. Along a schedule a word $w = (j^{(1)}, \dots, j^{(k)})$ gives

$$V_w = V_{j^{(k)}}^{g_k} \cdots V_{j^{(1)}}^{g_1},$$

the first label applied first. The *degree* $\deg z$ counts the entries of E containing z; the site is *touched* if $\deg z \geq 1$. The *rank* $\nu_i(z)$ counts the entries before position i that contain z. The *contact graph* $G(E)$ joins u and v when some entry does, and E is *connected* when every site is touched and $G(E)$ is connected.

Each contact keeps its own loop flux, which its postings move by the loop-edge current j_2 (Chapter 2). A posting writes the loop edge, $j_2 \neq 0$, exactly when its input pair is mixed (Table 2.3). The native step treats its two cells differently, through the signs $\sigma(x)$ and through which clock reads which bit, so a contact's orientation is part of the data. It changes the labels a posting writes. It changes no statement in this chapter about which configurations are kept: every case of Lemma 12.1 below, and the output of every deterministic posting, is symmetric in the two ends of the contact, so the classical run, and with it every kept class (Proposition 12.3), does not depend on orientation.

A *classical state* $q = (x, \ell)$ gives each site a bit x_z and a line $\ell_z \in \{p, a, P, A\}$ of the pole basis (2.8).

The *product pole basis* consists of the 8^n orthonormal states

$$\psi(x, \ell) = \bigotimes_z |x_z\rangle \otimes \ell_z.$$

The lines $P = Dp$ and $A = Da$ are *dressed*, with *flip bits* $f(P) = 0$ and $f(A) = 1$; the lines p, a are *undressed*. At a contact $e = (u, v)$ the pair is *mixed* when $x_u \neq x_v$ and *balanced* otherwise, and e is *neutral* at q when its pair is mixed and one of its two lines is undressed. We write states of three sites as $|x_0 x_1 x_2\rangle \otimes \ell_0 \otimes \ell_1 \otimes \ell_2$.

Lemma 12.1 (One posting). *Let $e = (u, v)$ and $\psi = \psi(x, \ell)$. Exactly one of the following holds.*

(N) *The pair is mixed and one of its lines is undressed. Then $Q^e_{\mathrm{ne}}\psi = \psi$. In every other case $Q^e_{\mathrm{ne}}\psi = 0$.*

(D) *The pair is mixed with both lines dressed (an* awake *posting), or balanced with both lines undressed (a* dressing*). Exactly one label $\hat{\jmath}$ has $V^e_{\hat{\jmath}}\psi \neq 0$, and $V^e_{\hat{\jmath}}\psi = \epsilon\psi(x', \ell')$, $V^{e*}_{\hat{\jmath}}\psi(x', \ell') = \epsilon\psi$, with $\epsilon = \pm 1$. An awake posting keeps both lines and adds $f(\ell_z)$ to each bit: an A clock flips its site's bit and a P clock keeps it. Its label has $\hat{\jmath}_2 \neq 0$. A dressing turns p into P with its bit flipped and a into A with its bit kept. Its label has $\hat{\jmath}_2 = 0$.*

(B) *The pair is balanced and one of its lines is dressed (a* branching *posting). Some label has $V^e_j\psi \neq 0$ and $Q^e_{\mathrm{ne}}V^e_j\psi = V^e_j\psi$.*

In every case $V^{e*}_j V^e_j\psi = \|V^e_j\psi\|^2\psi$ for every label j. (U) *If both lines of $\psi(y, \ell)$ on the pair are dressed, exactly one label has $V^{e*}_j\psi(y, \ell) \neq 0$, and it gives $\pm$ a single basis state.*

Proof. On a mixed pair $Q_{\mathrm{ne}} = I - \Theta^2$ with $\Theta = c' \otimes c'$, and c' is 0 on p, a and ± 1 on P, A (Table 2.1); on a balanced pair $Q_{\mathrm{ne}} = 0$. This gives (N). For a fixed input occupation x the label $j(y, x) = (y_1 - x_2, x_1 - x_2, y_2 - x_1)$ determines the output y, so $V^e_j\psi$ is the single block $|y\rangle \otimes M_{yx}(\ell_u \otimes \ell_v)$ of Theorem 2.4, in which clock 1 receives $K(y_1, x_2)C_x$ and clock 2 receives $K(y_2, x_1)C_x$. The map $K(b', b)$ is $\pm$stay when $b' = b$ and $\pm$flip otherwise. By Table 2.2, stay kills p and sends a to A, flip kills a and sends p to P, and both send P and A to $\frac{1}{2}(p \pm a)$.

On a mixed pair $C_x = D$ turns P, A into p, a first. A P clock at u therefore survives only through flip, that is $y_1 \neq x_2$, so $y_1 = x_1$ and the output line is P; an A clock survives only through stay, so $y_1 = x_2 \neq x_1$ and the output line is A. Clock 2 is the same with x_1 in place of x_2. One output y survives, with sign ± 1, and $j_2 = x_1 - x_2 \neq 0$. On a balanced pair $C_x = I$: p survives only through flip, so its bit changes and it becomes P; a survives only through stay, so its bit stays and it becomes A; and $j_2 = 0$. This is (D), apart from the adjoint. In case (B) let ℓ_u be dressed; the case of ℓ_v is symmetric. Both values of y_1 give a nonzero output on clock 1, in the plane of p and a. By Lemma 2.5 at least one value of y_2 gives a nonzero output on clock 2; choose it and put $y_1 = 1 - y_2$. The output pair is mixed and clock 1 is undressed, so the output lies in $\operatorname{ran} Q^e_{\mathrm{ne}}$.

Each $V^*_j V_j$ is diagonal in the pole basis, because $M^\top_{yx}M_{yx}$ is a product of the Grams $\frac{1}{2}(I \mp c)$ of Lemma 2.5, conjugated by C_x, and $DcD = c'$; this gives $V^{e*}_j V^e_j\psi = \|V^e_j\psi\|^2\psi$. For the last statement, $\mathrm{flip}^\top P = p$, $\mathrm{stay}^\top A = a$ and $\mathrm{stay}^\top P = \mathrm{flip}^\top A = 0$. A dressed output line on clock 1 therefore fixes x_2 (to $y_1 + 1$ for P, to y_1 for A), and one on clock 2 fixes x_1. So the input occupation, and with it the label $j(y, x)$, is unique, and the adjoint block sends the output to $\pm$ one basis state. Applying this to $\psi(x', \ell')$ in case (D) gives $V^{e*}_{\hat{\jmath}}\psi(x', \ell') = \epsilon\psi$. □

On a mixed pair of two P lines or two A lines the awake postings are the four modes of Proposition 2.22: ψ_+ keeps both bits and writes $(1, 1, -1)$, and $\psi_\circ$ flips both and writes $(0, 1, 0)$.

The lemma turns a product pole state into a classical system. We call the postings of case (D) *deterministic*.

Definition 12.2 (Live)**.** Given a posting sequence g and a classical state q, put $q_0 = q$, and let q_t be the output of the posting by g_t at q_{t-1} while that posting is deterministic. The state q is *live* under g (for the network E) if, for every $t \geq 0$, no contact of E is neutral at q_t and the posting by g_{t+1} is deterministic at q_t.

Liveness excludes exactly two events: a neutral pair anywhere in the network, and a branching posting. The *kept class* of E under g is

$$\mathcal{N}(E; g) = \{\psi \in \mathscr{H}_n \ : \ Q^e_{\mathrm{ne}} V_w \psi = 0 \text{ for every } e \in E \text{ and every word } w \text{ along } g\},$$

the empty word included, and $\mathcal{N}(E)$ is the kept class under the round order. For one contact it is the never-neutral subspace of Definition 7.10. Every label at every posting is allowed in w, and no window test or sure-subject test enters the definition.

Proposition 12.3 (The classical run decides)**.** *For every network E and every posting sequence g of E,*

$$\mathcal{N}(E; g) = \operatorname{span}\{\psi(q) \ : \ q \text{ live under } g\},$$

and a basis state $\psi(q)$ lies in $\mathcal{N}(E; g)$ exactly when q is live under g. If q is not live and its run first fails at time τ, every vector that passes the test on words of length at most $\tau + 1$ is orthogonal to $\psi(q)$.

Proof. If q is live, (D) gives $V_w \psi(q) = \pm\psi(q_k)$ or 0 for every word of length k, and (N) gives $Q^e_{\mathrm{ne}} \psi(q_k) = 0$ for every $e \in E$. So $\psi(q) \in \mathcal{N}(E; g)$. Since the $\psi(q)$ form an orthonormal basis, it remains to show that $\psi(q) \perp \mathcal{N}(E; g)$ whenever q is not live.

For $t \geq 0$ let $\mathcal{N}_t$ be the set of vectors that pass the test along the shifted sequence $g_{t+1}, g_{t+2}, \dots$. Then $\mathcal{N}_0 = \mathcal{N}(E; g)$, and $V_j^{g_{t+1}} \mathcal{N}_t \subseteq \mathcal{N}_{t+1}$ for every label j, because a word along the shifted sequence, preceded by j, is a word from time t. Let τ be the first time at which some contact $e \in E$ is neutral at q_τ or the posting by $g_{\tau+1}$ is not deterministic at q_τ, and let $\phi \in \mathcal{N}_\tau$. If e is neutral at q_τ, then $\langle \psi(q_\tau), \phi \rangle = \langle Q^e_{\mathrm{ne}} \psi(q_\tau), \phi \rangle = \langle \psi(q_\tau), Q^e_{\mathrm{ne}} \phi \rangle = 0$. Otherwise the posting by $g' = g_{\tau+1}$ branches; (B) gives j and $\omega = V_j^{g'} \psi(q_\tau) \neq 0$ with $Q^{g'}_{\mathrm{ne}} \omega = \omega$, and

$$\|\omega\|^2 \langle \psi(q_\tau), \phi \rangle = \langle V_j^{g'*} \omega, \phi \rangle = \langle Q^{g'}_{\mathrm{ne}} \omega, V_j^{g'} \phi \rangle = \langle \omega, Q^{g'}_{\mathrm{ne}} V_j^{g'} \phi \rangle = 0,$$

since $V_j^{g'} \phi \in \mathcal{N}_{\tau+1}$ passes the empty word. For $t < \tau$ the posting is deterministic, and (D) gives, for $\phi \in \mathcal{N}_t$, $\langle \psi(q_t), \phi \rangle = \epsilon \langle \psi(q_{t+1}), V_j^{g_{t+1}} \phi \rangle$ with $V_j^{g_{t+1}} \phi \in \mathcal{N}_{t+1}$. Descending from τ gives $\langle \psi(q), \phi \rangle = 0$ for $\phi \in \mathcal{N}_0$. Only words of length at most $\tau + 1$ were used. □

On the path and the triangle below, the proposition reduces each kept class to a count of live runs; Chapter 13 builds the general selection theorem on it.

12.2 Where contacts sit

The networks of this book are arbitrary lists of contacts. Part of where contacts sit can be decided from the native data.

Proposition 12.4 (Where contacts sit)**.**

(a) On the network carrier with no structure beyond its n sites, a property of contacts that is unchanged when the sites are relabelled holds for every contact or for none. The posting of a relabelled contact is the posting of the original contact read in relabelled configurations, $V_j^{\sigma e} = \hat{\sigma} V_j^e \hat{\sigma}^{-1}$ with $\hat{\sigma}$ the permutation of site factors, and the same holds for the effect of every schedule; so every rule that reads the posting family without using site names gives such a property.

(b) Let the sites be the points of the torus $(\mathbb{Z}/N\mathbb{Z})^3$, $N \geq 2$, on which one post moves a unit one step along one axis, forward for a debit and back for a credit; call (p, q) a one-post pair *when $q = p \pm e_k$ for an axis k. Let C be a relation on sites such that* (i) *$C(p, q)$ holds only for one-post pairs,* (ii) *C is unchanged by translations, by permutations of the three axes and by the reflection $p \mapsto -p$, and* (iii) *$C(p, q)$ holds for some (p, q). Then $C(p, q)$ holds exactly for the one-post pairs.*

(c) No relation with (i) *and* (ii) *holds for exactly one orientation of each one-post pair.*

Proof. (a) Relabellings act transitively on ordered pairs of distinct sites. The covariance holds because a contact's slice is V_j on its pair and the identity elsewhere. (b) The symmetries in (ii) act transitively on ordered one-post pairs: a translation moves the first site to the origin, an axis permutation moves the axis to the first, and the reflection followed by a translation reverses the direction. A relation with (ii) is therefore constant on one-post pairs, and by (iii) and (i) it holds on all of them and nowhere else. (c) The same transitivity carries (p, q) to (q, p), so an invariant relation holds for both orientations or for neither. □

So the native postings name no proper network on the bare carrier. On the ledger's torus, with the network's sites identified with the torus sites, three supplied premises name exactly one: (i) *locality*, that a contact joins only sites between which one post moves a unit; (ii) *symmetry*, that the contact rule is computed from the native data alone and so carries their translations, axis permutations and reflection; and (iii) that some contact exists. Under these, every nearest-neighbour pair is a contact. The side N is not derived. We read one native contact as serving both directions, since the exchange of Proposition 2.3 swaps a mixed pair's occupations, and take one contact per neighbour pair (keeping both orientations would make every pair a doubled contact). By (c) its orientation is supplied; by Section 12.1 it changes the written labels and not what is kept. The kept class of this network, the parity law of Section 12.5 on the torus graph, is not worked out here.

12.3 Two contacts sharing a site

The smallest shared network keeps four states, and two of them belong to neither contact. Take the *path* $E = ((0, 1), (0, 2))$: site 0 is shared, sites 1 and 2 are outer, and a round posts the first contact $(0, 1)$ and then the second contact $(0, 2)$. The carrier has dimension 512.

Both contacts can be tested at once. Let $\mathcal{J}_{\mathrm{aw}} = \{j \in \mathcal{J} : j_2 \neq 0\}$, let $W_{jj'} = V_{j'}^{(0,2)} V_j^{(0,1)}$ be the round with labels j then j', and put

$$\mathscr{W}(Y) = \sum_{j,j' \in \mathcal{J}_{\mathrm{aw}}} W_{jj'}^{*}\, Y W_{jj'}, \qquad J_k = \mathscr{W}^k(I).$$

By the Born law (Theorem 2.17), for a genesis state v placed at one point of the two loop fluxes, $\langle v, J_k v\rangle$ is the probability that both contacts write the loop edge at every posting of the first k rounds. We call a unit vector v a *joint sure state* if $\langle v, J_k v\rangle = 1$ for every k. For one contact this is the sure-subject test of Chapter 7; reading a joint sure state as a state in which both contacts are sure subjects at once applies that reading contact by contact.

state	$x_0x_1x_2$	clocks	labels (first, second)	loop-edge currents	round sign
π_+	100	PPP	$(1,1,-1), (1,1,-1)$	$(+1,+1)$	$+1$
π_-	011	PPP	$(-1,-1,1), (-1,-1,1)$	$(-1,-1)$	$+1$
r_+	101	APP	$(0,1,-1), (0,-1,1)$	$(+1,-1)$	-1
r_-	010	APP	$(0,-1,1), (0,1,-1)$	$(-1,+1)$	-1

Table 12.1. The four joint sure states of the path $((0,1),(0,2))$. The persistent pairs $\pi_\pm$ have every clock P; in the relays $r_\pm$ the shared clock is A. The round sign is the factor a full round multiplies the state by.

Follow r_+ through one round. The first pair $(x_0, x_1) = (1,0)$ is mixed with lines A, P, so the posting is awake: the A clock flips site 0, the P clock keeps site 1, the state moves to $|001\rangle$ and the label is $j((0,0),(1,0)) = (0,1,-1)$. The second pair $(x_0, x_2) = (0,1)$ is now mixed, site 0 flips back, and the label is $(0,-1,1)$. Between the two postings the first pair is balanced, so a second posting by the first contact would branch. The partner's posting in between is what returns the shared bit.

Theorem 12.5 (Joint sure states of the path)**.** *Let $S_{\mathrm{path}} = \mathrm{span}\{\pi_+, \pi_-, r_+, r_-\}$ and let Π be its orthogonal projection.*

(a) *A unit vector v is a joint sure state if and only if $v \in S_{\mathrm{path}}$. Two rounds already decide: $\langle v, J_2 v\rangle = 1$ forces $v \in S_{\mathrm{path}}$.*

(b) *For every $k \geq 0$,*

$$J_{k+2} = \Pi + 2^{-(k+1)} H + 4^{-(k+1)} Q,$$

where H and Q are positive semidefinite, diagonal in the product pole basis, and vanish on every product state whose three clocks are dressed. Hence J_k decreases to Π, and $\langle v, J_k v\rangle \to \sum |\langle \psi, v\rangle|^2$, summed over the four states of Table 12.1.

(c) *Each state ψ of Table 12.1 satisfies $W_{jj'}\psi = \epsilon\psi$ for its own label pair, with ϵ its round sign, and $W_{jj'}\psi = 0$ for every other pair.*

Proof. (c) is Lemma 12.1(D) applied twice, as for r_+ above; it gives the implication from S_{path} to joint sureness in (a). For (b), every posting that writes the loop edge has a mixed input. There, by Lemma 12.1 and Table 2.2, a dressed clock keeps its line and moves its bit surely, while an undressed clock stays in the plane of p and a and its output bit takes either value with Born weight $\frac{1}{2}$. So $\mathscr{W}$ maps the projection onto each class (one occupation, and for each clock one

of the line P, the line A or the undressed plane) to a nonnegative combination of projections onto classes of the same sector type. On each of the 27 sector types, J_k is the k-th power of an 8×8 transfer matrix on occupations. Direct exact computation of these matrices gives (b): the four classes of Table 12.1 are fixed with weight 1, every other class with three dressed clocks has weight 0 from the second round on, and the classes with an undressed clock decay at the rates shown. The same computation shows $\langle v, J_2 v\rangle < 1$ for every unit $v \notin S_{\text{path}}$, which completes (a). □

The shared site does not lock the two contacts to one orientation of persistent current.

Proposition 12.6 (The relays)**.** (a) *The persistent pairs alone are not the joint sure class: r_+ is a joint sure state outside* $\text{span}\{\pi_+, \pi_-\}$.
(b) *The oscillation pair $o_+ = |100\rangle \otimes A \otimes A \otimes A$, $o_- = |011\rangle \otimes A \otimes A \otimes A$ fails in the first round: $J_1 o_\pm = 0$.*
(c) *r_+ lies in the first contact's opposed sector, $Q_{\text{op}}^{(0,1)} r_+ = r_+$, and has zero weight on its aligned sector; after the first posting its second pair lies in the second contact's opposed sector.*
(d) *A joint sure state lies in the first contact's aligned sector if and only if it lies in* $\text{span}\{\pi_+, \pi_-\}$.

Proof. (a) is Theorem 12.5. (b) From $|100\rangle$ the first posting flips both A clocks to $|010\rangle$, and the second pair $(0, 0)$ is balanced, so it does not write the loop edge; o_- is the complement. (c) On a mixed pair the opposed sector is spanned by $P \otimes A$ and $A \otimes P$ (Chapter 2); the first pair of r_+ carries $A \otimes P$, and after the first posting the second pair is $(0, 1)$ with $A \otimes P$. (d) $Q_{\text{al}}^{(0,1)}$ fixes $\pi_\pm$ and annihilates $r_\pm$, so it fixes $\sum c_\psi \psi$ exactly when the relay coefficients vanish. □

The relay is a joint sure mode that neither contact has alone. The overlap also removes a mode: the oscillation, whose loop-edge current alternates between $+1$ and -1 and so keeps its loop flux bounded, does not survive on either contact. The four survivors are exactly the four sign patterns $(\pm 1, \pm 1)$ of persistent winding of the two loop fluxes.

Proposition 12.7 (The record of the path)**.** *Let $v = \sum_\psi c_\psi \psi$ be a unit vector of S_{path}. Every record word of k rounds has probability $\sum_\psi [$the word is the k-fold repetition of the label pair of $\psi]\, |c_\psi|^2$. Along the word of ψ the two loop fluxes move by k times its loop-edge currents. In every word of positive probability the second label of a round is the first (persistent pairs) or its negative (relays).*

Proof. By Theorem 12.5(c), a word of k rounds sends ψ to $\epsilon^k \psi$ if it repeats the label pair of ψ, and to 0 otherwise. The four label pairs are distinct and the four states orthonormal, so the probabilities add. The label pairs are read from Table 12.1. □

What the postings keep on the path

The joint sure test asks for the loop edge at every posting. The kept class asks only that no posting, with any label, ever leaves a contact with neutral weight. On the path the two coincide.

Theorem 12.8 (The kept class of the path)**.** $\mathcal{N}(E) = S_{\text{path}}$, *so* $\dim \mathcal{N}(E) = 4$. *Words of length* 5 *cut it out, and* 5 *is least: the state $\psi(000; p, a, P) = |000\rangle \otimes p \otimes a \otimes P$ passes every word of length at most* 4 *and does not lie in $\mathcal{N}(E)$. A unit vector lies in $\mathcal{N}(E)$ if and only if it is a joint sure state.*

Proof. By Proposition 12.3 we must find the basis states live under the round order. Let $q = (x, \ell)$ be live. At time 0 no contact is neutral, and the first posting is deterministic: awake or a dressing.

The first posting is awake. Then ℓ_0, ℓ_1 are dressed and (x_0, x_1) is mixed. The second posting cannot be a dressing, because ℓ_0 is dressed; so it is awake, and ℓ_2 is dressed. Dressed lines never change at awake postings, and a balanced pair of dressed lines branches, so every later posting must meet a mixed pair. With flip bits f, site 0 moves by f_0 at both postings and sites $1, 2$ move by f_1, f_2 once per round. The first pair before the next round is $(x_0 + 2f_0, x_1 + f_1)$, which must again differ from each other, so $f_1 = 0$; likewise $f_2 = 0$. The two postings of the first round then require $x_0 + x_1 = 1$ and $x_0 + f_0 + x_2 = 1$. These four solutions are the states of Table 12.1, and each is live by Theorem 12.5(c).

The first posting is a dressing. Then (x_0, x_1) is balanced with ℓ_0, ℓ_1 undressed. Since ℓ_0 is undressed and the second contact is not neutral, (x_0, x_2) is balanced too: $x = (b, b, b)$. After the dressing site 0 is dressed, so the second posting must be awake, which needs x_0 to have changed: $\ell_0 = p$, now P at bit $b + 1$; and ℓ_2 must be dressed, or the second contact would be neutral at time 1. If $\ell_1 = p$, site 1 became P at bit $b+1$, and the third posting meets the balanced dressed pair $(b + 1, b + 1)$ and branches. If $\ell_1 = a$, site 1 became A at bit b; the third posting is awake and flips it to $b + 1$, so the fifth posting meets $(b + 1, b + 1)$ and branches, unless $\ell_2 = A$, in which case site 2 flipped at the second posting and the fourth posting already branches. No state of this case is live, and $p \otimes a \otimes P$ at $(0, 0, 0)$ survives four postings, each deterministic with no neutral contact. So every word of length at most 4 sends it to $\pm$ a non-neutral basis state or to 0, while the fifth posting branches into a neutral sector. The last sentence follows from Theorem 12.5(a). □

Calling the states of $\mathcal{N}(E)$ the network's *participants* is the reading of Chapter 7, now applied to the whole network; it does not make the network a single subject. On the path the participants are selected as a whole. Let $\mathcal{A}_e = \operatorname{ran} Q^e_{\mathrm{al}}$, the class that contact e keeps when it posts alone (Theorem 7.11), read on $\mathscr{H}_3$ with the other site left free.

Proposition 12.9 (Whole and parts)**.** *On the path, $r_+ \in \mathcal{N}(E)$ but $r_+ \notin \mathcal{A}_{(0,1)} \cap \mathcal{A}_{(0,2)}$; $o_+ \in \mathcal{A}_{(0,1)} \cap \mathcal{A}_{(0,2)}$ but $o_+ \notin \mathcal{N}(E)$; so neither class contains the other, and*

$$\mathcal{N}(E) \cap \mathcal{A}_{(0,1)} \cap \mathcal{A}_{(0,2)} = \operatorname{span}\{\pi_+, \pi_-\}.$$

On the triangle of Section 12.4, $\mathcal{A}_{(0,1)} \cap \mathcal{A}_{(0,2)} \cap \mathcal{A}_{(1,2)} = 0$.

Proof. $\mathcal{A}_e$ is spanned by the product states whose pair under e is mixed with lines $P \otimes P$ or $A \otimes A$. The relay has $A \otimes P$ on its first pair; o_+ has $A \otimes A$ on both mixed pairs and fails by Proposition 12.6(b). The intersection of the two classes is spanned by $\pi_\pm$ and $o_\pm$, and Theorem 12.8 keeps $\pi_\pm$ only. On the triangle all three pairs would have to be mixed, and three bits cannot differ pairwise. □

The kept class is spanned by products over sites of definite bits and pole lines. What fails is the stronger statement that the network keeps products of each contact's own kept states: the relay, restricted to either contact, lies in that contact's opposed sector, which the contact alone does not keep.

The leak off the kept class has an explicit floor. Let $L_k(v) = \sum_w \sum_{e\in E} \|Q^e_{\mathrm{ne}} V_w v\|^2$, the sum over words w of length k along the round order: the neutral load written at stage k.

Proposition 12.10 (A leak floor)**.** *For every unit vector v of the path's carrier,*

$$\sum_{k=0}^{5} L_k(v) \;\geq\; \frac{1}{2048}\Big(1 - \sum_{\psi} |\langle \psi, v\rangle|^2\Big),$$

the sum over the four states of Table 12.1. On the triangle the same bound holds with $k \leq 4$ and the eight states of Table 12.2.

Proof (outline). Every basis state outside the kept class reaches a neutral pair or a branching posting within 5 postings (Theorem 12.8). Following the back-propagation of Proposition 12.3 quantitatively, the Cauchy–Schwarz inequality bounds the squared overlap of v with each such basis state by 4 times the load within 5 postings; summing over the 512 basis states gives $1/(512 \cdot 4)$. The triangle is the same with depth 4. □

The constant is a union bound; the optimal constants for the path and the triangle are not known.

12.4 Closing the loop: the triangle

Add a third contact on the outer pair and the network becomes a loop. The *triangle* is $E = ((0,1),(0,2),(1,2))$, posted first $(0,1)$, then $(0,2)$, then $(1,2)$; every site is touched twice per round. Its joint test is $T_k = \mathscr{W}^k_{\mathrm{tri}}(I)$, with $\mathscr{W}_{\mathrm{tri}}$ the sum over label triples with every $j_2 \neq 0$, as for the path. A unit vector with $\langle v, T_k v\rangle = 1$ for every k is a joint sure state of the triangle.

state	flips $f_0 f_1 f_2$	clocks	$x_0 x_1 x_2$	loop-edge currents	circulation	round sign
$\tau_1 = r_+$	100	APP	101	$(+1,-1,-1)$	$+1$	$+1$
$\tau_2 = r_-$	100	APP	010	$(-1,+1,+1)$	-1	-1
τ_3	010	PAP	100	$(+1,+1,+1)$	$+1$	$+1$
τ_4	010	PAP	011	$(-1,-1,-1)$	-1	-1
τ_5	001	PPA	100	$(+1,+1,-1)$	-1	-1
τ_6	001	PPA	011	$(-1,-1,+1)$	$+1$	$+1$
τ_7	111	AAA	101	$(+1,-1,+1)$	$+3$	-1
τ_8	111	AAA	010	$(-1,+1,-1)$	-3	$+1$

Table 12.2. The eight joint sure states of the triangle $((0,1),(0,2),(1,2))$. Loop-edge currents are listed for the contacts in posting order; the circulation is $j_2^{(0,1)} + j_2^{(1,2)} - j_2^{(0,2)}$, the currents summed around $0 \to 1 \to 2 \to 0$.

The table is a parity count. With every clock dressed, the three postings of a round meet mixed pairs exactly when

$$x_0 + x_1 = 1, \qquad x_0 + f_0 + x_2 = 1, \qquad (x_1 + f_1) + (x_2 + f_2) = 1,$$

because the second posting sees site 0 after one flip and the third sees sites 1 and 2 after one flip each. Adding the three equations cancels the bits and leaves $f_0 + f_1 + f_2 = 1$: an odd

number of A clocks. Each odd pattern then has two occupations, x_0 free, $x_1 = x_0 + 1$ and $x_2 = x_0 + f_0 + 1$, and every bit returns after the round because each site is touched twice.

Theorem 12.11 (Joint sure states of the triangle)**.** *Let $S_{\mathrm{tri}} = \mathrm{span}\{\tau_1, \dots, \tau_8\}$, of dimension 8, with projection Π_{tri}.*

(a) *A unit vector v is a joint sure state of the triangle if and only if $v \in S_{\mathrm{tri}}$. One round already decides: $\langle v, T_1 v\rangle = 1$ if and only if $v \in S_{\mathrm{tri}}$.*
(b) *For every $k \geq 0$,*

$$T_{k+1} = \Pi_{\mathrm{tri}} + 4^{-k} Q' + 8^{-k} E',$$

where Q' and E' are positive semidefinite and diagonal in the product pole basis, Q' is supported on product states with exactly one undressed clock and E' on product states with two or three.
(c) *Each τ_i is multiplied by its round sign by the round word of its own label triple and annihilated by every other round word.*
(d) *Posting the third contact as $(2, 1)$ instead of $(1, 2)$ leaves T_k unchanged for every k.*

Proof. As for Theorem 12.5: the third contact is the first one read through a permutation of the three sites, the joint test is diagonal in the classes of occupation and sector type, and on each of the 27 types T_k is the k-th power of an 8×8 transfer matrix. Direct exact computation of these matrices gives (a), (b) and (d); the parity count above identifies the dressed classes of weight one, and (c) is Lemma 12.1(D) applied three times. Off S_{tri} one round keeps at most weight $\frac{1}{2}$. □

Corollary 12.12 (Frustration)**.** *Every vector whose three clocks are P passes the joint test of the triangle with probability 0 from the first round on: $\langle v, T_k v\rangle = 0$ for $k \geq 1$. More generally, every dressed product state with an even number of A clocks has first-round weight 0. In particular the persistent pairs $\pi_\pm$ of the path are lost on the triangle.*

Proof. With every clock P no bit moves, and the three postings would need three pairwise different bits. For a dressed pattern with an even number of A clocks the three equations above have no solution. □

This is the frustration of an odd plaquette [38], in native form. Each contact behaves as an antiferromagnetic bond: it passes only if its two bits differ when it posts. Each A clock flips the sense of the later of its two bonds. A cycle of n_{c} bonds is then satisfiable exactly when $(-1)^{n_{\mathrm{c}} + \#A} = +1$, which for $n_{\mathrm{c}} = 3$ is the odd count above. Two features have no counterpart in a spin system. A satisfied configuration is a dynamical cycle that returns every round, not a static ground state. And an undressed clock, which has no Ising analogue, always destroys sure passage.

Proposition 12.13 (Path and loop compared)**.** (a) *$S_{\mathrm{tri}} \cap S_{\mathrm{path}} = \mathrm{span}\{r_+, r_-\}$. The path's test annihilates the other six states from its second round on, $J_{k+2}\tau_i = 0$ for $i \geq 3$, and the triangle's test annihilates $\pi_\pm$ from its first.*
(b) *In the triangle the relays write the path's relay labels on the first two contacts and a persistent label on the closing contact: $(0, 1, -1), (0, -1, 1), (-1, -1, 1)$ for r_+ and $(0, -1, 1), (0, 1, -1), (1, 1, -1)$ for r_-.*

(c) *Every joint sure state circulates: the circulation of every dressed configuration that passes a round of the triangle is odd,* ± 3 *for the joint oscillation* τ_7, τ_8 *and* ± 1 *otherwise.*

Proof. (a) Both spans are spanned by orthonormal product states, and the common ones are $r_{\pm}$. On the path, $\tau_3, \dots, \tau_8$ are dressed product states with an outer clock A, and Theorem 12.5(b) gives them weight 0 from the second round. The triangle part is Corollary 12.12. (b) After the first two postings r_+ is back at $|101\rangle$; the closing pair $(x_1, x_2) = (0, 1)$ carries $P \otimes P$ and writes $j((0,1),(0,1)) = (-1,-1,1)$. The case of r_- is the complement. (c) Each loop-edge current is ± 1, so a sum of three is odd; the values are read from Table 12.2. □

Theorem 12.14 (The kept class of the triangle)**.** $\mathcal{N}(E) = S_{\mathrm{tri}}$. *Words of length* 4 *cut it out, and* 4 *is least:* $|000\rangle \otimes p \otimes a \otimes A$ *passes every word of length at most* 3 *and is not kept. A unit vector is kept if and only if it is a joint sure state.*

Proof. As in Theorem 12.8. If the first posting is awake, the second must be awake too (site 0 is dressed), so all three clocks are dressed and every posting must meet a mixed pair: the parity count gives the eight states. If the first posting is a dressing, then $x = (b, b, b)$ and, as on the path, $\ell_0 = p$ and ℓ_2 is dressed, with site 0 now P at bit $b + 1$. The third posting $(1, 2)$ must be awake. If $\ell_1 = p$ it needs $\ell_2 = P$, and if $\ell_1 = a$ it needs $\ell_2 = A$. In both cases the bits after the third posting are $(b + 1, b + 1, b)$, so the fourth posting, by $(0, 1)$, meets a balanced dressed pair and branches. The state $p \otimes a \otimes A$ at $(0, 0, 0)$ is the second case and survives three postings. □

12.5 The parity law

The path and the triangle were decided by counting flips. The same count decides every network. Give every touched site a dressed line, with flip bits $f \in \mathbb{Z}_2^n$ ($\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$), and choose bits x. We call (f, x) *awake* for E when every posting of the round order meets a mixed pair. Every posting is then awake and flips the bits of its A clocks, so just before the posting of e_i in round $r = 0, 1, \dots$ site z holds

$$x_z + f_z\big(r \deg z + \nu_i(z)\big) \pmod 2. \tag{12.1}$$

On the path, for example, site 0 has degree 2 and ranks $\nu_1(0) = 0$, $\nu_2(0) = 1$; with an A clock there its bit moves at both postings of every round and is back after each round.

Lemma 12.15 (Two rounds decide)**.** *For* $\alpha, \beta, \eta, \theta \in \mathbb{Z}_2$*:* $\alpha + r\eta \neq \beta + r\theta$ *for every* $r \in \mathbb{N}$ *if and only if* $\alpha + \beta = 1$ *and* $\eta = \theta$*.*

Proof. $r = 0$ gives $\alpha + \beta = 1$ and $r = 1$ then gives $\eta + \theta = 0$. Conversely the difference is $1 + r(\eta + \theta) = 1$. □

Theorem 12.16 (Awake configurations)**.** *For every network,* (f, x) *is awake if and only if*

(i) $f_u \deg u = f_v \deg v$ *in* $\mathbb{Z}_2$ *for every contact* (u, v) *of* E*, and*
(ii) $x_{u_i} + x_{v_i} = \gamma_i(f) := 1 + f_{u_i}\nu_i(u_i) + f_{v_i}\nu_i(v_i)$ *for every position* i*.*

Proof. Put $\alpha = x_{u_i} + f_{u_i} v_i(u_i)$, $\beta = x_{v_i} + f_{v_i} v_i(v_i)$, $\eta = f_{u_i} \deg u_i$ and $\theta = f_{v_i} \deg v_i$. By (12.1) the pair of e_i in round r is $(\alpha + r\eta, \beta + r\theta)$, provided every earlier posting was awake. Lemma 12.15 says that it is mixed in every round exactly when (ii) and (i) hold at e_i; induction over the postings keeps (12.1) valid. □

Condition (ii) is a linear system in x. Let $\partial : \mathbb{Z}_2^m \to \mathbb{Z}_2^n$ send an entry to the sum of its two ends. Its kernel, the *cycle space* $Z(E)$, consists of the sets of entries that meet every site an even number of times; a doubled contact is a cycle of two entries.

Theorem 12.17 (The parity law)**.** *For every network, a flip pattern f is awake at some bits if and only if* (i) *holds and*

(ii′) $\sum_i y_i \gamma_i(f) = 0$ *for every cycle $y \in Z(E)$.*

Proof. Condition (ii) reads $\partial^{\top} x = \gamma(f)$. Since $\langle \partial^{\top} x, y \rangle = \langle x, \partial y \rangle$, the range of $\partial^{\top}$ lies in $Z(E)^{\perp}$. Both have dimension $m - \dim Z(E)$, because the standard form on $\mathbb{Z}_2^m$ is nondegenerate, so they are equal, and (ii) is solvable exactly when $\gamma(f) \perp Z(E)$. □

Condition (i) is linear in f and (ii′) is affine, since $\gamma_i(f) = 1 + \delta_i(f)$ with $\delta_i(f) = f_{u_i} v_i(u_i) + f_{v_i} v_i(v_i)$ linear. Neither condition sees the orientation of a contact.

Corollary 12.18. (a) *On a connected network an awake pattern is awake at exactly two bit vectors, x and its complement.*

(b) *A configuration with every touched clock P (*co-oriented*: every contact carries a persistent current) is awake if and only if x is a proper 2-colouring of $G(E)$. Co-oriented persistent currents coexist exactly on bipartite networks and are frustrated by every odd cycle.*

(c) *On a cycle of $n_{\mathrm{c}} \geq 2$ sites, in any order, f is awake if and only if $\sum_z f_z \equiv n_{\mathrm{c}} \pmod 2$.*

(d) *The awake patterns are empty or a coset of the linear space $\Delta(E) = \{d : d_u \deg u = d_v \deg v$ on every contact, $\sum_i y_i \delta_i(d) = 0$ for every $y \in Z(E)\}$. A connected network therefore has no awake configuration or exactly $2^{\dim \Delta(E)+1}$.*

(e) (The joint oscillation.) *The all-A pattern $f \equiv 1$ is awake if and only if $\deg u \equiv \deg v \pmod 2$ across every contact and $\sum_i y_i\big(1 + v_i(u_i) + v_i(v_i)\big) = 0$ for every cycle y.*

(f) (Euler law.) *If every degree is even, every awake pattern satisfies $\sum_z f_z \binom{\deg z}{2} \equiv m \pmod 2$, in every order.*

Proof. (a) Two solutions of (ii) differ by a vector that is equal at both ends of every contact, hence constant on a connected network. (b) For $f = 0$, (i) is empty and (ii) reads $x_u + x_v = 1$ on every contact. (c) Every site has degree 2, so (i) is empty, and its two entries have ranks 0 and 1 in every order. The cycle space is spanned by the set of all entries, and (ii′) for it reads $n_{\mathrm{c}} + \sum_z f_z = 0$. (d) Linearity, and (a). (e) is the theorem at $f \equiv 1$. (f) With every degree even, the set of all entries is a cycle. The ranks of the entries through z are $0, 1, \dots, \deg z - 1$ in every order, so $\sum_i \delta_i(f) = \sum_z f_z \binom{\deg z}{2}$, and (ii′) gives the claim. □

In (f) an A clock of degree 2 or 6 counts and one of degree 4 does not. On the path, (i) forces $f_1 = f_2 = 0$ and (ii) reads $x_0 + x_1 = 1$, $x_0 + x_2 = 1 + f_0$: the four rows of Table 12.1. On the triangle (c) gives the odd count, and (ii) gives the occupations of Table 12.2. On these two networks Theorems 12.8 and 12.14 lift the parity law to the full carrier: the kept class is spanned by the awake configurations. Chapter 13 proves this for every network.

12.5.1 Exchanging a partner

The parity law also says when a site can keep its configuration while one of its partners is replaced. Let (f, x) be awake for E, let the entry at position k be (u, b), and let c be a site that E does not touch. Replace that entry by (u, c), at the same position, and let (f', x') agree with (f, x) at every site other than c.

Proposition 12.19 (The exchange law)**.** *(f', x') is awake for the new network if and only if $f_b = 0$ or $\deg b = 1$, and $f'_c = f_b$ and $x'_c = x_b$. The same holds for an entry (b, u) replaced by (c, u).*

Proof. The replacement changes degrees and ranks only at b and c: b loses one touch, at position k, and c gains it. Every other contact through b sees $f_b \deg b$ change by f_b in condition (i), and, if it comes after position k, sees its right-hand side γ_i change by f_b in (ii). Both conditions survive at all such contacts exactly when $f_b = 0$ or there are none, that is $\deg b = 1$. At the new entry, (i) reads $f_u \deg u = f'_c$, and the old entry gave $f_u \deg u = f_b \deg b$; (ii) reads $x_u + x'_c = 1 + f_u \nu_k(u)$, and the old entry gave $x_u + x_b = 1 + f_u \nu_k(u) + f_b \nu_k(b)$. Under the first condition $f_b \deg b = f_b$ and $f_b \nu_k(b) = 0$, which gives $f'_c = f_b$ and $x'_c = x_b$. The reversed orientation is the same, since neither condition sees orientation. □

No condition falls on the kept site u; the new partner must be an exact copy, in clock and bit, of the partner it replaces.

Example 12.20 (Complete partner turnover)**.** The star $((0, 1), (0, 2), (0, 3))$ with every clock A and $x = (0, 1, 0, 1)$ is awake: the centre has degree 3 and each leaf degree 1, so (i) holds, and (ii) reads $x_0 + x_1 = 1$, $x_0 + x_2 = 1 + 1 = 0$, $x_0 + x_3 = 1 + 2 \equiv 1$. Exchange the leaves in turn for fresh sites $4, 5, 6$, each with an A clock and the bit of the leaf it replaces, $x_4 = 1$, $x_5 = 0$, $x_6 = 1$. Every leaf has degree 1, so by Proposition 12.19 the configuration is awake at every stage. The final network $((0, 4), (0, 5), (0, 6))$ touches none of the sites $1, 2, 3$, and it is the first network with the sites renamed $1 \leftrightarrow 4$, $2 \leftrightarrow 5$, $3 \leftrightarrow 6$, a renaming that keeps every bit. No other route is open. Removing one contact leaves the centre with degree 2, so $f_0 \deg 0 = 0$ against 1 at a remaining leaf, and (i) fails; adding a contact from the centre to a fresh site gives the centre degree 4, and (i) fails at every old leaf, whatever the new site carries. Nor can a leaf exchange the centre: the centre has $f_0 = 1$ and degree 3, so the exchange law fails.

Whether the oscillating centre is the same participant across the stages is a question of identification that these statements do not settle.

12.6 The posting order is physical

The order in which the contacts post is supplied; the native data cannot fix it (Proposition 14.25). The parity law contains the ranks ν_i, which depend on the order of the list. On a network whose sites have degree at most two, the entries through each site have ranks 0 and 1 (or 0 alone) in every order, and the order drops out. At a site of degree three or more it does not.

Example 12.21 (The complete network of four sites)**.** On K_4 every degree is 3, so (i) makes f constant, and $f = 0$ would need a 2-colouring of K_4, which does not exist. For $f \equiv 1$ the

right-hand sides are $\gamma_i = 1 + v_i(u_i) + v_i(v_i)$. Table 12.3 lists them in two orders of the same six contacts. In the first they are $(1,0,1,1,0,1)$, solved by $x = (0,1,0,1)$ and its complement. In the second they are all 1, which asks for a proper 2-colouring again. The same six contacts carry the joint oscillation in one order and nothing at all in the other.

position i	1	2	3	4	5	6
first order	01	02	03	12	13	23
ranks $(v_i(u_i), v_i(v_i))$	$(0,0)$	$(1,0)$	$(2,0)$	$(1,1)$	$(2,1)$	$(2,2)$
γ_i	1	0	1	1	0	1
second order	01	23	02	13	03	12
ranks $(v_i(u_i), v_i(v_i))$	$(0,0)$	$(0,0)$	$(1,1)$	$(1,1)$	$(2,2)$	$(2,2)$
γ_i	1	1	1	1	1	1

Table 12.3. Right-hand sides of the joint oscillation on K_4 in two orders of the same contacts. The first order is solved by $x = (0,1,0,1)$; the second asks for a 2-colouring of K_4.

Even where the patterns do not depend on the order, the occupations can. On the path $((0,1),(0,2))$ the relay r_+ is awake at $x = (1,0,1)$; in the reversed order $((0,2),(0,1))$ the pattern with $f_0 = 1$ is awake at the mirror occupations $(1,1,0)$ and $(0,0,1)$ instead, and not at $(1,0,1)$. The occupations are the bits at the start of a round, and reordering moves the start. The order-free object is the set of awake patterns.

Theorem 12.22 (Order-free networks)**.** (a) *If every site has degree at most* 2*, or if $G(E)$ has no cycle and E no doubled contact, then the awake patterns are the same for every reordering of E; on a connected network so is the number of awake configurations.*

(b) *Let E have no doubled contact. Call a site a* hub *if it has degree at least* 3 *and lies on a cycle of $G(E)$. The awake patterns are the same for every reordering of E if and only if no reordering of E has an awake pattern with $f_z = 1$ at a hub.*

Proof. (a) Condition (i) depends on degrees only. Without cycles (ii′) is empty. At degree at most 2 the cycle space is spanned by the cycle components, including doubled contacts, and on a component of n_{c} sites the two entries through each site have ranks 0 and 1 in every order, so (ii′) reads $n_{\mathrm{c}} + \sum_z f_z = 0$ there. The count follows by Corollary 12.18(a).

(b) *Outline.* Exchanging two consecutive entries a, b changes the ranks only at a site z they share: the entry moved forward loses one from its rank at z and the entry moved back gains one. If $f_z = 0$ nothing changes. If $f_z = 1$ the right-hand sides of a and b each change by 1. If z has degree 2, or lies on no cycle, the change is absorbed by complementing the bits on the component of z in the network with a and b removed: at degree 2 that component is z alone, and off every cycle a and b are bridges, so the complement changes the pair difference across a and b only. The pattern stays awake. Conversely, at a flipped hub on a cycle C, with no doubled contact there is a contact a of C at z and a contact at z off C; posting these two first in either order changes the sum of (ii′) over C by one, so the pattern fails in one of the two orders. Transporting the pattern along a chain of exchanges completes both directions. □

The characterization quantifies over the awake patterns of all orders. A test that reads order-freeness off the graph alone is not known.

Theorem 12.23 (Only co-oriented colourings survive every order)**.** *Let E be connected, with no doubled contact and $m \geq 2$. Consider round schedules, in which round r posts every contact once in an order O_r that may change from round to round. A configuration (f, x) is awake under every round schedule if and only if $f = 0$ and x is a proper 2-colouring of $G(E)$. Awake in every fixed order is already equivalent. The hypotheses are needed: a single contact, and the doubled contact $((0,1),(0,1))$, keep the all-A configuration at $x = (0,1)$ under every round schedule.*

Proof. A co-oriented colouring has no moving bit and meets a mixed pair at every posting, whatever the order. Conversely let (f, x) be awake in every fixed order and suppose $f \neq 0$. By (i), connectedness and $m \geq 2$, some A clock z has degree at least 2: an A site of degree 1 forces its neighbour to be an A site of odd degree, which is at least 3 unless the network is one contact. With no doubled contact, z has contacts $\{z, v\}$ and $\{z, v'\}$ with $v \neq v'$. Post these two first. Condition (ii) gives $x_z + x_{v'} = 1 + f_z = 0$ when $\{z, v\}$ goes first and $x_z + x_{v'} = 1$ when $\{z, v'\}$ goes first. So $f = 0$, and (ii) makes x a proper colouring. For the counterexamples, a lone contact and a doubled contact are awake at $(0,1)$ with two A clocks in their only orders, and their round schedules have nothing to reorder. □

Every configuration with an A clock, the joint oscillation included, fails at its occupations in some order. If occupations may depend on the order, more survives:

Proposition 12.24 (Patterns that survive reordering)**.** (a) *Let E have no doubled contact. A pattern f is awake, at bits that may depend on the order, in every reordering of E if and only if it is awake in one order and $f_z = 0$ at every hub. On paths, cycles and forests every awake pattern survives every reordering.*

(b) *On K_4 no pattern survives. Of the 720 orders of its six contacts exactly 192 admit an awake configuration, and in each of them the awake pattern is the joint oscillation $f \equiv 1$. An order admits it if and only if, at every site w, the ranks of the contacts at w, each taken at its other end, have an even sum $\Sigma(w)$.*

Proof. (a) follows from Theorem 12.22(b) and its proof. (b) On K_4, (i) makes f constant and $f = 0$ is impossible, so $f \equiv 1$. Its cycle space is spanned by the four triangles; let T_w be the triangle avoiding w. Over all six contacts the ranks sum to $4 \cdot (0 + 1 + 2) \equiv 0$, and over the three contacts at w to $(0 + 1 + 2) + \Sigma(w)$. So (ii′) for T_w reads $3 + 1 + \Sigma(w) \equiv \Sigma(w) = 0$. The count 192 is a direct exact computation over the 720 orders. □

Every site of K_4 is a hub, which is why no pattern survives every order there.

A schedule whose order varies from round to round is a different object from any fixed order. For a degree-balanced pattern (condition (i)) it adds nothing: (f, x) is awake under a round schedule if and only if it is awake, at the same bits, in every order the schedule uses. For a pattern that fails (i) it can add something.

Proposition 12.25 (An alternation keeps what no order keeps)**.** *On the path $((0,1),(0,2))$ the all-A pattern is awake in no fixed order. Under the round schedule that alternates the orders $((0,1),(0,2))$ and $((0,2),(0,1))$ round by round, starting with the first, it is awake at $x = (1,0,1)$ and at its complement $(0,1,0)$.*

Proof. At $f \equiv 1$, (i) fails at both contacts ($f_0 \deg 0 = 0$, $f_1 \deg 1 = 1$). Under the alternation the posting sequence is $a, b, b, a, a, b, b, a, \ldots$ with $a = (0,1)$, $b = (0,2)$. From $(1,0,1)$ the four postings of the first two rounds meet the mixed pairs $(1,0), (0,1), (1,0), (0,1)$ and pass through $(0,1,1), (1,1,0), (0,1,1)$ back to $(1,0,1)$; the sequence then repeats. The complement behaves the same way. □

Oscillating participants need a supplied schedule, and the class of schedules matters: fixed orders and alternations keep different configurations.

12.7 Without a supplied order

We now remove the order altogether and ask what the postings keep under every schedule, on the full carrier. For a network E let $\mathcal{N}_{\mathrm{free}}(E) = \bigcap_g \mathcal{N}(E; g)$, the intersection over all schedules of its contacts, not only round schedules. On the path the exchange of sites 1 and 2 swaps the two contacts and carries the class of one strict alternation onto the other. It fixes $\pi_\pm$ and sends the relays to the *mirror relays* $r'_+ = |110\rangle \otimes A \otimes P \otimes P$ and $r'_- = |001\rangle \otimes A \otimes P \otimes P$.

Theorem 12.26 (The path without an order)**.** $\mathcal{N}_{\mathrm{free}}(\mathit{path}) = \mathrm{span}\{\pi_+, \pi_-\}$. *The intersection over round schedules alone is the same, and so is the intersection of the classes of the two strict alternations,* $(0,1), (0,2), (0,1), \ldots$ *and* $(0,2), (0,1), (0,2), \ldots$. *The class of the second alternation is* $\mathrm{span}\{\pi_+, \pi_-, r'_+, r'_-\}$.

Proof. Every posting of either contact sends $\pi_\pm$ to $\pm$ itself with its one label and annihilates it with the others: both pairs are mixed and every clock is P. So $\pi_\pm$ lie in every $\mathcal{N}(E; g)$. The first alternation is the round order, whose class is S_{path} (Theorem 12.8); the site exchange gives the class of the second. Both are spans of orthonormal product states, so their intersection is the span of the common ones, $\pi_\pm$. Round schedules include both alternations. □

The rule of Proposition 12.3 decides every basis state under every schedule, and it pins down exactly which timing each mode needs.

Proposition 12.27 (The timing each mode needs)**.** *Let g be a schedule of the path.*

(a) $r_\pm \in \mathcal{N}(E; g)$ *if and only if g is the strict alternation starting with* $(0,1)$*;* $r'_\pm \in \mathcal{N}(E; g)$ *if and only if g is the strict alternation starting with* $(0,2)$.

(b) $o_+ \in \mathcal{N}(E; g)$ *if and only if* $g_{2r-1} = g_{2r}$ *for every* $r \geq 1$*: each contact posts in blocks of two. No round schedule keeps o_+; posting* $(0,1)$ *alone keeps it.*

Proof. (a) From r_+ at $|101\rangle$, a posting by $(0,2)$ meets the balanced dressed pair $(1,1)$ and branches, so the first posting is $(0,1)$, which leads to $|001\rangle$. There a second $(0,1)$ meets $(0,0)$ and branches, while $(0,2)$ returns to $|101\rangle$. By induction the run is live exactly under the strict alternation starting with $(0,1)$; r_- is the complement, and the mirror relays follow by the site exchange. (b) From o_+ at $|100\rangle$ either contact may post, flipping both of its bits; the other contact then meets a balanced pair and branches, while the same contact flips its bits back to $|100\rangle$. So the run is live exactly when the postings come in equal pairs. A round schedule posts two different contacts in each round, which here are the times $2r-1, 2r$. □

So the relays need exactly one strict alternation, and the joint oscillation needs the opposite timing, each contact twice in a row. The round alternation of Proposition 12.25, $a, b, b, a, a, b, \dots$, is a schedule of the second kind shifted by one posting. By Proposition 12.3 and the run in that proof, it keeps the all-A states $\psi(101; A, A, A)$ and $\psi(010; A, A, A)$ on the full carrier; by (a) it keeps none of the four relays.

Theorem 12.28 (The triangle without an order)**.** $\mathcal{N}_{\text{free}}(\textit{triangle}) = 0$*, and the intersection over round schedules alone is* 0*. In more detail:*

(a) *each of the six fixed orders of the three contacts keeps an* 8*-dimensional class, spanned by its awake configurations (Table A.11);*
(b) *two different fixed orders share a* 4*-dimensional class if they differ by exchanging two consecutive contacts of the round, and a* 2*-dimensional class otherwise;*
(c) *the three cyclic phases of one order, for example* $(0,1),(0,2),(1,2)$*, then* $(0,2),(1,2),(0,1)$*, then* $(1,2),(0,1),(0,2)$*, share nothing, and any four fixed orders share nothing;*
(d) *three is the least number of fixed orders whose classes intersect in* 0*.*

Proof. (a) For every fixed order, every basis state that is not an awake dressed configuration reaches a neutral pair or a branching posting within 4 postings, by the case analysis of Theorem 12.14, carried out for each order by direct exact computation. Proposition 12.3 then gives the span, and Theorem 12.16 gives the eight configurations of each order. (b) to (d) The classes are spans of orthonormal product states, so their intersections are spans of the common states, which are read from Table A.11: for $f = 100, 010, 001$ the six orders split into two triples with equal entries and for $f = 111$ into three pairs, every two orders agree in at least one row, no entry is shared by four orders, and the three cyclic phases agree in no row. The intersection over all schedules lies inside that of the three cyclic phases. □

On the path the timing-free class is the meet of whole and parts, $\operatorname{span}\{\pi_{\pm}\} = \mathcal{N}(E) \cap \mathcal{A}_{(0,1)} \cap \mathcal{A}_{(0,2)}$ (Proposition 12.9). Every whole-level participant of this chapter, relay or oscillation, exists only under a supplied timing.

Chapter 13

Selection and participation on every network

On every network of native contacts, posting in its round order, the postings by themselves keep exactly the product states that the parity law of Chapter 12 allows: every touched clock dressed, the flip pattern balanced across every contact and consistent around every cycle. No sure-subject test enters the definition, and no table is consulted in the proof. The selection is decided site by site, by a classical run of bits and pole lines, and yet what it keeps is a property of the whole network. Two further facts make that precise. The postings never move weight into the kept class or out of it, so composing contacts selects nothing the genesis did not hold. And each contact's own test, which watches only its own pair while every contact posts, already cuts out the class of its whole connected component.

Chapter 12 worked with configurations we chose to be dressed and with the path and the triangle. Here the question is what the postings keep when nothing is chosen, on any number of sites.

13.1 The network carrier, its postings and its classes

We use the network carrier, contacts, networks, posting sequences and product pole states of Section 12.1; an entry of E may repeat, and two entries may join the same pair. Two results of that chapter carry everything here: the one-posting Lemma 12.1, and Proposition 12.3, by which the kept class of every network under every posting sequence is spanned by the live basis states (Definition 12.2).

A record may read a posting's label, leave it unread, or read only whether it lies in a set S of labels. The effect of a finite record is the Born pullback of the identity through its postings, a read posting contributing $V_j^{e*} X V_j^e$, an unread one $\sum_j V_j^{e*} X V_j^e$, a coarse one the sum over S; its probability in a density ρ is $\mathrm{tr}(\rho E)$.

Definition 13.1 (The never-neutral classes)**.** For a network E and a posting sequence g of E, $\mathcal{N}(E; g)$ and $\mathcal{N}(E)$ are the kept classes of Section 12.1, $\mathcal{N}_L(E)$ is the round-order test over words of length at most L, and $\mathcal{N}_{\mathrm{free}}(E) = \bigcap_g \mathcal{N}(E; g)$ is the class kept by every posting sequence of E. For a contact e, the *own class*

$$\mathcal{N}_e(E; g) = \{\phi \,:\, Q^e_{\mathrm{ne}} V_w \phi = 0 \text{ for every word } w \text{ along } g\}$$

tests only e's neutral sector while the contacts of g post; $\mathcal{N}_e(E)$ is its round-order class. S_E denotes the orthogonal projection onto $\mathcal{N}(E)$.

As in Chapter 12, the states of $\mathcal{N}(E)$ are the network's *participants*; that is a reading of the class, and no theorem below uses the name.

13.2 The round order keeps the parity-law states

Recall from Chapter 12 that a flip pattern $f \in \mathbb{Z}_2^n$ and bits x are *awake* for E when every posting of the round order meets a mixed pair, and that this holds exactly when

$$\text{(i) } f_u \deg u = f_v \deg v \text{ for every contact } (u,v), \qquad \text{(ii) } x_{u_i} + x_{v_i} = 1 + f_{u_i} v_i(u_i) + f_{v_i} v_i(v_i) \text{ for every } i, \tag{13.1}$$

in $\mathbb{Z}_2$. Call a classical state $q = (x, \ell)$ a *parity-law state* of E when every touched clock is dressed and $(f(\ell), x)$ is awake; untouched sites carry any bit and any line. Three lemmas about live runs carry the proof. The first holds for every posting sequence.

Lemma 13.2 (An undressed clock in a live run)**.** *Let q be live under a posting sequence g of E, and let the clock of c be undressed in q. Until the first posting through c, every contact of E through c is balanced. That first posting is a dressing: its pair is balanced and both of its clocks are undressed. Afterwards c stays dressed, and every posting through c is awake on two dressed clocks.*

Proof. Before its first posting c keeps its undressed line, and a mixed contact through it would be neutral. At its first posting the pair is therefore balanced, and a deterministic posting at a balanced pair has both clocks undressed, by Lemma 12.1. The dressing makes c dressed. A deterministic posting through a dressed clock cannot be at a balanced pair, which would branch, so it is awake, and awake postings keep lines. □

The next two lemmas assume only that postings are deterministic. They do not ask whether a contact stands neutral between its postings; Section 13.5 uses that economy.

Lemma 13.3 (Two awake rounds give the parity law)**.** *Let every touched clock of q be dressed and let the first $2m$ postings of the round order be deterministic at q. Then $(f(\ell), x)$ is awake.*

Proof. Each of these postings touches dressed clocks, so it is awake and moves each of its bits by that site's flip bit. Just before posting i of round $r \in \{0,1\}$, site z has been touched $r \deg z + v_i(z)$ times and holds $x_z + f_z(r \deg z + v_i(z))$. With $\alpha = x_{u_i} + f_{u_i} v_i(u_i)$, $\beta = x_{v_i} + f_{v_i} v_i(v_i)$, $\eta = f_{u_i} \deg u_i$ and $\theta = f_{v_i} \deg v_i$, the pair is mixed at $r = 0$ and $r = 1$, so $\alpha + \beta = 1$ and $\alpha + \beta + \eta + \theta = 1$. These are (ii) and (i) of (13.1). □

Lemma 13.4 (No birth in the round order)**.** *If the first $3m$ postings of the round order are deterministic at q, every touched clock of q is dressed.*

Proof. Suppose a touched clock is undressed, and let $t_0 \le m$ be the first posting through it, by $e_i = (u, v)$. As in Lemma 13.2, this posting is a dressing of two undressed clocks on a balanced pair. It dresses them with flip bits f_u, f_v and moves their bits by $1 + f_u$ and $1 + f_v$ (p becomes P with its bit flipped, a becomes A with its bit kept). Every other entry posts once before e_i posts again at $t_0 + m$, touching u another $\deg u - 1$ times and v another $\deg v - 1$ times; each of these postings is deterministic through a dressed clock, hence awake. At $t_0 + m$ the pair therefore differs by

$$(1 + f_u) + (1 + f_v) + f_u(\deg u - 1) + f_v(\deg v - 1) = f_u \deg u + f_v \deg v.$$

By time m every touched site has been posted once and every touched clock is dressed, and the round order from time m is the round order again. Lemma 13.3, applied at time m to postings before $3m$, gives (i), so the difference is 0. Then e_i meets a balanced pair of dressed clocks at $t_0 + m \le 2m$, and that posting branches: a contradiction. □

Doubled contacts and repeated entries change nothing here: the counting is over list positions.

Theorem 13.5 (Selection on every network)**.** *For every network E, with no hypothesis on E,*

$$\mathcal{N}(E) = \operatorname{span}\{\psi(x,\ell) \,:\, (x,\ell) \text{ is a parity-law state of } E\}.$$

A basis state lies in $\mathcal{N}(E)$ exactly when its touched clocks are dressed and its flip pattern is awake at its bits. No basis state with an undressed touched clock lies in $\mathcal{N}(E)$, and words of length at most $3m$ cut out the class: $\mathcal{N}_{3m}(E) = \mathcal{N}(E)$.

Proof. For the empty network both sides are $\mathscr{H}_n$. Otherwise, by Proposition 12.3 with the round order, we must show that q is live exactly when it is a parity-law state. A parity-law state has every posting awake, hence deterministic, and no contact is ever neutral, since neutrality needs an undressed clock on a contact. Conversely, a live q has deterministic postings throughout, so Lemma 13.4 dresses its touched clocks and Lemma 13.3 makes its pattern awake. The two lemmas say more: if the first $3m$ postings are deterministic at q, then q is a parity-law state. So the run of any other q fails at some time $\tau < 3m$, and Proposition 12.3 kills $\psi(q)$ with words of length at most $\tau + 1 \le 3m$. □

The earlier classes are special cases, and the parity law's corollaries become statements about what the postings keep.

Corollary 13.6 (What the class looks like)**.** *(a) For one contact, $\mathcal{N}(E)$ is spanned by $P \otimes P$ and $A \otimes A$ at the two mixed occupations: the aligned sector of Theorem 7.11.*

(b) On the path $((0,1),(0,2))$ it is four-dimensional: the co-oriented pairs, every clock P at $x = (1,0,0)$ and $(0,1,1)$, and the relays*, shared clock A and outer clocks P at $x = (1,0,1)$ and $(0,1,0)$. On the triangle $((0,1),(0,2),(1,2))$ it is eight-dimensional: every clock dressed with an odd number of A clocks, at the two bit vectors each such pattern allows.*

(c) Some state with every clock P lies in $\mathcal{N}(E)$ if and only if $G(E)$ is 2-colourable.

(d) If E lists the contacts of a cycle of $n_c \ge 2$ sites in any order, a dressed pattern f lies in $\mathcal{N}(E)$ at some bits if and only if $\sum_z f_z \equiv n_c \pmod 2$.

Proof. (a) One contact has $\deg u = \deg v = 1$, so (i) reads $f_u = f_v$ and (ii) reads $x_u \ne x_v$. (b) On the path, (i) asks $2f_0 = f_1 = f_2$, so the outer clocks are P, and (ii) gives $x_0 + x_1 = 1$, $x_0 + x_2 = 1 + f_0$. The triangle is the case $n_c = 3$ of (d). (c) and (d) are the parity-law corollaries of Chapter 12 read through Theorem 13.5. □

How long the postings take. The bound $3m$ can be sharpened by the position of the last possible dressing. A position i of the round order is *fresh* when no earlier entry of the round touches either site of e_i; the first position is always fresh. Only a fresh position can dress, since a clock touched earlier in the round is already dressed or has already failed.

Theorem 13.7 (The depth of selection)**.** *Let E be nonempty, and let every fresh position be at most T. Then $\mathcal{N}_{2m+T}(E) = \mathcal{N}(E)$. On the star $((0,1),(0,2),\dots,(0,m))$ and on the chain $((0,1),(1,2),\dots,(m-1,m))$, $m \geq 1$, the least L with $\mathcal{N}_L(E) = \mathcal{N}(E)$ is exactly $2m+1$.*

Proof (outline of the upper bound; the lower bound in full). A state whose touched clocks are dressed and that is not a parity-law state fails within two rounds (Lemma 13.3). A dressing at a fresh position $t_0 \leq T$ moves each of its two bits by $1 + f$, f that site's new flip bit, and each of the two sites then takes $2\deg - 1$ further awake postings before the same entry posts again two rounds later; each bit moves by $1 + f + f(2\deg - 1) \equiv 1$, so the pair, balanced at the dressing, is balanced again at $t_0 + 2m$, and that posting branches. Proposition 12.3 turns each failure into a kill by words of length at most $2m + T$.

For the lower bound on the star, take the centre on p, the first leaf on a, every other leaf on P, and every bit 0. The first posting dresses the centre to P with bit 1 and the first leaf to A with bit 0. The centre's bit never moves again. The other leaves meet mixed pairs of P clocks. The first leaf's bit flips at posting $m + 1$, and at posting $2m + 1$ the first contact meets the balanced pair $(1, 1)$ of two dressed clocks and branches. So the state lies in $\mathcal{N}_{2m}(E)$ but not in $\mathcal{N}(E)$: here $f_0 \deg 0 = 0$ while $f_1 \deg 1 = 1$. On the chain the witness is the first site on a, the second on p, every other site on P, with bits $0, 0$ on the first two sites and $x_c = c \bmod 2$ beyond: the same dressing, the same return after two rounds. For $m = 1$ and $m = 2$ these are the depths 3 and 5 of Chapters 7 and 12. □

13.3 Without a supplied order

The round order is supplied. Take it away and ask what every posting sequence keeps. On a connected network with no doubled contact and at least two contacts, the answer is the static part of the class: co-oriented persistent currents if the network is bipartite, and nothing otherwise.

Theorem 13.8 (What no order removes)**.** *A* co-oriented colouring *of E is a classical state with every touched clock P and bits that differ across every contact.*

(a) On every network, every co-oriented colouring is live under every posting sequence of E.
(b) Let E be connected, with no doubled contact and $m \geq 2$. Then $\mathcal{N}_{\mathrm{free}}(E)$, the intersection of $\mathcal{N}(E')$ over all reorderings E' of E, and the intersection of $\mathcal{N}(E; g)$ over all sequences whose every round is a permutation of E, are all equal to the span of the co-oriented colourings. In particular $\mathcal{N}_{\mathrm{free}}(E) = 0$ exactly when $G(E)$ is not bipartite.

Proof. (a) At a co-oriented colouring every contact meets a mixed pair of two P clocks, which is awake and moves nothing. So every posting returns the same state, and nothing is ever neutral.

(b) Each of the three intersections lies between the span of the co-oriented colourings, by (a), and the intersection over reorderings, since a fixed order is one sequence and its rounds are permutations. By Proposition 12.3 each $\mathcal{N}(E')$ is spanned by a set of basis states, so their intersection is spanned by the basis states common to all of them, and by Theorem 13.5 such a state is dressed with (f, x) awake in every fixed order of E. Under the hypotheses of (b), Theorem 12.23 gives $f = 0$ with x a proper 2-colouring. □

Each hypothesis of (b) is needed (Theorem 12.23), and alternating the two orders of the path round by round keeps an all-A state that no fixed order keeps (Proposition 12.25): varying the order is not intersecting over orders.

13.4 Composition does not select

The selection theorem says which states are kept. The next theorem says that the postings never move weight into the kept class or out of it, whatever is recorded and whatever is left unread. A *round word* is $W = V_{j_m}^{e_m} \cdots V_{j_1}^{e_1}$, one label for each posting of one round.

Theorem 13.9 (The kept class is invariant in both directions)**.** *For every nonempty network E in its round order and every round word W,*

$$S_E W = W S_E.$$

Proof. Let $\mathcal{S}$ be the set of parity-law states and $q \in \mathcal{S}$. Its run is live, every posting of the first round is awake with label $\hat{\jmath}_i(q)$, and the round ends at $R(q) = (x + f\deg, \ell)$, the bits moved site by site. $R(q)$ is again a parity-law state, being the start of the second round of a live run, and $R(R(q)) = q$, since $2f\deg = 0$. By (D), $W\psi(q) = \pm\psi(R(q))$ when $j_i = \hat{\jmath}_i(q)$ for every i, and $W\psi(q) = 0$ otherwise. Hence W maps $\mathcal{N}(E)$ into itself: $WS_E = S_E W S_E$.

Now run the round backwards. Let $p = R(q)$, so that the first round of the run from p passes through states $p = p_0, p_1, \dots, p_m = q$, each with every touched clock dressed. Apply $W^* = V_{j_1}^{e_1 *} \cdots V_{j_m}^{e_m *}$ to $\psi(q)$. At the step that applies $V_{j_i}^{e_i *}$ the vector is a multiple of $\psi(p_i)$, whose two lines on e_i are dressed. By (U) exactly one label has a nonzero adjoint there, and by (D) the label $\hat{\jmath}_i(p)$, written by the forward posting from p_{i-1}, does, returning $\pm\psi(p_{i-1})$. So $W^*\psi(q)$ is $\pm\psi(p)$ or 0, and W^* also maps $\mathcal{N}(E)$ into itself: $W^* S_E = S_E W^* S_E$, that is, $S_E W = S_E W S_E$. Together, $S_E W = W S_E$. □

Corollary 13.10 (No selection by composition)**.** *Let ρ be a density on $\mathcal{H}_n$ and let the record consist of whole rounds, each label read, unread or coarsely read.*

(a) The kept weight $\operatorname{tr}(\rho S_E)$ *is the same after any number of rounds with every label unread. A genesis with no kept weight never acquires any.*
(b) Along every record h of k rounds with read labels, $S_E W_h \rho W_h^* S_E = W_h(S_E \rho S_E) W_h^*$: the kept block evolves by itself.
(c) The record law is the sum of the laws of the two blocks: $\operatorname{tr}(\rho E) = \operatorname{tr}(S_E \rho S_E E) + \operatorname{tr}((I - S_E)\rho(I - S_E)E)$ *for every such record effect E.*

Proof. Every product of round words commutes with S_E, hence so does every record effect built from them. Completeness of each contact's instrument gives $\sum_W W^* W = I$ over the round words, so $\sum_W W^* S_E W = \sum_W W^* W S_E = S_E$, which is (a) in the Heisenberg picture. (b) is the commutation applied on both sides. For (c) the cross terms vanish: $\operatorname{tr}(S_E \rho (I - S_E)E) = \operatorname{tr}(\rho(I - S_E)E S_E) = \operatorname{tr}(\rho(I - S_E)S_E E) = 0$. □

The kept weight is therefore fixed at genesis, exactly as the aligned weight of one cell is (Lemma 7.12).

The commutation holds posting by posting, which is stronger. Let S_t be the projection onto the states kept from time t of the round order; these are the parity-law states of the network whose list is rotated by t positions, and $S_0 = S_E$. The posting g_{t+1} of the round order satisfies $S_{t+1}V_j^{g_{t+1}} = V_j^{g_{t+1}}S_t$ for every label j, so every word of L postings from time t satisfies $S_{t+L}W = WS_t$, and a word of whole rounds commutes with S_E. In the backward step the awake predecessor of a state kept from time $t+1$ is mixed at the posting contact, because that contact posts again two rounds later and each of its sites is posted an odd number of times in between; it is therefore kept from time t. The round order cannot be dropped, since a timing whose dressing posting creates a participant exists (Proposition 13.21).

13.5 Each contact's own test sees its component

A contact can be tested in two ways, and they must not be confused. Its *isolated* class is the class of Theorem 7.11, the contact posting alone on its pair. Its *own* class $\mathcal{N}_e(E; g)$ tests only its own neutral sector while every contact of the network posts. Isolated classes and the network's class are not nested (Proposition 12.9). The own class always contains the joint one, $\mathcal{N}(E; g) \subseteq \mathcal{N}_e(E; g)$ for $e \in E$, since it tests less. The question is whether it is larger.

On a shared site it is not. Write *old* for $(0, 1)$ and *new* for $(0, 2)$ on the path, posting old, new, old, ….

Theorem 13.11 (Own tests on the path and the triangle)**.** *(a) On the path in alternation, old first, $\mathcal{N}_{\text{old}}(E) = \mathcal{N}_{\text{new}}(E) = \mathcal{N}(E)$, the four-dimensional class of Corollary 13.6(b). For either contact the least word length that cuts out its own class is* 5*.*

(b) On the triangle in its round order, each of the three contacts' own classes is the eight-dimensional class of the triangle, and the least cutting length is 4 *for each. Under each of the six fixed orders of the three contacts, each contact's own class is that order's joint class.*

Proof (mechanism). The containment $\supseteq$ holds since the own test tests less. For $\subseteq$, every basis state outside the joint class is killed by a chain of at most five postings (four on the triangle) that follows one branch per posting into the tested contact's neutral sector; the Born adjoint of one labelled posting returns a pole line, and where a branch lands on an undressed diagonal $p \pm a$ both pole components are followed. On the path the chain runs as follows. After a non-neutral old posting both old clocks are dressed. A balanced neighbour pair then branches the shared clock into the undressed plane with a bit that can be chosen to disagree with site 1, which makes the old pair neutral. A mixed neighbour pair with site 2 undressed splits site 2's bit, and the branch whose bit equals the shared bit meets a balanced pair at the neighbour's next posting, which is the first case. The chains are finite and are checked by direct exact computation over the 512 pole-line basis states of three sites; explicit witnesses, such as the path witness of Theorem 13.7, show that the lengths 5 and 4 are least. □

So no state on the path keeps one contact off its neutral sector while the other is exposed. In the language of noiseless subsystems [25], a subsystem stays quiet next to a noisy environment when the noise acts on a factor it does not share; here the neighbour's postings act on the shared site, which the old contact's sector reads, and the separation never occurs.

The general theorem needs one more notion. At a posting through a site c, call c *free* when it is not deterministic there: its clock is dressed at a balanced pair, or undressed at a mixed pair.

A free clock reaches both output bits, each with Born weight $\frac{1}{2}$, and lands on an undressed line $p \pm a$. A posting that is not deterministic has a free site.

Lemma 13.12 (Free clocks reach every contact)**.** *Let e be a contact and let every contact of E post at least once in every M consecutive postings of g. Let q be a classical state at time t, and let the posting g_{t+1} pass through a site c that is free there and is joined to e's first site by a path of at most d contacts of E. Then every vector that passes e's own test along $g_{t+1}, g_{t+2}, \ldots$ on words of length at most $1 + dM$ is orthogonal to $\psi(q)$. If g is only fair, every contact of E posting at arbitrarily late times, the same holds with some finite length in place of $1 + dM$.*

Proof (outline). Induction on d. By Lemma 12.1 and its Born adjoints, a basis state is killed from time t as soon as, for one choice of output bits at the free sites of the posting at t, every pole component of the resulting state is killed from $t + 1$. A strategy chooses each free clock's bit to fix the status of its next posting. If its next partner lies nearer to e, the bit is chosen to free that partner: a balanced pair against a dressed partner, a mixed pair against an undressed one. Otherwise the bit is chosen so that the clock meets a mixed pair while undressed and stays free. The partner's bit at that meeting is predicted by following the partner through its own deterministic postings; if the partner is itself free in between, it chooses its bit later, knowing the first clock's choice. Hence a free clock at distance $d + 1$ stays free until a contact joins it to a site at distance d, which happens within M postings, and frees that site. At distance 0 the free clock sits on e's first site, and choosing its bit to make e's pair mixed while it is undressed puts the state into e's neutral sector. □

Theorem 13.13 (Participation is joint on every connected network)**.** *Let E be connected and $e \in E$. Then $\mathcal{N}_e(E) = \mathcal{N}(E)$. If every site lies within D contacts of e's first site, words of length at most $3m + 1 + Dm$ cut out $\mathcal{N}_e(E)$; in particular $3m + 1 + nm$ always suffices.*

Proof. Call q *own-live* when e is never neutral along its run and every posting is deterministic. As in Proposition 12.3, with Lemma 13.12 handling the first non-deterministic posting (it has a free site, at finite distance from e since E is connected), $\mathcal{N}_e(E; g)$ is the span of the own-live basis states for every fair sequence g of E. In the round order, an own-live state has $3m$ deterministic postings, so Lemmas 13.4 and 13.3 make it a parity-law state; and parity-law states are live, hence own-live. So the two spans coincide, and Theorem 13.5 identifies them with $\mathcal{N}(E)$. For the depth, the run of a state that is not a parity-law state fails within $3m$ postings, and the round order has $M = m$, so Lemma 13.12 adds at most $1 + Dm$. □

By Theorem 13.9, every round word and its adjoint therefore carry each contact's own class $\mathcal{N}_e(E) = \mathcal{N}(E)$ of a connected network into itself. A network that is not connected separates its participants, and nothing else does.

Theorem 13.14 (Components)**.** *Let E be nonempty, $e \in E$, and let E_e be the sublist of the contacts in e's component of $G(E)$. Then $\mathcal{N}_e(E) = \mathcal{N}(E_e)$: the parity-law states of e's component, with any bit and any line at every other site. If $\mathcal{N}(E_e) \neq 0$, then $\mathcal{N}_e(E) = \mathcal{N}(E)$ exactly when every contact of E lies in e's component.*

Proof (outline). Postings off the component act on other sites. Kills are carried back through them with any output, and the component's postings, read in the order of E, are the round

order of E_e; Theorem 13.13 applies to E_e. For the last clause, if some contact lies outside, take a parity-law state of the component and give that contact a mixed pair with an undressed clock: e's test keeps the state, and the joint test fails at the empty word. □

On the path and the triangle these theorems recover the class equalities of Theorem 13.11 without tables, and they apply to every star and chain. We read the theorems as follows: a participant's boundary does not run through a connected network of native contacts, since whichever contact asks, its own test certifies the whole component, and only disconnection separates participants. This reading identifies a contact's own never-neutral test with what it is for that contact to participate; it makes no network or component a single subject.

13.6 The record decides on the path

Theorem 13.11 concerns subspaces. On the path the record also delivers the verdict, at a geometric rate, from either participant's own labels. Write $W_{jj'} = V_{j'}^{\mathrm{new}} V_j^{\mathrm{old}}$ for one round, call a label *awake* when $j_2 \neq 0$ (the posting started from a mixed pair), and define the effect of "the old contact posts an awake label at each of k rounds, its neighbour unread" by

$$A_k = \Phi_{\mathrm{old}}^k(I), \qquad \Phi_{\mathrm{old}}(X) = \sum_{j \text{ awake}} \sum_{j'} W_{jj'}^* X W_{jj'},$$

and the neighbour's effect $B_k = \Phi_{\mathrm{new}}^k(I)$, where Φ_{new} sums over every old label j and over the awake neighbour labels j'.

Theorem 13.15 (Either participant's record certifies the composite)**.** *Let $S = S_E$ on the path and $r = 1 - 2^{-17}$. For every vector ϕ and every k,*

$$0 \leq \langle \phi, A_k \phi \rangle - \|S\phi\|^2 \leq r^{\lfloor k/3 \rfloor} \, \|(I - S)\phi\|^2,$$

every matrix entry of $A_k - S$ has modulus at most $4r^{\lfloor k/3 \rfloor}$, and the same holds for B_k. Hence $A_k \to S$ and $B_k \to S$; for every density, the probability of the awake record tends to $\mathrm{tr}(\rho S)$; and $\phi \in \mathcal{N}(E)$ exactly when $\langle \phi, A_k \phi \rangle = \|\phi\|^2$ for every k. The neighbour's effect is one unread old posting followed by the image of the old effect under the exchange σ of sites 1 and 2: $B_k = \sum_j V_j^{\mathrm{old}*} \sigma(A_k) V_j^{\mathrm{old}}$, and the same unread posting carries $\sigma(S)$ back to S.

Proof (mechanism). One awake old posting is the mixed-input projection on the old pair. Every basis state outside $\mathcal{N}(E)$ meets a balanced old pair at the start of some round within three awake rounds, along a branch of Born weight at least 2^{-8}; with Parseval over the $512 = 2^9$ basis states this gives a loss of at least 2^{-17} of the off-class weight every three rounds, by direct exact computation. Theorem 13.9 keeps the kept block and the rest apart, so the losses iterate. The neighbour's statement follows from the regrouping and from the identity for $\sigma(S)$, which holds because one unread old posting carries the four kept states onto their mirror images and no other basis state onto them. □

The constant is a bound, not the sharp rate. The present carries the same verdict. For a density ρ on the path, let $m_k(\rho)$ be the mean over the old participant's records of k rounds, weighted by their probabilities, of $1 - \mathrm{tr}\, \sigma_h^2$, where σ_h is the normalized conditional state given the old record h with the neighbour's labels unread.

Theorem 13.16 (Off the kept class the present stays mixed)**.** *For every positive semidefinite ρ on the path and every $k \geq 3$,*

$$m_k(\rho) \;\geq\; 2^{-26}\,\mathrm{tr}\big(\rho(I - S)\big).$$

Consequently a density's mean mixedness tends to 0 *if and only if $S\rho S = \rho$.*

Proof (mechanism). After an unread neighbour posting the conditional state is mixed by at least $\mathrm{tr}(\tau(F - F^2))$, where τ is the state before the posting and F is the effect of the neighbour's posting landing on a mixed output; F is diagonal on pole-line products, because mixed and balanced outputs have orthogonal ranges for every label. Summed over the old records this bound survives every later round. A direct exact computation over the basis states shows that within two rounds every state off the kept class reaches a neighbour posting that spreads between mixed and balanced outputs, with constant 2^{-26}. The converse direction, that kept states purify, is the result of Chapter 16. □

13.7 Timing: where a contact's own class can be larger

The own-class theorems so far used the round order. Under other timings the own class can exceed the joint class, and under fair timing it can do so in one way only: a contact that posts before its neighbour can repair a clock that the neighbour's test would already have found neutral. Under unfair timing there is a second way, a neighbour that never posts and so leaves its private clock untested. We work on the path with the sequences of Table 13.1.

name	postings from time 1	fair
alternation	old, new, old, new, ...	yes
new first	new, old, new, old, ...	yes
round alternation	old, new, new, old, old, new, new, old, ...	yes
old only	old, old, old, ...	no
two neighbour postings	old, new, old, new, then old at every later time	no
old then pairs	old, new, old, old, new, new, old, old, new, new, ...	yes

Table 13.1. Posting sequences on the path; old is $(0, 1)$, new is $(0, 2)$. A sequence is fair when both contacts post at arbitrarily late times.

Proposition 13.17 (Own and joint classes under four timings)**.** *On the path:*

(a) New first: both contacts' own classes equal the joint class, spanned by the images of the four kept states under the exchange of sites 1 *and* 2*.*

(b) Round alternation: both contacts' own classes equal the joint class, spanned by the two co-oriented pairs and the all-A states at $x = (1, 0, 1)$ and $(0, 1, 0)$.

(c) Old only: the old contact's own class is spanned by the basis states with a mixed old pair whose two clocks are both P or both A, site 2 *arbitrary, and has dimension* 32*. The joint class has dimension* 20*: those of the former whose site-*2 *clock is dressed, and those whose site-*2 *clock is undressed while the shared clock is P and $x_0 = x_2$. The state with bits $(1, 0, 0)$ and lines (P, P, p) lies in the first and not the second.*

(d) Two neighbour postings, a sequence that is not fair: the old contact's own class equals the joint class, spanned by the two co-oriented pairs.

Proof. (c) by hand. When only the old contact posts, the neighbour's pair changes only through site 0. The old test keeps exactly its isolated class, the aligned sector of the old pair (Corollary 13.6(a)), with site 2 untouched: $2 \cdot 2 \cdot 8 = 32$ states. The joint test also needs the neighbour's pair never neutral. A dressed site 2 is never neutral (16 states); an undressed site 2 needs the neighbour's pair balanced at every time, hence a shared clock that never moves, P, with $x_0 = x_2$ (4 states). (a), (b) and (d) are direct exact computations over the 512 basis states, by the branch-following kill of Theorem 13.11. □

Two observations follow. Fairness is not what makes participation joint: in (d) the neighbour posts twice and stops, and already the old contact's class is joint. And a neighbour that stops at the outset does leave a gap, (c), because its private clock is then never tested by any posting. The next example shows that fairness does not exclude a gap either.

Proposition 13.18 (A fair timing in which the own class is larger)**.** *Under the fair sequence "old then pairs" of Table 13.1, the state with bits* $(0, 0, 1)$ *and lines* (a, a, A) *lies in the old contact's own class and not in the joint class. The neighbour's own class equals the joint class, so the two participants' own classes differ.*

Proof. At genesis the neighbour's pair $(x_0, x_2) = (0, 1)$ is mixed with an undressed shared clock: it is neutral, so the state fails the joint test at the empty word. The old pair $(0, 0)$ is balanced and not neutral. The first posting is old: a dressing that turns both a clocks into A with their bits kept, leaving every clock A at $(0, 0, 1)$. From then on every posting meets a mixed pair of A clocks and flips both bits: new gives $(1, 0, 0)$, old, old give $(0, 1, 0)$ and $(1, 0, 0)$, new, new give $(0, 0, 1)$ and $(1, 0, 0)$, and the pattern repeats with period four. Nothing is neutral after time 0 and every posting is deterministic, so the old test keeps the state. The neighbour posts second, and the next theorem gives its class. □

The gap is confined to the stretch before the other contact first posts.

Theorem 13.19 (Gaps close once the others have posted)**.** *(a) On the path, for every fair sequence g and each contact c: if $g_1 \neq c$, then $\mathcal{N}_c(E; g) = \mathcal{N}(E; g)$. For every m with $g_{m+1} \neq c$, every $\phi \in \mathcal{N}_c(E; g)$ and every word w of the first m postings, $V_w\phi$ lies in the joint class of the remaining sequence $g_{m+1}, g_{m+2}, \ldots$.*

(b) On the triangle, for every fair sequence g and each contact c with $g_1 \neq c$, $\mathcal{N}_c(E; g) = \mathcal{N}(E; g)$.

(c) On every connected network, for every fair sequence g of E, every contact e and every m such that each contact of E posts among the first m postings: for every $\phi \in \mathcal{N}_e(E; g)$ and every word w of the first m postings, $V_w\phi$ lies in the joint class of $g_{m+1}, g_{m+2}, \ldots$.

Proof (outline). By Lemma 13.12 and the argument of Theorem 13.13, the own class is spanned by the own-live basis states. An own-live state differs from a live one only by a contact that stands neutral while every posting stays deterministic, and that needs an undressed clock that has not yet been posted. Once every contact has posted, every touched clock on the run is dressed, and own-live and live agree from then on; this is (c). For the first halves of (a) and (b), take three sites and a tested contact that does not post first. The site k missed by the

first posting lies on the tested contact, and every contact through k also meets a site of the first posting, whose clocks are dressed from time 1 on. An undressed clock can be posted deterministically only beside another undressed one, so the clock at k is dressed from the start. A contact neutral at genesis then needs an undressed clock on the first posting's pair, which is therefore a balanced pair of two undressed clocks. The tested contact is balanced at genesis, since it is not neutral and one of its clocks is undressed; so the neutral contact, which joins k to the first posting's pair, is balanced too, a contradiction. After time 1 every clock that has been posted is dressed, and so is k, so no contact can stand neutral later. Hence on three sites own-live states are live. Finally, if $\phi \in \mathcal{N}_c(E; g)$ then $V_w\phi$ passes c's test along the shifted sequence $g_{m+1}, g_{m+2}, \dots$, which is fair and begins with the other contact when $g_{m+1} \neq c$; the first half of (a) applied to it gives the second half. □

The gap of Proposition 13.18 is therefore of one kind only, a genesis repair made before the neighbour first posts, and the repairing contact need not touch the failing one.

Proposition 13.20 (A repair two contacts away)**.** *On the chain* $((0,1),(1,2),(2,3))$*, let the middle contact post first and then the contacts post in the order* $(2,3),(1,2),(0,1)$ *repeated; this sequence is fair. The state* q *with bits* $(0,0,0,1)$ *and lines* (P,a,a,P) *lies in the own class of* $(0,1)$ *but not in the joint class, although* $(0,1)$ *shares no site with* $(2,3)$*, the contact that is neutral at genesis.*

Proof. At genesis $(2,3)$ has the mixed pair $(0,1)$ with an undressed clock at site 2: neutral. The first posting, $(1,2)$, dresses the two a clocks to A with bits kept, giving lines (P,A,A,P) at $(0,0,0,1)$, a parity-law state for the order $(2,3),(1,2),(0,1)$. Its run is $(0,0,1,1)$, $(0,1,0,1)$, $(0,0,0,1)$ and repeats, every posting awake. The pair of $(0,1)$ is balanced at genesis, so $(0,1)$ is never neutral. □

A fixed round order allows no such repair: Theorem 13.13 holds for every connected network because three rounds of deterministic postings already force the parity law. Timing can do more than enlarge an own class. It can create a participant.

Proposition 13.21 (A posting out of turn creates a participant)**.** *On the path, let* q *have bits* $(0,0,0)$ *and lines* (a,p,P)*, and let the old contact post twice before the alternation continues: old, old, new, old, new, ... Then* $\psi(q)$ *lies in the never-neutral class of this sequence but not in* $\mathcal{N}(E)$*. Its first posting writes the label* $(0,0,1)$ *and leaves the relay with bits* $(0,1,0)$ *and lines* (A,P,P)*, after which the sequence is the round order and keeps the relay.*

Proof. At genesis nothing is neutral: both pairs are balanced. The first posting dresses the balanced old pair: a becomes A with bit 0, p becomes P with bit 1, and the transition $(0,0) \to (0,1)$ writes $j = (y_1 - x_2, x_1 - x_2, y_2 - x_1) = (0,0,1)$. The resulting relay lies in $\mathcal{N}(E)$ (Corollary 13.6(b)), and from time 1 the sequence is the round order. In the round order itself, the second posting, new, would meet the balanced pair $(0,0)$ of the dressed clocks A and P and branch; and q has undressed touched clocks, so $q \notin \mathcal{N}(E)$ by Theorem 13.5. □

The record names the birth: in a live run the dressings are exactly the postings whose label has middle current $j_2 = 0$.

Chapter 14

Timing, gates and births

The round order of Chapter 13 keeps exactly the parity-law states, and nothing is born under it: a beginning with an undressed clock always fails. Every other timing changes this in one of two ways. It can dress undressed clocks in balanced pairs and so create participants, and it can keep states that the round order loses.

One mechanism runs through all of it. On dressed clocks, bits equal to 1 on A clocks behave as tokens (Lemma 14.3). They move inside a cluster of A clocks, enter only from a P neighbour of bit 1 and leave only to a P neighbour of bit 0. A cluster with no P neighbour conserves its token number. Births, fair keeping, gated participation, random laws and the admission of a beginning by a round order are all statements about these tokens.

We use the notation of Chapters 12 and 13: a network $E = (e_1, \dots, e_m)$ on sites $0, \dots, n-1$, classical states $q = (x, \ell)$ with product states $\psi(x, \ell)$, the lines p, a (undressed) and P, A (dressed, flip bits 0 and 1), posting sequences g, live states (Definition 12.2), the classes $\mathcal{N}(E; g)$ (Definition 13.1), and parity-law states (Section 13.2). A posting sequence is *fair* when every contact of E posts at arbitrarily late times. By Proposition 12.3, $\psi(q) \in \mathcal{N}(E; g)$ exactly when q is live under g, and $\mathcal{N}(E; g)$ is spanned by such $\psi(q)$; every statement below about live classical states is therefore a statement about never-neutral classes.

14.1 Births under a prefix

A posting out of turn can create a participant (Proposition 13.21). Exactly which beginnings are born this way has a three-part answer: no contact may be neutral at the start, the undressed clocks must pair up along contacts in a workable order, and the dressed state must be able to move its tokens to a parity-law state.

Definition 14.1 (Prefixes, dressings and births)**.** Let E be a nonempty network and $q = (x, \ell)$ a classical state.

(a) For a finite list $\pi = (\pi_1, \dots, \pi_k)$ of contacts of E, the *prefix sequence* g^π posts $\pi_1, \dots, \pi_k$ and then the round order from e_1: $g^\pi_t = \pi_t$ for $t \le k$ and $g^\pi_{k+t} = e_i$ for $t \equiv i \pmod m$.

(b) $U(q)$ is the set of touched sites whose clock is undressed. The state q is *born* if $U(q) \neq \emptyset$ and q is live under g^π for some prefix π; the state q_k at the end of the prefix is then a *born participant*.

(c) The *dressing* $q^\sharp$ dresses every clock of $U(q)$ as its first posting would (Lemma 12.1(D)): p becomes P with its bit flipped, a becomes A with its bit kept. Everything else is unchanged.

(d) A posting at q is *awake* if its pair is mixed and both of its clocks are dressed. We write $q \xrightarrow{\text{aw}} q'$ if a finite sequence of awake postings by contacts of E, in any order, leads from q to q'.

(e) A *dressing order* of q is a fixed-point-free involution μ of $U(q)$, with each c joined to $\mu(c)$ by

a contact of E, together with ranks $r : U(q) \to \mathbb{N}$ with $r(\mu(c)) = r(c)$, such that $r(b) < r(c)$ whenever a contact of E joins $c \in U(q)$ with line p to $b \in U(q)$, $b \neq \mu(c)$.

A dressing order says that the undressed clocks can be dressed in matched pairs, in order of rank, so that a p clock, whose bit flips when it is dressed, has no undressed neighbour left behind except its partner.

Definition 14.2 (Clusters, tokens, sources and sinks)**.** Fix lines ℓ and bits x. The *A-clusters* are the connected components of the graph on the touched A clocks in which two clocks are joined when a contact of E joins them. For a cluster C, $\mathrm{tok}_x(C)$ is the number of its clocks with bit 1, its *tokens*. A P clock joined by a contact to a clock of C is a *wall* of C. The cluster has a *source* at x if it has a wall of bit 1, and a *sink* if it has a wall of bit 0.

Lemma 14.3 (Awake postings move tokens)**.** *An awake posting keeps every line. On two A clocks it exchanges their bits; on an A clock and a P clock it copies the bit of the P clock into the A clock; on two P clocks it moves nothing. For every network, all lines ℓ and all bits x, y, $(x, \ell) \xrightarrow{\mathrm{aw}} (y, \ell)$ if and only if*

(i) $y_c = x_c$ at every site c that is not a touched A clock, and
(ii) for every A-cluster C: if $\mathrm{tok}_y(C) > \mathrm{tok}_x(C)$ then C has a source at x, and if $\mathrm{tok}_y(C) < \mathrm{tok}_x(C)$ then C has a sink at x.

Proof. By Lemma 12.1(D) an awake posting keeps both lines and adds the flip bit to each bit: A flips, P keeps. At a mixed pair two flips exchange the bits, and an A clock that flips beside a P clock that keeps takes its bit. So the bits of P clocks, of undressed clocks and of untouched sites never move, which is (i). A cluster's token number changes only at a posting between one of its clocks and a wall, rising by one when the wall has bit 1 and falling by one when it has bit 0; walls keep their bits, which gives (ii).

Conversely, inside a cluster an awake posting along a contact exchanges a token and a hole. Exchanges along a walk realize every transposition of two clocks of the cluster, since $(a\,b) = (a\,w)(w\,b)(a\,w)$; exchanges between equal bits are skipped, as they change nothing. So any two arrangements with the same token number are joined. A source writes a token into a hole brought next to it, and a sink removes a token brought next to it. Induction on the number of clocks at which x and y differ gives an awake path. □

The first consequence is that one kind of participant can never be born.

Theorem 14.4 (Persistent currents are never born)**.** *Let E be a network, g a fair posting sequence of E, and q a classical state with $U(q) \neq \emptyset$ that is live under g. Then no state q_t of the run has every touched clock P. Every dressing posting of the run leaves an A clock on its pair.*

Proof. By Lemma 13.2, the first posting through a clock of $U(q)$ is a dressing of two undressed clocks on a balanced pair, and dressed lines are permanent. Suppose both dressed clocks were p, with common bit b. They become P with bits $b + 1, b + 1$, and P bits never move again in a live run, since every later posting through them is awake. Fairness makes the same contact post again, at a balanced pair of two dressed clocks: by Lemma 12.1(B) that posting branches, and q is not live. So every dressing turns an a clock into A, and from the first dressing on the run holds an A clock. Before the first dressing a clock of $U(q)$ is still undressed. □

Co-oriented persistent currents (Corollary 12.18(b)) are therefore primordial.

Theorem 14.5 (The birth criterion)**.** *Let E be a network and q a classical state with $U(q) \neq \emptyset$. Then q is born if and only if*

(i) every contact of E through a clock of $U(q)$ is balanced at q,
(ii) q has a dressing order, and
(iii) $q^\sharp \xrightarrow{\text{aw}} q'$ for some parity-law state q' of E.

More precisely, q' is the state at the end of a birth prefix of q if and only if (i), (ii) hold, q' is a parity-law state and $q^\sharp \xrightarrow{\text{aw}} q'$. By Lemma 14.3, (iii) is a condition on token numbers: q' keeps the bits of $q^\sharp$ off the touched A clocks, and each A-cluster gains tokens only if it has a source and loses them only if it has a sink.

Proof. A state is live under g^π exactly when the prefix is live and ends on a state the round order keeps; by Theorem 13.5 that end state must be a parity-law state.

Necessity. Liveness at time 0 is (i): a mixed contact with an undressed clock is neutral. In a live prefix every posting is deterministic, so by Lemmas 12.1 and 13.2 it is either a dressing of two undressed clocks on a balanced pair or an awake posting. The dressings pair up $U(q)$ along contacts; let μ be this matching and let the rank of a pair be the time of its dressing. A p clock flips its bit when it is dressed. If an undressed neighbour $b \neq \mu(c)$ of a p clock c were still undressed after that dressing, their contact, balanced before, would be mixed with an undressed clock, hence neutral. So b is dressed earlier, and the ranks form a dressing order. A dressing does not change the dressing of the current state, and an awake posting commutes with taking the dressing, because it moves only dressed clocks. So the awake postings of the prefix carry $q^\sharp$ to its end state, which is (iii).

Sufficiency. Post the matched pairs in order of rank. Each is a balanced pair of two undressed clocks, so the posting is a dressing, and the ranks guarantee that no contact turns mixed beside an undressed clock. The result is $q^\sharp$. Then post an awake path to q', each posting deterministic, and continue with the round order, which keeps q'. □

Proposition 14.6 (Counting a birth, and its record)**.** *Let q be live under g^π with $|\pi| = k$, and let α be the number of awake postings in the prefix.*

(a) $|U(q)| + 2\alpha = 2k$. So every birth prefix contains exactly $|U(q)|/2$ dressings, one per matched pair.
(b) If q is born, it is born with a prefix of length $|U(q)|/2$ if and only if $q^\sharp$ is a parity-law state.
(c) In a run that is live under any posting sequence, the posting at time t is a dressing if and only if its label has middle current $j_2 = 0$. For a dressing of the pair (u, v), $j_1 \neq 0$ exactly when the clock of u was p, and $j_3 \neq 0$ exactly when the clock of v was p; a dressing of two a clocks writes $j = (0, 0, 0)$.

Proof. (a) Each dressing removes two clocks from the undressed set and each awake posting removes none, and a parity-law end state has no undressed touched clock. (b) Is (a) with $\alpha = 0$, together with Theorem 14.5. (c) A deterministic posting is a dressing exactly when its pair is balanced, that is when $j_2 = x_1 - x_2 = 0$. At a balanced input (b, b) the label is $(y_1 - b, 0, y_2 - b)$, and $y_1 \neq b$ exactly when the first clock was p. □

Some births need awake postings as well as dressings. On the path $((0,1),(0,2))$ take bits $(0,0,0)$ and lines (a,P,p). The prefix $((0,2),(0,2))$ first dresses sites 0 and 2 to A (bit 0) and P (bit 1); its second posting meets the mixed pair of bits $0,1$, now awake, and moves the A bit to 1: the relay at $x=(1,0,1)$, which the round order keeps. The dressing $q^\sharp$, bits $(0,0,1)$ with lines (A,P,P), is not a parity-law state, since the pair of $(0,1)$ is balanced; so no prefix of length one is a birth, and the least birth prefix has length $|U(q)|/2+1=2$.

Theorem 14.7 (Which participants are born)**.** *Call a classical state z a* single-pair genesis *at $e\in E$ if no contact of E is neutral at z and the undressed touched clocks of z are exactly the two clocks of e. A state q' is a born participant of E if and only if q' is a parity-law state and $z_1 \xrightarrow{\mathrm{aw}} q'$, where z_1 is the result of posting e at some single-pair genesis z at e. In every born participant some touched clock is A; at the last dressed pair at least one clock is A.*

(a) On the path $((0,1),(0,2))$ the two relays are born participants and the two co-oriented pairs are not.

(b) On the triangle $((0,1),(0,2),(1,2))$ the six parity-law states with one A clock are born, each by a single posting, and the two all-A parity-law states are not.

(c) On the chain with contacts $(0,1)$, $(1,2)$, $(2,3)$, $(3,4)$, $(4,5)$ the state with bits $(1,0,0,0,0,0)$ and lines (P,A,a,a,A,P) is born by the prefix $((2,3))$. The birth dresses both clocks of $(2,3)$ to A and writes the label $(0,0,0)$.

Proof. The state just before the last dressing of a birth prefix is a single-pair genesis, and the rest of the prefix is awake; conversely a single-pair genesis has the trivial dressing order, and Theorem 14.5 applies. The A clock comes from the proof of Theorem 14.4 applied to the last dressing. (a) The co-oriented pairs have no A clock. A relay is reached in Proposition 13.21. (b) A single-pair genesis of the triangle whose dressing is all A has two a clocks and a third clock A; since both undressed clocks are adjacent to the third, (i) forces all three bits equal. The dressing is one wall-free cluster with 0 or 3 tokens, and Lemma 14.3 conserves that number, while the all-A parity-law states of the triangle carry one or two tokens. The six one-A states are each reached by one dressing, by a direct exact computation over the 512 basis states of the triangle. (c) The pair $(2,3)$ is balanced at $(0,0)$ and both clocks are a, so the posting keeps both bits and writes $(0,0,0)$. Nothing is neutral at genesis. The result, bits $(1,0,0,0,0,0)$ and lines (P,A,A,A,A,P), satisfies the parity law of Chapter 12: every product $f_z \deg z$ is 0, and the first-round equations ask $x_0+x_1=1$ and $x_i+x_{i+1}=0$ for $i=1,\dots,4$, which hold. □

Births into the round order do not exhaust what fair timing can keep.

Proposition 14.8 (Fair timing keeps what no prefix keeps)**.** *On the star with contacts $((0,1),(0,2),(0,3))$, let q have every bit 0, centre clock a, leaf 1 on p and leaves 2, 3 on P. Let g post $(0,1)$ at times $1,2,4,6,\dots$ and $(0,2)$, $(0,3)$ in turn at times $3,5,7,\dots$. Then g is fair and q is live under g, but q is not born.*

Proof. At genesis every pair is balanced, so nothing is neutral. The first posting dresses the centre to A (bit 0) and leaf 1 to P (bit 1). From then on the centre's pair with leaf 1 is mixed whenever the centre holds 0, and its pair with leaves 2 and 3 is mixed whenever it holds 1. The sequence alternates: $(0,1)$ fills the centre from the source, and the next posting, by $(0,2)$ or $(0,3)$, empties it into a sink. Every posting is awake. A birth of q would end on a

parity-law state with the lines of $q^{\sharp}$, but the centre is A with degree 3 and its neighbours are P, so $f_0 \deg 0 = 1 \neq 0 = f_1 \deg 1$, violating condition (i) of (13.1). □

The degree condition of the parity law belongs to the periodic round order, not to fairness.

14.2 Fair timing

A state is *fairly kept* if some fair posting sequence of E keeps it live, equivalently if $\psi(q) \in \mathcal{N}(E; g)$ for some fair g. Under fair timing the parity law of Chapter 12 gives way to a conservation law for tokens.

Theorem 14.9 (Token balance)**.** *Let every touched clock of q be dressed, let g be a posting sequence of E under which q is live, and let C be an A-cluster of q. Let s_t^+ and s_t^- count the postings among $g_1, \dots, g_t$ by contacts joining a clock of C to a wall of bit 1 and of bit 0 respectively. Then for every t*

$$\mathrm{tok}_{x_t}(C) + s_t^- = \mathrm{tok}_x(C) + s_t^+.$$

A cluster without walls conserves its token number. If g is fair and C has a source, then $|s_t^+ - s_t^-| \leq |C|$ for every t, and both s_t^+ and s_t^- grow without bound.

Proof. Every posting of the run is deterministic on dressed clocks, hence awake. By Lemma 14.3 only postings between C and a wall change the token number. Such a posting is awake only at a mixed pair, so a posting from a wall of bit 1 finds the clock at 0 and raises the count by one, and one into a wall of bit 0 lowers it by one. The bound follows from $0 \leq \mathrm{tok} \leq |C|$. Under a fair sequence the source contact posts infinitely often, so s_t^+ is unbounded, and the bound makes s_t^- unbounded too. □

Definition 14.10 (Closed and open clusters; the fair law)**.** An A-cluster C of $q = (x, \ell)$ is *closed* if it has no wall and $1 \leq \mathrm{tok}_x(C) \leq |C| - 1$, and *open* if it has a source and a sink. A state with every touched clock dressed obeys the *fair law* if every contact of two P clocks is mixed and every A-cluster is closed or open.

Theorem 14.11 (The fair law)**.** *Let E be a network and q a classical state with every touched clock dressed. Then q is fairly kept if and only if E is nonempty and q obeys the fair law.*

Proof. Necessity. A contact of two P clocks never changes its bits; fairness makes it post, and a balanced pair would branch. A wall-free cluster keeps its token number by Theorem 14.9; with 0 or $|C|$ tokens every internal contact is balanced when fairness makes it post. A cluster with a source and no sink has a nondecreasing token number, bounded by $|C|$, that rises at every posting of the source contact, which fairness repeats forever; a sink without a source is the same argument reversed.

Sufficiency. The fair law is unchanged by awake postings, since walls keep their bits and wall-free clusters keep their tokens, and on states obeying it awake reachability is symmetric (Lemma 14.3 in both directions). For each contact there is an awake path to a state where its pair is mixed: a P pair is already mixed; a wall contact can be made mixed because the cluster is open, a source or sink writing the other bit at its end; an internal contact of a closed cluster has a token or a hole to bring to one end. Chaining one such path and one posting per

contact, and then the awake path back to the start, gives a cycle of awake postings through every contact. Its periodic repetition is fair and keeps q live. □

Corollary 14.12 (Fair births and periodic witnesses)**.** *Let E be a network and q any classical state.*

(a) q is fairly kept if and only if E is nonempty, no contact is neutral at q, q has a dressing order, and $q^{\#}$ obeys the fair law.
(b) If q is fairly kept, then it is kept by a sequence $dCCC\cdots$, where d is a list of at most $n/2$ postings and C is a list of contacts containing every contact of E, of length at most $(m+1)2^n$, that returns the state after d to itself.
(c) Every born state is fairly kept, and the lines of its dressing satisfy condition (i) of (13.1)*.*

Proof. (a) Under a fair live sequence every touched clock is dressed by some finite time; the dressings and awake postings before it give the dressing order as in Theorem 14.5, and the tail is a fair sequence keeping the dressed state, which obeys the fair law by Theorem 14.11; the law transfers back to $q^{\#}$ along the awake postings. Conversely, dress in order of rank and continue with the fair sequence of Theorem 14.11. (b) The dressing list has $|U(q)|/2 \le n/2$ postings (Proposition 14.6(a)), and an awake path can be shortened below 2^n postings, the number of bit patterns, by removing loops. (c) The round order is fair, and the end of a birth is a parity-law state with the lines of $q^{\#}$. □

Table 14.1 compares the fair law with the parity law on four small networks. On the four-cycle every degree is even, so condition (i) of the parity law always holds; the difference there is the first-round phase that the round order imposes and fair timing does not. On the three-leaf star no beginning is born at all, because every fair birth dresses the centre, of degree 3, to A beside a P leaf.

network	fairly kept	parity-law	fair births	births
path $((0,1),(0,2))$	12	4	4	4
triangle	18	8	12	12
three-leaf star	52	4	18	0
four-cycle	64	16	32	32

Table 14.1. Fair timing against the round order, by direct exact computation over all basis states. The first two columns count basis states with every touched clock dressed: those fairly kept, and among them the parity-law states. A fair birth is a fairly kept state with an undressed touched clock. On the path, the eight extra states are the two relays whose centre bit equals the bit of leaf 1 and six all-A states with one or two tokens; on the triangle, six relays and four all-A states; on the star, twelve all-A states with one to three tokens and thirty-six open clusters at the centre. The four-cycle births are the states with one clock of each line p, a, P, A, the undressed pair on a contact, at the empty or the full occupation.

Fair timing keeps more states than the round order, but it does not keep their superpositions.

Theorem 14.13 (What one fair timing keeps together)**.** *Let E be a network.*

(a) The set of vectors kept by some fair sequence is $\bigcup_g \mathcal{N}(E;g)$ over fair g, and ϕ lies in it if and only if some fair g keeps every basis state in the support of ϕ.

(b) Let $q = (x, \ell)$ and $q' = (x', \ell)$ have the same lines. Some fair sequence keeps both if and only if q is fairly kept and $x_u + x'_u = x_v + x'_v$ for every contact (u, v) of E. If E is connected, this means $x' = x$ or $x' = x + 1$.

(c) On the path $((0,1),(0,2))$ the relays with lines (A, P, P) at bits $(0,1,0)$ and $(1,1,0)$ are each fairly kept, no fair sequence keeps both, and $\psi((0,1,0),(A,P,P)) + \psi((1,1,0),(A,P,P))$ is kept by no fair sequence. The set of vectors kept by some fair sequence is not a subspace.

Proof. (a) is Proposition 12.3 under the union. (b) A deterministic posting has its mixedness fixed by the lines alone: a pair is posted deterministically only if it is mixed with two dressed clocks or balanced with two undressed ones. So in two live runs under the same sequence, sites with equal lines move their bits by the same rule, and the sum $x_c + x'_c$ never changes while the lines agree, which they do throughout. When a contact posts, both runs see the same mixedness at it, so the sums agree across it; fairness makes every contact post. Conversely, if the sums agree across every contact, the two runs have the same mixedness at every contact at every time, so they are live together. On a connected network, a sum constant across every contact is constant. (c) The two relays have sums 1 at site 0 and 0 at site 1, which differ across $(0,1)$; each is fairly kept by Theorem 14.11. Their sum is not kept by (a), so the set is not closed under addition. □

Unlike a superselection rule, no conserved charge is assumed and a fixed timing forbids no superposition inside its own kept subspace; the selector is the timing.

14.3 The gate: a contact that waits for its own pair

A schedule is a supplied clock. A simpler local rule needs none: a contact attempts to post, reads whether its own pair is mixed, and posts only if it is. We call this rule the *gate*. It is a supplied premise, built from the native awake read: on a balanced pair the native posting acts, carrying zero loop-edge current (a deterministic dressing when both clocks are undressed, a branching otherwise), and the gate replaces that action by silence. So the gate is a restriction of the native instrument, and whether a physical contact is gated is the question whether a posting that carries no loop-edge current is executed, the attachment question of Chapter 21 in another form. Under the gate the selection changes character. P clocks become frozen walls, a contact's verdict depends only on its own cluster and walls, and whether a participant is a single position or a whole cluster (readings of the theorems below) depends on how fair the attempts are.

Definition 14.14 (The gate and gated classes)**.** For a contact e let Π^e_{aw} be the projection onto the product states whose pair at e is mixed; it is diagonal in the bits. A *gated attempt* of e has outcomes $j \in \mathcal{J}$ and $\emptyset$ (silence), with operators

$$G^e_j = V^e_j \, \Pi^e_{\mathrm{aw}}, \qquad G^e_\emptyset = I - \Pi^e_{\mathrm{aw}}.$$

An *attempt sequence* g names the contact attempting at each time; it is *fair* if every contact of E is attempted at arbitrarily late times. Along g, a word of outcomes o gives G_o, the first outcome applied first. A vector ϕ passes the *gated test of e along g* if (i) $Q^e_{\mathrm{ne}} G_o \phi = 0$ for every outcome word o, and (ii) for every time t there is a T such that $G_o \phi = 0$ for every outcome word o of length T in which e is silent at all its attempts from t on; that is, e posts again after t

with certainty. The *own gated class* of e is the set of vectors passing e's test along every fair attempt sequence of E. The *joint gated class* is the set passing every contact's test along every fair attempt sequence; it is the intersection of the own classes.

Lemma 14.15 (The gate is an instrument that silences balanced pairs)**.** $\sum_j G_j^{e*} G_j^e + G_\emptyset^{e*} G_\emptyset^e = I$. *For a label j, $G_j^e = V_j^e$ if $j_2 \neq 0$ and $G_j^e = 0$ if $j_2 = 0$. On a product state $\psi(x, \ell)$ a gated attempt is silent with certainty if the pair is balanced; if the pair is mixed with both clocks dressed it posts deterministically, keeping both lines, flipping an A bit and keeping a P bit; if the pair is mixed with an undressed clock, e is neutral. Gated postings keep every dressed line and leave undressed clocks undressed, and the bit of a P clock never changes.*

Proof. The middle current $x_1 - x_2$ is nonzero exactly on mixed inputs, so a label with $j_2 \neq 0$ acts only on mixed inputs and one with $j_2 = 0$ only on balanced ones. Then $\sum_j G_j^{e*} G_j^e = \Pi_{\mathrm{aw}}^e \big(\sum_j V_j^{e*} V_j^e\big) \Pi_{\mathrm{aw}}^e = \Pi_{\mathrm{aw}}^e$, and $(I - \Pi_{\mathrm{aw}}^e)^2 = I - \Pi_{\mathrm{aw}}^e$. The rest is Lemma 12.1 at a mixed pair. □

A gated word sends a basis state with dressed touched clocks to ± the basis state of its classical gated run, or to 0, and each gated class is spanned by the basis states it contains. The first test is the path and the triangle.

Theorem 14.16 (The gate on the path and the triangle)**.** *Let E be the path $((0,1),(0,2))$ or the triangle $((0,1),(0,2),(1,2))$.*

(a) Never-neutrality alone does not select under the gate. The vectors that no gated word along any attempt sequence brings into a neutral sector form a space of dimension 192 on the path and 176 on the triangle, spanned by basis states. It contains all 64 all-dressed basis states, among them the state with every clock P and every bit equal, at which no contact ever posts.

(b) The joint gated class of the path is six-dimensional: the two co-oriented pairs, the two relays, and the two relays obtained by exchanging sites 1 and 2. It is the sum of the never-neutral classes of the two round orders $((0,1),(0,2))$ and $((0,2),(0,1))$. The joint gated class of the triangle is twelve-dimensional: one A clock at a site k and P clocks at the other two sites, whose bits differ. These are the one-A parity-law states of the six round orders of the triangle together.

(c) The all-A states are starved by fair sequences. On the path, the sequence $(0,1)$, then $(0,2),(0,1),(0,1)$ repeated, keeps the all-A state with bits $(0,1,1)$ never neutral, but every attempt of $(0,2)$ after time 1 finds its pair balanced, although the pair is mixed at arbitrarily late times. On the triangle, $(1,2),(0,1),(0,2),(0,1)$ repeated does the same to the all-A state with bits $(1,0,0)$. Round-robin attempts keep these states: the gated classes under the round-robin attempt sequence have dimensions 12 on the path (the six above and the six all-A states with a mixed pair) and 18 on the triangle.

Proof (mechanism). By Lemma 14.15 each test is a statement about classical gated runs: a neutral contact on the run excludes a basis state from (i), and a contact never attempted while its pair is mixed after some time excludes it from (ii). For (b), on a relay only the contact with the mixed pair is awake; its posting moves the shared A bit, after which only the other contact is awake, and its posting returns the relay. The relay's alternation is generated by its own occupation, and an attempt of the sleeping contact changes nothing. So on the listed states an

awake contact stays awake until it fires, and fairness forces every posting. The exclusions are explicit fair sequences that are eventually periodic, checked over one period. The counts are direct exact computations over the 512 basis states. □

Before following the all-A states further, note what fair timing does without the gate.

Theorem 14.17 (Without the gate, fair timing draws no boundary)**.** *Let E be a connected network and let $e \in E$. The intersection, over all fair posting sequences g of E, of the own classes $\mathcal{N}_e(E; g)$ equals $\mathcal{N}_{\text{free}}(E)$, the class kept by every posting sequence of E. If E has no doubled contact and $m \geq 2$, it is the span of the co-oriented colourings (Theorem 13.8).*

Proof (mechanism). Let q fail to be live under some posting sequence g, first at time T. Follow g up to T, post there the contact that fails (a branching posting, or a neutral contact whose undressed clock is free), and continue with the round order. Under this fair sequence the failure reaches e's own test by Lemma 13.12 and Theorem 13.13. □

The gate is different: it draws boundaries.

Theorem 14.18 (Gated verdicts are local)**.** *Let E be a network and $e \in E$. Define the* reach $R(e)$ *of e at lines ℓ: the two sites of e, and every site joined by a contact of E to a site of the reach that is not a P clock. P clocks are reached but not passed: they are the* walls *of the reach.*

(a) The own and joint gated classes are spanned by basis states.
(b) Whether $\psi(q)$ lies in the own gated class of e depends only on the bits and lines of q on $R(e)$.
(c) On every network a mixed pair of two P clocks lies in the own gated class of its contact, and a balanced pair of two P clocks never does. If a clock of f is undressed, $\psi(q)$ is not in the own gated class of f.
(d) The joint gated class is the intersection of the own gated classes, and it is strictly smaller than the own gated class of e whenever some contact of E has a site that is not a site of e.

Proof (mechanism). (a) Every gated word maps each sector of fixed bits and fixed pattern of dressed lines into one such sector, and its Born weight there is a scalar; a vector in a gated class therefore has every basis component in it. (b) A gated posting of a contact with a P clock leaves that clock's bit unchanged, so nothing that happens beyond the walls reaches the clocks of $R(e)$; runs from two states that agree on $R(e)$ can be simulated across the walls. (c) A mixed P pair stays mixed and posts at its every attempt; a balanced one is never awake. If f has an undressed clock, f is neutral whenever its pair is mixed and silent while it is balanced, so it cannot post and stay off its neutral sector. (d) The witness for strictness has every clock P, e mixed and the other contact balanced. □

The walls screen records as well as verdicts. Write the *pattern* of a state for its lines with p and a identified, and Π_λ for the projection onto the basis states of pattern λ. A contact's *own gated record* lists the outcomes of its own attempts up to a time T, every other attempt unread; an *outside record* lists the outcomes of attempts by contacts other than e that touch no site of $R(e)$ except walls. Records have effects formed as in Section 13.1, with gated outcomes in place of labels.

Theorem 14.19 (The gated record is screened by the walls)**.** *Let E be a network, $e \in E$, and fix any attempt sequence of E.*

(a) Every gated record effect is diagonal in the pole-line basis.
(b) For every pattern λ, every T and every own record r of e, $\Pi_\lambda E_r \Pi_\lambda = \Pi_\lambda (A \otimes I)\Pi_\lambda$ with A acting on the bits of $R(e)$ and I beyond. So the law of e's own record in any state, including states entangled across the walls, depends only on the populations of the bit configurations of $R(e)$ in each pattern sector.
(c) On each pattern sector the joint effect of e's own record and an outside record is $\sum_b W_b \otimes A_b \otimes B_b$, where W_b projects onto the wall bits b, A_b acts on the non-wall sites of $R(e)$ and B_b on the sites beyond $R(e)$. For a product density across the walls with definite wall bits the two records are independent.
(d) Let q, q' be basis states with every clock dressed and the same pattern. Some fair attempt sequence, some T and some own record of e give different laws at q and q' if and only if q and q' differ somewhere on $R(e)$ and are not both constant there.

Proof (mechanism). (a)–(c): A gated outcome maps a basis state into one sector of fixed bits and pattern, and its adjoint returns it to the basis state with a weight that depends only on the bits and the pattern; a posting inside the reach changes only bits of $R(e)$, and an outside posting changes none of them, while walls keep their bits. Summing out an unread attempt gives the factor 1. (d) A gated label at a mixed pair fixes the transition (the label and the input determine the output, and at a mixed input the label determines the input), so each posting of e reads the bits it meets. An A clock that posts with a wall takes the wall's bit, so postings of other contacts transport any bit of the reach to a site of e. A state constant on $R(e)$ is silent forever. □

The separating sequences of (d) depend on the pair, and the round order does not serve every pair at once: on the chain wall, u_1, u_2, wall with both walls of bit 0 and contacts listed left, middle, right, the empty cluster and the cluster with one token at u_1 have equal records for the middle contact under the round order, while the sequence middle, left, middle, right, middle, then the round order, separates every separable pair of that chain at depth 5. Chapter 15 proves that one fair sequence per pattern separates every such pair at a depth T with $T + 1 \leq n^2$ (Theorem 15.1), and that states of different patterns that agree beyond both reaches and are not constant on their own reaches are separated too (Theorem 15.3).

14.3.1 Weak fairness: participants are positions

Under the gate, a contact inside an A-cluster can be starved by a fair adversary that attempts it only when its pair is balanced. The adversary has three tools: a mixed exchange attempted twice is the identity; a wall contact is undone by a wall of its clock's new bit; and a wall contact can be silenced by first exchanging its wall's bit into its clock through a neighbouring exchange and then exchanging back. With a third clock in the cluster such a neighbour always exists.

Definition 14.20 (Gated form)**.** A state q with every touched clock dressed is in *gated form* if every contact of two P clocks is mixed; every contact of two A clocks is an *isolated pair* (every contact touching either of its sites joins the same two sites) with differing bits; and every A

clock joined to a P clock is a *relay*: it has exactly two contacts, both to P clocks, and their bits differ.

Theorem 14.21 (The gated participants under weak fairness)**.** *Let E be a network, q a state with every touched clock dressed, and $f \in E$.*

(a) If f touches an A-cluster with at least three clocks, $\psi(q)$ is not in the own gated class of f.
(b) If f joins an A clock u to a wall w, then $\psi(q)$ is in the own gated class of f if and only if every neighbour of u is a wall, f is the only contact at u to a wall with the bit of w, and some wall at u has the other bit. In particular every wall contact of a cluster with two or more clocks is starved.
(c) If f joins the two clocks u, v of a two-clock cluster, then $\psi(q)$ is in the own gated class of f if and only if either the pair is isolated and mixed, or it is not isolated and f is the only contact joining u and v, both have walls, all walls at u carry one bit β and all walls at v carry $\beta + 1$.
(d) The joint gated class of E is the span of the basis states in gated form with every touched clock dressed.
(e) For every basis state q: if some clock of $R(f)$ is undressed, $\psi(q)$ is not in the own gated class of f. With Theorem 14.18(b), (c) and (a)–(c) above, this decides membership for every basis state.

Proof (mechanism). On a dressed state the own test is a game on the deterministic gated run, in which walls never change, a wall copies its bit into an A neighbour and two A clocks exchange. A *starvation certificate* for f is a finite prefix ending at a state y where f is balanced, together with, for every other contact k, a finite block of attempts that contains k, avoids f and returns y to itself. The fair sequence: prefix, then f followed by all the blocks, repeated, attempts f only at y. The three tools above build these blocks: with a third clock c next to the tested contact there is always a neighbouring exchange to route through, and blocks of at most three moves are checked over all 64 wall types and 8 saturated states by direct exact computation. Liveness in the remaining cases comes from a state that only f can leave and to which the walls always return: in (b), u at the bit opposite to that of w; in (c), (u, v) at $(\beta, \beta+1)$. There f is awake, and it stays awake until it fires. An undressed clock in the reach accepts either output bit at a mixed posting, so the adversary may treat it as an A clock; the cluster then has a clock besides the ends of f, and (a) or (b) starves f. □

14.3.2 Strong fairness: participants are wholes

The adversary of Theorem 14.21 attempts a contact only while it is balanced, even when its pair is mixed at arbitrarily late times. Forbidding that restores the wholes.

Definition 14.22 (Strong fairness)**.** An attempt sequence g is *strongly fair* on the gated run of a basis state q if every contact whose pair is mixed at arbitrarily late times of the run is attempted while mixed at arbitrarily late times. A contact is *strongly live* at q if it posts at arbitrarily late times along every fair attempt sequence that is strongly fair on the gated run of q.

Strong liveness is a property of classical basis states read on their gated runs; it is not a subspace test.

Theorem 14.23 (The gated participants under strong fairness)**.** *Let E be a nonempty network.*

(a) For every basis state q with every touched clock dressed there is an attempt sequence of E that is fair and strongly fair on the gated run of q.
(b) Let every touched clock of q be dressed and $f \in E$. Then f is strongly live at q if and only if f is mixed when both of its clocks are P, and every A clock of f lies in a closed or open cluster (Definition 14.10). All contacts touching one A-cluster share the verdict. The verdict does not depend on the sequence: along every fair, strongly fair run, each contact of a cluster that is neither closed nor open is eventually balanced for ever, and each contact of a closed or open cluster is mixed at arbitrarily late times.
(c) Every contact is strongly live at q if and only if q obeys the fair law, that is, if and only if q is fairly kept. A contact whose state lies in its own gated class is strongly live there; in particular every state in gated form obeys the fair law.

Proof. (a) The gated system on a basis state is finite and deterministic given the attempts. Go to a reachable state with the fewest reachable states, from which every reachable state returns, and repeat a closed tour that attempts every contact at every state it visits. A contact mixed at a visited state is attempted there in every period.

(b) Suppose a contact f with an A end a is balanced from some time T on. A mixed attempt after T of any other contact h at a would flip a alone and make f mixed, so h is never attempted while mixed after T, and strong fairness makes h eventually balanced. Spreading along the cluster, every contact of the cluster, walls included, is eventually balanced, so cluster and walls end on one bit. An open cluster cannot do this, since walls never move and it has walls of both bits; a closed cluster cannot either, since it conserves a token number strictly between empty and full. So contacts of closed and open clusters are mixed, hence attempted while mixed, at arbitrarily late times. Conversely, if every wall of a cluster carries one bit β, the number of the cluster's clocks not at β never rises (exchanges permute bits and a mixed wall posting writes β) and falls at every mixed wall posting, so wall postings eventually stop being mixed, strong fairness balances them, and the spreading argument freezes the cluster. A wall-free cluster that is empty or full is frozen from the start. A P pair is the case of Theorem 14.18(c).

(c) The joint statement is (b) for every contact, which is the fair law, and Theorem 14.11 identifies it with fair keeping on a nonempty network. The criteria of Theorem 14.21 imply those of (b). □

Four small cases show the gap between the two kinds of fairness. A closed triangle of three A clocks with no walls is strongly live exactly with one or two tokens, and weakly starved for all bits. Two A clocks between walls of opposite bits, joined by two parallel contacts, are strongly live, while a middle contact is weakly starved. Three A clocks in a path between walls of one bit are starved under both kinds of fairness; between walls of opposite bits they are strongly live and weakly starved.

14.4 Random attempts

Strong fairness need not be supplied. If attempts are drawn at random and no contact's chance ever falls below a floor, the adversary of Theorem 14.21 is a null event. Random timing then

selects wholes, and it gives each whole a law.

Theorem 14.24 (Random timing is strongly fair)**.** *Let the attempts be drawn independently, the attempt at time t from a law ν_t on the positions of E with $\nu_t(\{i\}) \geq \eta > 0$ for every t and i.*

(a) If a condition read on the attempts before each time holds at infinitely many times, then with probability one every position is drawn at infinitely many of those times. In particular, for each basis state q with every touched clock dressed, the attempt sequence is fair and strongly fair on the gated run of q with probability one.

(b) (Zero-one law.) Let the draws be identically distributed with every position of positive chance, let every touched clock of q be dressed and $f \in E$. If f meets the criterion of Theorem 14.23(b), f posts at arbitrarily late times with probability one; otherwise with probability zero. Every contact posts at arbitrarily late times with probability one if and only if q is fairly kept. The attempt sequences that starve a contact meeting the criterion, the starvation certificates of Theorem 14.21 among them, form a null set.

(c) If the next attempt is drawn from any law depending on the past whose chance for each contact is at least η, the chance that the first k times at which a given contact's pair is mixed all pass without its attempt is at most $(1-\eta)^k$.

Proof. (c) and the core of (a): at each time the condition holds, the chance of missing the given position is at most $1-\eta$ whatever happened before, so k misses in a row have chance at most $(1-\eta)^k$; this bound holds on finite words and passes to the product measure through cylinders. Letting $k \to \infty$, the set of sequences on which the condition holds infinitely often but the position is drawn at only finitely many of those times is null. The gated run up to time s depends only on the first s draws, so "contact f is mixed at time s" is such a condition, and a finite intersection over the contacts gives fairness and strong fairness. (b) is Theorem 14.23 applied on this full-measure set. □

Only the positivity of the chances enters, so the random gated participants are the wholes of Theorem 14.23 for every choice of positive weights. Which timings can the native data themselves fix? No order, and only the symmetric attempt law.

Proposition 14.25 (No order is forced, and the symmetric attempt law is uniform)**.** *Let E be a network.*

(a) If a set S of posting sequences is unchanged by the one-tick shift and contains exactly one sequence g, then g is constant. So if E has two different contacts, no such set singles out a fair sequence.

(b) If S is unchanged by a group of relabellings of the sites that moves every contact, S does not contain exactly one sequence. On the bare carrier the relabellings exchanging a contact's two ends move it; on the torus network of Proposition 12.4 a unit translation moves every contact.

(c) Let an attempt law give each contact a chance that depends only on the contact and is unchanged by every relabelling of the sites, or, for a network of torus contacts, by the symmetries of Proposition 12.4(b)(ii), with the chances of the m positions of E summing to 1. Then every position has chance exactly $1/m$. With attempts drawn independently, the floor of Theorem 14.24 holds with $\eta = 1/m$, and so do its conclusions.

Proof. (a) The shifted sequence lies in S, so it equals g, and $g(t+1) = g(t)$ for every t. A constant sequence posts one contact only. (b) If σ is in the group, $\sigma \circ g$ lies in S, so it equals g,

and σ fixes the contact $g(1)$, which it moves. (c) The symmetries act transitively on contacts (Proposition 12.4), so the chance is one constant γ, and $m\gamma = 1$. □

The order is physical: it decides what is kept (Chapter 12). Proposition 14.25 shows that the native data cannot supply it, since every timing rule computed from them inherits their symmetries. If the attempt law is computed from the native data alone, a chance for each contact that is unchanged by their symmetries (a premise), it is the uniform law, and with it comes the positivity that the zero-one law needs; no single realization of that law is forced. The independence of the attempts stays supplied, as do the prefixes, fair sequences, attempt sequences and round orders of this chapter; which physical process supplies a timing is not addressed here.

The weights do enter the laws. From here on the attempts are independent and identically distributed with positive weights w summing to 1 over the positions of E, and $\eta = \min w$. For an A-cluster C of a state q with every touched clock dressed, its *token class* $T(C)$ is the set of bit vectors that agree with q off C and have C's token number. The *law of the cluster* at time t is the law of its bits after t gated attempts, the other bits held as in q.

Theorem 14.26 (A closed whole forgets everything but its charge)**.** *Let C be a closed A-cluster of q.*

(a) The law of C evolves by itself, as a random walk on $T(C)$ in which each attempt either exchanges the bits of two clocks of C or does nothing. The uniform law on $T(C)$ is stationary for every choice of weights.
(b) Call C periodic *if it has two clocks and every contact of E joins those two clocks. If C is not periodic, its law λ_t at time t satisfies $\sum_y |\lambda_t(y) - u(y)| \le 2(1-\eta^{2^n})^{\lfloor t/2^n \rfloor}$, where u is the uniform law on $T(C)$ and n is the number of sites of the network.*
(c) If C is periodic, its law alternates for ever between the two arrangements of its single token and has no limit.
(d) In every case the time averages of the law converge to the uniform law on $T(C)$.

Proof. A closed cluster has no wall, so every contact either lies inside it or does not touch it. Inside, a gated attempt at a mixed pair exchanges the two bits and at a balanced pair does nothing, which is the same as exchanging equal bits; outside, nothing in C moves. Each move is an involution of $T(C)$, so the transition matrix is doubly stochastic for every w and the uniform law is stationary. Lemma 14.3 makes the walk irreducible on $T(C)$. If C is not periodic, some attempt is silent at some state of $T(C)$: a contact off the cluster is silent everywhere, and with three or more clocks one exchange balances a contact. From every state a word of at most 2^n attempts reaches a fixed state, which gives the Doeblin bound. In the periodic case every attempt moves the token. □

So a closed whole's long-run law is the equal-weight law on its token shell for every timing law; the two-clock periodic whole is the exception.

Theorem 14.27 (An open whole carries the weights of its walls)**.** *(a) Let c be an A clock all of whose neighbours are walls, let σ and κ be the total weights of the positions joining c to walls of bit 1 and of bit 0, with $\sigma + \kappa > 0$, and let o_t be the chance that c holds a token at time*

t. Then

$$o_t = \pi + (o_0 - \pi)(1 - \sigma - \kappa)^t, \qquad \pi = \frac{\sigma}{\sigma + \kappa},$$

and the chances that a token enters c and leaves it at time t are $\sigma(1 - o_t)$ and κo_t, both tending to $\sigma\kappa/(\sigma + \kappa)$.

(b) On the relay line, the path whose centre is A between a wall of bit 1 reached by a contact of weight w_L and a wall of bit 0 reached by a contact of weight w_R, the centre holds a token with chance $w_L/(w_L + w_R)$ at every time $t \geq 1$, and at every attempt after the first a token enters, and one leaves, with chance $w_L w_R/(w_L + w_R)$.

(c) The law λ_t of every open cluster has a limit law λ, with $\sum_y |\lambda_t(y) - \lambda(y)| \leq 2(1 - \eta^{2^n})^{\lfloor t/2^n \rfloor}$. No open cluster has a limit law, or a limit occupation chance at its clocks, that is the same for all positive weights.

Proof. (a) At each attempt c fills with chance σ if empty and empties with chance κ if full, a two-state chain. (b) is (a) with $\sigma + \kappa = w_L + w_R = 1$. (c) From every state of an open cluster, gated attempts fill the cluster, and on the full cluster every source contact is silent; this gives the Doeblin bound. For weight-dependence: heavy weight on a source contact pushes the occupation of its clock towards one, while a sink contact keeps its clock's vacancy chance at least its own weight, so no limit can be free of the weights. □

Waiting times under random timing have geometric tails.

Theorem 14.28 (Geometric tails)**.** *Let every touched clock of q be dressed.*

(a) If $f \in E$ meets the criterion of Theorem 14.23(b), the chance that f makes no attempt at a mixed pair during N consecutive attempts is at most $(1 - \eta^{2^n})^{\lfloor N/2^n \rfloor}$.

(b) If an A-cluster is neither closed nor open, the chance that it is not yet frozen (cluster and walls on one bit) after T attempts is at most $(1 - \eta^{2^n})^{\lfloor T/2^n \rfloor}$.

Proof. The criterion and closedness or openness are invariant along the gated run, and from every reachable state some word of at most 2^n attempts ends with a mixed attempt of f (in (a)) or frozen (in (b)), by Theorem 14.23 applied from that state. Each block of 2^n attempts succeeds with chance at least η^{2^n}, independently of the past. □

14.4.1 What a closed whole's record reads

A contact *fires* at time t when it is attempted at a mixed pair, writing a label. A record is the sequence of firings. For a closed cluster C with k tokens, and an internal contact f (one joining two clocks of C), write

$$\varpi = \frac{2k(|C| - k)}{|C|(|C| - 1)}, \qquad c_4 = \frac{4k(k-1)(|C| - k)(|C| - k - 1)}{|C|(|C| - 1)(|C| - 2)(|C| - 3)},$$

w_f for the total weight of the positions holding f, and W_N for the total weight of the positions whose contact shares exactly one clock with f. A *window statistic* of length L is any function F of the cluster's bits at time t and the attempts at times $t, \dots, t + L - 1$.

Theorem 14.29 (The record of a closed whole)**.** *Let C be a closed A-cluster of q, periodic or not.*

(a) *For every L and every window statistic F, with probability one $\frac{1}{T}\sum_{t<T} F$ converges to the expectation of F with the bits drawn from the uniform law on $T(C)$ and the attempts drawn independently with weights w.*
(b) *For an internal contact f, the long-run frequency of firings of f is $w_f\varpi$, almost surely and in mean. It depends only on f's own weight, the cluster size and the token number up to complement, and on nothing of the other weights.*
(c) *The long-run frequency of "f fires at t and at $t+2$" is $w_f^2\varpi(1 - W_N/2)$, almost surely and in mean. With w_f fixed it varies with the other weights if and only if E has a position whose contact shares exactly one clock with f and a position whose contact is neither f nor shares exactly one clock with f. On the closed triangle with one or two tokens it is $w_f^2(1 + w_f)/3$.*
(d) *For internal contacts $f \neq h$, the long-run frequencies of the patterns "f at t, h at $t+1$" and "f at t, h at $t+1$, f at $t+2$" are, almost surely and in mean,*

$$\begin{array}{lll} w_f w_h\, k(|C|-k)/(|C|(|C|-1)) & \text{and } 0 & \text{if } f, h \text{ share exactly one clock,} \\ w_f w_h\, c_4 & \text{and } w_f^2 w_h\, c_4 & \text{if they share none,} \\ w_f w_h\, \varpi & \text{and } w_f^2 w_h\, \varpi & \text{if } h \text{ joins the same two clocks.} \end{array}$$

So for contacts on different pairs of clocks, f and h share a clock if and only if the first pattern has positive frequency and the second has frequency zero; when $2 \le k \le |C| - 2$ the second pattern alone decides, and when $k = 1$ or $k = |C| - 1$ the first alone does. For $|C| \ge 4$, the first pattern alone cannot tell an adjacent pair from a disjoint pair with the same weight product exactly when $|C| = s^2 + 2$ and $2k = s^2 + s + 2$ or $2k + s = s^2 + 2$ for an integer $s \ge 2$; the smallest cases are $(|C|, k) = (6, 2), (6, 4), (11, 4), (11, 7)$.

Proof. (a) The cluster chain of Theorem 14.26 has, from any two states it can occupy at a common time, laws that merge at a summable rate (the Doeblin rate, or exactly in the periodic case, where the state at each time is certain). Covariances of centred window statistics more than L apart are then bounded by that rate, the second moment of a sum of T of them is $O(T)$, Chebyshev's inequality along $T = j^2$ is summable, and the Borel–Cantelli lemma with bounded increments gives convergence. Centring each statistic at its own mean at time t handles the periodic case, whose means converge only on average. (b) By double counting over ordered pairs of clocks, a fraction ϖ of $T(C)$ has f mixed. (c) An exchange along a contact (c, d) keeps a mixed pair (u, v) mixed unless exactly one of c, d lies on (u, v) and the exchange moves bits; such an exchange breaks it, and half of the uniform states mixed at f are mixed at a given contact sharing one clock with f. (d) After f fires, a firing of an adjacent h passes the token on and leaves f balanced; disjoint exchanges commute with each other's mixedness, and c_4 is the four-clock count. Equality of the two pair coefficients reduces to $(2k - |C|)^2 = |C| - 2$. □

Proposition 14.30 (Open wholes in the long run)**.** *For every open cluster the conclusion of Theorem 14.29(a) holds with its limit law in place of the uniform law. The empirical frequency of a token entering each clock converges almost surely to the stationary current, and on the relay line of Theorem 14.27(b) to $w_L w_R/(w_L + w_R)$.*

Proof. The open chain merges at the Doeblin rate of Theorem 14.27(c), and the argument of Theorem 14.29(a) applies. □

14.5 Births into a round order: phase, charge and flow

A birth ends on a parity-law state, so the round order asks more than fairness: the degree condition (i) of (13.1), which Corollary 14.12(c) already records, and the first-round equations (ii). What the first-round equations add is decided cluster by cluster, and which beginnings some round order admits is decided by a counting law for tokens across cuts.

For a state z with every touched clock dressed, say that *z meets its phase* if some bit vector y solves the first-round equations (ii) of (13.1) with the flip pattern of z, agrees with z on every P clock, and gives every A-cluster without walls the token number it has in z.

Theorem 14.31 (What the round order adds to fair timing)**.** *For every network E and every classical state q with $U(q) \neq \emptyset$: q is born if and only if q is fairly kept, the lines of $q^\sharp$ satisfy condition (i) of* (13.1)*, and $q^\sharp$ meets its phase.*

Proof. By Theorem 14.5, q is born exactly when it is a fair birth (Corollary 14.12(a)) whose dressing reaches a parity-law state by awake postings. A parity-law state with the lines of $q^\sharp$ is condition (i) together with a solution y of (ii). Awake postings keep every P bit and the token number of every cluster without walls (Lemma 14.3); under the fair law every cluster with walls is open, so it has a source and a sink and its tokens may rise or fall freely. So y is reachable exactly when it keeps the P bits and the wall-free token numbers. □

Neither added condition is implied by the others.

Proposition 14.32 (Phase and charge both bind)**.** *(a) On the star with contacts* $(0,1)$*,* $(0,2)$*,* $(0,3)$*,* $(0,4)$*, the genesis with every bit* 0*, centre a, leaf* 1 *on p and leaves* $2,3,4$ *on P is a fair birth whose dressing satisfies condition (i), and it is not born. No beginning of this star is born in this order, while the eight beginnings with centre a, one leaf p, the other leaves P and all bits equal are degree-balanced fair births.*

(b) On the six-cycle with contacts $(0,1)$*,* $(1,2)$*,* $(2,3)$*,* $(3,4)$*,* $(4,5)$*,* $(5,0)$ *and lines a, a, A, A, A, A, the genesis with bits* $(0,0,0,1,1,0)$ *is a degree-balanced fair birth that is not born, while the genesis with bits* $(0,0,0,1,0,0)$*, which has the same dressed lines, is born.*

(c) On the four-cycle the 32 *fair births are exactly its births. The complete network on four sites has* 48 *fair births and no birth in either of the round orders of Example 12.21.*

Proof. (a) The dressing has the centre A with bit 0 and the leaves P with bits $1,0,0,0$. Condition (i) holds since the centre has degree 4. The centre's first-round equations ask $y_0 = x_{\mathrm{leaf}} + 1 + v_i(0)$ at its i-th contact, so the leaf bits must alternate along the centre's postings; three leaves of one bit and one of the other cannot. (b) The dressing is one wall-free cluster with 2 tokens, or 1 token for the second genesis. The all-A first-round solutions of this order are $(1,0,0,0,0,0)$ and $(0,1,1,1,1,1)$, with 1 and 5 tokens. The census statements of (a) and (c) are direct exact computations over all basis states. □

14.5.1 The phase is decided cluster by cluster

Fix a state z with every touched clock dressed and an A-cluster C. A contact joins two P clocks, a wall to a clock of C, or two clocks of one cluster, so no first-round equation couples two

clusters. The *internal equations* of C are the equations (ii) of the contacts inside C:

$$d_u + d_v = 1 + \nu_i(u) + \nu_i(v) \qquad \text{for every position } i \text{ joining two clocks } u, v \text{ of } C.$$

C is *cycle-consistent* if they have a solution d, an *offset* of C. Two offsets differ by a constant on C, since equal differences propagate along every contact. Given an offset d, C satisfies the *wall law* if $z_b + \nu_i(c) + d_c$ takes one value over all its wall contacts i joining a wall b to a clock c, and the *charge law* if $\mathrm{tok}_z(C)$ equals $\mathrm{tok}_d(C)$ or $|C| - \mathrm{tok}_d(C)$. Neither law depends on the offset chosen.

Theorem 14.33 (The phase is local)**.** *Let z have every touched clock dressed, and choose an offset for every cycle-consistent cluster. Then z meets its phase if and only if every contact of two P clocks is mixed, every A-cluster is cycle-consistent, every cluster with walls satisfies the wall law and every cluster without walls satisfies the charge law. On a state obeying the fair law the clusters with walls are the open ones and those without are the closed ones. So a beginning q is born if and only if it is a fair birth, the lines of $q^\sharp$ satisfy condition (i), every cluster of $q^\sharp$ is cycle-consistent, every open cluster satisfies the wall law and every closed cluster the charge law.*

Proof. A first-round solution restricts to each cluster; it solves the internal equations, and its wall equations read $y_c = z_b + 1 + \nu_i(c)$. Conversely solutions chosen cluster by cluster glue, because every equation lies inside one cluster or is a mixed P pair. On one cluster the solutions of the internal equations are d and $d + 1$; the wall equations fix the constant exactly when $z_b + \nu_i(c) + d_c$ is constant over the wall contacts, and for a wall-free cluster the two candidates carry $\mathrm{tok}_d(C)$ and $|C| - \mathrm{tok}_d(C)$ tokens. □

The four-leaf star fails the wall law for every offset, and the six-cycle fails the charge law. The order enters the local laws only through the ranks ν_i at the cluster's own clocks. On a cycle the laws can be solved in closed form for every order. Call a contact of a cycle C_L *extremal* in a round order if it posts after both neighbouring contacts or before both.

Theorem 14.34 (Cycles in every order)**.** *Let E be any round order of the $L \geq 3$ contacts $(j, j+1)$ of the cycle C_L (indices modulo L), and give every clock the line A.*

(a) The right-hand side of the first-round equation of a contact is 1 exactly when the contact is extremal. The solutions are λ, with $\lambda_v = 1$ exactly when the contact $(v-1, v)$ posts before $(v, v+1)$, and its complement $\lambda + 1$; they flip exactly across extremal contacts. So every cluster of every state of a cycle is cycle-consistent in every order.

(b) The all-A cycle meets its phase in this order exactly when its token number is t_E or $L - t_E$, where $t_E = \mathrm{tok}(\lambda)$, the number of sites whose left contact posts first. Over all orders t_E takes exactly the values $1, \dots, L-1$.

(c) A beginning of C_L is born in some round order of its contacts if and only if it is a fair birth, for every L. The births of C_L are the same in every round order exactly when $L \leq 4$.

(d) In the cyclic order $((0,1),(1,2),\dots,(L-1,0))$, $t_E = L-1$, so the admitted charges are 1 and $L-1$. For the beginnings with clocks $0, 1$ on a, every other clock A, bits 0 at sites $L-1, 0, 1, 2$ and at least one token, the beginning is a fair birth, and it is born exactly when it has one token. Fair birth plus condition (i) implies birth on C_L in the cyclic order exactly when $L \leq 5$.

(e) On the five-cycle, the beginning with one token at site 3 of the family in (d) is born in the order $(0,1)$, $(1,2)$, $(2,3)$, $(3,4)$, $(4,0)$ and not in the order $(0,1)$, $(1,2)$, $(4,0)$, $(3,4)$, $(2,3)$, whose all-A first-round solutions carry 2 and 3 tokens.

Proof. (a) At the position of $(j, j+1)$ the rank of j is 1 if $(j-1, j)$ posted earlier, and the rank of $j+1$ is 1 if $(j+1, j+2)$ posted earlier, so $1 + \nu(j) + \nu(j+1)$ is 1 exactly when both or neither posted earlier. The solution λ is checked equation by equation. (b) The charge law with offset λ; the order with only $(0, 1)$ first and $(t, t+1)$ last among extremal contacts has $t_E = t$. (c) A walled cluster of a fair birth on a cycle is an arc with two wall contacts of different bits, which meets the wall law in every order; a closed cluster is the whole cycle, admitted by an order with t_E equal to its token number. Order-freeness for $L \leq 4$ is the triangle and the four-cycle, by direct exact computation over all orders; for $L \geq 5$, (e) and its analogues give a one-token beginning born in the cyclic order and not in an order of charge 2. (d) A fair birth whose undressed pair is a, a keeps the pair and its two neighbours on one bit b, so its tokens are $4b$ plus those of the $L - 4$ free sites. For $L = 5$ that is one or four, the two admitted charges; for $L \geq 6$ the family with $b = 0$ and two tokens is a degree-balanced fair birth that is not born. The cases $L = 3, 5$ are direct exact computations over all basis states and $L = 4$ is (c) of Proposition 14.32. □

14.5.2 Some order: turns, half charge and token flow

Which beginnings does some round order of the same contacts admit? Write *born in some order* for born in some round order E' that lists the contacts of E in another order.

Proposition 14.35 (Orders glue cluster by cluster)**.** *The phase of an A-cluster depends on the round order only through the relative order of the contacts that touch the cluster, and no contact touches two clusters. If each cluster of $q^\sharp$ meets its phase in some order, one order does so for all clusters at once. So q is born in some order if and only if it is a fair birth, the lines of $q^\sharp$ satisfy condition (i), and every cluster of $q^\sharp$ is admitted by some order, which is decidable cluster by cluster.*

Proof. The internal equations, wall equations, walls and tokens of a cluster read only its contacts and their ranks at its clocks. An order is assembled by posting the chosen sublist of each cluster in turn and then the remaining contacts. Born in an order E' is fair birth for E (which depends only on the set of contacts), condition (i), and the phase in E', by Theorem 14.31. □

Cycle-consistency has a combinatorial form. Around a cycle of a cluster, a clock where the cycle arrives along one contact and leaves along another makes an *even turn* if the ranks of the two contacts at that clock have equal parity, that is, if an odd number of the clock's other contacts post between them.

Theorem 14.36 (Even turns, half charge and all charges)**.** *Let z have every touched clock dressed and let C be an A-cluster.*

(a) If no two contacts of E join the same two sites, C is cycle-consistent if and only if every cycle of C has an even number of even turns. A clock of degree at most 2 never makes an even turn, so trees and clusters of degree at most 2 are cycle-consistent in every order; at a clock of degree 3 a turn is even exactly when the third contact posts between the two. In the order $(0, 1), (2, 3), (0, 2), (1, 3), (0, 3), (1, 2)$ of the complete network on four sites, the triangle $0, 1, 2$ has exactly one even turn, so no all-A state is cycle-consistent there.

(b) Every network has a round order of its contacts in which every cluster of every state is cycle-consistent.

(c) Let C have no walls. In every order every offset carries between 1 and $|C|-1$ tokens. If every clock of C has odd degree, every offset carries exactly $|C|/2$ tokens, and some order admits C if and only if $2\,\mathrm{tok}_z(C) = |C|$. If every clock of C has even degree, then for every t with $1 \le t \le |C|-1$ some order has an offset with t tokens.

(d) If the maximum degree is at most 2 and no two contacts join the same two sites, a beginning is born in some order if and only if it is a fair birth whose dressed lines satisfy condition (i).

(e) If $n \le 4$ and no two contacts join the same two sites, the births are the same in every round order. With Theorem 14.34(e), the five-cycle is a smallest network whose births depend on the order.

Proof (mechanism). (a) The right-hand sides $1 + v_i(u) + v_i(v)$ summed around a closed walk regroup clock by clock into the number of even turns; vanishing sums around every cycle give a potential, and a closed walk reduces to cycles and back-and-forth steps. (c) Each contact has exactly one end at which $y + \nu$ equals 1, so a clock of odd degree d carries $(d-1)/2 + y_c$ such ends; summing gives $2\,\mathrm{tok} = |C|$. The first internal contact posted has rank 0 at both ends, so its ends differ, giving the bounds. For even degrees, a token schedule balances at every clock of even degree, a closed walk through all of the cluster's contacts exists by Hierholzer's construction, and a single token run along it with suitable splitting realizes every charge. (b) Every list of contacts has a reordering that is a token schedule. (d) The degree-balanced clusters at degree 2 are isolated contacts, arcs with two walls and whole cycles, handled by (a), the wall law at degree 2, and Theorem 14.34. (e) On four sites the only clusters whose phase could depend on the order are all-A four-cycles and complete graphs, and a non-neutral beginning with a matched undressed pair has one bit on every site of such a network, so its cluster is empty or full and not fair. □

Walled clusters need a flow of tokens.

Definition 14.37 (Token flows and cut counts)**.** Let C be an A-cluster of z with walls. A *token flow* of C directs every contact of E (the direction in which it passes its token) so that a contact from a wall of bit 1 to C points out of the wall, a contact from C to a wall of bit 0 points into the wall, and at every clock of C as many contacts point out as point in. For a set S of clocks of C let $\mathrm{src}(S)$ and $\mathrm{snk}(S)$ count the contacts joining a wall of bit 1, respectively of bit 0, to a clock of S, and $\mathrm{cut}(S)$ the contacts inside C with exactly one end in S. *S passes the cut count* if $\mathrm{src}(S) \le \mathrm{snk}(S) + \mathrm{cut}(S)$ and $\mathrm{snk}(S) \le \mathrm{src}(S) + \mathrm{cut}(S)$. The cluster has *balanced walls* if $\mathrm{src}(C) = \mathrm{snk}(C)$.

A first-round solution is a token schedule: every posting meets differing bits and passes one token across its contact. An A clock flips at each posting, so over a round it passes on as many tokens as it receives; a wall only writes (bit 1) or removes (bit 0).

Theorem 14.38 (Walled clusters: flows and cuts)**.** *Let z have every touched clock dressed and let C be an A-cluster with walls, every clock of even degree.*

(a) Some round order admits C if and only if C has a token flow. Every token flow is, on the contacts touching C, the token motion of a first-round solution in some round order, and the motion of every admitting order is a token flow.

(b) C has a token flow if and only if every set of its clocks passes the cut count. It suffices to test the sets S that are connected in C and whose complement in C is empty or connected; with balanced walls, one of the two inequalities for every S implies the other. For a single clock c the cut count reads $2\,\mathrm{src}(c) \le \deg c$ *and* $2\,\mathrm{snk}(c) \le \deg c$. *The cut law is decidable.*

(c) Balanced walls are necessary and not sufficient, and so are the single-clock bounds together with balanced walls, already at maximum degree 4 *with one contact per pair of sites (Appendix A.7).*

Proof. (a) Necessity: the token motion of a first-round solution balances at every clock of even degree, and a wall step leaves the wall exactly when it holds bit 1. Sufficiency: pair the sink and source wall contacts into walk steps, cover the cluster by one closed walk (Hierholzer), and run one token along it; the resulting order realizes exactly the given flow. (b) Balance at a clock c asks that exactly $\mathrm{need}(c) = \frac{1}{2}(\mathrm{int}(c) + \mathrm{src}(c) - \mathrm{snk}(c))$ of its internal contacts point out of it, where $\mathrm{int}(c)$ counts its internal contacts. By Hall's theorem [19], applied to the internal contacts against $\mathrm{need}(c)$ copies of each clock, tails with at most $\mathrm{need}(c)$ at each clock exist exactly when every set of internal contacts is at most the needs of the clocks it touches, which is $\mathrm{snk}(S) \le \mathrm{src}(S) + \mathrm{cut}(S)$; the whole-cluster count then makes every clock receive exactly its need. Disconnected parts add, and the complement of S gives the other inequality under balanced walls. □

Theorem 14.39 (The counting law of births)**.** *For every network E, parallel contacts allowed, and every classical state q with* $U(q) \neq \emptyset$*: q is born in some round order of the contacts of E if and only if*

(i) q is fairly kept,
(ii) the lines of $q^\sharp$ *satisfy condition (i) of* (13.1)*,*
(iii) every cluster of $q^\sharp$ *without walls whose clocks all have odd degree holds exactly half its clocks as tokens, and*
(iv) every cluster of $q^\sharp$ *with walls passes every cut count.*

The conditions are decidable.

Proof. By Proposition 14.35 it suffices to decide each cluster. Condition (ii) makes the degree parity constant on every cluster and even on every walled cluster, since a wall has flip bit 0. A walled cluster is then decided by Theorem 14.38. A closed cluster of a fair birth has between 1 and $|C| - 1$ tokens; if its degrees are odd, Theorem 14.36(c) asks half charge, and if they are even, some order admits its charge. □

All four conditions bind (Appendix A.7): the four-leaf star fails the cut count at its centre (walls $1, 0, 0, 0$, W2); a seven-site degree-balanced fair birth fails half charge (W4); and a nine-site one with balanced walls, half charge and even degrees fails the flow at a clock carrying three source walls among four contacts, which cannot alternate (W5). Each is born in no order.

14.5.3 The flow as a unit current, and its reach

A token flow is a current. Give a walled cluster C its *surplus* $s(c) = \mathrm{src}(c) - \mathrm{snk}(c)$ at each clock.

Theorem 14.40 (Unit currents on trees and cycles)**.** *Let z have every touched clock dressed and let C be an A-cluster with walls, every clock of even degree.*

(a) A token flow is the same as a direction of each internal contact, carrying one unit of current, with net outflow $s(c)$ at every clock c. At every clock, $s(c)$ has the parity of the number of internal contacts at c.

(b) Let no two contacts join the same two sites and let C be a tree. Removing an internal contact splits C into two sides; let Σ be the total surplus of a side. Then C has a token flow if and only if its walls balance and $|\Sigma| \le 1$ for both sides of every internal contact. The flow is then unique: the current leaves the side with $\Sigma = 1$.

(c) At maximum degree 2 (equivalently, under even degrees, at most 3), C is admitted by some order exactly when it has one source wall contact and one sink wall contact, that is, when its walls balance.

(d) Let no two contacts join the same two sites and let C be a cycle $c_0, \dots, c_{L-1}$, $L \ge 3$, with partial sums $P_j = s(c_0) + \cdots + s(c_j)$. Then C has a token flow if and only if $\sum_j s(c_j) = 0$ and $\max_j P_j - \min_j P_j \le 2$; equivalently, every $s(c_j)$ is 0 or ± 2 and the nonzero values alternate in sign around the cycle. The flow is unique if some surplus is nonzero; if all are zero there are exactly two, the two circulations.

Proof. (a) At a clock, outflow minus inflow over wall contacts is $\mathrm{snk}(c) - \mathrm{src}(c)$, so balance is net internal outflow $s(c)$; the degree is even. (b) Under any current, a side's net outflow across the single contact joining it to the rest is its surplus, which must be ± 1; conversely the sides are exactly the connected sets with connected complement, each crossed by one contact, so (b) is the cut law. (c) A cluster at maximum degree 2 has at most two wall contacts. (d) The current on the pair (c_j, c_{j+1}) is $\kappa + P_j$ for a constant κ, and each current is ± 1. □

Theorem 14.41 (Flows differ by cycles; how far a count must reach)**.** *Let z have every touched clock dressed, let C be an A-cluster with walls, and let E list no contact twice.*

(a) The internal contacts on which two token flows disagree split into edge-disjoint simple directed cycles of the first flow; reversing them gives the second. Reversing any directed cycle of a token flow gives a token flow. A flow is the only one on the contacts touching C if and only if it has no directed cycle, and then no flow has one. Any two flows are joined by a chain of single cycle reversals, all through token flows, of length at most the number of disagreeing contacts, and at most a third of it when no two contacts join the same two sites. Trees have no directed cycle; a cycle cluster has one exactly when all surpluses vanish.

(b) If every clock of C has even degree, C has a token flow if and only if its walls balance and every set of at most $|C|/2$ of its clocks passes the cut count.

(c) For every k there are walled clusters with one contact per pair of sites, maximum degree 4, even degrees and balanced walls, all of whose sets of at most k clocks pass the cut count, and with no token flow: a cycle of $4k + 4$ clocks and a path of $2k + 4$ clocks. On the path every set of at most $k + 1$ clocks passes and a set of $k + 2$, half the cluster, fails. For every k there are fair births with the same properties, half charge and balanced walls, born in no round order.

Proof (mechanism). (a) Both flows are currents with the same sources, so their disagreement is balanced at every clock and splits into closed walks through distinct clocks. (b) Under balanced walls a set passes exactly when its complement does. (c) The families are listed in Appendix A.7. On the path, the side of clock $k + 1$ at the middle contact has surplus

3, violating Theorem 14.40(b), while every shorter set has |surplus| at most its number of boundary contacts. On the cycle, $P_{k+1} = 4$ violates Theorem 14.40(d). Below degree four every clock of even degree has degree two, where balanced walls decide, so four is the least maximum degree at which no bounded count suffices. □

14.6 Which orders admit, and how often

The chance that a uniformly random round order admits a walled cluster is a product of local factors when the busy clocks share no loop, and every loop of shared contacts raises it by an exact factor.

Fix a state z with every touched clock dressed and an A-cluster C, and let E list no contact twice. A round order E' of the contacts of E *realizes* a token flow if some first-round solution in E' has, on the contacts touching C, exactly that flow as its token motion.

Theorem 14.42 (Alternation decides which flow an order realizes)**.** *(a) E' realizes a token flow of C if and only if at every clock c of C the contacts at c, read in the order in which E' posts them, alternate between pointing out of c and into c.*

(b) If C has walls, one order realizes at most one flow. If C has no walls, the motions of two first-round solutions in one order agree on the contacts of C or are reversed, and their token numbers are t and $|C| - t$.

(c) Let C have walls and every clock even degree. If E' realizes a flow and E'' realizes the flow with a directed cycle Γ reversed, then at some clock c of Γ of degree at least 4, with a contact off Γ, the contacts at c are posted in different relative orders by E' and E''.

(d) If every clock of C has degree 2, every order realizes every flow.

Proof. (a) In the first round the bit of a cluster clock c just before position i is $y_c + v_i(c)$, and a posting at differing bits passes the token out of c exactly when that bit is 1. So the postings at c leave and arrive alternately; conversely, an alternating pattern determines the parities, and these give a solution whose internal equations hold because exactly one end of each contact is left. (b) On a walled cluster two solutions differ by a constant that a wall equation fixes. (c) Every directed cycle of a walled cluster has a clock with a contact off the cycle, and two alternating readings of one length agree everywhere or nowhere; a clock of degree 2 on the cycle has no freedom. (d) At a clock of degree 2 any reading alternates. □

Call a clock of C of degree at least 4 a *many-contact clock*, and let the *sharing graph* join two many-contact clocks that share a contact.

Theorem 14.43 (The admission count on loop-free sharing)**.** *(a) Among the $m!$ orders of m items, a fixed set of $2r$ items of which r are marked is read in alternating marks in exactly a fraction $1/\binom{2r-1}{r}$ of the orders, also conditionally on the rank of any one item of the set.*

(b) Let C have walls, let no two contacts of E join the same two sites, let every clock of C have even degree, and let the sharing graph be a forest. Then the number N_{adm} of round orders of E that admit C and the number N_{flow} of token flows of C satisfy

$$N_{\mathrm{adm}} \prod_{c \in C} \binom{\deg c - 1}{\deg c / 2} = m!\, N_{\mathrm{flow}}.$$

(c) *If moreover C is a tree with every degree 2 or 4 and has a token flow, then $N_{\mathrm{adm}} \cdot 3^{N_4} = m!$, where N_4 is the number of four-contact clocks.*
(d) *The forest hypothesis is needed. Let three A clocks form a triangle, and join each of them to a wall of bit 1 and to a wall of bit 0 (one wall of each bit serves all three in Appendix A.7, W9; nine contacts, every clock of degree 4). Then $N_{\mathrm{flow}} = 2$, each flow is realized by 13536 of the $9!$ orders, and $N_{\mathrm{adm}} = 27072$, while the formula of (b) would give 26880.*

Proof. (a) Ranks of alternating readings are fixed by parity; for every order exactly $(r-1)!\,r!$ of the $(2r-1)!$ rearrangements of the set minus one item alternate. (b) By Theorem 14.42, N_{adm} is the sum over flows of the number of orders alternating at every clock, and an order alternates at one flow at most. On a forest of shared contacts some many-contact clock shares at most one contact with the rest; averaging over the rearrangements of its private postings removes its factor, and induction over such leaves gives the product. (c) A tree has one flow (Theorem 14.40(b)) and $\binom{3}{1} = 3$. (d) Direct exact computation over the $9!$ orders. □

On a closed cycle the order selects a charge, and its law over random orders is Eulerian. Write $A(n,k)$ for the number of permutations of n items with k descents.

Theorem 14.44 (The charge a random order admits on a cycle)**.** *Let every clock of the cycle C_L, $L \geq 3$, be A with t tokens, $1 \leq t \leq L-1$. The fraction of the $L!$ round orders of its contacts that admit it is*

$$\frac{2A(L-1,t-1)}{(L-1)!} \quad \text{if } 2t \neq L, \qquad \frac{A(L-1,t-1)}{(L-1)!} \quad \text{if } 2t = L.$$

Proof. Call v a cyclic descent of an order if $(v, v+1)$ posts before $(v-1, v)$. Then t_E of Theorem 14.34 plus the number of cyclic descents is L, so the order admits t exactly when the number of cyclic descents is t or $L-t$. Rotating the posting times preserves cyclic descents, and fixing the contact $(0,1)$ first leaves one more than the descents of the rest; so exactly $L \cdot A(L-1,k)$ orders have $k+1$ cyclic descents. □

For loops of shared contacts a continuous description is exact. Posting each contact of a round at an independent uniform time in $[0,1]$ gives a uniformly random round order, since the order of m independent uniform times is uniform over the $m!$ orders. Write $\tilde{B}_k(x) = B_k(x - \lfloor x \rfloor)$ for the periodic Bernoulli functions, so $\tilde{B}_2(x) = \{x\}^2 - \{x\} + \frac{1}{6}$. A posting at a many-contact clock is *shared* if its contact also meets another many-contact clock, and *private* otherwise. For a flow, the *alternation chance* α_c of a many-contact clock is the chance that its postings alternate, given the times of its shared postings, its private postings drawn afresh.

Theorem 14.45 (The admission law with loops)**.** *Let C have walls, let no two contacts join the same two sites, and let every clock of C have even degree.*

(a) *For every sharing pattern, $N_{\mathrm{adm}} = \sum_{\text{flows}} m!\,\mathbb{E}\big[\prod_c \alpha_c\big]$, the expectation over independent uniform times of the shared postings and the product over the many-contact clocks.*
(b) *Four postings p, q pointing out and r, s pointing in alternate exactly when exactly one of r, s is posted between p and q. A clock of degree $2r$ with all postings private but one has $\alpha_c = 1/\binom{2r-1}{r}$. With exactly two shared postings at gap δ,*

$$\alpha_c = \frac{2r-1}{r\binom{2r}{r}}\big(1 + (1-2\delta)^{2r-2}\big) \text{ or } \alpha_c = \frac{2r-1}{(r-1)\binom{2r}{r}}\big(1 - (1-2\delta)^{2r-2}\big),$$

according as the two shared postings point in different directions or the same. At degree 4 *these are* $\frac{1}{2} - \delta + \delta^2 = \frac{1}{3} + \tilde{B}_2(\delta)$ *and* $2\delta(1-\delta) = \frac{1}{3} - 2\tilde{B}_2(\delta)$.

(c) *Let every many-contact clock share at most two postings. Then the vertex sets of the cycles of the sharing graph, its* loops, *are pairwise disjoint, and*

$$N_{\mathrm{adm}} = \sum_{\text{flows } o} m! \prod_{c} \binom{\deg c - 1}{\deg c/2}^{-1} \prod_{\text{loops } \Gamma} \big(1 + \Lambda_\Gamma(o)\big), \qquad \Lambda_\Gamma(o) > 0.$$

At degree 4, *with* $\varepsilon_c = 1$ *when the two shared postings of* c *point in different directions and* $\varepsilon_c = -2$ *when they point the same way,*

$$\Lambda_\Gamma(o) = 3^L \Big(\prod_{c \in \Gamma} \varepsilon_c\Big) (-1)^{L-1} \frac{2^L B_{2L}}{(2L)!}, \qquad L = |\Gamma|;$$

the number of clocks with $\varepsilon_c = -2$ *on a loop is even. In general* $\Lambda_\Gamma(o)$ *is a finite sum of Bernoulli numbers equal to a Fourier series with nonnegative terms.*

(d) *Hence* $m!\, N_{\mathrm{flow}} \le N_{\mathrm{adm}} \prod_c \binom{\deg c - 1}{\deg c/2}$, *with equality exactly when* $N_{\mathrm{flow}} = 0$ *or the sharing graph is a forest.*

Proof (mechanism). (a) Coordinate permutations preserve the product measure, so each round order is a region of times of measure $1/m!$; given the shared times, the private postings of different clocks are independent. (b) The four-posting rule is a check over the rankings. For degree $2r$, exchangeability reduces α_c to the number of rearrangements of the private postings that alternate, which is fixed by whether the ranks of the two shared postings have equal parity; each private posting falls between them with chance δ, independently, so the sign of the parity has mean $(1-2\delta)^{2r-2}$. (c) Expanding $\prod_c(\mu_c + \psi_c)$, with μ_c the local fraction and ψ_c the centred part of α_c, a set of clocks that is not a union of loops integrates to 0, because some shared time is read by one clock only. Along a loop, the convolution identity $\tilde{B}_n * \tilde{B}_k = -\frac{n!\,k!}{(n+k)!}\tilde{B}_{n+k}$ merges the edges one shared time at a time and closes with a Bernoulli number. Around any closed walk of the sharing graph the flow switches direction at an even number of clocks, so every loop product is positive: loops are never frustrated. At degree $2r$ the centred chance is a combination of $\tilde{B}_2, \dots, \tilde{B}_{2r-2}$ whose cosine coefficients are nonnegative for different directions and nonpositive for the same, as the cosine transform of the decreasing convex weight $(1-s)^{2r-2}$. (d) Every factor $1 + \Lambda_\Gamma$ exceeds 1, and a forest has no loop. □

On the triangle of Theorem 14.43(d), one loop of three clocks with $\varepsilon = 1$ gives $1 + \Lambda = \frac{141}{140}$, so each flow is alternated with chance $\frac{1}{27} \cdot \frac{141}{140} = \frac{47}{1260}$, which is $9! \cdot \frac{47}{1260} = 13536$ orders. The same triangle of six-contact clocks, each joined to two walls of bit 1 and two of bit 0 (seven sites, fifteen contacts), has two flows, $\Lambda = \frac{9344}{1702701}$, and $N_{\mathrm{adm}} = 2629701120$ against 2615348736 from the product law.

It is open whether the inequality of (d), $m!\, N_{\mathrm{flow}} \le N_{\mathrm{adm}} \prod_c \binom{\deg c - 1}{\deg c/2}$, holds for every walled cluster with one contact per pair of sites and even degrees whatever the sharing, and open even when every many-contact clock has degree four. Two facts constrain it. Four postings alternate exactly when the linking number of the chord of the outgoing pair with the chord of the incoming pair is ± 1, so the alternation indicator is the square of a sum of sawtooth

functions. And termwise positivity fails: on the complete bipartite network $K_{2,3}$ of four-contact clocks, one term of the expansion of $\prod_c \alpha_c$ equals $-\frac{1}{467775}$. A second open question concerns reach: whether, for some fixed k, on every tree cluster with balanced walls, one contact per pair of sites and even degrees at most four, cut counts on all sets of at most $k(d+1)$ clocks, d the diameter of the cluster, imply a token flow. The expected answer, that no such k exists, is also unproved.

14.7 What an unrecorded timing leaves readable

So far timing decided which states survive. When the postings are not recorded, timing also decides what of the past the composite can still read. We take the smallest composite, the path with shared site c and outer sites a, b, contacts (c, a) and (c, b), and let both contacts post unread. The lasting content turns out to be the kept states of Chapters 12 and 13, read as statistics: what every timing keeps is the co-oriented pairs, strict alternation adds the relays, paired timing adds the all-A oscillations instead, and every site can tell from its own statistics which kind of timing occurred (a part *reads* a distinction when its statistics have different limits or laws under the two cases).

Statistics and their pullbacks. A *statistic* is an operator Y on the three-site carrier, read in a density ρ as $\mathrm{tr}(\rho Y)$. An unread posting of (c, a) pulls it back by $\Phi_{ca}(Y) = \sum_j V_j^{ca*} Y V_j^{ca}$, and likewise Φ_{cb}; on a product, $\Phi_{ca}(X \otimes B) = \Phi(X) \otimes B$ with Φ the free pull of Chapter 11 on the pair (c, a). A word $w = (w_1, \dots, w_k)$ of unread postings, w_1 posting first, pulls back by $\Phi_w = \Phi_{w_1} \circ \dots \circ \Phi_{w_k}$. On one clock the six *pointer operators* are

$$\kappa_0 = |p\rangle\langle p|,\ \kappa_1 = |a\rangle\langle a|,\ \kappa_2 = |p\rangle\langle a| + |a\rangle\langle p|,\ \kappa_3 = |P\rangle\langle P|,\ \kappa_4 = |A\rangle\langle A|,\ \kappa_5 = |P\rangle\langle A| + |A\rangle\langle P|,$$

the undressed and dressed pole projections and their coherences. The *readable statistics* of one site are the 12-dimensional span $\mathcal{S}_1$ of $|\beta\rangle\langle\beta| \otimes \kappa_k$, and the readable three-site statistics are the products: every readable statistic has unique coordinates in the $12^3 = 1728$ products of pointer statistics. Weight κ_2, κ_5 by 2 and the others by 1, and let $\langle \cdot, \cdot \rangle_\nu$ be the inner product of pointer coordinates with the product weights; a ν-orthogonal projection acts on real and imaginary parts separately. For a classical state q write $[q] = |\psi(q)\rangle\langle\psi(q)|$, sites in the order (c, a, b), and let $H = I \otimes L^2 \otimes L^2 \otimes L^2$ be the joint half-turn of the three clocks; L^2 is $+1$ on the undressed plane and -1 on the dressed plane, so H reads the parity of the number of dressed clocks. Name

$$\begin{aligned} \Pi_+ &= [(1,0,0),(P,P,P)], & \Pi_- &= [(0,1,1),(P,P,P)], \\ R_1 &= [(0,1,0),(A,P,P)], & R_2 &= [(1,0,1),(A,P,P)], \\ R_1' &= [(1,1,0),(A,P,P)], & R_2' &= [(0,0,1),(A,P,P)], \\ O_1 &= [(0,1,1),(A,A,A)], & O_2 &= [(1,0,0),(A,A,A)]. \end{aligned}$$

$\Pi_\pm$ are the co-oriented pairs of the path and R_1, R_2 its relays (Corollary 13.6(b)), R_1', R_2' the relays of the other phase, and O_1, O_2 two all-A oscillations.

Theorem 14.46 (One unrecorded contact keeps its classes)**.** *(a) A readable statistic of the pair (c, a) is fixed by Φ if and only if it is a combination of the five class projections $\Lambda_0, \dots, \Lambda_4$ of Definition 10.13, and reversed if and only if it is a multiple of the phase read $\Psi_\circ$. A readable*

three-site statistic Y has $\Phi_{ca}(Y) = Y$ if and only if $Y \in \mathrm{span}\{\Lambda_k \otimes B : B \in \mathcal{S}_1\}$, a space of dimension 60, and $\Phi_{ca}(Y) = -Y$ if and only if $Y \in \Psi_\circ \otimes \mathcal{S}_1$, of dimension 12. The same holds for (c, b) with a and b exchanged.

(b) Let k unread postings of (c, a) be followed by any later list of postings of the two contacts, each read or unread, and let E_k be the effect of the whole record. For basis states r, r' of site a, the slices $\langle r|E_k|r'\rangle$, operators on (c, b), converge as $k \to \infty$ along each parity of k. Their limits lie in the 60-dimensional span of the products of c's five statistics I, $|\beta\rangle\langle\beta| \otimes \kappa_3$, $|\beta\rangle\langle\beta| \otimes \kappa_4$ ($\beta = 0, 1$) with $\mathcal{S}_1$ of b, and two read postings of (c, b) as the later list already attain all 60 at each parity. There is one constant C such that every coordinate of every slice, for every later list and every $k \geq 3$, is within $C(1/2)^{\lfloor (k-3)/12 \rfloor}$ of its parity limit.

Proof (mechanism). (a) The unread posting is the free pull summand by summand, the class projections and the phase read are readable products, and Theorem 11.6(a) identifies the fixed and reversed reads; on three sites the statistic splits as $\sum_q X_q \otimes$ (pointer statistic q of b) with independent pointer statistics, so it is fixed or reversed exactly when every X_q is. (b) The core functionals of the unread posting, left eigenvectors dual to its fixed and reversed reads, give the parity limit; the rank of the limit slices is certified by direct exact computation. The uniform bound uses that every record effect satisfies $0 \leq E \leq I$, so its entries have modulus at most 1. □

Theorem 14.47 (Two unrecorded contacts: what the schedule keeps)**.** *(a) The readable statistics fixed by both Φ_{ca} and Φ_{cb} are exactly $\mathrm{span}\{I, H, \Pi_+, \Pi_-\}$. No nonzero readable statistic is reversed by both.*

(b) The round, (c, a) then (c, b), fixes exactly $\mathrm{span}\{I, H, \Pi_+, \Pi_-, R_1, R_2\}$ and reverses no nonzero readable statistic. Each relay is exchanged between the contacts: $\Phi_{cb}(R_i) = R_i'$ and $\Phi_{ca}(R_i') = R_i$. The powers of the round converge on every readable statistic to its ν-orthogonal projection onto these six, with $\|(\Phi_{ca}\Phi_{cb})^n(Y) - \lim\| \leq C(1/2)^{\lfloor n/N \rfloor}\|Y\|$ for some N, C; no readable statistic rotates.

(c) Let a word in which both contacts post consist of runs (maximal blocks of one contact) of lengths $n_1, \ldots, n_{2k}$, the first run of (c, a). Its fixed readable statistics are $\mathrm{span}\{I, H, \Pi_\pm, R_1, R_2\}$ if every run has length 1; $\mathrm{span}\{I, H, \Pi_\pm, O_1, O_2\}$ if every run has even length; and $\mathrm{span}\{I, H, \Pi_\pm\}$ otherwise. A word uv whose rotation vu is such a run word fixes exactly the images under Φ_u of the statistics vu fixes. The powers of every word in which both contacts post converge to the ν-orthogonal projection onto its fixed statistics, at a geometric rate.

Proof (mechanism). Each unread posting is a contraction for the ν-norm, certified by an exact sum-of-squares identity; a statistic fixed or reversed, or one whose orbit keeps its norm, is therefore isometric at every step, which forces it into the products of seven isometric pointer statistics per site, where the postings act by an explicit table. Exact rank certificates then cut the fixed and isometric spaces to the named statistics. A vector fixed by a ν-contraction is fixed by its adjoint, so the fixed space is reducing and the limit of convergent powers is the orthogonal projection. For runs, two postings of one contact land in that contact's unitary part, and a finite automaton of spans reads the run lengths: every even run keeps the oscillation pair, which each posting exchanges with a partner and the next posting of the same contact restores, and an odd run of length at least 3 followed by two more runs leaves only the common four. □

Theorem 14.48 (Random unrecorded timing)**.** *Let the contact of each posting be drawn from a law under which, given the past, each contact has chance at least $\eta > 0$; independent draws with a floor η are a special case.*

(a) With probability one, for every readable statistic Y, the pullback through the first n postings converges to the ν-orthogonal projection of Y onto $\mathrm{span}\{I, H, \Pi_+, \Pi\_\}$*, the same limit for almost every history, at a rate $C\rho^n$ with $\rho < 1$ fixed and C depending on the history. If each drawn contact posts twice, the limit is the projection onto* $\mathrm{span}\{I, H, \Pi_\pm, O_1, O_2\}$*. The powers of the averaged posting $r\Phi_{ca} + (1-r)\Phi_{cb}$, $0 < r < 1$, converge to the first projection.*

(b) Let blocks be drawn from a finite alphabet of unread words in which both contacts occur somewhere. With probability one the pullbacks converge to the projection onto the statistics that every block fixes, at an almost sure geometric rate. The limit depends only on which blocks are possible. For alternating runs with lengths drawn from a set: if every possible length is 1 *the limit is that of the round, for every history; if every possible length is even it is the projection onto* $\mathrm{span}\{I, H, \Pi_\pm, O_1, O_2\}$*; otherwise onto* $\mathrm{span}\{I, H, \Pi_\pm\}$*. Posting each drawn contact k times keeps the four for odd k and the six with O_1, O_2 for even k.*

(c) Some single contact, posted alone and unread, moves O_1; the alphabet of that one posting keeps exactly the statistics it fixes, yet its pullbacks of O_1 alternate for ever and have no limit. The hypothesis that both contacts occur cannot be dropped.

Proof (mechanism). A block in which both contacts post shrinks the ν-norm by a factor $s < 1$ off the space W it fixes. Given the floor, the chance that the first M aligned blocks of the history contain few copies of a fixed contracting block decays geometrically, uniformly in the past, so by Markov's inequality and the Borel–Cantelli lemma almost every history contracts at a geometric rate. A law with a conditional floor is dominated, event by event, by the finite-word law of its own conditional chances, which brings history-dependent laws under the same bound. Gluing: a concatenation of the possible blocks fixes exactly what every block fixes. For one contact alone, a statistic fixed by both postings would be common, which O_1 is not, while the doubled posting fixes it. □

Theorem 14.49 (Every site reads the timing class)**.** *Let P_C, P_M and P_R be the ν-orthogonal projections onto* $\mathrm{span}\{I, H, \Pi_\pm\}$, $\mathrm{span}\{I, H, \Pi_\pm, O_1, O_2\}$ *and* $\mathrm{span}\{I, H, \Pi_\pm, R_1, R_2\}$*: the limits of unpaired, paired and strictly alternating timing.*

(a) At each site the occupation contrast $(|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes I_{\mathrm{clock}}$, with the identity at the other sites, has three pairwise different images under P_C, P_M, P_R. For the centre, the statistic $|0\rangle\langle 0| \otimes \kappa_4 - |1\rangle\langle 1| \otimes \kappa_4$ at c has images 0, $O_1 - O_2$ and $R_1 - R_2$.

(b) Under alternating runs with lengths drawn from a finite set by a law with a conditional floor, the almost sure limit of such a statistic equals its P_R image if and only if every possible length is 1*, its P_M image if and only if every possible length is even, and its P_C image otherwise. Under k-fold postings it is the P_C image for odd k and the P_M image for even k.*

(c) For a part of the composite, its reading rank *for two limits is the dimension of the span of their differences on the statistics of that part (the identity on the other sites). For the pairs (P_M, P_C), (P_R, P_C), (P_M, P_R) the ranks are $2, 2, 2$ at the centre, $2, 2, 4$ at each leaf, $2, 2, 4$ for each of the pairs (c, a), (c, b), (a, b), and $2, 2, 4$ for the whole. The ranks on complex readable statistics equal these.*

Proof. P_M and P_R are P_C plus explicit rank-two corrections, along two pairs of statistics

orthogonal to the common four and to each other, computed in closed form from the ν-Gram table; the images in (a) are evaluated directly. (b) is Theorem 14.48(b) read through (a). For (c), upper bounds come from the correction directions and lower bounds from test statistics of the part whose differences are independent. On the centre's statistics the exchanged pair O_1, O_2 and the exchanged pair R_1, R_2 have the same profile, since both put c at κ_4 with bit 0 or 1; they differ only at the leaves. □

In particular the outer pair (a, b), which never interacts directly, reads the timing class.

Chapter 15 turns to what a participant's own record reads of the world around it: the shape of its network, its size, and the copies of itself it shares the world with.

Chapter 15

Reading one's world: shape, size and copies

A participant under the awake gate lives in a bounded world: the open clocks its contact can reach and the frozen clocks that wall them in. This chapter asks exactly what the participant's own record can learn about that world. Once the walls are known, one timing fixed in advance reads the world's populations within fewer than n^2 attempts, up to the two constant configurations. Without the walls, reading is possible exactly when every site has a safe exit, and contents and shape together need several copies of the world, whose number, with all records pooled, is a covering number of the routes a single token can take and not, in general, the largest family of mutually exclusive first exits. Under random attempts on a closed whole, one participant reads its pair's transport law, a connected family reads its incidence, and families with a disjoint pair or a star read size and filling up to complement, except on one explicit family of pairs of wholes.

15.1 The gated world and its records

We work on the network carrier of Chapter 12, with contacts $e = (u, v)$ placed on pairs of sites and the pole basis $\psi(x, \ell)$ of classical states. Throughout this chapter every clock is dressed, so $\ell_z \in \{P, A\}$ at every site; the all-dressed pattern is supplied as boundary data, and worlds with undressed clocks are outside the chapter. We call the assignment m of a dressed line to each site a *pattern*, the sites with $m_z = P$ the *walls* and those with $m_z = A$ the *open* clocks. A *token* is a bit 1 on an open clock. The *pattern sector* of m is the span of the basis states $\psi(x, m)$, $x \in \{0, 1\}^n$.

The gate and the attempt sequence. Following Chapter 14, contacts *attempt* postings, and an attempting contact posts only when its own pair is mixed; at a balanced pair the attempt is silent. An *attempt sequence* $g = (g_0, g_1, \dots)$ names the contact attempted at each time; it is *fair* on the network E when its values lie in E and every contact of E is attempted at arbitrarily late times. By Lemma 14.15 an attempt at a mixed pair of dressed clocks flips the bit of an A end and keeps that of a P end, and at a mixed pair flipping a bit is the same as copying the partner's. So, on all-dressed states,

$$\begin{array}{l}\text{an attempt at } (s, s') \text{ replaces the bit of each open end}\\ \text{by the bit of the other end, and changes nothing else.}\end{array} \tag{15.1}$$

(Lemma 14.3); at a balanced pair it changes nothing. Each attempt therefore moves original bits between positions, and a run along a list L of attempts sends the bit configuration x to $x \circ \pi_L$, where $\pi_L(c)$ is the site whose original bit sits at c after the run. We call this the *position*

bookkeeping of the run. Because a label and the output fix the input (Proposition 2.7), the outcome of an attempt at a mixed pair also tells both bits.

The bounded world. For a contact $e = (u, v)$ and a pattern m, the *world* $W_e(m)$ of e is the reach $R(e)$ of Theorem 14.18 at the lines m: the least set of sites containing u and v and every site joined by a contact of E to an open site of the set, walls reached but not passed. By Theorem 14.19, under every attempt sequence and at every depth e's own record effect on the sector of m is a matrix on the sites of $W_e(m)$ tensored with the identity beyond, diagonal in the pole basis.

Records. The *own record* of e to depth T under g is its own gated record (Section 14.3): the outcomes, a label or a silence, of e's attempts among the first T, every other attempt unread. On a basis state of an all-dressed pattern the gated run is deterministic, so the own record is a single list and its law is an indicator. A record *separates* two states when some record has different chances on them, and the participant *reads* whatever its record separates. That is the meaning of "read" in this chapter: a statement about record laws that attaches nothing to the participant beyond its record. Calling the contact e a participant, its world a bounded world and a family of contacts a composite are readings of these objects.

Two basis states $\psi(x, m)$, $\psi(x', m)$ of one pattern are *separable* when x and x' differ somewhere on $W_e(m)$ and are not both constant there. By Theorem 14.19(d) exactly these pairs are separated by e's own record under some fair sequence: states that agree on the world have the same own record law, and a constant world never posts.

15.2 One timing per boundary reads the bounded world

The separating sequences of Chapter 14 depend on the pair of states. The first result removes that dependence: once the walls are known, one sequence reads every separable pair at once, and its depth is polynomial.

Theorem 15.1 (One reader per pattern)**.** *Let E be a network of n sites, $e \in E$, and m an all-dressed pattern. There are a fair attempt sequence g with values in E and a depth T with $T + 1 \le n^2$ such that, for all basis states $\psi(x, m)$ and $\psi(x', m)$ of pattern m, some own record of e at depth T has different laws on them if and only if x and x' differ somewhere on $W_e(m)$ and are not both constant there.*

Proof. The "only if" direction holds for every sequence (Section 15.1). For the other direction we construct one finite list L of contacts of E and continue it by any fair sequence; continuation does not undo a separation, because records of dressed states are deterministic and a longer record extends a shorter one.

By (15.1) and the position bookkeeping, e's outcome at an attempt is silence exactly when the two original bits now at u and v are equal, and otherwise it tells both. Call a site *read* when the record so far determines whether its original bit equals that of u. A label determines both bits absolutely, so once every site of the world is read the record determines the whole configuration on the world, unless every comparison was silent, which happens only for the

two constant configurations. One attempt of e at the start reads u and v. Sort the sites of $W_e(m)$ into layers: layer 0 holds u, v, and layer $k+1$ holds the sites first reached from open sites of layer k; every site of the world lies within n layers. For each unread site c of layer $k+1$, choose a path of at most $k+1$ contacts from c to one of e's sites whose intermediate sites are open sites of earlier layers, and avoiding the other site of e. Attempt the contacts of the path in order and then attempt e. By (15.1) the path carries c's original bit to e's site, and the attempt of e compares it with a position already read, so c becomes read.

The invariant that makes this work is that no unread original bit is ever overwritten: read sites hold read positions and unread sites hold their own. A path moves only bits of open sites already read, and a wall is read in its own layer after the open neighbour that reaches it, so a wall's copy only ever lands on a read position. At the end every site of the world is read, which is the claimed separation. Each newly read site costs at most $D+1$ attempts when the world has D layers, so $|L| + (D+1) \le |W_e(m)|(D+1)$, and with $D < n$ and $|W_e(m)| \le n$ this gives $T = |L|$ with $T + 1 \le n^2$. □

The reader's record is not only separating; it is informationally complete for the populations of the world, sector by sector. For a vector v of the pattern sector of m and a bit configuration a on $W_e(m)$, let $\mathrm{pop}_v(a)$ be the total weight $\sum |v_x|^2$ over the basis states whose bits on $W_e(m)$ are a; it is the diagonal of v's reduced state on the world.

Theorem 15.2 (The reader's record is complete for populations)**.** *Under the sequence and depth of Theorem 15.1, two vectors v, v' of the pattern sector of m have the same own record law for every record if and only if $\mathrm{pop}_v(a) = \mathrm{pop}_{v'}(a)$ for every non-constant configuration a on $W_e(m)$ and $\mathrm{pop}_v(\mathbf{0}) + \mathrm{pop}_v(\mathbf{1}) = \mathrm{pop}_{v'}(\mathbf{0}) + \mathrm{pop}_{v'}(\mathbf{1})$. This holds for superpositions and for states entangled across the walls.*

Proof. The own record effect is diagonal in the pole basis and acts only on the world, so the chance of a record r is $\sum_a \mathrm{pop}_v(a)\,[\mathrm{rec}(a) = r]$, where $\mathrm{rec}(a)$ is the deterministic record of a basis state with world bits a. By Theorem 15.1, $\mathrm{rec}(a) = \mathrm{rec}(a')$ forces $a = a'$ or both constant. So the fibres of rec are single non-constant configurations and the constant pair, and equal laws for every r are exactly equal populations on each fibre. □

No gated record reads coherences, so the completeness is for the diagonal only, and it holds sector by sector. Across patterns it fails: on two sites joined by one contact, the balanced pair of two P clocks and the balanced pair of two A clocks, both with bits 0, have the same own record law under every attempt sequence, while their components in the sector of (P, P) differ. The next result shows that the pattern itself is still written in the record, with a sequence that depends on the pair.

Theorem 15.3 (The record reads the walls)**.** *Let $e \in E$, and let $\psi(x, m)$ and $\psi(x', m')$ be all-dressed basis states of different patterns that agree in bits and lines at every site outside both worlds, and whose bits are not constant on their own worlds. Then e's own record separates them under some fair sequence with values in E.*

Proof. Follow the world of m outward from e's sites. The two patterns first differ either at one of e's sites or at a contact f from an open site t of one world to a site s that is open in one

pattern and a wall in the other. In the first case the first label of e at a mixed state differs, because a label fixes the transition and the transition flips an open end and keeps a wall. In the second case, carry a mixed configuration to t and s by a prefix L_0, then compare the two lists L_0, f, L_t, e and L_0, f, f, L_t, e, where L_t carries t's bit to e. For the state in which s is a wall, the doubled attempt copies twice, which is the same as copying once, so the two lists give it the same record. For the state in which s is open, the doubled attempt exchanges twice, which undoes the exchange, so the two lists give it different records. Hence the two states have different records under at least one of the two lists. □

The cost of reading is transport, and transport through exchanges moves one bit one site per attempt.

Proposition 15.4 (Depth of reading)**.** *1. If every site of $W_e(m)$ is one of e's sites or is joined by a contact of E to an open site of e, one fair sequence reads every separable pair at a depth T with $T + 2 \le 2|W_e(m)|$.*

2. On the path of $n \ge 2$ sites with every clock open and e the end contact $(0, 1)$, a fair reader exists with $T + 1 \le n^2$, and every attempt sequence with values in the path, fair or not, that reads every separable pair at depth T satisfies $n(n-1) \le 2(T+n)$.

Proof. (1) is the construction of Theorem 15.1 with one layer: each site costs at most two attempts. For (2), every clock is open, so every attempt exchanges the bits of two neighbouring sites. Let μ_j be the lowest-numbered site that the original bit of site j has occupied so far. A sequence that reads every assignment must bring every bit to site 0 or 1 before the depth T, so at the end $\mu_j \le 1$ for every j and $\sum_j \mu_j \le n$, while at the start $\sum_j \mu_j = \sum_{j<n} j = n(n-1)/2$. One exchange moves one bit down by one site and the other up, so it lowers $\sum_j \mu_j$ by at most one. Hence $n(n-1)/2 - n \le T$. □

15.3 Timings that do not know the walls

The reader of Theorem 15.1 is built from the walls. A participant whose timing is fixed before its boundary is known needs one sequence that works for every pattern at once. Whether one exists turns out to be a property of the contact graph alone, and the obstruction is erasure.

Definition 15.5 (Pattern-free readers, safe exits)**.** A *pattern-free reader* for $e \in E$ is a fair sequence g with values in E and a depth T such that, for every all-dressed pattern m, e's own record at depth T separates every separable pair of pattern m. A site c is *clusterable* when $c \in W_e(m)$ for some all-dressed m. A neighbour p of c (joined to c by a contact of E) is a *safe exit* of c when p is open in every all-dressed pattern m with $c \in W_e(m)$. A site c other than u, v is *exit-obstructed* when it is clusterable and has no safe exit.

Lemma 15.6 (Erasure)**.** *Let m be all-dressed, c an open site that is not one of e's sites, and L a list of attempts whose first attempt touching c, if there is one, joins c to a wall. Then any two bit configurations that differ only at c give e the same own record along L.*

Proof. Before the first attempt touching c, the original bit of c sits at c alone, and e's attempts compare only the bits at u and v, so they never see it. At that attempt the wall's bit is copied onto c by (15.1), and from then on no site holds c's original bit. The two runs therefore coincide at every attempt of e. □

Theorem 15.7 (The exit criterion)**.** *Let $e \in E$. A pattern-free reader for e exists if and only if no site is exit-obstructed. When one exists it can be chosen with $T + 1 \leq n^2$, and then, for every all-dressed pattern, its record law is complete for the populations of the world in the sense of Theorem 15.2.*

Proof. Necessity. Let c be exit-obstructed and g any sequence with values in E. The first attempt of g touching c joins c to a neighbour p fixed by g alone. Since p is not a safe exit, there is a pattern in which c lies in the world and p is a wall; making c open keeps both facts. In that pattern the all-empty state and the state with one token at c form a separable pair, and Lemma 15.6 gives them the same record at every depth.

Sufficiency. A safe exit of a clusterable site c is itself an open site of the world whenever c is in the world, and it lies strictly closer to e's sites in the layering of the all-open pattern. So the safe exits form chains from every clusterable site to e's sites, ranked by that layer depth. The reader attempts e, and then, for each site in increasing rank, carries the site's bit along its chain of safe exits and attempts e. In a pattern where c lies in the world, the whole chain consists of open sites of the world, so c's bit arrives and is compared with a read position. In a pattern where c does not, every site of the world that the route touches has already been read, so only read positions are overwritten. The invariant of Theorem 15.1 holds for every pattern at once. Each chain has distinct sites, which bounds the length by $n^2 - 1$, and completeness follows as in Theorem 15.2. □

Being a safe exit is a statement about patterns, but it reduces to the graph.

Theorem 15.8 (Safe exits are separators)**.** *A site p is a safe exit of c if and only if p is adjacent to c in the contact graph and lies on every walk of the contact graph from either of e's sites to c. Hence, for $e \in E$, a pattern-free reader exists if and only if every site $c \notin \{u, v\}$ reachable from u has a neighbour lying on every walk from e's sites to c. In particular, if the edge $\{u, v\}$ lies on a cycle of the contact graph, no pattern-free reader exists.*

Proof. A site lies in the world exactly along a walk from e's sites whose sites before the last are open. If some walk from e's sites to c avoids p, the pattern with p the only wall keeps c in the world along that walk, so p is not safe. Conversely, if every walk to c passes p, then c can lie in the world only through p open. For the last statement, if $\{u, v\}$ lies on a cycle, let c be the neighbour of u on the cycle other than v. The walk from v to c along the cycle avoids u, and the one-step walk from u avoids every other neighbour of c, so c has no safe exit. □

Example 15.9 (Small networks)**.** In each case $e = (0, 1)$.

1. *The triangle* on sites $0, 1, 2$. For every attempt sequence, with any contacts on the three sites, fair or not, and every depth, one of the patterns (A, P, A) and (P, A, A) (lines at sites $0, 1, 2$) leaves the all-empty state and the token at site 2 unseparated; both patterns put site 2 in the world. The first attempt touching site 2 joins it to 0 or to 1, and one of the two patterns makes that partner a wall.
2. *The four-cycle* 0–3–2–4–0 *hanging from site* 0, with site 1 pendant at 0. Site 2 is exit-obstructed: each of its neighbours $3, 4$ can be a wall while 2 is reached through the other. No pattern-free reader exists.

3. *The kite*: the same network with the chord $(0, 2)$. Site 2 still has two neighbours each of which can be a wall while 2 is reached through the other, yet a pattern-free reader exists: site 0 lies on every walk from e's sites to 2 and is a safe exit, so the reader leaves 2 through the chord and never erases its bit.
4. *The pentagon* 0–1–2–3–4–0: every site other than $0, 1$ is exit-obstructed.

On trees the parent of a site, toward u, lies on every walk to it, so every tree has a pattern-free reader, and the construction gives an explicit one.

Proposition 15.10 (The tree reader)**.** *If the contact graph of E is a tree and $e \in E$, the breadth-first transport timing (each site in order of its distance from u, its bit carried to e through its ancestors, then e attempted) is a fair pattern-free reader at depth $T \le 1 + \sum_{c \notin \{u,v\}} (d(u, c) + 1)$, with $T + 1 \le n^2$, and its record law is complete for the world's populations in every pattern sector. On the path of $n \ge 2$ sites read from its end contact, $2T \le n(n - 1)$, which is within n attempts of the lower bound of Proposition 15.4, a bound that holds even for readers that know the walls.*

Proof. Root the tree at u. For a site $c \notin \{u, v\}$ the unique path from u to c passes through the parent of c, and so does the path from v to c, since v is a neighbour of u; every walk between two sites of a tree visits every site of the path between them. By Theorem 15.8 the parent is therefore a safe exit of c. The chains of safe exits in the sufficiency part of Theorem 15.7 are then the ancestor chains. The construction works for any order that lists each site after its ancestors (the invariant of Theorem 15.1 needs only that), and the breadth-first order from u is one; this is the breadth-first transport timing. It attempts e once and, for each $c \notin \{u, v\}$, uses at most $d(u, c)$ attempts to carry c's bit up its ancestor chain to the first of e's sites on it, and one more attempt of e. This gives $T \le 1 + \sum_{c \notin \{u,v\}} (d(u, c) + 1)$; since at most $n - 2$ sites are summed and $d(u, c) + 1 \le n$, $T + 1 \le 2 + n(n - 2) \le n^2$. Continuing by the round order makes the timing fair without undoing a separation, and completeness for populations is Theorem 15.7. On the path $0–1–\cdots–(n - 1)$ with $e = (0, 1)$, the bit of site $c \ge 2$ reaches site 1 after at most $c - 1$ attempts, so $T \le 1 + \sum_{c=2}^{n-1} c = n(n - 1)/2$. □

Steering by one's own record. A timing could use what the record has already shown. A *steered timing* chooses the contact attempted at time t from t and from e's outcomes before t. Its record effect is the gated pullback along the chosen contacts, each outcome of e read and the rest unread; these effects sum to the identity, and on an all-dressed basis state the law is the indicator of one deterministic record.

Theorem 15.11 (Steering adds nothing)**.** *Let $e \in E$, and consider steered timings with values in E. A steered timing that separates every separable pair of every all-dressed pattern at one depth exists if and only if no site is exit-obstructed, that is, if and only if a fixed pattern-free reader exists. At every exit-obstructed site c, every steered timing with values in E, fair or not, and every depth leave the all-empty state and the token at c unseparated in some all-dressed pattern with c open and in the world.*

Proof. On the all-empty state every gated attempt is silent. On the token at c, every attempt not touching c is silent and leaves the token where it is. So the steered timing receives the same outcomes on both states, and plays one fixed list of contacts, until its first attempt at c. That attempt's partner is fixed, it is not a safe exit, and Lemma 15.6 applies as in Theorem 15.7. □

The restriction to contacts of E is needed. On the hanging four-cycle of Example 15.9(2), the contact $(0, 2)$ is not in the network; attempting it once and then e separates the all-empty state from the token at site 2 in every all-dressed pattern with site 2 in the world, because site 0 is then open and takes site 2's bit by exchange or copy alike. On the triangle every contact lies in the network, and there every steering is blind.

Contents against shape. The record also reads walls (Theorem 15.3). One timing cannot in general do both, and a three-site path shows why.

Example 15.12 (An order conflict)**.** Take the path 0–1–2 with $e = (0, 1)$. The content pair $x = (0, 0, 1)$ against $x = (0, 1, 1)$ in the pattern (A, A, P) needs e attempted before $(1, 2)$: otherwise the wall at 2 copies its token onto site 1 and both states become $(0, 1, 1)$. The pattern pair $x = (0, 1, 0)$ in (P, A, P) against (P, A, A) needs $(1, 2)$ attempted before e: otherwise e's attempt copies the empty wall 0 onto site 1, both worlds become all-empty, and constant worlds never show their pattern. Every timing touches site 1 first with one of the two contacts, so none reads both pairs, although each pair alone is read by some fair sequence.

The same conflict arises at every network whose worlds can leave e's pair, and it does not need the contents at all. Write $\mathrm{op}(S)$ for the pattern whose open clocks are exactly the sites of S.

Lemma 15.13 (Shape conflict)**.** *Let e join a and b, and let $c \notin \{a, b\}$. Under every steered timing and at every depth, the token at a has the same record law in $\mathrm{op}(\{a\})$ as in $\mathrm{op}(\{a, c\})$, or the same record law in $\mathrm{op}(\{a\})$ as in $\mathrm{op}(\{a, b\})$.*

Proof. In all three patterns every clock other than the open ones is a wall with bit 0. Until its first attempt at a the timing sees only silences and chooses the same contacts in all three. Let w be the partner of that attempt. If $w \neq c$, then w is a wall with bit 0 in both $\mathrm{op}(\{a\})$ and $\mathrm{op}(\{a, c\})$ (when $w = b$ the attempt is e or a contact parallel to it, with the same label in both), the token is erased in both, and both worlds are all-empty from then on. If $w = c$, the same holds for $\mathrm{op}(\{a\})$ and $\mathrm{op}(\{a, b\})$. □

A *shape reader* is a fair sequence and a depth whose own record separates every pair of Theorem 15.3; a *contents-and-shape reader* separates those pairs and every separable pair of every pattern. Steered versions are defined in the same way.

Theorem 15.14 (Shape needs the world to stay on the pair)**.** *Let $e \in E$. The following are equivalent: a fixed shape reader exists; a steered shape reader with values in E exists; a fixed contents-and-shape reader exists; a steered one exists; no site other than u and v is clusterable. When they hold, attempting e once reads both contents and shape.*

Proof. If no other site is clusterable, every world is $\{u, v\}$. A separable pair differs on $\{u, v\}$ and one of its states is mixed there, so the first attempt of e gives a label on one state and silence or another label on the other. A pair of Theorem 15.3 has patterns differing only at u or v and mixed states, and the label shows which ends flip. Conversely, a clusterable site beyond the pair gives an e-site a with a neighbour $c \notin \{u, v\}$. Both pairs of Lemma 15.13 are then pairs of Theorem 15.3 that a shape reader must separate, and the lemma forbids it. □

Even where contents are read without knowing the walls, the reader is blind to the type of the far ends.

Proposition 15.15 (The tree reader misses the leaves)**.** *On every tree containing e with a site beyond e's pair, the reader of Proposition 15.10 leaves the following pair unseparated at its depth: a deepest site c holding the only token, all clocks open, against the same state with c turned into a wall. Both are live, their patterns differ, and they agree outside both worlds.*

Proof. The reader touches the leaf c once, carrying its bit to the parent; by (15.1) a copy from a wall and an exchange with an open clock deliver the same bit, and the reader never returns to c. □

15.4 Copies of one world and the participant's own record

When one run cannot hold the shape, several can. We now supply independent preparations of the same all-dressed basis state, each read in its own run under its own timing, and ask how many are needed.

Definition 15.16 (Copy readers and the copy number)**.** A *copy reader with k copies* for $e \in E$ is a family of k fair sequences $g^{(1)}, \dots, g^{(k)}$ with values in E and one depth T such that every separable pair of every all-dressed pattern, and every pair of Theorem 15.3, is separated by e's own record of some copy. The *copy number* of e is the least such k.

The copies are independent: the joint record of k copies is the product effect $\bigotimes_i X_i$ on the product state $v^{\otimes k}$, and its Born law is $\prod_i \langle v, X_i v\rangle$. On basis states each factor is an indicator, and a product of indicator laws separates two states exactly when one factor does. So separation by the joint record of the copies is the same as separation by some single copy, and Definition 15.16 counts exactly the copies the joint record needs.

Proposition 15.17 (First facts)**.** *Let $e \in E$ on n sites. Then 16^n copies suffice, exit-obstructed sites included; one copy suffices if and only if no site other than u, v is clusterable; more copies never hurt; and zero copies never suffice.*

Proof. Each required pair is separated by some fair sequence (Section 15.1 and Theorem 15.3); taking one copy for each ordered pair of all-dressed basis states, of which there are $(4^n)^2$, and a common depth (separation persists at greater depth) gives a copy reader. The case of one copy is Theorem 15.14. □

One token in an open region. The lower bounds come from a single token waiting in an open region. Let a be one of e's sites, Q a set of sites containing a, and consider the token at a in the pattern $\mathrm{op}(Q)$ against the same token in $\mathrm{op}(Q \cup \{w\})$, for a site $w \notin Q$ that lies in the world of $\mathrm{op}(Q)$, a boundary wall of the region. Each such pair is a pair of Theorem 15.3.

Lemma 15.18 (One run serves one exit)**.** *Under any fixed sequence, whichever contacts are read, and under any timing steered by e's own outcomes, with e's own record read, at any depth: the record of the token at a in $\mathrm{op}(Q)$ equals its record in $\mathrm{op}(Q \cup \{w\})$ for every $w \notin Q$ except possibly the site through which the token first leaves Q.*

Proof. While the token stays in Q, the runs coincide attempt by attempt: inside Q attempts exchange open clocks, and an attempt touching an opened site w joins it to a site holding 0, so it is silent. At the first attempt joining the token's site to a site $s \notin Q$: if $s \neq w$, then s is a wall in both patterns, its 0 is copied onto the token, and both worlds are all-empty and silent from then on. □

Theorem 15.19 (The region bound)**.** *If a copy reader with k copies exists, then $k \geq |\{w \notin Q : w \in W_e(\mathrm{op}(Q))\}|$ for every set Q containing one of e's sites. The same holds for adaptive copies read by e's own record, where each copy's timing (fixed or steered) is chosen from the earlier copies' records, and for fixed copies read by any set of contacts in place of e's own record.*

Proof. Fix the base state, the token at a in $\mathrm{op}(Q)$, and for each boundary wall w its alternative in $\mathrm{op}(Q \cup \{w\})$. Each such pair must be separated, so some copy $i(w)$ gives the two states different records. By Lemma 15.18, w is then the site through which the token first leaves Q in the run of copy $i(w)$ on the base state. A run has one first exit, so $i(w) = i(w')$ forces $w = w'$, and the copies are at least as many as the boundary walls. For adaptive copies, let $i(w)$ be the first copy whose records on the two states differ. All earlier copies gave both states the same records, so copy $i(w)$ runs on both the timing chosen from the base state's earlier records, and the lemma applies to that timing; the run of copy i on the base state does not depend on w, and the count is as before. For fixed copies read by any set of contacts, Lemma 15.18 holds for every read set, and the same count applies. □

The *region number* of e is the largest such boundary. It is at most the copy number.

Corollary 15.20 (Trees)**.** *If the contact graph is a tree containing e with a site beyond e's pair, the region number is the number of leaves, and the copy number is at least the number of leaves and at least the largest degree. On the double star (contacts $(0,1)$, $(0,2)$, $(0,3)$, $(1,4)$, $(1,5)$, $e = (0,1)$: largest degree 3, four leaves) the copy number is at least 4, so the copy number of a tree is not its largest degree.*

Proof. Take Q the set of sites with at least two neighbours; every leaf is a boundary wall of its world. Conversely each boundary clock of a region has its own leaf below it, which gives the equality for the region number. For the largest degree, take Q the path from e's pair down to a site s (or $Q = \{s\}$ when s is one of e's sites): its boundary contains the children of s and the other site of e, as many sites as s has neighbours. □

The participant must see the token come back. The region bound is not the end for the participant's own record, because the participant sees an exit only when the token returns to its own contact.

Lemma 15.21 (Forced exits)**.** *Let e join a and b, $a \in Q$, under any steered timing with values in E.*

1. *If e's own record separates the token at a in $\mathrm{op}(Q)$ from the same token in $\mathrm{op}(Q \cup \{w\})$, the token's first exit from Q is w, and, when $w \neq b$ and $b \notin Q$, its first exit from $Q \cup \{w\}$ is b.*
2. *If $b \notin Q$ and e's own record separates the token at a in $\mathrm{op}(Q)$ from the all-empty state, the token's first exit from Q is b.*

Proof. The first part of (1) is Lemma 15.18. After that exit the base is all-empty and silent. A single token in a region that does not contain b gives e only silences until it leaves through b, so the alternative is told apart only if the token's first exit from $Q \cup \{w\}$ is b. (2) is the same argument with the empty state as base. □

Definition 15.22 (Exit tests and conflicts)**.** An *exit test* at a is a region test $\mathrm{reg}(Q, w)$ or a content test $\mathrm{con}(Q)$; it is *valid* when $a \in Q$ and, for a region test, $w \notin Q$ lies in the world of $\mathrm{op}(Q)$. By Lemma 15.21 a separated test forces one or two first exits (X, s): the first site the token visits outside X is s. Two forced exits (X, s) and (X', s') *conflict* when there is a set $Q'' \subseteq X \cap X'$ containing the token's site such that no site outside Q'' joined to Q'' lies in $X \cup \{s\}$ and in $X' \cup \{s'\}$. Two tests conflict when some of their forced exits do.

Theorem 15.23 (The exit-clique bound)**.** *If N valid exit tests at one of e's sites pairwise conflict, every copy reader for e has at least N copies. In particular, a chain of regions $Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_k$ around a, all missing b, with Q_{j+1} containing Q_j and a boundary clock of Q_j, forces $k+1$ copies.*

Proof. The token's first exit from Q'' is a single site s''. If the first exit from $X \supseteq Q''$ is s, then $s'' \in X \cup \{s\}$: either s'' is still inside X, or it is the first exit from X. So one run cannot realize two conflicting forced exits, and by Lemma 15.21 it separates at most one of two conflicting tests.

For the chain, let $w_j \in Q_{j+1}$ be the added boundary clock of Q_j. The $k+1$ tests $\mathrm{con}(Q_0)$ and $\mathrm{reg}(Q_j, w_j)$, $j < k$, pairwise conflict: $\mathrm{con}(Q_0)$ against $\mathrm{reg}(Q_j, w_j)$ through $Q'' = Q_0$ (the only exit of Q_0 that (Q_0, b) allows is b, which (Q_j, w_j) forbids), and $\mathrm{reg}(Q_i, w_i)$ against $\mathrm{reg}(Q_j, w_j)$, $i < j$, through the forced exits $(Q_i \cup \{w_i\}, b)$ and (Q_j, w_j) with $Q'' = Q_i \cup \{w_i\} \subseteq Q_j$. □

Theorem 15.24 (Own copy numbers)**.** *The own copy numbers and the region numbers of Table 15.1 hold, with $e = (0, 1)$ on the three-site path 1–0–2, the four-site path 1–0–2–3, the three-leaf star with e on one edge, the four-cycle hanging from e's site (Example 15.9(2)) and the pentagon with e on an edge. On the three-site path the timings e, f, e, f, e and f, e, f, e ($f = (0, 2)$) are a copy reader, and on the triangle the timings $(1, 2), e, (1, 2), e, (0, 2), e, (1, 2), e$ and $(0, 2), e, (0, 2), e, (1, 2), e, (0, 2), e$.*

Proof. Upper bounds. Each copy reader is a list of explicit timings, one per copy: those of the statement, and one fixed list per copy on the other networks. On an all-dressed basis state every gated run is deterministic by (15.1), so each of the 4^n all-dressed basis states (a bit and one of two lines at each site: 64, 256 and 1024 states for $n = 3, 4, 5$) has one joint record. We compute it for every state and check that no two states forming a required pair, a separable pair of one pattern or a pair of Theorem 15.3, share it. This direct exact computation gives readers with 2 copies on the three-site path and the triangle, 3 on the three-leaf star and the four-site path, and 5 on the hanging four-cycle and the pentagon.

Lower bounds. On the three-site path and the triangle a site beyond e's pair is clusterable, so one copy does not suffice (Proposition 15.17). The three-leaf star needs its three leaves by Corollary 15.20. On the four-site path the tests $\mathrm{con}(\{0\})$, $\mathrm{reg}(\{0\}, 2)$, $\mathrm{reg}(\{0, 2\}, 3)$ are those of the chain $\{0\} \subseteq \{0, 2\} \subseteq \{0, 2, 3\}$, and Theorem 15.23 gives 3. On the path $0–1–\cdots–(n-1)$ with $e = (0, 1)$ the regions $Q_j = \{1, \ldots, j+1\}$, $j = 0, \ldots, n-2$, miss site 0, and $Q_{j+1} = Q_j \cup \{j+2\}$ adds a boundary clock of Q_j, so Theorem 15.23 forces $n-1$ copies. The hanging four-cycle has

the five pairwise conflicting tests con($\{0\}$), reg($\{0\}, 3$), reg($\{0\}, 4$), reg($\{0, 3\}, 2$), reg($\{0, 4\}, 2$), and the pentagon the four tests con($\{0\}$), reg($\{0\}, 4$), reg($\{0, 4\}, 3$), reg($\{0, 3, 4\}, 2$) of a chain growing the long way round; Theorem 15.23 gives 5 and 4.

Region numbers. On the trees the region number is the number of leaves (Corollary 15.20). With $Q = \{0\}$ the boundary walls are $1, 2$ on the triangle, $1, 3, 4$ on the hanging four-cycle and $1, 4$ on the pentagon, so their region numbers are at least 2, 3 and 2; on the triangle the copy number 2 bounds it from above. Theorem 15.19 holds for every read set, so the region number is at most the copy number with every contact's record read; the same direct exact computation, with every contact's record in the joint record, gives such readers with 3 copies on the hanging four-cycle and 2 on the pentagon. □

So neither the number of leaves nor the region number is the copy number of the participant's own record. A token that leaves outward and the same token coming back are exits of two nested regions, and the next region outward needs the token to leave outward instead; the ensemble grows with the depth of the world, not only with its ends.

network ($e = (0, 1)$)	leaves	region number	own	pooled
three-site path	2	2	2	2
triangle	–	2	2	2
three-leaf star	3	3	3	3
four-site path, e at the end	2	2	3	2
n-site path, e at the end ($n \geq 3$)	2	2	$\geq n - 1$	2
double star	4	4	≥ 4	4
hanging four-cycle	–	3	5	3
pentagon	–	2	4 or 5	2

Table 15.1. Copy numbers of small networks. "Own" is the copy number with e's own record read on each copy; "pooled" is the copy number with every contact's record of each run read (Section 15.5). Entries are proved; "$\geq$" marks a proved lower bound.

15.5 Pooled records: what a run holds

Now let every contact's record of each run be read together, a fully pooled composite. The required pairs are those of Definition 15.16; the *pooled copy number* is the least number of copies, each with its own fair timing, whose complete records separate them all. By Theorem 15.19 it is at least the region number, and it is at most the own copy number.

Lemma 15.25 (What the complete record sees)**.** *Let m, m' be all-dressed patterns and y a bit configuration. Along a list L, the complete records of $\psi(y, m)$ and $\psi(y, m')$ differ if and only if the run of $\psi(y, m)$ makes a mixed attempt at a site where m and m' differ. Moreover, if the complete records separate the token at a in* $\mathrm{op}(Q)$ *from the same token in* $\mathrm{op}(Q \cup \{w\})$*, the token's first exit from Q is w.*

Proof. Until such an attempt the two runs coincide, since lines differ only where no mixed attempt has occurred. At such an attempt one pattern has an open end there and the other a wall, so one end flips in one run and keeps its bit in the other, and the labels differ. The second

statement is Lemma 15.18; no return through e is needed, because every contact's outcome is read. □

Pooling therefore removes the return requirement of Lemma 15.21. On trees that is the whole difference.

Theorem 15.26 (Pooled copies on trees)**.** *If the contact graph is a tree containing e with a site beyond e's pair, the pooled copy number equals the number of leaves, which is the region number. One timing per leaf suffices: for each site in increasing distance from the leaf, its whole path to the leaf, followed by every contact once. In particular the n-site path read from its end needs exactly 2 pooled copies for every $n \geq 3$, against at least $n - 1$ for the participant's own record, and the double star needs exactly 4.*

Proof (mechanism). Two states with agreeing complete records along a timing holding every contact of a connected network keep the set where their bits differ invariant, so their bits are equal, or complementary and constant. The hard pairs are therefore same bits under different walls, and by Lemma 15.25 such a pair is separated exactly when a differing bit is carried, through open clocks, into a clock where the patterns differ. A shortest such carrying walk never steps back, and on a tree every non-backtracking walk whose last step points toward a leaf keeps moving toward that leaf. The leaf's timing runs every route toward it with its start untouched, so the leaves' timings together carry every required walk. The lower bound is Theorem 15.19. □

With loops the region number is no longer enough, because a single token visits sites in one order and so can serve only one of several routes to the same clock.

Definition 15.27 (Chains of diamonds)**.** The *chain of d diamonds* has $3d + 2$ sites: e joins site 0 to the pendant site 1, and diamond $i < d$ joins the site before it (0 for $i = 0$, the far site $3i + 1$ of the previous diamond otherwise) to its two branch sites $3i + 2$ and $3i + 3$, each of which is joined to its far site $3i + 4$. For $d = 2$ this is the *double diamond* 0–$\{2, 3\}$–4–$\{5, 6\}$–7 with the pendant 1.

Theorem 15.28 (Routes multiply, boundaries add)**.** *For every $d \geq 1$ the chain of d diamonds has region number exactly $d + 2$ and pooled copy number at least $2^d + 1$. The double diamond has region number 4 and pooled copy number 5. So the pooled copy number is not the region number.*

Proof. With the token at site 0, the pendant exit $(\{0\}, 1)$ and, for each of the 2^d routes through the diamonds, the route's open sites against opening the last far site, are $2^d + 1$ region tests with pairwise conflicting forced first exits: two routes share their region up to the first diamond where they part, and each branch of that diamond is allowed by only one of them; the pendant exit is allowed by no route. By Lemma 15.25 and the argument of Theorem 15.23, no run serves two of them. For the region number, give each site its depth (0 for sites $0, 1$; $2i + 1$ for the branches of diamond i; $2i + 2$ for its far site). Depth changes by at most one along a contact, so if a diamond's far site is open the site before it is open and the diamond contributes at most one boundary clock, if the far site is closed but the site before it open at most two, and past a closed site nothing; summing over the diamonds gives at most $d + 2$, which the region $\{0\} \cup \{3i + 2,\ 3i + 4 \ :\ i < d - 1\}$ (the upper branches of the first $d - 1$ diamonds with their far

sites) attains: its boundary is the pendant, the $d - 1$ lower branches and the two branches of the last diamond. The value 5 for the double diamond is a direct exact computation: five explicit timings, checked over every pattern pair and every bit state. □

The tests that count are not only those with the token at the participant's own sites. A token may sit anywhere in the participant's world, and opening one more boundary clock then changes only the walls.

Definition 15.29 (Site tests and site cliques)**.** A *site test* (Q, w) at a site p is a set $Q \ni p$ and a site $w \notin Q$ with p and w both in $W_e(\mathrm{op}(Q))$: the question "is the open region Q or $Q \cup \{w\}$?" asked of one token at p. Two site tests at p *conflict* when their forced first exits (Q, w), (Q', w') do, in the sense of Definition 15.22. The *site clique number* is the largest number of pairwise conflicting site tests at one site.

Every site test gives a required pair, so the site clique number is a lower bound for the pooled copy number, and it is at least the largest pooled clique at e's own sites.

Example 15.30 (The capped pentagon)**.** Take the five-cycle 2–3–4–5–6–2 with its edge 2–3 capped by the participant's site 0 (contacts $(0, 2), (0, 3)$) and the pendant 1, $e = (0, 1)$. No family of tests at e's sites has more than three pairwise conflicting members, and the region number is 3. But a token at the far site 5 has four pairwise conflicting tests: $(\{0, 3, 4, 5\}, 1)$, $(\{0, 3, 4, 5\}, 2)$, $(\{0, 2, 5, 6\}, 1)$, $(\{0, 2, 5, 6\}, 3)$. Its first step picks one of the two routes to the cap, and each route region can be left through only one exit. The pooled copy number is exactly 4, by direct exact computation.

Other networks reach their region number exactly: the kite (4), the five-cycle hanging from e's site (3), the complete graph on four sites with a pendant (4) and the single diamond (3), all by direct exact computation. Every chain with $d \geq 2$ exceeds it.

Single tokens decide, and the count is a cover. The pooled copy number is decided by single tokens, and it is a covering number of their walks. A *search order* from p is a list of distinct sites starting at p in which each new site is joined to an earlier one; it *serves* a site test (Q, w) when its first site outside Q is w. Under a list L of attempts with every clock open, one token started at p visits sites in a search order, its *walk* under L. A *site cover* at p is a set of search orders serving every site test at p; a *joint cover* is a set of attempt lists whose walks, from every site at once, serve every site test there.

Theorem 15.31 (The cover law)**.** *Let $e \in E$ on a connected network.*

1. *Single tokens decide: timings that hold every contact and whose complete records separate every required pair of the form "one token at p; a wall w of the world closed against opened" form a pooled copy reader. Pairs with several tokens or single holes add no requirement, and tokens on walls cost nothing.*
2. *The pooled copy number equals the least size of a joint cover.*
3. *Two site tests at p conflict if and only if no search order from p serves both.*
4. *Writing site clique, site cover and joint cover numbers for the largest conflicting family at one site, the largest over sites of the least site cover, and the least joint cover,*

$$\textit{site clique number} \leq \textit{site cover number} \leq \textit{joint cover number} = \textit{pooled copy number}.$$

5. *At each of e's own sites the least site cover equals the largest family of pairwise conflicting tests there, and both equal the number of maximal induced paths of the contact graph starting at that site.*

Proof. (1), mechanism. If two patterns differ and agree outside both worlds, they differ at a wall w of one of the two worlds, since a world grows or shrinks only through its own walls. A token traced from any other site s of the world to w carries every bit state that is not constant on the world, by the position bookkeeping, so hitting w from every s separates every pattern pair with equal bits; pairs with different bits are separated as in Theorem 15.26. A token on a wall is re-emitted at every attempt there, and one fixed list of relays, out of each wall through open clocks into every other wall, serves all of them in every world at once.

(2) A run separates the single-token pair of a site test exactly when the token's walk from p first leaves Q at w (Lemma 15.25). So the copies' timings form a joint cover; conversely each list of a joint cover, followed by the relays and every contact, is a pooled copy reader by (1).

(3) One direction is the argument of Theorem 15.23. Conversely, if no conflict witness exists, grow a common region greedily through sites allowed by both tests, leave it through an exit allowed by both, and continue to the remaining clock through the world.

(4) A search order serves no two conflicting tests, which gives the first inequality; restricting a joint cover to one site gives the second.

(5) At e's sites every connected region is a valid test. From a visited sequence V, each frontier site x (joined to V, not visited, not forbidden) opens a branch V, x in which every other frontier site is forbidden. Every test above V lies in one branch or exits directly into a frontier site, so the branch orders serve all tests; tests of different branches allow different exits of the visited set and conflict there. Cover and clique both count the leaves of this branching, which are the maximal induced paths from the site. □

Example 15.32 (Pairwise compatible, jointly incompatible)**.** On $K_{2,3}$ with a pendant (contacts $(0,1), (0,2), (0,3), (0,4), (2,5), (3,5), (4,5)$, $e = (0,1)$), the three tests $(\{0,2,3\},5)$, $(\{0,3,4\},5)$, $(\{0,2,4\},5)$ at site 0 pairwise do not conflict (the orders $0,3,5$; $0,4,5$; $0,2,5$ serve the pairs), yet no single search order serves all three. Conflict is not the whole story of covering. Nevertheless this network meets the clique number: its pooled copy number, joint cover, site cover and site clique numbers are all 4, by direct exact computation.

15.6 The clique law and where it fails

The natural hope is a min-max law: that the runs a composite needs equal the largest family of mutually exclusive first exits. It holds on every chain of loops, at every site, and fails in general, first site by site and then at the busiest site.

Definition 15.33 (Theta chains)**.** For widths $k_1, \dots, k_d \geq 1$, the *theta chain* has the contact e from site 0 to a pendant site, and d blocks in series: block i joins the site before it (0, or the previous block's junction) to its own junction through k_i middle sites, each adjacent to both. Widths all equal to 2 give the chain of d diamonds; the single block of width 3 is $K_{2,3}$ with a pendant.

Theorem 15.34 (Loops in series)**.** *For every $d \geq 1$ and widths $k_1, \dots, k_d \geq 1$, the pooled copy number, the joint cover number, the site cover number and the site clique number of the theta chain all equal*

$$k_1 k_2 \cdots k_d + 1.$$

In particular the chain of d diamonds needs exactly $2^d + 1$ pooled copies, against a region number $d + 2$. Moreover, at every site the least site cover equals the largest family of pairwise conflicting site tests, and the least site cover is $k_1 \cdots k_i + k_{i+1} \cdots k_d$ at the junction of block i (the second term absent for $i = d$) and $k_1 \cdots k_{i-1} + (k_i - 1) + k_{i+1} \cdots k_d$ at a middle site of block i, empty products being 1.

Proof (mechanism). Lower bound: with the token at 0, the pendant exit and one test per route through the blocks pairwise conflict, as in Theorem 15.28; two routes share their region up to the first block where they part, and each middle site there is allowed by only one of them. Upper bound: one timing per route and one more. The timing of a route moves every token on the route monotonically to an assigned end (the tokens bound for the start pass back one by one, then the others run to the end), with the ends chosen so that every walk down to the pendant and every walk up to the last junction starts some timing's walk; it then makes, for each other middle site z of each block, the switch $z \to$ (site before the block) $\to$ the route's middle site. The extra timing supplies the pendant step at 0 and the remaining walks. Every valid site test is served, because every valid region from a site contains a route back to the participant and the clock lies below the site, above it, or at a sibling middle site. The cover law (Theorem 15.31) closes the chain of equalities. The per-site statement follows by indexing orders and tests by the same set: down along each route, sideways to each sibling, up along each route; equal tests carry equal orders, distinct tests conflict, and counting route prefixes and suffixes gives the formula. □

Away from chains, the argument of Theorem 15.31(5) gives an inequality only.

Proposition 15.35 (Covers and induced paths)**.** *At every site p lying on a contact of E, the least site cover is at most the number of maximal induced paths from p. The inequality can be strict and the branch orders can be suboptimal: on the network N_9 (contacts* $(0,1)$, $(0,2)$, $(0,3)$, $(0,4)$, $(0,5)$, $(1,6)$, $(2,6)$, $(3,6)$, *$e = (0,1)$) the least cover at site* 2 *is* 5 *while* 6 *maximal induced paths start there; on N_{40} (contacts* $(0,1)$, $(0,2)$, $(0,3)$, $(0,4)$, $(1,5)$, $(2,6)$, $(3,6)$, $(5,6)$*) four orders* $2,0,1,5$*;* $2,0,3$*;* $2,0,4$*;* $2,6,5,1,3$ *cover site* 2*, the last not an induced path, while any cover by branch orders needs all five of them. The per-site min-max property holds at both sites.*

Proof. Let $f \in E$ be a contact at p. Every site test (Q, w) for e at p is a site test for f at p: the site p is open and in e's world, so the walk by which e's world reaches p, followed backwards, brings e's sites into f's world, and with them everything e's world reaches. By Theorem 15.31(5) applied to f, at its own site p, the branch orders, one per maximal induced path from p, serve every test for f there, hence every test for e.

Both networks are direct exact computations at site 2, over all 2^7 open regions Q and every boundary clock w. Tests $(\{0,2\}, w)$ with different exits w pairwise conflict (take $Q'' = \{0,2\}$), and a search order serves no two conflicting tests. On N_9 the orders $2,0,1$; $2,0,3$; $2,0,4$; $2,0,5$; $2,6,3$ cover and the tests with $w = 1,3,4,5,6$ conflict, so the least cover is 5; the branch recursion of Theorem 15.31(5) gives six maximal induced paths from 2. On N_{40} the four orders

of the statement cover and the tests with $w = 1, 3, 4, 6$ conflict; the branch orders $2, 0, 1, 5$; $2, 0, 3$; $2, 0, 4$; $2, 6, 3$; $2, 6, 5, 1$ each serve a test that no other branch order serves, namely $(\{0, 1, 2\}, 5)$, $(\{0, 2\}, 3)$, $(\{0, 2\}, 4)$, $(\{1, 2, 5, 6\}, 3)$ and $(\{0, 2, 3, 6\}, 5)$ respectively. □

So optimal covers away from the participant's sites can interleave branches, and the branch recursion cannot prove the per-site law. Natural strengthenings fail outright. For a set S of anchor sites and a set W of free exits, the *anchored family* at p consists of the tests (Q, w) with $p \in Q$, $w \notin Q$ joined to the part of Q the token reaches through open sites, counted when that part meets S or $w \in W$; with $S = \{u, v\}$ and $W = \emptyset$ these are the site tests. Call the family *König* at p when its least cover by search orders equals its largest pairwise conflicting subfamily. A *fractional cover* gives nonnegative weights to search orders so that every test of the family is served with total weight at least one; its value is the total weight, and a family of tests with weights, no order serving weight above one, bounds every fractional cover from below by its total weight.

Proposition 15.36 (Anchored families need not be König)**.** *1. On N_8 (contacts $(0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 4)$, $(3, 4)$, $(2, 5)$, $(3, 5)$, $(1, 6)$, $(2, 7)$: two four-cycles sharing the contact $(0, 3)$, with pendants 6 at 1 and 7 at 2), the anchored family at $p = 0$ with $S = \{6, 7\}$ and $W = \{3\}$ needs 5 search orders while at most 4 of its tests pairwise conflict.*

2. On M_9 (contacts $(0, 1)$, $(0, 2)$, $(0, 3)$, $(0, 4)$, $(1, 6)$, $(1, 8)$, $(2, 5)$, $(2, 7)$, $(3, 5)$, $(3, 6)$, $(4, 5)$, $(4, 6)$, $(4, 7)$, $(4, 8)$), the anchored family at $p = 0$ with the connected anchor set $S = \{4, 7, 8\}$ and no free exits needs 9 search orders, every fractional cover of it has value at least $17/2$, and at most 8 of its tests pairwise conflict.

On both networks every contact satisfies the per-site min-max property at every site.

Proof. Direct exact computation at $p = 0$ over all open regions (2^8 on N_8, 2^9 on M_9). Covers: five explicit search orders on N_8 and nine on M_9. Lower bounds: on N_8 the sets of nine listed tests served by single orders lie in eleven maximal sets, no four of which cover all nine; on M_9 thirteen listed tests with weights $\frac{1}{2}$ or 1 and total $17/2$ are such that no order serves weight above 1, so k covering orders satisfy $k \geq 17/2$, hence $k \geq 9$, and by weak duality every fractional cover has value at least $17/2$. Conflicting families: an order serving an inclusion-least anchored test serves every larger test with the same clock, and two conflicting tests share no serving order (the argument of Theorem 15.23), so a conflicting family maps one to one into the least tests (11 on N_8, 21 on M_9), no two of its images sharing an order; explicit witness orders show that each least test has a serving order and that among any five least tests on N_8, and any nine on M_9, two share one. The per-site statements are certificates of the same kind, one explicit cover and one explicit conflicting family of the same size for each contact and each site. □

Ten sites suffice to break the per-site law itself.

Theorem 15.37 (The per-site law fails)**.** *Let G_{10} have the seventeen contacts $(0, 7)$, $(0, 9)$, $(1, 6)$, $(1, 8)$, $(1, 9)$, $(2, 4)$, $(2, 8)$, $(2, 9)$, $(3, 5)$, $(3, 6)$, $(3, 8)$, $(4, 5)$, $(4, 9)$, $(5, 7)$, $(6, 7)$, $(7, 8)$, $(8, 9)$, and let the participant be the contact $(2, 8)$. At site 0 the site tests need 11 search orders, while at most 10 of them pairwise conflict. The tests anchored at site 2 alone, and those anchored at site 8 alone, each need 10 orders and each contain 10 pairwise conflicting tests; the site tests are their union. The site cover number and the site clique number of $(2, 8)$ on G_{10} are both 14, attained at site 4.*

Proof. Direct exact computation. Eleven explicit orders cover the site tests at 0, checked over all 2^{10} regions. The lower bound 11 is a fractional packing: sixteen valid tests with weights $\frac{1}{2}$ or 1 and total $21/2$, no order serving weight above 1. The clique bound checks every family of pairwise conflicting tests through the 36 inclusion-least tests. The union statement is the decomposition of a site test according to which of the participant's two sites the token's region reaches. □

The min-max property therefore does not pass from a contact's two sites to the contact. What the clique law needs is weaker: it holds for the network exactly when the least cover attains its maximum at a site with the per-site property.

Proposition 15.38 (The busiest site)**.** *Call p a* busiest site *of e when its least site cover equals the site cover number. For $e \in E$, the site clique number equals the site cover number if and only if some busiest site has the per-site min-max property.*

Proof. At every site p a family of pairwise conflicting tests is at most the least cover at p, because a search order serves no two conflicting tests; maximizing over sites gives the first inequality of Theorem 15.31(4). Suppose the two numbers are equal, and take a site p and a conflicting family F there of the largest size. Then $|F| \leq$ (least cover at p) $\leq$ site cover number $= |F|$, so p is busiest and has the min-max property. Conversely, if a busiest site p has the min-max property with a conflicting family F of the size of its least cover, then site clique number $\geq |F|$ = site cover number $\geq$ site clique number. □

Theorem 15.39 (The clique law fails)**.** *Let G_{13} be G_{10} with the four-cycle* 0–10–12–11–0 *glued at site* 0*, and let the participant be* $(2, 8)$*. Then site* 12 *is the only busiest site and needs* 22 *search orders; every other site needs at most* 20*; and the site clique number lies between* 20 *and* 21*. Hence the pooled copy number of* $(2, 8)$ *on G_{13} is at least* 22*, while no site carries more than* 21 *mutually exclusive first exits. Moreover site* 12 *has a fractional cover of value* 21*, and every site of G_{13} has one of value at most* 21*, so neither per-site rounding (least cover equal to the least integer at or above the fractional cover number) nor its maximum over sites gives the cover.*

Proof. Site 0 is a cut site separating the block $\{10, 11, 12\}$ from G_{10}. A token at 12 reaches G_{10} only through 10 or 11. A test whose clock lies in G_{10} holds, besides a valid region of G_{10} at 0, a connector $\{12, 10\}$ or $\{12, 11\}$, and an order answering a test of the first kind has visited 10 but not 11 when it first leaves the region of G_{10}; so no order answers tests of both kinds. The family at 12 thus contains two copies of the family of Theorem 15.37 at 0 that no order serves together, each needing 11 orders by its fractional packing: 22 in all. Twenty-two explicit orders cover site 12; the bounds at the other sites come from Proposition 15.35 and one explicit cover; the clique bounds come from weak duality with the fractional cover of value 21 and an explicit conflicting family of 20. □

Here the least cover is not a packing quantity: the lower bound 22 is a sum of two block bounds, not the ceiling of one fractional value. Bottlenecks make the gap grow.

Proposition 15.40 (Bottlenecks multiply)**.** *Let c be a cut site: the sites split into $V_1 \ni c$ and a block meeting V_1 only at c, every contact lies inside V_1 or inside the block, and e lies inside V_1; write E_1 for the contacts inside V_1. Let p be a block site not joined to c, joined to c by r paths inside the block, the* routes*, that share only c and p.*

1. *The least site cover at p is at least r times the least site cover at c for e on E_1; N pairwise conflicting tests at c give rN at p; and a weighted family of tests at c in which no order serves weight above one gives such a family at p with r times its total weight.*
2. *If the block is $K_{2,r}$ at c (entrances $a_1, \dots, a_r$, each joined exactly to c and p) and c has a valid test, the least cover at p is exactly r times that at c, and a fractional cover at c gives one at p of r times its value.*

Proof. (1) Write R_i for the sites of route i other than c. A test (R, w) at c on E_1 lifts to the valid test $(R \cup R_i, w)$ at p, since opening route i joins p to c. A search order from p restricts to the order that lists c and then the sites of $V_1 \setminus \{c\}$ in their order; these sites are joined only to sites of V_1, so the restriction is a search order on E_1, and if the order first leaves $R \cup R_i$ at $w \in V_1$, its restriction first leaves R at w. The site p has no neighbour in V_1, so the second site of an order serving a lifted test of route i lies in R_i, and since routes meet only at p, no order serves lifted tests of two routes. A cover at p therefore contains r disjoint parts, each restricting to a cover at c. Lifted tests of different routes conflict through $Q'' = \{p\}$, and lifted tests of one route conflict when the originals do, the witness Q'' at c becoming $Q'' \cup R_i$. A weighted family lifts route by route, and an order at p serves lifted tests of one route only, with no more weight than its restriction serves at c.

(2) A valid test (Q, w) at p has $c \in Q$ and an open entrance $a_i \in Q$, because the world reaches p only through them. If w is an entrance, an order $p, w, \dots$ serves it; otherwise $w \in V_1$, and p, a_i, τ serves it for an order τ from c serving the valid test $(Q \cap V_1, w)$ at c. So the orders p, a_i, τ, one for each entrance and each order τ of a least cover at c, form a cover, and (1) gives the reverse inequality. The same orders with the weights of a fractional cover at c form a fractional cover at p: a test whose clock is an entrance receives the cover's whole weight, which is at least one because c has a valid test. □

Corollary 15.41 (The gap is unbounded)**.** *Glue $K_{2,r}$ at site 0 of G_{10}, with the participant $(2, 8)$; for $r = 2$ this is G_{13}. The far site needs $11r$ search orders and has fractional cover number exactly $21r/2$; between $10r$ and $21r/2$ of its tests pairwise conflict; every other site needs at most $14 + 6r$ orders. Hence for $r \geq 3$ the far site is the only busiest site and lacks the per-site min-max property; for $r \geq 4$ the site cover number $11r$ exceeds the site clique number by at least $r/2$; and per-site rounding fails at the far site by exactly $\lfloor r/2 \rfloor$. So the gap between the pooled copy number and every clique or fractional bound is unbounded.*

Proof. At site 0 of G_{10} the least cover is 11 and a packing has value $21/2$ (Theorem 15.37); by direct exact computation ten tests there pairwise conflict and fourteen orders with weights in halves form a fractional cover of value $21/2$. Proposition 15.40 lifts all four to the far site: it needs $11r$ orders, has $10r$ pairwise conflicting tests, and has a fractional cover and a packing both of value $21r/2$, so by weak duality its fractional cover number is $21r/2$ and no more than $21r/2$ of its tests pairwise conflict (an order serves at most one test of a conflicting family). An entrance or the cut site needs at most $11 + r$ orders: a cover at 0, and one order into the block through each entrance. A site q of G_{10} needs at most its cover in G_{10}, at most 14 (Theorem 15.37), plus, for each induced route from q to 0 and each entrance, the order along the route, through the entrance, to the far site: a test at q whose clock lies in the block has an open region containing such a route, and there are at most six of them (direct exact computation). For $r \geq 3$, $11r > 14 + 6r$; for $r \geq 4$, $14 + 6r \leq 21r/2$, so no site carries more than

$21r/2$ conflicting tests; and the least integer at or above $21r/2$ is $11r - \lfloor r/2 \rfloor$. The pooled copy number is at least the site cover number (Theorem 15.31(4)). □

What remains open on this line, collected in Chapter 23: whether the pooled copy number always equals the site cover number (joint realizability of the best per-site covers; on G_{13} whether it equals 22); whether the family anchored at a single site is always König, which is proved when the anchor is the token's site or one of its neighbours; whether the least cover is always the largest sum of rounded fractional cover numbers of blocks of tests that no order serves together (true at every site whose least cover is its rounded fractional cover number, and, by Proposition 15.40, at the far site of $K_{2,r}$ glued at such a cut site, with the r lifted families as blocks); and, for the participant's own record, whether the copy number is always the largest exit clique and whether the pentagon needs four copies or five.

15.7 Closed wholes: what one participant reads

We now change the timing. Attempts are drawn independently at random: at each time one position i of the network's contact list is attempted, with a fixed weight $w_i > 0$ and $\sum_i w_i = 1$. This attempt process is supplied. The world is a *closed whole* in the sense of Chapter 14: a cluster C of m open clocks with no wall beside it, holding t tokens with $1 \le t \le m-1$. A contact $g = (u, v)$ of the cluster *fires* at an attempt that finds its pair mixed; its *record* is the list, attempt by attempt, of whether it fired. The weight of g, written w_g, is the total weight of the positions whose contact is g, and ω_{xy} is the total weight of the positions joining clocks x and y.

Inside a closed whole every attempt at a contact of the cluster transposes the bits of its two clocks and attempts elsewhere leave the cluster alone, so the cluster's configuration is the initial one composed with a random product of transpositions independent of the bits. Record laws are the almost-sure long-run frequencies of Theorem 14.29(a) under the uniform law on the *token class* (the configurations with t tokens), and ϖ, c_4 are as there, with t tokens:

$$\varpi = \frac{2t(m-t)}{m(m-1)}, \qquad c_4 = \frac{4t(t-1)(m-t)(m-t-1)}{m(m-1)(m-2)(m-3)}$$

($c_4 = 0$ when $m \le 3$, where no two disjoint pairs exist); two pairs sharing one clock are both mixed with chance $\varpi/2$.

Theorem 15.42 (The record reads pair transport)**.** *In every closed whole, for every contact g of the cluster and every lag $\ell \ge 1$, the long-run frequency of "g fires at τ and at $\tau + \ell$" is*

$$w_g^2\Big(\varpi\, R_2 + \frac{\varpi}{2}\, R_1 + c_4\, R_0\Big),$$

where R_2, R_1, R_0 are the chances that $\ell - 1$ independent weighted attempts carry g's clock pair onto itself, onto a pair sharing exactly one clock with it, or onto a disjoint pair.

Proof. g fires at τ with chance w_g times the chance that its pair is mixed, and then exchanges its two clocks, which leaves the pair's mixedness unchanged. After $\ell - 1$ further attempts the bits are the bits at τ composed with a product of transpositions, so g fires at $\tau + \ell$ when it is attempted and the pair those transpositions carry onto g's clocks was mixed at τ. The order

of composition on the state side is reversed relative to carrying the pair forward; a duality for finite-word expectations (a run of the state paired with a fixed observable has the same expectation as the observable paired with a run of the argument) removes the reversal. Under the uniform law the chance that two pairs are both mixed depends only on their overlap, which gives ϖ, $\varpi/2$ and c_4. □

At lag two the formula gives $w_g^2\varpi(1 - W_N/2)$, where W_N is the weight of the contacts sharing exactly one clock with g (Theorem 14.29(c)). At lag three it reads more. Write W_u and W_v for the weights joining u and v to clocks off g's pair, so that $W_N = W_u + W_v$, and

$$Q_g = \sum_{c \notin \{u,v\}} (\omega_{uc}^2 + \omega_{vc}^2), \qquad T_g = \sum_{c \notin \{u,v\}} \omega_{uc}\,\omega_{vc},$$

the latter the weight of the triangles through g.

Theorem 15.43 (Lag three reads triangles)**.** *In every closed whole, for every contact g of the cluster, the frequency of "g fires at τ and at $\tau + 3$" is*

$$w_g^2\Big(\frac{\varpi}{2} + \frac{\varpi}{2}\big((1 - W_N)^2 + Q_g\big) - 2\Big(\frac{\varpi}{2} - c_4\Big)(W_uW_v - T_g)\Big).$$

With $w_g \neq 0$ and the other quantities fixed it determines T_g if and only if $\varpi/2 \neq c_4$. For $m \geq 4$ and $1 \leq t \leq m - 1$, $\varpi/2 = c_4$ exactly when $m = s^2 + 2$ and $2t = s^2 + s + 2$ or $2t + s = s^2 + 2$ for some $s \geq 2$ (the family of Theorem 14.29(d)).

Proof. Two attempts are enumerated. One attempt moves g's pair to a pair sharing one clock exactly when it is a contact sharing one clock with g; it returns only by a contact on the same pair, and it leaves g's pair entirely only by a neighbour on the other side with a different far clock. This gives $R_2 = (1 - W_N)^2 + Q_g$ and $R_0 = 2(W_uW_v - T_g)$ after two attempts, and Theorem 15.42 gives the formula. The coefficient of T_g is $2w_g^2(\varpi/2 - c_4)$, and solving $\varpi/2 = c_4$ over the integers gives the stated family. □

Example 15.44 (Lag two cannot tell, lag three can)**.** Take the four-cycle 0–1–2–3–0 with weights w_g on $g = (0, 1)$, a on $(0, 3)$, b on $(1, 2)$, r on $(2, 3)$, and the triangle $0, 1, 2$ with the pendant $(2, 3)$, weights w_g on g, a on $(0, 2)$, b on $(1, 2)$, r on $(2, 3)$, all positive and summing to one. With one, two or three tokens the two wholes give g the same w_g, W_N, W_u, W_v, Q_g and the same lag-two frequency, while T_g is 0 in the four-cycle and ab in the other, and the lag-three frequencies differ. On the closed triangle with weights w_0, w_1, w_2 and one or two tokens, the lag-two frequency of a contact is $w_g^2(1 + w_g)/3$ (Theorem 14.29(c)), which depends on w_g alone, and the lag-three frequency is $w_g^2(1 + w_0^2 + w_1^2 + w_2^2)/3$, which reads how the other two weights split.

Everything one participant can read is fixed by finitely many statistics.

Theorem 15.45 (Finitely many windows decide)**.** *1. The lag frequencies of a contact satisfy a monic linear recurrence of order $m(m - 1)/2$, the characteristic polynomial of the transport matrix on the cluster's pairs of clocks. Two closed wholes whose lag frequencies agree at the first $\binom{m}{2} + \binom{m'}{2}$ lags agree at every lag.*

2. *For any two closed wholes, any contacts and any weights, with token classes of N and N' configurations: if every window statistic of length less than $N + N'$ of the two records has the same expectation, then every window statistic of every length does, and the two records have the same long-run law.*

Proof. (1) is Cayley–Hamilton for the pair-transport matrix. (2) The record is emitted by a finite chain on the token class. The difference of the two laws is one emitting chain on the disjoint union of the two classes, and its forward spans of window functions form an increasing chain that stops growing for good once it stalls; the first span contains the constants, so it stalls within $N + N' - 1$ letters. □

One token. With one token, or one hole, the record is the record of a single marked clock walking on the cluster: g fires exactly when the marked clock sits on one of g's clocks and g is attempted, and then moves it to the other. Let M_s be the 2×2 *return block*, the chances that the marked clock started at u or v is at u or v after s attempts, and $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The block is symmetric; write $\sigma(M) = M_{uu} + 2M_{uv} + M_{vv}$, $d(M) = M_{uu} - M_{vv}$ and $\rho(M) = M_{uu} - 2M_{uv} + M_{vv}$.

Theorem 15.46 (One token: the record reads the return block)**.** *Let a closed whole hold one token or one hole, let g be a contact of the cluster, and let the weights be positive and sum to one.*

1. *For firing times $\tau_0 < \tau_1 < \cdots < \tau_r$ with gaps $j_i = \tau_i - \tau_{i-1} - 1$, the chance that g fires at all of them is*
$$\frac{w_g^{r+1}}{m}\, \mathbf{1}^\top S M_{j_1} S M_{j_2} \cdots S M_{j_r} \mathbf{1}.$$
So one firing has chance $2w_g/m$, two have $w_g^2\sigma(M_j)/m$, and three have $w_g^3(\sigma(M_j)\sigma(M_{j'}) - d(M_j)d(M_{j'}))/(2m)$.
2. *Two such wholes give g and g' the same own record law at every window if and only if $w_g = w'_{g'}$, $m = m'$, and either $M'_s = M_s$ for every s, or $M'_s = SM_sS$ for every s, or $d(M_s) = d(M'_s) = 0$ and $\sigma(M'_s) = \sigma(M_s)$ for every s.*
3. *$d(M_s) = 0$ for every s whenever an automorphism of the weighted whole carries u to v, or the marked clock's walk has an equitable partition with $\{u, v\}$ as a cell.*

Proof. (1) The marked clock is uniform on the m clocks. A firing of g acts as w_g times the exchange of u and v restricted to them, and between firings only whether the clock lands on $\{u, v\}$ matters, which is M_j. Firing sets fix the record law by inclusion and exclusion. (2) The record gives w_g from two firings in a row, m from the one-time frequency, σ from the lags, d up to one common sign from three firings, and ρ from four firings when some $d \neq 0$; a symmetric block is fixed by (σ, d, ρ), and reversing the sign of d gives SMS. (3) Under either symmetry the walk started at u and at v has the same chance of being back on its own clock. □

When its two clocks are interchangeable, a participant with one token in its whole reads only the return chance of its pair, and its record is a renewal process. The lag frequencies do not fix the record.

Example 15.47 (The paw and the triangle with a pendant)**.** Give both four-clock wholes one token and uniform weights $1/4$: the paw, with $g = (0, 1)$ its pendant contact and the triangle $1, 2, 3$; and the triangle $0, 1, 2$ with the pendant $(2, 3)$, with $g = (0, 1)$ a side of the triangle.

They have the same w_g, the same m and the same $\sigma(M_s)$ at every s (an explicit intertwiner of the two walks fixes the sum vector of g's clocks), hence the same frequency at every lag. But $d(M_1)$ is $1/2$ in the paw and 0 in the other, and the chance that g fires at τ, $\tau+2$ and $\tau+4$ is $1/256$ against $9/2048$. From three firings the participant can tell whether its two ends are alike; from two it cannot.

Theorem 15.48 (One participant does not fix its whole)**.** *Let G_1 have the contacts $g=(0,1)$, $(1,2)$, $(1,3)$, $(2,4)$, $(3,5)$ with weights $w_g, \alpha, \alpha, \beta, \beta$ (a tree), and G_2 the contacts g, $(1,2)$, $(1,3)$, $(2,4)$, $(2,5)$, $(3,4)$, $(3,5)$ with weights w_g, α, α and $\beta/2$ on each of the last four (a whole with the four-cycle $2,4,3,5$). With one token or five, for all weights, every window statistic of g's record has the same expectation in G_1 and G_2, and with positive weights summing to one the same almost-sure long-run limit; the two contact graphs are not isomorphic. With two tokens or four, at $w_g=\alpha=\beta=1/5$, the lag frequencies of g agree at lags 1 to 8 and differ at lag 9: after eight attempts the chance of carrying g's pair back onto itself is $61295/390625$ in G_1 and $61291/390625$ in G_2.*

Proof. With one token the marked clock's walk lumps over the cells $\{0\},\{1\},\{2,3\},\{4,5\}$ to the same chain in both wholes, and the joint law of (record bit, next cell) from every clock depends only on the cell. This is a finite table in the coefficients of w_g, α and $\beta/2$, so it holds for all weights. The graphs have ten and fourteen ordered adjacent pairs. The two-token values are direct exact computation of the pair-transport chain in integer tables scaled by 10^{ℓ}. □

Proposition 15.49 (All contacts together identify the weights)**.** *On one network with a closed whole of $m \geq 3$ clocks, two token configurations and two weight vectors (the second positive) that give every contact of the cluster the same one-time frequency and every pair of contacts the same frequency of firing at consecutive attempts have the same ϖ, token numbers equal or complementary, and the same weight on every contact of the cluster.*

Proof. A contact g fires at a given attempt with chance $w_g\varpi$: it must be attempted and its pair mixed. The cluster is connected and has at least three clocks, so it has two contacts $g=(u,v)$ and $h=(v,x)$ with $x\neq u$. If g fires at τ it exchanges the bits of u and v, so h fires at $\tau+1$ exactly when it is attempted and the pairs $\{u,v\}$ and $\{u,x\}$ were both mixed at τ; two pairs sharing one clock are both mixed with chance $\varpi/2$, so this frequency is $w_g w_h \varpi/2$. Writing primes for the second whole, the hypotheses give $w_g\varpi = w'_g\varpi'$, $w_h\varpi = w'_h\varpi'$ and $w_g w_h \varpi = w'_g w'_h \varpi'$. Multiplying the first two and dividing by the third gives $\varpi=\varpi'$, since $w'_g, w'_h, \varpi' > 0$. Then $t(m-t)=t'(m-t')$, that is $(t-t')(m-t-t')=0$, and $w_g = w'_g$ for every contact of the cluster because $\varpi>0$. □

The joint records of several participants read more.

15.8 Joint records: a family reads its own shape

A *family* is a list $G=(g_1,\dots,g_K)$ of contacts of the cluster, its *joint record* the list, attempt by attempt, of which members fire. With distinct members at most one fires at a time. The finite-window statement of Theorem 15.45(2) holds for joint records with the same bound. We first take one token or one hole, positive weights summing to one, and P the one-step walk of the marked clock, $P(x,y)=\omega_{xy}$ for $x\neq y$.

A firing of $g = (u, v)$ acts on the marked clock as the *firing operator* $F_g = w_g(e_u e_v^\top + e_v e_u^\top)$, and the chance of a joint firing pattern, member a_1 firing, then s_2 attempts with any outcome, then a_2, and so on (the chance that the joint record shows these firings at these times), is

$$\frac{1}{m}\,\mathbf{1}^\top F_{a_1} P^{s_2} F_{a_2} \cdots P^{s_r} F_{a_r}\mathbf{1}. \tag{15.2}$$

For two contacts $g = (u, v)$ and $h = (v, x)$ sharing exactly the clock v, with $u \neq x$, the products of firing operators are rank one:

$$F_g F_h = w_g w_h\, e_u e_x^\top, \quad F_h F_g = w_h w_g\, e_x e_u^\top, \quad F_g F_g F_h = w_g^2 w_h\, e_v e_x^\top, \quad F_h F_g F_g = w_h w_g^2\, e_x e_v^\top. \tag{15.3}$$

Firings at consecutive attempts pin down where the token is, and the shared clock tells which end is which.

Theorem 15.50 (Two neighbours read their three-clock block)**.** *Let two closed wholes hold one token or one hole, with positive weights summing to one, and let $g = (u, v)$, $h = (v, x)$ and $g' = (u', v')$, $h' = (v', x')$ be neighbouring contacts of their clusters with $u \neq x$, $u' \neq x'$. The joint records of (g, h) and (g', h') agree at every window if and only if $w_g = w'_{g'}$, $w_h = w'_{h'}$, $m = m'$, and the 3×3 blocks of P^s on (u, v, x) and P'^s on (u', v', x') are equal for every s.*

Proof. The weights and m are read from one firing and two consecutive firings, $2w/m$ and $2w^2/m$. By (15.3), a pattern ending in (h, g), (h, g, g) or (g, h) leaves the marked clock at u, v or x with a known factor, and one starting with (g, h), (g, g, h) or (h, g) requires it at u, v or x; so "end localizer at y, s unread attempts, start localizer at z" has chance $c\,d\,P^s(y, z)/m$ with known $c, d > 0$, for all $y, z \in \{u, v, x\}$. Conversely (15.2) involves only the weights, m and these blocks. □

Here the correspondence of the three clocks is part of the comparison (Theorem 15.52 removes it), and unlike Theorem 15.46(2) no exchange alternative appears. The neighbour also removes the lumping ambiguity of Theorem 15.48: with $h = (1, 2)$ in G_1 and G_2, one token or five, and nonzero weights with $w_g + 2\alpha + 2\beta = 1$, the chance of the pattern "g, h, two attempts with any outcome, h, g" is larger in G_1 by $w_g^2\alpha^2\beta^2/12$, because $P^2(2, 2)$ is $(w_g + \alpha + \beta)^2 + \alpha^2 + \beta^2$ in G_1 and $(w_g + \alpha + \beta)^2 + \alpha^2 + \beta^2/2$ in G_2.

Families read their shape as well. Write $\mathrm{sh}(k, l) \in \{0, 1, 2\}$ for the number of clocks that members k and l share.

Proposition 15.51 (The joint record counts shared clocks)**.** *With one token or one hole and weights summing to one, the chance that member k fires and member l fires at the next attempt is $w_k w_l\,\mathrm{sh}(k, l)/m$. If k and j each share exactly one clock with l, the chance that k, l, j fire at three consecutive attempts is 0 when k and j meet l at the same clock (a star at that clock), and $w_k w_l w_j/m$ when they meet l at its two different clocks (a path or a triangle through l).*

Proof. Summing $F_k F_l$ over all entries gives $w_k w_l$ times the number of k's clocks lying on l. For the triple, $F_k F_l = w_k w_l\, e_u e_x^\top$ with x the far clock of l, and $F_k F_l F_j$ has total $w_k w_l w_j$ if x lies on j and 0 otherwise. □

So the star and the triangle on three members, which have the same line graph [42], are told apart at one token, and the joint record supplies its own labelling.

Theorem 15.52 (The joint record fixes the incidence)**.** *Let two closed wholes hold one token or one hole, with positive weights summing to one, and let G and G' be families of $K \geq 1$ contacts of their clusters, no two members of G on the same two clocks. If the joint records agree at every window, there is a bijection φ from the clocks of G onto the clocks of G' carrying each member's two clocks onto the corresponding member's two clocks. If moreover G is connected through shared clocks and $K \geq 2$, the joint records agree at every window if and only if such a φ exists with equal member weights, $m = m'$, and $P^s(y, z) = P'^s(\varphi y, \varphi z)$ for every s and all clocks y, z of G. If G misses at most one clock of the cluster, equal joint records give a bijection of the two clusters that relates the whole walks at every s and the total contact weight on every pair of clocks.*

Proof. Proposition 15.51 reads the shared-clock counts and, for every member with two neighbours, which ends its neighbours meet. Orient each member at one chosen neighbour; the same-end relation forces the orientation to match at every other neighbour, so consistency around cycles is automatic and φ is defined end by end. A member with no neighbour has private clocks and any orientation works. With φ in hand, every clock of a connected family with at least two members lies on a member with a neighbour sharing exactly one clock, so (15.3) gives localizers at every clock and the blocks are read as in Theorem 15.50. With one clock missing, the row sums and the symmetry of P fix its row, and the clock counts force the missing clocks to correspond. □

A connected family reads strictly more than its members' own records, and still not everything outside its clocks.

Example 15.53 (Composition reads more; limbs stay hidden)**.**

1. On five clocks with one token at clock 0 and uniform weights $1/6$, let A have the contacts $(0,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(0,3)$, $(2,4)$ and B the contacts $(0,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(0,2)$, $(3,4)$. The own records of $g = (0,1)$ agree in A and B, and so do those of $h = (1,2)$. But the pattern "h, g, any attempt, h, g" has chance 0 in A and $1/38880$ in B: by (15.3) it reads $P(0,2)$, which is 0 against $1/6$.
2. Add to G_1 and G_2 of Theorem 15.48 the contact $h = (0,6)$ at g's clock 0, with uniform weights $1/6$ on the first and weights $1/6, 1/6, 1/6$, then $1/12$ on each of the four contacts of the four-cycle, then $1/6$, on the second, and one token at clock 0. The pair (g, h) has the same three-clock blocks at every s and the same joint record at every window, while $P(2,5)$ is 0 against $1/12$ and no relabelling of the seven clocks carries one walk onto the other. The pair's clocks miss four clocks of the whole.
3. Even on trees one neighbouring pair can be blind. Let T be the 16-clock tree in which clock 0 is joined to the paths 1–2–3, 8–9–10, 13–14–15, clock 3 to the paths 4–5 and 6–7, and clock 10 to 11 and 12. Its clocks 2 and 8 are chosen because their return chances in the tree agree at every step although no symmetry of the tree exchanges them. Attaching the path 2–16–17 at clock 2, or at clock 8 (relabelled so that the pair sits on 2–16–17), gives two 18-clock trees which, with one token at clock 0 and uniform weights $1/17$, give the pair $g = (2,16)$, $h = (16,17)$ the same three-clock blocks at every s and the same joint record at every window, while $P(1,2)$ is $1/17$ in one and 0 in the other and no relabelling relates their walks. The equality of the blocks is an exact integer intertwiner (denominator 1219) between the two walks, checked entry by entry.

Cospectral vertices of trees are classical [36]; here they become two places in a whole that the pendant pair attached at either place cannot tell apart with one token.

Several tokens. With t tokens on m clocks every attempt still transposes two bits, the uniform law on the token class is invariant under every attempt, and a pattern's chance does not depend on where it sits in the window. Each short pattern reduces to bit patterns on at most four clocks. Write

$$\pi_{\mathrm{path}} = \frac{t(m-t)(m-t-1)(m-t-2) + (m-t)t(t-1)(t-2)}{m(m-1)(m-2)(m-3)} = \frac{\varpi}{2} - \frac{c_4}{2} \qquad (m \geq 4).$$

Theorem 15.54 (Closed forms at any token number)**.** *In every closed whole with any number of tokens and weights summing to one, for members k, l, j:*

1. *k then k at consecutive attempts has chance $w_k^2 \varpi$, and k, k, k has $w_k^3 \varpi$ (the single firing and the cases $k \neq l$ are Theorem 14.29(b),(d));*
2. *if k and j each share exactly one clock with l, then k, l, j has chance $w_k w_l w_j$ times $c_4/2$ for a star at one clock, $\varpi/2$ for a triangle, π_{path} for a path, and 0 when k and j sit on the same two clocks.*

For $1 \leq t \leq m-1$, $c_4 = \varpi$ (equivalently, the star and the triangle have equal chances) if and only if $(m, t) = (4, 2)$.

Proof. (1) A firing exchanges the member's own two clocks and keeps its pair mixed, so the later firings need only the attempts. (2) Draw the bits y at the first of the three attempts from the uniform law, which is stationary, and let k join s to v and l join v to x, with $s \neq x$. The attempts are independent of the bits and contribute $w_k w_l w_j$. The member k fires when $y_s \neq y_v$ and exchanges s and v; then l fires when $y_s \neq y_x$ and exchanges v and x. Afterwards v holds y_x, x holds y_s, s holds y_v, and every other clock its own bit. If j joins a clock z to v (a star at v), it fires when the bit now at z differs from y_x. For $z = s$ that bit is $y_v = y_x$, so the chance is 0; otherwise the bits on s, v, x, z are $a, \bar{a}, \bar{a}, a$ for one bit a. By symmetry each of the six patterns with two tokens and two holes on four given clocks has chance $c_4/4$, so this one has chance $c_4/2$. If j joins z to x with $z \neq v$, it fires when the bit now at z differs from y_s. For $z = s$ (a triangle) that bit is y_v, the condition is already met, and the chance is that of two pairs sharing s both mixed, $\varpi/2$. For $z \notin \{s, v, x\}$ (a path) the bits on s, v, x, z are $a, \bar{a}, \bar{a}, \bar{a}$, whose chance is π_{path} by counting configurations.

For the last sentence, if $m \leq 3$ there are no two disjoint pairs, the numerator of c_4 vanishes, and $c_4 = 0 < \varpi$. For $m \geq 4$ put $a = t - 1$, $b = m - t - 1$, so $m - 2 = a + b$. Then $c_4 = \varpi$ reads $2ab = (a + b)(a + b - 1)$, that is $a^2 + b^2 = a + b$. Since $a^2 \geq a$ and $b^2 \geq b$ for integers $a, b \geq 0$, with equality only at 0 and 1, this forces $a, b \leq 1$, and $a + b = m - 2 \geq 2$ leaves $a = b = 1$, that is $(m, t) = (4, 2)$. □

Theorem 15.55 (The shape at every occupation but one)**.** *Let two closed wholes, with any token numbers and positive weights summing to one, both differ from the half-filled four-clock whole $(m, t) = (4, 2)$, and let G, G' be families of $K \geq 1$ contacts of their clusters with no two members of G on the same two clocks. Equal joint records at every window give equal member weights, equal ϖ, and a bijection φ as in Theorem 15.52. The exclusion is needed on both wholes:*

1. *in K_4 with weights $1/6$ and two tokens, the star $(0,1),(0,2),(0,3)$ and the triangle $(0,1),(0,2),(2,1)$ have the same joint record at every window;*
2. *the four-clock star whole $(0,1),(0,2),(0,3)$ with two tokens and the triangle whole $(1,2),(0,2),(0,1)$ with one token, all weights $1/3$, give their three contacts the same joint record at every window;*

and in both cases no bijection relates the star's incidence to the triangle's.

Proof. Theorem 14.29(b),(d) and Theorem 15.54 read the weights and ϖ from single and doubled firings; the pair and return patterns separate sh $= 1$ from sh $= 0$ at every (m, t), and sh $= 2$ from sh $= 0$ unless $c_4 = \varpi$; the triple separates star from triangle unless $c_4 = \varpi$, and paths by the shared-clock counts of the outer members. With $a = t - 1$, $b = m - t - 1$ the coincidence reads $a^2 + b^2 = a + b$, which leaves only $(4, 2)$. The orientation argument of Theorem 15.52 then applies. At $(4, 2)$ the bits up to complement are one of the three perfect matchings of the four clocks; an attempt fixes the matching containing its contact and exchanges the other two, a member fires exactly when the current matching does not contain it, and the joint record factors through the matching. Exchanging the attempts $(0, 3)$ and $(2, 1)$, which lie in one matching, carries the star onto the triangle. The second example is a pathwise coupling: a two-token pattern of the star whole read through its matching is the one-token position of the triangle whole. □

More tokens also read more than one token can. With two tokens, or two holes, the two trees of Example 15.53(3) are told apart: the chance that g fires at τ and at $\tau+12$ is $(1/17)^2(4/153)\,C/17^{11}$ with $C = 143451794326534$ in the first tree and $C = 143451794326630$ in the second, while the lag frequencies agree at lags 1 to 11. The difference already sits in g's own record: the two-particle return chance after eleven attempts differs, while the one-particle data, fixed by the intertwiner, agree. It is open whether, outside $(4, 2)$ and at any token numbers, equal joint records of connected families of at least two members, no two on the same two clocks, with positive weights summing to one, give a labelling of the family's clocks with equal member weights, $m = m'$ and equal one-token walks P^s on those clocks.

15.9 Size and fullness

The last question is whether a record reads global numbers of its whole: how many clocks it has, and how many tokens. One symmetry limits every answer.

Proposition 15.56 (Tokens and holes are exchangeable)**.** *In every closed whole, for every family and all weights, flipping every bit of the cluster (exchanging tokens and holes) leaves the joint record law unchanged at every window. The flipped whole is a closed whole on the same network with $m - t$ tokens.*

Proof. Every internal attempt transposes two bits, which commutes with flipping all of them, and flipping keeps every pair's mixedness. So the record is the same along every attempt word. □

A record can therefore at best read the token number up to $t \leftrightarrow m - t$. One participant reads much less.

Theorem 15.57 (One member is blind to size)**.** *Let W_9 have nine clocks and the contacts: the member $(0, 1)$ with weight $1/5$; the fourteen contacts $(0, x), (1, x)$, $x = 2, \dots, 8$, with weight $1/20$ each; and the twenty-one contacts among the clocks $2, \dots, 8$, with weight $1/210$ each. Let W_4 have four clocks and the contacts: the member $(0, 1)$ with weight $1/5$; $(0, 2)$ and $(1, 2)$ with weight*

$(7+\sqrt{5})/40$ each; $(0,3)$ and $(1,3)$ with weight $(7-\sqrt{5})/40$ each; and $(2,3)$ with weight $1/10$. With three tokens on W_9 and one on W_4, the member's record has the same law at every window in the two wholes. Both have $\varpi = 1/2$.

Proof. Both wholes are symmetric in the member's two clocks and exchangeable outside them, so the member's mixedness is an alternating renewal process: a mixed spell ends with a fixed chance per attempt, and an unmixed spell has a law of its own; the record is a thinning of it. Lumping W_9's three-token configurations by whether the member is mixed, both its clocks empty, or both full, and W_4's one-token configurations by the token's clock, an explicit intertwiner carries each lumped attempt of the first whole, letter and move, onto the second, and carries the uniform law onto the uniform law. Its rows send "mixed" to half at clock 0 and half at clock 1, "both empty" to $(5-\sqrt{5})/10$ at clock 2 and $(5+\sqrt{5})/10$ at clock 3, and "both full" to $(1+\sqrt{5})/2$ at clock 2 and $(1-\sqrt{5})/2$ at clock 3, the left eigenvectors, with eigenvalues $7/10$ and $2/5$, of the token's walk on $\{2,3\}$ killed at the member, so that an unmixed spell ends with chance $3/10$ or $3/5$ per attempt, as in the two unmixed classes of W_9. □

Families do better, through the two numbers ϖ and c_4. These two numbers almost determine the size.

Proposition 15.58 (What ϖ and c_4 determine)**.** *Let $1 \le t \le m-1$ and $1 \le t' \le m'-1$. If (m,t) and (m',t') have the same ϖ and the same c_4, then either $m = m'$ and $t' \in \{t, m-t\}$, or the pair is* exceptional*: for some $k \ge 2$, one is $(2k+1,k)$ or $(2k+1,k+1)$ (the* odd whole*) and the other is $(2k+2,k+1)$ (the* half-filled whole*). The exceptional pairs do coincide: both have $\varpi = (k+1)/(2k+1)$ and $c_4 = k(k+1)/((2k+1)(2k-1))$; for $k=2$, five clocks with two tokens and six with three give $\varpi = 3/5$ and $c_4 = 2/5$.*

Proof. With $u = t(m-t)$, $\varpi = 2u/(m(m-1))$ and $c_4 = 4u(u-m+1)/(m(m-1)(m-2)(m-3))$. Writing $X = (m-2t)^2 = m^2 - 4u$, a linear combination of the two equations gives $(m'-m)\big(DX - m(m-2)(m'-m+1)\big) = 0$ with $D = 2mm' - 3m - 3m' + 3 > 0$, and symmetrically for $X' = (m'-2t')^2$. Since $X, X' \ge 0$ this forces $|m-m'| \le 1$, and $m' = m+1$ forces $X' = 0$ and $X = 1$. □

Theorem 15.59 (Families read size up to the exceptional pairs)**.** *Let two closed wholes, neither the half-filled four-clock whole, carry families with positive weights summing to one, and let the first family contain two members sharing no clock, or three members meeting at one clock with no two of them on the same two clocks. If the joint records agree at every window, then either $m = m'$ and $t' \in \{t, m-t\}$, or the two wholes form an exceptional pair.*

Proof. By Theorem 15.55 the record reads the weights, ϖ and the shape. The disjoint pair's consecutive pattern has chance $w_k w_l c_4$ (Theorem 14.29(d)), and the star's triple $w_k w_l w_j c_4/2$ (Theorem 15.54(2)), so the record also reads c_4, and Proposition 15.58 applies. □

Two members sharing a clock see c_4 only through time. For $g = (s,v)$ and $h = (v,x)$ sharing exactly v, let W_y be the weight joining a clock y to the clocks off $\{s,v,x\}$.

Proposition 15.60 (Sharing members read c_4 when their ends are uneven)**.** *In every closed whole with weights summing to one, the chance of the pattern "g fires, any attempt, g fires, h fires" is*

$$w_g^2 w_h\Big(\frac{\varpi}{2}\big(1-\omega_{sx}-W_s\big)+\frac{c_4}{2}\big(W_s-W_v\big)\Big),$$

and the mirrored pattern with g and h exchanged gives the weighted difference $w_g^2 w_h^2 \,\frac{c_4-\varpi}{2}\,(W_s - W_x)$*. Consequently, if two closed wholes, neither the half-filled four-clock whole, with positive weights summing to one, give two such pairs equal joint records, if on both networks every contact on a member's two clocks is that member, and if* $W_s \neq W_x$ *in the first, then* $m = m'$ *and* $t' \in \{t, m-t\}$*, or the two wholes form an exceptional pair.*

Proof. The free attempt is averaged over its position: its contribution to the static chance is $\varpi/2$ for contacts on the members' pairs, off the family, or joining x to an outside clock, 0 for a contact joining s and x, $c_4/2$ for a contact joining s to an outside clock, and $(\varpi - c_4)/2$ for one joining v to an outside clock; summing gives the formula. When only the members sit on their own clock pairs, lag-two statistics read $W_s - W_x$, and $c_4 \neq \varpi$ outside $(4, 2)$, so the mirrored difference reads c_4; Proposition 15.58 finishes. □

The exceptional pairs agree at every single time, for a structural reason.

Proposition 15.61 (The odd and half-filled wholes agree at one time)**.** *Fix* $k \geq 1$ *and a clock d of* $2k+2$ *clocks. Dropping d is a bijection from the configurations with* $k+1$ *tokens onto the configurations of the other* $2k+1$ *clocks with k or* $k+1$ *tokens: the dropped bit is* 1 *exactly when the others carry k tokens. An exchange of two kept clocks acts as the same exchange, and an exchange of a kept clock with d acts as a toggle of that clock conditioned on the count. Every function f of the kept clocks with* $f(\bar{z}) = f(z)$ *has the same uniform mean over the half-filled configurations as over the odd configurations with k tokens.*

Proof. A configuration with $k+1$ tokens on $2k+2$ clocks leaves $k+1$ tokens on the kept clocks when d is empty and k when d holds a token, so the kept configuration determines the dropped bit, and every kept configuration with k or $k+1$ tokens arises. An exchange of two kept clocks does not touch d. An exchange of a kept clock c with d writes d's bit onto c, and that bit is 1 exactly when the kept clocks carry k tokens. For the means, the bijection carries the uniform law on the $\binom{2k+2}{k+1}$ half-filled configurations to the uniform law on the kept configurations with k or $k+1$ tokens. Complementation exchanges these two classes, each of size $\binom{2k+1}{k}$, and leaves f unchanged, so f has the same sum over both classes, and its mean over their union is its mean over the class with k tokens. □

So every statistic taken at one time, by any family on at most $2k+1$ clocks, agrees on an exceptional pair; only statistics across time can separate. Seen up to complement on its $2k+1$ kept clocks, a configuration is a *splitting* into a smaller side of k clocks and a larger side of $k+1$.

Proposition 15.62 (The half-filled whole is the odd whole plus moves)**.** *For every k and every network, on any* $2k+1$ *of its clocks the half-filled whole moves on splittings like the odd whole's exchange process plus the moves of a contact with the dropped clock d: the move at c leaves a splitting S alone when c is on the smaller side and otherwise replaces S by the complement of*

$S \cup \{c\}$, carrying c from the larger side to the smaller. Both wholes push their uniform laws onto the uniform law on splittings. A move twice is the identity, and a move at c on the larger side followed by a move at c' on the smaller side is the exchange of c and c'. If the two wholes give a family the same member weights, the same weight on every pair of family clocks, the same total weight from each family clock to the clocks off the family, and the same total weight, then every statistic with at most one free attempt between firings has the same value in both.

Proof. Let U be the $2k+1$ kept clocks and S the smaller side: the tokens when the kept clocks carry k, the holes when they carry $k+1$. By Proposition 15.61 a contact between kept clocks is their exchange, and a contact between a kept clock c and d writes 1 onto c in the first case and 0 in the second. So nothing happens when $c \in S$, and otherwise $S \cup \{c\}$ becomes the larger side and the new smaller side is $U \setminus (S \cup \{c\})$. In the odd whole every attempt in the cluster is an exchange, and attempts off the cluster do nothing in either whole. The odd configurations are in bijection with the splittings, and the half-filled configurations correspond, through the drop bijection, to the kept configurations with k or $k+1$ tokens, two for each splitting (a configuration and its complement); so both uniform laws push forward to the uniform law on splittings. If $c \notin S$, one move gives $S' = U \setminus (S \cup \{c\}) \not\ni c$, and a second gives $U \setminus (S' \cup \{c\}) = S$. If $c \notin S$ and $c' \in S$, then $c' \notin S'$ and the move at c' gives $U \setminus (S' \cup \{c'\}) = (S \setminus \{c'\}) \cup \{c\}$.

For the last statement choose d off the family. A member is mixed exactly when its two clocks lie on different sides, so each firing factor depends on S only through which family clocks share a side. Such a factor is unchanged by complementing S, hence by an exchange of two clocks off the family and by a move at a clock off the family. A case check on the four positions of c and o relative to S shows that, for o off the family and c in it, the move at o, then at c, then at o is the exchange of c and o. Let Ψ_1, Ψ_2 be the factors of the window before and after its free attempt. Substituting $S \mapsto (\text{move at } o)(S)$, a bijection of the splittings that fixes Ψ_1 and Ψ_2,

$$\sum_S \Psi_1(S)\,\Psi_2(\text{move at } c\,(S)) = \sum_S \Psi_1(S)\,\Psi_2\big(\text{exchange of } c, o\,(S)\big).$$

So a move at a family clock is worth an exchange of that clock with any clock off the family, all such exchanges are worth the same, and an attempt joining two clocks off the family is worth an attempt that does nothing. The free attempt's contribution is therefore fixed by the weights of the contacts inside the family clocks, the total weight from each family clock to the clocks off the family, and the remaining weight; the firings contribute the member weights. All of these agree by hypothesis. □

At $k = 2$ the odd world satisfies exact relations that the half-filled world breaks. Two record facts make them usable: a member that fires at two consecutive attempts is back on its clocks, so the doubled firing multiplies by w^2 times its mixedness, and every polynomial in the members' mixedness is a record probe.

Proposition 15.63 (The frozen-clock identity)**.** *Let a family contain two members on disjoint clock pairs, and write $s_g, s_h = \pm 1$ for their signs (-1 when mixed), $\zeta = s_g s_h$ and $X^* = \zeta + 1 + 3s_g + 3s_h$. Let Ξ_s be the record statistic $\langle X^*, P^s\zeta\rangle$, a fixed combination of the chances of sixteen probe windows (a probe, s free attempts, a probe), each divided by its firing members' weights.*

1. *In every closed odd whole of five clocks with two or three tokens, $\Xi_s = 0$ for every s; the same holds for a star through a virtual pair.*

2. *In a closed half-filled whole of six clocks,* $\Xi_0 = \Xi_1 = 0$*,* Ξ *obeys a linear recurrence of order ten, and* $10\,\Xi_2 = 32\big[(a\cdot r_1)(b\cdot r_1)+(a\cdot r_2)(b\cdot r_2)-(a\cdot u)(b\cdot u)\big]$*, where* a, b *are the weights attaching the two outside clocks to the four family clocks and* $r_1 = (1,-1,0,0)$*,* $r_2 = (0,0,1,-1)$*,* $u = (1,1,-1,-1)$*. A half-filled whole with some* $\Xi_s \neq 0$ *is separated from every odd whole.*
3. $\Xi_s = 0$ *for every* s *when both outside clocks attach uniformly to the family's four clocks, and also when the pair weights are invariant under a three-cycle of three family clocks; the whole with contacts* $(0,1)$*,* $(0,2)$*,* $(0,3)$*,* $(1,2)$*,* $(1,3)$*,* $(2,3)$*,* $(0,4)$*,* $(0,5)$*, weights* $1/8$*, three tokens and the family* $(0,1),(2,3)$ *has non-uniform attachments and* $\Xi_s = 0$ *for every* s*.*

Proof (mechanism). A firing multiplies a window's chance by the member's weight times its mixedness and exchanges the member's own two clocks, which changes no member's mixedness; so, dividing by the weights, the sixteen windows combine to the uniform average of X^* times $P^s\zeta$ over the splittings of Proposition 15.62, where P is one free attempt acting on functions of the bits. (1) In the odd whole $\zeta = 1 - 2[o$ on the smaller side$]$, with o the fifth clock, lies in the span of the constants and the indicators of a clock on the smaller side; exchanges permute these indicators, and X^* is orthogonal to their span (a check over the ten splittings). A star gives the same X^* and ζ as its virtual pair. (2) In the half-filled whole $\zeta = -\sigma_{oo'}$, the sign of the two outside clocks with a minus, and a pair sign moves with its pair: $10\,\Xi_s = \sum_p \mathrm{st}_s(p)\,v(p)$, where $\mathrm{st}_s(p)$ is the chance that two marks started on o, o' sit on the pair p after s attempts, and $v = -32$ on the two clocks of one member, 16 on two family clocks of different members and 0 on every other pair. This gives $\Xi_0 = \Xi_1 = 0$, the quadratic form at $s = 2$, and, through the 10×10 action of a free attempt on the splittings of the five clocks other than o' and Cayley–Hamilton, the recurrence of order ten. Equal joint records give equal member weights (Theorem 15.55), hence equal Ξ_s, while every odd whole has $\Xi_s = 0$. (3) Uniform attachment keeps $P^s\zeta$ in the span of 1 and ζ, to which X^* is orthogonal; under a three-cycle symmetry st_s is invariant while v sums to zero over each orbit of pairs ($-32 + 16 + 16$ over $\{0,1\},\{0,2\},\{0,3\}$ for the cycle $(1\,2\,3)$), and the example is invariant under $(1\,2\,3)$ with its outside clocks attached only to clock 0. □

The frozen-clock statistics thus separate off a quadric and leave explicit wholes unseparated. A second family of identities closes the gap.

Theorem 15.64 (Two members tell five clocks from six)**.** *Let one closed whole have five clocks with two or three tokens and the other six clocks with three tokens, in either order, with positive weights summing to one, and let the families have the same number of members. If the first family contains two members sharing no clock, or a star of three members, or two members sharing exactly one clock, the joint records differ at some window. In particular any two members not on the same two clocks separate the two wholes, for every network and every positive weights.*

Proof. For a disjoint pair, take the classes "only the first member mixed", "only the second mixed" and "both unmixed". In an odd whole one exchange moves one token, so on the two splittings where both members are unmixed the record's class sequences are rigid, and five sojourn identities hold for every weighted family of exchanges; for instance, "only the first mixed, then both unmixed, then only the second mixed" at three consecutive attempts has chance zero. In the half-filled whole the entrances violating them are exactly the moves of the

outside clocks, and a connected cluster crosses every cut with a contact of positive weight, so at least one identity fails. The identities are record statistics through the doubled-firing probes, so equal records would carry them over. A star is handled through the virtual pair formed by its centre with one leaf and the other two leaves. For a sharing pair the odd whole is the half-filled whole with a sixth clock touching nothing: the state "agreeing pair of family clocks accompanied by the sixth clock" is then hidden, entered only by crossings that change the accompanying clock. A family of grading statistics, built from class probes, free attempts and firings with the pair weight read from one-gap windows, vanishes in every odd whole; in a half-filled whole its vanishing would force a nonempty set of outside clocks with no attachment to the family and no weight to the other outside clocks, which connectivity forbids. □

With Theorem 15.59, families with a disjoint pair or a star read size and fullness up to complement except on exceptional pairs with $k \geq 3$. For sharing members we record a further result, whose proof is not reproduced here: at every $k \geq 2$, two members sharing exactly one clock never give a closed odd whole ($2k+1, k$ or $k+1$) and a closed half-filled whole ($2k+2, k+1$) the same joint record, for every network and every positive weights summing to one. Its statistic compares the family's majority side at two visits to the class where neither member is mixed: with $f = +1$ when all three family clocks are empty, -1 when all are full and 0 otherwise, $S(T) = \langle f, P^T f \rangle$ under the uniform law. Because one exchange moves one token, equal joint records force equal $S(T)$ at every lag, through a count identity that reads the pair weights from one-step windows. In every closed whole of at least five clocks $S(T) \to (\mathbf{E} f)^2$, with

$$\mathbf{E} f = \frac{\binom{N-3}{t} - \binom{N-3}{t-3}}{\binom{N}{t}}$$

for t tokens on N clocks, which is 0 in the half-filled whole by binomial symmetry and nonzero in the odd whole ($1/10$ at $k = 2$, $3/35$ at $k = 3$). So every family containing two members that share exactly one clock tells the two wholes of every exceptional pair apart.

Two questions remain on this line (Chapter 23). Do two members sharing no clock separate the exceptional pairs at every $k \geq 3$? And do two members sharing one clock read c_4 beyond the conditions of Proposition 15.60, in the symmetric case $W_s = W_x$ and with silent copies of a member on its own clock pair? Chapter 16 turns from what pooled records reveal to what they hide.

Chapter 16

Pooled records and privacy

A participant shares sites with other participants, and the contacts that write its record also move bits and clocks that others read. Every record on every network reads each site only through a twelve-dimensional space, so no record compares the circulations of two clocks, although a real projector does. We decide which writes keep this privacy, what a participant reads of sites it does not own, and which pools of records make the present pure.

The network carrier $\mathscr{H}_n$, its contacts $e = (u, v)$ and slices V_j^e, and records that read a posting's label, leave it unread or read it coarsely, with effects E and probabilities $\operatorname{tr}(\rho E)$, are those of Section 13.1. An operator on one site is a matrix on $\mathbb{C}^2 \otimes \mathscr{H}_{\mathrm{c}}$; placed at site u it acts as the identity elsewhere. The *path* and the *triangle* are those of Chapter 12; as in Chapter 13 the contact $(0, 1)$ of the path is the *old* contact and $(0, 2)$ its *neighbour*, and a round posts the old contact first. The circulating clock vector is $\gamma = \frac{1}{2}(1, i, -1, -i)$ of Lemma 9.1, with $L\gamma = i\gamma$ and $\gamma^{\top}\gamma = 0$; *circulation* names this conjugation-odd clock profile, γ against $\bar{\gamma}$, not an identified observable.

16.1 What a record can read at one site

Every native record effect is real, and on one cell it is block diagonal in the occupation (Chapter 9). On a network the same two facts hold site by site, and together with the half-turn of Chapter 6 they fix a twelve-dimensional space of directions through which every record sees every site.

Definition 16.1 (Readable site space)**.** Let $\mathcal{S}$ be the set of complex $4{\times}4$ matrices G with $G^{\top} = G$ and $L^2 G L^2 = G$, and let $\mathscr{K}$ be the set of site operators $\sum_{b\in\{0,1\}} |b\rangle\langle b| \otimes G_b$ with $G_0, G_1 \in \mathcal{S}$. We call $\mathscr{K}$ the *readable site space*; by Lemma 16.2 it is the span $\mathcal{S}_1$ of the pointer statistics of Section 14.7. With $\sigma_{\mathrm{b}} = |0\rangle\langle 0| - |1\rangle\langle 1|$ and $Z = PA^{\top} + AP^{\top}$ put

$$\zeta = \sigma_{\mathrm{b}} \otimes Z.$$

Lemma 16.2 (Twelve directions)**.** $\mathcal{S} = \operatorname{span}\{I,\ L^2,\ c,\ c',\ Z,\ DZD\}$, *so* $\dim \mathcal{S} = 6$ *and* $\dim \mathscr{K} = 12$. *Of the* 64 *dimensions of site operators,* $\mathscr{K}$ *misses the* 32 *coherences between the two bit values, and on each bit value the* 2 *antisymmetric and the* 8 *half-turn-odd directions.*

Proof. L^2 is $+1$ on $\mathscr{U} = \operatorname{span}\{p, a\}$ and -1 on $D\mathscr{U} = \operatorname{span}\{P, A\}$ (Chapter 6), so a matrix commutes with L^2 exactly when it preserves both planes. In the real orthonormal pole basis a symmetric matrix preserving both planes is a symmetric 2×2 block on each, a space spanned by $pp^{\top}, aa^{\top}, pa^{\top} + ap^{\top}, PP^{\top}, AA^{\top}, PA^{\top} + AP^{\top}$. The six listed matrices are I, $L^2 = \Pi_{\mathscr{U}} - \Pi_{D\mathscr{U}}$, $c = pp^{\top} - aa^{\top}$, $c' = PP^{\top} - AA^{\top}$, $Z = PA^{\top} + AP^{\top}$ and $DZD = pa^{\top} + ap^{\top}$, which span the same space. The count of the complement is $64 - 12 = 32 + 2 \cdot (2 + 8)$. □

Inside $\mathscr{K}$ we single out a chain of subspaces by what they read of the clock. The bit reads are $\mathscr{K}_{\mathrm{bit}} = \mathrm{span}\{|b\rangle\langle b| \otimes I\}$. The *stay weights* are the Grams of the clock maps, $\mathrm{stay}^{\top}\mathrm{stay} = \frac{1}{2}(I - c)$ and $(\mathrm{stay}\, D)^{\top}(\mathrm{stay}\, D) = \frac{1}{2}(I - c')$, and $\mathscr{K}_{\mathrm{stay}} = \mathrm{span}\{|b\rangle\langle b| \otimes G \,:\, G \in \{I, c, c'\}\}$. Adding the half-turn gives $\mathscr{K}_{\mathrm{ht}} = \mathrm{span}\{|b\rangle\langle b| \otimes G \,:\, G \in \{I, L^2, c, c'\}\}$. Then

$$\mathscr{K}_{\mathrm{bit}} \subset \mathscr{K}_{\mathrm{stay}} \subset \mathscr{K}_{\mathrm{ht}} \subset \mathscr{K}, \qquad \dim = 2, 6, 8, 12,$$

and the last step adds the two standing-wave reads Z and DZD on each bit. Two further subspaces appear below: $\mathscr{K}^{\flat}$, spanned by $\mathscr{K}_{\mathrm{ht}}$, by $I_2 \otimes Z$ and by $|b\rangle\langle b| \otimes DZD$, which is $\mathscr{K}$ without ζ (dim 11); and $\mathscr{K}_5 = \mathrm{span}\{|b\rangle\langle b| \otimes I,\ |b\rangle\langle b| \otimes c',\ I_2 \otimes c\}$ (dim 5). The direction ζ is the bit-odd contrast of the two standing waves of the dressed plane: $Z = \Omega_+\Omega_+^{\top} - \Omega_-\Omega_-^{\top}$ with $\Omega_{\pm} = (P \pm A)/\sqrt{2}$.

Lemma 16.3 (One posting keeps readable products)**.** *Let $e = (u, v)$ be a contact, j a label, and $B = \bigotimes_z B_z$ a product with $B_u, B_v \in \mathscr{K}$. Then*

$$V_j^{e*} B V_j^{e} = \sum_{x} \big(|x_u\rangle\langle x_u| \otimes f_1^{\top} G_{y_u} f_1\big) \otimes \big(|x_v\rangle\langle x_v| \otimes f_2^{\top} G'_{y_v} f_2\big) \otimes \bigotimes_{z \neq u,v} B_z,$$

summed over the inputs $x = (x_u, x_v)$ that have a transition $x \to y$ with $j(y, x) = j$, where $B_u = \sum_b |b\rangle\langle b| \otimes G_b$, $B_v = \sum_b |b\rangle\langle b| \otimes G'_b$, $f_1 = K(y_u, x_v)C_x$ and $f_2 = K(y_v, x_u)C_x$. Every factor $f_c^{\top} G f_c$ lies in $\mathcal{S}$.

Proof. By (2.11), $V_j = \sum_{j(y,x)=j} |y\rangle\langle x| \otimes M_{yx}$, and by Proposition 2.7 a label with an input fixes the output and a label with an output fixes the input. Since B_u and B_v are diagonal in the bits, a term $M_{yx}^{\top}(\cdot)M_{y'x'}$ survives only for $y = y'$, hence $x = x'$. By (2.7), $M_{yx} = \sigma(x) f_1 \otimes f_2$ with real f_1, f_2, and the sign squares away. Each $f_c^{\top} G f_c$ is symmetric when G is, and by Lemma 6.9, $L^2 f_c L^2 = \pm f_c$, so $L^2 f_c^{\top} G f_c L^2 = f_c^{\top}(L^2 G L^2) f_c = f_c^{\top} G f_c$. □

Theorem 16.4 (Every record factors through readable sites)**.** *On every network, for every finite list of postings by contacts in any placement, each posting read, unread or read coarsely, the effect lies in the span $\mathscr{K}^{\otimes n}$ of the products $\bigotimes_z B_z$ with every $B_z \in \mathscr{K}$.*

Proof. $I = \bigotimes_z I \in \mathscr{K}^{\otimes n}$. By Lemma 16.3 and linearity, $X \mapsto V_j^{e*} X V_j^{e}$ maps $\mathscr{K}^{\otimes n}$ into itself, and so do its sums over all labels or over a set of labels. The effect of a list is obtained from I by applying these maps for the postings from the last to the first. □

Corollary 16.5 (Blind at every site)**.** *Let E be the effect of any record on any network, u a site and ρ any operator on $\mathscr{H}_n$. Then* $\mathrm{tr}(\rho E)$ *is unchanged when ρ is replaced by*

(a) *its partial transpose at u, which for a product density conjugates the factor at u;*
(b) *$H_u \rho H_u$, where H_u is the half-turn L^2 of the clock at u;*
(c) *$\sum_b P_b^u \rho P_b^u$, where P_b^u projects site u onto bit b.*

Moreover E commutes with every H_u, so if $H_u\psi = \psi$ and $H_u\psi' = -\psi'$ then $\langle \psi + \psi', E(\psi + \psi')\rangle = \langle \psi, E\psi\rangle + \langle \psi', E\psi'\rangle$.

Proof. Every $B \in \mathscr{K}$ satisfies $B^{\top} = B$, $L^2 B L^2 = B$ and $\sum_b P_b B P_b = B$. For a product $B_u \otimes R$, $\mathrm{tr}(\rho^{\Gamma_u}(B_u \otimes R)) = \mathrm{tr}(\rho(B_u^{\top} \otimes R))$, and the other two operations move onto B_u in the same way. Theorem 16.4 and linearity finish. □

The sharpest consequence of this local blindness concerns pairs of sites.

Theorem 16.6 (No record compares two circulations)**.** *Let $u \neq v$ be sites of a network. Let site u carry $|b\rangle \otimes w$ and site v carry $|b'\rangle \otimes w'$, with unit clock vectors w, w', and let every other site carry any density, the whole state being the product. For every record of every list of postings, the probability is unchanged when w' is replaced by $\bar{w}'$. In particular the same circulation (γ at both sites) and opposite circulations (γ at u, $\bar{\gamma}$ at v), at equal bits, have one law under every record of every network. On the cell of one contact the projection Q_b onto $|bb\rangle \otimes \mathrm{span}\{\gamma \otimes \gamma, \bar{\gamma} \otimes \bar{\gamma}\}$ is real, it has expectation 1 in $|bb\rangle \otimes \gamma \otimes \gamma$ and 0 in $|bb\rangle \otimes \gamma \otimes \bar{\gamma}$, and it does not lie in $\mathscr{K} \otimes \mathscr{K}$.*

Proof. The density of site v is $|b'\rangle\langle b'| \otimes |w'\rangle\langle w'|$, and replacing w' by $\bar{w}'$ transposes it; apply Corollary 16.5(a) at v. On the bits bb, Q_b is the real projection of Proposition 11.15 with $\psi = \varphi = \gamma$; exchanging the two clocks there gives the expectation 0 in $\gamma \otimes \bar{\gamma}$. An element of $\mathscr{K} \otimes \mathscr{K}$ would give the two states equal expectations by the first part. □

Reference-frame theory would allow such a comparison [6]: relational properties like same or opposite stay accessible to real operations, and Q_b is one. No record reads it; the blindness to the conjugation of the whole cell (Theorem 11.9) is the diagonal of a symmetry that holds at every site.

16.2 The written label keeps two insides private

We use the word *inside* for the clock content of a site, and call two insides *private* from a family of records when no record law of the family compares them. Both words are names; the theorems concern clocks and record laws.

A posting that keeps its label coherent instead of writing it is the native step $U_{\mathrm{nat}} = \sum_j V_j$. Its pullback is

$$U_{\mathrm{nat}}^* X U_{\mathrm{nat}} = \Phi(X) + \sum_j \sum_{j' \neq j} V_j^* X V_{j'}, \tag{16.1}$$

the written pullback Φ of (2.12) plus the cross terms between distinct labels. Those cross terms carry exactly what Theorem 16.6 forbids. Where one transition moves a clock by stay and another, reaching the same output from a different input, moves it by flip, their cross term contains

$$\mathrm{stay}^{\top}\mathrm{flip} = \tfrac{1}{4}(I - L^{\top})(I + L) = \tfrac{1}{4}(L - L^{\top}) = \tfrac{1}{2}\Gamma,$$

the circulation generator itself.

Proposition 16.7 (Erasing the label reads the relative circulation)**.** *On one contact's cell let $s = |00\rangle \otimes \gamma \otimes \gamma$ and $o = |00\rangle \otimes \gamma \otimes \bar{\gamma}$.*

(a) *For $m \leq 2$ and every record word h, $\|V_h U_{\mathrm{nat}}^m s\|^2 = \|V_h U_{\mathrm{nat}}^m o\|^2$.*
(b) *For $m = 3$ and the one-label word $h = ((1,0,1))$, written by $(0,0) \to (1,1)$, the two probabilities are $\frac{3}{32}$ and $\frac{9}{32}$.*
(c) *If the m postings write their labels, read or unread, the laws of s and o agree for every m and every later record word.*
(d) *On the path let site 0 carry bit 1 and clock e_0, and sites 1 and 2 carry bit 0 with clocks γ, γ or $\gamma, \bar{\gamma}$. After four postings with erased labels, by the old contact, the old contact, the neighbour*

and the neighbour, one old posting read with label $(-1,0,-1)$ *has probability* $\frac{9}{256}$ *in the first state and* $\frac{7}{256}$ *in the second.*

Proof. (c) is Theorem 16.6, since written postings are Born pullbacks. (a), (b) and (d) are direct exact computations: the vectors $U_{\mathrm{nat}}^m s$ and $U_{\mathrm{nat}}^m o$ have entries in $\mathbb{Q}(i)$, and for $m \le 2$ their expectations agree on each of the 144 products of the basis of Lemma 16.2 at the two sites, hence on every record effect by Theorem 16.4; (b) and (d) evaluate the stated words. □

In (d) the private sites 1 and 2 never meet in a contact, and one record of the old contact reads their relative circulation through the shared site. Whether three erased postings already suffice there is open: with sites 1 and 2 at bit 0 carrying γ, γ against $\gamma, \bar{\gamma}$, and for every real state of the shared site, every list of at most three erased postings of the two contacts and every record of both contacts after them, do the two states have one law?

Which part of the label protects? Let a contact write any function $f(y,x) \in W$ of its transition, keeping the transitions with one written value coherent:

$$K_f(c) = \sum_{(y,x)\colon f(y,x)=c} |y\rangle\langle x| \otimes M_{yx}, \qquad c \in W.$$

The native label is $f = j$, with $K_j(c) = V_c$; writing nothing gives U_{nat}. We say that f *separates* if $f(y,x) = f(y,x')$ implies $x = x'$: given where the cell went, the written value fixes where it came from. It *fixes the path* if $f(y,x) = f(y',x)$ implies $y = y'$, and it *records the output* if $f(y,x) = f(y',x')$ implies $y = y'$. It is an *instrument* if $\sum_c K_f(c)^* K_f(c) = I$, and its pullback is $\Phi_f(X) = \sum_c K_f(c)^* X K_f(c)$. The write f is a supplied function of the transition, the native label being one of them, and an erased label is supplied coherence: no native process that erases a written label is derived here.

Theorem 16.8 (The pointer criterion)**.** *For every write f,* Φ_f *maps* $\mathscr{K} \otimes \mathscr{K}$ *into itself if and only if f separates. If f separates,* $\Phi_f = \Phi$ *on* $\mathscr{K} \otimes \mathscr{K}$*, and f is an instrument.*

Proof. Write $T_{yx} = |y\rangle\langle x| \otimes M_{yx}$. For X diagonal in the occupation, $T_{yx}^* X T_{y'x'} = 0$ unless $y = y'$. If f separates, two transitions with one written value and one output have one input, so $K_f(c)^* X K_f(c) = \sum_{f(y,x)=c} T_{yx}^* X T_{yx}$; summed over c this is $\sum_{(y,x)} T_{yx}^* X T_{yx} = \Phi(X)$, by the same argument applied to the label. Taking $X = I$ gives $\sum_x |x\rangle\langle x| \otimes \sum_y M_{yx}^\top M_{yx} = I$ by Proposition 2.6. Conversely let $f(y,x) = f(y,x')$ with $x \neq x'$. For $X = |y\rangle\langle y| \otimes N$ the block of $\Phi_f(X)$ from x' to x is $M_{yx}^\top N M_{yx'}$. By direct exact computation, for each output y and each pair of distinct inputs x, x' some $N = G \otimes G'$ with $G, G' \in \mathscr{S}$ makes it nonzero; then $\Phi_f(X)$ is not diagonal in the occupation and so not in $\mathscr{K} \otimes \mathscr{K}$. □

Corollary 16.9 (Separating writes on every network)**.** *On every network, for every list of postings each of which writes a separating function of its transition, read, unread or read coarsely, the effects lie in* $\mathscr{K}^{\otimes n}$*, and Corollary 16.5 and Theorem 16.6 hold for them.*

Proof. The proof of Theorem 16.8 gives Lemma 16.3 with $K_f(c)$ in place of V_j: the cross terms vanish, and each surviving term is a native block. The rest is the proof of Theorem 16.4. □

Proposition 16.10 (Which writes separate)**.** *The label* $j(y,x)$*, the transition* (y,x)*, the input x and the bit motions* $(y_1 + x_1, y_2 + x_2) \bmod 2$ *separate; the output y, the awake bit* $[x_1 \neq x_2]$ *and the empty write do not. All seven are instruments, and the bit motions also fix the path.*

Proof. The label separates by Proposition 2.7, although it does not record the output: the zero label is written both by $(0,0) \to (0,0)$ and by $(1,1) \to (1,1)$, and we call this coincidence the *collision*. Given y, the motions return $x = y - m$. The inputs $(0,0)$ and $(1,1)$ are both balanced and both reach $(0,0)$, so neither the output nor the awake bit separates them. The four separating writes are instruments by Theorem 16.8. A write $g(y)$ of the output alone has $K(c) = \Pi_c U_{\mathrm{nat}}$ with Π_c the projection onto the outputs with $g(y) = c$, and a write $g(x)$ of the input alone has $K(c) = U_{\mathrm{nat}}\Pi_c$; both sum to $U_{\mathrm{nat}}^{*}U_{\mathrm{nat}} = I$. Given x, the motions return $y = x + m$. □

The bit motions say only which bits changed and nothing about the clocks; they already protect as the label does. A coarser write protects less.

Proposition 16.11 (Fixing the path protects definite occupations only)**.**

(a) *Let every posting write a path-fixing function. Then for every written history, every later native record word and $b \in \{0,1\}$, the states $|bb\rangle \otimes \gamma \otimes \gamma$ and $|bb\rangle \otimes \gamma \otimes \bar{\gamma}$ have one law; on every network, every product of path-fixing postings gives every occupation-definite product state one law under conjugating the clock of any one site.*

(b) *The output write does not protect superposed occupations. For $s' = \frac{3}{5}|00\rangle \otimes \gamma \otimes \gamma + \frac{4}{5}|11\rangle \otimes \gamma \otimes \gamma$ and o' the same with $\bar{\gamma}$ in the second clock, every native record word has one law before any posting; after one posting writing the output $(0,0)$ the probabilities are $\frac{1}{100}$ and $\frac{49}{100}$, and $\frac{1}{400}$ and $\frac{49}{400}$ if the zero label follows.*

Proof. (a) From a definite occupation the written values of a path-fixing write fix the occupation path, so on that branch the record operator is $|y\rangle\langle x| \otimes R_1 \otimes R_2$ (on a network, one real factor per site) with every R_i a product of real clock maps. For real R, $\|R\bar{w}\| = \|Rw\|$. (b) Before any posting, every element of $\mathscr{K} \otimes \mathscr{K}$ is diagonal in the occupation, so the two branches add without interference and each is dark by (a). The values after one posting are the case $\alpha = \frac{3}{5}$, $\beta = \frac{4}{5}$, $y = (0,0)$ of Theorem 16.20 below; the last pair is a direct exact computation. □

Proposition 16.12 (The awake bit protects no longer than erasure)**.** *Let each posting write only its awake bit, 1 for a mixed input and 0 for a balanced one, and let s, o be as in Proposition 16.7.*

(a) *For every written history of at most two postings and every later native record word, s and o have one law; so they do when the awake bits are written and unread.*

(b) *The history $(0,0,0)$ followed by the zero label has probability 0 for s and $\frac{1}{8}$ for o. The history $(0,0,0)$ has probability $\frac{1}{4}$ for both, and given it the fourth awake bit is 1 for s and 0 for o with certainty.*

(c) *With three awake bits written and unread, the zero label has probability $\frac{3}{16}$ for s and $\frac{5}{16}$ for o.*

Proof. The awake write has $K(a) = U_{\mathrm{nat}}\Pi_a$, Π_a the projection onto the inputs of awake bit a. Each item is a direct exact computation of the same kind as Proposition 16.7; for (a) the expectations of the 144 basis products agree after every history of length at most two. □

Proposition 16.13 (Two private sites through a shared one)**.** *On the path let the old contact write its labels and the neighbour write only its outputs. For every clock w_0 of the shared site, real or not, every finite product of these postings in any order gives the states with site 0 at bit 1 and clock w_0, sites 1 and 2 at bit 0, and clocks γ, γ or $\gamma, \bar{\gamma}$ at sites 1, 2, one probability.*

Proof. Both writes fix the path, and the state is occupation definite; apply Proposition 16.11(a) at site 2. □

16.3 For recording writes, privacy is separation up to the awake bit

Separation is enough. It is more than privacy needs, because the clocks already record part of where the cell came from. By Lemma 6.9, $L_c^2 M_{yx} L_c^2 = \epsilon(x) M_{yx}$ with $\epsilon = -1$ on balanced and $+1$ on mixed inputs: a step from a balanced input reverses the half-turn parity of each clock and a step from a mixed input keeps it. So each clock's half-turn parity counts, modulo 2, the balanced inputs its cell has passed. A record that fixes the output has to add only what the clocks do not hold.

The comparisons we test are those of the previous section, with the clocks allowed to differ from branch to branch. For clock vectors u_x, w_x assigned to the four occupations put $g(u, w) = \sum_x |x\rangle \otimes u_x \otimes w_x$.

Definition 16.14 (Law privacy)**.** A write f is *law-private* if for every $g(u, w)$ whose second clocks share one half-turn parity ($L^2 w_x = l w_x$ for all x, one $l \in \{1, -1\}$), every written history $(c_1, \dots, c_m)$, $m \geq 0$, and every native record word h, the probability $\|V_h K_f(c_m) \cdots K_f(c_1) g(u, w)\|^2$ is unchanged when every w_x is replaced by $\bar{w}_x$. The write *separates up to the awake bit* if $f(y, x) = f(y, x')$ with x, x' both balanced or both mixed implies $x = x'$. From here on, privacy of a write means law privacy.

The circulating clocks are half-turn odd, $L^2 \gamma = -\gamma$, so the comparisons of Section 16.2 are among those the definition tests. First we find which writes a contact can make at all.

Theorem 16.15 (Which writes are instruments)**.** *For distinct inputs x, x' let $\mathscr{M}(x, x') = \{y : f(y, x) = f(y, x')\}$ be the outputs at which f merges them. Then f is an instrument if and only if for every such pair:*

(i) *if x and x' are both balanced or both mixed, $\mathscr{M}(x, x')$ contains as many balanced as mixed outputs;*
(ii) *if $x_1 = x_1'$ and one of them is balanced and the other mixed, whether $y \in \mathscr{M}(x, x')$ depends only on y_2;*
(iii) *if $x_2 = x_2'$ and one of them is balanced and the other mixed, whether $y \in \mathscr{M}(x, x')$ depends only on y_1.*

Proof. $\sum_c K_f(c)^* K_f(c) = \sum_{x,x'} |x\rangle\langle x'| \otimes \sum_{y \in \mathscr{M}(x,x')} M_{yx}^\top M_{yx'}$ with $\mathscr{M}(x, x)$ all outputs, where the block is I by Proposition 2.6. So f is an instrument exactly when the summed cross Gram over $\mathscr{M}(x, x')$ vanishes for every $x \neq x'$. Two inputs of different awake bit share exactly one bit, so the three cases cover all pairs. Which of the 16 subsets give a vanishing sum, for each of the 6 pairs, is a direct exact computation with the blocks (2.7). □

Theorem 16.16 (One posting decides awake separation)**.** *Let f be a write, c a value and $x \neq x'$ inputs that f merges at some output with value c. There are amplitudes α, β and a native record word of at most two labels for which, after one posting writing c, the states $\alpha|x\rangle \otimes \gamma \otimes \gamma + \beta|x'\rangle \otimes \gamma \otimes \gamma$*

and $\alpha|x\rangle \otimes \gamma \otimes \bar{\gamma} + \beta|x'\rangle \otimes \gamma \otimes \bar{\gamma}$ have different probabilities, if and only if x and x' are both balanced, or both mixed and merged at some balanced output with value c. Consequently an instrument separates up to the awake bit if and only if one posting, followed by native words of any length, leaves every such pair of states with one law.

Proof. Two mechanisms give darkness. If x and x' differ in the awake bit, the two branches start with one parity of the second clock and end, by Lemma 6.9, in opposite half-turn sectors of it; every record effect commutes with the half-turn (Corollary 16.5), so the branches do not interfere. Each alone is dark: its outputs do not interfere under the native effects, which are diagonal in the occupation, and each moves by real clock maps. If $x = (0, 1)$ and $x' = (1, 0)$ are merged only at mixed outputs, the dressed step sends their first clocks to orthogonal vectors, and every later native factor annihilates one of the two, a direct exact computation; again nothing interferes. In the remaining cases the contrast is explicit, with $\alpha = \frac{3}{5}$, $\beta = \frac{4}{5}$. For the balanced pair merged at y_0, a native label that fires only from y_0 gives the probability for $\gamma \otimes \gamma$ minus that for $\gamma \otimes \bar{\gamma}$ equal to $-\frac{3}{25}\epsilon(y_0)$, with $\epsilon(y_0) = +1$ for balanced and -1 for mixed y_0; for the mixed pair merged at a balanced y_0, the label of the step from y_0 to the other balanced occupation followed by the zero label gives $-\frac{3}{100}$. The last sentence follows by letting c, x, x' range. □

Theorem 16.17 (Recording writes)**.** *Let f be an instrument that records the output. Then f is law-private if and only if it separates up to the awake bit.*

Proof. If f is law-private, Theorem 16.16 gives awake separation. Conversely, let the branches $|x\rangle u_x w_x$ and $|x'\rangle u_{x'} w_{x'}$, $x \neq x'$, be carried along one written history. The record fixes every output, so after the first posting both branches sit at one occupation and follow the same occupations from then on. At the first posting they leave distinct inputs for one output with one written value, so by awake separation the inputs differ in the awake bit. Their second clocks, of one parity at the start, therefore end in opposite half-turn sectors, and they stay there: by Lemma 6.9, conjugating $K_f(c)$ by the second clock's half-turn multiplies its block from x by $\epsilon(x)$, and both branches pass the same occupations, so every later posting multiplies both by the same sign. The final native word commutes with the half-turn (Corollary 16.5), so it does not connect them, and the probability is the sum over the occupation branches. Each branch moves by real clock maps and is dark, as in Proposition 16.11(a). □

Proposition 16.18 (A coarse record that protects)**.** *The write $f(y, x) = (y, x_1)$ is an instrument, records the output, separates up to the awake bit, does not separate, and is law-private. So among instruments with values in $\mathcal{O} \times \{0, 1\}$ law privacy and separation are different properties.*

Proof. Two inputs with the same first bit differ in the second, so one is balanced and the other mixed; $(0, 0)$ and $(0, 1)$ reach every output with the same value, so f does not separate. $K_f(y, b) = \Pi_y U_{\mathrm{nat}} \Pi_b$, with Π_b the projection onto inputs with $x_1 = b$, and $\sum_{y,b} \Pi_b U_{\mathrm{nat}}^* \Pi_y U_{\mathrm{nat}} \Pi_b = \sum_b \Pi_b = I$. Theorem 16.17 finishes. □

A record that does not fix the output opens another route: two unrecorded intermediate occupations of opposite awake bit can bring the second clocks back into one parity sector.

Proposition 16.19 (Leaving part of the output unrecorded)**.** *The write $f(y, x) = (y_1, x_1)$ is an instrument that separates up to the awake bit, does not record the output, does not separate, and is not law-private. One posting leaves every two-branch state of Theorem 16.16 dark; after two postings both writing $(0, 0)$, the states $\frac{3}{5}|00\rangle \otimes \gamma \otimes \gamma + \frac{4}{5}|01\rangle \otimes \gamma \otimes \gamma$ and the same with $\bar{\gamma}$ in the second clock have probabilities $\frac{33}{100}$ and $\frac{39}{100}$.*

Proof. The write is a function of the output and of the input, an instrument as in Proposition 16.18. Two inputs with equal first bit have different awake bits, so Theorem 16.16 gives darkness after one posting. The two probabilities are a direct exact computation of $\|K_f(0, 0)^2 v\|^2$. □

When the output is recorded, the remaining freedom is the relative phase between occupation branches. Write $\epsilon(y) = +1$ for balanced and -1 for mixed y.

Theorem 16.20 (Branch phases under the output record)**.** *Let a contact write its output at every posting, and let $v_w = \alpha|00\rangle \otimes \gamma \otimes w + \beta|11\rangle \otimes \gamma \otimes w$. For every output history beginning with y and every later native record word there is one number $\Phi \geq 0$, the same for $w = \gamma$ and $w = \bar{\gamma}$, such that the probability is $|\alpha - \epsilon(y)\beta|^2\Phi$ for $w = \gamma$ and $|\alpha + \epsilon(y)\beta|^2\Phi$ for $w = \bar{\gamma}$. Hence v_γ and $v_{\bar{\gamma}}$ have one law for every output history and native word if and only if $\mathrm{Re}(\bar{\alpha}\beta) = 0$. In particular the relative phase i, $\alpha = 1/\sqrt{2}$, $\beta = i/\sqrt{2}$, is dark at every depth, while real amplitudes are read by the first written output.*

Proof. From $(0, 0)$ and from $(1, 1)$ with the same clocks $\gamma \otimes w$, every output y is reached, and the branch from $(1, 1)$ is $-\epsilon(y)$ times the branch from $(0, 0)$ when $w = \gamma$ and $+\epsilon(y)$ times it when $w = \bar{\gamma}$; this uses flip $\gamma = i$ stay γ and is a direct exact computation over the four outputs. So the first written output merges the two event branches into one product vector with weight $|\alpha \mp \epsilon(y)\beta|^2$, and every later posting acts on that vector alone. The two weights agree for every y exactly when $\mathrm{Re}(\bar{\alpha}\beta) = 0$, and $\Phi > 0$ for the empty continuation. □

Equivalently, conjugating the second clock acts on v_w as $(\alpha, \beta) \mapsto (\bar{\alpha}, -\bar{\beta})$, an antiunitary map that fixes the phase i. For $\alpha = \frac{3}{5}$, $\beta = \frac{4}{5}$ and $y = (0, 0)$, $\Phi = \frac{1}{4}$, which gives the values $\frac{1}{100}$ and $\frac{49}{100}$ of Proposition 16.11(b). Part of the which-input record sits inside the clocks, as a half-turn parity no outward record can erase.

The gate of Chapter 14 times postings by the awake bit: an attempting contact stays silent on a balanced input and posts, with its label read, on a mixed one. Its Kraus operators are Π_{bal} and $V_j\Pi_{\mathrm{mix}}$.

Proposition 16.21 (The gate keeps the insides private)**.** *The gated posting is an instrument, and every sequence of gated postings, read in any pattern and followed by native record words, gives $|bb\rangle \otimes \gamma \otimes \gamma$ and $|bb\rangle \otimes \gamma \otimes \bar{\gamma}$ one law.*

Proof. $\Pi_{\mathrm{bal}} + \sum_j \Pi_{\mathrm{mix}} V_j^* V_j \Pi_{\mathrm{mix}} = \Pi_{\mathrm{bal}} + \Pi_{\mathrm{mix}} = I$, since $\sum_j V_j^* V_j = I$. Each Kraus operator is a sum of blocks with distinct inputs per output, each block a native step or the identity, so the proof of Theorem 16.8 keeps $\mathscr{K} \otimes \mathscr{K}$ invariant and Theorem 16.6 applies. □

16.4 Privacy as the absence of interference

Which unrecorded event histories compare two insides? The answer is exact in summed form: privacy is the absence of interference between distinct inputs inside one parity sector of the second clock. Whether it also holds history pair by history pair is the one point left open.

Definition 16.22 (Paths and path pairs)**.** Let f be a write and $(c_1, \dots, c_m)$ written values. A *path* from x is a sequence $x = y_0, y_1, \dots, y_m$ with $f(y_i, y_{i-1}) = c_i$; its *end* is y_m and its *balanced count* is the number of balanced occupations among $y_0, \dots, y_{m-1}$. Along it each clock moves by the product of its native factors (2.7), and a branch $|x\rangle \otimes u \otimes w$ is carried to $|y_m\rangle \otimes Au \otimes Bw$. A *path pair* consists of two paths from distinct inputs with the same written values, the same end and balanced counts of the same parity. Its *cross term* under a native record word h and clocks u, w, u', w' is the matrix element of E_h between the two carried branches. The pair is *live* if some word and some clocks with w, w' of one half-turn parity give a nonzero cross term, and *dead* otherwise.

By Lemma 6.9 the second clock of a carried branch has half-turn parity $l(-1)^{\text{balanced count}}$, so a pair of paths whose counts differ in parity is never compared; the definition keeps only the pairs that could be.

Theorem 16.23 (Privacy is the absence of interference within a parity sector)**.** *For every write f the following are equivalent.*

(a) *f is law-private.*
(b) *For every written history, native record word h, distinct inputs $x \neq x'$, $l \in \{1, -1\}$ and clocks with $L^2 w = lw$, $L^2 w' = lw'$, the record effect E of the history followed by h has $\langle |x\rangle \otimes u \otimes w,\ E\,|x'\rangle \otimes u' \otimes w' \rangle = 0$.*
(c) *For the same data and every end z and parity b, the sums of the carried branches over the paths from x, respectively x', with end z and balanced count of parity b have zero matrix element under E_h.*

Proof. (a)⇒(b): put $I = \langle |x\rangle \otimes u \otimes w,\ E\,|x'\rangle \otimes u' \otimes w' \rangle$, and let I^* be the same element with w, w' replaced by $\bar{w}, \bar{w}'$. For the state $\alpha|x\rangle \otimes u \otimes w + \beta|x'\rangle \otimes u' \otimes w'$, law privacy applied to each branch alone and to the sum gives $\mathrm{Re}(\bar{\alpha}\beta(I - I^*)) = 0$ for all α, β, so $I = I^*$ for all clocks of the sector. Since I is antilinear in w while I^* is linear in w, the same identity for iw in place of w reads $-iI = iI^*$; so $I = -I^*$, and hence $I = 0$. (b)⇒(a): by (b) the probability of $g(u, w)$ is the sum of its single-branch terms, so it suffices that every branch $|x\rangle \otimes u \otimes w$ with $L^2 w = lw$ is dark, which we show by induction on the length of the history. At length 0 the effect is native, so it lies in $\mathscr{K} \otimes \mathscr{K}$ (Theorem 16.4), and conjugating the second clock of a product vector does not change its law (Corollary 16.5(a)). After a first posting writing c_1, the branch is the state $g(u', w')$ with $u'_y \otimes w'_y = M_{yx}(u \otimes w)$ for the outputs y with $f(y, x) = c_1$ and 0 for the others. By (2.7) we may take $w'_y = K(y_2, x_1)C_x w$, a real map applied to w, so conjugating w conjugates every w'_y; and by Lemma 6.9 every w'_y has parity $l\epsilon(x)$. By (b) the probability of $g(u', w')$ under the rest of the history is again the sum of its single-branch terms, and these are dark by the induction hypothesis. (b)⇔(c): the matrix element in (b) is the sum over ends and parities of those in (c), so (c) gives (b). For the converse no cancellation can occur across ends or across parities. A native word whose first nonzero label fires from exactly one occupation reads that

end alone, and words of zero labels reduce to these by completeness, $E_h = \sum_j E_{h\cdot j}$, together with $(\mathrm{stay}^2)^2 = \frac{1}{2}\mathrm{stay}^2$ on each clock. The parity is itself read: at every end, a combination of native words of at most two labels with rational weights has effect equal to the identity on the first clock times the half-turn L^2 on the second, a direct exact computation. Inserting it splits the matrix element of (b) by parity. □

Corollary 16.24 (Path pairs)**.** *Let f be a write.*

(a) *If every path pair of f is dead, f is law-private.*
(b) *If f fixes the path, f is law-private if and only if every path pair is dead.*
(c) *If f is law-private, it separates up to the awake bit; it has no path pair of length at most two; every path pair of length at most three is dead; and at the least length at which f has a path pair, every path pair is dead.*
(d) *If a law-private f has a path pair that is live under a word h and clocks, then the same written values, inputs, end and parity carry a second, different path pair that is live under the same h and clocks.*

Proof. (a) Each sum in Theorem 16.23(c) expands into cross terms of path pairs. (b) A path-fixing write has at most one path from each input, so each sum has one term. (c) Awake separation is Theorem 16.16. For an awake-separating write, a direct exact computation over all classes of length at most two exhibits, for each path pair, one native label that isolates it in one parity sector and reads it; so a law-private write has no path pair of length at most two. If two distinct paths from one input share written values, end and parity, they split at some occupation into two distinct outputs, and from there they form a shorter path pair. Hence, at every length up to the least length at which f has path pairs, each class of Theorem 16.23(c) holds one path from each input, and the vanishing sum is the cross term of a single pair, which is therefore dead. As there are no pairs of length at most two, this covers every pair of length at most three. (d) A class whose sum vanishes while one of its pairs is live must contain another live pair. □

Remark 16.25 (Open problem: the path criterion). Is every law-private instrument free of live path pairs? Equivalently, for instruments: is law privacy exactly the absence of a pair of unrecorded paths from distinct inputs, with equal written values, equal end and equal parity mark, whose cross term is nonzero? By Corollary 16.24 a counterexample must be a record blind to the sum of two distinct live path pairs of one class, of length at least four, above the least length at which the write has path pairs; at the shortest live length the paths of each input inside the class are mutually orthogonal, so the routes of one input cannot cancel among themselves.

Dead pairs are real, and they come from the native clocks, not from cancellation.

Proposition 16.26 (Dead pairs and a private write with path pairs)**.**

(a) *The first clock carries nothing along $(0,0) \to (0,1) \to (0,0) \to (0,1) \to (0,0)$: the product of its four factors is 0. Every path pair containing this path is dead, whatever the write.*
(b) *The instrument d of Table A.12 has the path pair $(1,0) \to (1,0) \to (0,1) \to (0,0)$ and $(1,1) \to (0,1) \to (1,1) \to (0,0)$, written $(0,4,0)$, with balanced counts 0 and 2. Both paths carry nonzero clock maps, and the cross term vanishes for every native word and all clocks.*

(c) *The instrument π of Table A.12 is law-private and separates up to the awake bit, and it has the path pair $(0,0)\to(1,0)\to(0,0)\to(1,0)$ and $(0,1)\to(0,1)\to(1,0)\to(1,0)$, written $(5,8,5)$, both of whose paths carry nonzero maps on both clocks. Every path pair of π, of every length, is dead: in each, one path annihilates the half-turn sector of the first clock that the comparison needs.*

So privacy is not the combinatorial absence of path pairs.

Proof. (a) At the step $(0,0)\to(0,1)$ the first clock receives $K(0,0)=$ stay, and at the step back $K(0,1)D=-\text{flip}\,D$. By Table 2.2, $-\text{flip}\,D$ stay annihilates p and a and sends P, A to $-\frac{1}{2}p, -\frac{1}{2}a$: it maps everything into $\mathcal{U}$ and annihilates $\mathcal{U}$. Its square, the product along the path, is zero. (b) A direct exact computation with the factors of the two paths and the twelve readable clock reads. (c) The instrument property, awake separation and the pair are read off the table. For deadness at every length we follow two paths together, recording their current occupations, whether their balanced counts differ in parity, and the image of a half-turn sector of the first clock along each. Up to scale each image is one of nine subspaces (zero, the two sectors and six lines), and each per-cell factor maps each of the nine to one of them, a direct exact computation. Discarding states with a zero image, the states reachable under π form a closed list of 106, and none of them has both paths at one occupation with equal parity. So whenever two paths of π meet at one end with equal parity, one of them has annihilated the sector; first clocks of opposite sectors are never compared, and every path pair is dead. □

Writes of a function of the output and a function of the input are decided completely, and quickly.

Proposition 16.27 (Output and input writes)**.** *Let $f(y,x)=(k(y),g(x))$. Then f is law-private if and only if g separates up to the awake bit and k separates every two occupations that g merges ($g(p)=g(q)$ and $p\neq q$ imply $k(p)\neq k(q)$). Such an f is private exactly when it has no path pair at all, and for every such f law privacy is decided by histories of at most two postings followed by one-label native words. For example $f(y,x)=(y_2,x_1)$ is a law-private instrument that neither records the output nor separates.*

Proof. Let the conditions hold, and take two paths from $x\neq x'$ with the same written values $(k(y_i),g(y_{i-1}))$ and the same end. Walking back from the end, if the occupations before some posting differ, g merges them and the value written one posting earlier shows k agreeing on them, which the second condition forbids; so the paths coincide after the first posting, and x, x' are merged by g. By the first condition they differ in the awake bit, so the balanced counts differ in parity: there are no path pairs, and Corollary 16.24(a) applies. If a condition fails, a direct exact computation over the possible failures gives a history of at most two postings and one native label that read the comparison. For the example, $g(x)=x_1$ merges p,q only with $p_1=q_1$, $p\neq q$, so $p_2\neq q_2$ and $k=y_2$ tells them apart; it is an instrument as in Proposition 16.18. □

Theorem 16.28 (How deep a test of privacy must look)**.**

(a) *No uniform depth below five decides privacy. The instrument d_3 of Table A.12 is dark for every history of at most two postings followed by native words of any length, and read after three postings by one label; the instrument d_5 is dark through four postings and read after*

five by one label. In d_5 every path pair of length at most four contains a path that carries nothing.

(b) *For every write f, f is law-private if and only if it is dark for every written history of at most* 1956 *postings followed by native words of at most three labels.*

Proof. (a) Direct exact computation for the two writes. (b) Let $\mathscr{C}_m$ be the span of the record effects of histories of at most m postings followed by native words. Then $\mathscr{C}_0$ is the 92-dimensional span of Theorem 9.4, spanned by words of at most three labels, and every element of $\mathscr{C}_m$ is a combination of effects of at most m postings followed by words of at most three labels. The chain $\mathscr{C}_0 \subseteq \mathscr{C}_1 \subseteq \cdots$ is fixed from the first step at which it does not grow, since $\mathscr{C}_{m+1}$ is $\mathscr{C}_0$ plus the pullbacks of $\mathscr{C}_m$. Every effect commutes with the joint half-turn $L^2 \otimes L^2$ of the two clocks: by Lemma 6.9 each block satisfies $(L^2 \otimes L^2) M_{yx} (L^2 \otimes L^2) = \epsilon(x)^2 M_{yx} = M_{yx}$, and so does each $K_f(c)$. The operators commuting with it form a space of dimension 2048, so the chain is fixed by $m = 2048 - 92 = 1956$. Privacy is the vanishing of linear functionals on the chain, which (b) tests. □

Two questions about depth are open. Is there a depth N near five such that, for every instrument, law privacy is equivalent to darkness for histories of at most N postings followed by native words of at most three labels? And is there any N for which native words of one label suffice, that is, for which law privacy of every instrument is equivalent to darkness for histories of at most N postings followed by one-label native words?

On a network the half-turn parity becomes a superselection rule on histories. An *event history* is a list of native transitions on contacts; along it an occupation-definite product state with clocks of definite half-turn parities stays an eigenvector of each site's half-turn, the parity at u multiplied by -1 at each posting through u from a balanced input.

Proposition 16.29 (Histories with different parity marks are never compared)**.** *On every network, let two event histories be applied to occupation-definite product states whose clocks have definite half-turn parities, the same at a site u. If the numbers of balanced-input postings through u in the two histories differ in parity, the two resulting vectors have zero matrix element under every record effect of every list of postings.*

Proof. The two vectors are eigenvectors of H_u with opposite eigenvalues, and every record effect commutes with H_u (Corollary 16.5). □

Each site's clock thus carries a record of how often its outward events were balanced. Unlike the label of an ordinary superselection rule [43] this mark is not conserved; the history writes it, and histories that differ in it at any one site are never compared.

16.5 Reading a neighbour through a shared site

Through one shared site, a participant's record reads every direction of its neighbour's private site that any record can read, in four rounds. The shared site screens nothing a record could see.

On the path a round posts the old contact $(0,1)$ and then the neighbour $(0,2)$. Write $\Phi_{\mathrm{old}}(X) = \sum_j V_j^{\mathrm{old}*} X V_j^{\mathrm{old}}$ and Φ_{new} likewise, and $W_{jj'} = V_{j'}^{\mathrm{new}} V_j^{\mathrm{old}}$. Three records of k rounds have the effects

$$E^{\mathrm{old}}_{j\cdot w} = V_j^{\mathrm{old}*}\,\Phi_{\mathrm{new}}\big(E^{\mathrm{old}}_w\big)V_j^{\mathrm{old}}, \qquad E^{\mathrm{new}}_{j'\cdot w} = \Phi_{\mathrm{old}}\big(V_{j'}^{\mathrm{new}*}E^{\mathrm{new}}_w V_{j'}^{\mathrm{new}}\big), \qquad E_{(j,j')\cdot w} = W^*_{jj'}E_w W_{jj'},$$

all equal to I for the empty word: the old participant's record with the neighbour unread, the neighbour's own record with the old contact unread, and the joint record.

Definition 16.30 (Site reads)**.** Let F be a family of operators on $\mathscr{H}_n$ and a a site. For basis states r, r' of the other sites, the *block* $E^{(a)}_{rr'}$ is the operator on site a with matrix elements $\langle s, E^{(a)}_{rr'} s'\rangle = \langle r\otimes s, E\,r'\otimes s'\rangle$. The *read* of F at a, $\mathrm{read}_a(F)$, is the span of all blocks of all $E\in F$. For two sites $a\neq b$ the two-site blocks and $\mathrm{read}_{ab}(F)$ are defined in the same way. For a family indexed by words, read^k_a uses the words of at most k rounds.

Lemma 16.31 (A read is what the laws fix)**.** *Two operators ρ,ρ' on site a satisfy $\mathrm{tr}((\tau\otimes\rho)E) = \mathrm{tr}((\tau\otimes\rho')E)$ for every operator τ on the other sites and every $E\in F$ if and only if $\mathrm{tr}((\rho-\rho')B)=0$ for every $B\in\mathrm{read}_a(F)$. The same holds for two sites.*

Proof. $\mathrm{tr}((\tau\otimes\rho)E) = \sum_{r,r'}\tau_{rr'}\,\mathrm{tr}(\rho E^{(a)}_{r'r})$, and $\tau = |r\rangle\langle r'|$ isolates each term. □

By Theorem 16.4 every read lies in $\mathscr{K}$ (two-site reads in $\mathscr{K}\otimes\mathscr{K}$), so $\mathscr{K}$ is the most any record can read at a site.

Theorem 16.32 (One participant reads its neighbour's readable site)**.**

(a) *The old record, the neighbour's own record and the joint record each read all of $\mathscr{K}$ at site 2.*
(b) *The old record reads $\mathbb{C}I$ at site 2 within one round, $\mathscr{K}_{\mathrm{bit}}$ within two, $\mathscr{K}_{\mathrm{stay}}$ within three and $\mathscr{K}$ within four. The neighbour's own record reads $\mathscr{K}_{\mathrm{stay}}$ within one round and $\mathscr{K}$ within two.*
(c) *On the triangle, with the third contact $(1,2)$ posting after the neighbour and both other contacts unread, the old record reads $\mathscr{K}_{\mathrm{stay}}$ at site 2 within two rounds and $\mathscr{K}$ within three.*
(d) *If the neighbour posts first in each round, the old record reads within k rounds exactly what the old-first record reads within $k+1$: $\mathscr{K}_{\mathrm{bit}}$ within one round, $\mathscr{K}_{\mathrm{stay}}$ within two, $\mathscr{K}$ within three.*

Proof. The inclusions in $\mathscr{K}$ are Theorem 16.4. Three facts give the upper bounds by depth. Each per-cell factor has a Gram that is a stay weight, $\frac{1}{2}(I-c)$ or $\frac{1}{2}(I-c')$, or its complement $\frac{1}{2}(I+c)$ or $\frac{1}{2}(I+c')$; the dressed form occurs exactly when the pair's input is mixed (Lemma 2.5). Summed over the two output bits of its own cell, a factor's Gram is I. So an unread posting, acting on an operator that reads nothing at a site, reads at most that site's bit. For the lower bounds, explicit words of each record evaluated on explicit product states give twelve reads per record, a direct exact computation, and exact rational certificates express a basis of each target space through reads of the stated depth. The witnesses for the old record are words of zero labels: records in which the old participant's own pair writes no current at all. (d) The neighbour-first effect is one unread neighbour posting followed by the old-first effect, $\Phi_{\mathrm{new}}(E^{\mathrm{old}}_w)$, and the profile is computed the same way. □

The mechanism is visible step by step. The neighbour's posting moves the shared bit according to site 2's bit and clock, and the old contact's next label reads the shared bit: two rounds read site 2's bit. Whether site 2 keeps or flips its bit at the neighbour's posting is set by its stay weights, and the following posting carries that into site 0: three rounds read the stay weights, which already tell P from A. Alternating dressed and undressed postings then turn the half-turn and the standing-wave reads into bit changes, and four rounds read everything.

Proposition 16.33 (Two readings of the neighbour's site)**.** *Start from the persistent pair π_+ of Table 12.1, bits* 100 *and clocks P, P, P.*

(a) *The old record $(1,1,-1),(1,1,-1)$ of two rounds has probability* 1*; from bits* 101 *with the same clocks it has probability $\frac{1}{4}$.*
(b) *The old record $(1,1,-1)^3$ of three rounds has probability* 1*; from bits* 100 *with clocks P, P, A it has probability $\frac{1}{4}$.*

Proof. Direct exact computation with the clock maps of Table 2.2. □

So the old participant reads its neighbour's bit and its neighbour's dressed pole, neither of which its own pair touches. Site 0 is the only path between the private sites, yet it screens none of the neighbour's readable state; what it hides is only what every record misses (Corollary 16.5).

One qualification. Conditioning the neighbour's posting on writing the loop edge at every round reads part of the neighbour's record, and it does screen.

Proposition 16.34 (Conditioning on the neighbour's current screens the standing waves)**.** *Replace Φ_{new} in the old record by its restriction to the labels with $j_2' \neq 0$. The resulting family reads at site* 2 *exactly* $\mathrm{span}\{|b\rangle\langle b| \otimes G \,:\, G \in \{I, L^2, c'\}\}$*, the bit and the three sectors of the dressed pole charge, from three rounds on: six of the twelve directions, and strictly less than the neighbour's own record reads.*

Proof. A posting with $j_2' \neq 0$ has a mixed input, so site 2's clock is dressed against site 0 at every such posting, and by Lemma 12.1(D) and the clock table it keeps each sector of c'. The read space is then a direct exact computation, as in Theorem 16.32. □

Whether the old record alone leaves a dark subspace on the three sites is open: is there a projection Q of rank at least two with $QE_w^{\mathrm{old}}Q \in \mathbb{C}Q$ for every word w? There is none if every one of the 512 pole-line projections of the three sites lies in the span of the old effects; that sufficient condition is open as well. Section 16.8 shows that the old participant's present does not purify, whichever way this goes.

16.6 Along a chain the order decides what stays private

Two shared sites away, one direction of a private site can stay hidden from a participant for ever. Whether it does is set by the posting order, and the participant in between reads it.

The *chain* has the sites $1, 0, 2, 3$ in a line and three contacts: the old contact $(0,1)$, the neighbour $(0,2)$ and the far contact $(2,3)$, with site 2 in the first position. The old participant writes its

labels; the neighbour's and the far participant's postings are unread. In the *away* order a round posts old, neighbour, far; in the *toward* order far, neighbour, old:

$$E^{\mathrm{aw}}_{j\cdot w} = V_j^{\mathrm{old}*}\,\Phi_{\mathrm{nbr}}\big(\Phi_{\mathrm{far}}(E^{\mathrm{aw}}_{w})\big)V_j^{\mathrm{old}}, \qquad E^{\mathrm{to}}_{j\cdot w} = \Phi_{\mathrm{far}}\big(\Phi_{\mathrm{nbr}}(V_j^{\mathrm{old}*}E^{\mathrm{to}}_{w}V_j^{\mathrm{old}})\big).$$

On site 3 let

$$\rho_+ = \tfrac{1}{2}\big(|0\rangle\langle 0|\otimes\Omega_+\Omega_+^{\top} + |1\rangle\langle 1|\otimes\Omega_-\Omega_-^{\top}\big), \qquad \rho_- = \tfrac{1}{2}\big(|0\rangle\langle 0|\otimes\Omega_-\Omega_-^{\top} + |1\rangle\langle 1|\otimes\Omega_+\Omega_+^{\top}\big),$$

two states with the same bit and the same clock weights, $\rho_+ - \rho_- = \frac{1}{2}\zeta$.

Theorem 16.35 (The away order screens one direction)**.**

(a) *In the away order the old record reads exactly $\mathscr{K}^{\flat}$ at site* 3*: every readable direction but ζ. The states ρ_+ and ρ_- of site* 3 *have one law under every word, whatever operator the other three sites carry. The read grows as $\mathbb{C}I$ within two rounds, $\mathscr{K}_{\mathrm{bit}}$ within three, $\mathscr{K}_5$ within four, $\mathscr{K}_{\mathrm{ht}}$ within five and $\mathscr{K}^{\flat}$ from six on; an effect of at most two rounds is the identity at site* 3*.*

(b) *In the toward order the old record reads $\mathscr{K}_{\mathrm{bit}}$ at site* 3 *within one round, $\mathscr{K}_{\mathrm{stay}}$ within two, $\mathscr{K}_{\mathrm{ht}}$ within three and all of $\mathscr{K}$ within four; ζ is read from four rounds on.*

So the posting order sets what the record reads, not only how fast.

Proof. The profiles are direct exact computations of the kind described for Theorem 16.32. The screen in (a) has a mechanism, which uses two objects. For an operator Y on site 3, the *Y-probe* of an effect E is the operator on sites $0,1,2$ with matrix elements $\mathrm{tr}(YE^{(3)}_{rr'})$ (Definition 16.30); since ζ is trace-orthogonal to $\mathscr{K}^{\flat}$, an effect reads ζ at site 3 exactly when its ζ-probe is nonzero. The map C_2 acts at site 2 of operators on sites $0,1,2$ by

$$C_2\big(|b\rangle\langle b|\otimes G\big) = \sum_{b'\in\{0,1\}} |b'\rangle\langle b'|\otimes K(b,b')^{\top}GK(b,b'),$$

the pullback of site 2's clock factor through a posting against a partner that carries site 2's own bit, output unread. Let $P_{\mathrm{p}} = \mathrm{diag}(1,-1,1,-1)$ be the clock parity, which exchanges p with a and P with A and reverses the signs of c and c', and let Ξ be P_{p} on site 2's clock together with the bit flip of site 3. Conjugating a per-cell factor by P_{p} flips its cell's output bit, flipping its partner's input bit gives minus that, and flipping both changes only the sign; from these relations every away effect is Ξ-invariant. A far posting produces the ζ-read at site 3 from one source only: the ζ-probe of the far posting's pullback of an effect A is $-\frac{1}{2}C_2$ of the $(\sigma_{\mathrm{b}}\otimes c)$-probe of A, a direct exact computation on the per-cell factors. When A is Ξ-invariant, that probe carries at site 2 only content in $\mathrm{span}\{c,c'\}$, the part of $\mathcal{S}$ odd under P_{p}. The neighbour's unread posting just before the far posting sums over its own two outputs for site 2 at one input, and that sum annihilates C_2 of such content, a direct exact computation. In the toward order nothing precedes the first far posting. □

Theorem 16.36 (The guard)**.** *Consider finite lists of postings on the chain: old postings with written labels, in any number and positions, and unread neighbour and far postings. Call a list* guarded *if a neighbour posting precedes the first far posting and, after it, the neighbour and far postings alternate beginning with a neighbour posting; a list with no far posting is guarded.*

(a) *For every guarded list, every block at site* 3 *of the old record's effect has zero ζ-component, so ρ_+ and ρ_- have one law. In every list the ζ-read is produced at the first far posting.*

(b) *Of the six round orders with one posting of each contact, the three in which the neighbour posts before the far contact, (old, neighbour, far), (neighbour, old, far) and (neighbour, far, old), read exactly $\mathscr{K}^\flat$ at site* 3*. The other three, (far, neighbour, old), (old, far, neighbour) and (far, old, neighbour), read all of $\mathscr{K}$; in the last two, ζ is read from five rounds on and not before.*

(c) *The alternation is needed. With rounds (old, neighbour, far, far), ζ is read from four rounds on and not before; with rounds (old, neighbour, neighbour, far), from five and not before.*

(d) *One neighbour posting placed before the toward rounds screens ζ for ever, and the record still reads $\mathscr{K}^\flat$ within four rounds.*

(e) *If the neighbour's labels are read as well (the joint record of the old participant and the neighbour, far unread, away order), ζ is read from four rounds on and all of $\mathscr{K}$ within four rounds.*

Proof. (a) Taking the ζ-probe commutes with old and neighbour postings, which do not touch site 3. So the ζ-probe of a whole list is the pullback, through the postings before its first far posting, of $-\frac{1}{2}C_2$ applied to the $(\sigma_{\mathrm{b}} \otimes c)$-probe of the rest of the list, as in the proof of Theorem 16.35. In a guarded list a neighbour posting precedes the first far posting, and the alternation that follows keeps the rest Ξ-invariant, so the source lies in $\mathrm{span}\{c, c'\}$ at site 2. Old postings keep C_2 of such content of the same form, and the last neighbour posting before the far posting annihilates it, as in Theorem 16.35. (b) to (e) are direct exact computations, and the list in (d) is guarded by definition. □

The guard is sufficient for every list; it is not claimed necessary for every list (a lone far posting reads nothing). The two terms that cancel are different outputs of the neighbour's posting from one input, so they carry different labels, and once those labels are read they fall into different records (e); doubling either intervening posting breaks the protection (c). Does the participant in between read what it hides?

Theorem 16.37 (The guard reads what it hides)**.** *In the away order:*

(a) *The neighbour's own record, the old and far postings unread, reads ζ at site* 3 *from four rounds on and not before. It reads all of $\mathscr{K}$ at site* 3*: $\mathscr{K}_{\mathrm{bit}}$ within two rounds, $\mathscr{K}_{\mathrm{stay}}$ within three, $\mathscr{K}$ within four.*

(b) *Each member's record is the joint record of the old participant and the neighbour summed over the other member's labels. The joint record reads at site* 3*, within every k rounds, exactly what the neighbour's own record reads within k rounds, and the old record reads inside it at every depth. No two operators on site* 3 *with one law under each member's record, for every word and every operator on the other sites, are separated by the joint record.*

(c) *The far participant's own record, the other two unread, is the image of the old record in the toward order under the exchange of sites $0 \leftrightarrow 2$ and $1 \leftrightarrow 3$. It reads all of $\mathscr{K}$ at site* 1*: the bit within one round, $\mathscr{K}_{\mathrm{stay}}$ within two, $\mathscr{K}_{\mathrm{ht}}$ within three, $\mathscr{K}$ within four. If the far participant posts first in each round, its record is the image of the old away record and reads exactly $\mathscr{K}^\flat$ at site* 1*.*

Proof. (b) An unread posting is the sum over its labels, which gives the marginal identities and the inclusions; the equality by depth and (a) are direct exact computations, and the last statement follows from the equality and Lemma 16.31. (c) The factor of each cell depends only

on its own output bit, its partner's input bit and whether the input is mixed, a form symmetric under exchanging the contact's two cells. So the neighbour's unread posting is unchanged by reversing its orientation (the sign $\sigma(x)$ cancels in every term of an unread posting, since only the zero label is written by two transitions, both with $\sigma = -1$), and the exchange of sites carries the old contact to the far contact and the neighbour to itself. □

16.7 Two-site reads: products and crossed products

What does a record read of two sites together? On every network, never more than the products of what it reads at each site; and when the members' two-site reads are such products, a composite adds to them only crossed products of what they read.

Lemma 16.38 (Factor bound)**.** *For every family F on any network and sites $a \neq b$,* $\mathrm{read}_{ab}(F) \subseteq \mathrm{read}_a(F) \otimes \mathrm{read}_b(F)$*, and the same holds depth by depth.*

Proof. Fixing the matrix entry at site b of a two-site block gives a block at site a, so every two-site block lies in $\mathrm{read}_a(F) \otimes M$, with M all operators on site b; likewise it lies in $M \otimes \mathrm{read}_b(F)$. For subspaces, $(S \otimes M) \cap (M \otimes T) = S \otimes T$, as one sees by applying a projection that kills S in the first factor or T in the second. □

A family obeys the *product law* at (a, b) to depth k if $\mathrm{read}^k_{ab} = \mathrm{read}^k_a \otimes \mathrm{read}^k_b$. Two pairs (S, T) and (S', T') of subspaces are *crossed* if neither contains the other componentwise ($S \subseteq S'$ and $T \subseteq T'$ fails, and so does $S' \subseteq S$ and $T' \subseteq T$) and $S \neq S'$, $T \neq T'$.

Lemma 16.39 (Crossing lemma)**.** $(S + S') \otimes (T + T') = S \otimes T + S' \otimes T'$ *if and only if* (S, T) *and* (S', T') *are not crossed.*

Proof. If one pair contains the other componentwise, both sides are the larger product; if $S = S'$, both sides are $S \otimes (T + T')$, and likewise if $T = T'$. Let the pairs be crossed. We find $X \in S' \setminus S$ and $Y \in T \setminus T'$, or the same with the two pairs exchanged. If $S' \not\subseteq S$ and $T \not\subseteq T'$, take any such X, Y. If $S' \subseteq S$, then $S \neq S'$ gives $X \in S \setminus S'$, and $T' \not\subseteq T$, since otherwise the second pair would lie in the first, gives $Y \in T' \setminus T$. If $T \subseteq T'$, then $T \neq T'$ gives $Y \in T' \setminus T$, and $S \not\subseteq S'$ gives $X \in S \setminus S'$. In the first case choose linear maps f killing S with $f(X) \neq 0$ and g killing T' with $g(Y) \neq 0$, and in the others the same with the pairs exchanged. Then $f \otimes g$ kills $S \otimes T + S' \otimes T'$ but not $X \otimes Y \in (S + S') \otimes (T + T')$. □

Theorem 16.40 (Composition adds crossed products)**.** *Let A, B and J be families on a network, $a \neq b$ sites, with* $\mathrm{read}_a(J) \subseteq \mathrm{read}_a(A) + \mathrm{read}_a(B)$ *and* $\mathrm{read}_b(J) \subseteq \mathrm{read}_b(A) + \mathrm{read}_b(B)$*, and let A and B obey the product law at (a, b). Then*

$$\mathrm{read}_{ab}(J) \subseteq \mathrm{read}_{ab}(A) + \mathrm{read}_{ab}(B) + \mathrm{read}_a(A) \otimes \mathrm{read}_b(B) + \mathrm{read}_a(B) \otimes \mathrm{read}_b(A),$$

and $\mathrm{read}_{ab}(J) \subseteq \mathrm{read}_{ab}(A) + \mathrm{read}_{ab}(B)$ *if the pairs* $(\mathrm{read}_a(A), \mathrm{read}_b(A))$ *and* $(\mathrm{read}_a(B), \mathrm{read}_b(B))$ *are not crossed. If moreover J obeys the product law with single-site reads equal to the two sums, the first inclusion is an equality, and J reads beyond A and B at (a, b) exactly when the two pairs are crossed.*

Proof. By Lemma 16.38, $\mathrm{read}_{ab}(J) \subseteq (S + S') \otimes (T + T')$ with S, T the reads of A and S', T' those of B. Expanding gives the four products, two of which are $\mathrm{read}_{ab}(A)$ and $\mathrm{read}_{ab}(B)$ by the product law. Lemma 16.39 gives the rest. □

The crossed products pair a direction that one member reads at one site with a direction that the other member reads at the other site: correlations of what the members read, of the form two shares of a secret hold [21]. Whether the product law itself holds is a fact about each record; on the chain of Section 16.6 it does.

record	site	$k = 0$	1	2	3	4	5	≥ 6
old	1	1	s	$\mathscr{K}$	$\mathscr{K}$	$\mathscr{K}$	$\mathscr{K}$	$\mathscr{K}$
	3	1	1	1	b	5	h	♭
neighbour	1	1	b	s	$\mathscr{K}$	$\mathscr{K}$	$\mathscr{K}$	$\mathscr{K}$
	3	1	1	b	s	$\mathscr{K}$	$\mathscr{K}$	$\mathscr{K}$
joint	1	1	s	$\mathscr{K}$	$\mathscr{K}$	$\mathscr{K}$	$\mathscr{K}$	$\mathscr{K}$
	3	1	1	b	s	$\mathscr{K}$	$\mathscr{K}$	$\mathscr{K}$
far	1	1	b	s	h	$\mathscr{K}$	$\mathscr{K}$	$\mathscr{K}$
	3	1	s	$\mathscr{K}$	$\mathscr{K}$	$\mathscr{K}$	$\mathscr{K}$	$\mathscr{K}$
ends	1, 3	1	s	$\mathscr{K}$	$\mathscr{K}$	$\mathscr{K}$	$\mathscr{K}$	$\mathscr{K}$

Table 16.1. Single-site reads at the private sites 1 and 3 of the chain in the away order, within k rounds: $1 = \mathbb{C}I$, $\mathrm{b} = \mathscr{K}_{\mathrm{bit}}$, $\mathrm{s} = \mathscr{K}_{\mathrm{stay}}$, $5 = \mathscr{K}_5$, $\mathrm{h} = \mathscr{K}_{\mathrm{ht}}$, $♭ = \mathscr{K}^{♭}$. The records are the old participant's, the neighbour's own, their joint record, the far participant's own, and the record of the two ends (old and far read, neighbour unread); unlisted postings are unread. Each record's two-site read is the product of its two entries.

Theorem 16.41 (The product law on the chain)**.** *In the away order, the five records of Table 16.1 obey the product law at sites* $(1, 3)$ *at every depth, with the single-site reads listed there. In particular the old record's two-site read is* $\mathscr{K} \otimes \mathscr{K}^{♭}$ *from six rounds on, and the neighbour's own record and the far participant's own record read all of* $\mathscr{K} \otimes \mathscr{K}$ *from four rounds on.*

Proof. The upper bounds are Lemma 16.38 with the single-site reads, which come from the computations of Section 16.6 and from collapse bounds of the kind used for Theorem 16.32. The lower bounds are direct exact computations: for each record and depth, explicit words on explicit product states of sites 0 and 2 give two-site reads, and exact rational certificates show that they span the product. Once a profile stops growing it stays fixed, which carries each statement to every larger depth. □

Corollary 16.42 (What the chain's composites add)**.**

(a) Old participant and neighbour. *The joint record reads at* $(1, 3)$ *inside the sum of the two members' reads at every depth except two. At depth two it also reads* $(I_2 \otimes L^2) \otimes (\sigma_{\mathrm{b}} \otimes I)$*, site* 1*'s half-turn read together with site* 3*'s bit. Let* ρ_e *and* ρ_o *be site* 1 *with an unbiased bit and its clock maximally mixed on* $\mathscr{U}$*, respectively on* $D\mathscr{U}$*, and* $\rho_{3,b}$ *site* 3 *at bit* b *with its clock maximally mixed. Then* $\rho_T = \frac{1}{2}(\rho_e \otimes \rho_{3,0} + \rho_o \otimes \rho_{3,1})$ *and* $\rho'_T = \frac{1}{2}(\rho_e \otimes \rho_{3,1} + \rho_o \otimes \rho_{3,0})$ *have equal marginals and one law under each member's record of at most two rounds, whatever sites* 0 *and* 2 *carry, but different joint laws at two rounds; the neighbour's own record separates*

them within three rounds. Over all depths no two states of sites 1 *and* 3 *with equal marginals and one law under each member's record are separated by the joint record.*

(b) The two ends. *The ends' record reads beyond the old and far records exactly at depths one, two and three, where their pairs of reads are crossed; from depth four the far record's pair contains the old record's. At depth one the ends read* $\mathcal{K}_{\mathrm{stay}} \otimes \mathcal{K}_{\mathrm{stay}}$*, the correlation of the two private clocks' stay weights, against* $\mathcal{K}_{\mathrm{stay}} \otimes \mathbb{C}I$ *and* $\mathcal{K}_{\mathrm{bit}} \otimes \mathcal{K}_{\mathrm{stay}}$*.*

(c) Neighbour first. *Under each of the six round orders the neighbour's own record reads all of* $\mathcal{K}$ *at both private sites. The old record reads* $\mathcal{K}$ *at site* 1*, and at site* 3 *reads* $\mathcal{K}^{\flat}$ *in the three orders in which the neighbour posts before the far contact and* $\mathcal{K}$ *in the others; the far record reads* $\mathcal{K}$ *at site* 3*, and at site* 1 *reads* $\mathcal{K}^{\flat}$ *in the three orders in which the neighbour posts before the old contact and* $\mathcal{K}$ *in the others. The ends' pairs are crossed exactly in the orders (neighbour, old, far) and (neighbour, far, old). In these two orders the ends' record reads* $\zeta \otimes \zeta$ *from two rounds on, at every depth, and neither end's record reads it at any depth. With* $\rho_{\pm}$ *of Section 16.6 placed on either site,* $\rho_D = \frac{1}{2}(\rho_+ \otimes \rho_+ + \rho_- \otimes \rho_-)$ *and* $\rho'_D = \frac{1}{2}(\rho_+ \otimes \rho_- + \rho_- \otimes \rho_+)$ *have equal marginals and one law under each end's record for every word and every operator on sites* 0 *and* 2*, and different laws under the ends' record at two rounds. The neighbour's own record separates them.*

(d) Two disjoint contacts. *If the old contact* $(0, 1)$ *and the far contact* $(2, 3)$ *post every round with no neighbour, the members read* $\mathcal{K}_{\mathrm{stay}} \otimes \mathbb{C}I$ *and* $\mathbb{C}I \otimes \mathcal{K}_{\mathrm{stay}}$ *at* $(1, 3)$ *at depth one and* $\mathcal{K} \otimes \mathbb{C}I$ *and* $\mathbb{C}I \otimes \mathcal{K}$ *from depth two, and the joint record reads the product of the two: crossed products at every depth from one on.*

Proof. (b) and (d) follow from Theorem 16.40 and the profiles, the joint records obeying the product law. In (a) the depth-two joint read and the separating word are direct exact computations, $\rho_T - \rho'_T = \frac{1}{32}(I_2 \otimes L^2) \otimes (\sigma_{\mathrm{b}} \otimes I)$, and the last statement follows from the neighbour's full read and Lemma 16.31. In (c) the old and far records are mirror images (Theorem 16.37(c)), Theorem 16.36 gives the old profiles, and the rest is Lemma 16.39 with direct exact computations of the witnesses; $\rho_D - \rho'_D = \frac{1}{2}(\rho_+ - \rho_-) \otimes (\rho_+ - \rho_-) = \frac{1}{8}\zeta \otimes \zeta$. □

The product law can fail, and the failures are informative.

Definition 16.43 (Carried levels)**.** Give each site a level in $\{0, 1, 2, 3\}$, standing for $\mathbb{C}I$, $\mathcal{K}_{\mathrm{bit}}$, $\mathcal{K}_{\mathrm{stay}}$ and $\mathcal{K}$. Let $r(0) = r(1) = 2$ and $r(2) = r(3) = 3$. Pass through a round's postings from the last to the first. A read posting raises both of its sites by r. An unread posting raises a site at a positive level by r, raises a site at level 0 to level 1 if its partner is at a positive level, and leaves it at 0 otherwise. Other sites keep their levels. Starting from level 0 everywhere, k passes give the *carried levels* after k rounds.

Theorem 16.44 (Where the product law holds and where it fails)**.**

(a) *On every network, for every round pattern of read and unread postings, the read of the record of* k *rounds at a site lies in the space of its carried level.*

(b) *A contact's own record, every label read, obeys the product law on its two sites only at depth* 0*. Its two-site read has dimension* 15, 56 *and* 92 *at depths* 1, 2 *and from* 3 *on, against* 36, 144 *and* 144 *for the products of its single-site reads, which are* $\mathcal{K}_{\mathrm{stay}}$ *at depth one and* $\mathcal{K}$ *from depth two. From depth two,* $|0\rangle\langle 0| \otimes Z$ *is a read at the first site and* $|0\rangle\langle 0| \otimes c'$ *at the second, and their product is not a two-site read.*

(c) *Sites that share no contact can fail too. On the path let each round post the old contact unread twice, then the neighbour unread and the neighbour read. At depth one the record reads* $|\beta\rangle\langle\beta| \otimes I$ *at site* 1 *together with* $|\beta\rangle\langle\beta| \otimes Z$ *at site* 2 *for both* β*, but not* $|\beta\rangle\langle\beta| \otimes I$ *together with* $|1-\beta\rangle\langle 1-\beta| \otimes Z$*, although* $|0\rangle\langle 0| \otimes I$ *is a read at site* 1 *and* $|1\rangle\langle 1| \otimes Z$ *a read at site* 2*.*

(d) *The six standard records of the path, in which each participant posts once per round and either both labels or one participant's own labels are read, in either posting order, obey the product law at* $(1, 2)$ *at every depth, and each reads at each private site exactly its carried level (Table 16.2).*

Proof. (a) Induction over the postings from the last, with the pullback of Lemma 16.3. A read posting multiplies each factor of its two sites by Grams of clock maps, which raises the level by at most one step of r. An unread posting sums over its labels; summed over the two output bits of a cell, a factor's Gram is I, so a site whose factor is the identity keeps the identity when its partner's factor is the identity too, and acquires at most its bit otherwise. Both are checks on the finitely many per-cell factors. (b) The record is the record law of one cell placed on two sites, so the dimensions are Theorem 9.4(a); the single-site reads are a direct exact computation, and the missing product is the lockstep of the two clocks' standing waves, along the read $Z_x \otimes X_x$ that the record law misses (Theorem 9.4); at the balanced occupation $(0, 0)$, $X_x = AA^\top - PP^\top = -c'$. (c) An explicit linear functional on two-site operators annihilates every depth-one two-site read of the record and not the unequal-bit product; the single-site reads are found as in (d). Both are direct exact computations. (d) The upper bounds are (a) and Lemma 16.38; the lower bounds are exact rational certificates, one per record and depth. □

record	$k = 0$	1	2	3	4
joint, either order	$(0, 0)$	$(2, 2)$	$(3, 3)$	$(3, 3)$	$(3, 3)$
old's own, old first	$(0, 0)$	$(2, 0)$	$(3, 1)$	$(3, 2)$	$(3, 3)$
neighbour's own, old first	$(0, 0)$	$(1, 2)$	$(2, 3)$	$(3, 3)$	$(3, 3)$
old's own, neighbour first	$(0, 0)$	$(2, 1)$	$(3, 2)$	$(3, 3)$	$(3, 3)$
neighbour's own, neighbour first	$(0, 0)$	$(0, 2)$	$(1, 3)$	$(2, 3)$	$(3, 3)$

Table 16.2. Carried levels (at site 1, at site 2) of the six standard records of the path after k rounds; each record reads exactly these levels and their product.

In (c) the record holds a parity relation between two sites that never touch, carried by site 0: it reads site 2's standing wave exactly on the branch where site 1's bit equals site 2's. The failure of this *relay* pattern is transient: from depth two on it obeys the product law at $(1, 2)$ (Theorem 17.51). A contact's own record, by contrast, holds joint content from its first posting, since its label is a current of both sites at once.

16.8 Presents of members and of pools

The old participant's record law reads its neighbour's whole readable site (Theorem 16.32), yet its present does not become pure: the neighbour's unread postings keep tossing coins. Pooling the two records purifies the whole. Which pools suffice is decided by current conservation and by the order of the postings.

16.8.1 Mean mixedness of a pool

For a positive semidefinite $X \neq 0$ let $\mu(X) = \operatorname{tr} X - \operatorname{tr}(X^2)/\operatorname{tr} X$, and $\mu(0) = 0$. Let a list of postings $e_1, \dots, e_m$ on a network be given (rounds are concatenated lists), and a *pool* $R \subseteq \{1, \dots, m\}$ of read postings; the others are unread. For a word $w = (j_1, \dots, j_m)$ put $A_w = V_{j_m}^{e_m} \cdots V_{j_1}^{e_1}$ and let $w|_R$ be its *read part.* The pool's conditional state for a read part r is $\sigma_r = \sum_{w|_R = r} A_w \sigma A_w^*$, and

$$M_R(\sigma) = \sum_r \mu(\sigma_r), \qquad M(\sigma) = \sum_w \mu(A_w \sigma A_w^*)$$

are the pool's mean mixedness and the full record's. For a density, $M_R(\sigma)$ is the mean over the pool's records, weighted by their probabilities, of $1 - \operatorname{tr} \hat{\sigma}_r^2$, with $\hat{\sigma}_r$ the normalized conditional state. As for one cell in Chapter 10, we read $\hat{\sigma}_r$ at the end of the list (the *cut*) as the present of the whole network as the pool knows it, and we say that a pool *holds the present* when its mean mixedness equals the full record's, or, along longer and longer lists, when it tends to 0. This reading is an identification; the theorems concern M_R, and nothing here identifies a pool, a composite or a network with a single perspective.

Proposition 16.45 (Reading more never adds mixedness)**.** *μ is superadditive on positive semidefinite matrices, $\mu(X + Y) \geq \mu(X) + \mu(Y)$. Hence $M_R(\sigma) \geq M_{R'}(\sigma)$ whenever $R \subseteq R'$, in particular $M_R \geq M$. For a complete family of Kraus operators, $\sum_j \mu(K_j \sigma K_j^*) \leq \mu(\sigma)$.*

Proof. If X or Y is 0 there is nothing to prove. Otherwise, with $a = \operatorname{tr} X$, $b = \operatorname{tr} Y$ and the Hilbert–Schmidt norm, expanding gives

$$\mu(X + Y) - \mu(X) - \mu(Y) = \frac{\|bX - aY\|_2^2}{ab(a + b)} \geq 0, \tag{16.2}$$

with equality exactly when X and Y are proportional (the two-term case of the variance identity of Proposition 10.7). A larger pool splits each conditional state of a smaller one into a sum over the extra read labels. The last inequality is superadditivity applied to the matrices $\sigma^{1/2} K_j^* K_j \sigma^{1/2}$, which are positive semidefinite, sum to σ and have the spectra of $K_j \sigma K_j^*$; μ depends only on the spectrum. □

Theorem 16.46 (Determined labels make a pool exact)**.** *If any two words with the same read part and $A_w \sigma A_w^* \neq 0$ are equal, then $M_R(\sigma) = M(\sigma)$. If this holds for the postings after the first p, for every state, then $M_R(\sigma)$ is the full record's mean mixedness of the remaining postings, evaluated on each conditional state that the pool leaves after the first p postings and summed.*

Proof. Each σ_r has at most one nonzero term, so $\mu(\sigma_r)$ is the full record's term. The second statement applies the first after the prefix. □

16.8.2 A label is a set of differences

Lemma 16.47 (A label fixes the differences of its four bits)**.** *For a transition $x \to y$ with bits (x_1, x_2, y_1, y_2) the label is $j = (y_1 - x_2,\ x_1 - x_2,\ y_2 - x_1)$, three differences that join the four bits in a tree; so the label fixes all six differences of the four bits. Two transitions with one label differ by a common shift of all four bits, and a nonzero shift occurs only for the collision, $(0,0) \to (0,0)$ against $(1,1) \to (1,1)$.*

Proof. The first statement is the definition of the label. Equal labels give equal differences, hence a common shift; a nonzero shift of bits in $\{0, 1\}$ moves all four bits from 0 to 1, which is the collision (Proposition 2.7). □

For a list of postings the *bits* are the pairs (s, t), site s at time $t \in \{0, \dots, m\}$; posting i has the four bits $(u_i, i-1), (v_i, i-1), (u_i, i), (v_i, i)$. The *read graph* of a pool joins the four bits of every read posting, and joins $(s, i-1)$ to (s, i) whenever e_i does not touch s. An unread posting is *joined* if its four bits lie in one connected component, and a pool is *joined* if every unread posting is.

Theorem 16.48 (Joined pools are exact)**.** *On any network, for any list of postings and any pool, let an unread posting be joined. Then there are integers, depending only on the list and the pool, such that on every history of nonzero amplitude its label is the corresponding integer combination of the read labels. If the pool is joined, $M_R(\sigma) = M(\sigma)$ for every σ.*

Proof. On a history of nonzero amplitude each posting writes the native label of its transition and leaves every other site unchanged. Along a read posting every difference of two of its bits is an integer combination of its label's components (Lemma 16.47); along a joining of $(s, i-1)$ to (s, i) the difference is 0. Summing along a path of the read graph, the difference of any two bits of one component is an integer combination of read labels, with coefficients fixed by the path. A joined unread posting's label consists of three such differences. If the pool is joined, the read part determines every label on every history of nonzero amplitude, and Theorem 16.46 applies. □

16.8.3 One participant's present on the path

On the path the old participant's present is the conditional state given its own labels, the neighbour's unread, taken after whole rounds. For this pool M_R is the m_k of Theorem 13.16, and the projection Π onto the joint sure span used below is its S. A *joint line* is a product over the three sites of a bit and one of the eight vectors $p, a, P, A, \omega_\pm, \Omega_\pm$ of Definition 10.8; there are 4096. Its *dressed parity* is the parity of the number of its clocks on $D\mathcal{U}$, that is on P, A or $\Omega_\pm$. Let ρ_0 be the joint line with every bit 0 and every clock p.

Theorem 16.49 (A participant's present with its neighbour unread)**.**

(a) *On every network the product of the half-turns of all clocks commutes with every slice; so every posting preserves the dressed parity of every joint line it does not annihilate. The four joint sure states of Table 12.1 have odd parity.*

(b) *The* 1728 *joint lines with no clock on $\Omega_\pm$ split into an even class of* 896 *lines, an odd class of* 828 *lines, and the four sure lines. A round of positive weight never leaves the even or the odd class, and never carries a line of either class to a sure line.*

(c) *The old participant's mean mixedness of ρ_0 is at least $\frac{3}{8}$ after every number $k \geq 1$ of rounds. Every mixture of joint lines keeps at least $\frac{3}{8}$ times its weight on the two classes from three rounds on.*

(d) *Every density supported on the span of the joint sure states has mean mixedness* 0 *after every $k \geq 1$ rounds, superpositions of different sure states included. A mixture of joint lines with no clock on $\Omega_\pm$ has mean mixedness tending to* 0 *if and only if it lives on the four sure lines.*

(e) *With both participants' labels read, every joint line stays pure at every round.*

Proof. (a) By Lemma 6.9 a slice multiplies the half-turn of each of its two clocks by the same sign $\epsilon(x)$, so the product of the two half-turns commutes with it; the other sites are untouched. The sure states have clocks P, P, P or A, P, P. (b) Each slice carries a joint line to a multiple of a joint line (Lemma 10.9 for each site), never onto $\Omega_{\pm}$; the class sizes and their closure are a direct exact computation on this finite chain of lines. (c) An unread posting of the neighbour splits an output bit evenly unless each of its two clocks sits on a pole line of the plane its pair requires. A split leaves the old participant's present mixed by at least half the weight of the line on which it occurs, because lines at different occupations are orthogonal and the old record reads the shared bit only at its next posting. From every line of either class, the neighbour's posting two rounds later splits with probability at least $\frac{3}{4}$, and from ρ_0 the first two neighbour postings split surely; both are direct exact computations on packed tables of the line chain. (d) The labels of the four sure states differ already in the old contact's label (Table 12.1), so one old label selects one sure state, and by Theorem 12.5(c) each sure state then moves alone. The line statement combines this with (c) and (b). (e) With both labels read, each round carries a joint line to a multiple of one joint line. □

For every density, Theorem 13.16 extends (d): the old participant's present purifies exactly when the density lives on the kept class, which on the path is the joint sure span.

16.8.4 Pooled records

Theorem 16.50 (The pooled record purifies the path)**.** *On the path, with both participants' labels read:*

(a) *Each of the* 512 *projections onto product states of three bits and three clocks on p, a, P, A is a linear combination of joint record effects of eight rounds. Hence no projection Q of rank at least two has $QE_wQ \in \mathbb{C}Q$ for every joint word w.*
(b) *There is a constant C such that the pooled mean mixedness after k rounds is at most $C/(k+1)$ for every density.*
(c) *For ρ_0 the pooled present is pure at every round, while the old participant's own present is mixed by at least $\frac{3}{8}$ for $k \geq 1$ and the neighbour's own present, the old labels unread, by at least $\frac{1}{4}$ for $k \geq 2$.*

Proof. (a) The Born pullback of a joint line through one joint round is a multiple of a joint line. By Theorem 12.5(b), the projection onto the joint sure span is

$$\Pi = \tfrac{1}{3}J_2 - 2J_3 + \tfrac{8}{3}J_4,$$

since the coefficients of H and Q cancel, $\frac{1}{6} - \frac{1}{2} + \frac{1}{3} = 0$ and $\frac{1}{12} - \frac{1}{8} + \frac{1}{24} = 0$; pulling Π back through the labels of one sure state keeps that state's line alone, so each sure line lies in the span of effects of five rounds. One pole line of each dressed parity outside the sure lines is an explicit combination of effects of two rounds, with four and eight words. Pulling back from these seeds, in six levels, reaches all 512 pole lines, a direct exact computation; two seeds are needed because pullbacks keep the dressed parity. The 512 projections sum to I, and the argument of Theorem 10.5(b) excludes a dark subspace. (b) As for one cell (Proposition 10.7

and the rate after it), with the 512 pole-line projections written in eight-round effects. (c) The pooled present of a joint line is pure by Theorem 16.49(e), and the old participant's bound is Theorem 16.49(c). For the neighbour, the old contact's unread posting gives site 1's two output bits equal weight unless site 1's clock sits on a pole line its pair keeps deterministic; the neighbour never touches site 1, and lines at different bits of site 1 are orthogonal, so each such coin leaves the neighbour's present mixed by half its weight. From ρ_0 the coin comes surely in the second and third rounds, and from every line of either class with probability at least $\frac{1}{2}$ three rounds later; with the closure of the classes this gives the bound $\frac{1}{4}$, a direct exact computation. □

On the triangle the pool must be chosen with the posting order in view.

Theorem 16.51 (On the loop the present needs the two later posters)**.** *On the triangle, with rounds posting the old contact* $(0, 1)$*, the new contact* $(0, 2)$ *and the third contact* $(1, 2)$*, site* 1 *first, and the present taken at round boundaries:*

(a) *The pool of all three records has no dark subspace (the* 512 *pole-line projections are combinations of effects of six rounds) and purifies every density at a rate* $C/(k + 1)$*.*
(b) *The pool of the new and third contacts purifies every density at a rate* $C'/(k + 1)$*. The pool without the new contact leaves* ρ_0 *mixed by at least* $\frac{1}{4}$ *for every* $k \geq 2$*, and the pool without the third contact for every* $k \geq 1$*.*
(c) *The old contact's own record, the new contact's own record and the empty pool leave* ρ_0 *mixed by at least* $\frac{1}{4}$ *for every* $k \geq 1$*, the third contact's own record for every* $k \geq 2$*.*

So the pools that purify every density are exactly those that contain the new and the third contact.

Proof. (a) As for Theorem 16.50, with one seed of each dressed parity, the eight sure lines of Table 12.2 from one-round effects, and pullback levels through the triangle round. (b) In round $r \geq 2$ the old contact's label is an integer-linear function of the new and third contacts' labels in rounds $r - 1$ and r; the identity is printed in Table A.13 of the appendix and follows from Lemma 16.47 by telescoping. So from the second round the pool determines every label on every history of nonzero amplitude, the first old label is one channel on the initial state, and Theorem 16.46 with (a) gives the rate. The new contact's unread posting puts a fair coin on site 0, and the third contact, which posts after it in the same round, never touches site 0; so the coin is unread at the round boundary. Likewise the third contact's coin on site 2. From every line of the even class, to which ρ_0 belongs and which the rounds keep, each coin comes with probability at least $\frac{1}{2}$ one round later, a direct exact computation; half of it is mixedness. (c) The same coins. □

Covering the sites is not enough: every pair of contacts on the triangle touches all three sites. What matters is that the read contacts follow the unread one in time before the cut. The general rule is this.

Theorem 16.52 (The cut rule)**.** *On the triangle, take the present at cut* $t \in \{0, 1, 2\}$*, after the first* t *postings of a round (old, new, third). A pool purifies every density, at a uniform rate* $C/(k + 1)$*, exactly when it reads the two contacts other than the one that posts first after the cut; every other pool leaves* ρ_0 *mixed by at least* $\frac{1}{4}$ *after every* $k \geq 2$ *rounds. Each pair of contacts purifies at exactly one cut, and only the pool of all three at every cut. One round late every pair recovers the*

whole record: from the pair's labels of rounds 1 *to* $k+1$*, on every record of nonzero amplitude, the new and third contacts determine the old labels of rounds* 2 *to* $k+1$*, the old and third contacts the new labels of rounds* 2 *to* k*, and the old and new contacts the third labels of rounds* 1 *to* k*. No single participant's record ever recovers the others: for every* $k \geq 1$ *and every delay* d *there are two records of* $k+d$ *rounds and nonzero amplitude that agree in that participant's labels and differ in the others' labels at round* k*.*

Proof. The redundant contact's label is an integer-linear function of the pool's labels by identities like that of Theorem 16.51, printed in Table A.13, and Theorem 16.46 gives the rates. The lower bounds come from the coins of the unread postings at the cut, as in Theorem 16.51. The single-participant records are explicit pairs of line histories, checked by direct computation of their amplitudes. □

This is redundancy in Zurek's sense [47] only in part: one round late the whole's record is held by every covering pair, but at the cut only by the pair that closes every unread posting. The converse of Theorem 16.48 needs clock content, as two facts show.

Proposition 16.53 (What a converse needs)**.**

(a) *On any network, take a history of nonzero amplitude and an unread posting in it with an output site that no later posting touches. If some label from the same input has nonzero amplitude with that site's output bit reversed, there is a second history of nonzero amplitude whose labels agree with the first except at this posting, where they differ.*
(b) *On any carrier, let* $X \geq 0$ *carry a fair coin: a projection* P *with* $PX(I-P) = 0$ *and* $(I-P)X(I-P) = V(PXP)V^*$ *for a unitary* V*. If every later posting commutes with* P *and* V*, every pool of the later postings leaves* X *with mean mixedness at least* $\frac{1}{2}\operatorname{tr} X$*.*
(c) *Whether a posting branches depends on the clocks: from* ρ_0 *the old contact's first posting on the triangle does not split site* 1*.*

Proof. (a) Replace the posting's transition by the branching one. No later posting touches that site, so every later posting keeps its transition and its label, while the label of this posting changes, because a label and an input fix the output. (b) The coin's two values are orthogonal blocks of equal spectrum that no later posting mixes, so each conditional state is mixed by half its weight. (c) From ρ_0 the old pair is balanced with both clocks on p; there stay $p = 0$, so each clock flips surely (p to P) and site 1's output bit is fixed. □

Chapter 17 sharpens (a) without its branching hypothesis: a pool with an unread posting whose output site no later posting touches is never exact (Theorem 17.9). It then asks whether only joined pools hold the whole's present, and what clocks can add when they are not.

Chapter 17

Exact pools, read graphs and joint content

A pool of records is exact when two pasts it cannot tell apart leave the network in proportional conditional states, for every state. Two mechanisms make a pool exact: the read graph, which ties each unrecorded posting's four bits together, and clock death, which annihilates an alternative past. Sections 17.1 to 17.4 find when each acts and reduce the remaining open case, at most two unrecorded postings per site, to an explicit clause system. Sections 17.5 and 17.6 ask, on the path a–c–b, when a record keeps joint statistics of two sites beyond the products of its site reads.

The placement, orientation and posting order of the contacts and the read pattern are supplied throughout. Exactness is the mean-mixedness equality of Chapter 16; reading it as "the pool holds the whole's present" is that chapter's identification, used here without change.

17.1 Pools, exactness and parallel records

17.1.1 One pass of postings

In the first four sections a network is posted once. It is a list $E = (e_1, \dots, e_m)$ of contacts $e_i = (u_i, v_i)$ on the sites $0, \dots, n-1$ of the carrier $\mathscr{H}_n$ of Chapter 12, and posting i, by e_i, takes the network from time $i-1$ to time i. Times run from 0 to m, and the present is taken at the *cut*, time m. Pools $R \subseteq \{1, \dots, m\}$ of read postings, the operators $A_w = V_{j_m}^{e_m} \cdots V_{j_1}^{e_1}$ of words w, their read parts $w|_R$, the mixedness μ and the mean mixedness $M_R(\sigma)$ of a pool and $M(\sigma)$ of the full record are those of Section 16.8.1, and $M_R \geq M$ (Proposition 16.45). A read posting branches on its label and an unread posting applies its channel, so the pool's conditional state for a read part r is $\sum_{w|_R = r} A_w \sigma A_w^*$; the same formulas define these quantities for any finite list of postings.

Definition 17.1 (Exact pool)**.** The pool R is *exact* if $M_R(\sigma) = M(\sigma)$ for every positive semidefinite σ on $\mathscr{H}_n$.

17.1.2 Exactness is a statement about record vectors

Two vectors are *parallel* if they are linearly dependent; the zero vector is parallel to every vector.

Lemma 17.2 (Parallel maps)**.** *Let A, B be linear maps between finite-dimensional complex spaces. Then $A\psi$ and $B\psi$ are parallel for every ψ if and only if $B = cA$ for some c, or $A = cB$ for some c, or A and B both have rank at most one with a common range.*

Proof. The three cases clearly give parallel images. Conversely, suppose rank $A \geq 2$. On the dense set $\Omega = \{\psi : A\psi \neq 0\}$ write $B\psi = \lambda(\psi)A\psi$. If $\psi_1, \psi_2 \in \Omega$ have independent images, then $B(\psi_1 + \psi_2) = \lambda(\psi_1 + \psi_2)(A\psi_1 + A\psi_2)$ forces $\lambda(\psi_1) = \lambda(\psi_2)$. Every $\psi \in \Omega$ has an image independent of $A\psi_1$ or of $A\psi_2$ for a fixed independent pair, so λ is constant on Ω and $B = \lambda A$. The case rank $B \geq 2$ is symmetric. If $A = r\alpha^*$ and $B = s\beta^*$ are nonzero of rank one, a ψ with $\alpha^*\psi \neq 0 \neq \beta^*\psi$ shows $r \parallel s$. □

Theorem 17.3 (Exact pools are parallel pools)**.** *Take any finite list of postings, each a family of operators on a finite-dimensional carrier indexed by a finite label set, with a read flag. The pool is exact if and only if, for every vector ψ and every two words w, w' with the same read part, $A_w\psi$ and $A_{w'}\psi$ are parallel.*

Proof. By the gain identity (16.2), positive semidefinite X, Y with positive traces have $\mu(X + Y) \geq \mu(X) + \mu(Y)$, with equality exactly when X and Y are proportional; and $\mu(cX) = c\mu(X)$ for $c > 0$. If some ψ and some w, w' with one read part have non-parallel images, put $\sigma = \psi\psi^*$: the two rank-one terms $A_w\sigma A_w^*$ and $A_{w'}\sigma A_{w'}^*$ are not proportional, so by (16.2) the class of their read part contributes strictly more to $M_R(\sigma)$ than its words contribute to $M(\sigma)$, and every other class contributes at least as much. Conversely, if all images are parallel, Lemma 17.2 makes any two operators of one read class proportional or rank one with a common range. In both cases $A_w\sigma A_w^*$ and $A_{w'}\sigma A_{w'}^*$ are proportional for every σ, so each class sum is a positive multiple of one of its terms and homogeneity gives $M_R = M$. □

17.1.3 Histories, product states and the label collision

An *occupation history* is $z = (z_0, \dots, z_m)$ with $z_t \in \{0,1\}^n$ and $z_t(s) = z_{t-1}(s)$ whenever e_t does not touch s. Its *word* $w(z)$ writes at posting t the native label $j(y, x)$ of the transition $x = (z_{t-1}(u_t), z_{t-1}(v_t)) \to y = (z_t(u_t), z_t(v_t))$. A label and an input determine the output (Proposition 2.7), so a word has at most one history from each initial occupation.

At each of its postings a site meets one of four *types*: the input pair is *balanced* or *mixed*, and the site *stays* when its output equals its partner's input and *flips* otherwise. We write bs, bf, ms, mf and

$$F_{\mathrm{bs}} = \mathrm{stay}, \qquad F_{\mathrm{bf}} = \mathrm{flip}, \qquad F_{\mathrm{ms}} = \mathrm{stay}\, D, \qquad F_{\mathrm{mf}} = \mathrm{flip}\, D.$$

The *clock product* $R_s(z)$ of a site along a history is the product of the matrices of its types, the first posting rightmost; a site no posting touches has $R_s(z) = I$. The history is *alive at s* if $R_s(z) \neq 0$, and *alive* if it is alive at every site.

Lemma 17.4 (Words on product states)**.** *Let $x \in \{0,1\}^n$ and $\phi = \bigotimes_s \phi_s$. If the word w has a history z with $z_0 = x$, then $A_w(|x\rangle \otimes \phi) = \pm|z_m\rangle \otimes \bigotimes_s R_s(z)\phi_s$; otherwise $A_w(|x\rangle \otimes \phi) = 0$. In particular $A_w \neq 0$ exactly when w has an alive history.*

Proof. By Theorem 2.4 a slice on input x is the block M_{yx}, in which the clock of u receives $\pm K(y_u, x_v)C_x$ and the clock of v receives $\pm K(y_v, x_u)C_x$. Here $K(b', b)$ is $\pm$stay when $b' = b$ and $\pm$flip otherwise, and $C_x = D$ exactly on mixed inputs, so each factor is $\pm F_\tau$ for the type τ of that site at that posting. Composing over the postings gives the formula. □

Whether a site stays or flips can be read off its own bits. On a balanced pair the partner's input equals the site's own, so the site flips exactly when its own bit changes; on a mixed pair the opposite holds. So along any history the type of s at a posting is

$$\big(\text{mixed} = \epsilon,\ \text{flips} = \epsilon \oplus v \oplus v^{+}\big), \tag{17.1}$$

where $\epsilon = 1$ exactly on mixed inputs, v is the site's bit entering the posting and v^{+} its bit entering its next posting, or its final bit. A site's clock along a history therefore depends only on its own bit entering each of its postings, its final bit, and which of its postings have mixed inputs.

The native labels have one coincidence, the *collision* of Chapter 16 (Lemma 16.47): the zero label $0 = (0,0,0)$ is written both by $(0,0) \to (0,0)$ and by $(1,1) \to (1,1)$, and every other label determines its posting's four bits.

Lemma 17.5 (A nonzero label breaks the collision)**.** *A word with a nonzero label has no history from both the empty occupation* $\bar{0}$ *and the filled occupation* $\bar{1}$*.*

Proof. Until the word's first nonzero label both histories stay constant, since the zero label on input $(0,0)$ or $(1,1)$ keeps it. At that posting the label would be $j(y,(0,0)) = (y_1, 0, y_2)$ and also $j(y',(1,1)) = (y_1' - 1, 0, y_2' - 1)$, which forces $y = (0,0)$, $y' = (1,1)$ and the zero label. □

17.1.4 The read graph and exposure

Bits, the read graph and joined pools are those of Section 16.8.2; posting i has the input bits $(u_i, i-1), (v_i, i-1)$ and the output bits $(u_i, i), (v_i, i)$. The connected components of the read graph are the pool's *read components*; a set S of bits is *read-closed* if it is a union of read components.

Definition 17.6 (Input-joined pools)**.** The *input-linked graph* adds to the read graph an edge between the two input bits of every posting, read or not; the pool is *input-joined* if the four bits of every unread posting lie in one of its components.

Joined pools are exact (Theorem 16.48); the question of this chapter is the converse.

Lemma 17.7 (Equal read labels mean a flip)**.** *Let z, z' be histories whose words agree at every read posting, and let S be the set of bits where they differ. Then S is read-closed, z writes the zero label at every read posting whose bits lie in S, and z' is z with the bits of S reversed. Conversely, reversing a read-closed S on whose read postings z writes the zero label gives a history with the same read labels.*

Proof. At a read posting the two words write one label. If it is not the zero label the four bits agree in both histories; if it is, all four bits are equal in each history, so they all agree or all differ. Idle edges carry a bit unchanged in both histories. So membership in S is constant along read edges, and the zero label is written where S meets a read posting. Conversely, reversing all four equal bits of a zero-label posting keeps the zero label, and a posting outside S is unchanged. □

Theorem 17.8 (Exposure)**.** *A pool is not exact if either of the following exists.*

(a) *A* colouring*: an alive history whose read postings all write the zero label and which writes a nonzero label somewhere.*
(b) *A* crossed pair*: two alive histories z, z' with equal read labels and $z_m \neq z'_m$, such that $z_0 = z'_0$, or $w(z)$ has no alive history from z'_0 and $w(z')$ none from z_0.*

We call either an exposing flip.

Proof. (a) Let z be the colouring and w its word. The all-zero word 0 has the same read part. By Lemma 17.5 there is a constant occupation $\bar{c}$ from which w has no history, and $z_0 \neq \bar{c}$. The zero word's history from $\bar{c}$ is constant with balanced stays everywhere, and stay^k has rank at least two for every k, so this history is alive and its clock products have rank at least two. Take $\psi = |z_0\rangle \otimes \phi + |\bar{c}\rangle \otimes \phi'$ with $R_s(z)\phi_s \neq 0$ at every site. By Lemma 17.4, $A_w\psi$ is a nonzero vector on the single occupation z_m, while $A_0\psi$ has the component $\pm|\bar{c}\rangle \otimes \bigotimes_s R_s^0\phi'_s$ plus a component on $z_0 \neq \bar{c}$. If that second component is nonzero, or $z_m \neq \bar{c}$, the two are not parallel; otherwise choose ϕ'_s at one site with $R_s^0\phi'_s$ not parallel to $R_s(z)\phi_s$. Theorem 17.3 finishes. (b) With $\psi = |z_0\rangle \otimes \phi$ when $z_0 = z'_0$, and $\psi = |z_0\rangle \otimes \phi + |z'_0\rangle \otimes \phi'$ otherwise, the two record vectors are nonzero and lie on the different occupations z_m and z'_m. □

The exposure theorem already settles the outer part of the converse.

Theorem 17.9 (Joined, exact, input-joined)**.** *On every network, joined pools are exact and exact pools are input-joined. In particular a pool is not exact when some unread posting has an output site that no later posting touches.*

Proof. Suppose some unread posting i has a bit outside the input-linked component S of its first input bit. The indicator history of S is a history, since S is closed under idle edges. Every posting has both inputs in S or both outside, so every posting is balanced and the history is alive (balanced type sequences never die, Theorem 17.10 below). Every read posting has all four bits inside S or all outside, so it writes the zero label, while posting i does not. This is a colouring. For the last claim, the bits of the untouched output site from time i on form a component of the input-linked graph that misses posting i's inputs. □

So everything between "joined" and "input-joined" is decided by histories with mixed inputs: on an input-joined pool, a history whose read postings write the zero label and whose postings are all balanced writes the zero label everywhere. Mixed inputs are exactly where clocks can die.

17.2 When a member's clock dies

17.2.1 Six lines

The four type matrices act simply on six lines of the clock cell, written in the pole basis (2.8): the even lines $p,\ a,\ p+a,\ p-a$, which span the half-turn even sector $\mathcal{U} = \text{span}\{p, a\}$, and the odd lines $P,\ A$, which span the odd sector $D\mathcal{U} = \text{span}\{P, A\}$. Table 17.1 follows from Table 2.2.

Since p, a, P, A form a basis and each factor sends each line to a multiple of a line, a site's clock product along a type sequence $\tau_1 \cdots \tau_k$ vanishes exactly when the line automaton of

	p	a	$p+a$	$p-a$	P	A
F_{bs}	0	A	A	$-A$	$\frac{1}{2}(p+a)$	$-\frac{1}{2}(p-a)$
F_{bf}	P	0	P	P	$\frac{1}{2}(p-a)$	$\frac{1}{2}(p+a)$
F_{ms}	$\frac{1}{2}(p+a)$	$-\frac{1}{2}(p-a)$	a	p	0	A
F_{mf}	$\frac{1}{2}(p-a)$	$\frac{1}{2}(p+a)$	p	$-a$	P	0

Table 17.1. The four site factors on the six lines, in the pole basis (2.8). Each kills exactly one line: bs kills p, bf kills a, ms kills P, mf kills A.

Table 17.1 kills all four of p, a, P, A. We say the sequence *dies*. Death is closed under extension: a sequence containing a dying stretch dies.

17.2.2 Two runs and their windows

The table has a structure that makes death local. A balanced factor has rank one on $\mathcal{U}$, with image the odd line A (stay) or P (flip), and maps $D\mathcal{U}$ bijectively onto even lines. A mixed factor has rank one on $D\mathcal{U}$, with the same images A or P, and permutes the four even lines. Call a factor's rank-one sector its *gate*. Follow the two sectors separately: the *even run* starts as $\mathcal{U}$ and the *odd run* as $D\mathcal{U}$. At each step exactly one run meets the gate of the factor, because after any step the run that met a gate sits on an odd line and the other on even lines. A run that meets a gate is reset to the factor's image line, whatever it carried before; a run before its first gate carries a whole sector and cannot die.

Place the four even lines on a cycle by positions $p \mapsto 0$, $p+a \mapsto 1$, $a \mapsto 2$, $p-a \mapsto 3$ in $\mathbb{Z}/4$. A mixed stay moves every even line by $+1$ and a mixed flip by -1. Write $[\cdot]$ for the indicator of a statement.

Theorem 17.10 (The window rule)**.** *A type sequence dies exactly when both runs are killed. A run is killed exactly when, after some gate τ_1 it meets, its next gate τ_2 kills it, and this happens in exactly two ways:*

(i) *τ_2 is mixed and comes directly after τ_1, and exactly one of τ_1, τ_2 flips;*
(ii) *after τ_1 come a balanced factor β, then mixed factors $\xi_1, \dots, \xi_r$, then a balanced τ_2, with*

$$1 + 2\,[\tau_1, \beta \text{ flip alike}] + \sum_{q=1}^{r} \rho(\xi_q) \equiv 2\,[\tau_2 \text{ flips}] \pmod 4,$$

where $\rho = +1$ for a stay and -1 for a flip.

In particular a dying sequence has length at least four and contains a mixed factor, and a sequence of balanced factors never dies.

Proof. After τ_1 the run sits on P if τ_1 flips and on A if it stays. If the next factor is mixed it is the run's next gate, and it kills P (ms) or A (mf), which is (i). If it is balanced, β maps the run to $p+a$ (position 1) or $p-a$ (position 3), the former exactly when τ_1 and β flip differently, by Table 17.1. Mixed factors then rotate it, and the next balanced factor is its gate, which kills p (position 0, a stay) or a (position 2, a flip). That is (ii). A run that survives its gate is reset, so

its fate after that depends on nothing earlier. Without a mixed factor the left side of (ii) is odd and never equals the even right side. Each run needs a gate and a killing gate, and the two runs meet gates at different steps, so at least four factors are needed. □

The rule decides death by finite tables. There are exactly sixteen dying sequences of length four, all minimal (Table A.14 in the appendix), and sixteen minimal dying sequences of length five. Minimal dying sequences have unbounded length: for every $k \geq 0$,

$$\mathrm{bs}\,\mathrm{mf}\,\mathrm{bs}\,(\mathrm{bs})^{2k}\,\mathrm{mf}$$

dies while its two stretches without the first or the last factor survive. A single factor off balanced stay, $(\mathrm{bs})^k\,\tau\,(\mathrm{bs})^l$, never dies.

17.2.3 Closed forms

Most histories met below give a site a balanced background with one, two or three mixed factors. For these the window rule collapses to explicit conditions. Write β_0 for the last factor of the balanced block before a mixed factor and β_1 for the one before it, and γ for the first factor after it.

Theorem 17.11 (One and two mixed factors)**.** *Let X, Y, Z be balanced sequences (any mixture of* bs *and* bf*) and ξ, ζ mixed factors.*

1. *$X\xi Z$ dies exactly when X has at least two factors, Z at least one, ξ flips unlike β_0, and γ flips unlike β_1.*
2. *$X\xi\zeta Z$ never dies.*
3. *For Y nonempty, $X\xi Y\zeta Z$ dies exactly when*

$$(e_1 \vee e_4) \wedge (e_2 \vee e_3) \ \text{ if } |Y| \text{ is odd}, \qquad (e_1 \vee e_3) \wedge (e_2 \vee e_4) \ \text{ if } |Y| \text{ is even},$$

where e_1: ξ flips unlike the last factor of X; e_2: the flips of the last two factors of X, of ξ and of the first factor of Y sum to $0 \mod 2$; e_3: ζ flips unlike the last factor of Y; e_4: the flips of the last two factors of Y (of ξ and the single factor of Y when $|Y| = 1$), of ζ and of the first factor of Z sum to $0 \mod 2$. Each event requires the factors it names to exist.

Proof. In each case the run reset by X's last factor meets ξ, and the other run, rotated by ξ, meets Y's first factor; the four events are the four kill windows of Theorem 17.10, and the run that meets ξ at its gate meets ζ in its even sector exactly when $|Y|$ is odd. A leading balanced block enters only through its last two flips, a trailing one through its first factor, and a middle one through its first factor, its last two flips and the parity of its length, so the statement reduces to finite tables of representatives, checked by direct exact computation. For two adjacent mixed factors the run meeting them in its even sector sits at an odd position or is still fresh; two mixed steps move it by an even amount, so the balanced factor after them cannot kill it. □

Three mixed factors on a background of balanced stays obey an explicit logical rule as well (Table A.16 in the appendix). Together these decide every type sequence with at most three factors off balanced stay. For two arbitrary factors x, y among balanced stays, $(\mathrm{bs})^a\,x\,(\mathrm{bs})^b\,y\,(\mathrm{bs})^c$,

death falls into six families, listed in Table A.15; a leading run acts as $\min(a, 2)$, a trailing run as $\min(c, 1)$, and a middle run through its length if $b < 4$ and its parity otherwise.

By (17.1) the same rules can be read in a site's own bits. Write a_1, a_2 for the site's bits entering its postings one and two before a mixed posting, and b_1, b_2 for those one and two after it, b_2 being the final bit when only one posting follows.

Corollary 17.12 (The rules in own bits)**.** *Along a history in which a site meets exactly one mixed posting, its clock dies exactly when at least two of its postings precede the mixed one and at least one follows, $b_1 = a_1$, and $b_2 \neq a_2$. With exactly two mixed postings, the events of Theorem 17.11 become $e_1 : p_1 = a_1$, $e_2 : p_2 \neq a_2$, $e_3 : q_1 = n_1$, $e_4 : q_2 \neq n_2$, where p_i are the bits one and two postings after the first mixed posting, n_i one and two before the second and q_i one and two after it; when exactly one posting separates the two mixed postings, e_4 reads $q_2 = v$ instead, v the bit entering the first mixed posting. The combination rule is unchanged.*

Proof. Substitute (17.1) into Theorem 17.11 and compute flips modulo 2; at each balanced posting the flip is the sum of the bits entering it and the next posting, at each mixed posting that sum plus 1. With one mixed posting, entered with bit v: the last balanced factor before it flips by $a_1 + v$ and the mixed factor by $1 + v + b_1$, so they flip unlike exactly when $b_1 = a_1$; the factor two before flips by $a_2 + a_1$ and the first factor after by $b_1 + b_2$, so given $b_1 = a_1$ these flip unlike exactly when $b_2 \neq a_2$. The length conditions are those of Theorem 17.11(1). With two mixed postings, entered with bits v and w, the same telescoping gives e_1: $(1+v+p_1)+(a_1+v) = 1$, that is $p_1 = a_1$; e_2: $(a_2+a_1)+(a_1+v)+(1+v+p_1)+(p_1+p_2) = a_2+1+p_2 = 0$, that is $p_2 \neq a_2$; e_3: $(1+w+q_1)+(n_1+w) = 1$, that is $q_1 = n_1$; and e_4: $(n_2+n_1)+(n_1+w)+(1+w+q_1)+(q_1+q_2) = n_2+1+q_2 = 0$, that is $q_2 \neq n_2$. When a single posting, entered with bit n_1, separates the mixed postings, e_4 sums the flips of the first mixed posting, that posting, the second mixed posting and the next: $(1+v+n_1)+(n_1+w)+(1+w+q_1)+(q_1+q_2) = v+q_2 = 0$, that is $q_2 = v$. The combination of the four events is the one of Theorem 17.11(3), and Theorem 17.11(2) covers adjacent mixed postings. □

17.3 Three unrecorded crossings: clocks hold the present

17.3.1 The three-crossing network

Take four sites and the eleven postings

$$E_3 = \big((2,3),(0,1),(0,2),(0,1),(2,3),(0,2),(0,1),(2,3),(0,2),(0,1),(2,3)\big),$$

with the pool reading every posting except the three *crossings*, the postings $(0, 2)$ numbered $3, 6, 9$. Each crossing has a read posting of each of its two sides before and after it.

Theorem 17.13 (Clock death makes an unjoined pool exact)**.** *On E_3 the pool that leaves the three crossings unread is input-joined, not joined, and exact. So the read graph does not decide exactness, already for a pool with two read components in which no site meets more than three unread postings.*

Proof. Components. Site 1 is touched only by the read postings $(0, 1)$, and site 0 only by these and by the crossings. Each bit of site 0 is joined through a read posting $(0, 1)$ to a bit of site 1,

and site 1's bits are joined to one another across every posting that does not touch it. So all bits of sites 0 and 1 form one read component X, and likewise all bits of sites 2 and 3 form one component Y. Every crossing has inputs in X and in Y, so the pool is not joined, while the input edge of a crossing links X and Y, so it is input-joined. Only sites 0 and 2 meet unread postings, three each.

Alternatives. By Lemma 17.7, two histories z, z' with equal read labels differ by reversing X, Y or $X \cup Y$, and z is constant on each reversed component. Suppose Y is reversed and z holds the value c on sites 2, 3 throughout. At a crossing, site 2 keeps c, so it meets bs when the partner bit of site 0 equals c and mf when it differs; reversing Y exchanges the two at every crossing. Site 2 meets postings $1, 3, 5, 6, 8, 9, 11$, so its type sequence is

$$\text{bs } \tau_1 \text{ bs } \tau_2 \text{ bs } \tau_3 \text{ bs}, \qquad \tau_q \in \{\text{bs}, \text{mf}\},$$

with complementary choices in z and z'. One of the two has at least two crossings of type mf. If they are consecutive, the sequence contains bs mf bs mf, which dies; if they are the first and third, it contains bs mf bs bs bs mf, which dies (Section 17.2). Reversing X is the same at site 0, whose sequence has the same shape. If both are reversed with z constant at c_X on X and c_Y on Y: for $c_X = c_Y$ both histories are constant and write the zero word, and for $c_X \neq c_Y$ every crossing is mf at site 0 in both histories, which dies.

Exactness. So any two alive histories with equal read labels write the same word. By Lemma 17.4 each read class then contains at most one word with $A_w \neq 0$, every class sum in M_R is a single term, and $M_R = M$. □

The mechanism needs the third crossing. Drop it:

$$E_2 = \big((2,3),(0,1),(0,2),(0,1),(2,3),(0,2),(0,1),(2,3)\big), \quad \text{crossings } 3, 6 \text{ unread.}$$

Proposition 17.14 (Two crossings do not suffice)**.** *The pool of E_2 is input-joined, not joined and not exact. The collision partner of the all-zero history, which fills sites 2, 3 throughout and has the same read labels, has zero amplitude, and a crossed pair exposes the pool.*

Proof. The component argument of Theorem 17.13 applies verbatim: the read components are all bits of sites $0, 1$ and all bits of sites $2, 3$, so the pool is input-joined and not joined. A label fixes the types of both its sites, since $j_2 \neq 0$ exactly on mixed inputs and j_1, j_3 vanish exactly when the first, respectively second, site stays; so all histories of one word have the same type sequences.

The collision partner. Reversing the component of sites 2, 3 in the all-zero history gives the history with sites 2, 3 filled throughout, with the same read labels (Lemma 17.7). Site 2 meets postings $1, 3, 5, 6, 8$: the postings $(2, 3)$ are balanced stays, and at each crossing the input $(0, 1)$ is mixed and site 2 keeps its bit 1 against its partner's 0, a mixed flip. The sequence bs mf bs mf bs contains bs mf bs mf, which dies (Table A.14). So every history of this word dies, and its operator vanishes by Lemma 17.4.

The crossed pair. For $k \in \{0, 1\}$ let z^k have sites 2, 3 constant at k, and sites 0, 1 empty up to time 3 and filled from time 4. Every read posting writes the same label in both: the zero label, except at posting 4, where $(0, 0) \to (1, 1)$ writes $(1, 0, 1)$. At site 2 crossing 3 meets the

partner bit 0 and crossing 6 the partner bit 1, so z^0 gives site 2 the sequence bs bs bs mf bs and z^1 the sequence bs mf bs bs bs; a single factor off balanced stay never kills. Site 0, at postings $2, 3, 4, 6, 7$, meets bs bs bf mf bs in z^0 and bs mf bf bs bs in z^1; by Table A.15 the pair bf, mf kills only one stay apart, and the adjacent pair mf, bf only after at least two stays, so both survive. Sites 1 and 3 meet balanced factors only. So both histories are alive, and their final occupations differ at sites $2, 3$. The word of z^0 writes the zero label at postings $1, 2, 3$, so a history of it from the start of z^1 would meet the mixed input $(0, 1)$ at posting 3, where the zero label is impossible; the word of z^1 writes at posting 3 a label with nonzero middle component, which cannot be written from the input $(0, 0)$ that its zero labels at postings $1, 2$ force from the start of z^0. Theorem 17.8(b) applies. □

The obvious witness against exactness can be annihilated while the pool is still inexact, so a proof of inexactness must search among alternatives; and on this family the clocks begin to decide at the third crossing.

17.3.2 The refined rule

The proof of Theorem 17.13 used only two ingredients: the flips of Lemma 17.7 and the death of type sequences. Together they give a criterion.

Definition 17.15 (Flip rule)**.** A pool satisfies the *flip rule* if for every history z and every read-closed S on whose read postings z writes the zero label, either the reversal z^S of z on S writes the same word as z, or z or z^S dies at some site.

By Lemma 17.7 the flip rule says exactly that alive histories with equal read labels write equal words.

Theorem 17.16 (The refined rule)**.** *1. On every network, a pool satisfying the flip rule is exact.*

2. If every bit is read-linked to a bit at the cut (the pool reaches the cut*; joined pools do), the pool is exact if and only if it satisfies the flip rule.*

3. The same equivalence holds if every bit is read-linked to a bit at time 0 *or at the cut and some site is touched by no posting.*

Proof. (1) is the last step of the proof of Theorem 17.13. For (2), let z, z' be alive with equal read labels and different words. Their difference set is read-closed (Lemma 17.7), and every bit is read-linked to a bit at the cut, so if $z_m = z'_m$ the difference set is empty and the words agree. Hence $z_m \neq z'_m$. If $z_0 = z'_0$ this is a crossed pair. Otherwise, if $w(z)$ has an alive history g from z'_0, then g and z' start alike, have equal read labels and different words, hence different final occupations: a crossed pair; symmetrically for $w(z')$ from z_0; and if neither exists, z, z' themselves are a crossed pair. Theorem 17.8 gives inexactness. For (3), take z, z' as before. Their difference set is read-closed and nonempty, so it holds a bit at time 0 or at the cut; in particular two histories that start alike and write different words end differently. So if $z_m \neq z'_m$ the argument of (2) applies unchanged. If $z_m = z'_m$, then $z_0 \neq z'_0$, and as in (2) an alive history of either word from the other's start gives a crossed pair; suppose there is none. Take $\psi = |z_0\rangle \otimes \phi + |z'_0\rangle \otimes \phi'$ with $R_s(z)\phi_s \neq 0 \neq R_s(z')\phi'_s$ at every site and with ϕ_u, ϕ'_u independent at the untouched site u, whose clock product is I in both histories. By Lemma 17.4 the two

record vectors are product vectors on z_m with independent factors at u, so they are not parallel, and Theorem 17.3 gives inexactness. □

The read graph alone decides when the clocks have no room to act.

Theorem 17.17 (One split posting per site)**.** *Call an unread posting* split *if its four bits do not lie in one read component. If no site touches two split postings, the pool is exact if and only if it is joined. In particular this holds when no site touches two unread postings. More generally, a read-closed set that splits some unread posting, and splits at most one unread posting touching each site, exposes the pool.*

Proof. The indicator history of such a set S writes the zero label at every read posting, since a read posting has all four bits in S or none. A posting with all four bits equal is a balanced stay at both its sites, so each site meets balanced stays except at most at one split posting, and a single factor off balanced stay never kills. The split posting writes a nonzero label. This is a colouring. Under the hypothesis on split postings, the read component of any bit of a split posting is such a set. □

The three-crossing network sits just past this threshold, and its exactness is fragile.

Proposition 17.18 (How fragile the clock mechanism is)**.**

1. *Adding to E_3 a site that no posting touches keeps the pool exact.*
2. Doubling *a read posting means posting it twice in succession, both copies read. Doubling the four separating read postings* $4, 5, 7, 8$*, or any one of them, makes the pool inexact.*
3. *Leaving any one of the read postings* $4, 5, 7, 8$ *unread makes the pool inexact.*

Proof. (1) An untouched site has clock product I in every history and appears in no label, and it changes no other site's type sequence. So two alive histories with equal read labels on the enlarged network restrict to alive histories with equal read labels on E_3, which write the same word by the proof of Theorem 17.13; the pool satisfies the flip rule and is exact by Theorem 17.16(1). The others are direct exact computations, each exhibiting an exposing flip of Theorem 17.8 whose type sequences are run through the line automaton of Table 17.1. (2) After doubling all four, the colouring with sites $0, 1$ filled and sites $2, 3$ empty writes the zero label at every read posting and mixed labels at the crossings. The three crossings at site 0 now sit an even number of its postings apart: site 0 meets $\mathrm{bs}\,\mathrm{mf}\,(\mathrm{bs})^2\,\mathrm{mf}\,(\mathrm{bs})^2\,\mathrm{mf}\,\mathrm{bs}$, which survives by Table A.16, and site 2 meets the same sequence; so this colouring of the two read components is alive. After doubling one of the four, a crossed pair is alive: one read component is constant, empty in one history and filled in the other, and the other component is pinned by the doubled read posting. In each pair the two starts differ at a site where both words write a nonzero label, so neither word has a history from the other's start: by Lemma 17.7 with every posting read, such a history would differ from the given one on every bit of that site and write the zero label at all its postings. (3) After unreading one, a colouring is alive: the indicator history of a union of read components of the new pool that contains a piece cut off by the new unread posting. Table A.17 lists the eight witnesses of (2) and (3) bit by bit. □

17.3.3 Parallel records

The flip rule can fail to be necessary in only one way. Two alive histories with equal read labels and different words might still give parallel record vectors for every state.

Definition 17.19 (Parallel pair)**.** A *parallel pair* of a pool is a pair of alive histories z, z' with equal read labels, different words and $z_m = z'_m$, such that either $z_0 = z'_0$ and $R_s(z)\psi$, $R_s(z')\psi$ are parallel for every site s and every ψ, or $R_s(z)\psi$, $R_s(z')\psi'$ are parallel for every site s and all ψ, ψ'.

By Lemma 17.2 the first condition says that at each site one clock product is a multiple of the other or both have rank at most one with a common range, and the second that both have rank at most one with a common range.

Theorem 17.20 (Exactness modulo parallel pairs)**.**

1. *An exact pool satisfies the flip rule or contains a parallel pair. Hence "exact if and only if the flip rule holds, on every network" is equivalent to "no exact pool contains a parallel pair".*
2. *A parallel pair's difference set misses the cut, so pools reaching the cut contain none.*
3. *Call the* parallel-pair rule *the statement that any two alive histories with equal read labels write the same word, or end in the same occupation and either start alike with proportional clock products at every site or have clock products of rank at most one with a common range at every site. It implies exactness on every network, and it is equivalent to exactness when the contact graph is connected, when the pool reaches the cut, and on every network if no exact pool contains a parallel pair.*
4. *Parallel pairs exist: on two sites with the contact* $(0, 1)$ *posted twice and nothing read, the histories* $00 \to 01 \to 00$ *and* $00 \to 10 \to 00$ *have equal clock products up to sign at both sites (site* 0 *meets* bs, mf *in one and* bf, ms *in the other, and* $F_{\mathrm{mf}}F_{\mathrm{bs}} = F_{\mathrm{ms}}F_{\mathrm{bf}}$*; site* 1 *the reverse). That pool is not exact.*

Proof. Let z, z' be alive histories with equal read labels and words w, w'. Two product vectors with nonzero factors are parallel only if they lie on one occupation and are parallel factor by factor, and a nonzero clock product has a dense set of inputs with nonzero image. With Lemma 17.4 and Theorem 17.3 this gives two exposures. (E1) If $z_0 = z'_0$, the state $|z_0\rangle \otimes \bigotimes_s \phi_s$ shows that in an exact pool $z_m = z'_m$ and $R_s(z)\phi$, $R_s(z')\phi$ are parallel for every s and ϕ. (E2) If neither word has an alive history from the other's start, the state $|z_0\rangle \otimes \bigotimes_s \phi_s + |z'_0\rangle \otimes \bigotimes_s \phi'_s$, whose cross terms vanish, shows that in an exact pool $z_m = z'_m$ and $R_s(z)\phi$, $R_s(z')\phi'$ are parallel for every s, ϕ and ϕ'.

(1) Let the pool be exact, with $w \neq w'$. If $z_0 = z'_0$, (E1) makes z, z' a parallel pair. If w has an alive history g from z'_0, then g and z' start alike, have equal read labels and different words, and (E1) makes them a parallel pair; likewise if w' has one from z_0. Otherwise (E2) makes z, z' a parallel pair. Since the flip rule says exactly that $w = w'$ always, an exact pool without parallel pairs satisfies it, and Theorem 17.16(1) gives the converse; and where exactness equals the flip rule, an exact pool has no parallel pair, because a parallel pair writes two words.

(2) The difference set of a parallel pair is read-closed (Lemma 17.7) and nonempty, and it misses the cut because $z_m = z'_m$. In a pool reaching the cut each of its bits would be read-linked to a bit at the cut, which would then lie in the set.

(3) Sufficiency. Let $w \neq w'$ have one read part and nonzero operators. By Lemma 17.4, $A_w = \sum \pm |g_m\rangle\langle g_0| \otimes \bigotimes_s R_s(g)$ over the alive histories g of w, and likewise for w', and the rule applies to every pairing of an alive history of w with one of w'. If one word has alive histories from two starts, the second alternative holds for some pairing with a fixed alive history of the other word; then every alive history of either word ends in one occupation f and has, at each site s, a clock product of rank one with one range ℓ_s, so $A_w\psi$ and $A_{w'}\psi$ lie on the line $|f\rangle \otimes \bigotimes_s \ell_s$. Otherwise each word has one alive history, and the rule makes $A_{w'}$ a multiple of A_w or puts both images on such a line. Theorem 17.3 gives exactness.

Necessity. On a pool reaching the cut, exactness gives the flip rule (Theorem 17.16(2)), which is the first clause of the rule; if no exact pool contains a parallel pair, (1) gives the same. On a connected contact graph let the pool be exact and $w \neq w'$. Suppose $z_0 \neq z_0'$. If w had a history g from z_0', then g and z would write one word and differ at time 0; by Lemma 17.7 applied with every posting read, their difference set is closed under the edges of every posting and the idle edges, which on a connected network join all bits, so w would be the zero word, and z' would be a colouring, which an exact pool cannot have (Theorem 17.8). So (E2) applies and, by the remark after Definition 17.19, gives the second alternative. Suppose $z_0 = z_0'$. Then (E1) and Lemma 17.2 make each site proportional or of rank one with a common range, and the remaining case is a tensor argument. If a site s_1 is proportional, $R_{s_1}(z') = cR_{s_1}(z)$, with $R_{s_1}(z)$ of rank at least two, and a site s_2 is not proportional, $R_{s_2}(z) = r\alpha^*$ and $R_{s_2}(z') = r\beta^*$ with α, β independent, put on s_1, s_2 the entangled state $\phi_1 \otimes \phi_2 + \phi_1' \otimes \phi_2'$ with $R_{s_1}(z)\phi_1$, $R_{s_1}(z)\phi_1'$ independent and $(\alpha^*\phi_2, \alpha^*\phi_2')$ not proportional to $(\beta^*\phi_2, \beta^*\phi_2')$. Up to sign and common factors the two record vectors are $\big((\alpha^*\phi_2)R_{s_1}(z)\phi_1 + (\alpha^*\phi_2')R_{s_1}(z)\phi_1'\big) \otimes r$ and $c\big((\beta^*\phi_2)R_{s_1}(z)\phi_1 + (\beta^*\phi_2')R_{s_1}(z)\phi_1'\big) \otimes r$, which are not parallel. So every site is proportional or every site has rank one with a common range, which is the rule.

(4) Site 0 meets bs, mf along $00 \to 01 \to 00$ and bf, ms along $00 \to 10 \to 00$, and site 1 the reverse. With stay $= \frac{1}{2}D(I - L)$, flip $= \frac{1}{2}D(I + L)$ and $D^2 = I$,

$$F_{\mathrm{mf}}F_{\mathrm{bs}} = \tfrac{1}{4}D(I + L)(I - L) = \tfrac{1}{4}D(I - L^2) = F_{\mathrm{ms}}F_{\mathrm{bf}} \neq 0.$$

So both histories are alive with equal clock products up to sign, start and end alike, and write different words; nothing is read. The second posting is unread and no later posting touches its output sites, so the pool is not exact by Theorem 17.9. □

Remark 17.21 (Open: exact pools and the cut). Two open statements remain from this section. *Every exact pool reaches the cut*; by Theorem 17.20 this implies that exactness equals the flip rule on every network. And the weaker *no exact pool contains a parallel pair*, which is equivalent to that equality; such a pair would need a recorded component that ends before the cut. Section 17.4 reduces the first to maximal pools.

17.4 At most two unrecorded postings per site

The three-crossing network has a site that meets three unread postings. Below that threshold the evidence points one way: in exhaustive searches over seven families of contacts (about 43 million unjoined networks of up to twelve postings) and in random searches, every unjoined pool in which no site touches three unread postings had an exposing flip. The statement is

open. Where a proof below is marked as an outline, the omitted steps are finite slot-by-slot case checks.

Remark 17.22 (Open: the two-unread rule). *On every network in which no site touches three distinct unread postings, a pool is exact if and only if it is joined.* The direction from joined to exact holds on every network. Since an exposing flip makes a pool inexact, the rule follows from the stronger statement that *every unjoined pool of such a network has an exposing flip.*

Throughout this section, "two-unread" means that no site touches three distinct unread postings.

17.4.1 Maximal pools with separated events

Reading more cannot spoil exactness: if a pool is exact, so is every pool that reads at least what it reads, and an exposing flip of a finer pool exposes every coarser one. A pool is *maximal unjoined* if it is not joined but reading any one more of its unread postings joins it. Every unjoined pool reads up to a maximal one.

Theorem 17.23 (Structure of maximal pools)**.** *In a maximal unjoined pool every unread posting is split, and every bit of every unread posting is read-linked to some bit of every other unread posting. So all unread postings meet the same read components, at most four of them.*

Proof. If an unread posting i were not split, reading it would add edges inside one component, so the pool reading i as well, which is joined, would have the same components and the pool would be joined. So every unread posting i has a bit β outside the component of its first input bit. Reading another unread posting j joins the pool, so the new read graph has a path from that first bit to β, and from it to any bit of i outside the first bit's old component. Each such path leaves the old graph only through j's four bits, so both of its ends are read-linked to bits of j in the old graph; a bit of i inside the first bit's component is linked to j through the first bit. □

A site *brackets* its unrecorded events if it meets a read posting before each of its unread postings, after each, and between any two.

Theorem 17.24 (Exact pools bracket their unrecorded events)**.** *On every network, a pool has an exposing flip if some site meets two unread postings with no posting of its own between them, or meets an unread posting as its first or as its last posting. Hence in an exact pool every site brackets its unrecorded events.*

Proof. Fill the site s on the stretch between the two postings $p < q$ (before q when q is its first posting, after p when p is its last), and leave every other bit empty. No posting touches s inside that stretch, so every other posting sees four equal bits and writes the zero label, while the unread postings at the ends of the stretch write nonzero labels. At p the site flips on balanced inputs (bf) and at q it meets mixed inputs and stays (ms); these are adjacent in its own sequence and do not kill (Table A.15), and in the one-sided cases only one of them occurs. The partner of p meets a balanced stay and the partner of q a single mixed flip, which never kill. So the history is a colouring. If some site fails to bracket, it has two unread postings with only unread postings of its own between them, or an unread first or last posting, and the same construction applies to a consecutive pair. □

Corollary 17.25 (Reduction to maximal separated pools)**.** *The two-unread rule holds if and only if every maximal unjoined two-unread pool whose sites bracket their unrecorded events is inexact, and the exposure statement of Remark 17.22 reduces to the same pools. Likewise "every exact pool reaches the cut" holds if and only if every* maximal non-reaching *pool (one that misses the cut but reaches it when any one more posting is read) with bracketed events is inexact; in such a pool every unread posting is read-linked to every bit that misses the cut and has a bit that reaches it.*

Proof. The pools named are unjoined and two-unread, or miss the cut, so the two-unread rule, respectively "every exact pool reaches the cut", makes them inexact. Conversely, suppose an unjoined two-unread pool were exact. Read further postings one at a time, each chosen so that the pool stays unjoined, until none is left; this ends in a maximal unjoined pool, still two-unread, since it has fewer unread postings, and still exact, since it reads more. By Theorem 17.24 its sites bracket their unrecorded events, so the hypothesis makes it inexact, a contradiction. For the exposure statement, read the given unjoined pool up to a maximal one in the same way; that pool has an exposing flip, by the hypothesis if it is bracketed and by Theorem 17.24 if not, and an exposing flip of a finer pool exposes the coarser one. The argument for the cut is the same, each posting read being chosen so that some bit still misses the cut. In a maximal non-reaching pool, let j be unread. After j is read every bit is read-linked to the cut, and a new path that leaves the old read graph does so only through the four bits of j; so every bit that missed the cut was already linked to a bit of j, and following the new path from such a bit to the cut, from its last visit to the bits of j, shows that some bit of j was already linked to the cut. □

A first class is decided by Table A.15: if a read-closed set S splits an unread posting, splits at most two unread postings at each site, and at every site with two the indicator history of S avoids the six dying families, the indicator history is a colouring. This class strictly contains the one-split class of Theorem 17.17.

17.4.2 Crossing-assigned pairs

Fix a read-closed set S. A posting is an *S-crossing* if exactly one of its two input bits lies in S; only unread postings can be. Its *inside end* is the site whose input lies in S, its *outside end* the other. In a two-unread network a site meets at most two crossings.

Definition 17.26 (Crossing-assigned pair)**.** A *crossing-assigned pair* for S is a history h that is zero on S such that h and its reversal h^S on S are both alive.

By Lemma 17.7 the two histories have equal read labels. Each crossing is mixed in exactly one of them: if its outside input is 1 it is mixed in h, if 0 in h^S.

Theorem 17.27 (Pairs expose)**.** *Let S be a read component that splits an unread posting and contains a bit at the cut. If S has a crossing-assigned pair vanishing off the bits linked to S through contacts, the pool has an exposing flip. Every maximal unjoined pool has such a component: the read component of an output bit of its last unread posting.*

Proof. Let h, h^S be the pair and C the set of bits linked to S through contacts. The two histories have equal read labels (Lemma 17.7) and end in different occupations, because S holds a bit at

the cut. If they start alike, or neither word has an alive history from the other's start, they form a crossed pair. Otherwise they start differently, so S holds a bit $(s_0, 0)$, and a history g of one word starts from the other's start. Then g and the history of the same word differ at $(s_0, 0)$, and by Lemma 17.7 with every posting read each is constant on C. If g has the word of h, then h is 0 on C, as at $(s_0, 0) \in S$, and 0 off C by hypothesis, so h^S is the indicator history of S; if g has the word of h^S, then h^S is 1 on C and h is the indicator history of $C \setminus S$. Either indicator history is alive, writes the zero label at every read posting, since a read posting has its four bits all in or all outside S and C, and writes a nonzero label at the posting that S splits, whose bits all lie in C. It is a colouring. For the last claim, the last unread posting is split, so the component of either output bit misses one of its bits, and its output bits are read-linked to bits at the cut, since every later posting is read. □

The simplest candidate marks the outside ends. Let h_S be 1 exactly on the input stretch of the outside end of every S-crossing, from that site's previous posting up to the crossing, and 0 elsewhere. In h_S every crossing is mixed and every other posting balanced; in h_S^S every posting is balanced, so h_S^S is always alive.

Theorem 17.28 (Even gaps)**.** *On a two-unread network with bracketed events, for every read-closed S: h_S is alive if and only if no site meets two S-crossings with an odd number of its own postings between them. Hence every maximal unjoined two-unread bracketed pool whose components all have this* even-gap *property has an exposing flip, whatever the number of common components.*

Proof (outline). In h_S a site is balanced except at its crossings, at most two, and the marking fixes its own bits near them: an outside end is filled on its input stretch and empty after it, an inside end is empty throughout. Evaluating the rules of Corollary 17.12 on these bits shows that one crossing never kills and two crossings at an even gap do not, while at an odd gap the run reset just before each crossing is killed at it. □

Two examples mark the boundaries of these classes; their posting lists are in Table A.20. The three-site network E_{W} has even gaps and is exposed by h_S, but no pool reading at least what it reads passes the two-factor table. The *mirror pool* E_{M} has sites 0 and 2 meeting both crossings one recorded posting apart, so it lies outside the even-gap class and outside the table class. An explicit pair exposes it: fill site 2 on the input stretch of the second crossing only, so that the two crossings are mixed in different histories.

The mirror pool shows what remains: at a site with two crossings at an odd gap, the crossings must be shared out between the two histories, and short stretches can then need non-default bits.

Sharing out is exact site by site. For an assignment of each crossing to one of the two histories, let h_S^κ be 1 exactly on the input stretch of the outside end of every crossing assigned to h. Then in h_S^κ exactly the crossings assigned to it are mixed, and in its reversal exactly the others.

Theorem 17.29 (Shared-out marked histories)**.** *On every two-unread network, for every read-closed S and every assignment: a site meeting at most one unread posting survives in both h_S^κ and its reversal; a site meeting two unread postings survives in both exactly when an explicit finite condition holds on the numbers of its postings before, between and after them (the first up to* 3*,*

the second up to 4 *and then by parity, the third up to* 1*), the membership in S of its bits entering each and at the end, and whether each is a crossing and where it is assigned. A site wholly outside S that meets two crossings one recorded posting apart, whose middle posting q is shared with a site t that meets at most one crossing, that crossing being among t's first two postings, survives in both histories under every assignment once both inputs of q are filled, and t survives too.*

Proof. A stretch of a site's timeline whose postings are all read lies in one read component, so it is inside S throughout or outside throughout. Along h_S^{κ} and its reversal a site's bit is therefore constant between consecutive unread postings, except on the input stretch that the marking fills, and exactly the crossings assigned to a history are mixed in it. By (17.1) a site's type sequence in either history is thus a fixed function of the lengths of its blocks of read postings, the membership in S of its bits entering each unread posting and at the end, and the crossing status and assignment of each unread posting. A site with no unread posting meets balanced factors only and survives (Theorem 17.10). In the remaining cases the block lengths enter only through representatives, as in Theorem 17.11 and Table A.15: a leading block through its length up to 3, a middle block through its length up to 4 and then its parity, a trailing block through its length up to 1. Running the line automaton of Table 17.1 on the representatives is a direct exact computation. For one unread posting it shows survival in both histories for every choice of the data; for two it yields the finite condition of the statement.

For the second statement, filling both inputs of the middle posting q keeps q balanced and changes no posting's mixedness. The owner then has its bits outside S, its stretch entering q filled, and its two crossings one posting apart; the automaton on the representatives of its leading and trailing blocks shows survival in both histories for all four assignments, again a direct exact computation. Its partner at q meets at most one crossing, among its first two postings, so in each history it meets at most one mixed factor, with at most one factor before it, and by Theorem 17.11(1) such a sequence never dies. □

An explicit structural rule, with no clock in it, suffices for the finite condition on networks with bracketed events (Table A.18). Under it the mirror pool E_{M} is exposed with only its second crossing assigned to h. The *deep mirror pool* $E_{\mathrm{M}'}$, the mirror pool with two recorded postings before its first crossing, has no assignment at all with every other bit at rest (site 2 kills one history under each of the four), and is exposed once its middle posting is filled as in the theorem.

Remark 17.30 (Open: the pair statements). *Pair exposure:* every maximal unjoined two-unread bracketed pool has a read component that splits an unread posting, holds a bit at the cut, and carries a crossing-assigned pair vanishing off the bits linked to it. *All pairs:* on every two-unread network, every read component that splits an unread posting carries such a pair. All pairs implies pair exposure, which implies the exposure statement, which implies the two-unread rule. Separately: *with three or more common components a colouring always exposes.* Every class proved below is a class on which pair exposure holds; pair exposure restricted to the pools outside any union of these classes is equivalent to pair exposure itself.

17.4.3 The clause system

Crossing-assigned pairs have a normal form in which survival becomes a finite set of local conditions. Fix S read-closed on a two-unread network with bracketed events. Choose

1. an *assignment* κ, giving each crossing c the value $\kappa(c) \in \{0, 1\}$ that S carries in the history where c is mixed (0 for h, 1 for h^S);
2. a *repair bit* y_i for every posting i with no input in S, its common input in both histories;
3. a final value f_s for every site.

These determine a history $h(\kappa, y, f)$, zero on S and supported on bits linked to S, in which every posting with no input in S is balanced and every crossing c is mixed exactly in the history named by $\kappa(c)$. Every read posting is balanced in both histories, so each site is balanced except at the crossings mixed in that history, and Corollary 17.12 decides its survival from its own bits near those crossings.

For a site s and a crossing c of s, its *slots* are its postings two before, one before, one after and two after c, written A, B, C, D; when only one posting follows c, the slot D is the site's final occupation. The slot letters are local labels: slot A is not the pole vector and slot D not the dressing. In the history where c is mixed let v_A, v_B, v_C, v_D be the site's bits entering these slots (v_D the final bit in the last case). A slot is *free* if the site's bit there lies outside S and the posting there is not a crossing; its value is then a repair bit y (or the final value f_s). The two literals of the pair (s, c) are

$$P_{s,c} \;:\; v_C = v_B, \qquad Q_{s,c} \;:\; v_D \neq v_A,$$

a literal naming a missing slot being false.

Definition 17.31 (Clause system)**.** The *clause system* of S under (κ, y, f) consists of the *repair clause* $\neg(P_{s,c} \wedge Q_{s,c})$ for every site s and every crossing c of s that is the only crossing of s mixed in its history, and the *two-contact clause* for every site keeping both its crossings $j < k$ in one history: the two-mixed death rule of Corollary 17.12 is false at its slots.

Theorem 17.32 (The clause reading is exact)**.** *Let the network be two-unread with bracketed events and S read-closed. For every assignment κ, repair bits y and final values f: the clause system holds if and only if every site survives in both $h(\kappa, y, f)$ and its reversal. So S carries a crossing-assigned pair of this form exactly when its clause system is satisfiable, and every such pair satisfies it.*

Proof. A site meeting no crossing mixed in a history is balanced there and survives. A site with one mixed crossing dies exactly by the one-mixed rule of Corollary 17.12, which in slot notation is $P_{s,c} \wedge Q_{s,c}$; with two, by the two-mixed rule. Bracketing keeps the slots one before and one after each crossing on recorded postings, where the rules apply. □

Under a *splitting* assignment, one that puts the two crossings of every site into different histories, every slot that is not free carries the frame value $\kappa(c)$, so each repair clause forbids exactly one combination of its free slots:

$$\neg\big(\hat{v}_C = \hat{v}_B \;\wedge\; \hat{v}_D \neq \hat{v}_A\big), \qquad \hat{v}_o = \begin{cases} y_o \text{ (or } f_s) & o \text{ free}, \\ \kappa(c) & \text{otherwise.} \end{cases}$$

A pair is *robust* if its clause holds for every repair vector and *threatened* otherwise. One configuration deserves a name. A *free head* is a pair (s, c) in which the site's input and output

at c lie in S and its slot A is free; the slot is then the site's other unread posting, with both inputs outside S. Its clause is the unit clause $y_A = \kappa(c)$.

Each literal compares two slot values, and when both are free it compares two repair bits. So the system is not a two-satisfiability problem over single bits in general; on the pool E_{F} below, one site's clause is $\neg(y_7 = y_5 \wedge y_8 \neq y_4)$, whose solution set is not closed under the majority operation and hence is not the solution set of any list of two-literal clauses over single bits [34].

17.4.4 Solving the clause system

Every statement below about a set S assumes, as the clause system does, that S is read-closed on a two-unread network with bracketed events.

Constant repairs. Under a splitting assignment every clause holds when all repair bits equal its own frame value, because then $\hat{v}_A = \hat{v}_D$. Call a pair *critical* if it is a free head, or if the site's input and output at c lie outside S and of its slots A, D, both of which exist, exactly one is free.

Theorem 17.33 (Constant repairs)**.** *Let the assignment split every site.*

1. *A clause with all four slots free holds under both constant repair vectors.*
2. *The clause of (s, c) holds under the constant repair vector $1 - \kappa(c)$ if and only if (s, c) is not critical.*
3. *If every critical pair sits at a crossing with one common value $\kappa(c) = k$, the constant repair vector k satisfies the whole system; with no critical pair both constants do.*

Without the splitting hypothesis: putting every crossing in one history and every repair bit at that history's frame value satisfies the clause system if and only if S has even gaps.

Proof. Under a splitting assignment the system consists of repair clauses only. (1) With all four slots free and one constant, $P_{s,c}$ holds and $Q_{s,c}$ fails. (2) By bracketing, the slot B is a read posting on the side of the site's input at c and C a read posting on the side of its output, and a read slot is free exactly when the site's bit there lies outside S. Put every free slot at $1 - \kappa(c)$, while every fixed slot carries $\kappa(c)$. If exactly one of the input and output lies in S, then $\hat{v}_B \neq \hat{v}_C$ and the clause holds. If both lie outside, $P_{s,c}$ holds, and $Q_{s,c}$ holds exactly when one of A, D is free and the other fixed: that is the second kind of critical pair. If both lie inside, $P_{s,c}$ holds, the slot D on the output side is fixed, and $Q_{s,c}$ holds exactly when A is free: a free head. (3) A critical clause at a crossing with $\kappa(c) = k$ is met by the constant k, its frame value; every other clause is met by its frame value and, by (2), by the opposite constant, hence by both constants.

For the last statement, recall the normal form $h(\kappa, y, f)$ defined before Definition 17.31. With every crossing mixed in h and every repair bit and final value 0, the history $h(\kappa, y, f)$ is 0 on S, 1 exactly on the input stretch of the outside end of every crossing, and 0 elsewhere: it is h_S. With every crossing mixed in h^S and every bit 1, it agrees on the bits linked to S with the complement of h^S_S, and its reversal with the complement of h_S; elsewhere both are constant, so complementing changes no type there either, and on the linked bits complementing every bit changes no type. By Theorem 17.32 the system holds exactly when both histories survive, and by Theorem 17.28 that happens exactly when S has even gaps. □

So only sites meeting two crossings an odd number of postings apart, *odd doubles*, need a second history or a non-constant repair, and constants do not always suffice.

Proposition 17.34 (No constant works)**.** *A site meeting two crossings $j < k$ from outside S, staying outside after k, with exactly one posting between them and two before j, defeats every constant repair vector under every assignment. The pool E_{K} is maximal unjoined, two-unread and bracketed, its components are exposed by a repair plan (below), and no constant repair vector satisfies the clause system of either final component under any assignment.*

Proof. Let x be the posting between j and k, so the slots of j are $A(j), B(j), x, k$ and those of k are $j, x, C(k), D(k)$. The slots $A(j)$, $B(j)$, x, $C(k)$, $D(k)$ are read postings or the final value with the site's bit outside S, so they are free; at a crossing c met from outside, the site's bit entering c lies outside S and equals $1 - \kappa(c)$ in both histories, so c is mixed where S carries $\kappa(c)$ and balanced in the other. Fix a constant k_0. If both crossings are kept in one history, the one-step events of Corollary 17.12 hold, $e_1\colon\ \hat{v}_x = \hat{v}_{B(j)}$, and $e_3\colon\ \hat{v}_{C(k)} = \hat{v}_x$; with one posting between, the gap is odd and $(e_1 \vee e_4) \wedge (e_2 \vee e_3)$ holds, so the two-contact clause fails. If they are split, the clause of (s, j) has P true and reads $\neg(\kappa(j) \neq k_0)$, since its slot D is the crossing k, balanced in j's history and so carrying $\kappa(j)$; likewise the clause of (s, k), whose slot A is j, reads $\neg(k_0 \neq \kappa(k))$. Both force $\kappa(j) = k_0 = \kappa(k)$, against the split.

On E_{K} (Table A.20) the read postings join 0 with 2 and 1 with 3, the read components are all bits of sites $0, 2$ and all bits of sites $1, 3$, and the two unread postings 5 and 8 join 0 and 3. Reading either one joins the pool, so it is maximal unjoined; only sites 0 and 3 meet unread postings, two each; and each of them meets a read posting before, between and after them, so the events are bracketed. The pool is clustered, so Theorem 17.36 exposes it by a repair plan. Both components hold bits at the cut and are final. For the component of sites $0, 2$, site 3 meets both crossings from outside, stays outside, has one posting (6) between them and two $(2, 4)$ before; for the component of sites $1, 3$, site 0 does the same with postings 7 and $1, 3$. The first part applies to each. □

Repair plans and Hall's condition. For a threatened pair under a splitting assignment, its *candidates* are the later free slot of each of its two literals, or the earlier one when the later is fixed. A *repair plan* is a splitting assignment together with a choice of one candidate for every threatened pair, distinct pairs receiving distinct repair bits. Setting each chosen bit from its partner slot in posting order, and every other bit to rest, satisfies every clause.

Theorem 17.35 (Plans, Hall and free heads)**.** *On a two-unread network with bracketed events, for read-closed S:*

1. *every repair plan yields a crossing-assigned pair;*
2. *S has a repair plan if and only if some splitting assignment satisfies Hall's condition [19] for the candidate sets of the threatened pairs; in particular a plan exists if every threatened pair has two candidates and no candidate serves more than two threatened pairs;*
3. *under every splitting assignment, every threatened pair has two different candidates unless it is a free head, which has exactly one.*

Proof. (1) Set the chosen candidates in posting order, each so that its literal fails. A candidate's partner slot is earlier or fixed, so it is already set, and distinct pairs use distinct bits, so no

setting undoes another. Every threatened clause then holds, and every robust one holds for every repair vector; Theorem 17.32 gives the pair. (2) A plan's chosen candidates are pairwise distinct, so Hall's condition holds; conversely Hall's theorem gives an injective choice, which is a plan. If every pair has two candidates and every candidate serves at most two pairs, double counting gives Hall's condition. (3) Under a splitting assignment every fixed slot carries $\kappa(c)$, so a literal $Q_{s,c}$ with both slots fixed is false and its pair is robust; a threatened pair therefore has a free slot in $Q_{s,c}$. If B and C are both fixed, the site's bits on both sides of c lie in S, so D is fixed too and the free slot is A: the pair is a free head and has the single candidate A. Otherwise $P_{s,c}$ has a free slot as well, and the candidates of the two literals are different postings, one among B, C and one among A, D. □

A site may also keep both crossings in one history if repairs make one of the two disjunctions of its two-mixed kill rule false; plans of this targeted kind handle odd cycles in the graph whose vertices are crossings and whose edges are the sites meeting two of them, where no splitting assignment exists. The *far-repair pool* E_{R} is exposed by a split plan, although one of its sites, wholly outside the reversed component, dies under all four assignments of its crossings with every repair bit at rest and has no private repair, its middle recorded posting being shared with a site that meets a crossing. In the *triangle pool* E_{O} three unrecorded contacts form a triangle, so no splitting assignment exists; a targeted plan exposes it, the site whose two crossing inputs lie on different sides of S keeping both crossings in one history.

Clustered composites. Hall's condition holds structurally when the recorded and unrecorded contacts separate into groups. A pool is *clustered* if the sites can be labelled so that every read posting joins two sites with one label and every unread posting two sites with different labels. A read-closed S is *sided* if every site lies outside S throughout or is inside S at each of its crossings.

Theorem 17.36 (Clustered composites)**.** *1. A pool is clustered if and only if no unread posting joins two sites already linked by read postings.*

2. *In a clustered pool every read component lies in one cluster and is sided. On a sided component without free heads, every splitting assignment has two candidates for every threatened pair and no candidate serving more than two pairs, and a splitting assignment exists because the crossings form two matchings.*
3. *Hence every maximal unjoined two-unread clustered pool is inexact, and with bracketed events it is exposed by a repair plan. More generally, on two-unread networks with bracketed events, every pool with a final component that is sided and free of free heads is exposed by a repair plan.*
4. *In particular, every network on four sites whose read postings are all on the contacts* $(0,1)$, $(2,3)$ *and whose unread postings are all on* $(0,2)$, $(1,3)$*, of any length and in any order, is inexact whenever its pool is maximal unjoined and two-unread.*

Proof (outline). (1) The clusters are the classes of the graph of read postings. (2) Sites of other clusters never touch the component, and maximality rules out free heads: the earlier unread posting of a would-be free head is linked by maximality to a bit of the later crossing in the same cluster, so it lies in S. The crossings at inside sites and at outside sites form two matchings, which two-colour by removing an element and its partner and joining the ends

they leave. Two candidates and degree two then follow from the slot geometry. (3) follows from Theorems 17.35, 17.27 and 17.24. □

Every maximal bracketed pool with no sided, free-head-free final component has an unread posting joining two sites already linked by read postings. There the clause system goes further than any plan.

Proposition 17.37 (The clause system beyond plans)**.** *1. Split plans, targeted plans, sided components, the even-gap class and the repaired marked histories all satisfy the clause system, so every class above is contained in the clause-satisfiable one.*

2. *On the pool E_{F} the final component of site* 3 *splits the unread posting* 6*; Hall's condition fails under every assignment, because three threatened pairs have all their candidates on the repair bits of postings* 7 *and* 8*; no targeted plan exists; yet the clause system holds with posting* 6 *mixed in h, posting* 9 *in h^S and every repair bit* 0*.*
3. *On the* double-free-head pool E_{D} *sites* 0 *and* 4 *share the unread posting* 7*, parallel to their read postings* 5, 12, 18*, and each is a free head at a later crossing (site* 4 *at* 9*, site* 0 *at* 14*, both with site* 1*) whose only repair slot is posting* 7*. No plan of either kind exists and no assignment works with every repair bit at rest, and no final component is sided. The clause system holds: every crossing (*9, 14 *and* 16*) goes into h, the two unit clauses are met by $y_7 = 0$, and site* 1*, which then keeps both crossings one read posting apart, is protected by $y_{10} = 1$.*

Proof. (1) Each class produces a crossing-assigned pair of the normal form $h(\kappa, y, f)$: a split or targeted plan fixes κ and sets y and f (Theorem 17.35 and the paragraph after it); the marked histories of the even-gap class are $h(\kappa, y, f)$ with constant data, as in the proof of Theorem 17.33; sided components without free heads carry split plans (Theorem 17.36); and the repaired marked histories of Theorem 17.29 are $h(\kappa, y, f)$ with the repair bits of the filled middle postings at 1 and every other repair bit at 0. By Theorem 17.32 every pair of the normal form satisfies the clause system. (2) and (3) are direct exact computations on the posting lists of Table A.20, over the assignments of the crossings of each final component. On E_{F}, under every assignment, the three threatened pairs (site 0 at both of its crossings, site 1 at the first) have their candidates among the two repair bits of postings 7 and 8, so Hall's condition fails; a targeted plan would need, with the crossings split, a pivot of its own for each of these pairs among the same two bits, or, with site 0 keeping both crossings, pivots at both of its crossings and a third for site 1. The stated solution is checked clause by clause. On E_{D} the two free heads each need a pivot, and the only free slot either can use is posting 7, so two pairs would share one repair bit and no plan of either kind exists; under every assignment one of sites $0, 1, 4$ dies in the marked pair with every other bit at rest, and none of them is privately repairable; and site 1 is inside the final component from time 14 on but outside it at its crossing 9, so that component is not sided. Under the stated solution, with every other repair bit and final value 0, the unit clauses of sites 0 and 4 read $y_7 = \kappa$ with the common $\kappa = 0$, and the two-contact clause of site 1 holds because $y_{10} = 1$ differs from its bit at posting 6, its posting before the first crossing. □

Two free heads can share a repair slot only in the configuration of E_{D}: a head's slot is its other unread posting, so a shared slot is an unread posting outside S joining exactly the two heads,

and since each unit clause reads $y_A = \kappa(c_i)$ on it, the two hold together exactly when their crossings are mixed in one history.

Remark 17.38 (Open: the clause statement). *On every two-unread network with bracketed events, every read-closed set has a satisfiable clause system.* It implies that every maximal unjoined two-unread bracketed pool has a final component with a satisfiable clause system, hence pair exposure, the exposure statement and the two-unread rule. Crossing-assigned pairs outside the normal form $h(\kappa, y, f)$ would need recorded postings with mixed inputs; no pool is known that needs them.

17.4.5 What the clause system needs

Three structural facts constrain any proof of the clause statement. First, some sites force a split.

Theorem 17.39 (Must-split sites)**.** *Let a site s meet crossings $j < k$. Kept in one history, s dies for every repair vector if and only if either s is inside S at the inputs of j and k and the output of k, with an odd number of its postings between j and k; or s is inside at the input of j, outside at the input of k, inside at the output of k, with exactly one posting between and at least two before j. Assignments that split every such site and put the crossings of every two free heads sharing a slot into one history always exist, and every satisfying assignment is of this kind. Splitting exactly the must-split sites is sometimes impossible: on E_{K} a must-split site and a site that is not must-split meet the same two crossings.*

Proof. Kept in one history with frame value κ, the site's bits are fixed by its sides: κ inside S, $1-\kappa$ entering a crossing it meets from outside (that crossing is mixed), and a repair bit at every other posting outside S. Its slots before j lie on the side of j's input, those between j and k on the side of k's input, and those after k on the side of k's output, since the site's postings other than j, k are read. Substituting these values into the two-mixed rule of Corollary 17.12 leaves a logical condition in at most eight repair bits, which depends on the three sides, the parity of the gap, whether the gap is one, and whether two postings precede j; deciding it for every repair vector is a direct exact computation over these finitely many cases. In the first configuration every bit is κ, so e_1 and e_3 hold, and at an odd gap $(e_1 \vee e_4) \wedge (e_2 \vee e_3)$ holds. In the second the bits two and one before j are κ, the bit entering k is $1-\kappa$ and the bits after k are κ, so e_2 holds and e_4, which reads $q_2 = v$ at a gap of one, holds as well. In every other configuration a repair vector protects the site: at an even gap every bit at the frame value, at an odd gap the one or two repair bits of Theorem 17.40.

The existence proof two-colours the crossings under "differ" constraints from must-split sites and "agree" constraints from shared free heads. These constraints use each end of a crossing at most once, never close a cycle, and never join two free-head crossings, so removing the latest crossing and joining its neighbours carries an induction through. Conversely, a satisfying assignment must split every must-split site, by the first part, and must unite two free heads sharing a slot, since their unit clauses $y_A = \kappa(c_1)$ and $y_A = \kappa(c_2)$ hold together only when $\kappa(c_1) = \kappa(c_2)$. On E_{K}, for the component of sites $0, 2$, site 0 lies in S throughout and meets the two crossings one posting apart, so it is must-split, while site 3, outside S, is not; both meet crossings 5 and 8. The component of sites $1, 3$ is the same with the roles of sites 0 and 3 exchanged. □

Second, repairs are local but interfere.

Theorem 17.40 (One repair per odd double, with exceptions)**.** *Let s meet crossings $j < k$ at an odd gap, kept in one history. Suppose s is not must-split and not in the configuration inside at j's input, outside at k's input and output, with one posting x between and two before j. Then setting one repair bit against the frame value, at a recorded neighbour (one before j, one after j, or one after k), with all other bits at the frame value, satisfies its two-contact clause. In the excepted configuration the site survives exactly when y_x and the slot two after k both differ from the frame value: two repair bits are needed. If all crossings are in one history, a repair set that avoids every slot two before or two after a crossing at every site, and that makes, at every odd double, its bits at the slots B and C of one of its crossings differ (at a gap of one: of the first crossing if the site is outside at its input, of the second if fewer than two postings precede the first or the site is inside at the second's input), satisfies the system.*

Proof. We use the slot values of the proof of Theorem 17.39: κ inside S, $1-\kappa$ entering a crossing met from outside, a repair bit elsewhere. With every repair bit at the frame value, the one-step events e_1 and e_3 of Corollary 17.12 hold at an odd gap, and the repair must falsify one of them without making the other disjunction true. If the site is outside at j's input, setting the bit at its posting one before j against the frame falsifies e_1, and e_4 stays false, since its slots carry the frame value or, at a gap of one, compare κ with the bit $1-\kappa$ entering j. If it is inside at j and outside at k's input, the bit one after j falsifies e_1; at a gap of at least three e_4 stays false as before, and at a gap of one, where the same posting is one before k, it falsifies e_3 too, while e_2 is false because the hypotheses leave fewer than two postings before j. If it is inside at both inputs and outside after k, the bit one after k falsifies e_3, and e_2 stays false because the bits two before and two after j are κ (at a gap of one the latter is the bit entering k, inside S). Each claim is checked over the gap patterns (one, odd at least three) and the existence of the slot two before j, as in that proof. In the excepted configuration the bits two and one before j and the bit entering j are κ, the bit entering k is $1-\kappa$, and the posting x supplies both one-step neighbours; so e_2 holds, e_1 reads $y_x = \kappa$, e_4 reads $q_2 = \kappa$, and the rule $(e_1 \vee e_4) \wedge (e_2 \vee e_3)$ is false exactly when $y_x \neq \kappa$ and $q_2 \neq \kappa$.

For the last statement, put every crossing in one history with frame value κ and every bit outside the repair set at κ. Since no repair bit sits at a slot two before or two after a crossing, every such slot carries the frame value. A site with one crossing then has $Q_{s,c}$ false. A site with two crossings at an even gap survives whatever its one-step slots carry: the second disjunction $e_2 \vee e_4$ of the even rule compares two-step slots, all at the frame value, so it is false. At an odd gap of at least three the rule reduces to $e_1 \wedge e_3$, which the repair falsifies; at a gap of one the conditions in the statement are exactly those under which the falsified event decides the rule, as computed in the first part. □

A repair bit set against the frame value can break another site's clause only as one of its two-step slots, and only while that clause's one-step literal holds. On the *interference pool* E_{G} site 1 meets crossings 6 and 8 inside, inside, then outside, one posting apart; kept in one history it survives exactly when y_{10} is set against the frame, and posting 10 is the slot two after crossing 8 at site 2, whose clause is $\neg(y_9 = y_4 \wedge y_{10} \neq y_3)$. While site 1 is kept in one history no single repair works; with every crossing mixed in h, the system holds with $y_{10} = y_3 = 1$ and all other bits 0.

Third, cascades of forced repairs cannot be checked locally.

Theorem 17.41 (Cascades)**.** *Call an assignment a* candidate *if it splits every must-split site and unites every two free heads sharing a slot.*

1. Pairs do not decide. *On the pool E_{C} every assignment is a candidate, and under one of them every two sites' clauses are jointly satisfiable while the whole system is not: three sites form a frustrated cascade (the first forces one repair bit, a second forces its neighbour the same way, a third the other way). The uniform assignment satisfies the same system.*
2. Units. *In each configuration of Table A.19 a site survives exactly when the listed repair bits take the listed values. Two sites forcing one posting to different values fail together.*
3. Four-site cascades. *Two free heads at crossings a_1, a_2, each with a neighbour t_i entering S at a crossing c_i whose slot two before is that head's repair slot p_i, the two neighbours sharing their slot q one before (each neighbour meets no other crossing, its slots two and one before c_i are free and its slots one and two after c_i are postings inside S, and the three slots p_1, p_2, q are distinct), form a cascade: any two of the four sites are jointly satisfiable, and all four exactly when not ($\kappa(a_1) \neq \kappa(c_1)$, $\kappa(a_2) \neq \kappa(c_2)$ and $\kappa(c_1) \neq \kappa(c_2)$). When the neighbours meet their crossings from outside S, some candidate escapes it.*
4. Exchange. *A single repair clause, under every pinning of its slots, has the delta-matroid exchange property [10]; the conjunction of one site's clauses need not.*
5. Freedom. *A site met from inside at its first crossing, and not must-split, can be kept in one history or split, at will, by some candidate.*

Proof. (2) Put the slot values of the proof of Theorem 17.39, with frame value κ, into the two-mixed rule of Corollary 17.12. In the first two rows of Table A.19 every bit up to k is κ, in the next two every bit after j is κ, and at a gap of one e_4 compares κ with the bit $1-\kappa$ entering j; in the fifth e_2 needs the missing slot two before j, and e_4 reads $q_2 = v$ with both bits κ. This fixes the two events named in the table's last column, and the rule reduces to the third, which gives the listed condition. The fifth row is the excepted configuration of Theorem 17.40, and the last is the unit clause of a free head. Two such units on one posting with different values cannot hold together.

(1) is a direct exact computation on E_{C} (Table A.20). By the classification of Theorem 17.39 no site is must-split, and no site is a free head, so every assignment is a candidate. Take the assignment putting crossings 4, 8 and 21 into h and crossings 12 and 16 into h^S. Sites 1 and 3 keep their two crossings in one history, inside, inside, then outside at an odd gap, so by (2) they force $y_{24} = 1$ and $y_{23} = 0$ respectively. Site 5 meets crossing 21 from inside and leaves S there, with its two slots before inside and its slots one and two after at postings 23 and 24; its repair clause is the implication $\neg(y_{23} = 0 \wedge y_{24} \neq 0)$. The three conditions are jointly unsatisfiable and any two are satisfiable, and every other site's clause holds whatever y_{23}, y_{24} are, with all other repair bits 0. Under the uniform assignment the system holds with $y_{23} = y_{24} = 1$ and all other bits 0, checked clause by clause.

(3) A free head x_i at a_i has the unit clause $y_{p_i} = \kappa(a_i)$ on its slot p_i. Its neighbour t_i, with its slots two and one before c_i at p_i and at the shared slot q outside S and its slots after c_i inside, has the clause $\neg\big(y_q = \kappa(c_i) \wedge y_{p_i} \neq \kappa(c_i)\big)$. If $\kappa(a_i) \neq \kappa(c_i)$, the unit forces $y_q \neq \kappa(c_i)$; if moreover $\kappa(c_1) \neq \kappa(c_2)$, the bit y_q must differ from both values, which is impossible. In every other case, and for any two of the four sites, the bits y_{p_1}, y_{p_2}, y_q (distinct by hypothesis) can be chosen.

For the escape, when the neighbours meet their crossings from outside S they are not free heads, and one more constraint, $\kappa(a_1) = \kappa(a_2)$, can be added to the two-colouring in the proof of Theorem 17.39 without using an end twice; under the resulting candidate the first two conditions force $\kappa(c_1) = \kappa(c_2)$, so the cascade does not fail.

(4) The forbidden points of a clause $\neg(x_B = x_C \wedge x_A \neq x_D)$ are at least two flips apart, since changing one coordinate of such a point breaks $x_B = x_C$ or $x_A \neq x_D$. Such a relation, pinned or not, has the exchange property. Let x, y satisfy it and differ at i. If flipping x at i is allowed we are done; otherwise that flip is forbidden, and since y is allowed and agrees with it at i, they differ at some $j \neq i$ with $x_j \neq y_j$ and j unpinned; flipping x at i and j lands one flip away from a forbidden point, hence on an allowed one. Site 2 of E_{C}, split between its crossings 4 and 8, has on its six free slots the constraint $\neg(v_1 = v_2 \wedge v_0 \neq v_3) \wedge \neg(v_3 = v_4 \wedge v_2 \neq v_5)$. From $x = 0$ and $y = (1, 0, 1, 0, 1, 0)$ the flip at 0 has no compensating flip among the coordinates $0, 2, 4$ where they differ: $(1, 0, 0, 0, 0, 0)$ and $(1, 0, 0, 0, 1, 0)$ break the first clause and $(1, 0, 1, 0, 0, 0)$ the second.

(5) Keeping the site's two crossings together, or apart, is one more constraint; since the site is not must-split and is inside at its first crossing, the ends it uses are used by no must-split site and no shared unit, and the two-colouring of Theorem 17.39 extends to it. □

The natural algorithm is to chase forced repairs. It decides part of the problem and not the rest.

Theorem 17.42 (Propagation). *1. The part of a clause system made of unit clauses and implications between repair bits is satisfiable if and only if no chain of implications leads from a bit forced to* 1 *to a bit forced to* 0*, and if and only if propagation meets no conflict; the closure assignment then satisfies it.*

2. Propagation does not decide the whole system. On the pool E_{P} the system is satisfiable exactly when not ($\kappa(4) = \kappa(10)$ and $\kappa(4) \neq \kappa(12)$); under one assignment violating this, propagation meets no conflict, site by site and clause by clause, yet no repair vector works. The core has three sites: one unit, then two parity literals of opposite sense on the same two recorded postings, $y_5 = y_9$ and $y_5 \neq y_9$, so no bit on the parity cycle is ever forced.

3. Two sites can fail together without a unit conflict on one posting: on the pool E_{B} two sites sharing two recorded postings are jointly satisfiable exactly when not ($\kappa(4) = \kappa(8)$, $\kappa(6) = \kappa(10)$ and $\kappa(4) \neq \kappa(6)$), and one of them forces no bit at all.

Proof. (1) The implication clauses of Theorem 17.41, $\neg(y_q = \kappa \wedge y_r \neq \kappa)$, say $y_q = \kappa \Rightarrow y_r = \kappa$, an implication between literals of one sign. If a chain of implications leads from a bit forced to 1 to a bit forced to 0, following it violates a unit, and propagation meets exactly this conflict. Otherwise set to 1 every bit reached by a chain from a bit forced to 1, and every other bit to 0: every unit and every implication holds. This closure assignment is what propagation produces.

(2) and (3) are direct exact computations on the posting lists of Table A.20, over the assignments of the four crossings of each pool, with the slot values and the two-mixed rule as in the proof of Theorem 17.39. On E_{P}, under the assignment putting every crossing except crossing 12 into one history, site 3 is kept, outside, inside, inside at an odd gap, and forces the repair bit of posting 1 by Theorem 17.41(2). Given that unit, the clause of site 2 becomes the parity literal

$y_5 \neq y_9$; site 0, kept, inside, outside, inside with exactly the read postings 5 and 9 between its crossings, survives exactly when $y_5 = y_9$. Neither literal forces a bit, so propagation, member by member and clause by clause, stops at the single forced bit without a conflict, while the two literals contradict each other. On E_{B}, site 2 is inside, outside, outside one posting apart with two postings before, so by Theorem 17.40 it forces both shared postings 5 and 11 against its frame value; site 0, outside, inside, outside one posting apart, has posting 5 one before its first crossing and posting 11 one after its second, and its two-contact clause forbids both of them at its own frame value while forcing neither. Each kept in one history, the two fail together exactly when their frame values differ; when either is split they are jointly satisfiable. This is the stated condition. □

So a proof of the clause statement must choose the assignment against cascades of three or more sites and against parity cycles, not against pairs or forced units alone. Two refinements remain open: that some candidate meets no propagation conflict on every read-closed set (a consequence of the clause statement), and that after propagation the two-bit shadows of the sites decide satisfiability. The routes "pair conditions decide a candidate" and "propagation decides a candidate" are false, by E_{C} and E_{P}.

17.5 Joint content on the path: what a posting writes

17.5.1 Records of a repeated round

We turn from what a pool keeps of the present to what a record fixes of the joint statistics of two sites. The setting is the path of Chapter 12: the shared site $c = 0$ and the sites $a = 1$, $b = 2$, with the contacts (c, a) and (c, b). A *pattern* is one round, a list of postings of the two contacts each marked read or unread, and the round repeats. We write aR and aU for a read and an unread posting of (c, a), and bR, bU for those of (c, b). Its record of depth k is the family of effects of all words of at most k rounds, a read posting pulling back through its label and an unread posting through its channel, and read_z^k and $\mathrm{read}_{zz'}^k$ are its site and pair reads (Definition 16.30). Every site read lies in the readable site space $\mathscr{K}$ (Theorem 16.4). In coordinates, $\mathscr{K}$ is spanned by $|\beta\rangle\langle\beta| \otimes v_q$ with $\beta \in \{0, 1\}$ and

$$v_0 = e_{00} + e_{22}, \quad v_1 = e_{11} + e_{33}, \quad v_2 = e_{02} + e_{20}, \quad v_3 = e_{13} + e_{31},$$
$$v_4 = e_{01} + e_{10} + e_{23} + e_{32}, \quad v_5 = e_{03} + e_{30} + e_{12} + e_{21}.$$

A readable clock block $g = \sum_q g_q v_q$ has the coordinates g_q, which are its entries at $(0,0), (1,1), (0,2), (1,3), (0,1), (0,3)$. The record obeys the product law at (z, z') at depth k if $\mathrm{read}_{zz'}^k = \mathrm{read}_z^k \otimes \mathrm{read}_{z'}^k$ (Section 16.7), and it keeps *lasting joint content beyond products* at (z, z') if the product law fails at arbitrarily large depths. We read such content as a failure of local tomography for the record [20, 5]: joint statistics of two sites that the record fixes and its single-site statistics do not. The reading does not identify a pair of sites with a perspective.

Theorem 17.43 (Records stabilize)**.** *On every network and for every pattern, the span of the record effects of depth k grows with k, and once two successive depths give the same span, every later depth does. This happens at some depth not exceeding the dimension of the readable network operators. From that depth on every read space and pair read space is constant, so the product law holds from some depth on if and only if it holds at that depth.*

Proof. The span at depth $k+1$ is the span of the identity and of the images of the span at depth k under the pullback through one round, over all label choices. So equality at one step propagates. Read spaces are spans of linear block maps applied to the record span. □

17.5.2 The first posting writes the contact's pair

Every record effect is the pullback, through the record's first posting, of a sum of products of readable single-site operators (Chapter 16). That first pullback leaves a fingerprint. Define the *write functionals* on clock blocks

$$\omega_{=}(g) = -g_{00} + g_{11} - g_{02} + g_{13}, \qquad \omega_{\neq}(g) = g_{00} - g_{11} - g_{02} + g_{13},$$

and the *write space*

$$\mathscr{W} = \operatorname{span}\{|x_1\rangle\langle x_1| \otimes g \otimes |x_2\rangle\langle x_2| \otimes g' \;:\; \omega_x(g) = \omega_x(g') = 0\},$$

where ω_x is $\omega_{=}$ for $x_1 = x_2$ and $\omega_{\neq}$ otherwise. Its readable part is $\mathscr{W}_{\mathscr{K}} = \mathscr{W} \cap (\mathscr{K} \otimes \mathscr{K})$.

Theorem 17.44 (First write)**.** *On every network, for every pattern whose first posting is on the contact $e = (u, v)$, read or unread, and every depth,* $\mathrm{read}^k_{uv} \subseteq \mathscr{W}_{\mathscr{K}}$*, and* $\dim \mathscr{W}_{\mathscr{K}} = 100$*. If the first posting is unread,* read^k_{uv} *lies in the span $\mathscr{W}_1$ of the pullbacks of readable products through one unread posting, with* $\dim \mathscr{W}_1 = 49$ *and $\mathscr{W}_1 \subsetneq \mathscr{W}_{\mathscr{K}}$. Both bounds are attained on the path: the pattern* $[a\mathrm{R}, b\mathrm{R}]$ *reads exactly $\mathscr{W}_{\mathscr{K}}$ at (c, a) from depth* 3*, and* $[a\mathrm{U}, b\mathrm{R}]$ *reads exactly $\mathscr{W}_1$ there from depth* 4*.*

Proof. A posting on input x with label j contributes, at its two sites, clock blocks $F^{\top}GF$ with F the clock factor of the block M_{yx} and G readable. A direct exact computation over the per-cell factors shows $\omega_x(F^{\top} v_q F) = 0$ for every q, so the first pullback lands in $\mathscr{W}$; the identity lies in $\mathscr{W}$ as well. For each of the four occupations x the readable blocks killed by ω_x form a five-dimensional space, which bounds the dimension by $4 \cdot 5 \cdot 5 = 100$; the attainment below gives equality. The unread bound is the same argument with the label summed; its dimension and the two attainments are rank computations on explicit word slices. The full space $\mathscr{W}$ also contains non-readable blocks and has dimension larger than 100. □

The bound misses an explicit product: $|0\rangle\langle 0| \otimes v_0 \otimes |0\rangle\langle 0| \otimes I \notin \mathscr{W}$. So the first-posting contact keeps lasting joint content beyond products as soon as its sites read the factors $|0\rangle\langle 0| \otimes v_0$ and $|0\rangle\langle 0| \otimes I$, whatever the network posts afterwards. On the path's six standard records of Chapter 16, in which each round posts each contact once and reads one or both, direct exact computation on their word slices gives the following profile. The contact posting first in the round fails the product law at every depth $k \geq 1$. The contact posting second obeys it from depth 2 on, with both of its sites reading all of $\mathscr{K}$, when its label is read, and fails it at depth 1; when its label is unread it obeys the product law at every depth.

17.5.3 The second contact: two positions

The contact that posts second can also keep lasting content, through a write that nothing rewrites before it is read; whether only so is open (Remark 17.48).

For a pair operator B on (c,b) write $B[(\beta,p),(\beta',q)]$ for its entry at the representative positions of $|\beta\rangle\langle\beta|\otimes v_p$ at c and $|\beta'\rangle\langle\beta'|\otimes v_q$ at b; on $\mathscr{K}\otimes\mathscr{K}$ these are the expansion coefficients. For coefficient vectors $\lambda,\rho\in\mathbb{Q}^6$ and a bit y let

$$\psi_y[\lambda,\rho](B)=\sum_{p,q,\beta'}\lambda_p\,(-1)^{\beta'}\,\rho_q\,B[(y,p),(\beta',q)],$$

the product of c at bit y read by λ with b read by ρ signed by its bit. Put $\rho^{\neq}=(1,-1,-1,1,0,0)$, the coordinates of $\omega_{\neq}$, and $\alpha_0=(1,-1,0,0,0,0)$, $\alpha_1=(0,0,1,-1,0,0)$.

Theorem 17.45 (The second contact's writes)**.** *Let the pattern begin with an unread posting of (c,b) directly after an unread posting of (c,a).*

1. Position three. *If the pattern is $[a{*},a\mathrm{U},b\mathrm{U},\dots]$ (the first posting read or not), then at every depth and for every continuation $\mathrm{read}^{k}_{cb}\subseteq\mathscr{W}_{\mathrm{III}}=\bigcap_{y,i}\ker\psi_y[\alpha_i,\rho^{\neq}]$. The readable part of $\mathscr{W}_{\mathrm{III}}$ has dimension 140 and misses $|0\rangle\langle0|\otimes v_0\otimes|0\rangle\langle0|\otimes v_0$. The pattern $[a\mathrm{R},a\mathrm{U},b\mathrm{U}]$ reads exactly that readable part from depth 4, obeys the product law at (c,b) at depths $0,1,2$, and fails it at every depth from 3 on.*
2. Position two. *If the pattern is $[a\mathrm{U},b\mathrm{U},\dots]$, then $\mathrm{read}^{k}_{cb}\subseteq\mathscr{W}_{\mathrm{II}}$, the common kernel of twenty explicit integer functionals (Table A.22), each vanishing on every branch of the unread (c,a) posting summed over the unread (c,b) posting. The readable part of $\mathscr{W}_{\mathrm{II}}$ has dimension 124, and $\mathscr{W}_{\mathrm{II}}$ lies strictly inside the space cut out by the four functionals $\psi_y[\gamma_i,\rho^{\neq}]$ with $\gamma_0=e_4$, $\gamma_1=e_5$. The pattern $[a\mathrm{U},b\mathrm{U},b\mathrm{R}]$ reads exactly the readable part of $\mathscr{W}_{\mathrm{II}}$ from depth 3 and fails the product law at (c,b) at every depth from 3 on.*

The same holds with the two contacts, and a and b, exchanged.

Proof. Every record effect of such a pattern is the pullback, through the round's first postings, of a sum of products of readable single-site operators (Chapter 16), so it suffices that the functionals vanish on every such pullback, whatever the products and the continuation. The mechanism is a cancellation at the shared clock. The functional $\rho^{\neq}$ at b is the unequal-bit write functional, which vanishes on the writes of unequal-bit steps, so it sees the unread (c,b) posting only through its equal-bit steps, whose two branches, signed by b's bit, count alike. The unread (c,a) posting directly before sums c's two clock branches into one operator with no stay coordinates v_4,v_5; this gives the four functionals $\psi_y[\gamma_i,\rho^{\neq}]$. One more native factor at c also kills that operator's v_0-v_1 and v_2-v_3 coordinates, which gives the four functionals of position three. The other sixteen functionals of position two express the same branch-by-branch vanishing on c's pole-line coordinates. Each of these vanishings is an identity on the finitely many branch blocks of the postings involved, checked by direct exact computation. The product $|0\rangle\langle0|\otimes v_0\otimes|0\rangle\langle0|\otimes v_0$ has the single coefficient $B[(0,0),(0,0)]=1$, so $\psi_0[\alpha_0,\rho^{\neq}]$ takes the value 1 on it and it is missing from $\mathscr{W}_{\mathrm{III}}$. In the same way $|0\rangle\langle0|\otimes v_4\otimes|0\rangle\langle0|\otimes v_0$ has $\psi_0[\gamma_0,\rho^{\neq}]=1$, so it is missing from the four-functional space and hence from $\mathscr{W}_{\mathrm{II}}$; the strict inclusion is witnessed by $|0\rangle\langle0|\otimes v_1\otimes|0\rangle\langle0|\otimes v_0$, which the four functionals kill and $\mathscr{W}_{\mathrm{II}}$ does not contain. The dimensions 140 and 124, and the equality of the stated reads with the readable parts from depths 4 and 3, are rank computations on explicit word slices of one round. Every pair read lies in $\mathscr{K}\otimes\mathscr{K}$ and a product $\mathrm{read}_c\otimes\mathrm{read}_b$ of two subspaces of the 12-dimensional $\mathscr{K}$ has dimension a product of two numbers at most 12; neither 140 nor 124 is such a product, so the product law fails at (c,b) wherever the pair reads these spaces. At

depth 3 the position-three pattern fails it through an explicit functional, and at depths $0, 1, 2$ its reads are checked to be products. The exchanged statements follow from the reflection of the path that exchanges a with b and the two contacts. □

Call a pattern of these two shapes, or their exchanges, a pattern of *rule shape*. Lasting content follows once the sites read everything. A pattern is *read-saturating* if, when it uses both contacts and reads some label, every site reads all of $\mathscr{K}$ from some depth on. A pattern's repeated posting stream *begins up to extra reads* with a list O of postings if O has the same contacts as the first postings of the stream and every posting that O reads is read there too.

Lemma 17.46 (Domination)**.** *If the first postings of k rounds of a pattern's stream begin up to extra reads with O, then every site read and pair read of one round of O lies in the corresponding read of the pattern at depth k, and hence at every later depth.*

Proof. An unread posting's effect is the sum of the corresponding read posting's label effects, and a word on the first postings of the stream has the sum of the effects of its extensions as effect. So every record effect of one round of O is a sum of record effects of the pattern at depth k, and read spaces are spans of block maps of record effects. Reads grow with depth by Theorem 17.43. □

Theorem 17.47 (The second contact's rule, proved part)**.**

1. *Every read-saturating pattern of rule shape that reads a label keeps lasting joint content beyond products at the second contact's pair.*
2. *Read saturation holds for every pattern whose repeated posting stream begins, up to extra reads, with one of the fifteen openings of at most five postings of Table A.21, or whose exchange does; on these patterns (1) is unconditional.*
3. *No lasting content at the second contact's pair arises in patterns using one contact only, in patterns reading no label, or in patterns whose stream, or its exchange, begins up to extra reads with one of the seven openings of at most five postings outside the rule's shape of Table A.21 (from the opening's depth on the second contact's pair reads all* 144 *products).*

Proof. (1) From some depth on every site reads all of $\mathscr{K}$, so $\mathrm{read}_c^k \otimes \mathrm{read}_b^k = \mathscr{K} \otimes \mathscr{K}$, while by Theorem 17.45 the pair read at (c, b) lies in $\mathscr{W}_{\mathrm{III}}$ or $\mathscr{W}_{\mathrm{II}}$ (or their exchanges), each of which misses a product of two readable statistics (its proof names them). So the product law fails at every depth from then on. (2) For each of the fifteen openings, a rank computation on explicit word slices shows that one round of it, at the depth certified, reads all of $\mathscr{K}$ at each of the three sites (direct exact computation). Lemma 17.46 carries this to every pattern of the class, and the exchange to the mirrored openings. With (1) this makes the lasting content unconditional on the class. (3) If only one contact posts, the second contact's pair contains a site that no posting touches; that site reads only the identity, and every pair read is a site read of c tensored with the identity, a product. If no label is read, every word's effect is the identity and every read is spanned by it. For each of the seven openings a rank computation shows that the second contact's pair reads all 144 products of $\mathscr{K} \otimes \mathscr{K}$ from the opening's depth on; a pair read equal to $\mathscr{K} \otimes \mathscr{K}$ obeys the product law, with both site reads equal to $\mathscr{K}$, and Lemma 17.46 carries this to every pattern of the class. □

Remark 17.48 (Open: the exact second-contact rule). *The second contact keeps lasting joint content beyond products exactly when the pattern has rule shape and reads some label.* It follows

from two open statements: *every path pattern is read-saturating*, and *every pattern with lasting content at the second contact has rule shape and reads a label.* Both reduce to patterns with a single read. A record keeping one of a pattern's reads is dominated by the pattern's record, so read saturation is equivalent to its single-read case; and the converse follows from its single-read case, in which, outside the rule's shape, the second contact's pair reads all 144 products at some depth. Exhaustive computation over all two-contact patterns of length six with a read, and over all single-read patterns of lengths seven and eight, agrees with the rule; that is evidence, not proof.

17.5.4 Unread runs stop shrinking

What a run of unrecorded postings leaves readable can be computed exactly.

Theorem 17.49 (Run writes)**.** *On every network, if a pattern begins with m unread postings of one contact $e = (u, v)$, then* $\mathrm{read}^k_{uv} \subseteq \mathscr{W}_m$ *at every depth, where $\mathscr{W}_m$ is the span of the m-fold pullbacks of readable products through the unread posting ($\mathscr{W}_1$ is the space of Theorem 17.44). Moreover*

$$\dim \mathscr{W}_1 = 49, \quad \dim \mathscr{W}_2 = 37, \quad \dim \mathscr{W}_m = 31 \ \ (m \geq 3),$$

with $\mathscr{W}_m = \mathscr{W}_3$ for $m \geq 3$ and the unread posting acting bijectively on $\mathscr{W}_3$. The relay pattern $[a\mathrm{U}, a\mathrm{U}, b\mathrm{U}, b\mathrm{R}]$ *reads exactly $\mathscr{W}_2$ at (c, a) from depth* 3*.*

Proof. The bound is the argument of Theorem 17.44 applied to the first m postings: every record effect is the m-fold pullback, through the unread posting, of a sum of products of readable operators, and its (u, v) blocks lie in the span of such pullbacks. Write Φ_e for the pullback through one unread posting of e. Then $\mathscr{W}_{m+1} = \Phi_e(\mathscr{W}_m)$, with $\mathscr{W}_0 = \mathscr{K} \otimes \mathscr{K}$; since the pullback of a readable product is again readable (Chapter 16), $\mathscr{W}_1 \subseteq \mathscr{W}_0$, and $\mathscr{W}_{m+1} \subseteq \mathscr{W}_m$ by induction. The dimensions 49, 37, 31 and $\dim \mathscr{W}_4 = 31$ are rank computations in the coordinates of $\mathscr{K} \otimes \mathscr{K}$ (direct exact computation). Since $\mathscr{W}_4 \subseteq \mathscr{W}_3$ have equal dimension, $\Phi_e(\mathscr{W}_3) = \mathscr{W}_3$; a linear map of a finite-dimensional space onto itself is bijective, and $\mathscr{W}_m = \mathscr{W}_3$ follows for every $m \geq 3$. The statement about the relay is a rank computation on its word slices at depth 3, with read spaces growing with depth (Theorem 17.43) and bounded by $\mathscr{W}_2$. □

The stable space can be filled by one read. Let $L_* = [b\mathrm{U}, a\mathrm{U}, b\mathrm{U}, b\mathrm{U}, a\mathrm{R}]$ and $\pi_k = (a\mathrm{U})^k L_*$.

Theorem 17.50 (A saturating family with one read)**.** *For every $k \geq 3$, one round of π_k reads exactly $\mathscr{W}_3$ at (c, a) and reads all of $\mathscr{K}$ at every site. Hence every pattern whose stream, or its exchange's, begins up to extra reads with π_k for some $k \geq 3$ is read-saturating. This class contains single-read patterns of every run length and patterns outside the fifteen-opening class of Theorem 17.47.*

Proof. For $k = 3$, a rank computation on the word slices of one round of π_3 shows that it reads all of $\mathscr{W}_3$ at (c, a) and all of $\mathscr{K}$ at each site (direct exact computation). The proof then transfers reads through the run: the site b, which the run does not touch, reads at least after k postings what it read after three, because the kernels of the run's powers are equal from the third on, and the pair (c, a) reads the image under the run of what it read after three. That image is the

image of $\mathscr{W}_3$ under $k-3$ unread postings, which is $\mathscr{W}_3$ by Theorem 17.49, and the reads of π_k at (c,a) lie in $\mathscr{W}_k = \mathscr{W}_3$; so they equal $\mathscr{W}_3$, and the sites c and a, whose reads are spanned by slices of the pair's, read all of $\mathscr{K}$ as for $k = 3$. Lemma 17.46 carries this to every pattern of the class, and the exchange is symmetric. Each π_k reads a single label, and the class contains patterns outside the fifteen openings, checked against their list. □

17.6 Between untouching sites

The sites a and b never share a contact. What a record fixes jointly about them passes through c.

Theorem 17.51 (Recovery at the untouching pair)**.** *1. Patterns using one contact only, and patterns reading no label, obey the product law at (a,b) at every depth.*

2. The relay $[a\mathrm{U}, a\mathrm{U}, b\mathrm{U}, b\mathrm{R}]$, the pattern $[a\mathrm{U}, a\mathrm{U}, a\mathrm{U}, b\mathrm{R}]$ and the pattern $[a\mathrm{U}, b\mathrm{R}, a\mathrm{U}]$ fail the product law at (a,b) at small depths and obey it from depths 2, 2 and 3 on respectively, these depths being least; there both a and b read all of $\mathscr{K}$ and the pair reads all 144 products.

3. If, in the first four postings of the repeated stream, each contact posts exactly twice and its second posting is read, then from depth 4 on a and b read all of $\mathscr{K}$ and the product law holds at (a,b).

Proof. (1) If only (c,a) posts, b is never touched, so it reads only the identity and every read at (a,b) is a site read of a tensored with the identity; the case of (c,b) alone is the same with the roles exchanged. If no label is read, every word's effect is the identity. (2) At the stated depths a rank computation on explicit word slices finds all 144 products at (a,b), so both sites read all of $\mathscr{K}$ and the pair read is $\mathscr{K} \otimes \mathscr{K}$, which persists at every later depth (Theorem 17.43); at the depth before, an explicit functional that annihilates the pair read but not the product of the site reads shows the failure (direct exact computation; the one for $[a\mathrm{U}, b\mathrm{R}, a\mathrm{U}]$ is described below). (3) The four postings in question form one of six lists, the two postings of each contact in some order, with at least the second posting of each contact read. A rank computation shows that one round of each of the six lists, reading exactly those second postings, reads all 144 products at (a,b). Four rounds of the pattern contain the first four postings of the stream, so by Lemma 17.46 the pattern reads all 144 products from depth 4 on; a pair read equal to $\mathscr{K} \otimes \mathscr{K}$ gives both site reads equal to $\mathscr{K}$ and the product law. □

The failure of $[a\mathrm{U}, b\mathrm{R}, a\mathrm{U}]$ at depth 2 is purely relational: both sites already read everything, and the missing product is detected by an explicit functional pairing a's bit-0 pole difference $v_1 - v_0$ with b's pole-line difference $v_1 - v_0 + v_2 - v_3$.

Remark 17.52 (Open: transience between untouching sites). *Every path pattern obeys the product law at (a,b) from some depth on.* By Theorems 17.43 and 17.51 it remains open only for patterns using both contacts, reading a label, and not opening twice in the sense of (3). All 1302 path patterns of length at most five with a read stabilize with the product law at (a,b); that is evidence, not proof.

17.6.1 After a long unrecorded run

The run family π_k shows why a uniform argument is hard.

Theorem 17.53 (Undamped core and faint content)**.** *1. The readable (c,a) operators fixed by the unread (c,a) posting form a five-dimensional space inside $\mathscr{W}_3$, and one further operator of $\mathscr{W}_3$ is reversed by it. Off this six-dimensional core, twelve unread postings at least halve every vector's coordinates in an explicit basis of $\mathscr{W}_3$, in the maximum norm.*

2. The shared-site restrictions of the undamped operators lie in the five-dimensional space F_5 of readable statistics $x = \sum x_{\beta q}\,|\beta\rangle\langle\beta| \otimes v_q$ of c with

$$x_{01} = x_{00},\ x_{03} = x_{02},\ x_{05} = -x_{04},\ x_{11} = x_{10},\ x_{12} = x_{13} = x_{00} + x_{02} - x_{10},\ x_{15} = -x_{14}.$$

3. For every pattern L, every one-round record word s of L and every index at a there is a constant C such that for every $k \geq 3$ the (c,b) slice of the record word $(a\mathrm{U})^k s$ lies, coordinate by coordinate, within $C(1/2)^{\lfloor (k-3)/12 \rfloor}$ of the span of the products of five statistics of c spanning F_5 with the twelve readable statistics of b, a space of dimension at most 60.

4. The patterns π_3 and π_4 read all 144 products at (c,b) at depth 2. Hence every pattern whose stream, or its exchange's, begins up to extra reads with two rounds of π_3 or of π_4 is read-saturating and keeps no lasting content at the second contact's pair, in accordance with the rule.

Proof. (1) In an explicit basis of $\mathscr{W}_3$, thirty-one operators, the unread posting acts by an explicit integer matrix divided by 16 (Theorem 17.49 makes it a bijection of $\mathscr{W}_3$). A direct exact computation finds five operators of the basis span that it fixes and one that it reverses. A readable operator fixed by the unread posting equals its own threefold pullback, so it lies in $\mathscr{W}_3$, and the fixed space there is five-dimensional. Let Π be the projection onto the six-dimensional period-two part along the complementary invariant subspace, and T the matrix of the unread posting. Every row of $T^{12} - \Pi$ has absolute values summing to at most $\frac{1}{2}$ (direct exact computation), so for every coordinate vector r with $\Pi r = 0$ the maximum norm obeys $\|T^{12} r\| = \|(T^{12} - \Pi) r\| \leq \frac{1}{2}\|r\|$. (2) The shared-site restrictions of the six core operators are computed directly and satisfy the seven linear relations displayed, which cut F_5 out of the twelve-dimensional readable space of c. (3) The (c,a) blocks of the effect of s are readable, so after $k \geq 3$ unread postings they lie in $\mathscr{W}_3$; there split the coordinates of the $(k-3)$-fold image into the period-two part, on which the posting acts by ± 1, and the rest, whose maximum norm (1) bounds by a constant times $(1/2)^{\lfloor (k-3)/12 \rfloor}$, the constant covering the fewer than twelve remaining postings. The (c,b) slice of the period-two part is a sum of products of shared-site restrictions of core operators, in F_5 by (2), with readable statistics of b, so it lies in the span of the products of five statistics spanning F_5 with the twelve readable statistics of b, of dimension at most $5 \cdot 12 = 60$. The remainder gives the constant C. (4) The two rank computations are direct exact computations. A pattern of the class reads all 144 products at the second contact's pair from depth 2 on, by Lemma 17.46; such a pair read obeys the product law, and the pattern is read-saturating by Theorem 17.50. Its stream begins with a leading run of unread first-contact postings, so it is not of rule shape, and keeping no lasting content agrees with the rule. □

The free directions of F_5 are the identity, the half-turn $v_2 + v_3$ and $v_4 - v_5$ at bit 0, and the identity and $v_4 - v_5$ at bit 1. The other 84 products are read for $k = 3, \ldots, 15$ (Remark 17.54), with weights that decay geometrically; numerically the slowest mode decays by about 0.78 per posting.

Remark 17.54 (Open: pair saturation after every run). *For every $k \geq 3$, the second contact's pair of π_k reads all* 144 *products at some depth.* Computation modulo a large prime finds rank 144 at depth 2 for every k from 3 to 15. By Theorem 17.53 a proof cannot pass through the limit of long runs, which reads at most 60 products; it must control exponentially small terms at every k, a non-vanishing problem for sums of decaying exponentials of Skolem type. Leading-run invariance in general, that a pattern's sites and second pair saturate after $k \geq 3$ leading unread postings whenever they do after three, is open as well; at depth one it holds for the untouched site, for the pair (c, a) up to the run's image, and for c and a whenever the round after three postings fills $\mathscr{W}_3$.

Chapter 18 turns from what records hold to what survives inside one cell when it is coupled to the photon field.

Chapter 18

Protected insides I: survival under the relay

Chapter 7 found the participants of one native cell by asking which states the postings keep. A posting is a discrete event. Between postings a cell is coupled to a field, and nothing in the posting law says that the coupling respects the aligned sector. This chapter asks whether a sure subject survives such a coupling, for one interaction written down in full: the *relay*, in which an occupation hops between two sites while two clocks drive the transverse photon field. The relay is supplied; Proposition 18.1 shows which of its parts the native posting accounts for. Section 18.1 states it and says exactly what is supplied.

The question is what keeps a sure subject a sure subject while the relay acts. The relay alone keeps none: for $\varepsilon \neq 0$ every unit sure subject leaves the aligned sector at a short-time rate of at least $(15-\sqrt{5})\varepsilon^2$. Frequent projective readings of the sector, an operation we supply, hold every sure subject. Native postings that erase their record hold one only when the relay emits no photon, and then only the oscillation: the persistent currents leave and return for ever. Native postings that write their record hold every sure subject at every finite time, and the written word selects its mode. Under the relay, what keeps a sure subject is its own record.

18.1 The relay and its sure subjects

The relay acts on the native cell together with a photon field. We state the operator in full.

18.1.1 The photon field

Let $\mathfrak{h}$ be the one-photon space of the transverse photon field on the cubic lattice, the square-integrable transverse amplitudes on the Brillouin cube, and let $\mathscr{F}$ be its symmetric Fock space, with vacuum vac. The free photon energy H_F is the second quantization of the lattice frequency $\omega(k) = 2C_E \operatorname{arsinh}\big(\tfrac{1}{2}\sqrt{2\sum_a(1-\cos k_a)}\big)$ for a constant $C_E \geq 0$. It is self-adjoint on its maximal domain, and $H_F\,\mathrm{vac} = 0$. Inner products are conjugate-linear in the first argument. For $g \in \mathfrak{h}$ the Weyl displacement $W(g)$ is unitary on $\mathscr{F}$, and

$$W(g)W(h) = e^{-i\,\mathrm{Im}\langle g,h\rangle}\,W(g+h), \qquad \langle W(g)\mathrm{vac}, W(h)\mathrm{vac}\rangle = e^{-\frac{1}{2}\|g\|^2-\frac{1}{2}\|h\|^2+\langle g,h\rangle}. \tag{18.1}$$

For $x \in \mathbb{R}$, τ_x translates $\mathfrak{h}$ by x lattice units along the axis (multiplication by e^{ixK}, K the axial momentum), and $\Gamma(\tau_x)$ translates every photon of $\mathscr{F}$ in the same way. Both are unitary, and $\Gamma(\tau_x)\mathrm{vac} = \mathrm{vac}$. For $g \neq 0$ the translate $\tau_{-1}g$ differs from g, because $e^{-iK} \neq 1$ almost everywhere on the Brillouin cube.

18.1.2 The eight-state clock

The relay's clocks are compressions of one Hermitian circulant 8×8 matrix,

$$\Omega_{ij} = \tfrac{1}{2}\big(7\,[i=j] - [i \neq j]\big) + \tfrac{i}{2}\big(\xi_{i-j} + \sqrt{2}\,\zeta_{i-j}\big), \qquad i, j \in \mathbb{Z}/8,$$

with $\xi = (0,1,1,-1,0,1,-1,-1)$ and $\zeta = (0,1,0,1,0,-1,0,-1)$ indexed by $i-j \bmod 8$. Its spectrum is $\{0,1,\dots,7\}$. For $b \in \{0,1\}$ and $j \in \{0,1,2,3\}$ put

$$f_j^b = \tfrac{1}{\sqrt{2}}\big(e_{2j} + (-1)^b\, i\, \varsigma_j\, e_{2j+1}\big), \qquad \varsigma = (-1,1,1,-1).$$

The eight vectors f_j^b form an orthonormal basis of $\mathbb{C}^8$, and the *clock blocks* are the four 4×4 blocks of Ω in this basis, $B(b',b)_{jk} = \langle f_j^{b'}, \Omega f_k^{b}\rangle$, acting on the clock cell $\mathscr{H}_{\mathrm{c}}$; $B(b',b)^* = B(b,b')$. Table A.23 lists their action on P and A. Four of its numbers carry most of the chapter:

$$\begin{aligned} &\langle A, B(b,b)A\rangle = 4, \qquad \|B(b,b)A - 4A\|^2 = \tfrac{9}{4} + \sqrt{2}, \\ &\|B(1-b,b)A\|^2 = \tfrac{9}{4} - \sqrt{2}, \qquad \langle P, B(b,b)A\rangle = -\tfrac{i}{2}. \end{aligned} \tag{18.2}$$

18.1.3 The relay operator

The relay carrier is $\mathscr{H} \otimes \mathscr{F}$: each of the 64 coordinates of the native cell carries a photon state. Fix a photon profile $z \in \mathfrak{h}$, write $\tilde{z} = \tau_{-1} z$ for its unit translate, and fix real numbers ϕ (a momentum phase), ε (the coupling), ν_0 and $\Delta\nu$ (energy offsets). The *relay interaction* is the bounded operator

$$V = \varepsilon\,(K_1 + K_2 + R) + (2\varepsilon + \nu_0)\, I + \Delta\nu\big(|11\rangle\langle 11| \otimes I\big) \tag{18.3}$$

on $\mathscr{H} \otimes \mathscr{F}$, whose three terms act as follows.

- K_1 moves the first occupation bit and the first clock:

$$K_1\big(|b\,x_2\rangle \otimes u \otimes u' \otimes g\big) = \sum_{b'} |b'\,x_2\rangle \otimes B(b',b)u \otimes u' \otimes W_1(b',b)g,$$

  with $W_1(b,b) = I$, $W_1(1,0) = W(z)$ and $W_1(0,1) = W(-z)$.
- K_2 does the same on the second bit and the second clock, with $W_2(b,b) = I$, $W_2(1,0) = W(-\tilde{z})$ and $W_2(0,1) = W(\tilde{z})$.
- R exchanges the two mixed occupations through the dressing parity and a recoil of the field. With the unitary $\varrho = W(\tau_{-2}z)\,\Gamma(\tau_{-3})$,

$$\begin{aligned} R\big(|10\rangle \otimes w \otimes g\big) &= 2i\,e^{-i\phi}\,|01\rangle \otimes (D \otimes D)w \otimes \varrho\, g, \\ R\big(|01\rangle \otimes w \otimes g\big) &= -2i\,e^{i\phi}\,|10\rangle \otimes (D \otimes D)w \otimes \varrho^{-1} g, \end{aligned}$$

  and R vanishes on the balanced occupations.

The *relay* is the Hamiltonian $H = H_0 + V$ with $H_0 = I \otimes H_F$. Since V is bounded and symmetric, H is self-adjoint on the domain of H_0 and generates the unitary group $U(t) = e^{-itH}$. The *prepared source* is

$$\psi_{\mathrm{rel}} = \psi_\circ \otimes \mathrm{vac} = |10\rangle \otimes A \otimes A \otimes \mathrm{vac}.$$

The relay is supplied, and we say once what that means. It is an operator for a hopping charge coupled to the photon field, constructed in companion work on that field, and no result of this book selects it from other possible dynamics. Part of it is tied to the native posting. Write K_1°, K_2° and R_0 for the cell operators by which K_1, K_2 and R act on $\mathscr{H} \otimes \mathrm{vac}$ at $z = 0$ and $\phi = 0$, so that the interaction there is $V_0^{\mathrm{nat}} = \varepsilon(K_1^\circ + K_2^\circ + R_0) + (2\varepsilon + \nu_0)I$ when $\Delta\nu = 0$ (the matrix V_0 of Theorem 18.17 at $\phi = 0$, $\Delta\nu = 0$).

Proposition 18.1 (What the native posting generates)**.**

(a) $U_{\mathrm{nat}} = \exp\!\big(-\frac{i\pi}{4}(K_1^\circ + K_2^\circ)\big)\exp\!\big(-\frac{i\pi}{4}R_0\big)$*: the posting's clock factor and its exchange are eighth-turns, of one common duration* $\pi/4$*, of the relay's clock terms and its reciprocal term.*

(b) For all real ε*,* ν_0 *and* t*, with* $s = t/n$*,*

$$\Big(e^{-is(2\varepsilon+\nu_0)}\exp\!\big(-is\varepsilon(K_1^\circ + K_2^\circ)\big)\exp\!\big(-is\varepsilon R_0\big)\Big)^n \longrightarrow e^{-itV_0^{\mathrm{nat}}} \qquad (n \to \infty),$$

and at $\varepsilon = 1$*,* $\nu_0 = -2$*,* $s = \pi/4$ *the factor in brackets is* U_{nat}*. On* $u \otimes \mathrm{vac}$ *the left side therefore converges to the relay's own evolution* $U(t)(u \otimes \mathrm{vac})$ *at* $z = 0$*,* $\phi = 0$*,* $\Delta\nu = 0$*.*

(c) The photon-free interaction at $z = 0$ *equals* $\varepsilon(K_1^\circ + K_2^\circ + R_0) + (2\varepsilon + \nu_0)I$ *if and only if* $\Delta\nu = 0$ *and either* $\varepsilon = 0$ *or* $\phi \in 2\pi\mathbb{Z}$*. Otherwise its evolution differs at some time from the limit in (b).*

(d) For every profile z*, the photon factor of each clock term on each single-site transition* $x \to y$ *is* $W\big(\sum_e j(y,x)_e\, w_e\big)$*, the Weyl map of the posting's own current read through the edge profiles* $w = (z, -(z+\tilde{z}), -\tilde{z})$*. The reciprocal factor* $\varrho = W(\tau_{-2}z)\Gamma(\tau_{-3})$ *has the displacement this incidence assigns to its transition only when* $z + \tau_{-1}z + \tau_{-2}z = 0$*, and its recoil* $\Gamma(\tau_{-3})$ *has no counterpart in any clock transition.*

Proof. (a) A direct exact computation in the occupation chart, with this mechanism. The clock step of one eight-state channel is the cyclic shift of its slots. The discrete Fourier transform diagonalizes both the shift, with eigenvalue $e^{-i\pi k/4}$ on its k-th vector, and Ω, with eigenvalue k, so the shift is $\exp(-\frac{i\pi}{4}\Omega)$; read in the chart vectors f_j^b, which are the columns of the occupation chart of Chapter 2, the shift is $-i$ times the clock map and Ω is the matrix of clock blocks $B(b', b)$. The exchange is the identity on balanced occupations and on mixed ones a real matrix J with $J^2 = -1$, so it is $\exp(\frac{\pi}{2}J) = \exp(-\frac{i\pi}{4}R_0)$ with $R_0 = 2iJ$ there and 0 on balanced occupations, which is the reciprocal term at $z = 0$, $\phi = 0$. The factor $(-i)^2 = -1$ of the two clock maps, together with the signs of the exchange, gives the signs $\sigma(x)$ of the blocks M_{yx} (Theorem 2.4). (b) is the Lie–Trotter product formula for the two matrices [12]; the scalar contributes a global phase, and the free photon energy vanishes on vac. (c) The diagonal entry at the doubly occupied cell carries $\Delta\nu$ and the reciprocal entry carries $e^{-i\phi}$, so the equality forces $\Delta\nu = 0$ and $\varepsilon(e^{-i\phi} - 1) = 0$; conversely these give equality. If $e^{-itH_1} = e^{-itH_2}$ for every t, then $H_1 = H_2$. (d) A check of the eight single-site transitions against the label components, and of the reciprocal transition $(1,0) \to (0,1)$, whose incidence displacement is $-(z + \tilde{z})$. □

So the posting is the product of eighth-turns of the photon-free clock and reciprocal terms, and alternating them at a common rate generates the photon-free relay. The posting fixes these generators only up to the branch of the logarithm: (a) takes for each clock the branch Ω, with spectrum $\{0, \dots, 7\}$, and for the exchange the branch R_0, with spectrum $\{-2, 0, 2\}$; these are the branches whose common duration $\pi/4$ reproduces both factors. What the posting does not

supply is the Coulomb difference Δv, the momentum phase ϕ off $2\pi\mathbb{Z}$, the translated profile and recoil of the reciprocal factor, the common rate ε of the two factors, and the reading of fractional postings as the dynamics between postings. Whether a field-coupled native posting produces the reciprocal photon factor, its translated profile and recoil included, and whether the Coulomb difference and the momentum phase have native origins, are open (Section 23.1). The clock Ω, the displacements, the recoil and the parameters $z, \phi, \varepsilon, \nu_0, \Delta v, C_E$ are part of the operator as supplied. Every theorem of this chapter is a theorem about this operator, valid for every value of the parameters unless a condition is stated, and its time t is dimensionless and uncalibrated. The relay states $\rho_\pm$ of the networks of Chapter 12 are a different object; only the word is shared.

Definition 18.2 (Sure subjects of the relay carrier)**.** Let $\Pi = Q_{\mathrm{al}} \otimes I$ and $\Pi^\perp = I - \Pi$. A vector $x \in \mathscr{H} \otimes \mathscr{F}$ is a *sure subject* when $\Pi x = x$. Equivalently,

$$x = \psi_+ \otimes g_+ + \psi_- \otimes g_- + \psi_\circ \otimes g_\circ + \psi_\circ' \otimes g_\circ',$$

with photon *payloads* $g_m \in \mathscr{F}$, and then $\|x\|^2 = \sum_m \|g_m\|^2$. The *loss* of a vector y is $\|\Pi^\perp y\|^2$, its weight outside the sure-subject sector.

The name rests on Theorem 7.16: a cell state passes every field-current window surely exactly when it lies in the aligned sector, and Π asks this of every photon component. Two facts about Π are used throughout. It commutes with H_0, which acts on the photons alone, and with $X \otimes I$ for every matrix X on $\mathscr{H}$ that commutes with Q_{al}. Among these are every slice V_j and the native step $U_{\mathrm{nat}} = \sum_j V_j$ (Lemma 8.1).

18.2 The source is the oscillation, and the relay moves it out

The relay's prepared source is a sure subject, the occupation oscillation of Chapter 2 with no photon present. The relay's first response already has a part of squared norm $13\varepsilon^2$ outside the sector. The native step, applied to the same state, does not.

Theorem 18.3 (The first response)**.** *For all $z, \phi, \varepsilon, \nu_0, \Delta v$:*

(a) ψ_{rel} *is a sure subject, it lies in the domain of H, and* $H\psi_{\mathrm{rel}} = V\psi_{\mathrm{rel}}$.
(b) $\Pi V \psi_{\mathrm{rel}} = (10\varepsilon + \nu_0)\,\psi_{\mathrm{rel}}$.
(c) The departure $\Pi^\perp V \psi_{\mathrm{rel}}$ *is, occupation by occupation,*

occupation	*departure*	*squared norm*
$(0,0)$	$\varepsilon\, B(0,1)A \otimes A \otimes W(-z)\mathrm{vac}$	$\varepsilon^2(\frac{9}{4} - \sqrt{2})$
$(1,1)$	$\varepsilon\, A \otimes B(1,0)A \otimes W(-\tilde{z})\mathrm{vac}$	$\varepsilon^2(\frac{9}{4} - \sqrt{2})$
$(0,1)$	$2i\varepsilon e^{-i\phi}\, a \otimes a \otimes W(\tau_{-2}z)\mathrm{vac}$	$4\varepsilon^2$
$(1,0)$	$\varepsilon\big((B(1,1)A - 4A) \otimes A + A \otimes (B(0,0)A - 4A)\big) \otimes \mathrm{vac}$	$(\frac{9}{2} + 2\sqrt{2})\varepsilon^2$

and $\|\Pi^\perp V \psi_{\mathrm{rel}}\|^2 = 13\varepsilon^2$. *For $\varepsilon \neq 0$ the first response leaves the sector.*

(d) The native step returns the source to the sector: its block $M_{(01),(10)}$ equals $(\mathrm{stay} \otimes \mathrm{stay})(D \otimes D)$, *with* $(D \otimes D)(A \otimes A) = a \otimes a$ *and* $(\mathrm{stay} \otimes \mathrm{stay})(a \otimes a) = A \otimes A$. *More generally, if the posting moves the photon field by an arbitrary linear map $E(y, x)$ attached to each block M_{yx}, it sends ψ_{rel} to $\psi_\circ' \otimes E\big((01),(10)\big)\mathrm{vac}$, a sure subject.*

Proof. (a) $\psi_\circ$ is a mode, and $H_0\psi_{\mathrm{rel}} = \psi_\circ \otimes H_F\mathrm{vac} = 0$.

(b), (c) Apply each term of (18.3) to $|10\rangle \otimes A \otimes A \otimes \mathrm{vac}$. The first clock turns bit 1 into bit b': for $b' = 0$ it writes $B(0,1)A$ on the first clock with $W(-z)$ on the field, landing at $(0,0)$; for $b' = 1$ it writes $B(1,1)A$ and stays at $(1,0)$. The second clock turns bit 0 into b': for $b' = 1$ it lands at $(1,1)$ with $B(1,0)A$ and $W(-\tilde{z})$; for $b' = 0$ it stays with $B(0,0)A$. The recoil sends the state to $(0,1)$ with $(D \otimes D)(A \otimes A) = DA \otimes DA = a \otimes a$ and $\varrho\,\mathrm{vac} = W(\tau_{-2}z)\mathrm{vac}$. The diagonal term multiplies by $2\varepsilon + \nu_0$. The balanced occupations lie outside the sector, and $a \otimes a$ is orthogonal to $P \otimes P$ and $A \otimes A$, so the $(0,0)$, $(1,1)$ and $(0,1)$ parts are departures. At $(1,0)$ the aligned read of $\varepsilon(B(1,1)A \otimes A + A \otimes B(0,0)A) + (2\varepsilon + \nu_0)A \otimes A$ is $(4+4)\varepsilon + 2\varepsilon + \nu_0 = 10\varepsilon + \nu_0$ on $A \otimes A$ and 0 on $P \otimes P$, by (18.2) and $\langle P, A\rangle = 0$. This gives (b) and the $(1,0)$ departure. The squared norms follow from (18.2): Weyl displacements are unitary, $\|A\| = \|a\| = 1$, and the two terms at $(1,0)$ are orthogonal because $\langle A, B(b,b)A - 4A\rangle = 0$. The sum is $2(\frac{9}{4} - \sqrt{2}) + 4 + 2(\frac{9}{4} + \sqrt{2}) = 13$.

(d) By the block formula of Chapter 2, $M_{(01),(10)} = \sigma(1,0)\,K(0,0)D \otimes K(1,1)D = -(\mathrm{stay}\,D) \otimes (-\mathrm{stay}\,D)$. Next, $DA = a$ and $\mathrm{stay}\,a = \frac{1}{2}D(I - L)a = Da = A$, since $La = -a$. The other three blocks out of $(1,0)$ annihilate $A \otimes A$ (Proposition 2.22). With a field map $E(y,x)$ attached to each block, the only surviving block carries $E\big((01),(10)\big)$ to the vacuum. □

Part (d) locates the difference exactly. The native step *composes* the reciprocal parity $D \otimes D$ with the clock factor stay ⊗ stay, and the composite returns $A \otimes A$ to itself. The relay's generator *adds* the reciprocal term and the clock terms, and its reciprocal term stops at $a \otimes a$, whose dressed pole charge vanishes on both cells. So sure subjecthood, as the window certifies it, is a property of the posted dynamics at posting resolution. The continuous motion between postings does not keep it.

18.3 The relay keeps no sure subject

Under the relay's own evolution, every sure subject starts to leave the sector at once, quadratically in time, at a rate that depends on the state but never falls below $(15 - \sqrt{5})\varepsilon^2$. For the prepared source the rate is its energy variance.

18.3.1 The prepared source drains at its energy variance

Theorem 18.4 (Short-time loss of the source)**.** *Let $\psi = \psi_{\mathrm{rel}}$, $\psi(t) = U(t)\psi$, and $S = \|H^2\psi\|$, which is finite. For all $z, \phi, \varepsilon, \nu_0, \Delta\nu$ and $C_E \geq 0$:*

(a) $\langle\psi, H\psi\rangle = 10\varepsilon + \nu_0$, $\|H\psi\|^2 = (10\varepsilon + \nu_0)^2 + 13\varepsilon^2$, *and* $\|H\psi - (10\varepsilon + \nu_0)\psi\|^2 = 13\varepsilon^2$.

(b) For $0 \leq t \leq 1$,

$$\Big|\,\|\Pi\psi(t)\|^2 - \big(1 - 13\varepsilon^2 t^2\big)\Big| \leq \big(\sqrt{13}\,|\varepsilon|\,S + \tfrac{1}{4}S^2\big)\,t^3,$$

so the loss satisfies $\|\Pi^\perp\psi(t)\|^2/t^2 \to 13\varepsilon^2$ as $t \to 0^+$. The survival of the source itself obeys the same law: $\big(1 - |\langle\psi, \psi(t)\rangle|^2\big)/t^2 \to 13\varepsilon^2$.

(c) The loss splits by occupation: for each occupation y, $\|(|y\rangle\langle y| \otimes I)\Pi^\perp\psi(t)\|^2/t^2$ tends to the squared norm of the y part of the departure in Theorem 18.3(c). The occupation-changing part $(\frac{17}{2} - 2\sqrt{2})\varepsilon^2$ is smaller than the in-place part $(\frac{9}{2} + 2\sqrt{2})\varepsilon^2$.

(d) For $\varepsilon > 0$ the probability of the occupation $(0,1)$ is $4\varepsilon^2 t^2 + o(t^2)$, while the weight of $\psi_\circ' \otimes \mathcal{F}$ is $o(t^2)$.

Proof. (a) $H\psi = V\psi$ (Theorem 18.3(a)), so $\langle \psi, H\psi\rangle = \langle \psi, \Pi V\psi\rangle = 10\varepsilon + v_0$ by part (b) of that theorem, and Pythagoras with part (c) gives the rest. The vector $V\psi$ lies in the domain of H_0, since its photon states are displaced vacua, which have finite free energy; so S is finite.

(b) Since ψ is in the domain of H^2, $\|\psi(t) - (\psi - itH\psi)\| \le \frac{1}{2}St^2$. The projection $\Pi^\perp$ is a contraction and kills ψ, so with $\alpha = \|\Pi^\perp\psi(t)\|$ and $\beta = \|\Pi^\perp H\psi\|\, t = \sqrt{13}|\varepsilon|t$ we get $|\alpha - \beta| \le \frac{1}{2}St^2$ and

$$|\alpha^2 - \beta^2| \le |\alpha - \beta|\,(2\beta + |\alpha - \beta|) \le \sqrt{13}|\varepsilon|St^3 + \tfrac{1}{4}S^2t^4.$$

For $t \le 1$ this is the bound, and the limit follows. For the survival, replace $\Pi^\perp$ by $I - |\psi\rangle\langle\psi|$: both projections send $H\psi$ to $H\psi - (10\varepsilon + v_0)\psi$, because the aligned part of $H\psi$ is a multiple of ψ.

(c) The same argument with the contraction $(|y\rangle\langle y| \otimes I)\Pi^\perp$ in place of $\Pi^\perp$. The occupation-changing part is $2(\frac{9}{4} - \sqrt{2}) + 4$, and $\frac{9}{2} + 2\sqrt{2} > \frac{17}{2} - 2\sqrt{2}$ because $\sqrt{2} > 1$.

(d) The occupation $(0,1)$ is empty in ψ, so its weight in $\psi(t)$ is the squared norm of the $(0,1)$ part of $\psi(t) - \psi = -itV\psi + O(t^2)$. That part is $-it$ times the $(0,1)$ departure $2i\varepsilon e^{-i\phi} a \otimes a \otimes W(\tau_{-2}z)\mathrm{vac}$, of squared norm $4\varepsilon^2$, and it has no component on $A \otimes A$ because $\langle A, a\rangle = 0$. □

Two features stand out. The rate is the energy variance, which is the Mandelstam–Tamm and Zeno short-time law [31, 32]; here the loss from the whole sure-subject sector, not only from the prepared state, has this rate, because the aligned part of $H\psi$ is the mean-energy multiple of ψ. And more than half of the loss, $(\frac{9}{2} + 2\sqrt{2})/13 \approx 56\%$, is internal clock spread at the prepared occupation, with the field still in its vacuum: nothing is emitted and no occupation changes. A readout of occupations or photons alone sees only the smaller part. Part (d) adds that the continuous motion does not perform the hop: what reaches the other mixed occupation at second order is $a \otimes a$, not the oscillation's partner $\psi_\circ'$.

18.3.2 Every sure subject leaves, at a least rate

The prepared source is not special. Every sure subject in the domain of H leaves at a positive rate when $\varepsilon \neq 0$, and the least rate over all of them has a closed form. First, inside the sector the relay acts mode by mode.

Proposition 18.5 (The compressed interaction)**.** *Write $e_+ = e_- = 8\varepsilon + v_0$ and $e_\circ = e_\circ' = 10\varepsilon + v_0$. Then*

$$\Pi V \Pi = \sum_m e_m\, |\psi_m\rangle\langle\psi_m| \otimes I,$$

the sum over the four modes. No compressed coupling joins two modes, and the photon payload is untouched.

Proof. The diagonal term contributes $2\varepsilon + v_0$ on every mode. A clock term changes the occupation, which leaves the sector, or applies $B(b,b)$ to one cell while the other cell keeps its P or A; the aligned read of $B(b,b)u \otimes u$ for $u \in \{P, A\}$ is $\langle u, B(b,b)u\rangle$ on $u \otimes u$ and $\langle u', B(b,b)u\rangle\langle u', u\rangle = 0$

on the other aligned vector $u' \otimes u'$. By Table A.23, $\langle P, B(b,b)P\rangle = 3$ and $\langle A, B(b,b)A\rangle = 4$, so the two clocks add 6ε on the persistent modes and 8ε on the oscillation modes. The recoil sends $P \otimes P$ and $A \otimes A$ to $p \otimes p$ and $a \otimes a$, whose aligned part is zero. □

Theorem 18.6 (No sure subject is kept)**.** *For a unit sure subject x in the domain of H let $\ell(x) = \|\Pi^{\perp} H x\|^2$. Let $\theta = \operatorname{Im}\langle z, \tilde{z}\rangle$ and*

$$\gamma(k) = 14 - \tfrac{\sqrt{5}}{2} - \big(\tfrac{\sqrt{5}}{2} - 1\big)|k|.$$

For all $z, \phi, \varepsilon, v_0, \Delta v$ and $C_E \geq 0$:

(a) $\ell(x) = \|\Pi^{\perp} V x\|^2$, and $\|\Pi U(t)x\|^2 = 1 - \ell(x)\, t^2 + o(t^2)$ as $t \to 0$.

(b) The least value of ℓ over unit sure subjects in the domain of H is attained and equals $\varepsilon^2 \gamma(\cos\theta)$. It depends on z only through θ, and not at all on ϕ, v_0, Δv or C_E.

(c) $(15 - \sqrt{5})\,\varepsilon^2 \leq \varepsilon^2 \gamma(\cos\theta) \leq (14 - \tfrac{\sqrt{5}}{2})\,\varepsilon^2$, and the lower bound is attained at $z = 0$. Hence for $\varepsilon \neq 0$ every unit sure subject in the domain has $\ell(x) \geq (15 - \sqrt{5})\varepsilon^2 > 0$ and has loss $\|\Pi^{\perp} U(t)x\|^2 > 0$ at all small $t \neq 0$.

(d) A unit sure subject with $\Pi^{\perp} H x = 0$ exists if and only if $\varepsilon = 0$.

(e) For $\varepsilon \neq 0$ the prepared source is not a least-loss state: $\varepsilon^2 \gamma(\cos\theta) < 13\varepsilon^2 = \ell(\psi_{\mathrm{rel}})$.

Proof. (a) H_0 commutes with Π and $\Pi^{\perp} x = 0$, so $\Pi^{\perp} H_0 x = 0$. The expansion follows from the derivative of $U(t)x$ at $t = 0$, as in Theorem 18.4(b).

A floor first. The recoil sends the persistent and oscillation parts at one mixed occupation to $p \otimes p$ and $a \otimes a$ at the other. Each clock term acts on one cell and leaves the other cell's P or A in place, and p, a are orthogonal to P, A; so the $p \otimes p$ and $a \otimes a$ components of Vx come from the recoil alone, lie outside the sector, and have total squared norm $4\varepsilon^2\|x\|^2$. So $\ell(x) \geq 4\varepsilon^2$ before any clock is computed.

(b) Write x with its payloads $g_+, g_\circ, g_-, g_\circ'$ (Definition 18.2). Computing $\Pi^{\perp} V x$ occupation by occupation from (18.3) and Table A.23 gives four forms in the payloads. At $(1,0)$ the departure has squared norm

$$\varepsilon^2\big(\tfrac{9}{2}\|g_+\|^2 + (\tfrac{9}{2} + 2\sqrt{2})\|g_\circ\|^2 - \operatorname{Re}\langle g_+, g_\circ\rangle + 4\|g_-\|^2 + 4\|g_\circ'\|^2\big),$$

the in-place clock spread of its own modes plus the recoil from $(0,1)$; at $(0,1)$ the mirror form. At $(0,0)$ it is $\varepsilon^2\kappa\big(W(-z)g_+, W(-z)g_\circ, W(\tilde{z})g_-, W(\tilde{z})g_\circ'\big)$ and at $(1,1)$ it is $\varepsilon^2\kappa\big(W(-\tilde{z})g_+, W(-\tilde{z})g_\circ, W(z)g_-, W(z)g_\circ'\big)$, where

$$\begin{aligned}\kappa(h_1,h_2,h_3,h_4) = {} & \tfrac{13}{4}\big(\|h_1\|^2 + \|h_3\|^2\big) + (\tfrac{9}{4} - \sqrt{2})\big(\|h_2\|^2 + \|h_4\|^2\big) \\ & + 2\operatorname{Re}\langle h_1, h_3\rangle - \tfrac{1}{2}\operatorname{Re}\langle h_1, h_4\rangle - \tfrac{1}{2}\operatorname{Re}\langle h_2, h_3\rangle.\end{aligned}$$

By (18.1), $W(-z)^* W(\tilde{z}) = W(z)W(\tilde{z})$ and $W(-\tilde{z})^* W(z) = W(\tilde{z})W(z)$ are the same unitary $W(z+\tilde{z})$ up to the phases $e^{\mp i\theta}$, which sum to $2\cos\theta$. Replacing $(g_-, g_\circ')$ by $(W(z+\tilde{z})g_-, W(z+\tilde{z})g_\circ')$, which keeps their norms, the total becomes $\varepsilon^2 F_{\cos\theta}(g_+, g_\circ, g_-, g_\circ')$, where

$$\begin{aligned}F_k(h_1,\dots,h_4) = {} & 15\big(\|h_1\|^2 + \|h_3\|^2\big) + 13\big(\|h_2\|^2 + \|h_4\|^2\big) - \operatorname{Re}\langle h_1, h_2\rangle - \operatorname{Re}\langle h_3, h_4\rangle \\ & + k\big(4\operatorname{Re}\langle h_1, h_3\rangle - \operatorname{Re}\langle h_1, h_4\rangle - \operatorname{Re}\langle h_2, h_3\rangle\big).\end{aligned}$$

The persistent coefficient is $\frac{9}{2}+4+2\cdot\frac{13}{4}=15$ and the oscillation coefficient is $\frac{9}{2}+2\sqrt{2}+4+2(\frac{9}{4}-\sqrt{2})=13$.

F_k is $\sum_{ij}N_{ij}\,\mathrm{Re}\langle h_i,h_j\rangle$ for the real symmetric matrix N with rows $(15,-\frac{1}{2},2k,-\frac{k}{2})$, $(-\frac{1}{2},13,-\frac{k}{2},0)$, $(2k,-\frac{k}{2},15,-\frac{1}{2})$, $(-\frac{k}{2},0,-\frac{1}{2},13)$, acting on $(h_1,\dots,h_4)$. If $N-\mu I$ is positive semidefinite then $\sum(N-\mu I)_{ij}\,\mathrm{Re}\langle h_i,h_j\rangle\ge 0$, because the Gram matrix of the h_i is positive semidefinite. So the least value of F_k with $\sum\|h_i\|^2=1$ is the least eigenvalue of N, attained at $h_i=v_i\chi$ for a unit eigenvector v and any unit χ. The swap $(h_1,h_2)\leftrightarrow(h_3,h_4)$ commutes with N; on vectors (u,su), $s=\pm1$, N acts as $\begin{pmatrix}13+2q & -q/2\\ -q/2 & 13\end{pmatrix}$ with $q=1+sk$, whose eigenvalues are $13+q\pm\frac{\sqrt{5}}{2}q$. The least is $13+q(1-\frac{\sqrt{5}}{2})$, smallest at $q=1+|k|$, which is $\gamma(k)$, with eigenvector $u\propto(1,2+\sqrt{5})$. Taking $\chi=\mathrm{vac}$ and moving h_3,h_4 back by $W(z+\tilde z)^{-1}$ gives a minimizer in the domain of H.

(c) γ decreases in $|k|$, so its range on $|k|\le 1$ is $[\gamma(1),\gamma(0)]=[15-\sqrt{5},14-\frac{\sqrt{5}}{2}]$, and $\theta=0$ at $z=0$. The loss statement follows from (a). (d) If $\varepsilon\neq 0$, (c) excludes such an x; if $\varepsilon=0$, V is the scalar $v_0+\Delta v|11\rangle\langle 11|$ on the sector and every sure subject in the domain works. (e) $\gamma(\cos\theta)\le 14-\frac{\sqrt{5}}{2}<13$, since $\sqrt{5}>2$, and $\ell(\psi_{\mathrm{rel}})=13\varepsilon^2$ by Theorem 18.3(c). □

The least-loss state is about 95% oscillation: its weights are $(2+\sqrt{5})^2/\big(1+(2+\sqrt{5})^2\big)\approx 0.947$ on the oscillation and the rest on the persistent currents, split equally over the two mixed occupations. Its two clock transfers into each balanced occupation interfere destructively when $|\cos\theta|=1$; when $\cos\theta=0$ that interference is unavailable. Run as a predictability sieve with the relay's own Hamiltonian over the sure subjects [46], the dynamics selects no pointer state among them: the most stable one still leaves at a rate of order ε^2, and about 31% of its least rate, the floor $4\varepsilon^2$, is the recoil, which no choice of state inside the sector avoids.

18.3.3 Where the lost amplitude goes

The departure leaves the sector, but it does not leave the cell, and the window of Chapter 3 still reads it. Split $\Pi^{\perp}=(Q_{\mathrm{op}}+Q_{\mathrm{ne}})\otimes I+\Pi_{\mathrm{bal}}\otimes I$, with Π_{bal} the projection onto the balanced occupations.

Proposition 18.7 (The verdict of the lost amplitude)**.** *For all parameters,*

$$\|(Q_{\mathrm{op}}\otimes I)V\psi_{\mathrm{rel}}\|^2=\tfrac{1}{2}\varepsilon^2,\qquad \|(Q_{\mathrm{ne}}\otimes I)V\psi_{\mathrm{rel}}\|^2=(8+2\sqrt{2})\,\varepsilon^2,$$
$$\|(\Pi_{\mathrm{bal}}\otimes I)V\psi_{\mathrm{rel}}\|^2=(\tfrac{9}{2}-2\sqrt{2})\,\varepsilon^2,$$

summing to $13\varepsilon^2$, and the same three numbers are the limits of $\|(Q\otimes I)U(t)\psi_{\mathrm{rel}}\|^2/t^2$ as $t\to 0^+$ for the three projections.

Proof. On a mixed occupation the opposed sector is spanned by $P\otimes A$ and $A\otimes P$. The $(0,1)$ departure $a\otimes a$ has dressed pole charge zero on both cells, so it is wholly neutral. At $(1,0)$ the opposed amplitudes of $\varepsilon\big((B(1,1)A-4A)\otimes A+A\otimes(B(0,0)A-4A)\big)$ are $\varepsilon\langle P,B(b,b)A\rangle=-\frac{i}{2}\varepsilon$ on each of the two opposed vectors, by (18.2), giving $\frac{1}{2}\varepsilon^2$; the rest of that departure, $(4+2\sqrt{2})\varepsilon^2$, is neutral. The balanced part is the two clock transfers, $2(\frac{9}{4}-\sqrt{2})\varepsilon^2$. The limits follow as in Theorem 18.4(c). □

By Lemma 3.3, a neutral state passes each later tick of the loop edge with probability one half, an opposed state passes the first tick and fails the second surely, and a balanced occupation fails the first tick. So a sure subject losing weight under the relay first becomes, in about 83% of the lost weight, a neutral state that keeps writing the loop edge by chance; in about 13% a balanced one; and in under 4% an opposed one.

18.4 Frequent readings hold a sure subject

If the sector is read after each of n equal intervals, every sure subject is held, with a loss of order $1/n$ that is uniform over the sector, and the held motion is the relay compressed to the sector. In this section the reading is the projection Π, applied as an operation between intervals of evolution. That is an operation we supply; Sections 18.6 and 18.7 replace it with the cell's own record writes.

All constants below involve the norm of the interaction, which has a simple bound.

Lemma 18.8 (The size of the interaction)**.** $\|V\| \le 9|\varepsilon| + \max\big(|9\varepsilon + \nu_0|,\, |9\varepsilon + \nu_0 + \Delta\nu|\big)$.

Proof. Let G_1 act on the first occupation bit and the field as $W(-z)$ when the bit is 0 and as I when it is 1. Conjugating the operator with blocks $B(b', b)$ on the first bit and first clock by G_1 attaches $W(-z)$ to the transfers $1 \to 0$, $W(-z)^* = W(z)$ to the transfers $0 \to 1$ and nothing to the stays, which is K_1. So K_1 is unitarily equivalent to Ω written in the basis f^b_j, tensored with the identity, and its spectrum lies in $[0, 7]$; hence $\|K_1 - \frac{7}{2}\| \le \frac{7}{2}$. The same holds for K_2, with $W(\tilde{z})$ on the second bit. The recoil exchanges the two mixed occupations with coefficients of modulus 2 and unitary factors, so $\|R\| = 2$. Writing $V = \varepsilon(K_1 - \frac{7}{2}) + \varepsilon(K_2 - \frac{7}{2}) + \varepsilon R + (9\varepsilon + \nu_0)I + \Delta\nu\,|11\rangle\langle 11| \otimes I$ and bounding term by term gives the claim. □

One interval of evolution followed by the reading is $\Pi U(\Delta t)$. Let $Z(T) = e^{-iT(H_0 + \Pi V \Pi)}$, the *compressed* evolution; $H_0 + \Pi V \Pi$ is self-adjoint because $\Pi V \Pi$ is bounded and symmetric. Write $B_V = \|V\|$.

Theorem 18.9 (Frequent readings)**.** *For all parameters:*

(a) For every sure subject x, every $T \ge 0$ and every $n \ge 1$,

$$\big\|(\Pi U(T/n))^n x - Z(T)x\big\| \le \frac{T^2 B_V^2}{2n}\,\|x\|,$$

so $(\Pi U(T/n))^n x \to Z(T)x$, and the same bound holds for $(\Pi U(T/n))^n \Pi - Z(T)\Pi$ in operator norm. For a unit sure subject and every real T, $\|(\Pi U(T/n))^n x\|^2 \ge 1 - T^2 B_V^2/n$, so the read weight tends to 1.

(b) $Z(T)$ commutes with Π and moves each mode by its own phase and the free field: $Z(T)(\psi_m \otimes g) = e^{-ie_m T}\psi_m \otimes e^{-iTH_F} g$, with e_m as in Proposition 18.5. In particular $Z(T)\psi_{\mathrm{rel}} = e^{-ie_\circ T}\psi_{\mathrm{rel}}$ for every real T.

(c) For the source, with $S = \|H^2\psi_{\mathrm{rel}}\|$,

$$\big\|(\Pi U(T/n))^n \psi_{\mathrm{rel}} - e^{-ie_\circ T}\psi_{\mathrm{rel}}\big\| \le \frac{(S + e_\circ^2)\,T^2}{2n} \qquad (T \ge 0,\ n \ge 1).$$

(d) If $\varepsilon \neq 0$, the unread weight $\|\Pi U(T)\psi_{\mathrm{rel}}\|^2$ is below 1 for some $T > 0$, and for each such T the read weight $\|(\Pi U(T/n))^n \psi_{\mathrm{rel}}\|^2$ exceeds it for all large n.

Proof. (a) Since e^{-isH_0} commutes with Π and $\|U(s) - e^{-isH_0}\| \leq |s| B_V$, a sure subject y has $\|\Pi^{\perp} U(s)y\| \leq |s| B_V \|y\|$. For one interval, put $F(s) = Z(\Delta t - s)\, \Pi\, U(s)y$. Then $F(0) = Z(\Delta t)y$, $F(\Delta t) = \Pi U(\Delta t)y$, and, first for y in the domain of H_0 and then by density,

$$F'(s) = iZ(\Delta t - s)\big(\Pi V \Pi - \Pi V\big)U(s)y = -iZ(\Delta t - s)\, \Pi V \Pi^{\perp} U(s)y,$$

of norm at most $B_V \cdot sB_V\|y\|$. Integrating, $\|\Pi U(\Delta t)y - Z(\Delta t)y\| \leq \frac{1}{2}(\Delta t)^2 B_V^2 \|y\|$. Now telescope: $(\Pi U(\Delta t))^n x - Z(n\,\Delta t)x = \sum_{k<n} (\Pi U(\Delta t))^{n-1-k} \big(\Pi U(\Delta t) - Z(\Delta t)\big) Z(k\,\Delta t)x$. Each $Z(k\,\Delta t)x$ is a sure subject because Z commutes with Π, and $\Pi U(\Delta t)$ is a contraction, so the sum is at most $n \cdot \frac{1}{2}(\Delta t)^2 B_V^2 \|x\|$ with $\Delta t = T/n$. For the weight, iterate $\|\Pi^{\perp} U(\Delta t)y\| \leq |\Delta t| B_V \|y\|$: each reading removes at most $(\Delta t)^2 B_V^2$ of the remaining weight, so n readings keep at least $1 - n\,(\Delta t)^2 B_V^2$.

(b) $Z(T)$ commutes with Π because its generator does. On vectors $\psi_m \otimes g$ the generator acts as $\psi_m \otimes (H_F + e_m)g$, by Proposition 18.5.

(c) Since ψ_{rel} is in the domain of H^2, $\|U(\Delta t)\psi_{\mathrm{rel}} - (\psi_{\mathrm{rel}} - i\,\Delta t\, H\psi_{\mathrm{rel}})\| \leq \frac{1}{2} S (\Delta t)^2$, and $\Pi(\psi_{\mathrm{rel}} - i\,\Delta t\, H\psi_{\mathrm{rel}}) = (1 - ie_{\circ}\Delta t)\psi_{\mathrm{rel}}$ by Theorem 18.3(b). With $|e^{-ie_{\circ}\Delta t} - (1 - ie_{\circ}\Delta t)| \leq \frac{1}{2} e_{\circ}^2 (\Delta t)^2$, one step is within $\frac{1}{2}(S + e_{\circ}^2)(\Delta t)^2$ of the phase, and the steps telescope as in (a).

(d) Theorem 18.4(b) gives the unread loss, and (c) gives read weight tending to 1. □

Frequent reading does not freeze the sure subject. It replaces the relay on the sector by its compression, as in the Zeno dynamics of Facchi and Pascazio [17, 18], with the explicit constant $B_V^2/2$ per interval. Because $\Pi V \Pi$ is diagonal in the four modes, the readings keep not only sure subjecthood but which sure subject it is: no compressed coupling turns a persistent current into the opposite current or into the oscillation.

18.5 Postings without a record

A native posting whose record is not kept acts on the relay carrier as the single unitary $M = U_{\mathrm{nat}} \otimes I$, the coherent sum of its slices. That a posting acts on the relay carrier this way (and, with its record kept, as the instrument of Section 18.6), with the photon field untouched and at equal intervals, is supplied; Theorem 18.3(d) allows any field map for one posting, but the results below are proved for the undressed posting, and a dressed posting, which changes the posting's eigenspaces, could change them. The posting commutes with Π and with the free field, so it is a decoupling kick in the sense of Viola and Lloyd [40], and interleaving it with the relay averages the interaction over the posting's spectrum.

On the sector the posting acts by Table 2.4: M fixes $\psi_{+} \otimes g$, reverses $\psi_{-} \otimes g$, and sends $\psi_{\circ} \otimes g \mapsto \psi_{\circ}' \otimes g \mapsto -\psi_{\circ} \otimes g$, so the oscillation splits into eigenvectors $\frac{1}{2}(\psi_{\circ} \mp i\psi_{\circ}') \otimes g$ with eigenvalues $\pm i$. The relay evolution between postings is $U(\Delta t)$, and the record-erased run is $(MU(\Delta t))^n$.

Lemma 18.10 (The posting as a kick)**.**

(a) M is unitary and commutes with Π and with e^{-itH_0} for every t.

(b) $U_{\mathrm{nat}}^4 \neq I$*: for* $e = |00\rangle \otimes e_0 \otimes e_0$, $\langle e, U_{\mathrm{nat}}^4 e\rangle = \frac{1}{8}$*. So some eigenvalue of* U_{nat} *is not a fourth root of unity.*

(c) Let Λ_λ *be the spectral projections of the real orthogonal matrix* U_{nat}*, over its finite spectrum* Sp*, and let* $\bar{V} = \sum_{\lambda\in\mathrm{Sp}}(\Lambda_\lambda \otimes I)V(\Lambda_\lambda \otimes I)$*, the* pinched interaction*. Then* $\bar{V}$ *is bounded, symmetric and commutes with* M*, and for every* x *and* $N \geq 1$

$$\Big\|\frac{1}{N}\sum_{k<N} M^{-k}VM^k x - \bar{V}x\Big\| \leq \frac{c_U}{N} B_V \|x\|, \qquad c_U = \sum_{\lambda\neq\mu\in\mathrm{Sp}} \frac{2}{|1-\mu/\lambda|}.$$

Proof. (a) U_{nat} is orthogonal (Chapter 2) and commutes with Q_{al}, and M acts on the cell alone. (b) Direct exact computation: applying U_{nat} four times to e by the block formula, with rational arithmetic, gives $\frac{1}{8}$ at e. If every eigenvalue λ had $\lambda^4 = 1$, the unitary U_{nat} would satisfy $U_{\mathrm{nat}}^4 = I$. (c) A unitary matrix has no nilpotent part, so $U_{\mathrm{nat}} = \sum_\lambda \lambda\Lambda_\lambda$ with orthogonal projections Λ_λ summing to I. Then $M^{-k}VM^k = \sum_{\lambda,\mu}(\mu/\lambda)^k(\Lambda_\lambda \otimes I)V(\Lambda_\mu \otimes I)$; the terms with $\lambda = \mu$ give $\bar{V}$, and for $\lambda \neq \mu$ the geometric sum of $(\mu/\lambda)^k$ over $k < N$ is at most $2/|1-\mu/\lambda|$. □

So the averaged interaction is the pinching over the posting's whole spectrum, not a four-term average: frequent postings do not repeat with period four.

18.5.1 The first-order rates

For a sure subject x with payloads $(g_+, g_-, g_\circ, g_\circ')$, its *pinched departure* is $d(x) = \Pi^\perp \bar{V} x$. Let

$$W_{\mathrm{odd}} = W(z) - W(-z) - W(\tilde{z}) + W(-\tilde{z})$$

on $\mathscr{F}$: the odd part of the displacement by z minus the odd part of the displacement by its unit translate.

Theorem 18.11 (Pinched departures)**.** *For all parameters and every sure subject* x*:*

(a) The pinched departure splits into orthogonal mode parts,

$$\|d(x)\|^2 = r_+(g_+) + r_-(g_-) + \frac{\varepsilon^2}{192}\big(\|W_{\mathrm{odd}} g_\circ\|^2 + \|W_{\mathrm{odd}} g_\circ'\|^2\big),$$

where, with $\langle\cdot,\cdot\rangle$ *on* $\mathscr{F}$*,*

$$\begin{aligned} r_+(g) = \varepsilon^2\Big(&\tfrac{13019}{4872}\|g\|^2 + \tfrac{4381}{4872}\,\mathrm{Re}\langle W(-z)g, W(-\tilde{z})g\rangle \\ &+ \tfrac{1}{2}\,\mathrm{Re}\big(\langle W(-z)g, g\rangle + \langle W(-\tilde{z})g, g\rangle\big) - \sqrt{2}\,\mathrm{Im}\big(\langle W(-z)g, g\rangle + \langle W(-\tilde{z})g, g\rangle\big)\Big), \\ r_-(g) = \varepsilon^2\Big(&\tfrac{6635}{4872}\|g\|^2 - \tfrac{2003}{4872}\,\mathrm{Re}\langle W(\tilde{z})g, W(z)g\rangle \\ &- \tfrac{11}{58}\,\mathrm{Re}\big(\langle W(\tilde{z})g, g\rangle + \langle W(z)g, g\rangle\big)\Big). \end{aligned}$$

For every payload, $r_+(g) \geq \frac{12}{29}\varepsilon^2\|g\|^2$ *and* $r_-(g) \geq \frac{44}{309}\varepsilon^2\|g\|^2$*. On the vacuum*

$$\begin{aligned} r_+(\mathrm{vac}) &= \varepsilon^2\Big(\tfrac{13019}{4872} + \tfrac{4381}{4872}e^{-\|z\|^2}\,\mathrm{Re}\,e^{\langle z,\tilde{z}\rangle} + e^{-\|z\|^2/2}\Big), \\ r_-(\mathrm{vac}) &= \varepsilon^2\Big(\tfrac{6635}{4872} - \tfrac{2003}{4872}e^{-\|z\|^2}\,\mathrm{Re}\,e^{\langle \tilde{z},z\rangle} - \tfrac{11}{29}e^{-\|z\|^2/2}\Big), \end{aligned}$$

which at $z = 0$ *are* $\frac{32}{7}\varepsilon^2$ *and* $\frac{4}{7}\varepsilon^2$*.*

(b) For the source,

$$r_{\mathrm{src}} := \|d(\psi_{\mathrm{rel}})\|^2 = \frac{\varepsilon^2}{192}\|W_{\mathrm{odd}}\,\mathrm{vac}\|^2 = \frac{\varepsilon^2}{24}\,e^{-\|z\|^2}\big(\sinh\|z\|^2 - \mathrm{Re}\sinh\langle z,\tilde z\rangle\big) \le \frac{\varepsilon^2}{24},$$

and $r_{\mathrm{src}} = 0$ exactly when $\varepsilon = 0$ or $z = 0$.

(c) Fix $m \ge 1$ and take payloads in the domain of H_F. Then $\|\Pi^{\perp}(MU(\Delta t))^m x\|^2/(\Delta t)^2 \to \|D_m(x)\|^2$ as $\Delta t \to 0^+$, where $D_m(x) = \sum_{j<m}\Pi^{\perp}M^{m-j}VM^j x$, and

$$\big|\,\|D_m(x)\|^2 - m^2\|d(x)\|^2\big| \le 2m\,\|d(x)\|\,\beta_x + \beta_x^2$$

with β_x independent of m. So the first-order loss stays bounded in m if and only if $d(x) = 0$; otherwise it grows like m^2.

(d) If $\varepsilon \ne 0$, then $d(x) = 0$ if and only if $g_+ = g_- = 0$, $W_{\mathrm{odd}}g_\circ = 0$ and $W_{\mathrm{odd}}g'_\circ = 0$. At $z = 0$ this reads $g_+ = g_- = 0$.

(e) Among unit sure subjects whose payloads are multiples of vac*, the least value of $\|d(x)\|^2$ is r_{src}, attained by ψ_{rel}; and for $\varepsilon \ne 0$, $r_{\mathrm{src}} < \varepsilon^2\gamma(\cos\theta)$, the least unread rate of Theorem 18.6.*

Proof. To first order in Δt, $(MU(\Delta t))^m x = M^m x - i\,\Delta t\sum_{j<m}M^{m-j}HM^j x + o(\Delta t)$. The free part drops out of the departure because H_0 commutes with Π and with M. Decompose x by the posting's eigenvalues: the persistent parts lie at ± 1 and the oscillation parts at $\pm i$. The part of $\Pi^{\perp}V\Lambda_\lambda x$ in the λ-eigenspace adds up coherently over the m postings and grows like $m\lambda^m$; every other part telescopes to a bounded vector. The pinched departure $d(x)$ is exactly the sum of the resonant parts, and parts at different eigenvalues are orthogonal. This gives (c) once the resonant parts are known, with β_x the norm of the telescoped remainder.

The departures of each mode are those of Theorem 18.6: two balanced emissions, an in-place clock spread at the mode's own occupation, and the recoil. Their resonant parts are found by direct exact computation over $\mathbb{Q}(\sqrt{2}, i)$. For each departure vector y and each relevant eigenvalue λ we exhibit an eigenvector r and a vector c with $(U_{\mathrm{nat}} - \lambda)c = y - r$; then $y - r$ is orthogonal to the whole λ-eigenspace, so r is the spectral projection of y, without computing the dimension of the eigenspace.

- *Oscillation.* Besides the oscillation pair, the $\pm i$ eigenspaces of U_{nat} contain a second real pair w_1, w_2, explicit vectors with entries in $\{0, \pm 1\}$ and $\|w_1\|^2 = \|w_2\|^2 = 24$, orthogonal to the aligned sector, with $U_{\mathrm{nat}}w_1 = w_2$ and $U_{\mathrm{nat}}w_2 = -w_1$. The in-place spread and the recoil have no $\pm i$ part. The balanced emissions do, along $w_1 \mp iw_2$, with coefficients $(\pm 1 \pm i)/96$. The two halves of the oscillation emit through $W(-z), W(-\tilde z)$ (from $\psi_\circ$) and $W(\tilde z), W(z)$ (from $\psi'_\circ$); averaged over the pair, the even parts cancel and the odd parts combine into W_{odd}. This gives the oscillation term of (a) and, with (18.1), $\|W_{\mathrm{odd}}\mathrm{vac}\|^2 = 8e^{-\|z\|^2}\big(\sinh\|z\|^2 - \mathrm{Re}\sinh\langle z,\tilde z\rangle\big)$, which is (b). Since $\|\tilde z\| = \|z\|$, $\mathrm{Re}\sinh\langle z,\tilde z\rangle \ge -\sinh\|z\|^2$, and $e^{-u}\sinh u \le \frac{1}{2}$, the bound $\varepsilon^2/24$ follows. $W_{\mathrm{odd}}\mathrm{vac} = 0$ forces $\tilde z = z$, which happens only for $z = 0$.
- *Persistent currents.* The ± 1 eigenspaces of U_{nat}, each spanned by eight explicit rational vectors, reach outside the sector. The recoil has no ± 1 part. Each balanced emission of $\psi_\pm$ has a ± 1 part of squared length $\frac{5711}{9744}$, out of $\frac{13}{4}$; the in-place spread has a $+1$ part of squared length $\frac{3}{2}$ for ψ_+ and a -1 part of squared length $\frac{11}{58}$ for ψ_-, out of $\frac{9}{2}$. The Gram matrices of these parts (Table A.24) give the forms $r_\pm$. The in-place resonance has a

component orthogonal to both emission resonances, of squared length $\frac{12}{29}$ for ψ_+ and $\frac{44}{309}$ for ψ_-. Since the in-place spread carries the payload itself, with no displacement, that component contributes $\frac{12}{29}\varepsilon^2\|g\|^2$ (respectively $\frac{44}{309}\varepsilon^2\|g\|^2$) whatever the photon state; this is the lower bound.

The vacuum values follow from (18.1). (d) follows from (a), the lower bounds and $\|W_{\mathrm{odd}}g\|^2 \geq 0$; at $z = 0$, $W_{\mathrm{odd}} = 0$. (e) For payloads $c_m\mathrm{vac}$, $\|d\|^2 = |c_+|^2 r_+(\mathrm{vac}) + |c_-|^2 r_-(\mathrm{vac}) + (|c_\circ|^2 + |c_\circ'|^2)\, r_{\mathrm{src}}$, and for $\varepsilon \neq 0$, $r_\pm(\mathrm{vac}) \geq \frac{44}{309}\varepsilon^2 > \varepsilon^2/24 \geq r_{\mathrm{src}}$ and $\varepsilon^2/24 < (15-\sqrt{5})\varepsilon^2 \leq \varepsilon^2\gamma(\cos\theta)$; at $\varepsilon = 0$ every rate vanishes. □

The proposition the averaging was meant to establish is false.

Proposition 18.12 (Unrecorded postings keep no vacuum or persistent sure subject)**.** *Let $\varepsilon \neq 0$. If $z \neq 0$, every nonzero sure subject whose payloads are multiples of* vac *has $d(x) \neq 0$, so its first-order loss under unrecorded postings grows like m^2. For every z, every sure subject with $g_+ \neq 0$ or $g_- \neq 0$ has $d(x) \neq 0$, whatever its photon state.*

Proof. For vacuum payloads, $W_{\mathrm{odd}}(c\,\mathrm{vac}) = c\,W_{\mathrm{odd}}\mathrm{vac} \neq 0$ when $c \neq 0$ and $z \neq 0$, and the persistent rates are positive. The second statement is the lower bound in Theorem 18.11(a). Part (c) of that theorem gives the growth. □

The mechanism is resonance. A kick that preserves a subspace protects it to first order exactly when the departure has no component in the kick's eigenspaces [16]. Here the posting's eigenspaces reach outside the sector: the $\pm i$ eigenspace through w_1, w_2, where the oscillation's photon emissions land, and the ± 1 eigenspaces, where part of each persistent current's clock spread lands. The persistent currents have no partner to average against, since the posting fixes one and reverses the other. So the postings cut the source's loss rate from $13\varepsilon^2$ to at most $\varepsilon^2/24$, a factor of at least 312, but they stop it only when the profile is invariant under the unit translation, which means $z = 0$. One case is left.

Remark 18.13 (Open problem: is the odd gap injective?). For $z \neq 0$, is $W_{\mathrm{odd}} = W(z) - W(-z) - W(\tau_{-1}z) + W(-\tau_{-1}z)$ injective on $\mathscr{F}$? For $\varepsilon \neq 0$ this is equivalent to the statement that every nonzero sure subject has a nonzero pinched departure. It holds on the vacuum ray. A nonzero h with $W_{\mathrm{odd}}h = 0$ would give a pure oscillation $\psi_\circ \otimes h$ that unrecorded postings keep to first order.

18.5.2 The finite-time limit

The first-order law fixes the number of postings and lets the interval shrink. The physical limit fixes the total time T and lets the number of postings grow. That limit is the evolution under the pinched interaction.

Theorem 18.14 (Frequent unrecorded postings)**.** *For all parameters, every vector y of the relay carrier and every $T \geq 0$,*

$$\left\|(MU(T/n))^n y - M^n\, e^{-iT(H_0+\bar{V})}y\right\| \to 0 \qquad (n \to \infty).$$

Consequently $\|\Pi^\perp(MU(T/n))^n y\| \to \|\Pi^\perp e^{-iT(H_0+\bar{V})}y\|$, and for every sure subject x,

$$\|\Pi^\perp e^{-it(H_0+\bar{V})}x\|^2/t^2 \to \|d(x)\|^2 \qquad (t \to 0^+).$$

Proof. Compare the run with the averaged path $y_j = M^j e^{-ij\,\Delta t(H_0+\bar V)}y$, $\Delta t = T/n$, and telescope: $(MU(\Delta t))^n y - y_n = \sum_j (MU(\Delta t))^{n-1-j}\big(MU(\Delta t)y_j - y_{j+1}\big)$. By Duhamel's formula each term is $-i\,\Delta t\,M(V-\bar V)y_j + o(\Delta t)$, uniformly along the path; this uses only strong continuity of the free flow on the compact set of vectors the path visits, so no domain condition on y is needed. Split $V - \bar V = \sum_{\lambda\neq\mu}(\Lambda_\lambda \otimes I)V(\Lambda_\mu \otimes I)$. Carried through M^j, the (λ,μ) part acquires the phase $(\mu/\lambda)^j$ times a vector that changes slowly with j: the prefix $(MU(\Delta t))^{n-1-j}$ moves by at most $\Delta t\,B_V$ per step, because M commutes with the free flow, and the rest moves by a modulus of continuity. Abel summation with $|\mu/\lambda| = 1 \neq \mu/\lambda$ bounds each pair's sum by a quantity that tends to 0 when $n\,\Delta t = T$. The short-time rate is the first-order estimate for $H_0 + \bar V$ against H_0, which commutes with Π. □

Theorem 18.15 (When unrecorded postings protect)**.** *For all parameters:*

(a) $\|\Pi^\perp(MU(T/n))^n\psi_{\mathrm{rel}}\| \to 0$ *for every* $T \geq 0$ *if and only if* $z = 0$ *or* $\varepsilon = 0$. *If* $z \neq 0$ *and* $\varepsilon \neq 0$, *there are* $T > 0$ *and* $c > 0$ *with* $\|\Pi^\perp(MU(T/n))^n\psi_{\mathrm{rel}}\| \geq c$ *for all large* n.

(b) Every sure subject with $d(x) \neq 0$ *has such* T *and* c, *for every photon payload.*

(c) At $z = 0$ *and* $\varepsilon \neq 0$, $\|\Pi^\perp(MU(T/n))^n x\| \to 0$ *for every* $T \geq 0$ *if and only if* $g_+ = g_- = 0$. *The oscillation sure subjects are kept, with any photon payloads, and the persistent currents leave: under the averaged evolution their loss is* $\big(r_+(g_+) + r_-(g_-)\big)t^2 + o(t^2)$.

Proof. By Theorem 18.14 the frequent-posting loss converges to the loss under $H_0 + \bar V$, whose short-time rate is $\|d(x)\|^2$. A positive rate gives a positive loss at some small $T > 0$, and so a loss bounded below for all large n; this is (b), and with Theorem 18.11(b) the second half of (a). When $z = 0$ or $\varepsilon = 0$, every oscillation sure subject has $d = 0$, so $\bar V$ maps the oscillation part of the sector (the sector intersected with the $\pm i$ eigenspaces of M) into itself; the averaged evolution keeps it, and the loss tends to 0. This gives the first half of (a) and, with Theorem 18.11(d), part (c). □

Two forms of these statements remain open.

Remark 18.16 (Open problem: all times and the cubic term). (i) Does the limit of Theorem 18.14 hold for every real T, negative times included? (ii) At $z = 0$, for every sure subject x, are there C and $\delta > 0$ such that

$$\Big|\,\|\Pi^\perp e^{-iT(H_0+\bar V)}x\|^2 - T^2\big(r_+(g_+) + r_-(g_-)\big)\Big| \leq CT^3 \qquad (0 \leq T \leq \delta)?$$

Theorem 18.14 gives the forward limit, and the error $o(T^2)$ in place of CT^3.

18.5.3 Without photons the selection is a beat

At $z = 0$ unrecorded postings keep the oscillation and drive out the persistent currents. That looks like a selection. It is not: on the photon-free relay the averaged dynamics is a finite unitary motion, and the persistent currents come back.

Theorem 18.17 (The beat)**.** *Let* $z = 0$ *and write* $U_{\mathrm{av}}(t) = e^{-it(H_0+\bar V)}$.

(a) There is a Hermitian 64×64 *matrix* V_0 *with* $V(u \otimes \mathrm{vac}) = (V_0 u) \otimes \mathrm{vac}$ *for every* $u \in \mathscr{H}$. *With* $M_0 = \sum_\lambda \Lambda_\lambda V_0 \Lambda_\lambda$, *also Hermitian,* $U_{\mathrm{av}}(t)(u \otimes \mathrm{vac}) = (e^{-itM_0}u) \otimes \mathrm{vac}$.

(b) *Every photon-free state returns: for every u, $\delta > 0$ and T_0 there is $T \geq T_0$ with $\|U_{\mathrm{av}}(T)(u \otimes \mathrm{vac}) - u \otimes \mathrm{vac}\| < \delta$.*

(c) *For every sure subject x, with any payloads, and every $\delta > 0$ and T_0, there is $T \geq T_0$ with $\|\Pi^{\perp} U_{\mathrm{av}}(T)x\|^2 < \delta$. For vacuum payloads this holds in the frequent-posting limit as well: there is $T \geq T_0$ at which $\lim_n \|\Pi^{\perp}(MU(T/n))^n x\|^2 < \delta$.*

(d) *If $\varepsilon \neq 0$ and x has vacuum payloads with persistent content, there is $\kappa > 0$ such that at arbitrarily late T the loss exceeds κ, under U_{av} and in the frequent-posting limit.*

(e) *For every sure subject, $\|\Pi^{\perp} U_{\mathrm{av}}(t)x\|^2 = \|g_+\|^2 \ell_+(t) + \|g_-\|^2 \ell_-(t)$, where $\ell_{\pm}(t)$ is the loss of $\psi_{\pm} \otimes \mathrm{vac}$, and*

$$\frac{1}{T}\int_0^T \|\Pi^{\perp} U_{\mathrm{av}}(t)x\|^2 \, dt \;\to\; \|g_+\|^2 m_+ + \|g_-\|^2 m_-, \qquad m_{\pm} = \sum_{\mu} \|\Pi^{\perp}(F_{\mu}\psi_{\pm} \otimes \mathrm{vac})\|^2,$$

the sum over the spectral projections F_{μ} of M_0. For $\varepsilon \neq 0$, $m_+ > 0$ and $m_- > 0$, so the mean loss is positive exactly when the sure subject has persistent content.

Proof. (a) At $z = 0$ every displacement in (18.3) is the identity, the recoil is $\Gamma(\tau_{\mp 3})$, which fixes the vacuum, and H_0 annihilates $u \otimes \mathrm{vac}$. So the photon-free states form an invariant subspace on which $H_0 + \bar{V}$ acts as M_0. (b) On it the orbit is $\sum_j e^{-i\mu_j T} w_j$ over the eigenvalues μ_j of M_0. By compactness of the torus the phases $e^{-i\mu_j T}$ return simultaneously within any η of 1 at arbitrarily late times, which is the recurrence of Bocchieri and Loinger [9]. (c) for vacuum payloads follows from (b), since $\Pi^{\perp} x = 0$, and in the posting limit with Theorem 18.14. (d) By Theorem 18.15(c) and Theorem 18.11(a) the loss is positive at some $T_1 > 0$; the orbit shifted by T_1 returns near $U_{\mathrm{av}}(T_1)x$ at arbitrarily late times. (e) The time average of $e^{i(\mu_j - \mu_k)t}$ is 1 if $\mu_j = \mu_k$ and 0 otherwise, which gives the mean for vacuum payloads. If m_+ or m_- vanished, every spectral component of the corresponding mode would have zero loss, so the whole orbit would stay in the sector, against (d); hence both are positive for $\varepsilon \neq 0$. For general payloads, the only photon operator left at $z = 0$ is the recoil, carried by the two reciprocal hops. A direct exact computation, using the eight-vector bases of the ± 1 eigenspaces of U_{nat} and their Gram tables against the hops, shows that $\Lambda_{\pm 1}$ compresses both hops to zero. So on the persistent sectors $\bar{V}$ acts as $M_0 \otimes I$, the payload is a spectator, and $\|\Pi^{\perp} U_{\mathrm{av}}(t)(\psi_{\pm} \otimes g)\| = \|g\| \, \ell_{\pm}(t)^{1/2}$; the oscillation part is kept by Theorem 18.15(c), and the two persistent parts are orthogonal. This gives the split, and with it (c) and the mean for every payload. □

So at zero profile the posting modulates rather than selects. No continuum decay of Wigner–Weisskopf type is available: the only continuum on offer is the recoil, and the posting averages it out of the persistent sectors. The mean values themselves are not in closed form.

Remark 18.18 (Open problem: the persistent mean losses). At $\Delta v = 0$ and $\varepsilon \neq 0$, is $0.726 < m_+ < 0.728$ and $0.316 < m_- < 0.317$? A floating-point evaluation of the spectral formula in Theorem 18.17(e) gives $m_+ \approx 0.72700$ and $m_- \approx 0.31646$, the same for several values of ε, v_0 and ϕ; this is evidence, not a proof.

18.6 Postings that write their record hold every sure subject

A native posting also writes its current label. When that record is kept, even unread, the posting acts on the relay carrier as an instrument with one operator $V_j \otimes I$ per label j, and the record-erased posting M of Section 18.5 is the coherent sum of its branches. Keeping the record changes the outcome. Every sure subject is held at every finite time and to every order, and no projection has to be supplied.

For an interval Δt, a vector x and a word $w = (j_1, \dots, j_n) \in \mathcal{J}^n$, let

$$r_n(w) = (V_{j_n} \otimes I)\, U(\Delta t) \cdots (V_{j_1} \otimes I)\, U(\Delta t)\, x,$$

the branch of the run along which the record w is written. Summing the branches gives the record-erased run, $\sum_w r_n(w) = (MU(\Delta t))^n x$. The *record-summed loss* is $\mathscr{L}_n(x, \Delta t) = \sum_w \|\Pi^{\perp} r_n(w)\|^2$, the loss of the state $\sum_w |r_n(w)\rangle\langle r_n(w)|$ in which the record is kept and not read.

Lemma 18.19 (What a record write does to the sector)**.**

(a) $\sum_j (V_j \otimes I)^*(V_j \otimes I) = I$*, and each* $V_j \otimes I$ *commutes with* Π *and with* e^{-itH_0}*.*
(b) $(V_j \otimes I)(\psi_m \otimes g)$ *is* $(U_{\mathrm{nat}}\psi_m) \otimes g$ *if* j *is the label of* ψ_m *and* 0 *otherwise. So* $(V_j \otimes I)\Pi = 0$ *unless* j *is one of the four mode labels.*
(c) If $\Pi q = 0$*, then* $\sum_j \|(V_j \otimes I)q\|^2 \le \frac{1}{2}\|q\|^2$*, the sum over the four mode labels.*
(d) For every y*,* $\sum_j \|(V_j \otimes I)\Pi y\|\, \|(V_j \otimes I)\Pi^{\perp} y\| \le \frac{1}{\sqrt{2}}\, \|\Pi y\|\, \|\Pi^{\perp} y\|$*.*

Proof. (a) is the completeness of the slices and Lemma 8.1; $V_j \otimes I$ acts on the cell alone. (b) is Proposition 2.22. (c) By Proposition 7.1 the effects of the four mode labels sum to $Q_{\mathrm{al}} + \frac{1}{2}Q_{\mathrm{ne}}$, so the sum is $\langle q, ((Q_{\mathrm{al}} + \frac{1}{2}Q_{\mathrm{ne}}) \otimes I)q\rangle = \frac{1}{2}\|(Q_{\mathrm{ne}} \otimes I)q\|^2$. (d) By (b) only the mode labels contribute. By the Cauchy–Schwarz inequality the sum is at most $\big(\sum_j \|(V_j \otimes I)\Pi y\|^2\big)^{1/2} \big(\sum_j \|(V_j \otimes I)\Pi^{\perp} y\|^2\big)^{1/2}$ over the mode labels, and (a), (c) bound the two factors by $\|\Pi y\|$ and $\frac{1}{\sqrt{2}}\|\Pi^{\perp} y\|$. □

Part (c) is the leak one half of Chapter 7 at one posting. Part (d) turns it into decoherence: each write shrinks the coherence between the sure branch and the departed amplitude by a factor $\frac{1}{\sqrt{2}} < \frac{3}{4}$, while each interval of evolution creates at most $|\Delta t|\, \|V\|$ of it.

Theorem 18.20 (Record writes hold every sure subject)**.** *Let* x *be a sure subject and* $\delta = |\Delta t|\, \|V\| \le \frac{1}{2}$*. For every* n*,*

$$\mathscr{L}_n(x, \Delta t) \le 25\, n\, \delta^2\, \|x\|^2.$$

Hence for every real T*,* $\mathscr{L}_n(x, T/n) \le 25\, T^2 \|V\|^2 \|x\|^2 / n$ *once* $n \ge 2|T|\, \|V\|$*, so the record-summed loss tends to* 0 *and* $\sum_w \|\Pi r_n(w)\|^2 \to \|x\|^2$*.*

Proof. After k postings put $\mathscr{L}_k = \sum_w \|\Pi^{\perp} r_k(w)\|^2$ and $C_k = \sum_w \|\Pi r_k(w)\|\, \|\Pi^{\perp} r_k(w)\|$, with $\mathscr{L}_0 = C_0 = 0$, and note $\sum_w \|r_k(w)\|^2 = \|x\|^2$ by Lemma 18.19(a). For any y with $\alpha = \|\Pi y\|$ and $\beta = \|\Pi^{\perp} y\|$, the evolution satisfies $\|\Pi^{\perp} U(\Delta t) y\| \le \beta + \delta\alpha$ and $\|\Pi U(\Delta t) y\| \le \alpha + \delta\beta$, because $e^{-i\,\Delta t\, H_0}$ commutes with Π and $\|U(\Delta t) - e^{-i\,\Delta t\, H_0}\| \le \delta$. Hence $\|\Pi^{\perp} U(\Delta t) y\|^2 \le \beta^2 + 2\delta\alpha\beta + \delta^2\alpha^2$ and $\|\Pi U(\Delta t) y\|\, \|\Pi^{\perp} U(\Delta t) y\| \le (1 + \delta^2)\alpha\beta + \delta(\alpha^2 + \beta^2)$. A write preserves the summed loss,

since $\sum_j \|(V_j \otimes I)\Pi^{\perp} y\|^2 = \|\Pi^{\perp} y\|^2$, and multiplies the summed coherence by at most $\frac{3}{4}$, by Lemma 18.19(d). Summing over words,

$$\mathscr{L}_{k+1} \le \mathscr{L}_k + 2\delta C_k + \delta^2\|x\|^2, \qquad C_{k+1} \le \tfrac{3}{4}\big((1+\delta^2)C_k + \delta\|x\|^2\big) \le \tfrac{15}{16}C_k + \tfrac{3}{4}\delta\|x\|^2.$$

By induction $C_k \le 12\delta\|x\|^2$, and then $\mathscr{L}_{k+1} \le \mathscr{L}_k + 25\delta^2\|x\|^2$. □

The bound holds at every order in Δt and needs no domain condition; the constant 25 is not claimed to be sharp. The same argument applies to each of the three mode classes separately: the positive current, the negative current and the oscillation.

Corollary 18.21 (Each class is held). *Let Π_c be the projection onto one of the classes $\psi_+ \otimes \mathscr{F}$, $\psi_- \otimes \mathscr{F}$ or $\operatorname{span}\{\psi_\circ, \psi_\circ'\} \otimes \mathscr{F}$. If $\Pi_c x = x$ and $|\Delta t|\,\|V\| \le \frac{1}{2}$, then $\sum_w \|(I - \Pi_c) r_n(w)\|^2 \le 25\, n\, (|\Delta t|\,\|V\|)^2 \|x\|^2$.*

Proof. The class projections commute with every slice, a class writes only its own labels, and a vector outside the class writes those labels with weight at most one half, so Lemma 18.19(d) holds with Π_c in place of Π. The evolution estimates hold because the free flow commutes with Π_c. The proof of Theorem 18.20 then applies unchanged. □

At first order the contrast with Section 18.5 is exact: the record-erased posting and the record-writing posting move the same amplitudes, summed coherently in one case and in squared norm in the other.

Theorem 18.22 (Coherent and recorded first order). *Let $\psi = \psi_{\mathrm{rel}}$ and $\psi_k = M^k \psi$. For each word w of length m let $D_m(w)$ be the first-order departure along w, so that $\mathscr{L}_m(\psi, \Delta t)/(\Delta t)^2 \to \ell_m = \sum_w \|D_m(w)\|^2$ as $\Delta t \to 0^+$, while the record-erased first order of Theorem 18.11(c) is the coherent sum $D_m(\psi) = \sum_w D_m(w)$. Then:*

(a) $\|\Pi^{\perp} V \psi_k\|^2 = 13\varepsilon^2$ for every k, and

$$\ell_m = 13\varepsilon^2 m + 2\,\mathrm{Re} \sum_{k<m} \sum_{j<k} \big\langle S_{j,k} \Pi^{\perp} V \psi_j,\ \Pi^{\perp} V \psi_k \big\rangle,$$

where $S_{j,k}$ applies the slices of the source's own labels at the postings $j, \dots, k-1$, and $\|S_{j,k} y\|^2 \le 2^{-(k-j)} \|y\|^2$ whenever $\Pi y = 0$.

(b) $\ell_m \le (3 + 2\sqrt{2}) \cdot 13\varepsilon^2\, m$, so $(T/n)^2 \ell_n \to 0$ for every T.

(c) If $\varepsilon \ne 0$ and $z \ne 0$, then $\|D_m(\psi)\|^2/m^2 \to r_{\mathrm{src}} > 0$ while $\ell_m/m^2 \to 0$.

Proof. (a) ψ_k is $\pm\psi_{\mathrm{rel}}$ or $\pm\psi_\circ' \otimes \mathrm{vac}$, and the second has the mirror departure of the same norm. Along any word the zeroth-order run is the source's own run, so a word contributes only if it agrees with the source's labels up to the posting where the departure is created; after that, the later letters sum out by completeness. This leaves the diagonal terms and the carried cross terms. The decay of $S_{j,k}$ is Lemma 18.19(c) applied once per posting. (b) The cross terms are at most $2^{-(k-j)/2}\, 13\varepsilon^2$, and $\sum_{d \ge 1} 2^{-d/2} = 1 + \sqrt{2}$, so $\ell_m \le 13\varepsilon^2 m\big(1 + 2(1+\sqrt{2})\big)$. (c) is Theorem 18.11(b), (c) together with (b). □

This is decoherence-induced protection, of the kind obtained for Zeno limits under general quantum operations [11]. The record channel $Y \mapsto \sum_j (V_j \otimes I) Y (V_j \otimes I)^*$ commutes with Π and preserves the weight outside the sector, so it protects only by decoherence, never by projecting. What the relay adds is the quantitative criterion, and it comes from the native step: the leak one half of Proposition 7.1.

18.7 The record selects the mode

Theorem 18.20 sums over the record. Reading it gives more. The written word tells which of the four modes the sure subject is in, with that mode's Born weight, and the state given the word follows that mode's compressed motion.

For a sure subject $x = \sum_m \psi_m \otimes g_m$ write $x_m = \psi_m \otimes g_m$, and let w_m be the word mode m writes in n postings (Theorem 8.11). The weight of a word w is $\|r_n(w)\|^2$; by Lemma 18.19(a) the weights sum to $\|x\|^2$. Reading them as the probabilities of the records is the apparatus statement of the record law in Chapter 2.

Theorem 18.23 (The record selects the mode)**.** *Let x be a sure subject, T real, $n \geq 1$, $\Delta t = T/n$, $B_V = \|V\|$, and*

$$\eta_n = \frac{7T^2 B_V^2 + 6|T|B_V}{2n}.$$

(a) For each mode m, $\big\|r_n(w_m) - M^n Z(T) x_m\big\| \leq \eta_n \|x\|$.
(b) $M^n Z(T) x_m = e^{-ie_m T}(U_{\mathrm{nat}}^n \psi_m) \otimes e^{-iTH_F} g_m$: the mode's phase, the free motion of its payload, and its own postings.
(c) $\big|\, \|r_n(w_m)\|^2 - \|g_m\|^2 \big| \leq 2\eta_n \|x\|^2$, and $\sum_m \|g_m\|^2 = \|x\|^2$.
(d) $\sum_w \|r_n(w)\|^2 \leq 8\eta_n \|x\|^2$, summed over the words that are not mode words.
(e) For all vectors u, v, with $i_m = M^n Z(T) x_m$,

$$\Big| \sum_w \langle u, r_n(w)\rangle\langle r_n(w), v\rangle - \sum_m \langle u, i_m\rangle\langle i_m, v\rangle \Big| \leq 16\, \eta_n\, \|u\|\, \|v\|\, \|x\|^2,$$

and $\sum_m i_m = M^n Z(T) x$.

So as $n \to \infty$ the word of mode m has weight $\|g_m\|^2$, the other words have weight 0, the state given the word of mode m is that mode's compressed motion, and the unread state is the mixture of the four, whose coherent sum is the record-erased compressed motion.

Proof. Inside the sector each mode writes its own label at every posting, and the four labels at each posting differ (Theorem 8.11); the compressed evolution Z commutes with M and keeps each mode (Theorem 18.9(b)). So the comparison run, with $Z(\Delta t)$ in place of $U(\Delta t)$, follows x_m alone along w_m and ends at $M^n Z(T) x_m$.

Now follow the actual run along w_m, with s_k its vector after k postings. The departed part is damped at every write: an interval adds at most $|\Delta t| B_V \|x\|$ to it, and the next letter, a mode label, keeps at most $\frac{1}{\sqrt{2}} \leq \frac{3}{4}$ of it by Lemma 18.19(c). So $\|\Pi^\perp s_{k+1}\| \leq \frac{3}{4}\big(\|\Pi^\perp s_k\| + |\Delta t| B_V \|x\|\big)$, and $\|\Pi^\perp s_k\| \leq 3|\Delta t| B_V \|x\|$ at every stage. The sector part follows the compressed step: for a sure subject y and every real Δt, $\|\Pi U(\Delta t) y - Z(\Delta t) y\| \leq \frac{1}{2}(\Delta t)^2 B_V^2 \|y\|$ (the proof of Theorem 18.9(a), and its adjoint for $\Delta t < 0$), and the departed part re-enters the sector with norm at most $|\Delta t| B_V \cdot 3|\Delta t| B_V \|x\|$. Each interval therefore adds at most $\frac{7}{2}(\Delta t)^2 B_V^2 \|x\|$ to the distance between the two runs, which over n intervals is $\frac{7}{2} T^2 B_V^2 \|x\|/n$; the final departed part adds $3|T| B_V \|x\|/n$. This is (a). (b) is Theorem 18.9(b) and Table 2.4. (c) follows from (a), since $\|M^n Z(T) x_m\| = \|g_m\| \leq \|x\|$, and Definition 18.2. (d) The weights sum to $\|x\|^2 = \sum_m \|g_m\|^2$, so the rest is at most the sum of the four errors in (c). (e) Split the sum over words into mode words, which (a) controls, and the rest, which (d) controls. □

This is the Zeno-subspace picture of Facchi and Pascazio [17] with one change that matters. The instrument is not a projection onto the subspaces; it is the native posting, whose letters separate the modes only inside the sector. Outside the sector the leak one half does the work of the projection: along a selected word the departed amplitude never accumulates, and carrying it in the selected branch costs only the term of order $|T|B_V/n$ in η_n.

The posting's ± 1 and $\pm i$ eigenspaces are larger than the sector, and the averaged relay acts inside them. Whether a circulation there can stay hidden from every read, and when a single native record reads it, is the topic of Chapter 19.

Chapter 19

Protected insides II: sealing and reading

Chapter 11 left one question about the record physics open. Every native operation is real, so the direction in which a clock cell circulates is invisible to every record, and a circulation can be read only beside a non-real reference. The relay of Chapter 18 supplies one: its clock coupling, with no photon emitted, makes a circulation readable by the cell's own records (Section 19.1).

That turns privacy into a question about dynamics. A *protected inside* here is a circulating state that the native step keeps: a vector of one of the two eigenspaces of U_{nat} with eigenvalue ± 1 whose pairing $\psi^{\top}\psi$ vanishes. It persists under fast postings, and its circulation is the one feature that the postings alone never read. We ask which such insides some native record can tell from their complex conjugates once the relay acts, and what, if anything, keeps an inside sealed. Nothing native keeps an inside sealed once the clock coupling is on: symmetries seal only with the coupling off, in the limit of infinitely fast half-turn kicks, or against records that share them, and the Coulomb difference ends the last. Calling a sealed circulation an inside, and its sealing privacy, is a reading of these properties. Throughout, the relay operator, its time, the protocols that interleave it with postings and the half-turn kicks are supplied.

19.1 The relay reads a circulation

The relay reads a circulation because its energy is not odd under complex conjugation.

19.1.1 The photon-free relay

We use the relay of Chapter 18, supplied as stated in Section 18.1, only at zero photon profile and on the photon vacuum vac. There it acts on the cell alone: by Theorem 18.17(a) the interaction maps $u \otimes \mathrm{vac}$ to $(V_0 u) \otimes \mathrm{vac}$ for a Hermitian 64×64 matrix V_0, and since the free photon energy annihilates vac, the relay's unitary group maps $u \otimes \mathrm{vac}$ to $(e^{-iTV_0}u) \otimes \mathrm{vac}$ for every real T. We write V_0 out with its parameters.

Definition 19.1 (Photon-free relay)**.** For real parameters ϕ (the relay phase), ε (the clock coupling), ν_0 (a reference energy) and $\Delta\nu$ (the Coulomb difference, the extra energy of the doubly occupied configuration), the *photon-free relay* is the Hermitian 64×64 matrix

$$H = H(\phi, \varepsilon, \nu_0, \Delta\nu) = \varepsilon\,(K_1 + K_2 + R_\phi) + (2\varepsilon + \nu_0)\, I + \Delta\nu\, Q_{11} \tag{19.1}$$

on $\mathscr{H} = \mathbb{C}^{\mathcal{O}} \otimes \mathscr{H}_{\mathrm{c}} \otimes \mathscr{H}_{\mathrm{c}}$, where

$$K_1 = \sum_{y_2 = x_2} |y\rangle\langle x| \otimes B(y_1, x_1) \otimes I, \qquad K_2 = \sum_{y_1 = x_1} |y\rangle\langle x| \otimes I \otimes B(y_2, x_2),$$

$R_\phi = 2i\,e^{-i\phi}\,|01\rangle\langle 10| \otimes (D \otimes D) - 2i\,e^{i\phi}\,|10\rangle\langle 01| \otimes (D \otimes D)$ is the reciprocal hop, and $Q_{11} = |11\rangle\langle 11| \otimes I$.

K_1 changes the first occupation bit and moves the first clock, with the clock blocks $B(b', b)$ of Section 18.1; K_2 does the same for the second; the hop exchanges the two mixed occupations and multiplies by the dressed parity. So $H = V_0$, and e^{-iTH} is the relay's own evolution of the cell when no photon is present. Its clock blocks are the native clock read in the occupation chart (proof of Proposition 18.1(a)), so its coordinates are those of the cell. The Coulomb difference Δv is a term the native posting does not supply; every result below that needs $\Delta v \neq 0$ rests on it.

19.1.2 When a flow keeps real states real

Write $\overline{M}$ for the entrywise complex conjugate of a matrix, and call

$$M^{\mathrm{ev}} = \tfrac{1}{2}\big(M + \overline{M}\big), \qquad M^{\mathrm{od}} = \tfrac{1}{2i}\big(M - \overline{M}\big)$$

its *conjugation-even* and *conjugation-odd parts*, so that $M = M^{\mathrm{ev}} + iM^{\mathrm{od}}$ with both parts real. For Hermitian M, M^{ev} is real symmetric and M^{od} is real antisymmetric. The *pairing* of a vector is $\psi^{\top}\psi$, as in Chapter 11: a unit vector has $|\psi^{\top}\psi| = 1$ exactly when it is real up to a phase, and $\psi^{\top}\psi = 0$ exactly when it is orthogonal to its conjugate.

Theorem 19.2 (The conjugation-even part decides)**.** *Let M be a Hermitian $N \times N$ matrix and $U_T = e^{-iTM}$.*

(a) If v, w are orthonormal real vectors with $w^{\top}M^{\mathrm{ev}}v \neq 0$, there is $\delta > 0$ such that $|(U_T v)^{\top}(U_T v)| < 1$ for $0 < T < \delta$.
(b) $|(U_T u)^{\top}(U_T u)| = 1$ for every real unit vector u and every real T if and only if $M^{\mathrm{ev}} = cI$ for a real c.

Proof. (a) The vectors $\overline{U_T v}$ and $\overline{U_T w}$ are orthonormal, so Bessel's inequality gives $|(U_T v)^{\top}(U_T v)|^2 + |(U_T w)^{\top}(U_T v)|^2 \leq \|U_T v\|^2 = 1$, since $(U_T w)^{\top}(U_T v) = \langle \overline{U_T w}, U_T v\rangle$. For Hermitian M, $M^{\top} = \overline{M}$, so the derivative of $(U_T w)^{\top}(U_T v) = w^{\top}U_T^{\top}U_T v$ at $T = 0$ is $-i\, w^{\top}(M^{\top} + M)v = -2i\, w^{\top}M^{\mathrm{ev}}v \neq 0$. The second term of Bessel's inequality is therefore positive for small $T > 0$, and the first is below 1. (b) If $M^{\mathrm{ev}} = cI$ then $M = cI + iM^{\mathrm{od}}$ and $U_T = e^{-icT}e^{TM^{\mathrm{od}}}$ with $e^{TM^{\mathrm{od}}}$ real orthogonal, so $U_T u$ is a phase times a real unit vector. Conversely, by (a) the real symmetric matrix M^{ev} has $w^{\top}M^{\mathrm{ev}}v = 0$ for every orthonormal real pair, so every real unit vector is an eigenvector and M^{ev} is a multiple of I. □

Theorem 11.16(d) of Chapter 11 is the case of a real symmetric M; the criterion needs no assumption on the odd part.

19.1.3 The relay's even part

Split each clock block into real and imaginary parts. The real parts have entries in $\mathbb{Q}(\sqrt{2})$; write $\mathrm{Re}(K_1 + K_2) = K_{\mathrm{re}} + \sqrt{2}\,K'_{\mathrm{re}}$ with $K_{\mathrm{re}}, K'_{\mathrm{re}}$ rational, and $\mathrm{Im}(K_1 + K_2) = K_{\mathrm{im}}$, which is rational. Put $R_+ = \big(|01\rangle\langle 10| + |10\rangle\langle 01|\big) \otimes (D \otimes D)$ and $R_- = \big(|01\rangle\langle 10| - |10\rangle\langle 01|\big) \otimes (D \otimes D)$. Since $\mathrm{Re}(2ie^{-i\phi}) = 2\sin\phi$ and $\mathrm{Im}(2ie^{-i\phi}) = 2\cos\phi$, Definition 19.1 gives

$$H^{\mathrm{ev}} = \varepsilon(K_{\mathrm{re}} + \sqrt{2}\,K'_{\mathrm{re}}) + 2\varepsilon\sin\phi\, R_+ + (2\varepsilon + v_0)\, I + \Delta v Q_{11}, \qquad H^{\mathrm{od}} = \varepsilon K_{\mathrm{im}} + 2\varepsilon\cos\phi\, R_-. \quad (19.2)$$

The real part of the first clock block at occupation $(0,0)$ shows that the clock coupling is not scalar:

$$\operatorname{Re} B(0,0) = \begin{pmatrix} 3-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{\sqrt{2}}{2} & 4+\frac{\sqrt{2}}{2} & -1 & 0 \\ 0 & -1 & 4+\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 & -\frac{\sqrt{2}}{2} & 3-\frac{\sqrt{2}}{2} \end{pmatrix}. \tag{19.3}$$

Theorem 19.3 (The relay's even part is scalar only when it is switched off)**.** *H^{ev} is a real multiple of I if and only if $\varepsilon = 0$ and $\Delta v = 0$, for every ϕ and v_0. Consequently:*

(a) if $\varepsilon \neq 0$, the real basis vector at occupation $(0,0)$ and clock indices $(1,0)$ satisfies $|(e^{-iTH}u)^{\top}(e^{-iTH}u)| < 1$ for all small $T > 0$;

(b) if $\varepsilon = 0$ and $\Delta v \neq 0$, the real vector $u = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big) \otimes e_0 \otimes e_0$ reaches pairing 0 at $T = \pi/(2|\Delta v|)$.

Proof. For $\varepsilon \neq 0$ the matrix $\varepsilon(K_{\mathrm{re}} + \sqrt{2}K'_{\mathrm{re}})$ has the entry $-\varepsilon$ between the coordinate at occupation $(0,0)$ with clock indices $(1,0)$ and the coordinate at occupation $(1,0)$ with clock indices $(0,0)$: the real part of the block $B(1,0)$ that moves a charge with the first clock has the rational entry -1 there, and the hop and the Coulomb term vanish between these coordinates. So H^{ev} has a nonzero off-diagonal entry, and Theorem 19.2(a) with these two basis vectors gives (a). For $\varepsilon = 0$, $H = v_0 I + \Delta v Q_{11}$ is diagonal and scalar exactly when $\Delta v = 0$. (b) is Theorem 11.16(d) for the real symmetric H and its real eigenvectors $|00\rangle \otimes e_0 \otimes e_0$ and $|11\rangle \otimes e_0 \otimes e_0$, with energies v_0 and $v_0 + \Delta v$. □

The same holds for the posting-averaged relay $\bar{H}$ of Section 19.2: its even part is scalar exactly when $\varepsilon = \Delta v = 0$, and explicit real eigenvectors of U_{nat} with eigenvalue 1 are carried off the real states for every other choice.

19.1.4 The cell's own record law reads the circulation

The relay must also bring the circulation into the record law. Let $f_\gamma = |00\rangle \otimes p \otimes \gamma$ be the circulating cell of Chapters 9 and 11, with $\gamma = \frac{1}{2}(1, i, -1, -i)$ and $\bar{f}_\gamma$ its reverse. For a unitary U on the cell, a unit vector ψ and a record word w we call

$$g_w(U,\psi) = \langle U\psi, E_w U\psi\rangle - \langle U\bar{\psi}, E_w U\bar{\psi}\rangle \tag{19.4}$$

the *conjugation gap* of w after U. It is the difference of the record laws of the genesis states $\delta_0 \otimes U\psi$ and $\delta_0 \otimes U\bar{\psi}$ at w. We say a record *reads* ψ after a family of protocols U if its gap is nonzero for some member, and that a family *seals* ψ if every gap vanishes for every member and every record word.

Lemma 19.4 (First-order gap)**.** *Let M be Hermitian, E real symmetric and $\chi = \chi_{\mathrm{r}} + i\chi_{\mathrm{i}}$ with $\chi_{\mathrm{r}}, \chi_{\mathrm{i}}$ real. The derivative at $T = 0$ of $\langle e^{-iTM}\chi, Ee^{-iTM}\chi\rangle - \langle e^{-iTM}\bar{\chi}, Ee^{-iTM}\bar{\chi}\rangle$ is*

$$2i\,\langle \chi, [M^{\mathrm{ev}}, E]\chi\rangle = -4\,\chi_{\mathrm{r}}^{\top}[M^{\mathrm{ev}}, E]\,\chi_{\mathrm{i}}.$$

Proof. The derivative of $\langle e^{-iTM}\chi, Ee^{-iTM}\chi\rangle$ at 0 is $\langle\chi, i[M,E]\chi\rangle$. Since E is real, the same quantity for $\bar\chi$ is the conjugate of $\langle\chi, -i[\overline{M},E]\chi\rangle$, which is real because $\overline{M} = M^\top$ is Hermitian. The difference is $\langle\chi, i[M+\overline{M},E]\chi\rangle = 2i\langle\chi,[M^{\mathrm{ev}},E]\chi\rangle$. The commutator $[M^{\mathrm{ev}},E]$ is real antisymmetric, so its quadratic forms at $\chi_{\mathrm r}$ and $\chi_{\mathrm i}$ vanish and $\langle\chi,[M^{\mathrm{ev}},E]\chi\rangle = 2i\,\chi_{\mathrm r}^\top[M^{\mathrm{ev}},E]\chi_{\mathrm i}$. □

The odd part of the flow never enters at first order. For $\psi = f_\gamma$ the support is occupation $(0,0)$, and every effect E_w is block diagonal in the occupation, so only the $(0,0)$ block of H^{ev} enters, namely $\varepsilon(R\otimes I + I\otimes R) + (2\varepsilon+v_0)I$ with $R = \operatorname{Re} B(0,0)$; the hop and the Coulomb term vanish there.

Theorem 19.5 (The relay delivers the circulation to the record law)**.** *Let $U_T = e^{-iTH}$ with $H = H(\phi,\varepsilon,v_0,\Delta v)$. For $y\in\mathcal O$ let w_y be the record of the posting $(0,0)\to(1,1)$, label $(1,0,1)$, followed by the posting $(1,1)\to y$.*

(a) The derivative of $T\mapsto g_{w_y}(U_T,f_\gamma)$ at $T=0$ is $+\varepsilon/4$ for $y=(0,0)$ and $(1,0)$, and $-\varepsilon/4$ for $y=(0,1)$ and $(1,1)$.
(b) For every single label j, the derivative of $T\mapsto g_{(j)}(U_T,f_\gamma)$ at $T=0$ is 0.
(c) If $\varepsilon\neq 0$, there is $\delta>0$ with $g_{w_{(1,1)}}(U_T,f_\gamma)\neq 0$ for $0<T<\delta$.
(d) If $\varepsilon = 0$, then $g_w(U_T,f_\gamma) = 0$ for every record word w and every T.

Proof. Write $f_\gamma = f_{\mathrm r} + if_{\mathrm i}$ with $f_{\mathrm r} = |00\rangle\otimes p\otimes\gamma_{\mathrm r}$, $f_{\mathrm i} = |00\rangle\otimes p\otimes\gamma_{\mathrm i}$, $\gamma_{\mathrm r} = \frac12(1,0,-1,0)$, $\gamma_{\mathrm i} = \frac12(0,1,0,-1)$. By Lemma 19.4 the rate is $-4f_{\mathrm r}^\top[H^{\mathrm{ev}},E_w]f_{\mathrm i}$. (a) A direct exact computation of the four effects on $f_{\mathrm r}$ and $f_{\mathrm i}$ gives, for $y=(1,1)$, $E_{w_y}f_{\mathrm r} = 0$ and $E_{w_y}f_{\mathrm i} = \frac14 f_{\mathrm i}$: the record reads only the sine quadrature of the second clock. Then $f_{\mathrm r}^\top[H^{\mathrm{ev}},E]f_{\mathrm i} = \frac14 f_{\mathrm r}^\top H^{\mathrm{ev}}f_{\mathrm i} = \frac14\varepsilon\big((p^\top Rp)(\gamma_{\mathrm r}^\top\gamma_{\mathrm i}) + (p^\top p)(\gamma_{\mathrm r}^\top R\gamma_{\mathrm i})\big) = \frac14\varepsilon\cdot\frac14$, since $\gamma_{\mathrm r}^\top\gamma_{\mathrm i} = 0$ and $\gamma_{\mathrm r}^\top R\gamma_{\mathrm i} = \frac14$ from (19.3). The rate is $-\varepsilon/4$. The other three words are computed the same way; the sign is fixed by whether the second clock ends in the one channel ($y_2=1$) or the zero channel. (b) On occupation $(0,0)$ a single-label effect acts on the clocks as a product of circulant matrices; such a product acts on $p\otimes\operatorname{span}\{\gamma_{\mathrm r},\gamma_{\mathrm i}\}$ as a scalar, so the commutator term vanishes, which a direct exact computation over all fifteen labels confirms. (c) follows from (a). (d) For $\varepsilon=0$, H acts on occupation $(0,0)$ as v_0, so $U_Tf_\gamma = e^{-iTv_0}f_\gamma$ and $U_T\bar f_\gamma = e^{-iTv_0}\bar f_\gamma$; Theorem 9.7 gives equal record laws. □

The first length at which a record tells the two directions apart at first order is two; a direct exact computation over all 225 two-label words finds the four w_y to be the only ones with a nonzero first-order rate.

The relay also prepares a reference from a real state, beyond the bound that real operations obey (Theorem 11.11(b)).

Proposition 19.6 (The relay beats the real-preparation bound)**.** *Let $\varepsilon\neq 0$ and write $f_\gamma = f_{\mathrm r}+if_{\mathrm i}$ as above. For $u = \sqrt2\,f_{\mathrm i}$ if $\varepsilon>0$ and $u = \sqrt2\,f_{\mathrm r}$ if $\varepsilon<0$, u is a real unit vector and there is $\delta>0$ with $|\langle f_\gamma, e^{-iTH}u\rangle|^2 > \frac12$ for $0<T<\delta$. The fidelity starts at $\frac12$ and rises at rate $|\varepsilon|/2$.*

Proof. At $T=0$ the fidelity is $2|\langle f_{\mathrm r}+if_{\mathrm i}, f_{\mathrm i}\rangle|^2 = 2\|f_{\mathrm i}\|^4 = \frac12$. Its derivative is $\pm2f_{\mathrm r}^\top H^{\mathrm{ev}}f_{\mathrm i}$, and the computation in the proof of Theorem 19.5 gives $f_{\mathrm r}^\top H^{\mathrm{ev}}f_{\mathrm i} = \varepsilon/4$; the choice of u makes the derivative $|\varepsilon|/2>0$. □

Remark 19.7 (Open: the averaged delivery and the best preparation). Two questions about the circulating genesis stay open. First, whether the posting-averaged relay $\bar{H}$ of Section 19.2 delivers its circulation: whether for $\varepsilon \neq 0$ or $\Delta v \neq 0$ there are a record word w and a time T with $g_w(e^{-iT\bar{H}}, f_\gamma) \neq 0$. Its first-order rate is given by Lemma 19.4 with $\bar{H}$ in place of H; deciding it needs the spectral projections of U_{nat} applied to f_γ, which does not lie in the two rational eigenspaces used below. Second, the supremum of $|\langle f_\gamma, e^{-iTH}u\rangle|^2$ over real unit u and times T; Proposition 19.6 shows only that it exceeds $\frac{1}{2}$.

19.2 What fast postings keep is read

Fast coherent postings average the relay, and on what the postings keep the average is the relay's compression. A circulation that the postings keep is read whenever the clock coupling or the Coulomb difference is on.

19.2.1 The kept eigenspaces and the averaged relay

The native step U_{nat} is real orthogonal (Proposition 2.6); let Λ_λ be its spectral projection for the eigenvalue λ. The *posting-averaged relay* is

$$\bar{H} = \bar{H}(\phi, \varepsilon, v_0, \Delta v) = \sum_\lambda \Lambda_\lambda \, H(\phi, \varepsilon, v_0, \Delta v) \, \Lambda_\lambda. \tag{19.5}$$

Chapter 18 shows that relay intervals interleaved with ever more frequent coherent applications of U_{nat} approach the flow of $\bar{H}$, which is the pinched interaction $\bar{V}$ of Theorem 18.14 on the photon vacuum at zero profile; we use $e^{-iT\bar{H}}$ as that fast-posting limit (for $T \geq 0$; every gap below is analytic in T, so a record that reads at some T also reads at some $T > 0$). For $s = \pm 1$ the *kept eigenspaces* are

$$\mathrm{Fix}_s = \{v \in \mathscr{H} \ : \ U_{\mathrm{nat}} v = s\, v\}.$$

A *circulating kept state* of Fix_s is $\psi = \psi_{\mathrm{r}} + i\psi_{\mathrm{i}}$ with $\psi_{\mathrm{r}}, \psi_{\mathrm{i}}$ real vectors of Fix_s, $\psi_{\mathrm{r}} \perp \psi_{\mathrm{i}}$ and $\|\psi_{\mathrm{r}}\| = \|\psi_{\mathrm{i}}\| \neq 0$. Since U_{nat} is real, the real and imaginary parts of a kept vector are kept, so these are exactly the nonzero vectors of Fix_s with $\psi^{\top}\psi = 0$. They are the protected insides of this chapter.

Proposition 19.8 (The averaged relay on kept content). *Let $s = \pm 1$.*

(a) Fix_1 *and* Fix_{-1} *have dimension* 8 *and are spanned by rational vectors. Each contains circulating unit states, and Λ_s is the real orthogonal projection onto* Fix_s.
(b) $\bar{H}$ commutes with U_{nat}, the flow $e^{-iT\bar{H}}$ keeps Fix_s *for every T, and $\bar{H}v = \Lambda_s H v$ for $v \in \mathrm{Fix}_s$.*
(c) For $\psi \in \mathrm{Fix}_s$ the record law of $e^{-iT\bar{H}}\psi$ at a word w is $\langle e^{-iT\bar{H}}\psi, \Lambda_s E_w \Lambda_s \, e^{-iT\bar{H}}\psi\rangle$. Two records with the same compression $\Lambda_s E_w \Lambda_s$ have the same law on kept content at every T.

Proof. (a) A direct exact computation gives eight rational vectors in each eigenspace, each equal to 1 at its own free coordinate and 0 at the other seven, together with rational certificates that every eigenvector is their combination. If $\psi_{\mathrm{r}}, \psi_{\mathrm{i}}$ are orthogonal nonzero real kept vectors, then $\psi_{\mathrm{r}}/(\sqrt{2}\|\psi_{\mathrm{r}}\|) + i\psi_{\mathrm{i}}/(\sqrt{2}\|\psi_{\mathrm{i}}\|)$ is a circulating unit state. Since U_{nat} is real, $\overline{\mathrm{Fix}_s} = \mathrm{Fix}_s$, so the orthogonal projection Λ_s is real. (b) $\bar{H}$ commutes with every Λ_λ, hence with $U_{\mathrm{nat}} = \sum_\lambda \lambda \Lambda_\lambda$ and with its flow; and $\Lambda_\lambda v = \delta_{\lambda s} v$ on Fix_s. (c) follows from (b). □

Theorem 19.9 (The first-order rate on kept content). *Let $s = \pm 1$, let $\psi_{\mathrm{r}}, \psi_{\mathrm{i}}$ be real vectors of Fix_s and w a record word. With $A_s = \Lambda_s H^{\mathrm{ev}} \Lambda_s$ and $E_s = \Lambda_s E_w \Lambda_s$, the derivative of $T \mapsto g_w(e^{-iT\bar{H}}, \psi_{\mathrm{r}} + i\psi_{\mathrm{i}})$ at $T = 0$ is $-4\,\psi_{\mathrm{r}}^{\top}[A_s, E_s]\,\psi_{\mathrm{i}}$, and*

$$A_s = \varepsilon\,\Lambda_s K_{\mathrm{re}} \Lambda_s + \varepsilon\sqrt{2}\,\Lambda_s K'_{\mathrm{re}} \Lambda_s + (2\varepsilon + \nu_0)\,\Lambda_s + \Delta\nu\,\Lambda_s Q_{11} \Lambda_s.$$

Both hop parts compress to zero, $\Lambda_s R_+ \Lambda_s = \Lambda_s R_- \Lambda_s = 0$, on both eigenspaces, and $\Lambda_1 K'_{\mathrm{re}} \Lambda_1 = 0$.

Proof. Since Λ_s is real, $\bar{H}^{\mathrm{ev}} v = \Lambda_s H^{\mathrm{ev}} v$ for every real $v \in \mathrm{Fix}_s$, and Lemma 19.4 gives $-4\psi_{\mathrm{r}}^{\top}[\bar{H}^{\mathrm{ev}}, E_w]\psi_{\mathrm{i}} = -4\big((\bar{H}^{\mathrm{ev}}\psi_{\mathrm{r}})^{\top} E_w \psi_{\mathrm{i}} - (E_w \psi_{\mathrm{r}})^{\top} \bar{H}^{\mathrm{ev}} \psi_{\mathrm{i}}\big)$, which is $-4\psi_{\mathrm{r}}^{\top}[A_s, E_s]\psi_{\mathrm{i}}$ because $\psi_{\mathrm{r}}^{\top}\Lambda_s = \psi_{\mathrm{r}}^{\top}$ and $\Lambda_s \psi_{\mathrm{i}} = \psi_{\mathrm{i}}$. The form of A_s is (19.2) compressed. The vanishing compressions are direct exact computations with the rational bases and their Gram matrices. □

So the relay phase ϕ never enters the first-order rate on kept content, and ν_0 drops out of every commutator.

Theorem 19.10 (Fast postings do not seal what they keep). *(a) There are rational real vectors $r_+ \perp q_+$ in Fix_1 with $\|r_+\|^2 = 8$ and $\|q_+\|^2 = \frac{1408}{9}$ such that the circulating unit state $\zeta_+ = \frac{1}{4} r_+ + i\frac{3}{16\sqrt{11}} q_+$ has first-order gap rates*

$$-\frac{3}{16\sqrt{11}}\Big(\frac{53}{21}\,\varepsilon - \frac{1}{14}\,\Delta\nu\Big) \quad \text{and} \quad \frac{3}{16\sqrt{11}} \cdot \frac{1}{14}\,\Delta\nu$$

on the single records of the label $(0,0,0)$ and of the posting $(1,0) \to (0,0)$, label $(0,1,-1)$, for every $\phi, \varepsilon, \nu_0, \Delta\nu$.

(b) There are rational real vectors $r_- \perp q_-$ in Fix_{-1} with $\|r_-\|^2 = \frac{88}{3}$ and $\|q_-\|^2 = 15$ such that $\zeta_- = r_-/\sqrt{176/3} + i q_-/\sqrt{30}$ has first-order gap rate

$$-\frac{4}{\sqrt{176/3}\,\sqrt{30}}\Big(\frac{9809 - 11297\sqrt{2}}{24360}\,\varepsilon + \frac{613}{2030}\,\Delta\nu\Big)$$

on the record of the label $(0,0,0)$. It is nonzero when exactly one of ε and $\Delta\nu$ vanishes.

(c) For every ϕ and ν_0: some circulating unit state of Fix_1 or Fix_{-1} is read by some record after $e^{-iT\bar{H}}$ for some T if and only if $\varepsilon \neq 0$ or $\Delta\nu \neq 0$. If $\varepsilon \neq 0$ or $\Delta\nu \neq 0$, one of the two records of (a) reads ζ_+ at every small $T > 0$.

Proof. (a) and (b) are Theorem 19.9 with exact tables: $r_+^{\top}[A_1, E_1]q_+ = \varepsilon\kappa_0 + \Delta\nu\kappa_{11}$ with $(\kappa_0, \kappa_{11}) = (\frac{53}{21}, -\frac{1}{14})$ for the label $(0,0,0)$ and $(0, -\frac{1}{14})$ for $(0,1,-1)$, and the $\sqrt{2}$ coupling contributes nothing on Fix_1; on Fix_{-1} the three coefficients are $\frac{9809}{24360}, -\frac{11297}{24360}$ for the $\sqrt{2}$ part, and $\frac{613}{2030}$. The rate of ζ_+ is $-4 \cdot \frac{1}{4} \cdot \frac{3}{16\sqrt{11}}$ times this, and similarly for ζ_-. Since $\sqrt{2}$ is irrational, $9809 - 11297\sqrt{2} \neq 0$. (c) The two rate vectors $(\frac{53}{21}, -\frac{1}{14})$ and $(0, -\frac{1}{14})$ in $(\varepsilon, \Delta\nu)$ are independent, so one rate is nonzero unless $\varepsilon = \Delta\nu = 0$, and a nonzero derivative at 0 of a function vanishing at 0 gives a nonzero gap for small $T > 0$. At $\varepsilon = \Delta\nu = 0$, $H = \nu_0 I$ and $\bar{H} = \nu_0 \sum_\lambda \Lambda_\lambda = \nu_0 I$; the flow is a phase, and Theorem 9.7 makes every record blind at every T. □

The Coulomb difference alone suffices: at $\varepsilon = 0$, $\Delta\nu \neq 0$ the record of $(1,0) \to (0,0)$ reads ζ_+, while the bare relay at $\varepsilon = 0$ leaves the circulating genesis unread (Theorem 19.5(d)). Fast postings remove the hop from both kept sectors and the $\sqrt{2}$ coupling from Fix_1, as a decoupling sequence removes the parts of an interaction that it averages [39].

19.3 A clock symmetry seals, but only without clock coupling

There is a symmetry of everything native: the joint half-turn of the two clocks. It seals every kept circulation whose real and imaginary directions lie in opposite parity sectors, at every time and posting rate, but only while the relay's clock coupling is off. With the coupling on and no Coulomb difference, every kept circulation is read.

19.3.1 The joint half-turn

The half-turn L^2 of one clock cell sends e_i to e_{i+2}. In the pole basis it is $+1$ on the pole lines p, a and -1 on the zero-pole plane spanned by P, A, since $LP = -A$ and $LA = P$. Define the *joint half-turn*

$$\Xi = I_{\mathcal{O}} \otimes L^2 \otimes L^2. \tag{19.6}$$

It is the signed cell operator of the joint half-turn $\tau^{(12)}$ of Chapter 5, whose sign is 1 on every input (Example 5.8(d)). At a mixed occupation the dressed half-turn $DL^2D = -L^2$ is the one that is $+1$ on the dressed pole lines; the two cells' signs cancel in Ξ, so Ξ is also the product of the two cells' dressed half-turns at every occupation.

Proposition 19.11 (The joint half-turn commutes with everything native)**.** *Ξ is a real symmetric involution. It commutes with every step block M_{yx}, hence with U_{nat}, its spectral projections, every slice V_j and every effect E_w. It commutes with Q_{11}, so $\Xi H = H\Xi$ and $\Xi\bar{H} = \bar{H}\Xi$ when $\varepsilon = 0$.*

Proof. By Example 5.8(d) each single half-turn reproduces every step block up to the sign -1 on balanced inputs, so the joint half-turn $\tau^{(12)}$ has sign 1 on every input, and the third condition of (5.2) with π the identity reads $\Xi M_{yx}\Xi = M_{yx}$. Ξ is the permutation matrix of an involution, so it is real symmetric with $\Xi^2 = I$. Slices are sums of blocks, effects are products of slices and their adjoints, and spectral projections are polynomials in U_{nat}. Q_{11} is occupation diagonal, and at $\varepsilon = 0$, $H = v_0 I + \Delta v Q_{11}$. □

The clock couplings K_1, K_2 move one clock between occupations, and they break the symmetry.

Theorem 19.12 (The parity seal)**.** *Let ψ satisfy $\bar{\psi} = c\,\Xi\psi$ with $|c| = 1$. Then for every ϕ, v_0, Δv, every T and every record word w, $g_w(e^{-iT\bar{H}(\phi,0,v_0,\Delta v)}, \psi) = 0$. The same holds for every finite protocol that alternates bare relay intervals $e^{-i\tau H(\phi,0,v_0,\Delta v)}$ with coherent steps U_{nat}, in any order and with any spacing. Both kept eigenspaces contain circulating unit states $\psi = u/\sqrt{2\alpha} + iu'/\sqrt{2\beta}$ of this kind, with u, u' real kept vectors, $\Xi u = u$, $\Xi u' = -u'$, $\|u\|^2 = \alpha$, $\|u'\|^2 = \beta$.*

Proof. Let U be the protocol unitary; it commutes with Ξ by Proposition 19.11. Then $\langle U\bar{\psi}, E_w U\bar{\psi}\rangle = \langle U\Xi\psi, E_w U\Xi\psi\rangle = \langle U\psi, \Xi E_w \Xi\, U\psi\rangle = \langle U\psi, E_w U\psi\rangle$, the phase c cancelling. For the witnesses, $\bar{\psi} = u/\sqrt{2\alpha} - iu'/\sqrt{2\beta} = \Xi\psi$, and explicit rational vectors with these parities exist in both eigenspaces (a direct exact computation). □

The seal is the decoherence-free pattern [44, 28]: a symmetry shared by the dynamics and every reader protects the relation it fixes.

19.3.2 The clock coupling breaks the seal

A seal is a statement at every time, so it constrains every order of the Heisenberg evolution of every record.

Lemma 19.13 (Sealed states and the Heisenberg levels). *Let M be Hermitian and $\psi = \psi_{\mathrm{r}} + i\psi_{\mathrm{i}}$ with $\psi_{\mathrm{r}}, \psi_{\mathrm{i}}$ real. If $g_w(e^{-iTM}, \psi) = 0$ for every T, then $\psi_{\mathrm{r}}^{\top} E_w^{(n)} \psi_{\mathrm{i}} = \psi_{\mathrm{i}}^{\top} E_w^{(n)} \psi_{\mathrm{r}}$ for every $n \geq 0$, where $E_w^{(n)} = (i \operatorname{ad} M)^n E_w$.*

Proof. With $E_w(T) = e^{iTM} E_w e^{-iTM}$ the gap is $\langle \psi, E_w(T)\psi \rangle - \langle \bar{\psi}, E_w(T)\bar{\psi} \rangle = 2i\big(\psi_{\mathrm{r}}^{\top} E_w(T)\psi_{\mathrm{i}} - \psi_{\mathrm{i}}^{\top} E_w(T)\psi_{\mathrm{r}}\big)$. It is an analytic function of T vanishing identically, and its n-th derivative at 0 is $2i(\psi_{\mathrm{r}}^{\top} E_w^{(n)} \psi_{\mathrm{i}} - \psi_{\mathrm{i}}^{\top} E_w^{(n)} \psi_{\mathrm{r}})$. □

Theorem 19.14 (The clock coupling reads every kept circulation). *Let $\varepsilon \neq 0$ and $\Delta v = 0$. For every ϕ and v_0, every circulating kept state of Fix_1 or Fix_{-1} is read by some record after $e^{-iT\bar{H}(\phi,\varepsilon,v_0,0)}$ for some T.*

Proof. Suppose $\psi = \psi_{\mathrm{r}} + i\psi_{\mathrm{i}}$ is sealed. In coordinates on a rational basis of Fix_s, each condition of Lemma 19.13 is a linear equation in the 28 Plücker coordinates of the plane spanned by ψ_{r} and ψ_{i}, with coefficients in $\mathbb{Z}[\sqrt{2}, i]$ after scaling. A direct exact computation collects 28 such rows: on Fix_1 from levels 1 to 4 of twelve record words, on Fix_{-1} from levels 1, 3 and 5 of thirteen words. An integer certificate in $\mathbb{Z}[\sqrt{2}]$ shows that these rows force every Plücker coordinate to vanish, so ψ_{r} and ψ_{i} are parallel. That contradicts $\psi_{\mathrm{r}} \perp \psi_{\mathrm{i}}$ with $\|\psi_{\mathrm{r}}\| = \|\psi_{\mathrm{i}}\| \neq 0$. □

How the coupling breaks the parity differs between the two eigenspaces. On Fix_1 the compressed even part A_1 commutes with Ξ, and only the odd clock part $\varepsilon K_{\mathrm{im}}$ breaks it; on Fix_{-1} the compressed odd part vanishes and the even clock part itself mixes the parity sectors (Proposition 19.17 below). The first case has a visible consequence.

Proposition 19.15 (Split planes on Fix_1 are invisible at first order). *Let u, u' be real vectors of Fix_1 with $\Xi u = u$ and $\Xi u' = -u'$. For every $\phi, \varepsilon, v_0, \Delta v$ and every record word, the first-order rate of $g_w(e^{-iT\bar{H}}, u + iu')$ at $T = 0$ is 0.*

Proof. By Theorem 19.9 the rate is $-4u^{\top}[A_1, E_1]u'$ with $A_1 = \varepsilon \Lambda_1 K_{\mathrm{re}} \Lambda_1 + (2\varepsilon + v_0)\Lambda_1 + \Delta v \Lambda_1 Q_{11} \Lambda_1$. A direct exact computation shows that $\Lambda_1 K_{\mathrm{re}} \Lambda_1$ commutes with Ξ, and so do Λ_1, $\Lambda_1 Q_{11} \Lambda_1$ and E_1 by Proposition 19.11. So $[A_1, E_1]$ commutes with the symmetric Ξ, and $u^{\top}[A_1, E_1]u' = u^{\top}\Xi[A_1, E_1]u' = u^{\top}[A_1, E_1]\Xi u' = -u^{\top}[A_1, E_1]u'$. □

Such planes are read only from the second order on, through the odd clock part. The kept record content is also small. The compressions to Fix_s of fifteen records are linearly independent on each eigenspace: the twelve single labels other than $(0, 0, -1)$, $(0, -1, 1)$ and $(1, 1, 0)$, together with the two-posting words $\big((0,0,0),(-1,0,-1)\big)$, $\big((0,0,0),(1,0,0)\big)$, $\big((0,1,-1),(0,0,0)\big)$ on Fix_1 and $\big((0,0,0),(-1,0,-1)\big)$, $\big((0,0,0),(0,0,1)\big)$, $\big((-1,-1,0),(0,0,0)\big)$ on Fix_{-1}, each word written in chronological order. A direct exact computation, not part of the proofs, finds that every compressed effect lies in their span: the records see 15 of the 36 real symmetric dimensions of a kept eigenspace.

Remark 19.16 (Open: every clock coupling reads). Theorem 19.14 is proved at $\Delta v = 0$. The open statement is that for every ϕ, v_0, Δv and every $\varepsilon \neq 0$, no circulating kept state of Fix_1 or Fix_{-1} is sealed by $\bar{H}(\phi, \varepsilon, v_0, \Delta v)$. Under it, a sealed kept circulation exists exactly when $\varepsilon = 0$. In a direct exact computation the greatest common divisor of three maximal minors of the constraint matrix, as polynomials in $t = \Delta v/\varepsilon$, is t^4 on both eigenspaces, which points to $t = 0$ as the only candidate exception; this is evidence, not a proof.

19.4 Half-turn kicks seal exactly the parity-split insides

The seal of Section 19.3 needs the clock coupling off. A different arrangement keeps the coupling on and still seals: kick both clocks by a half-turn between short relay intervals, fast enough that the relay is averaged over Ξ. The averaged relay still couples to the clocks on kept content, and the insides it seals are exactly the parity-split ones. No native process applies the kicks; they are supplied.

19.4.1 The half-turn average

For a matrix O put $O^{\Xi} = \frac{1}{2}(O + \Xi O \Xi)$, the *half-turn average*. For Hermitian M, its *half-turn parts* are $M_+ = M^{\Xi}$ and $M_- = M - M^{\Xi}$, so that $\Xi M \Xi = M_+ - M_-$. A state ψ is *parity-split* if $\tilde{\psi} = c\,\Xi\psi$ for some $|c| = 1$. A circulating kept state is parity-split exactly when it is a phase times $u + iu'$ with u a parity-even and u' a parity-odd real kept vector, both nonzero. The kept eigenspaces have rational bases adapted to Ξ: four parity-even vectors followed by four parity-odd ones, on Fix_1 and on Fix_{-1}.

Proposition 19.17 (What the half-turn removes). *Write $O_s = \Lambda_s O \Lambda_s$ for a matrix O. By direct exact computation in the adapted bases:*

(a) on Fix_1, $(K_{\mathrm{im}})_1$ is nonzero and anticommutes with Ξ; $(K_{\mathrm{re}})_1$ and $(Q_{11})_1$ commute with Ξ; $(K'_{\mathrm{re}})_1 = 0$;

(b) on Fix_{-1}, $(K'_{\mathrm{re}})_{-1}$ is nonzero and anticommutes with Ξ; $(Q_{11})_{-1}$ commutes with Ξ; $(K_{\mathrm{re}})_{-1}$ has nonzero parts of both kinds; $(K_{\mathrm{im}})_{-1} = 0$.

Consequently $\bar{H}^{\Xi}$ commutes with Ξ, U_{nat} and every Λ_λ, and with $N = \varepsilon K_{\mathrm{re}} + \Delta v Q_{11} + (2\varepsilon + v_0)I$,

$$\Lambda_1 \bar{H}^{\Xi} \Lambda_1 = \Lambda_1 N \Lambda_1, \qquad \Lambda_{-1} \bar{H}^{\Xi} \Lambda_{-1} = \left(\Lambda_{-1} N \Lambda_{-1}\right)^{\Xi}.$$

Proof. The parity statements are read off the Gram matrices of K_{re}, K'_{re}, K_{im} and Q_{11} on the adapted bases. Ξ commutes with U_{nat} and the Λ_λ, so averaging commutes with compression, $(\Lambda_s O \Lambda_s)^{\Xi} = \Lambda_s O^{\Xi} \Lambda_s$. On Fix_s, $\bar{H}$ acts as $\Lambda_s H \Lambda_s = \Lambda_s(H^{\mathrm{ev}} + iH^{\mathrm{od}})\Lambda_s$; the hops compress to zero (Theorem 19.9), $(K'_{\mathrm{re}})_1 = 0$ and $(K_{\mathrm{im}})_{-1} = 0$, and averaging removes the odd $(K_{\mathrm{im}})_1$ and, on Fix_{-1}, the odd $(K'_{\mathrm{re}})_{-1}$. □

So averaging over the half-turn does not detach the relay; it removes exactly the part of the coupling that is odd under Ξ. The kick itself is not something the postings can do.

Proposition 19.18 (The half-turn is not a posting). *Ξ is not a polynomial in U_{nat}.*

Proof. A polynomial $f(U_{\mathrm{nat}})$ acts on each kept eigenspace Fix_s as the scalar $f(s)$, while Ξ has both eigenvalues ± 1 on Fix_1 (its adapted basis has four vectors of each parity). □

The half-turn needs two ticks of each clock's own shift L.

19.4.2 The exact seal of the averaged relay

Theorem 19.19 (The half-turn average seals exactly the parity-split insides)**.** *(a) For all values of ϕ, ε, v_0 and Δv, every parity-split state ψ is sealed by $\bar{H}^{\Xi}$: $g_w(e^{-iT\bar{H}^{\Xi}}, \psi) = 0$ for every T and every record word w.*

(b) Let $\varepsilon \neq 0$, $\Delta v = 0$, $s = \pm 1$, and let ψ be a circulating kept state of Fix_s. Then ψ is sealed by $\bar{H}^{\Xi}(\phi, \varepsilon, v_0, 0)$ if and only if it is parity-split. Both eigenspaces contain sealed states; a circulation inside the parity-even part of Fix_1 is read.

(c) There is a finite set $S \subset \mathbb{R}$ with $S \cap [-\frac{1}{20}, \frac{1}{20}] = \emptyset$ such that the equivalence of (b) holds at $\Delta v = t\varepsilon$ for every $t \notin S$, every ϕ, v_0 and every $\varepsilon \neq 0$. S is the set of real roots of two explicit integer polynomials of degree 10.

Proof. (a) is the argument of Theorem 19.12, since $\bar{H}^{\Xi}$ commutes with Ξ for all parameters. (b) Let $\psi = \psi_{\mathrm{r}} + i\psi_{\mathrm{i}}$ be sealed. Every level $E_w^{(n)} = (i \operatorname{ad} \bar{H}^{\Xi})^n E_w$ of Lemma 19.13 commutes with Ξ, so in the adapted basis its antisymmetric part has no entries between the parity sectors, and the conditions involve only the $6 + 6$ Plücker coordinates of the parity-even parts of ψ_{r} and ψ_{i} and of their parity-odd parts. Ten rows at level 1 and two at level 3, from the record words of Section 19.3, form an exact integer table, and an integer certificate shows that they force these twelve coordinates to vanish. So the parity-even parts of ψ_{r} and ψ_{i} are parallel, and so are the parity-odd parts: $\psi = \lambda_E r_E + \lambda_O r_O$ with r_E, r_O real of opposite parity and $\lambda_E, \lambda_O \in \mathbb{C}$. Since $r_E \perp r_O$, $0 = \psi^{\top}\psi = \lambda_E^2 \|r_E\|^2 + \lambda_O^2 \|r_O\|^2$. Neither part vanishes, for otherwise ψ would be real up to a phase and not circulating; so $|\lambda_E| \|r_E\| = |\lambda_O| \|r_O\|$ and $\lambda_E^2/|\lambda_E|^2 = -\lambda_O^2/|\lambda_O|^2$. Then $c = \bar{\lambda}_E/\lambda_E = -\bar{\lambda}_O/\lambda_O$ has $|c| = 1$ and $\bar{\psi} = \bar{\lambda}_E r_E + \bar{\lambda}_O r_O = c(\lambda_E r_E - \lambda_O r_O) = c\,\Xi\psi$. The converse is (a); the witnesses of Theorem 19.12 are parity-split, and an explicit circulation in the parity-even part of Fix_1 is not, so it is read. (c) The same certificate with the Coulomb term carried as a polynomial in t gives $N(t)M(t) = d(t)I$ on the twelve coordinates; the two determinants are integer polynomials of degree 10, nonzero on $|t| \leq \frac{1}{20}$ by explicit bounds, and the argument of (b) runs at every t where both are nonzero. □

The averaged relay is the limit of frequent kicks.

Theorem 19.20 (The kick limit)**.** *Let M be Hermitian, $T \geq 0$ and $x \in \mathscr{H}$. Then $\|(\Xi e^{-i(T/n)M})^n x - \Xi^n e^{-iTM^{\Xi}} x\| \to 0$ as $n \to \infty$. Consequently, for $M = \bar{H}(\phi, \varepsilon, v_0, \Delta v)$ and a parity-split ψ, the conjugation gap of every record word after $(\Xi e^{-i(T/n)M})^n$ tends to 0.*

Proof. Let $U_\tau = e^{-i\tau M}$ with $\tau = T/n$, and $U_\tau' = \Xi U_\tau \Xi = e^{-i\tau(M_+ - M_-)}$. Then $(\Xi U_\tau)^{2m} = (U_\tau' U_\tau)^m$, and the first-order product estimate $\|e^{-i\tau(M_+ - M_-)} e^{-i\tau(M_+ + M_-)} - e^{-2i\tau M_+}\| \leq \frac{1}{2}\tau^2 \|[M_+ - M_-, M_+ + M_-]\| = \tau^2 \|[M_+, M_-]\|$ gives $\|(U_\tau' U_\tau)^m - e^{-2im\tau M_+}\| \leq m\tau^2 \|[M_+, M_-]\|$, which is of order T^2/n. For odd n one extra factor ΞU_τ appears, and $U_\tau \to I$. Record laws are continuous in the state, and Theorem 19.19(a) makes the limit gap vanish. □

19.4.3 At zero clock coupling a degeneracy seals more

Without clock coupling the parity seal is not the whole story. A degenerate eigenplane of the compressed Coulomb projector seals an inside that is not parity-split.

Proposition 19.21 (A seal by degeneracy). *$\Lambda_1 Q_{11} \Lambda_1$ has a plane W of eigenvectors with eigenvalue $\frac{1}{4}$ inside the parity-odd part of* Fix_1. *A circulating kept state $\psi_W \in W$ exists with $\Xi\psi_W = -\psi_W$ and $\bar{\psi}_W \notin \mathbb{C}\psi_W$, so ψ_W is not parity-split. It is sealed by $\bar{H}(\phi, 0, v_0, \Delta v)$ for every ϕ, v_0 and Δv, and it is read by $\bar{H}^{\Xi}(\phi, \varepsilon, v_0, 0)$ for every $\varepsilon \neq 0$.*

Proof. At $\varepsilon = 0$, $\bar{H}$ acts on W as $v_0 + \Delta v/4$ (a direct exact computation of $\Lambda_1 Q_{11} \Lambda_1$ on W), so $e^{-iT\bar{H}}\psi_W$ and $e^{-iT\bar{H}}\bar{\psi}_W$ carry the same phase and the gap is the gap at $T = 0$, which is 0 (Theorem 9.7). If $\bar{\psi}_W = c\Xi\psi_W = -c\psi_W$, ψ_W would be real up to a phase, against $\psi_W^{\top}\psi_W = 0$. Theorem 19.19(b) gives the reading. □

A direct exact computation, not part of the proofs, finds the full sealed set at $\varepsilon = 0$, $\Delta v \neq 0$. The records split each kept eigenspace into blocks of dimensions 1, 3 and 4, the last being the parity-odd part, and a plane is sealed exactly when its components in the three-dimensional block are parallel and its components in the four-dimensional block are parallel or span the $\frac{1}{4}$-eigenplane of the compressed Coulomb projector.

19.5 Finite kick rates leak, and no finite word seals

The seal of Theorem 19.19 is a limit. At every finite kick rate the half-turn leaves a leak, at an explicit order, and it is read. More strongly, no finite pattern of kicks spread over a relay time seals any parity-split kept circulation.

19.5.1 Alternating and palindromic kicks

Fix a Hermitian M, with half-turn parts $M_{\pm}$, and write $U_t = e^{-itM}$, $S_T = e^{-iTM_+}$ and

$$G_{\Xi} = -\frac{i}{2}[M_+, M_-].$$

The *alternating protocol* is $\mathscr{K}_n(T) = (\Xi U_{T/n})^n$, n relay intervals each followed by a kick. The *palindromic protocol* is $\mathscr{Q}_m(T) = \left(U_{\tau/2}\,\Xi\,U_{\tau}\,\Xi\,U_{\tau/2}\right)^m$ with $\tau = T/(2m)$: half step, kick, step, kick, half step. Norms of matrices are operator norms.

Theorem 19.22 (The alternating leak). *Let ψ be parity-split and w a record word.*

(a) For even n, $|g_w(\mathscr{K}_n(T), \psi)| \leq 2T^2 \|[M_+, M_-]\|\,\|\psi\|^2/n$. For odd n, $|g_w(\mathscr{K}_n(T), \psi)| \leq 2\big(T^2\|[M_+, M_-]\| + 2|T|\,\|M_-\|\big)\|\psi\|^2/n$.

(b) Along even n, $n\, g_w(\mathscr{K}_n(T), \psi) \to c_1 = 2T\mathrm{Re}\langle S_T\psi, E_w i[M_-, S_T]\psi\rangle$, and

$$c_1 = 2T\,\frac{d}{d\delta}\Big|_{\delta=0}\big\langle e^{-iT(M_+ + \delta G_{\Xi})}\psi,\; E_w e^{-iT(M_+ + \delta G_{\Xi})}\psi\big\rangle.$$

(c) Along odd n, $n\, g_w(\mathscr{K}_n(T), \psi) \to c_1^{\mathrm{odd}} = 2T\mathrm{Re}\langle S_T\psi, E_w(-i)\{M_-, S_T\}\psi\rangle$, and $c_1^{\mathrm{odd}} - c_1 = -4T\mathrm{Re}\langle S_T\psi, E_w i M_- S_T\psi\rangle$. When the two limits differ, $n\, g_w(\mathscr{K}_n(T), \psi)$ has no limit.

(d) If n is even and $n \geq 4(T\|M\|)^2/\eta$ for some $\eta > 0$, then $|g_w(\mathscr{K}_n(T), \psi)| \leq \eta\|\psi\|^2$.

Proof. For a parity-split state and any unitary U, the phase drops out and $g_w(U, \psi)$ is the law of $U\psi$ minus the law of $\Xi U \Xi\psi$, so $|g_w(U, \psi)| \leq 2\|\psi\|^2\|U - \Xi U \Xi\|$, effects being contractions. With $U'_\tau = \Xi U_\tau \Xi$, $\mathscr{K}_{2m} = (U'_\tau U_\tau)^m$ and $\Xi \mathscr{K}_{2m} \Xi = (U_\tau U'_\tau)^m$. Then $\|(U'_\tau U_\tau)^m - (U_\tau U'_\tau)^m\| \leq m\|[U'_\tau, U_\tau]\| \leq m\tau^2\|[M_+ - M_-, M_+ + M_-]\| = 2m\tau^2\|[M_+, M_-]\|$, which is (a) for even n with $\tau = T/n$; the odd case adds one unpaired step, whose two versions differ by at most $2|\tau|\,\|M_-\|$. (b) The exact identity $(U'_\tau U_\tau)^m = U_{-\tau/2}\, W_\tau^m\, U_{\tau/2}$ with the palindromic period $W_\tau = U_{\tau/2} U'_\tau U_{\tau/2}$, a symmetric product formula $\|W_\tau - e^{-2i\tau M_+}\| = O(\tau^3)$ [12], and the expansion $U_{-\tau/2} S_T U_{\tau/2} = S_T + \frac{i\tau}{2}[M, S_T] + O(\tau^2)$ give $(U'_\tau U_\tau)^m = S_T + \frac{i\tau}{2}[M, S_T] + O(\tau^2)$. The parity-odd part of $\frac{i\tau}{2}[M, S_T]$ is $\frac{i\tau}{2}[M_-, S_T]$, and only it contributes to the gap of a parity-split state, which gives c_1 with an explicit $O(1/m)$ remainder. The derivative form follows because G_Ξ is the time derivative of M_- under M_+ up to the factor $-\frac{1}{2}$: $i[M_+, M_-] = -2G_\Xi$, and Duhamel's formula expands $e^{-iT(M_+ + \delta G_\Xi)}$ to first order in δ. (c) For $n = 2m+1$, $U_\tau (U'_\tau U_\tau)^m$ lies within $O(\tau^2)$ of $S_T - \frac{i\tau}{2}\{M_-, S_T\}$, and the same computation gives c_1^{odd}; the difference is algebra. Two subsequences with different limits exclude a limit. (d) $\|[M_+, M_-]\| \leq 2\|M\|^2$ in (a). □

Theorem 19.23 (No large finite rate seals)**.** *Let $M = \bar{H}(\phi, \varepsilon, v_0, 0)$ with $\varepsilon \neq 0$, $s = \pm 1$, and ψ a parity-split circulating kept state of* Fix_s.

(a) There are a record word w and a time T with $c_1 \neq 0$, so there is N with $g_w(\mathscr{K}_{2m}(T), \psi) \neq 0$ for every $m \geq N$: the leak is exactly of order $1/n$.

(b) For the parity-split witness of Theorem 19.12 in Fix_1, *$c_1^{\mathrm{odd}} \neq c_1$ for some w and T, so the full sequence $n\, g_w(\mathscr{K}_n(T), \psi)$ diverges.*

Proof. (a) If $c_1 = 0$ for every T, every derivative at $T = 0$ of the smooth function $T \mapsto \mathrm{Re}\langle S_T\psi, E_w i[M_-, S_T]\psi\rangle$ vanishes. Each derivative is $\mathrm{Re}\langle\psi, \mathrm{Y}_n\psi\rangle$ for an explicit level operator Y_n built from M_+, M_- and E_w that anticommutes with Ξ. Writing ψ as a phase times $u + iu'$ with u parity-even and u' parity-odd, these conditions are linear in the 16 Plücker coordinates of $u \wedge u'$ that pair the two sectors. Exact rows (on Fix_1 levels 1 and 3, on Fix_{-1} levels 2 and 4, from ten record words) and a certificate in $\mathbb{Z}[\sqrt{2}]$ force all sixteen to vanish, impossible for $u, u' \neq 0$. A nonzero limit of $2m\, g_w$ gives nonzero gaps for large m. (b) If $\mathrm{Re}\langle S_T\psi, E_w i M_- S_T\psi\rangle$ vanished for all $T > 0$, all its derivatives at 0 would vanish; for one record word its first derivative at 0 is nonzero, by a direct exact computation (its value at 0 is the first-order term, which vanishes on Fix_1). □

Time reversal explains which levels speak. On Fix_1, $M_- = i\varepsilon(K_{\mathrm{im}})_1$ is imaginary antisymmetric while M_+, G_Ξ and the records are real symmetric, so only odd levels contribute and c_1 grows like T^2 at small T; on Fix_{-1}, M_- is real and only even levels contribute.

Theorem 19.24 (The palindromic leak)**.** *Let M be Hermitian, ψ parity-split, $m \geq 1$.*

(a) $|g_w(\mathscr{Q}_m(T), \psi)| \leq \frac{3}{4}|T|^3\big(\|[M_+, [M_+, M_-]]\| + \|[M_-, [M_+, M_-]]\|\big)\|\psi\|^2/m^2 \leq 6|T|^3\|M\|^3\|\psi\|^2/m^2$.

(b) $m^2 g_w(\mathscr{Q}_m(T), \psi) \to c_3 = \frac{T^2}{4}\,\mathrm{Re}\langle S_T\psi, E_w i[G_\Xi, S_T]\psi\rangle$.

(c) For $M = \bar{H}(\phi, \varepsilon, v_0, 0)$ with $\varepsilon \neq 0$ and every parity-split circulating kept ψ of Fix_1 *or* Fix_{-1}, *$c_3 \neq 0$ for some w and T; so the palindromic protocol reads ψ at every large m, with leak exactly of order $1/m^2$.*

Proof. (a) The symmetric product has no second-order term, so each period differs from its parity image at third order, with the stated constants from the third-order product formula [12]. (b) The parity-odd part of the third-order term of each period is a multiple of $[M_+, [M_+, M_-]]$, which is a time derivative of G_Ξ under M_+; so its accumulation over the m periods telescopes to the boundary term $\frac{i\tau^2}{4}[G_\Xi, S_T]$, and this gives c_3. (c) The certificate method of Theorem 19.23 with G_Ξ in place of M_-: full rank 16 from levels 2 and 4 on Fix_1 and levels 1 and 3 on Fix_{-1}. □

A palindromic pattern buys one order and no more on these states, and its leading coefficient is a boundary term that does not grow with T. This is the dynamical-decoupling picture [39] on the boundary's own clocks.

19.5.2 Every finite kick word reads

A finite pattern of kicks might cancel every order at once. It does not.

Definition 19.25 (Kick words)**.** A *kick word* $\mathscr{W}$ is a finite list of relay fractions $f_1, \dots, f_r$ in time order, each followed by a half-turn or not. It is *admissible* if every $f_j \geq 0$ and $\sum_j f_j = 1$. Its protocol at relay time T is $U_{\mathscr{W}}(T) = \Xi^{k_r} e^{-if_r TM} \cdots \Xi^{k_1} e^{-if_1 TM}$, with $k_j \in \{0, 1\}$ recording whether a kick follows interval j. The *plain word* is the single interval $f_1 = 1$ with no kick: $U(T) = e^{-iTM}$.

Admissibility matters: a word all of whose fractions vanish runs no relay and trivially reads nothing.

Definition 19.26 (Toggling frame, pattern, top frequency)**.** Moving each kick of a word to the end gives the *toggling frame* $U_{\mathscr{W}}(T) = \Xi^K \prod_j e^{-if_j T(M_+ + s_j M_-)}$, where K is the number of kicks and the *frame* $s_j = \pm 1$ of interval j changes at each kick that precedes it. The *first block* is the earliest maximal run of intervals in one frame, and its total length $F(\mathscr{W}) \in (0, 1]$ is the *first-block fraction*; the rest, of length $1 - F(\mathscr{W})$, begins in the other frame. On Fix_s let v_α be eigenvectors of the relay coordinates A_s, with eigenvalues λ_α, and let a_α, b_α be the coefficients of ψ and of $\Xi\psi$. The *pattern* of ψ is the Hermitian array $\Delta_{\alpha\beta} = \bar{a}_\alpha a_\beta - \bar{b}_\alpha b_\beta$. The gap of a word is a finite exponential sum in T whose frequencies, in units of ε, are sums of fractions f_j times eigenvalue differences, each term carrying an entry of the pattern; the *top frequency* is the largest frequency among the terms whose pattern entry is nonzero.

$F(\mathscr{W}) = 1$ exactly when the word never changes frame, and then $U_{\mathscr{W}}(T) = \Xi^K e^{-iTM}$ has the gaps of the plain word, since Ξ commutes with every effect.

Theorem 19.27 (No finite kick word seals)**.** *Let $M = \bar{H}(\phi, \varepsilon, v_0, 0)$ with $\varepsilon \neq 0$, $s = \pm 1$, and let ψ be a parity-split circulating kept state of Fix_s. For every admissible kick word there are T and a record word w with $g_w(U_{\mathscr{W}}(T), \psi) \neq 0$.*

Proof. In the toggling frame (Definition 19.26) the gap of a parity-split state is the record law of $\prod_j e^{-if_j T(M_+ + s_j M_-)}\psi$ minus that of the same product with every sign s_j reversed. Expanded in the eigenbases of the two frames it is a finite sum $\sum_i \kappa_i e^{i\Omega_i T}$; if it vanishes for every real T, the coefficients grouped by frequency vanish. *A hidden involution.* On Fix_s, $M = \varepsilon A_s + (2\varepsilon + v_0)$ in the rational basis, and A_s commutes with a rational involution Y_s whose eigenspaces, the

blocks, of dimensions 3 and 5, are invariant under the relay in both frames and under Ξ; the records do not commute with Y_s. Section 19.6 identifies Y_s as the compression of a relabelling Y of the cell. *The top frequency.* Let $F = F(\mathscr{W})$. Among the pairs (α, β) with $\Delta_{\alpha\beta} \neq 0$, the top frequency has the form $F(\lambda_\alpha - \lambda_\beta) + (1 - F)\big(\lambda_{\mathrm{top}(b)} - \lambda_{\mathrm{bot}(b')}\big)$, where b and b' are the blocks containing α and β and $\lambda_{\mathrm{top}(b)}$, $\lambda_{\mathrm{bot}(b')}$ are the top and bottom eigenvalues in them. Diagonal pairs cannot carry it. If the pattern vanished off the diagonal of a block, its diagonal entries there would satisfy $-\Delta_{\gamma\gamma} = \sum_\nu |\langle v_\nu, \Xi v_\gamma\rangle|^2 \Delta_{\nu\nu}$; at the entry of largest modulus the weights sum to 1 and include $|\langle v_\gamma, \Xi v_\gamma\rangle|^2 > 0$, which forces that entry, and with it the whole diagonal of the block, to vanish. So a nonzero diagonal entry comes with a nonzero entry $\Delta_{\gamma\delta}$, $\gamma \neq \delta$, in the same block, and since the pattern is Hermitian the ordering with $\lambda_\gamma > \lambda_\delta$ has a strictly larger frequency than every diagonal pair of the block. Within one pair of blocks the maximizer is unique because the eigenvalue gaps are distinct. Ties between the four pairs of blocks are broken by four records whose matrix elements between top and bottom eigenvectors are linearly independent, certified by interval arithmetic with outward rounding. So the top coefficient is nonzero and the gap is not identically zero. If the word never changes frame, it is the plain relay, which reads ψ by Theorem 19.14. □

The proof never expands in T, so it covers words that cancel any finite number of moments. At small T the gap of every word starts as $4\mu_0 T \mathrm{Re}\langle\psi, E_w(-iM_-)\psi\rangle$ with $\mu_0 = \sum_j s_j f_j$, and on Fix_1 this first-order term vanishes for every word and record. Exact protection by these clocks exists only in the infinitely fast limit of Theorem 19.19.

19.6 The relay's mirror symmetry

The involution behind the three-plus-five split in the proof of Theorem 19.27 is native. It is the compression of a relabelling of the cell that the posting does not detect: exchange the sites while complementing both occupations, and exchange the clocks while advancing each by half a cycle. The relay respects it, the records are permuted by it, and the Coulomb difference breaks it.

19.6.1 Which relabellings commute with the step

Chapter 5 found sixteen cell relabellings (π, τ) among the posting symmetries (Corollary 5.13): π is one of the occupation maps of the identity, the complement σ^{co}, the swap σ^{sw} and $\sigma^{\mathrm{co}}\sigma^{\mathrm{sw}}$, and τ composes the corresponding clock map with one of the four half-turns. Each carries an input sign $\epsilon(x)$. Its *signed cell operator* is

$$C_\sigma\big(|x\rangle \otimes e_i\big) = \epsilon(x)\,|\pi x\rangle \otimes e_{\tau i}.$$

Theorem 19.28 (Four relabellings commute with the step). *$C_\sigma U_{\mathrm{nat}} = U_{\mathrm{nat}} C_\sigma$ if and only if $\epsilon \equiv 1$, and $C_\sigma U_{\mathrm{nat}} = -U_{\mathrm{nat}} C_\sigma$ if and only if $\epsilon \equiv -1$. Exactly four relabellings commute: the identity, Ξ, the operator X_{cs} of $\sigma^{\mathrm{co}}\sigma^{\mathrm{sw}}$, and $Y = X_{\mathrm{cs}}\Xi$. Exactly four anticommute, those composing one of σ^{co}, σ^{sw} with one single half-turn, and each maps Fix_s onto Fix_{-s}. The other eight do neither.*

Proof. By the third condition of (5.2), the block of $C_\sigma U_{\mathrm{nat}} C_\sigma^{\top}$ from πx to πy is $\epsilon(y)\epsilon(x)$ times the relabelled M_{yx}, which is $\epsilon(y) M_{\pi y, \pi x}$. So $C_\sigma U_{\mathrm{nat}} = \Delta_\epsilon U_{\mathrm{nat}} C_\sigma$, where Δ_ϵ is diagonal and equals

$\epsilon(y)$ on the occupation πy. Every output occupation receives a nonzero block, so commutation holds exactly when $\epsilon \equiv 1$ and anticommutation exactly when $\epsilon \equiv -1$. The signs of Example 5.8 multiply: σ^{co} and σ^{sw} each have $\epsilon = -1$ on mixed inputs only, a single half-turn has $\epsilon = -1$ on balanced inputs only. The products with $\epsilon \equiv 1$ use both or neither of $\sigma^{\mathrm{co}}, \sigma^{\mathrm{sw}}$ and both or neither half-turn; those with $\epsilon \equiv -1$ use exactly one of each. If $C_\sigma U_{\mathrm{nat}} = -U_{\mathrm{nat}} C_\sigma$ and $U_{\mathrm{nat}} v = sv$, then $U_{\mathrm{nat}} C_\sigma v = -sC_\sigma v$. □

Explicitly,

$$Y\big(|x_1 x_2\rangle \otimes e_{i_1} \otimes e_{i_2}\big) = |1 - x_2,\ 1 - x_1\rangle \otimes e_{i_2+2} \otimes e_{i_1+2}. \tag{19.7}$$

Write labels as $j = (j_1, j_2, j_3) = (y_1 - x_2,\ x_1 - x_2,\ y_2 - x_1)$. The label map of Y is the product of those of the complement and the swap,

$$\kappa_Y(j_1, j_2, j_3) = (-j_3,\ j_2,\ -j_1),$$

applied letter by letter to a record word.

Theorem 19.29 (The mirror symmetry of the relay). *(a) Y is a real permutation matrix with $Y^2 = I$ and $Y^\top = Y$. It commutes with U_{nat}, with Ξ, and with $H(\phi, \varepsilon, v_0, 0)$ and $\bar{H}(\phi, \varepsilon, v_0, 0)$ for all ϕ, ε, v_0.*
(b) $Y E_w Y = E_{\kappa_Y(w)}$ for every record word w.
(c) Y keeps Fix_s. Its compression has trace 2, with eigenspaces of dimension 5 (eigenvalue $+1$) and 3 (eigenvalue -1); these are the two blocks of the proof of Theorem 19.27.
(d) $Y Q_{11} Y = |00\rangle\langle 00| \otimes I$. For $\Delta v \neq 0$, $\bar{H}(\phi, \varepsilon, v_0, \Delta v)$ fails to commute with Y on Fix_1 and on Fix_{-1}.

Proof. (a) Y has sign 1 and an involutive cell map, and Theorem 19.28 gives commutation with U_{nat}; Ξ and Y are both relabellings that commute as maps. A direct exact check shows that each of K_{re}, K'_{re}, K_{im}, R_+, R_- is invariant under conjugation by Y, entry by entry; so Y commutes with $H(\phi, \varepsilon, v_0, 0)$, and with each Λ_λ, hence with $\bar{H}$. (b) A posting symmetry relabels each slice: $Y V_j Y = V_{\kappa_Y(j)}$, the sign being 1; take products. (c) A direct exact computation of Y in the rational bases of Fix_s, together with the relay's eigenvectors there. (d) Y exchanges the occupations $(1, 1)$ and $(0, 0)$; the compressions of Y and of Q_{11} fail to commute on both eigenspaces (exact tables), while $\bar{H}(\phi, \varepsilon, v_0, 0)$ commutes with Y. □

The site exchange X_{cs} and the joint half-turn Ξ each commute with the step, but neither commutes with the relay's clock coupling; their product Y does. Because Y relabels the currents, it moves individual records while the set of records stays symmetric. That is why records can break the three-plus-five split of the relay while the relay keeps it.

19.6.2 Coulomb ratios

Write $\Delta v = t\varepsilon$, with t the Coulomb ratio.

Theorem 19.30 (No word seals outside finitely many ratios). *There is a finite set $S_* \subset \mathbb{R}$ with $0 \notin S_*$ such that for every $t \notin S_*$, every ϕ, v_0, every $\varepsilon \neq 0$, every $s = \pm 1$, every parity-split circulating kept state ψ of Fix_s and every admissible kick word, some record word reads ψ under $\bar{H}(\phi, \varepsilon, v_0, t\varepsilon)$ at some T. For $t \notin S_* \cup \{0\}$, every relay eigenvector on Fix_s has nonzero overlap with the parity image of every other: the kept eigenspace is a single block.*

Proof. At $t = 0$ this is Theorem 19.27. For $t \neq 0$ run the top-frequency argument with the whole kept eigenspace as one block. It needs distinct relay eigenvalues, distinct gaps, nonzero overlaps $\langle v_\gamma, \Xi v_\beta \rangle$, every pair of distinct eigenvectors joined by some record, and a nonzero pattern; the last holds for every parity-split circulating state, because a vanishing pattern would make $\Xi\psi$, hence $\bar{\psi}$, a multiple of ψ. Each of the other conditions is the nonvanishing of a polynomial in t: resultants of the characteristic polynomial of the relay on Fix_s, and two-level resultants over ordered pairs of its roots, evaluate to products of the required differences and matrix elements, so no eigenvalue is ever computed. One exact witness at $t = 2$, with rigorous interval enclosures of the eight eigenvalues, their eigenvectors, all 64 parity overlaps and all 64 combined record forms, shows each polynomial is nonzero. A nonzero polynomial has finitely many roots. □

Theorem 19.31 (The two kept eigenspaces are inequivalent)**.** *Let $\varepsilon \neq 0$ and $\Delta v = 0$. No linear map J from Fix_1 into Fix_{-1} with trivial kernel satisfies $J\bar{H}v = \bar{H}Jv$ for all $v \in \mathrm{Fix}_1$. In particular none of the four anticommuting relabellings of Theorem 19.28 intertwines the relay on the two kept eigenspaces.*

Proof. In the rational bases, $\bar{H}$ acts on $\mathrm{Fix}_{\pm 1}$ as $\varepsilon A_\pm + cI$ with $c = 2\varepsilon + v_0$, A_+ rational and A_- over $\mathbb{Q}(\sqrt{2})$. Exact sums give

$$\operatorname{tr} A_+^2 = \frac{748743113}{1854405}, \qquad \operatorname{tr} A_-^2 = \frac{149535229}{370881} + \frac{9952}{3045}\sqrt{2},$$

which differ since $\sqrt{2}$ is irrational. An injective J between spaces of dimension 8 is invertible and conjugates the two restrictions, so they have equal traces and equal traces of squares. Equal traces give $\operatorname{tr} A_+ = \operatorname{tr} A_-$, and then $\operatorname{tr}(\varepsilon A_\pm + cI)^2 = \varepsilon^2 \operatorname{tr} A_\pm^2 + 2\varepsilon c \operatorname{tr} A_\pm + 8c^2$ forces $\operatorname{tr} A_+^2 = \operatorname{tr} A_-^2$, a contradiction. □

Theorem 19.32 (An exceptional ratio in the window)**.** *There is $t_2 \in S_*$ with $-\frac{193}{5000} \leq t_2 \leq -\frac{77}{2000}$ such that for every ϕ, v_0 and $\varepsilon \neq 0$, the relay $\bar{H}(\phi, \varepsilon, v_0, t_2\varepsilon)$ on Fix_{-1} has two equal gaps between distinct pairs of its eigenvalues. So the top-frequency argument does not apply at t_2, and S_* meets $[-\frac{1}{20}, \frac{1}{20}]$.*

Proof. On the window, centred exact tables for each of the eight eigenvalues of the relay coordinates on Fix_{-1}, as polynomials in the window variable, isolate each eigenvalue in a box over the whole window; each box holds exactly one eigenvalue, which therefore moves continuously. With the eigenvalues $\lambda_1 < \cdots < \lambda_8$ labelled by their boxes, the difference $(\lambda_3 - \lambda_2) - (\lambda_7 - \lambda_6)$ of two gaps is positive at one end of the window and negative at the other, and the intermediate value theorem gives t_2. □

Remark 19.33 (Open: Coulomb-tuned seals). No seal has been found at any ratio, and two statements stay open. (i) For every t with $|t| \leq \frac{1}{20}$, no admissible kick word seals a parity-split circulating kept state of Fix_1 or Fix_{-1} under $\bar{H}(\phi, \varepsilon, v_0, t\varepsilon)$, for any ϕ, v_0 and $\varepsilon \neq 0$; this holds outside S_* and is equivalent to its restriction to the points of S_* in the window, t_2 among them. It contains the narrower open statement that no admissible word seals at a ratio of $[-\frac{193}{5000}, -\frac{77}{2000}]$ at which the two gaps of Theorem 19.32 coincide; whether a word seals at t_2 is undecided. (ii) The same at every real t, equivalent to its restriction to S_*. Similarly, the equivalence of Theorem 19.19(b) at the finitely many real roots of its two degree-10 polynomials is undecided.

19.7 Records that share the symmetry

A symmetry shared by the dynamics and a class of readers hides exactly what it fixes. The records that Y fixes are blind to exactly the mirror-real insides at zero Coulomb difference, and the Coulomb difference, which breaks Y, exposes every one of them to the same records.

Definition 19.34 (Mirror-fixed records and mirror-real states)**.** A record word w is *Y-fixed* if $YE_w = E_wY$, equivalently $E_{\kappa_Y(w)} = E_w$. A state ψ is *mirror-real* if $\bar{\psi} = cY\psi$ for some $|c| = 1$.

Proposition 19.35 (Which records Y fixes)**.** *A single label j is Y-fixed if and only if $j_1 = -j_3$. These are five labels: $(0,0,0)$, written by the balanced stays $(0,0) \to (0,0)$ and $(1,1) \to (1,1)$, and the labels $(-1,-1,1)$, $(1,1,-1)$, $(0,1,0)$, $(0,-1,0)$ of the four postings between mixed occupations, $(0,1) \to (0,1)$, $(1,0) \to (1,0)$, $(1,0) \to (0,1)$, $(0,1) \to (1,0)$. A word whose letters are all Y-fixed is Y-fixed. The converse fails: the word $\big((0,0,0),(-1,-1,0)\big)$, a balanced stay followed by $(0,1) \to (0,0)$, has effect 0 and is Y-fixed although its second letter is moved.*

Proof. By Theorem 19.29(b), j is Y-fixed when $\kappa_Y(j) = j$, which is $j_1 = -j_3$. For the other ten labels an explicit pair of cell vectors separates $YE_{(j)}$ from $E_{(j)}Y$ (a direct exact computation). The labels of the four mixed postings follow from $j(y,x)$. Products of Y-fixed effects are Y-fixed. After the stay the occupation is balanced, so the next posting cannot start at $(0,1)$. □

Theorem 19.36 (Symmetric records are blind to mirror-real states)**.** *Let M be a Hermitian matrix commuting with Y, such as $\bar{H}(\phi,\varepsilon,v_0,0)$, let w be a Y-fixed record word, and let ψ be mirror-real, or more generally $\psi = \alpha u + \beta u'$ with u, u' real, $Yu = u$, $Yu' = -u'$ and $\alpha, \beta \in \mathbb{C}$. Then for every kick word, admissible or not, and every T, $g_w(U_{\mathscr{W}}(T), \psi) = 0$.*

Proof. Y commutes with M and with Ξ, hence with $U = U_{\mathscr{W}}(T)$, and with E_w. For mirror-real ψ, $\langle U\bar{\psi}, E_wU\bar{\psi}\rangle = \langle UY\psi, E_wUY\psi\rangle = \langle U\psi, YE_wYU\psi\rangle = \langle U\psi, E_wU\psi\rangle$. In the second form Uu and Uu' lie in the ±1 eigenspaces of Y, which E_w does not connect, so the law of $U\psi$ is $|\alpha|^2\langle Uu, E_wUu\rangle + |\beta|^2\langle Uu', E_wUu'\rangle$, and $U\bar{\psi} = \bar{\alpha}Uu + \bar{\beta}Uu'$ has the same law. □

Theorem 19.37 (Exactly the mirror-real insides are hidden)**.** *Let $\varepsilon \neq 0$, $\Delta v = 0$, $s = \pm1$, and let ψ be a circulating kept state of Fix_s. The following are equivalent: (i) no Y-fixed record reads ψ under the plain word at any T; (ii) no Y-fixed record reads ψ under any kick word at any T; (iii) ψ is mirror-real; (iv) $\psi = \alpha u + \beta u'$ as in Theorem 19.36. Both kept eigenspaces contain parity-split circulating states with $\bar{\psi} = -Y\psi$. Every admissible word reads each of them, but only through records that Y moves.*

Proof. (iii) or (iv) implies (ii) by Theorem 19.36, and (ii) implies (i). (iii) and (iv) are equivalent for circulating kept states by the algebra in the proof of Theorem 19.19(b), with Y in place of Ξ. For (i) implies (iii), apply Lemma 19.13 to the five Y-fixed single labels. In a rational basis of Fix_s adapted to Y (its -1 eigenspace spanned by one parity-even and two parity-odd vectors, its $+1$ eigenspace by three and two), the levels up to 4 on Fix_1 and up to 5 on Fix_{-1} give exact rows, and a certificate in $\mathbb{Z}[\sqrt{2}]$ forces the $3 + 10$ Plücker coordinates inside the two eigenspaces of Y to vanish. So the components of ψ_{r} and ψ_{i} in each eigenspace of Y are parallel, which is (iv). The witnesses are explicit rational combinations; by Theorem 19.27 some record reads each of them under every admissible word, and by Theorem 19.36 no Y-fixed record does. □

This is the decoherence-free pattern in exact form [44, 28], with the symmetry now a property of the relay rather than of the reader alone.

Theorem 19.38 (The Coulomb difference exposes the mirror-real insides)**.** *(a) For every t with $0 < |t| \le \frac{1}{20}$, every ϕ, v_0 and $\varepsilon \neq 0$, at $\Delta v = t\varepsilon$ every mirror-real circulating kept state of* Fix_1 *or* Fix_{-1} *is read by some Y-fixed record under the plain word. The same holds at every $t \neq 0$ outside a finite set.*

(b) There is a finite set $S_Y \subset \mathbb{R}$ with $0 \notin S_Y$, the same for every word, such that for every $t \neq 0$ outside S_Y, Y-fixed records read every parity-split circulating kept state of Fix_1 *and* Fix_{-1} *under every admissible kick word, for every ϕ, v_0 and $\varepsilon \neq 0$. At $t = 0$ this fails.*

(c) If an admissible word seals a parity-split circulating kept state at ratio t, then $t \in S_* \cap S_Y$. Some ratio in $[-\frac{193}{5000}, -\frac{77}{2000}]$ at which two gaps of the relay on Fix_{-1} *coincide lies in $S_* \cap S_Y$.*

Proof. (a) For a mirror-real state the Plücker coordinates inside the eigenspaces of Y vanish, so only the 15 coordinates across them remain, and at $t = 0$ every row on them vanishes. Expanding the levels of $\bar{H}(\phi, \varepsilon, v_0, t\varepsilon)$ as polynomials in t, the rows are t times a first-order row plus higher orders. The first-order rows have full rank 15, by a certificate in $\mathbb{Z}[\sqrt{2}]$, which gives the finite-set statement. For the explicit radius, exact tables of every order in t and centre certificates with an approximate inverse and positive weights prove diagonal dominance on the windows: one centred at 0 of radius $\frac{1}{20}$ on Fix_1, three on Fix_{-1} (centre 0, radius $\frac{1}{50}$; centres $\pm\frac{7}{200}$, radius $\frac{3}{200}$). (b) In the argument of Theorem 19.30 the records enter only through their joining factor, so it runs with the Y-fixed records alone; one interval check at $t = 2$ on each eigenspace shows that the Y-fixed combined form joins all 64 ordered pairs of relay eigenvectors. At $t = 0$ Theorem 19.37 gives unread states. (c) A seal needs both arguments to fail. The condition polynomials of both arguments contain the relay's gap factor, which vanishes at every ratio where two gaps coincide, in particular at the ratio of Theorem 19.32. □

So the privacy of a mirror-real inside from the symmetric records belongs to one exact symmetry at one ratio, and by (b) half-turn kicks do not restore it.

Remark 19.39 (Open: symmetric reading at every ratio). Three statements stay open. (i) Theorem 19.38(a) at every $t \neq 0$; by (a) it is equivalent to its restriction to $|t| > \frac{1}{20}$. (ii) Theorem 19.38(b) at every $t \neq 0$, equivalent to its restriction to the points of S_Y. (iii) Every Y-fixed record word with nonzero effect has all its letters Y-fixed. A direct exact computation over the 225 two-letter words, of which 162 have zero effect, supports (iii).

19.8 One record reads every parity-split inside

Once the symmetry is broken, coarseness gives no privacy: a single native record, one label read once, already reads every parity-split inside under every finite pattern of kicks, outside finitely many Coulomb ratios.

Besides the fifteen labels read once we use five *two-posting words*, each in chronological order:

$$\begin{aligned} &w_A = \big((0,0,0),(-1,0,-1)\big), \quad w_B = \big((0,0,0),(1,0,0)\big), \quad w_C = \big((0,1,-1),(0,0,0)\big), \\ &w_D = \big((0,0,0),(0,0,1)\big), \quad w_E = \big((-1,-1,0),(0,0,0)\big) : \end{aligned} \tag{19.8}$$

a balanced stay followed by $(1,1) \to (0,0)$, by $(0,0) \to (1,0)$ or by $(0,0) \to (0,1)$, and the postings $(1,0) \to (0,0)$ or $(0,1) \to (0,0)$ followed by a stay. These are the twenty *single records* of this chapter.

Theorem 19.40 (One record reads every parity-split inside)**.** *There is one finite set $S_1 \subset \mathbb{R}$ with $0 \notin S_1$ such that for every $t \neq 0$ outside S_1, every ϕ, v_0, every $\varepsilon \neq 0$ with $\Delta v = t\varepsilon$, every $s = \pm 1$, every parity-split circulating kept state ψ of Fix_s and every admissible kick word, each of the twenty single records alone reads ψ at some relay time. In particular each of the five Y-fixed labels alone reads, outside a finite set of ratios, everything that the Y-fixed records read together; at $t = 0$ each of them fails.*

Proof. Run the argument of Theorem 19.30 with the one-record family. The record enters only through its matrix elements between relay eigenvectors, collected in a joining factor that is a polynomial in t; the relay's own conditions are unchanged. Forty rigorous interval checks at $t = 2$, one for each single record on each kept eigenspace, show that each record's matrix elements join all 64 ordered pairs of relay eigenvectors there, so every joining factor is a nonzero polynomial. Take S_1 to be the union of the finitely many nonzero real roots. At $t = 0$ Theorem 19.37 applies. □

The relay's dynamics carries one coarse record across the whole kept eigenspace, much as one observable under sufficiently generic dynamics is informationally complete [37]. Every single-record condition polynomial on Fix_{-1} carries the relay's gap factor and so vanishes somewhere in $[-\frac{193}{5000}, -\frac{77}{2000}]$.

Remark 19.41 (Open: one symmetric record near the symmetric point). Whether some single Y-fixed label reads, under the plain word, every mirror-real circulating kept state of Fix_1 and Fix_{-1} at every ratio $0 < |t| \leq \frac{1}{20}$ is open. Outside each record's finite set its parity-split members are read, and at $t = 0$ the statement fails.

19.9 At the symmetric point the kept compression decides

At zero Coulomb difference privacy against one record is still, in the main, a matter of symmetry, but the symmetry that counts is the one the record shares with the relay on the kept eigenspace, not on the whole cell. One record adds a different mechanism: it misses one relay mode entirely.

Definition 19.42 (Kept-mirror-fixed records)**.** A record word w is *kept-Y-fixed* on Fix_s if its compression $\Lambda_s E_w \Lambda_s$ commutes with Y. Since Y commutes with Λ_s, this says that E_w and $E_{\kappa_Y(w)}$ have the same compression to Fix_s.

Theorem 19.43 (The mirror identity)**.** *Let M be Hermitian with $YM = MY$, let ψ be mirror-real and $U = U_{\mathscr{W}}(T)$ for a kick word. Then for every record word v,*

$$g_v(U, \psi) = \langle U\psi, E_v U\psi \rangle - \langle U\psi, E_{\kappa_Y(v)} U\psi \rangle.$$

Proof. As in Theorem 19.36, $\langle U\bar{\psi}, E_v U\bar{\psi} \rangle = \langle U\psi, Y E_v Y U\psi \rangle$, and $Y E_v Y = E_{\kappa_Y(v)}$. □

record	posting(s)	mirror image κ_Y	class
$(0,0,0)$	$(0,0)\to(0,0)$, $(1,1)\to(1,1)$	itself	Y-fixed
$(-1,-1,1)$	$(0,1)\to(0,1)$	itself	Y-fixed
$(1,1,-1)$	$(1,0)\to(1,0)$	itself	Y-fixed
$(0,1,0)$	$(1,0)\to(0,1)$	itself	Y-fixed
$(0,-1,0)$	$(0,1)\to(1,0)$	itself	Y-fixed
$(-1,-1,0)$	$(0,1)\to(0,0)$	$(0,-1,1)$	kept-Y-fixed
$(0,-1,1)$	$(0,1)\to(1,1)$	$(-1,-1,0)$	kept-Y-fixed
$(0,1,-1)$	$(1,0)\to(0,0)$	$(1,1,0)$	kept-Y-fixed
$(1,1,0)$	$(1,0)\to(1,1)$	$(0,1,-1)$	kept-Y-fixed
$(0,0,1)$, $(-1,0,0)$	$(0,0)\to(0,1)$, $(1,1)\to(0,1)$	each other	reader
$(1,0,0)$, $(0,0,-1)$	$(0,0)\to(1,0)$, $(1,1)\to(1,0)$	each other	reader
$(1,0,1)$, $(-1,0,-1)$	$(0,0)\to(1,1)$, $(1,1)\to(0,0)$	each other	reader
$w_A,\dots,w_E$	two postings, (19.8)		reader

Table 19.1. The twenty single records at zero Coulomb difference. Y fixes five on the cell: the balanced stays and the postings between mixed occupations. It fixes nine on both kept eigenspaces: those five and the four postings from a mixed to a balanced occupation, whose mirror pairs have equal kept compressions. The other eleven, the six postings that leave a balanced occupation for a different one and the five two-posting words, are the *readers*.

A mirror-real state is therefore read by a record exactly as far as the record differs from its mirror image on the states the dynamics reaches. Table 19.1 sorts the twenty single records by this test.

Theorem 19.44 (Single records at the symmetric point)**.** *Let* $\Delta v = 0$, $\varepsilon \neq 0$ *and* $s = \pm 1$.

(a) Y fixes exactly five of the twenty records on the cell, and exactly the nine records of the first two groups of Table 19.1 are kept-Y-fixed, on Fix_1 *and on* Fix_{-1} *alike.*

(b) Every kept-Y-fixed record leaves every mirror-real state of Fix_s *unread, under every kick word and at every relay time. Mirror-real parity-split circulating kept states exist in both eigenspaces, so the four labels* $(-1,-1,0)$, $(0,-1,1)$, $(0,1,-1)$, $(1,1,0)$ *fail to read some protected inside although Y moves them on the cell.*

(c) Each of the eleven readers reads every parity-split circulating kept state of Fix_s *under the plain word.*

(d) For every one of the twenty records and every s, except the label $(-1,-1,1)$ *on* Fix_1*: a parity-split circulating kept state of* Fix_s *is unread by the record under the plain word if and only if the record is kept-Y-fixed and the state is mirror-real.*

Proof. (a) Proposition 19.35 for the cell; for the kept statement, a direct exact check that the coordinate matrix of Y preserves the Gram form of each of the nine compressions on each eigenspace, and the failure for the others. (b) For $x \in \mathrm{Fix}_s$, $\langle Yx, E_w Yx\rangle = \langle x, E_w x\rangle$ when w is kept-Y-fixed; kick words keep Fix_s and commute with Y, so the argument of Theorem 19.36 runs on the compression. The states are those of Theorem 19.37. (c), (d) Under the plain word the gap is $\sum_{\alpha,\beta}\Delta_{\alpha\beta}(E_w)_{\alpha\beta}\,e^{i\varepsilon(\lambda_\alpha-\lambda_\beta)T}$ in the relay eigenbasis v_α of Fix_s, with the pattern $\Delta_{\alpha\beta}$ of Definition 19.26. The 56 ordered gaps are distinct, so a blind state has $\Delta_{\alpha\beta}(E_w)_{\alpha\beta} = 0$ for $\alpha \neq \beta$.

Interval certificates show that each reader has $(E_w)_{\alpha\beta} \neq 0$ for all 28 pairs of distinct modes; then the pattern vanishes off the diagonal, which no parity-split circulating state allows. A kept-Y-fixed record has $(E_w)_{\alpha\beta} = 0$ between the two eigenspaces of Y, and the certificates show it joins every pair inside each, apart from the one exception; a blind state then has no relative phase inside either eigenspace of Y, which makes it mirror-real. □

The exception is a record that cannot see one relay mode.

Proposition 19.45 (A dark relay mode). *In the parity-adapted basis $b_0, \dots, b_7$ of Fix_1 (four parity-even vectors, then four parity-odd), let $w_1 = \sum_k \omega_k b_k$ with $\omega = (-i, i, i, 0, 3, -1, -3, -1)$. Then $\bar{H}(\phi, \varepsilon, v_0, 0)\, w_1 = \big(\frac{23}{3}\varepsilon + 2\varepsilon + v_0\big) w_1$ for all ϕ, ε, v_0, and the record of $(0,1) \to (0,1)$, label $(-1,-1,1)$, satisfies $\langle x, E_{(-1,-1,1)} w_1 \rangle = 0$ for every $x \in \mathrm{Fix}_1$. There is a parity-split circulating kept state $\psi_5 \in \mathrm{Fix}_1$ with $Y\psi_5 = -\psi_5$, not mirror-real, that this record leaves unread under the plain word for all ϕ, v_0 and $\varepsilon \neq 0$.*

Proof. Both identities are exact computations in the rational basis. The relay mode w_1 lies in the three-dimensional eigenspace of Y. Under the plain word the record sees only the pattern of ψ between relay modes it joins, and in that eigenspace it joins only the two modes other than w_1. Choosing the parity-odd part of ψ_5 in the -1 eigenspace of Y to satisfy one real linear condition makes the pattern between those two modes vanish; the plain-word expansion of the proof of Theorem 19.44 then has no nonzero term of nonzero frequency, so the gap is constant in T and equal to its value 0 at $T = 0$. Since $Y\psi_5 = -\psi_5$ and $\psi_5^{\top}\psi_5 = 0$, $\bar{\psi}_5 = cY\psi_5$ would make ψ_5 real up to a phase, so ψ_5 is not mirror-real. □

So at the symmetric point single-record privacy has two sources: the kept-compression symmetry, which hides mirror-real insides from nine records, and one dark mode, which hides a further family from one record under the plain relay.

19.10 Timing the half-turns against one record

Can a pattern of half-turns, timed against the relay, hide an inside from one record that the plain relay lets the record read? Off finitely many timings it cannot. At the remaining tie timings it cannot for the nine kept-Y-fixed records, it cannot for any reader along a single half-turn, and what is left is a rigid core of timings with two or more half-turns. Throughout this section $\Delta v = 0$.

19.10.1 First blocks and ties

We use the toggling frame, first-block fraction $F(\mathscr{W})$, modes v_α with eigenvalues λ_α of A_s, and pattern of Definition 19.26, with the modes taken in the two eigenspaces of Y, of dimensions 3 and 5; a *class* is a pair of these eigenspaces, so there are four. For a pair of distinct modes (α, β) with α in the eigenspace b and β in b', its *top level* at fraction F is

$$\ell_F(\alpha, \beta) = F(\lambda_\alpha - \lambda_\beta) + (1 - F)\big(\lambda_{\mathrm{top}(b)} - \lambda_{\mathrm{bot}(b')}\big),$$

the highest frequency, in units of ε, at which the pair can appear in the gap; $\mathrm{top}(b)$ and $\mathrm{bot}(b')$ are the highest and lowest modes of the eigenspaces. The *tie set* $\mathscr{T}_s$ is the set of F at which two pairs of distinct modes in different classes have equal top levels.

Proposition 19.46 (Ties are few). *$\mathscr{T}_1$ and $\mathscr{T}_{-1}$ are finite. $\mathscr{T}_1$ contains a fraction in $(0,1)$, approximately 0.7329. In $(0,1)$ the ties come from 854 ordered pairs of pairs on Fix_1 and 1062 on Fix_{-1}, and each tie fraction is enclosed in an interval of width below 10^{-15}.*

Proof. For fixed pairs, ℓ_F is affine in F, and two such functions with distinct relay gaps meet at most once; there are finitely many pairs. The rest is exact enumeration with rigorous interval arithmetic on the eigenvalue enclosures. □

Theorem 19.47 (Kicks add no privacy off the ties). *Let $\varepsilon \neq 0$, $s = \pm 1$, let r be one of the twenty single records, ψ a parity-split circulating kept state of Fix_s, and $\mathscr{W}$ an admissible word.*

(a) If $F(\mathscr{W}) \notin \mathscr{T}_s$, or if $Y\psi = \pm\psi$, and r leaves ψ unread under $\mathscr{W}$, then r leaves ψ unread under the plain word.
(b) If $F(\mathscr{W}) < 1$ and $F(\mathscr{W}) \notin \mathscr{T}_s$, then r leaves ψ unread under $\mathscr{W}$ if and only if r is kept-Y-fixed and ψ is mirror-real. There is no exception for the label $(-1,-1,1)$.
(c) If $Y\psi = \pm\psi$ and $F(\mathscr{W}) < 1$, every one of the twenty records reads ψ under $\mathscr{W}$.
(d) The label $(-1,-1,1)$ reads the state ψ_5 of Proposition 19.45 under every word with $F(\mathscr{W}) < 1$, in particular under one half-turn at mid relay time, while the plain word never does.

Proof. The top-frequency argument of Theorem 19.27 runs with one record: the top coefficient of a class is the pattern entry at its extreme pair times two nonzero chain amplitudes times the record's matrix element between the top mode of one eigenspace and the bottom mode of the other. A record blind under $\mathscr{W}$ therefore has a vanishing pattern on every class whose top frequency is attained in that class alone. Two classes share the top frequency only if $F(\mathscr{W}) \in \mathscr{T}_s$, and never for a state inside one eigenspace of Y, which occupies a single class. Off the ties the vanishing classes are exactly those the record joins at their extreme modes, and certificates show that every record joins the extreme modes of every class that its kept compression does not cut off by Y; that gives (a) and (b). (c) A state inside one eigenspace of Y has its pattern in a single class, so no tie arises, and the top term is carried by the extreme modes of that eigenspace, which every record joins; for the label $(-1,-1,1)$ on Fix_1 these are the two modes of the three-dimensional eigenspace other than the dark mode w_1. (d) is (c) for ψ_5; one half-turn at mid relay time is the admissible word with fractions $\frac{1}{2}, \frac{1}{2}$ and a kick between them, so $F = \frac{1}{2}$. □

19.10.2 The tie timings

At a tie two classes share the top, and the top-frequency method alone does not decide; a second level does.

Theorem 19.48 (The nine symmetric records at every timing). *Let r be one of the nine kept-Y-fixed records, $\varepsilon \neq 0$, $s = \pm 1$, and ψ a parity-split circulating kept state of Fix_s. For every admissible word $\mathscr{W}$, ties included: if r leaves ψ unread under $\mathscr{W}$, it leaves ψ unread under the plain word; and if $F(\mathscr{W}) < 1$, r leaves ψ unread under $\mathscr{W}$ if and only if ψ is mirror-real.*

Proof. A kept-Y-fixed record sees only the classes inside one eigenspace of Y. Blindness at $-\varepsilon$ is blindness at ε, so take $\varepsilon > 0$. A top tie between the two inner classes forces a short rest. At the frequency one relay step below the top of one eigenspace, taken at the shortest rest

intervals, only single deviations of the tied pair contribute, and their summed coefficient is nonzero by exact bracket certificates. On Fix_1 two distinct pairs of modes in one eigenspace of Y, not both diagonal, have gaps at least $\frac{21}{250}$ apart, which gives a uniform window; on Fix_{-1} the relay eigenvectors are real, so a parity-split pattern has zero diagonal, and the window is checked pair by pair. So no tie between the inner classes can carry a blind state, and the off-tie argument applies. □

For the readers, rank the four classes by their width $\lambda_{\mathrm{top}(b)} - \lambda_{\mathrm{bot}(b')}$; the order is strict on both eigenspaces (at $\varepsilon = 1$ the widths are approximately 5.2455, 5.2439, 0.8102, 0.8085 on Fix_1 and 5.6879, 5.0289, 3.4909, 2.8319 on Fix_{-1}, certified in order). A state is *led by an extreme entry* if, for some class, its pattern is nonzero at that class's extreme pair, the top mode of the first eigenspace with the bottom mode of the second, and zero on every wider class.

Theorem 19.49 (The readers). *Let r be one of the eleven readers, $\varepsilon \neq 0$, $s = \pm 1$, and ψ a parity-split circulating kept state of Fix_s. Then r reads ψ under every admissible word in each of the following cases:*

(a) ψ is mirror-real;
(b) ψ is led by an extreme entry;
(c) $\varepsilon > 0$ and the word's relay time runs in exactly two frames, a first block and one rest interval, at any first-block fraction, ties included.

Proof. (b) Every pair's top level is at most F times its own gap plus $(1 - F)$ times its class width, with equality only at the class's extreme pair; since the widths are strictly ordered, the extreme pair of the leading class is the unique top at every fraction. (a) On Fix_1, a second-level probe as in Theorem 19.48, with brackets for the eleven readers. On Fix_{-1}, every mirror-real parity-split state has $\Xi\psi = Y\psi$ or $\Xi\psi = -Y\psi$. If $\Xi\psi = -Y\psi$, determinant certificates put the lowest mode of the three-dimensional eigenspace and one of the two top modes of the five-dimensional one in the support, and the corresponding cross pair lies above every pair of the other cross class at every fraction. If $\Xi\psi = Y\psi$, the support of the pattern leaves one case in which a single extreme pair is on top at every fraction, which reads as in (b), and five tie fractions, listed in Table A.25. At a tie the mechanism is the second-level probe of Theorem 19.48: a frequency just below the tied top that only one term carries. Four ties close by that probe together with (c) for a single rest interval; at the fifth the two tied top terms have different moduli for every number of rest intervals, so they cannot cancel. (c) Off the ties this is Theorem 19.47. At a tie, a blind record has two support pairs of different classes tied at the top, and the tie is one of those of Proposition 19.46. For each of the $854 + 1062$ listed ties an exact check finds a probe: one of the tied pairs and a chain one relay step from its extreme chain whose frequency no other term at most as far from the extreme chain reaches. Its coefficient is a product of the pattern entry, two nonzero parity overlaps and a nonzero record entry, so the record law has a component at that frequency. □

No symmetry of the pulses is available to hide an inside from a reader; only a coincidence of relay frequencies could, and by (c) with one pulse it never does. This agrees with the decoupling picture, in which exact decoupling by pulses needs a symmetry of the pulse group [39].

19.10.3 What remains is rigid

Theorem 19.50 (Blindness coarsens). *(a) If $\sum_i \kappa_i e^{i\Omega_i T} = 0$ for every real T, and Ω' satisfies $\Omega'_i = \Omega'_j$ whenever $\kappa_i, \kappa_j \neq 0$ and $\Omega_i = \Omega_j$, then $\sum_i \kappa_i e^{i\Omega'_i T} = 0$ for every T.*
(b) Consequently a record blind along a toggling-frame list is blind along every list with the same frames whose frequency coincidences, between terms with nonzero coefficients, include those of the first list, for any real fractions.

Proof. (a) Group the terms by the frequencies Ω; each group's coefficient sum vanishes, and each group has a single frequency under Ω', so the new sum vanishes group by group. (b) In the expansion of the gap along a list, the coefficients depend only on the frames, and the frequencies are linear in the fractions; apply (a). □

Call a list *rigid* if its fractions are fixed, up to a common scale, by its own frequency coincidences. A blind list that is not rigid can be slid along its coincidence-preserving fractions until one interval vanishes, then shortened by merging equal frames, and it stays blind by Theorem 19.50. Since without a kick or off the ties a blind list is impossible for a reader, only rigid tie lists can still hide an inside from one. Every list with a first block at a tie and a single rest interval is rigid, and Theorem 19.49(c) shows that such lists hide nothing.

Theorem 19.51 (The open core is rigid). *The following statements are equivalent:*

(i) for every one of the twenty records, every s, every parity-split circulating kept state of Fix_s, every ϕ, ν_0, $\varepsilon \neq 0$ and every admissible word, a state left unread under the word is left unread under the plain word (kicks add no privacy);
(ii) the eleven readers read every parity-split circulating kept state of Fix_1 and Fix_{-1} under every admissible word;
(iii) the rigid core of Remark 19.52 holds.

If they hold, then for every record and s, except the label $(-1,-1,1)$ on Fix_1, a parity-split circulating kept state is left unread under some admissible word, for some ϕ, ν_0 and $\varepsilon \neq 0$, exactly when the record is kept-Y-fixed and the state is mirror-real.

Proof. Theorems 19.47 and 19.48 leave only the readers at tie words. Theorem 19.49 removes the mirror-real insides, the insides inside one eigenspace of Y and those led by an extreme entry. The reduction above removes the lists that are not rigid, and Theorem 19.49(c) removes the rigid lists with a single rest interval; blindness at $-\varepsilon$ is blindness at ε. Conversely a list of the core, with a kick after every interval, is the toggling frame of an admissible word. The last sentence combines (i) with Theorem 19.44(b) and (d). □

Remark 19.52 (Open: the rigid tie core). The open statement is: for each of the eleven readers, each $s = \pm 1$, each parity-split circulating kept state ψ of Fix_s that is not mirror-real, satisfies $Y\psi \neq \pm\psi$ and is not led by an extreme entry, each ϕ, ν_0 and $\varepsilon > 0$, and each toggling-frame list with a first block of fraction $F \in \mathscr{T}_s$ and a rest of at least two positive intervals alternating in frame, the earliest in the frame opposite to the first block, with total length 1 and rigid: the reader is not blind to ψ along the list. No counterexample was found. A counterexample would need every grouped coefficient of the record law to cancel at every frequency at once, and no symmetry beyond Y that would force this is known; the first open case is a rest of exactly two intervals.

Chapter 20

Record writes, capture and control

A record is something the dynamics wrote. We study one candidate for the elementary act of writing, a *capture*: two cells exchange places and a blank record pair takes a tag naming their occupation difference. The capture, its record pair and the carrier that holds them are supplied constructions (Section 20.1). We ask whether the carrier's own passive dynamics can capture, and if not, what kind of control can.

The passive answer is no, for two separate reasons: the exchange breaks a charge that the passive holding conserves, and the write keeps that charge but moves a record energy that the holding also conserves. So each half, and the write in each of its eight nonzero difference sectors, is beyond every passive flow at every flux, every value of $\hbar$ and every time. Under control the answer is graded. A literal serial alphabet that conserves total charge at every instant cannot reach capture even as a limit; concurrent bundles of the same tokens, priced and paid, reach large algebras of payload directions, prepare records from supplied blanks and transfer a matter cell into a record cell to any accuracy. Every constructive result is an existence statement on supplied schedules, strengths and blanks.

20.1 The capture and its carrier

A capture acts on two native cells and a record.

20.1.1 Cells, tags and the capture permutation

A *cell* is a basis label $c = (x, i)$ of the native cell $\mathscr{H}$ of Chapter 2: an occupation $x \in \mathcal{O}$ and an index $i \in \mathbb{Z}_4 \times \mathbb{Z}_4$ of its two clock cells, so there are 64 cells. Write $\operatorname{occ} c = x$. A *capture label* is a pair $d \in \{0,1,2\}^2$. For bits a, b put $\delta(a,b) = a - b + 1$, and for occupations $\delta(u,v) = (\delta(u_1,v_1), \delta(u_2,v_2))$; the label $(1,1)$ is *zero difference*. A *record pair* is $y \in \mathbb{Z}_8 \times \mathbb{Z}_8$, two eight-state record slots. The *tag chart* places the nine labels at $\{0,1,2\}^2 \subset \mathbb{Z}_8^2$, and the *blank* is $(1,1)$, the tag of zero difference. The capture carrier is

$$X_{\mathrm{cap}} = \mathrm{Cell} \times \mathrm{Cell} \times \mathbb{Z}_8^2, \qquad |X_{\mathrm{cap}}| = 64^3.$$

Definition 20.1 (Exchange, write, capture)**.** For $(c_1, c_2, y) \in X_{\mathrm{cap}}$:

1. the *exchange* is $\mathrm{Ex}(c_1, c_2, y) = (c_2, c_1, y)$;
2. the *write* is $\mathrm{Wr}(c_1, c_2, y) = (c_1, c_2, s_d y)$ with $d = \delta(\operatorname{occ} c_2, \operatorname{occ} c_1)$, where s_d is the transposition of the blank and the tag d in $\mathbb{Z}_8^2$;
3. the *sector write* Wr_d applies s_d to the basis states whose difference is d and fixes all others;
4. the *capture* is $\mathrm{Cap} = \mathrm{Wr} \circ \mathrm{Ex}$.

On a blank record the capture reads

$$\mathrm{Cap}(c_1, c_2, \mathrm{blank}) = \big(c_2, c_1, \delta(\mathrm{occ}\, c_1, \mathrm{occ}\, c_2)\big) \ :$$

the cells swap and the record keeps the occupation change at the second position, new minus old plus one in each bit. Take c_1 with occupation $(1, 0)$ and c_2 with occupation $(0, 0)$. The record receives $\delta((1, 0), (0, 0)) = (2, 1)$: the first bit rose, the second stayed. Since $s_{(1,1)}$ is the identity, a zero difference leaves the record blank.

Proposition 20.2 (Capture is a reversible completion)**.** Ex, Wr *and every* Wr_d *are involutions of* X_{cap}*, and* Cap *is a permutation; its permutation matrix* C *satisfies* $C^*C = CC^* = I$ *on all of* $\mathbb{C}^{X_{\mathrm{cap}}}$*, including every record value outside the tag chart. On inputs with a blank record,* C *restricts to the isometry*

$$e_{(c_1,c_2)} \longmapsto e_{(c_2,\, c_1,\, \delta(\mathrm{occ}\, c_1, \mathrm{occ}\, c_2))}$$

from $\mathbb{C}^{4096}$ *into* $\mathbb{C}^{4096 \cdot 9}$*, and the map* $(c_1, c_2) \mapsto (c_2, c_1, \delta(\mathrm{occ}\, c_1, \mathrm{occ}\, c_2))$ *is injective. The sector writes commute pairwise,* $\mathrm{Wr}_{(1,1)}$ *is the identity, and* Wr *is the product of the eight sector writes with* $d \neq (1, 1)$*, in any order.*

Proof. Ex is an involution. Wr fixes both cells and hence d, and applies the involution s_d to the record; so does Wr_d. A permutation matrix is unitary. On the blank, s_d(blank) is the tag d, and the map is injective because the output lists both cells. Sector writes with $d \neq d'$ act on disjoint sets of basis states and commute, and on a state of difference d the product reduces to $\mathrm{Wr}_d = \mathrm{Wr}$. □

Three things here are chosen, not derived, and we name them once. The capture permutation itself is a supplied target operation. The nine capture labels are a chart of their own: they are not identified with the fifteen current labels of the native step. And the blank $(1, 1)$ is a supplied initial record value. Nothing in this section prepares a blank, selects the cells that meet, or says what physical process enacts the permutation.

One physical feature is already fixed. Let a capture replace occupation b by occupation a at a cell carrying the edge flux (2.10). Its current is $\kappa(a, b) = (a_1 - b_1,\ 0,\ a_2 - b_2)$, and

$$m(n, b; c, d) + \kappa(a, b) = m(n, a; c, d),$$

while the reference charge $c + d - x_1 - x_2$ changes by $-(\kappa_1 + \kappa_3)$. The loop flux n does not move. A capture rearranges the tree edges and carries no loop current; the native step, by contrast, moves n by j_2.

20.1.2 The joined carrier

The passive dynamics lives on a larger carrier that holds the capture together with the references its charges need. Its building block is the *slot* $\mathbb{C}^8$: one of the two eight-state channels of Chapter 2, an occupation bit together with one clock cell, so that a native cell is two slots. On a slot let S be the diagonal sign $\mathrm{diag}(1, -1, -1, 1, -1, 1, 1, -1)$ and F the permutation $t \mapsto t \oplus 1$ that flips the low bit of the index. The *local charge* is $q = iSF$, with $q^2 = 1$, and the *occupation* is the projection $N = \frac{1}{2}(1 - q)$.

The *joined carrier* is $(\mathbb{C}^8)^{\otimes 14}$, fourteen slots in four groups:

slots	role	charge weight
0, 1, 2, 3	matter sources	$+1, +1, -1, -1$
4, 5, 6, 7	references of the matter sources	$-1, -1, +1, +1$
8, 9, 10	record sources of record pairs 0, 1, 2	$+1, +1, +1$
11, 12, 13	references of record pairs 0, 1, 2	$-1, -1, -1$

Every slot carries a nonzero weight w_k, and the *total joined charge* is $Q_{\mathrm{tot}} = \sum_k w_k N_k$. It is the Gauss charge of a single shared boundary: all seven references (four matter, three record) are placed at one root, and their summed charge is priced once. The pairing of each source with its own reference, and the root placement, are construction choices of this carrier, and we treat them as supplied.

The capture's six slots sit at positions 0, 1 (first cell), 6, 7 (second cell) and 8, 11 (record pair 0). A fixed unitary chart identifies two slots with the basis of a native cell, and the record pair with $\mathbb{Z}_8^2$ through the *record frame*, in which every tag, the blank included, is a vector of occupation zero in both record slots. Write $\iota(A) = A \otimes 1$ for an operator A on the six slots, placed there and extended by the identity on the other eight. Then

$$W = \iota(\mathrm{Wr}), \qquad E = \iota(\mathrm{Ex}), \qquad C = \iota(\mathrm{Cap}) = WE,$$

and likewise $W_d = \iota(\mathrm{Wr}_d)$. These are self-adjoint involutions, and C is their product.

Proposition 20.3 (Capture conserves the total charge)**.** *W, E, C and every W_d commute with Q_{tot}. On the six capture slots Q_{tot} restricts to $N_0 + N_1 + N_6 + N_7 + N_8 - N_{11}$, and the record pair charge $N_8 - N_{11}$ vanishes on the blank and on every tag.*

Proof. The joined charge splits as $Q_{\mathrm{same}} \otimes 1 + 1 \otimes Q_{\mathrm{res}}$, with $Q_{\mathrm{same}} = N_0 + N_1 + N_6 + N_7 + N_8 - N_{11}$ on the capture slots (the two cells count with weight $+1$) and Q_{res} on the rest. Any six-slot operator commuting with Q_{same} commutes with Q_{tot} after placement. The exchange permutes the two cells, which carry the same weights. The write fixes both cells and acts on the record only within the span of the blank and the tags, where the record pair charge is zero; the number operators of the cells are diagonal and untouched. The same holds for each Wr_d. □

So capture does not violate charge. What it moves are the separate charges of individual source/reference pairs, and that is where the passive dynamics enters.

20.2 Writing a record is not a passive flow

20.2.1 The holding and its conserved quantities

The *native clock* on a slot is the Hermitian circulant

$$\Omega_{st} = \tfrac{1}{2}\big(7[s = t] - [s \neq t]\big) + \tfrac{i}{2}\big(\rho(s - t) + \sqrt{2}\,\theta(s - t)\big), \qquad s, t \in \mathbb{Z}_8,$$

with $\rho = (0, 1, 1, -1, 0, 1, -1, -1)$ and $\theta = (0, 1, 0, 1, 0, -1, 0, -1)$ indexed by $s - t \bmod 8$. Let B be the 8×4 isometry onto the occupation-zero half of a slot ($qB = B$), with columns

$Be_j = \frac{1}{\sqrt{2}}\big(e_{2j} - i\,D_{jj}\,e_{2j+1}\big)$, $D = \mathrm{diag}(1,-1,-1,1)$ the dressing, and $B' = FB$ the occupied half. The *record clock* and the *reference clock* are the 4×4 compressions

$$\Omega_{\mathrm{rec}} = B^{*}\Omega B, \qquad \Omega_{\mathrm{ref}} = \tfrac{1}{2}\big(B^{*}\Omega B + B'^{*}\Omega B'\big).$$

Both are Hermitian with trace 14, and both are explicit; Table 20.1 prints them.

$$\Omega_{\mathrm{rec}} = \begin{pmatrix} 3-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}-i}{2} & -\frac{i}{2} & \frac{i}{2} \\ \frac{\sqrt{2}+i}{2} & 4+\frac{\sqrt{2}}{2} & -1-\frac{i}{2} & \frac{i}{2} \\ \frac{i}{2} & -1+\frac{i}{2} & 4+\frac{\sqrt{2}}{2} & \frac{-\sqrt{2}+i}{2} \\ -\frac{i}{2} & -\frac{i}{2} & \frac{-\sqrt{2}-i}{2} & 3-\frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\Omega_{\mathrm{ref}} = \begin{pmatrix} \frac{7}{2} & \frac{\sqrt{2}-i}{2} & -\frac{i}{2} & \frac{-1+i}{2} \\ \frac{\sqrt{2}+i}{2} & \frac{7}{2} & \frac{-1-i}{2} & \frac{i}{2} \\ \frac{i}{2} & \frac{-1+i}{2} & \frac{7}{2} & \frac{-\sqrt{2}+i}{2} \\ \frac{-1-i}{2} & -\frac{i}{2} & \frac{-\sqrt{2}-i}{2} & \frac{7}{2} \end{pmatrix}$$

Table 20.1. The record clock and the reference clock: compressions of the native clock Ω to the occupation-zero half of a slot, and the average of the compressions to its two halves. Tags use the indices $0, 1, 2$; the blank sits at index 1 in both slots.

For a 4×4 matrix A write $\mathrm{pay}(A) = BAB^{*} + B'AB'^{*}$, the same A on both occupation halves of a slot; it commutes with N and with F. Let $h_{\mathrm{d}} = \frac{1}{2}(\Omega + q\Omega q) = N\Omega N + (1-N)\Omega(1-N)$ be the charge-diagonal part of the native clock. The *record pair holding* acts on a record source slot (first factor) and its reference slot (second factor):

$$H_{\mathrm{pair}} = h_{\mathrm{d}}\otimes 1 + N\Omega(1-N)\otimes NF(1-N) + (1-N)\Omega N\otimes(1-N)FN + 1\otimes\mathrm{pay}(\Omega_{\mathrm{ref}}) + 7. \quad (20.1)$$

The source clock is lifted so that every change of the source's occupation comes with an equal change of the reference's occupation, so H_{pair} commutes with the pair charge $N_{\mathrm{ref}} - N_{\mathrm{src}}$; it is positive semidefinite. Since $NB = 0$ and $\mathrm{pay}(A)B = BA$, on the block where both slots have occupation zero it reads

$$(B\otimes B)^{*}\,H_{\mathrm{pair}}\,(B\otimes B) = \Omega_{\mathrm{rec}}\otimes 1 + 1\otimes\Omega_{\mathrm{ref}} + 7. \quad (20.2)$$

The *joined holding* at integer flux m is

$$H_m = H_m^{\mathrm{mat}}\otimes 1 + 1\otimes\sum_{r=0}^{2} H_{\mathrm{pair}}^{(r)} + H_{\mathrm{bdry}},$$

with three parts.

1. H_m^{mat} is the native holding of the four matter source/reference pairs on slots 0 to 7: the clocks of the matter cells with their hopping, the electric energy of the three-edge axis

at fixed integer flux m, and the four references, each attached to its own matter slot as in (20.1), so that every change of that slot's occupation is accompanied by an equal change of its reference's occupation, and each carrying $\mathrm{pay}(\Omega_{\mathrm{ref}}) + 7$. The flux enters only through the diagonal electric energy $\mathrm{el}(m) + \mathrm{el}(m - q_L) + \mathrm{el}(Q - m)$ of the edge fields $(m,\ m - q_L,\ Q - m)$, where q_L and Q are the left-site and total matter charges and $\mathrm{el}(j) = 4\big(j\,\mathrm{arsinh}(j/2) - \sqrt{j^2+4} + 2\big)$. No term changes m: the holding is a fixed-flux family. Here m is one integer flux of the joined carrier, the field on its first edge; it is not the edge-flux vector $m(n, x; c, d)$ of Chapter 2.

2. $H_{\mathrm{pair}}^{(r)}$ is H_{pair} placed on record pair r.
3. $H_{\mathrm{bdry}} = \mathrm{el}(Q_{\mathrm{tot}})$ is the shared boundary electric energy, the single root edge priced once at the total charge. It vanishes on neutral states.

Each part is positive semidefinite, and H_m commutes with Q_{tot}. On the neutral prepared states the seven references contribute the baseline $7 \cdot 7 = 49$. We do not print the four-slot matter holding, the clocks with their hopping, which is an explicit construction supplied with the carrier; the arguments below use only that it is positive semidefinite, that the flux enters it only through its electric diagonal, and that on the matter slots 0 and 1, compressed through the matter cell frame, its clock is h_{d}. The *passive flow* of H_m is $U_{m,\hbar}(t) = e^{-itH_m/\hbar}$. We call an operation *passive* when some passive flow performs it; this is the only sense of the word used below.

Two observables matter. The *source pair charge* is the summed separate charge of the first two matter pairs,

$$Q_{\mathrm{src}} = (N_0 - N_4) + (N_1 - N_5),$$

whose source part $N_0 + N_1$ is the occupation of the capture's first cell. The *record pair energy* is the full holding of record pair 0, the pair the capture writes,

$$R_0 = 1 \otimes H_{\mathrm{pair}}^{(0)}.$$

For record values y, y' let $e(y, y')$ be the matrix entry of H_{pair} between the record frame vectors of y' and y, and $e(y) = e(y, y)$. By (20.2), on tags $y = (a, b)$, $y' = (a', b')$,

$$e(y, y') = \Omega_{\mathrm{rec}}(a, a')[b = b'] + [a = a']\Omega_{\mathrm{ref}}(b, b') + 7[y = y']. \tag{20.3}$$

Lemma 20.4 (What the holding conserves)**.** *For every integer m, H_m commutes with Q_{tot}, with each separate matter pair charge $N_j - N_{4+j}$ $(j = 0, \dots, 3)$, hence with Q_{src}, and with R_0. So do all passive flows $U_{m,\hbar}(t)$.*

Proof. Each separate matter pair charge $N_j - N_{4+j}$ commutes with H_m^{mat}, because every term that changes the occupation of slot j changes that of slot $4 + j$ equally and the other terms change neither; it does not act on the record pairs, and it commutes with Q_{tot} and hence with H_{bdry}. For R_0: $H_{\mathrm{pair}}^{(0)}$ commutes with each $H_{\mathrm{pair}}^{(r)}$ (equal or on different slots) and trivially with $H_m^{\mathrm{mat}} \otimes 1$; and H_{pair} commutes with the pair charge, so R_0 commutes with Q_{tot} and with H_{bdry}. An operator commuting with H_m commutes with its exponential. □

20.2.2 Each half breaks one conserved quantity

Theorem 20.5 (The passivity split)**.** *On the joined carrier:*

(a) $C = WE$*;*
(b) $[Q_{\mathrm{src}}, W] = 0$ *and* $[Q_{\mathrm{src}}, E] \neq 0$*;*
(c) $[R_0, E] = 0$ *and* $[R_0, W] \neq 0$*; hence* $[R_0, C] \neq 0$*;*
(d) $[W, H_m] \neq 0$ *for every integer* m*;*
(e) for every integer m*, every real* $\hbar \neq 0$ *and every real* t*,* $U_{m,\hbar}(t) \neq W$*,* $U_{m,\hbar}(t) \neq E$ *and* $U_{m,\hbar}(t) \neq C$*.*

On the probe with first cell at occupation $(1, 0)$*, second cell at* $(0, 0)$ *and a blank record, the write puts the tag* $(0, 1)$ *on record pair* 0*, and*

$$e\big((0,1)\big) - e(\mathit{blank}) = \Omega_{\mathrm{rec}}(0,0) - \Omega_{\mathrm{rec}}(1,1) = -1 - \sqrt{2}.$$

Proof. (a) holds already on the six slots (Proposition 20.2), and placement is multiplicative. (b) Q_{src} is the first cell's occupation number minus the reference number $N_4 + N_5$ of the residual slots. The write fixes both cells, so it commutes with the first; every placed operator commutes with the second. The exchange carries a basis state whose first cell is at $(1, 0)$ and whose second is at $(0, 0)$ to one whose first cell is at $(0, 0)$, which lowers $N_0 + N_1$ by one while fixing $N_4 + N_5$; so $Q_{\mathrm{src}}E - EQ_{\mathrm{src}}$ has the entry -1 between these two states. (c) The exchange does not act on the record pair, so it commutes with R_0. For the write, use the probe: product vectors $\psi(c_1, c_2, y)$ built from the two cells, the record frame vector of y on record pair 0 and fixed vectors elsewhere, on which W acts as Wr on basis states. Their R_0-expectation, normalized, is $e(y)$. If W commuted with R_0, then, W being a self-adjoint involution, it would preserve these expectations, so $e(\mathrm{blank}) = e((0,1))$. But $e(y) = \Omega_{\mathrm{rec}}(a,a) + \Omega_{\mathrm{ref}}(b,b) + 7$ for a tag $y = (a, b)$; the blank and $(0, 1)$ share $b = 1$; and $\Omega_{\mathrm{rec}}(0,0) - \Omega_{\mathrm{rec}}(1,1) = -1 - \sqrt{2} \neq 0$. Then $[R_0, C] = [R_0, W]E \neq 0$. (d) If W commuted with H_m it would preserve the H_m-expectation of the probe. The write commutes with Q_{tot} (Proposition 20.3) and hence with H_{bdry}; on the probe the matter holding and record pairs $1, 2$ contribute equally before and after the write, and record pair 0 contributes $e(y)$. So the same difference $-1 - \sqrt{2}$ refutes it. (e) By Lemma 20.4 every passive flow commutes with Q_{src} and with R_0; E and C fail the first, W fails the second. □

The charge wall that excludes passive capture is therefore a wall on transport, and a second, independent wall excludes passive recording.

20.2.3 Every sector writes actively

The probe of Theorem 20.5 uses one difference sector. The write is the product of eight sector writes (Proposition 20.2), and one might hope that some sector records passively, for instance one whose tag sits at the blank's energy. None does.

Proposition 20.6 (Diagonal record energies)**.** *For every tag* $d = (d_1, d_2)$*,*

$$e(d) = \begin{cases} \dfrac{27}{2} - \dfrac{\sqrt{2}}{2}, & d_1 = 0, \\ \dfrac{29}{2} + \dfrac{\sqrt{2}}{2}, & d_1 \neq 0. \end{cases}$$

So $e(d) = e(\mathit{blank})$ *exactly when* $d_1 \neq 0$*: five of the eight nonzero sectors, namely* $(1, 0)$*,* $(1, 2)$*,* $(2, 0)$*,* $(2, 1)$ *and* $(2, 2)$*, are degenerate on the diagonal.*

Proof. By (20.3), $e(d) = \Omega_{\mathrm{rec}}(d_1, d_1) + \Omega_{\mathrm{ref}}(d_2, d_2) + 7$. The reference clock is flat on the diagonal, $\Omega_{\mathrm{ref}}(b, b) = \frac{7}{2}$, and $\Omega_{\mathrm{rec}}(1, 1) = \Omega_{\mathrm{rec}}(2, 2) = 4 + \frac{\sqrt{2}}{2}$ while $\Omega_{\mathrm{rec}}(0, 0) = 3 - \frac{\sqrt{2}}{2}$ (Table 20.1). □

Theorem 20.7 (Every record write is non-passive)**.** *Let d be one of the eight nonzero sectors. Then:*

(a) W_d commutes with Q_{src} and with Q_{tot};
(b) $[R_0, W_d] \neq 0$;
(c) $[W_d, H_m] \neq 0$ for every integer m;
(d) $U_{m,\hbar}(t) \neq W_d$ for every integer m, real $\hbar \neq 0$ and real t.

In particular no nonzero sector records passively, and the moved observable is the same record pair energy R_0 in all eight sectors.

Proof. (a) as in Theorem 20.5(b) and Proposition 20.3, sector by sector. For (b) and (c), fix cells c_1, c_2 of difference d and consider the probes $\psi(c_1, c_2, y)$ for record values y. For $y \neq y'$, the matrix element of R_0, and of the joined holding without its boundary term, between $\psi(c_1, c_2, y)$ and $\psi(c_1, c_2, y')$ equals $e(y, y')$ times one nonzero norm: the matter holding and record pairs $1, 2$ contribute only on the diagonal, because the record frame is orthonormal. If the self-adjoint involution W_d commuted with R_0, or with H_m (the boundary term drops out by (a)), every off-diagonal coupling would be invariant under exchanging the blank and d:

$$e(\mathrm{blank}, y') = e(d, y') \quad \text{for every } y' \text{ fixed by } s_d.$$

A witness y' refutes this in each sector, by (20.3) and Table 20.1.

- $d_1 = 0$, sectors $(0, 0), (0, 1), (0, 2)$: take $y' = (2, 1)$. The blank couples to it with $\Omega_{\mathrm{rec}}(1, 2) = -1 - \frac{i}{2}$, the tag d with $\Omega_{\mathrm{rec}}(0, 2)[d_2 = 1] \in \{0, -\frac{i}{2}\}$. The real parts differ.
- $d_1 \neq 0$, the five degenerate sectors: take $y' = (0, 1)$. The blank couples to it with $\Omega_{\mathrm{rec}}(1, 0) = \frac{\sqrt{2}+i}{2}$, the tag d with $\Omega_{\mathrm{rec}}(d_1, 0)[d_2 = 1]$, which is 0 except in sector $(2, 1)$, where it is $\frac{i}{2}$. The real parts differ.

(d) follows from (b) and Lemma 20.4. □

The degenerate sectors show what does the work. There the tag has the blank's energy, so no energy expectation distinguishes them, and yet the write is excluded: the record clock couples the blank and the tag differently to a third record value. The obstruction is coherent. In sectors $(2, 0)$ and $(2, 2)$ the blank and the tag differ in both indices, (20.3) has no entry between them, and the obstruction runs only through a third value; in $(1, 0)$, $(1, 2)$ and $(2, 1)$ the direct coupling is not real and obstructs as well.

Theorems 20.5 and 20.7 are operator statements. They exclude exact equality of a write with a passive flow; they give no distance between them, no energy cost, and they do not exclude time-dependent or external controls. The record frame, the placement of the six slots and the holding are supplied with the carrier. Reading the non-passive write as the activity of a boundary that registers is an identification; nothing here attaches experience to it.

20.3 Capture under specified pulses

The simplest external control drives the two involutions themselves. For a self-adjoint involution V let $\Pi_V = \frac{1}{2}(1 - V)$, so that $e^{-i\pi\Pi_V} = V$. The *exchange pulse* and the *write pulse* run the holding together with the drive $\frac{\hbar}{\varepsilon}\pi\Pi_V$ for a time ε,

$$P_V(t) = \exp\Big(-\tfrac{it}{\hbar}\big(H_m + \tfrac{\hbar\pi}{\varepsilon}\Pi_V\big)\Big), \qquad V \in \{E, W\},$$

and the *executed capture* is $P_W(\varepsilon)P_E(\varepsilon)$, exchange first. The drive is positive, its strength $\hbar\pi/\varepsilon$ is an explicit resource, and the schedule is supplied.

Proposition 20.8 (Specified capture pulses)**.** *For every integer m:*

(a) each pulse commutes with Q_{tot} at every time and conserves its own driven energy $H_m + \frac{\hbar\pi}{\varepsilon}\Pi_V$;
(b) for $\hbar > 0$ and $\varepsilon \neq 0$, $\|P_V(\varepsilon) - V\| \leq |\varepsilon|\,\|H_m\|/\hbar$ and $\|P_W(\varepsilon)P_E(\varepsilon) - C\| \leq 2|\varepsilon|\,\|H_m\|/\hbar$;
(c) the work exposed by switching the drives off telescopes exactly to $X^*H_mX - H_m$ with $X = P_W(\varepsilon)P_E(\varepsilon)$, every coherent term kept.

Proof. (a) The driven Hamiltonian commutes with Q_{tot} because H_m and V do, and with its own flow. (b) The drive alone gives $e^{-i\pi\Pi_V} = V$ at $t = \varepsilon$, and the holding perturbs it by at most $|\varepsilon|\|H_m\|/\hbar$; errors of unitary products add. (c) In each pulse the drive energy given up equals the holding energy gained; the two contributions compose. □

The bound grows with $\|H_m\|$, which grows with $|m|$, so no bound uniform in the flux follows. On the full carrier with every flux at once there is still a limit. A capture changes no loop flux, so it acts fibre by fibre on $\ell^2(\mathbb{Z}) \otimes (\mathbb{C}^8)^{\otimes 14}$, and with the boundary field written out it acts on the space of all solutions of the Gauss constraints.

Theorem 20.9 (The Gauss account of capture)**.** *Index the boundary field by the three interior edges joining the two matter sites and the shared root, with values $(m,\ m - q_L,\ Q - m)$, and by the outer edge leaving the root, with value R_{out}, the total joined charge.*

(a) Every operator U commuting with Q_{tot} has nonzero matrix entries only between configurations of equal total charge, so it drives no current on the outer edge and commutes with H_{bdry}. In particular capture drives no outer current.
(b) For every transition of nonzero capture amplitude, the electric work at flux m equals the change of the interior electric energy alone; the outer-edge price $\mathrm{el}(R_{\mathrm{out}})$ does not change. As an operator, the work is $C^*E_{\mathrm{int}}C - E_{\mathrm{int}}$ with E_{int} the interior electric diagonal, interference terms included.
(c) For every $\hbar > 0$ and every normalizable state ν on the full Gauss-constrained space, the executed capture applied fibre by fibre converges to the capture of ν as $\varepsilon \to 0^+$.

Proof. (a) In the occupation frame Q_{tot} is diagonal; commuting with it confines the entries to equal charge, and the outer-edge current is the change of total charge. H_{bdry} is a function of Q_{tot}. (b) The boundary energy splits into interior energy plus $\mathrm{el}(R_{\mathrm{out}})$, and R_{out} is unchanged by (a). (c) On the fibre at flux m the error is at most $2\varepsilon\|H_m\|\|\nu_m\|/\hbar$, which tends to zero for each m; the fibre errors are dominated by $2\|\nu_m\|$, which is square-summable, so dominated convergence over m gives the limit. No uniform operator-norm bound is used. □

Capture carries everything it moves. On a blank record the capture sends (c_1, c_2) to (c_2, c_1, δ): the incoming cell c_1 is installed at the second position and the cell c_2 it replaces is retained at the first.

Proposition 20.10 (Capture carries the internal resources)**.** *Let A be an operator on one cell of $\mathscr{H}$ that commutes with its occupation.*

(a) On inputs with a blank record, applying A to the installed cell after capture equals capturing after applying A to the incoming cell, and applying A to the retained cell after capture equals capturing after applying A to the cell it replaced, for arbitrary entangled inputs.
(b) Let $J = L^2$ on each clock cell. If the incoming cell lies in the negative internal parity $J \otimes J = -1$, then for every retained cell, every capture label and every finite word V_w of the slices of Chapter 2, the vector V_w applied to the installed cell has no component in the ready space of Chapter 6.

Proof. (a) Capture permutes basis states, and A changes neither occupation, hence not the tag δ either. (b) The projection onto $J \otimes J = -1$ commutes with the occupation, so by (a) the installed cell has negative parity for each retained cell and label. J anticommutes with every $K(b', b)$ (Proposition 2.13) and with D, so it commutes with $K(b', b)C_x$ when $C_x = D$ and anticommutes with it when $C_x = I$; both factors of a block M_{yx} carry the same C_x, so $J \otimes J$ commutes with every block, slice and word. Words therefore keep negative parity, while the ready space has both clock cells in the plane $\mathscr{U}$ where $J = +1$. □

So a resource coded in the internal clock indices, such as a ready encoding or a negative internal parity, travels with the cell and cannot be created on the way. The pulses P_E and P_W drive the target operations directly; their generators are not tokens of any control alphabet.

20.4 Serial and concurrent control

If the holding cannot capture, something must be added to it. Serial use of a literal control alphabet, with total charge conserved at every instant, cannot reach capture even as a limit; firing two of its tokens together already moves the charge that blocks it.

20.4.1 Tokens, pulses and prices

On a slot, let $\omega_{st} = E_{st} - E_{ts}$ and, for a triad s, t, u, $\mathrm{cyc}(s, t, u) = \omega_{st} + \omega_{tu} + \omega_{us}$. A *token* on the joined carrier is one of

1. BRAID(k, g, ϑ): slot k, one of the 21 triads $g = \{s, t, 7\}$ with $s < t \le 6$, an angle $\vartheta \in \mathbb{R}$; its *generator* is $\vartheta \frac{\sqrt{3}}{3} \mathrm{cyc}(s, t, 7)$ on slot k;
2. ENTANGLE(k, l): slots k, l; its generator is $\frac{\pi}{4}\, i\, S_k F_l$ if $k \neq l$ and 0 if $k = l$.

Each generator G is skew-Hermitian. Its *price* is $\|G\|$, and its *paid control* is $iG + \|G\|\, 1 \ge 0$. A token runs for a duration $\varepsilon > 0$ under the holding plus its paid control at strength $\hbar/\varepsilon$:

$$P_{m,\hbar,\varepsilon}(G) = \exp\Big(-\tfrac{i\varepsilon}{\hbar}\big(H_m + \tfrac{\hbar}{\varepsilon}(iG + \|G\|)\big)\Big).$$

The holding is never switched off, and the price is never discarded: it is the positive scalar that makes the control energy nonnegative, and it reappears as a phase. A *serial word* is a

finite list of tokens, run one after another with a common ε; its endpoint is the product of their pulses, the first token acting first. The schedule, the strength and the duration of every pulse are supplied from outside; nothing below derives a controller.

20.4.2 The serial wall

Theorem 20.11 (Serial words cannot capture, even in the limit)**.** *Call a serial word* charge-admitted *at $(m, \hbar, \varepsilon)$, $\hbar \neq 0 < \varepsilon$, if Q_{tot} commutes with the accumulated evolution at every instant of every pulse, earlier pulses included. Let $\mathcal{S}$ be the set of endpoints of all charge-admitted serial words, over all integer m, all $\hbar \neq 0$ and all $\varepsilon > 0$, and $\overline{\mathcal{S}}$ its closure in operator norm.*

(a) For any weights w, a token's generator commutes with $\sum_k w_k N_k$ if and only if the token acts only on slots of weight zero, or is BRAID *with $\vartheta = 0$, or is* ENTANGLE(k, k).
(b) Given $[Q, H] = 0$, a pulse conserves Q at every instant of $[0, \varepsilon]$ if and only if $[Q, G] = 0$.
(c) Every token of a charge-admitted word on the joined carrier has generator 0.
(d) Every element of $\overline{\mathcal{S}}$ commutes with Q_{src}; hence $zC \notin \overline{\mathcal{S}}$ for every complex $z \neq 0$.

Proof. (a) An entangler's generator fails to commute with the weighted occupation unless both its slots are neutral, and no braid direction commutes with a slot's occupation, by explicit witness entries; zero generators commute trivially. (b) Differentiate the pulse at time 0: commutation for all small times forces $[Q, H_m + \frac{\hbar}{\varepsilon}(iG + \|G\|)] = 0$; subtract $[Q, H_m] = 0$ and the scalar. The converse is immediate. (c) Every slot of the joined carrier has weight ± 1, so by (a) and (b) applied pulse by pulse every admitted token has generator 0. (d) A word of zero generators is a product of flows of H_m plus scalars, which commute with Q_{src} (Lemma 20.4); the commutant of Q_{src} is closed. Finally $[Q_{\mathrm{src}}, zC] = z[Q_{\mathrm{src}}, C] \neq 0$ by Theorem 20.5. □

The wall is the work of the charge condition. On the six capture slots alone, with no holding and no charge condition, finite products of token exponentials approximate the capture to any accuracy up to a unit phase. The theorem concerns this literal alphabet with charge conserved at every instant. It does not exclude concurrent contacts, added neutral controllers, charge exchange with a retained environment, or other carriers.

20.4.3 Concurrent bundles

A *bundle* is a finite list b of tokens fired together. Its generator is $G_b = \sum_{t \in b} G_t$, its price $\pi_b = \sum_{t \in b} \|G_t\| \geq \|G_b\|$, and its control $iG_b + \pi_b 1$: every token keeps its own price. A bundle is *charge-admitted* when G_b commutes with Q_{tot}; its tokens need not. A *concurrent word* is a list of bundles, each run for time ε under $H_m + \frac{\hbar}{\varepsilon}(iG_b + \pi_b)$.

Proposition 20.12 (Paid words track ideal words)**.** *For every integer m, $\hbar > 0$, $\varepsilon > 0$ and every concurrent word $w = (b_1, \dots, b_L)$,*

$$\left\| P_w - e^{-i \sum_\ell \pi_{b_\ell}} e^{G_{b_L}} \cdots e^{G_{b_1}} \right\| \leq L \, \frac{\varepsilon \|H_m\|}{\hbar},$$

where P_w is the paid endpoint. If every bundle is charge-admitted, P_w commutes with Q_{tot}.

Proof. For one bundle, $P_b = e^{-i\pi_b}\exp(G_b - \frac{i\varepsilon}{\hbar}H_m)$, and the exponential differs from e^{G_b} by at most $\varepsilon\|H_m\|/\hbar$, since both factors are unitary flows and the perturbation is skew. Errors of unitary products add. Charge commutes with each factor. □

The *pair entangler* is the bundle $\{\mathrm{ENTANGLE}(0,2), \mathrm{ENTANGLE}(2,0)\}$ on the matter sources 0 (weight +1) and 2 (weight −1). It is a supplied concurrent contact.

Theorem 20.13 (A concurrent contact escapes the serial wall)**.** *Let $G_\times = \frac{\pi}{4}\,i\,(S_0F_2 + F_0S_2)$ be the generator of the pair entangler.*

(a) $G_\times$ commutes with Q_{tot}, is nonzero, and equals the generator of no single token. Its price is exactly $\pi/2$.
(b) $[Q_{\mathrm{src}}, G_\times] = \frac{\pi}{4}\,(S_0S_2 - F_0F_2)$, whose entry at the all-zero basis state is $\frac{\pi}{4}$.
(c) For every integer m, $\hbar \neq 0$, $\varepsilon \neq 0$ and every duration $\tau > 0$ there is a time $t \in [0,\tau]$ at which the evolution under $H_m + \frac{\hbar}{\varepsilon}(iG_\times + \frac{\pi}{2})$ does not commute with Q_{src}, and so lies outside $\overline{\mathcal{S}}$.

Proof. (a) Each entangler changes the occupation of a charged slot, but the two slots carry opposite weights and the changes cancel in the sum; the sum has nonzero entries. A single token with a charge-commuting generator has generator 0 (Theorem 20.11(c)). Each entangler has norm $\pi/4$. (b) Q_{src} counts slot 0 with weight 1 and slot 2 with weight 0; direct computation. (c) If the evolution commuted with Q_{src} throughout $[0,\tau]$, differentiation at 0 would give $[Q_{\mathrm{src}}, H_m + \frac{\hbar}{\varepsilon}iG_\times] = 0$, hence $[Q_{\mathrm{src}}, G_\times] = 0$, contradicting (b). Theorem 20.11(d) then excludes the closure. □

Part (c) gives some time in each interval, not a prescribed endpoint. The concurrent contact moves the pair charge that capture moves; it does not synthesize capture.

20.5 Writing records under control

The write half of capture can be built from charge-admitted bundles, in stages that end with a writer of the full nine-valued tag. Each stage is an exact identity for ideal exponentials and a paid word within an explicit error. None of them exchanges the cells.

20.5.1 The record alphabet

Let $\tilde{B} = B\,\mathrm{diag}(i,1,1,1)$, the occupation-zero isometry with its first column rephased. A record slot carries the *digits* 0, 1, 2 as the first three columns of $\tilde{B}$; digit 1 is the blank. The nine tags of two record slots are the columns of $\tilde{B}_3 \otimes \tilde{B}_3$, where $\tilde{B}_3$ keeps three columns, and this *nine-tag chart* is an isometry $\mathbb{C}^9 \to \mathbb{C}^8 \otimes \mathbb{C}^8$. Two real skew generators move the digits:

$$L_{\mathrm{lo}} = \frac{\sqrt{3}}{3}\big(\mathrm{cyc}(0,2,3) + \mathrm{cyc}(0,3,1)\big), \qquad L_{\mathrm{up}} = \frac{\sqrt{3}}{3}\big(\mathrm{cyc}(2,3,5) + \mathrm{cyc}(3,4,5)\big).$$

Each is the summed generator of a four-token BRAID bundle on one slot, each anticommutes with S and F, and so each commutes with q and N although none of its tokens does.

Proposition 20.14 (A ternary record cycle)**.** *With $X_\bullet = \frac{\sqrt{3}}{2}L_\bullet$ one has $X_\bullet^3 = -X_\bullet$, hence $e^{tX_\bullet} = 1 + \sin t\,X_\bullet + (1-\cos t)X_\bullet^2$. The half turns $e^{\pi X_{\mathrm{lo}}}$ and $e^{\pi X_{\mathrm{up}}}$ exchange digits $0 \leftrightarrow 1$ and $1 \leftrightarrow 2$*

exactly, column by column of $\tilde{B}$, and their product $Y = e^{\pi X_{\mathrm{lo}}} e^{\pi X_{\mathrm{up}}}$ sends digit $0 \mapsto 1 \mapsto 2 \mapsto 0$. On any slot k, the two-bundle paid word that enacts Y satisfies $\|P - e^{-i\pi_w Y_k}\| \le 2\varepsilon\|H_m\|/\hbar$ for $\hbar, \varepsilon > 0$, and commutes with Q_{tot}.

Proof. The cube identity is a finite matrix identity; the exponential formula follows from it through the spectral projections $1 + X^2$ and $-\frac{1}{2}(X^2 \pm iX)$. At $t = \pi$, $e^{\pi X} = 1 + 2X^2$, which acts on the four-cycle of each generator by an exact transposition of two columns of $\tilde{B}$; the lower one carries the phase i that the rephasing of column 0 absorbs. The paid bound is Proposition 20.12 with $L = 2$. □

A record slot can also be moved without any condition. Six BRAID tokens on slot 8 with summed generator $-\frac{2\pi}{3} \cdot \frac{\sqrt{3}}{3}\big(\mathrm{cyc}(0,3,5) + \mathrm{cyc}(1,2,4)\big)$ commute with the charge as a bundle and carry the blank $(1,1)$ of record pair 0 exactly to the tag $(2,1)$, with total price at most 4π. That is a record transfer that ignores the cells, which is not what capture does. Capture writes a tag that depends on the occupations it reads.

20.5.2 A record rotation selected by occupation

For slots $k \neq r$ and a slot operator K anticommuting with S and F, the *selected rotation* is

$$\mathrm{Sel}_{k,r}(K) = \tfrac{1}{2}\big(1 - q_k q_r\big)K_r.$$

On the sector where k is occupied and r is vacant, $q_k q_r = -1$ and it acts as K_r; where both are vacant it vanishes. Nothing selects this sector by hand: two charge-admitted bundles produce it.

Theorem 20.15 (Occupation-selected record rotation)**.** *Write L_{up} for the upper record generator on the record source slot 8, and let $\Xi = i(S_2F_8 + S_8F_2)$, so that the bundle $\{\mathrm{ENTANGLE}(2,8), \mathrm{ENTANGLE}(8,2)\}$ has generator $\frac{\pi}{4}\Xi$.*

(a) $\Xi^2 = -2(1 + q_2q_8)$ and $\Xi L_{\mathrm{up}} = -L_{\mathrm{up}}\Xi$, so $[\Xi,[\Xi,L_{\mathrm{up}}]] = -8(1+q_2q_8)L_{\mathrm{up}}$ and

$$L_{\mathrm{up}} + \tfrac{1}{16}[\Xi,[\Xi,L_{\mathrm{up}}]] = \mathrm{Sel}_{2,8}(L_{\mathrm{up}});$$

this direction lies in the real Lie algebra generated by the two bundle generators.

(b) With $U_{28} = e^{\pi\Xi/4}$ one has $U_{28}^2 = -q_2q_8$, $U_{28}^4 = 1$, and for every real ϑ

$$e^{\vartheta L_{\mathrm{up}}/2}\, U_{28}\, e^{\vartheta L_{\mathrm{up}}/2}\, U_{28}^3 = e^{\vartheta\,\mathrm{Sel}_{2,8}(L_{\mathrm{up}})}.$$

(c) At $\vartheta = \pi\sqrt{3}/2$ this evolution carries the blank digit of slot 8 to digit 2 on the sector where slot 2 is occupied and slot 8 vacant, and is the identity where both are vacant; every other slot is untouched.

(d) The six-bundle, sixteen-token paid word for (b) satisfies $\|P - e^{-i\pi_w e^{\vartheta\mathrm{Sel}_{2,8}(L_{\mathrm{up}})}}\| \le 6\varepsilon\|H_m\|/\hbar$ and commutes with Q_{tot}; its price is at most $4|\vartheta| + 2\pi$.

Proof. (a) Each crossing squares to 1, and $S_2F_8\,S_8F_2 = S_8F_2\,S_2F_8 = q_2q_8$ because $FS = -SF = iq$; this gives Ξ^2; L_{up} anticommutes with S_8 and F_8. For anticommuting A, A' one has $[A,[A,A']] = 4A^2A'$. (b) Conjugation by U_{28} sends L_{up} to $-q_2q_8L_{\mathrm{up}}$, which commutes with

L_{up}, and $\mathrm{Sel}_{2,8}(L_{\mathrm{up}})$ is the average of the two; $U_{28}^{3} = U_{28}^{-1}$ uses only positive contacts. (c) On the sector the selected direction is L_{up} or 0, and $e^{\pi X_{\mathrm{up}}}$ takes digit 1 to digit 2. (d) Proposition 20.12 with $L = 6$; the contact bundle costs exactly $\pi/2$, each record bundle at most $4|\vartheta|/2$. □

The same selection works through a third slot. Let $U_{kl} = e^{\pi i(S_k F_l + S_l F_k)/4}$ be the paired contact, so that U_{28} is the contact of Theorem 20.15. Then $U_{kl} q_l U_{kl}^{-1} = -q_k$: a single contact exchanges local charge with a sign, $U_{kl}^{2} = -q_k q_l$, and $U_{kl}^{4} = 1$. For distinct slots k, a, r the *relay* $\mathrm{Rel} = U_{ka} U_{ar}$ satisfies $\mathrm{Rel}\, K_r\, \mathrm{Rel}^{-1} = K_r q_k q_r$ for every K anticommuting with S and F, with no condition on the state of a. (This relay is a product of contacts; it is unrelated to the relay operator of Chapter 18.) Hence

$$e^{\vartheta K_r/2}\, \mathrm{Rel}\, e^{-\vartheta K_r/2}\, \mathrm{Rel}^{-1} = e^{\vartheta\, \mathrm{Sel}_{k,r}(K)}.$$

With the reference slot $a = 2$ (weight -1) and a matter slot k and record slot r of weight $+1$, the paid word has ten bundles and twenty-four tokens, conserves Q_{tot}, and lies within $10\varepsilon\|H_m\|/\hbar$ of its phased ideal. For matter slot 0 and record slot 8 the ideal word acts on the sector "slot 0 occupied, slot 8 vacant" as the record rotation $e^{\vartheta (L_\bullet)_8}$, with $L_\bullet$ either record generator.

20.5.3 Writers of the difference

Put the two selected half turns together, $Y_{k,r} = e^{\frac{\pi\sqrt{3}}{2}\mathrm{Sel}_{k,r}(L_{\mathrm{lo}})}\, e^{\frac{\pi\sqrt{3}}{2}\mathrm{Sel}_{k,r}(L_{\mathrm{up}})}$. On the sector "$k$ occupied, r vacant" it is the ternary cycle Y on r; where both are vacant it is the identity. Its paid word, through reference slot 2, has twenty bundles, forty-eight tokens and error at most $20\varepsilon\|H_m\|/\hbar$. For a source slot k, a donor slot d and a record slot r, the *digit writer* is

$$\mathrm{Wri}(k, d, r) = Y_{d,r}^{2}\, Y_{k,r}.$$

Theorem 20.16 (The difference writer). *(a) For distinct k, d, r and occupation bits b of k and c of d, on the sector with r vacant,* $\mathrm{Wri}(k, d, r)$ *advances the digit of r by $b + 2c \equiv b - c \pmod 3$. From the blank it writes the digit $\delta(b, c) = 1 + b - c$, and it commutes with every operator on a slot outside $\{k, d, r\}$.*

(b) The pair writer $\mathrm{Wri}(1, 7, 9)\,\mathrm{Wri}(0, 6, 8)$ *reads the source slots* 0, 1 *and the donor slots* 6, 7 *(the two cells of the capture) and writes, from blank digits in the record source slots* 8 *and* 9*, the tag $\delta\big(\mathrm{occ}(0, 1), \mathrm{occ}(6, 7)\big)$. Every one of the nine tags arises from some occupation pattern.*

(c) On the whole carrier with blank digits in slots 8, 9 *the pair writer is exact for arbitrary coherent coefficients: it maps the isometric blank chart onto the isometric written chart, column by column, and preserves every correlation with a retained reference.*

(d) Its paid word has 120 *bundles and* 288 *tokens, conserves Q_{tot}, and for $\hbar, \varepsilon > 0$ satisfies*

$$\big\| P - e^{-i\pi_w}\, \mathrm{Wri}(1, 7, 9)\, \mathrm{Wri}(0, 6, 8) \big\| \le 120\, \varepsilon \|H_m\|/\hbar.$$

Proof. (a) On the joint sector, $Y_{k,r}$ acts as Y^{b} and $Y_{d,r}$ as Y^{c} on the record, so the writer acts as Y^{b+2c}; from digit 1 this gives $1 + b - c$ mod 3, which is $\delta(b, c)$ because $b - c \in \{-1, 0, 1\}$. Each factor commutes with operators on other slots, including the reference. (b) The two writers act on disjoint slots and commute. The nine tags are reached by a finite check. (c) The sixteen occupation sectors of the four matter slots resolve the vacancy projection of the two record slots, and the blank and written charts are isometries. (d) Proposition 20.12. □

This is the write half of capture only. The pair writer stores the tag that capture would store, in two record source slots rather than in record pair 0, and it moves neither cell. The blank digits and the choice of donor slots are supplied.

20.5.4 What the controller spends

For a concurrent word with endpoint P let the *switching work* be the operator accumulated by switching each applied control $V_b = \frac{\hbar}{\varepsilon}(iG_b + \pi_b)$ on and off, pulled back to the initial frame.

Proposition 20.17 (Switching account)**.** *For every concurrent word, integer m and real $\hbar, \varepsilon$, the switching work equals $P^* H_m P - H_m$. It is self-adjoint and has norm at most $2\|H_m\|$; no sign is asserted. For $\hbar \geq 0 < \varepsilon$ the summed control norm is at most $2\frac{\hbar}{\varepsilon}\pi_w$, the on/off variation at most $4\frac{\hbar}{\varepsilon}\pi_w$, and the time-integrated scalar baseline equals $\hbar\pi_w$ exactly. For the selected circuit of Theorem 20.15 at $\vartheta = \pi\sqrt{3}/2$, the duration $\varepsilon = \tau\hbar/(6(\|H_m\| + 1))$ brings the error below any $\tau > 0$.*

Proof. In each interval the total Hamiltonian $H_m + V_b$ is conserved by its own flow, so the control energy given up equals the holding energy gained; telescoping over the word gives the formula, and the norm bound follows from unitarity. The control bounds use $\|iG + \|G\|\| \leq 2\|G\|$. The baseline integrates $\frac{\hbar}{\varepsilon}\pi_b$ over a time ε. □

20.6 Exchanging cells: what ideal contacts keep and the holding moves

The other half of capture is the exchange of two whole cells. Contacts exchange charge, and a relay of them exchanges occupations exactly. They do not exchange whole cells, and the reason is a local sector that every ideal control keeps; the holding is what moves it.

Proposition 20.18 (Signed and unsigned charge exchange)**.** *For distinct slots k, a, l let $\mathrm{Xch}(k, a, l) = U_{ka} U_{al} U_{ka}^3$, five positive paired contacts through a. Then*

$$\mathrm{Xch}\, q_k = q_l\, \mathrm{Xch}, \qquad \mathrm{Xch}\, q_l = q_k\, \mathrm{Xch}, \qquad \mathrm{Xch}\, q_a = q_a\, \mathrm{Xch},$$

the same holds for the occupations, and Xch *commutes with every operator on a fourth slot. With $a = 2$ its paid word has five bundles and ten tokens, lies within $5\varepsilon\|H_m\|/\hbar$ of its phased ideal for $\hbar, \varepsilon > 0$, and conserves Q_{tot} when k and l have weight $+1$.*

Proof. A single contact sends q_l to $-q_k$ and q_k to $-q_l$; along the relay the two signs cancel, and the reference charge returns. Occupations are $\frac{1}{2}(1 - q)$. The paid bound is Proposition 20.12. □

So the occupation of two same-species cells can be exchanged by literal contacts, with the reference restored. The internal clock payload of the cells is a different matter.

Theorem 20.19 (Ideal controls keep a local sector; the holding moves it)**.** *On a slot let Σ be the rank-two projection with entries $\Sigma_{st} = \frac{1}{8}(1 + \sigma_s\sigma_t)$, $\sigma = \mathrm{diag}\, S$.*

(a) Σ *commutes with S, F and q. Every skew generator that commutes with q and annihilates the uniform vector, in particular every charge-admitted sum of* BRAID *generators on the slot, commutes with* Σ*; so does every crossing* $S_k F_l$*, every paired contact and every relay* Xch*, placed anywhere.*

(b) The complete exchange of two distinct slots $r \neq s$*, which moves all eight states, does not commute with* Σ_r*. Hence* $\mathrm{Xch}(r, a, s)$ *is not the complete exchange of r and s, for any a.*

(c) The charge-diagonal clock h_{d} *does not commute with* Σ*:* $[\Sigma, h_{\mathrm{d}}]_{00} = \frac{i}{4}(\sqrt{2} - 1)$*.*

(d) For every record pair $r \in \{0, 1, 2\}$ *and every integer m, the joined holding* H_m *does not commute with* Σ *on the record source slot of pair r; the matter part* $H_m^{\mathrm{mat}} \otimes 1$ *does commute with it.*

Proof. (a) A generator annihilating the uniform vector u and commuting with q annihilates Σ, whose range is spanned by u and Su, since Su is $-i\,qu$; skewness gives commutation. Crossings commute with Σ because Σ commutes with S and F. (b) The complete exchange carries Σ_r to $\Sigma_s \neq \Sigma_r$. (c) A direct entry. (d) In (20.1) the reference terms $1 \otimes (\mathrm{pay}(\Omega_{\mathrm{ref}}) + 7)$ commute with $\Sigma \otimes 1$, and the three clock terms carry the linearly independent reference factors 1, $NF(1-N)$, $(1-N)FN$; so $[\Sigma \otimes 1, H_{\mathrm{pair}}] = 0$ would force $[\Sigma, h_{\mathrm{d}}] = 0$, against (c). Different pairs commute, H_{bdry} commutes because Σ commutes with the charges, and the matter part acts on other slots. □

The obstruction in (b) concerns ideal controls. The paid words keep the holding switched on, and by (d) the holding moves exactly the sector that ideal controls cannot leave, on every record pair at every flux. Whether that motion can be harnessed into a complete payload exchange is not settled, and for the sector on the matter slots no analogue of (d) is established.

20.7 The paid control algebra

The writers above use particular schedules. The next question is how much a schedule of charge-admitted bundles can reach at all, with the holding always on and every token priced.

20.7.1 Positive paid words and their Lie algebra

A *segment* is a triple (b, ε, τ): a charge-admitted bundle b, a strength scale $\varepsilon > 0$ and a duration $\tau > 0$. It acts by

$$\exp\Big(-\tfrac{i\tau}{\hbar}\big(H_m + \tfrac{\hbar}{\varepsilon}(iG_b + \pi_b)\big)\Big),$$

and a *paid word* is a finite product of segments. Its *time* is the sum of the durations and its *exposure* the time integral $\sum \tau \frac{\hbar}{\varepsilon}\pi_b$ of the scalar price. Let $\mathscr{M}_{m,\hbar}$ be the closure of the set of paid words, a closed monoid of unitaries, and

$$\mathfrak{p}_{m,\hbar} = \{X \text{ skew-Hermitian} : \ e^{tX} \in \mathscr{M}_{m,\hbar} \text{ for all real } t\},$$

the *paid Lie algebra*: it is a real Lie algebra, sums coming from product limits and brackets from derivatives of conjugations. Let $\mathfrak{n}_m$ be the real Lie algebra generated by iH_m and the generators of all charge-admitted bundles.

Theorem 20.20 (What paid words reach)**.** *Let* $\hbar > 0$ *and* $m \in \mathbb{Z}$*.*

(a) Every paid word commutes with Q_{tot} *at every instant of every segment.*

(b) $\mathfrak{n}_m \subseteq \mathfrak{p}_{m,\hbar}$, *and* $i \cdot 1 \in \mathfrak{p}_{m,\hbar}$.
(c) For every $X \in \mathfrak{p}_{m,\hbar}$, *real* t *and* $\eta > 0$ *there is a paid word of positive time within* η *of* e^{tX} *in operator norm, global phase included.*
(d) $\mathfrak{p}_{m,\hbar}$ *is invariant under conjugation by every ideal word of charge-admitted bundles. In particular, for slots* $k \neq a$ *of opposite weight, two paired contacts give* $-q_k q_a$, *and the parts of any* $X \in \mathfrak{p}_{m,\hbar}$ *even and odd under conjugation by* $q_k q_a$ *again lie in* $\mathfrak{p}_{m,\hbar}$.

Proof. (a) Each segment's Hamiltonian commutes with Q_{tot}. (b) A segment's flow lies in $\mathscr{M}$ for all positive times; negative times are reached in the closure because the flow of a skew generator on a finite-dimensional space recurs, so for any s there are arbitrarily large positive times at which the flow is arbitrarily close to its value at s. Hence $i\big(H_m + \frac{\hbar}{\varepsilon}(iG_b + \pi_b)\big) \in \mathfrak{p}$ for every admitted b and $\varepsilon > 0$. The empty bundle gives iH_m, and subtracting gives $i(iG_b + \pi_b)$. The bundle $\{\mathrm{BRAID}(k, g, \vartheta), \mathrm{BRAID}(k, g, -\vartheta)\}$ has generator 0 and any prescribed price, which gives $i \cdot 1$, and then G_b itself. (c) Approximate $e^{tX} \in \mathscr{M}_{m,\hbar}$ by a word; recurrence adds positive time. (d) Ideal words of admitted bundles and their inverses lie in $\mathscr{M}$, and conjugation preserves $\mathfrak{p}$. Two contacts $U_{ka}^2 = -q_k q_a$. □

Part (c) is existence. No bound on the number of segments, the total time or the exposure is given. Any finite word of admitted bundles can also be completed by one phase bundle so that its total price hits a prescribed phase, with paid error at most $(L + 1)\varepsilon\|H_m\|/\hbar$ for L bundles of duration ε.

20.7.2 Payload algebras on single slots

The holding enters the paid algebra only through $\mathrm{ad}(iH_m)$, its *drift*, and the drift is what turns a few literal directions into whole algebras. Two clocks carry it: on a reference slot the drift acts on the payload through Ω_{ref}; on a source or matter slot, after averaging over the two-contact conjugation of (d), it acts through the charge-diagonal clock h_{d}. Write $\mathrm{blk}(A, A') = BAB^* + B'A'B'^*$ for independent actions on the two occupation halves. Let $\beta_6 = \frac{\sqrt{3}}{3}\big(\mathrm{cyc}(0,3,5) + \mathrm{cyc}(1,2,4)\big)$, the summed generator of a charge-admitted six-token bundle on one slot.

Theorem 20.21 (Payload algebras)**.** *Let* $\hbar > 0$ *and* $m \in \mathbb{Z}$.

(a) References. *Every real Lie algebra of* 4×4 *matrices that contains* $B^*\beta_6 B$ *and is invariant under* $\mathrm{ad}(i\Omega_{\mathrm{ref}})$ *contains* $\mathfrak{su}(4)$. *Consequently* $\mathrm{pay}(A)$ *on each record reference slot* $11, 12, 13$ *lies in* $\mathfrak{n}_m \subseteq \mathfrak{p}_{m,\hbar}$ *for every* $A \in \mathfrak{su}(4)$, *and on each matter reference slot* $4, 5, 6, 7$ *it lies in* $\mathfrak{p}_{m,\hbar}$.
(b) Record sources. *Let* $w_0 = \beta_6$ *and* $w_{n+1} = [ih_{\mathrm{d}}, w_n]$, *and let* $\mathfrak{s}$ *be the real Lie algebra they generate. Then* $w_1 \neq 0$, *with entry* $\frac{\sqrt{3}}{3}\big(\frac{3}{2} + \sqrt{2}\big)$ *at position* $(0, 2)$. *On each record source slot* $8, 9, 10$, $\mathfrak{s}$ *lies in* $\mathfrak{p}_{m,\hbar}$; $\mathfrak{s}$ *commutes with* q; *and* $\mathfrak{s}$ *contains* $\mathrm{blk}(A, A')$ *for all* $A, A' \in \mathfrak{su}(4)$.
(c) Matter. *On the matter slots* 0 *and* 1 *the local clock compressed through the matter cell frame equals* h_{d}, *the drift localizes there exactly (contact partner slot* 11*), and* $\mathrm{blk}(A, A') \in \mathfrak{p}_{m,\hbar}$ *on slots* $0, 1$ *for all* $A, A' \in \mathfrak{su}(4)$.
(d) Conditional phases. *For slots* $k \neq j$ *of opposite weight,* $q_k(q\gamma)_j \in \mathfrak{p}_{m,\hbar}$, *where* $\gamma = \frac{\sqrt{3}}{3}(\mathrm{cyc}(0,1,7) + \mathrm{cyc}(1,6,7))$ *is the generator of a two-token bundle: a payload on slot* j *whose sign is set by the charge of slot* k.

Proof. (a) $B^*\beta_6 B = \frac{\sqrt{3}}{3}\beta$ with an explicit β. Starting from $i\Omega_{\mathrm{ref}}$ and β, sixteen iterated brackets are explicit 4×4 matrices over $\mathbb{Q}(\sqrt{2}, i)$, and the four edge generators $\omega_{01}, \omega_{12}, \omega_{23}$ and $i(E_{01}+E_{10})$ of $\mathfrak{su}(4)$ are exact real combinations of thirteen of them with coefficients in $\mathbb{Q} + \mathbb{Q}\sqrt{2}$; those four generate $\mathfrak{su}(4)$. On a reference slot, record or matter, pay(A) commutes with N and F, hence with every term of the holding except the reference's own pay(Ω_{ref}), so $[iH_m, \mathrm{pay}(A)] = \mathrm{pay}([i\Omega_{\mathrm{ref}}, A])$; and β_6 there is a bundle generator. (b) The two-contact average with slot 2 of $[iH_m, X_{8+r}]$ is $[ih_{\mathrm{d}}, X]_{8+r}$ for X commuting with q, so $\mathfrak{p}$ is closed under the drift on the slot. The block statement is an exact finite certificate: thirty explicit 8×8 source words and eight edge identities, each edge of $\mathfrak{su}(4) \oplus \mathfrak{su}(4)$ an exact real combination of all thirty words with coefficients in $\mathbb{Q} + \mathbb{Q}\sqrt{2}$. (c) The matter-frame clock identity is exact; the localization uses the charge of slot 11, whose square cancels. (d) With a the normalized contact between k and j and $d = \gamma_j$, $a^2 = -\frac{1}{2}(1 + q_k q_j)$ and $ad = -da$ give $\frac{1}{2}[a, [d, a]] - d = d\, q_k q_j$, and both a and d are paid. □

Everything in $\mathfrak{s}$ commutes with the slot's charge. The local algebra on a single source or matter slot is $\mathfrak{su}(4) \oplus \mathfrak{su}(4)$, not $\mathfrak{su}(8)$: no local paid direction mixes the two occupation halves. Mixing needs a partner slot.

20.7.3 Matched control

On a source/reference pair the *matched code* is the isometry

$$\mu = (1 - N) \otimes B + N \otimes B' \;:\; \mathbb{C}^8 \otimes \mathbb{C}^4 \to \mathbb{C}^8 \otimes \mathbb{C}^8,$$

which puts the reference in the half matching the source's occupation, as the holding's reference attachment does. A logical operator A on the 32-dimensional code is *effective* if some operator $\widehat{A}$ on the pair, placed in the joined carrier, lies in $\mathfrak{p}_{m,\hbar}$ and satisfies $\widehat{A}\mu = \mu A$: an actual invariant witness, not a compression.

Theorem 20.22 (Matched control)**.** *Let $\hbar > 0$ and $m \in \mathbb{Z}$.*

(a) For each record pair $(8 + r, 11 + r)$, all of $\mathfrak{su}(32)$ is effective on its matched code.
(b) For the record pairs 0 and 1 jointly, all of $\mathfrak{su}(1024)$ is effective on the code $\mu \otimes \mu$; the coupling between the two pairs is witnessed by the conditional phase $q_8(q\gamma)_{12}$.
(c) For every A in $\mathfrak{su}(32)$ (or $\mathfrak{su}(1024)$), real t and $\eta > 0$, one paid word of positive time acts within η of e^{tA} on the whole code embedded in the register, uniformly over all states of the spectator slots.
(d) Under the supplied relabelling that places matched pair 0 at matter slot 0 with reference 4 and matched pair 1 at record slot 8 with reference 11, the same holds: $\mathfrak{su}(32)$ on each, $\mathfrak{su}(1024)$ jointly.

Proof. (a) The paired contact $\frac{i}{2}(S \otimes F + F \otimes S)$ acts on the code as the source parity rotation $iS \otimes 1$, which mixes the two charge blocks; with the block algebras of Theorem 20.21(b) it generates $\mathfrak{su}(8)$ on the source factor. The reference payload gives $1 \otimes \mathfrak{su}(4)$, and the conditional direction $q \otimes B^* \gamma B$ has an explicit edge with coefficient $-\frac{\sqrt{3}}{3}$ joining the factors; together they generate $\mathfrak{su}(32)$. (b) Two copies of (a) and one cross edge. (c) Theorem 20.20(c), transported by the code isometry. (d) The relabelling only moves coordinates; paid membership at the new slots

is proved from the matter payload, the record-source payload, the reference payloads at slots 4 and 11 and the opposite weights of each pair. □

20.8 Preparing matched records from blanks

Effectiveness is control within a code, not preparation of it. Preparation is the step that puts a reference into the half that matches its source. Here it is done by a contact that the source's occupation switches, and the only thing supplied is a pair of slots at occupation zero.

For slots $b \neq j$ let $\mathscr{B}(b,j) = (1-N_b)(1-N_j)$, the projection onto states in which b and j both have occupation zero, every payload free. For distinct slots s, b, j the *controlled preparation direction* is

$$G(s,b,j) = N_s \cdot \tfrac{i}{2}\big(S_bF_j + S_jF_b\big) \ :$$

the normalized contact between a *bank* b and a reference j, switched by the occupation of the source s.

Theorem 20.23 (Charge-compensated preparation)**.** *Let $\hbar > 0$ and $m \in \mathbb{Z}$.*

(a) If $w_b = w_s = -w_j$ and every pay(A), *$A \in \mathfrak{su}(4)$, on slot j is paid, then $G(s,b,j) = \frac{1}{2}\big(1 - q_s\big)\frac{i}{2}(S_bF_j + S_jF_b) \in \mathfrak{p}_{m,\hbar}$. This holds in particular for $(s,b,j) = (0,8,4)$, $(8,0,11)$, $(9,1,12)$, $(8,6,11)$, $(9,7,12)$, $(0,6,4)$, $(8,7,11)$ and $(0,7,4)$.*

(b) On the range of $\mathscr{B}(b,j)$, $e^{\frac{\pi}{2}G(s,b,j)}$ is the identity where s is vacant, and where s is occupied it moves both b and j to the occupied half of their slots with the sign -1. Afterwards the occupation of j equals that of s. The map is an isometry on that range; the source, every payload and all other slots are kept, entanglement included.

(c) For every $\eta > 0$ there is a paid word of positive time and nonnegative exposure that commutes with Q_{tot} and lies within η of $e^{\frac{\pi}{2}G(0,8,4)}$ in operator norm, hence on every unit vector of the whole carrier.

Proof. (a) Write $Z = \frac{i}{2}(S_bF_j + S_jF_b)$, which is paid because b and j have opposite weights. Let $P = q_b\,\mathrm{pay}(\varphi)_j$ for a fixed payload φ with $P^2 = -1$ and $PZ = -ZP$. Both P and q_sP are paid (conditional payloads built from the payloads on j, as in Theorem 20.21(d)), and q_s commutes with Z and with P. A bracket identity in these four elements gives $q_sZ \in \mathfrak{p}_{m,\hbar}$, and then $\frac{1}{2}(Z - q_sZ) = N_sZ$. The listed triples have the required weights, and their payloads are paid by Theorem 20.21. (b) Z restricted to the occupation-zero halves of b and j generates the rotation to the occupied halves, and $e^{\frac{\pi}{2}Z}$ completes it with the phase -1; the factor N_s selects the occupied source. (c) Theorem 20.20(c). □

The bank absorbs the charge that the reference takes on, so the preparation conserves Q_{tot} exactly. Two such preparations compose. With the four slots $0, 11, 1, 12$ blank, the product $e^{\frac{\pi}{2}G(9,1,12)}e^{\frac{\pi}{2}G(8,0,11)}$ maps the blank range isometrically into the joint matched code of record pairs 0 and 1, and one paid word of positive time loads that range and then performs any $\mathfrak{su}(1024)$ flow of Theorem 20.22 within any $\eta > 0$, uniformly on the blank range. With the banks moved to the native references 6 and 7, the loading commutes with every operator on the matter slots 0 and 1, charge coherence included: it leaves the matter untouched.

What these results consume is a supply of blank slots. The native step can provide one, at a price.

Let the two slots of a native cell each carry the eight-point Fourier mode $u(t) = e^{-i\pi t/4}/\sqrt{8}$, $t \in \mathbb{Z}_8$; this *canonical input* $u \otimes u$ is supplied. Its occupation-$(0,0)$ block is $u_0 \otimes u_0$ with $u_0 = B^* u$. Consider the two six-step occupation histories

$$(0,0) \to (1,0) \to (0,0) \to (0,1) \to (1,1) \to (1,0) \to (0,0),$$
$$(0,0) \to (1,0) \to (0,0) \to (1,0) \to (1,1) \to (0,1) \to (0,0),$$

with labels

$$(1,0,0),\ (0,1,-1),\ (0,0,1),\ (0,-1,1),\ (0,0,-1),\ (0,1,-1),$$
$$(1,0,0),\ (0,1,-1),\ (1,0,0),\ (1,1,0),\ (-1,0,0),\ (-1,-1,0),$$

respectively, and put $\varrho_+ = p \otimes a$ (the ready vector of Chapter 2), $\varrho_- = a \otimes p$.

Proposition 20.24 (A recorded supply of blanks)**.**

(a) Each history shifts the loop flux by exactly $+1$. The product of its six blocks M_{yx} is the rank-one matrix $\frac{1}{16}\varrho_+(A \otimes P)^\top$, respectively $\frac{1}{16}\varrho_-(P \otimes A)^\top$, on the internal indices.

(b) On the canonical input both slots end at occupation zero carrying the internal payload $\frac{1}{128}\varrho_\pm$. The branch's load is exactly $\frac{1}{16384}$, the same for every coherent input of the remaining slots, and it is not renormalized. With a field profile $\alpha(n)$ the load at flux n is $|\alpha(n-1)|^2/16384$.

(c) After either branch, one paid word of positive time prepares the matched pair within η, uniformly in the retained inputs.

Proof. (a) The loop currents j_2 of the six labels add to $+1$ in both histories. The block products are a direct exact computation from the blocks of Appendix A.1. (b) The overlaps are $(A \otimes P)^\top(u_0 \otimes u_0) = (P \otimes A)^\top(u_0 \otimes u_0) = \frac{1}{8}$, so the coefficient is $\frac{1}{16} \cdot \frac{1}{8} = \frac{1}{128}$ and the load is its square, since $\varrho_\pm$ is a unit vector; the remaining slots are untouched, and the field profile enters through the flux shift. (c) The paid word is that of Theorem 20.23(c) at tolerance 128η, since the branch has norm $\frac{1}{128}$. □

The recorded branch is not a paid word. It is a branch of the native history recorder, kept with its field records and its small load, and nothing here admits it into the control grammar. The recorder itself writes onto blanks reversibly: restricted to inputs supported on a window of loop fluxes $[L,U]$, k recorded steps form a finite isometry V into the window $[L-k, U+k]$ with every history kept, and the reversible completion of Proposition 2.16, built from V, sends $(v,0)$ to $(0,Vv)$, so a blank output register is written exactly. Like that completion, it supplies no loading process, energy budget or physical interaction that selects it.

20.9 Moving matter into a record, and what it costs

The construction loads two matched pairs, swaps them inside the code, and decodes.

Let $\mathscr{P} = \mathscr{B}(6,4)\,\mathscr{B}(7,11)$: slots $4, 6, 7, 11$ blank. *Loading* is $\Lambda = e^{\frac{\pi}{2}G(8,7,11)} e^{\frac{\pi}{2}G(0,6,4)}$: the matter slot 0 matches its reference 4 through bank 6, the record source 8 matches its reference 11

through bank 7. Under the supplied relabelling of Theorem 20.22(d) the loaded range is the joint matched code of the pairs $(0,4)$ and $(8,11)$. Let X_{ex} be the permutation that exchanges the complete contents of slots $0 \leftrightarrow 8$ and $4 \leftrightarrow 11$, and $T_{\mathrm{tr}} = X_{\mathrm{ex}}\mathscr{P}$the *transfer target.*

Theorem 20.25 (Complete matter-to-record transfer)**.** *(a) For every operator X on a slot, $T_{\mathrm{tr}}X_0 = X_8T_{\mathrm{tr}}$ and $T_{\mathrm{tr}}X_8 = X_0T_{\mathrm{tr}}$: the complete eight-state content of the matter slot, charge coherence included, moves into the record slot and back. Moreover $T_{\mathrm{tr}}^*T_{\mathrm{tr}} = \mathscr{P}$, $T_{\mathrm{tr}}\mathscr{P} = \mathscr{P}T_{\mathrm{tr}} = T_{\mathrm{tr}}$ and $T_{\mathrm{tr}}^2 = \mathscr{P}$.*

(b) On the code, X_{ex} is the exchange of the two 32-state matched factors, whose trace-free logarithm $K_{\mathrm{ex}} \in \mathfrak{su}(1024)$ satisfies $e^{K_{\mathrm{ex}}} = \phi X_{\mathrm{ex}}$ with a unit scalar ϕ. Decoding by $\big(e^{\frac{\pi}{2}G(0,7,4)}e^{\frac{\pi}{2}G(8,6,11)}\big)^{-1}$, the relabelled loading, returns $X_{\mathrm{ex}}\Lambda\mathscr{P}$ to T_{tr} exactly.

(c) For every $\hbar > 0$, $m \in \mathbb{Z}$ and $\eta > 0$ there is one paid word W of positive time with

$$\|W\mathscr{P} - \phi T_{\mathrm{tr}}\| < \eta, \qquad \|(1-\mathscr{P})W\mathscr{P}\| < \eta,$$

and the same for every family of inputs in the range of $\mathscr{P}$, with arbitrary retained correlations.

(d) Two exact transfers restore every input in the range of $\mathscr{P}$, and one exact transfer returns the four blanks to that range.

Proof. (a) X_{ex} is a permutation of slots, so it conjugates X_0 to X_8; it commutes with $\mathscr{P}$, since it exchanges the blank conditions on 4 and 11 and fixes those on 6 and 7, and it is a self-inverse permutation. (b) Each loading pulse matches a reference to its source; on the matched code X_{ex} is the logical exchange of the two factors. After the logical exchange the banks are crossed, and the relabelled loading undoes them. (c) Loading, the logical exchange $e^{K_{\mathrm{ex}}}$ and decoding are each in $\mathscr{M}_{m,\hbar}$ (Theorems 20.23 and 20.22), so their product is approximated by one paid word; leakage out of the range is bounded by the same error because $(1-\mathscr{P})T_{\mathrm{tr}} = 0$. (d) From (a). □

This is a physical permutation of four slots, achieved on the range where four blanks are supplied, up to a known phase and to any accuracy. It is not the capture: it moves the whole matter slot into the record slot without writing a tag of an occupation difference, and it is not shown to equal the write after the exchange, or any canonical capture with coherent references.

20.9.1 Work

Proposition 20.26 (The work of a transfer)**.** *(a) For every paid word W, the switching work of its segments equals $W^*H_mW - H_m$; it is self-adjoint with norm at most $2\|H_m\|$, and $2\|H_m\| - (W^*H_mW - H_m) \geq 0$. These bound the net work only, not the peak control energy, the time or the exposure.*

(b) The exact work *of the transfer is $\mathscr{W}_m = T_{\mathrm{tr}}^*H_mT_{\mathrm{tr}} - \mathscr{P}H_m\mathscr{P}$. For every $\eta > 0$ one paid word meets the transfer and leakage bounds of Theorem 20.25(c) and has range work within η of $\mathscr{W}_m$.*

(c) $H_m - H_0$ is the difference of electric energies, diagonal in the occupation frame. On the product state with slot 0 occupied and every other slot empty, which lies in the blank range and is carried by T_{tr} to slot 8 occupied, the expectation of $\mathscr{W}_1 - \mathscr{W}_0$ is $4c$, with

$$c = 4\big(\operatorname{arsinh}\tfrac{1}{2} - \sqrt{5} + 2\big) > 0.$$

Hence $\mathscr{W}_0 \neq 0$ or $\mathscr{W}_1 \neq 0$.

Proof. (a) Each segment conserves its own total Hamiltonian, so the control energy given up equals the holding energy gained, and the pieces compose in the initial frame. The norm bound uses unitarity. (b) $\|A^*HA - B^*HB\| \leq (\|A\| + \|B\|)\|H\|\|A - B\|$ with $\|W\mathscr{P}\|, \|T_{\mathrm{tr}}\| \leq 1$, and the unit phase drops out of $T_{\mathrm{tr}}^* H_m T_{\mathrm{tr}}$. (c) Only the electric term depends on m. The flux contrast of the electric energy is $-c$ on the initial state and $3c$ on its image. □

Part (c) is a disjunction. It refutes the claim that the transfer costs no work at every flux, and it does not say at which flux the work is nonzero or what its sign is. The positive operator in (a) is an algebraic allowance, not a supply of work.

20.9.2 One fixed interaction

A schedule of many segments can be replaced, for the occupations, by one interaction held on for a fixed time. For distinct slots k, a, l let the four tokens ENTANGLE(k,a), ENTANGLE(a,k), ENTANGLE(a,l) and ENTANGLE(l,a) be on together as one bundle; its price is exactly π and its generator is $\frac{\pi}{\sqrt{2}}\,\nu$, where $\nu = \frac{i}{2\sqrt{2}}\big(S_kF_a + S_aF_l + F_kS_a + F_aS_l\big)$ satisfies $\nu^3 = -\nu$.

Proposition 20.27 (A static relay)**.** *(a) $e^{\pi\nu} = \mathscr{X}$ with $\mathscr{X} = -\frac{1}{2}\big(q_kq_a + S_kS_l + F_kF_l + q_aq_l\big)$, which exchanges q_k and q_l (hence the occupations of k and l) and commutes with q_a.*

(b) Run at strength $\hbar/\varepsilon$ for the time $\sqrt{2}\varepsilon$ with the holding on, the evolution U_{st} lies within $\sqrt{2}\varepsilon\|H_m\|/\hbar$ of $e^{-i\sqrt{2}\pi}\mathscr{X}$; it conserves Q_{tot} when $w_a = -w_k = -w_l$, and it conserves holding plus interaction exactly, so that $U_{\mathrm{st}}^* H_m U_{\mathrm{st}} - H_m = V - U_{\mathrm{st}}^* V U_{\mathrm{st}}$ with $V \geq 0$ the interaction.

(c) On states with a definite occupation of the middle slot a, the interaction energy is exactly $\pi\hbar/\varepsilon$ per unit norm. For a budget $E > 0$ the choice $\varepsilon = \pi\hbar/E$ gives $\|V\| \leq 2E$ and an error at most $\sqrt{2}\pi\|H_m\|/E$.

Proof. (a) The two path quadratures square to 2 and commute, which gives the cube identity and $e^{\pi\nu} = 1 + 2\nu^2 = \mathscr{X}$; the conjugation identities are direct. (b) Proposition 20.12 with the rescaled time, and conservation of the fixed Hamiltonian. (c) On a sharp sector of a the off-diagonal part of the generator vanishes and only the price survives. □

Here the material work is balanced exactly by the energy that one fixed interaction gives up, with no switching terms to sum; but the interaction, its strength and the sector condition are supplied, and only the occupations are exchanged, not the eight-state payload. Replacing the constant control by a quantum source adds a backreaction error that does not shrink with ε; we do not develop it here.

The open questions this chapter leaves are stated in Section 23.6. The next chapter turns from how a record is written to which posts on the ledger count as writes at all.

Chapter 21

Event attachment

Every verdict in this book counts ticks, and a tick is something booked on a ledger. We identify the physical writes of a posting with the unit changes its own current makes; the padded encoding of Chapter 2 books those and adds a padding post for every silent axis. Soundness and completeness force the physical writes, the padded encoding is complete and not sound, and the attachment narrows to one question: whether a padding post is a physical write. It decides every executed count and every one-post pass, and not the field-current window. If each posting is booked at least cost, a step-level premise we name in Section 21.8, padding is not a write and every count agrees.

21.1 Write rules and the two clocks

An attachment is a rule that says what a posting books.

As in Chapter 3, the ledger has observed accounts $1, 2, 3$, which carry the three current components (account 2, the field-current account, carries j_2), a padding account $\circ$, and two further accounts that neither encoding touches. A signed post $(a, +)$ or $(a, -)$ adds $+1$ or -1 to the potential of account a. The physical writes $\mathrm{phys}(j)$ of a label j are the signed unit posts of Chapter 3, and the padded encoding of Chapter 2 is the list

$$\mathrm{pad}(j) = \big[e_1(j), e_2(j), e_3(j)\big], \qquad e_i(j) = \begin{cases} (i, \operatorname{sgn} j_i) & j_i \neq 0, \\ (\circ, +) & j_i = 0, \end{cases}$$

with $i = 1, 2, 3$ in this order in both. The post $p_\circ = (\circ, +)$ is the *padding post*. Axis i is *silent* at j when $j_i = 0$; write $\mathrm{sil}(j)$ for the number of silent axes, so that $\mathrm{phys}(j)$ has $3 - \mathrm{sil}(j)$ entries.

Definition 21.1 (Write rules)**.** A *write rule* W assigns to each label j a finite list $W(j)$ of signed posts. It is *sound* if every entry of $W(j)$ is an entry of $\mathrm{phys}(j)$, *complete* if every entry of $\mathrm{phys}(j)$ is an entry of $W(j)$, and *count-sound* if $W(j)$ is a sub-multiset of $\mathrm{phys}(j)$, for every j. It lies in the *membership bracket* if $\mathrm{phys}(j) \subseteq W(j) \subseteq \mathrm{pad}(j)$ as sets of entries, and in the *multiset bracket* if these inclusions hold as multisets. *Padding is a write* under W if $p_\circ \in W(j)$ for every label j with a silent axis; *padding is not a write* under W if $p_\circ \notin W(j)$ for every j.

The two readings of padding are not complementary: a rule may pad at some silent labels and not at others.

Definition 21.2 (Clocks under a rule)**.** Let W be a write rule, C a set of accounts (a *cut*) and $h = (j^{(1)}, \dots, j^{(k)})$ a record.

(a) On the *native-step clock*, posting t is a tick of C under W when $W(j^{(t)})$ has an entry on C. Eight consecutive postings pass when all eight are ticks.

(b) On the *one-post clock*, the posts of h under W are the list $W(j^{(1)})\,W(j^{(2)})\cdots W(j^{(k)})$, and each post is a tick of C when it lands on C. Eight consecutive posts pass when all eight land on C.
(c) C is *post-passable under* W when some record written by an occupation path from genesis has at least eight posts under W and its last eight posts pass; it is *step-passable under* W when some such record of length at least eight has its last eight postings pass on the native-step clock.

For $W = \mathrm{phys}$ these are the clocks of Chapter 3, and Theorem 3.20 classifies the passable cuts. The cut, the clock and the window length are supplied there, and we keep them supplied here.

Lemma 21.3 (Entries of the padded encoding)**.**

(a) A post lies in $\mathrm{pad}(j)$ *if and only if it lies in* $\mathrm{phys}(j)$*, or it is* $p_\circ$ *and* j *has a silent axis. The padding post is never a physical write.*
(b) $\mathrm{pad}(0,0,0) = [p_\circ, p_\circ, p_\circ]$ *while* $\mathrm{phys}(0,0,0)$ *is empty. Exactly two labels have no silent axis,* $(1,1,-1)$ *and* $(-1,-1,1)$*, and for them* $\mathrm{pad}(j) = \mathrm{phys}(j)$*. The other thirteen labels carry padding.*
(c) Both encodings change the potential of observed account i *by* j_i*. Under* pad *the potential of* $\circ$ *grows by* $\mathrm{sil}(j)$ *at a posting with label* j*; under* phys *it never changes.*

Proof. (a) Each $e_i(j)$ is either the physical post on i or $p_\circ$, the latter exactly when $j_i = 0$; physical posts land only on observed accounts. (b) A label without a silent axis has $j_2 \neq 0$, so its input is mixed (Proposition 2.7). From $x = (1,0)$ the label is $(y_1, 1, y_2 - 1)$, with no zero component only for $y = (1,0)$; from $x = (0,1)$ it is $(y_1 - 1, -1, y_2)$, with no zero component only for $y = (0,1)$. (c) The first claim is Proposition 2.11(a). Each padding post is a debit on $\circ$, and $\mathrm{pad}(j)$ holds $\mathrm{sil}(j)$ of them. □

21.2 Soundness and completeness force the physical writes

The padded encoding books every physical write and books other things too. The other things are enough to make cuts pass with nothing physical behind them.

Theorem 21.4 (The padded encoding is complete and not sound)**.**

(a) pad *is complete and not sound.* phys *is sound and complete, and every sound and complete rule* W *has the entries of* phys*:* $p \in W(j)$ *if and only if* $p \in \mathrm{phys}(j)$*.*
(b) Native-step clock. Under pad*, for each observed account* i *the cut* $\{i, \circ\}$ *passes on every eight labels. Under* phys*,* $\{i, \circ\}$ *passes on eight labels exactly when* $j_i \neq 0$ *at all eight. Under every sound rule, eight labels that include* $(0,0,0)$ *fail for every cut; in particular no cut passes on every eight labels under* phys*.*
(c) Every sound and complete rule gives, for every cut and every eight labels, the native-step verdict of phys*.*
(d) One-post clock. Under pad*, a record with three consecutive zero labels has eight consecutive posts equal to* $p_\circ$*, on which* $\{\circ\}$ *passes. Under every sound rule,* $\{\circ\}$ *passes on no eight consecutive posts of any record.*
(e) Every eight consecutive posts of every record pass C *under* phys *if and only if* $\{1,2,3\} \subseteq C$*, and under* pad *if and only if* $\{1,2,3,\circ\} \subseteq C$*.*

Proof. (a) Completeness of pad is Lemma 21.3(a); pad$(0,0,0)$ has entries and phys$(0,0,0)$ has none, so pad is not sound. For a sound and complete W the two inclusions are the two definitions. (b) Each $e_i(j)$ lands on i or on $\circ$, so every posting is a tick of $\{i,\circ\}$ under pad. Under phys no post lands on $\circ$ and a post lands on i exactly when $j_i \neq 0$. A sound rule books nothing at the zero label, so that posting is a tick of no cut; the eight postings $(0,0) \to (0,0)$ show that no cut passes always. (c) Whether a posting is a tick of C depends only on which entries it books, and these are the entries of phys(j) by (a). (d) Under pad the posts come in blocks of three, one per posting, so three zero labels give nine consecutive padding posts. A sound rule never books $p_\circ$. (e) If $\{1,2,3\} \subseteq C$, every physical post lands on C. Conversely, the record $(1,1,-1)^3$ posts on $1,2,3,1,2,3,1,2,3$ under phys, and any eight consecutive of these posts touch all three accounts. Under pad every post lands on $\{1,2,3,\circ\}$; the record $\big((1,1,-1),(0,0,0),(1,1,-1)\big)$ posts on $1,2,3,\circ,\circ,\circ,1,2,3$, and its first eight posts touch all four accounts. □

The zero-label posting, where the padded encoding books three posts and a sound rule none, is a real operation, with blocks $M_{(0,0),(0,0)} = M_{(1,1),(1,1)} = -\text{stay} \otimes \text{stay} \neq 0$ (Proposition 2.11(c)). The silence of a sound rule there is a statement about an operation that occurs, not about a missing step.

Theorem 21.5 (One identification decides the attachment)**.** *Let W lie in the membership bracket.*

(a) W has the entries of pad *if and only if padding is a write under W. W has the entries of* phys *if and only if padding is not a write under W. The two cannot both hold.*
(b) If padding is a write under W, then on the native-step clock under W every cut $\{i,\circ\}$, $i \in \{1,2,3\}$, passes on every eight labels.
(c) If padding is not a write under W, then on the native-step clock under W eight labels that include $(0,0,0)$ fail for every cut.

Proof. (a) By Lemma 21.3(a) every entry of pad(j) is a physical post, which $W(j)$ contains, or $p_\circ$ at a label with a silent axis. So $W(j)$ contains every entry of pad(j) exactly when it contains $p_\circ$ at every such label; the reverse inclusion is the bracket. Likewise every entry of $W(j)$ is physical or $p_\circ$, and $p_\circ$ is not physical, so $W(j)$ has only physical entries exactly when it never contains $p_\circ$. The zero label has a silent axis, so the two readings conflict there. (b) and (c) follow from (a) and the step-clock verdicts of Theorem 21.4(b), since a tick depends only on the entries booked. □

The ledger itself carries the decisive observable: the potential of account $\circ$, which gains the number of silent axes of the history under the padding reading and never moves under the silence reading (Lemma 21.3(c)). The discrepancy is not confined to the zero label. The occupation path $(0,0) \to (1,0) \to (1,1)$ writes $(1,0,0)$ and then $(1,1,0)$, three silent axes in all. Under pad the cut $\{3,\circ\}$ receives two entries at the first posting and one at the second; under phys it receives none at either.

The native step of Chapter 2 retains the zero label as one operation, and the question is whether that operation books anything on the ledger, and where. The ledger facts of Theorem 21.22(e) do not decide it; Section 21.8 names a step-level premise that does.

21.3 Executed counts

A count of events read off a history agrees with the net-current count exactly when padding is not a write. The excess is the padding account's gain.

For a record $h = (j^{(1)}, \dots, j^{(k)})$ and a rule W let

$$N(h) = \sum_{t=1}^{k} \bigl|\mathrm{phys}(j^{(t)})\bigr|, \qquad X_W(h) = \sum_{t=1}^{k} \bigl|W(j^{(t)})\bigr|, \qquad N^{\circ}_W(h) = \sum_{t=1}^{k} \#\bigl\{p_\circ \text{ in } W(j^{(t)})\bigr\},$$

the *net-current count*, the *executed count* and the *padding count*. Since every component of a label is 0 or ± 1, $N(h) = \sum_t \sum_i (j_i^{(t)})^2$.

Theorem 21.6 (The executed count)**.**

(a) $X_{\mathrm{phys}} = N$*, and* $X_{\mathrm{pad}}(h) = 3k$ *for every record of length* k*. The difference* $X_{\mathrm{pad}}(h) - N(h) = \sum_t \mathrm{sil}(j^{(t)})$ *is the gain of account* $\circ$ *along* h *under* pad*.*

(b) For every rule W*:* $X_W = N$ *on every record if and only if* $|W(j)| = |\,\mathrm{phys}(j)|$ *for every label* j*.*

(c) If W *lies in the multiset bracket, then* $|W(j)| = |\,\mathrm{phys}(j)| + \#\{p_\circ \text{ in } W(j)\}$*, so* $X_W(h) = N(h) + N^{\circ}_W(h)$ *for every record, and* $X_W = N$ *on every record if and only if padding is not a write under* W*.*

(d) If W *is complete and* $X_W = N$ *on every record, then padding is not a write under* W*. No upper bound on* W *is needed.*

(e) If W *lies in the membership bracket and no* $W(j)$ *repeats an entry, then* $X_W = N$ *on every record if and only if padding is not a write under* W*, if and only if* W *has the entries of* phys*. The hypothesis of no repeats cannot be dropped: the doubled rule* $\mathrm{dbl}(j) = \mathrm{phys}(j)\,\mathrm{phys}(j)$ *lies in the membership bracket, padding is not a write under it, and* $X_{\mathrm{dbl}} = 2N$*.*

Proof. (a) $\mathrm{pad}(j)$ has three entries, $\mathrm{sil}(j)$ of them padding posts (Lemma 21.3(c)). (b) A record's count is a sum over its postings, and each label j is the whole of the one-posting record (j). (c) Remove the copies of $p_\circ$ from the multiset chain $\mathrm{phys}(j) \subseteq W(j) \subseteq \mathrm{pad}(j)$. What remains of $\mathrm{pad}(j)$ is $\mathrm{phys}(j)$ (Lemma 21.3(a)), and $\mathrm{phys}(j)$ has no $p_\circ$, so what remains of $W(j)$ is exactly $\mathrm{phys}(j)$. Sum over the postings and use (b). (d) If $p_\circ \in W(j)$, then $W(j)$ contains the distinct entries of $\mathrm{phys}(j)$ and also $p_\circ$, so $|W(j)| > |\,\mathrm{phys}(j)|$, against (b). (e) Without repeats, the membership bracket is the multiset bracket, since $\mathrm{pad}(j)$ holds each physical post once and $p_\circ$ at least once when j has a silent axis. Then (c) and Theorem 21.5(a) apply. For dbl, every count doubles. □

A count of how much has happened along a history therefore has to say whether it counts executed posts or net current.

The same holds for the probability-weighted counts, on any field the postings drive.

Definition 21.7 (Field carriers and loads)**.** A *field carrier* is a complex inner-product space $\mathscr{Y}$ with an isometry U_j of $\mathscr{Y}$ for each label j. For an input $v \in \mathscr{H} \otimes \mathscr{Y}$ and a record h of length k, the *load* of h is

$$L_v(h) = \bigl\| (V_{j^{(k)}} \otimes U_{j^{(k)}}) \cdots (V_{j^{(1)}} \otimes U_{j^{(1)}})\, v \bigr\|^2 .$$

The *net-count read* and the *executed read* of a rule W are

$$\mathscr{A}_k(v) = \sum_{h \in \mathscr{J}^k} N(h)\, L_v(h), \qquad \mathscr{X}_{W,k}(v) = \sum_{h \in \mathscr{J}^k} X_W(h)\, L_v(h).$$

The loads at length k add up to $\|v\|^2$, by Proposition 2.12 and because each U_j is an isometry. With $\mathscr{Y} = \mathbb{C}$ and every $U_j = 1$, $L_v(h) = \|V_h v\|^2$ is the record law of a cell vector, and $\mathscr{A}_k(v)$ is its expected net-current count. At one posting, $\mathscr{A}_1(v)$ is the sum over the three axes of the load carried by the labels with current on that axis. Which apparatus reads $\mathscr{A}_k$ is not derived here.

Theorem 21.8 (The reads see the padding)**.** *Let W lie in the multiset bracket and let $v \in \mathscr{H} \otimes \mathscr{Y}$.*

(a) $\mathscr{X}_{\mathrm{phys},k}(v) = \mathscr{A}_k(v)$*, and*

$$\mathscr{X}_{W,k}(v) = \mathscr{A}_k(v) + \sum_{h \in \mathscr{J}^k} N_W^\circ(h)\, L_v(h), \qquad \mathscr{A}_k(v) \le \mathscr{X}_{W,k}(v) \le \mathscr{X}_{\mathrm{pad},k}(v) = 3k\|v\|^2.$$

(b) If $\mathscr{Y} \neq 0$, every label $j = j(y,x)$ carries positive one-posting load from a basis input $|x\rangle \otimes e \otimes \varphi$, with e a basis vector of $\mathscr{H}_{\mathrm{c}} \otimes \mathscr{H}_{\mathrm{c}}$ and $\varphi \neq 0$.

(c) If $\mathscr{Y} \neq 0$, then $\mathscr{X}_{W,k}(v) = \mathscr{A}_k(v)$ for all k and v if and only if padding is not a write under W; and $\mathscr{X}_{W,k} = \mathscr{X}_{\mathrm{pad},k}$ for all k and v if and only if each $W(j)$ is a rearrangement of pad(j)*.*

(d) The premise $\mathscr{Y} \neq 0$ cannot be dropped: on $\mathscr{Y} = 0$ every load vanishes, so $\mathscr{X}_{\mathrm{pad},k} = \mathscr{A}_k$ although padding is a write under pad*.*

Proof. (a) Weight Theorem 21.6(a),(c) by the loads, which are nonnegative and add up to $\|v\|^2$; and $X_W(h) \le X_{\mathrm{pad}}(h)$ in the multiset bracket. (b) The component of $(V_j \otimes U_j)(|x\rangle \otimes e \otimes \varphi)$ at $|y\rangle \otimes e$ is a diagonal entry of M_{yx}, which is $\pm\frac{1}{4}$ by Lemma 4.2, times $U_j\varphi$, which has the norm of φ. (c) By (a) the difference $\mathscr{X}_{W,k}(v) - \mathscr{A}_k(v)$ is a sum of nonnegative terms. At $k = 1$ it is $\sum_j \#\{p_\circ \text{ in } W(j)\}\, L_v((j))$, and (b) supplies for each j an input with $L_v((j)) > 0$. The second equivalence is the same argument applied to $\mathscr{X}_{\mathrm{pad},1}(v) - \mathscr{X}_{W,1}(v) = \sum_j (3 - |W(j)|) L_v((j))$, together with the fact that a sub-multiset of pad(j) with three entries is a rearrangement of it. (d) is immediate. □

So detection is complete after one posting: for each label some basis input loads it, and a padding post at that label then moves the read.

For an ensemble statement, take the uniform cell. For a function F on records of length k write

$$\overline{\mathbb{E}}_k[F] = \sum_{h \in \mathscr{J}^k} F(h)\, \frac{\operatorname{tr} E_h}{64},$$

the expectation under the record law of a genesis state with cell density $I/64$. On any field carrier, the inputs $b \otimes \varphi$ with b running over the 64 basis vectors of $\mathscr{H}$ and φ a unit vector give the same expectations, because $(V_h \otimes U_h)(b \otimes \varphi) = V_h b \otimes U_h \varphi$ and $\|U_h\varphi\| = 1$.

Theorem 21.9 (The uniform cell)**.**

(a) At every posting, the label j has probability $w(j)$ under the uniform cell, with $w(0,0,0) = \frac{1}{8}$ and $w(j) = \frac{1}{16}$ for the other fourteen labels. Equivalently, $\overline{\mathbb{E}}_k[f(j^{(t)})] = \frac{1}{16}\sum_{(y,x)} f\big(j(y,x)\big)$ for every function f on labels and every $t \le k$, the sum running over the sixteen transitions.

(b) Each component of $j(y,x)$ vanishes on exactly eight of the sixteen transitions: j_1 when $y_1 = x_2$, j_2 when x is balanced, j_3 when $y_2 = x_1$. So each axis is silent with probability $\frac{1}{2}$ at every posting.

(c) $\overline{\mathbb{E}}_k[N] = \frac{3}{2}k$ and $\overline{\mathbb{E}}_k[X_{\mathrm{pad}}] = 3k = 2\,\overline{\mathbb{E}}_k[N]$.

(d) For W in the multiset bracket, $\frac{3}{2}k \le \overline{\mathbb{E}}_k[X_W] \le 3k$. Moreover $\overline{\mathbb{E}}_k[X_W] = \frac{3}{2}k$ for every k if and only if padding is not a write under W, and $\overline{\mathbb{E}}_k[X_W] = 3k$ for every k if and only if each $W(j)$ is a rearrangement of $\mathrm{pad}(j)$.

Proof. (a) A label and an input determine the output ($y_1 = j_1 + x_2$, $y_2 = j_3 + x_1$), so $V_j V_j^* = \sum |y\rangle\langle y| \otimes M_{yx} M_{yx}^\top$ over the transitions with label j, and $\sum_j V_j V_j^* = \sum_y |y\rangle\langle y| \otimes \sum_x M_{yx} M_{yx}^\top = I$ because U_{nat} is orthogonal (Proposition 2.6). In $\operatorname{tr} E_h$, summing over the labels after posting t with $\sum_w E_w = I$ (Proposition 2.12), and over those before it with $\sum_j V_j V_j^* = I$, leaves $\operatorname{tr}(V_j V_j^*)/64 = \operatorname{tr}(E_j)/64$ for $j = j^{(t)}$. Each block has $\operatorname{tr}(M_{yx}^\top M_{yx}) = 4$ (sixteen columns of squared norm $\frac{1}{4}$, Lemma 4.2), and the zero label is the label of two transitions, every other label of one (Proposition 2.7). (b) is read off $j(y, x) = (y_1 - x_2,\ x_1 - x_2,\ y_2 - x_1)$. (c) By (a) and (b) each posting has $\frac{1}{2} \cdot 3$ nonzero components on average; $X_{\mathrm{pad}} = 3k$ on every record. (d) By Theorem 21.6(c), $\overline{\mathbb{E}}_k[X_W] = \frac{3}{2}k + \overline{\mathbb{E}}_k[N_W^\circ]$ with $0 \le N_W^\circ \le N_{\mathrm{pad}}^\circ = X_{\mathrm{pad}} - N$. At $k = 1$, $\overline{\mathbb{E}}_1[N_W^\circ] = \sum_j w(j)\, \#\{p_\circ \text{ in } W(j)\}$ with every $w(j) > 0$, and $\overline{\mathbb{E}}_1[X_{\mathrm{pad}} - X_W] = \sum_j w(j)(3 - |W(j)|)$. □

21.4 The one-post clock under padding

If padding is a write, every posting books three entries. A pass on the one-post clock then needs the padding account or all three observed accounts, and the field-current cut, which passes under the physical writes, never passes.

Theorem 21.10 (Padded passes on the one-post clock)**.**

(a) Under pad*, if eight consecutive posts of any record land on C, then $\circ \in C$ or $\{1, 2, 3\} \subseteq C$. This holds for every record, whether or not an occupation path writes it.*

(b) Under pad*, C is post-passable if and only if $\circ \in C$ or $\{1, 2, 3\} \subseteq C$. The same classification holds for each of the six orders of the three entries within a posting.*

(c) On the native-step clock, pad *and* phys *give the same ticks on every cut without $\circ$. Under* pad*, C is step-passable if and only if it holds $\circ$ or an observed account.*

(d) C is post-passable under phys *and not under* pad *if and only if $\circ \notin C$, C holds* 2 *or both* 1 *and* 3*, and C does not hold all of* 1, 2, 3*. It is post-passable under* pad *and not under* phys *if and only if $\circ \in C$, $2 \notin C$ and $\{1, 3\} \not\subseteq C$. In particular the field-current cut $\{2\}$ is post-passable under* phys *and not under* pad*, and $\{\circ\}$ under* pad *and not under* phys*.*

Proof. (a) The posts come in blocks of three, one block per posting, so any five consecutive posts contain a whole block. If that posting has a silent axis, its padding post lands on C; if not, its three posts land on 1, 2 and 3. (b) Necessity is (a), and the argument does not use the order within a block. Sufficiency: eight postings $(0, 0) \to (0, 0)$ post only $p_\circ$, and eight postings $(1, 0) \to (1, 0)$, each writing $(1, 1, -1)$, post only on 1, 2, 3. Both records are written by occupation paths. (c) Padding posts land only on $\circ$ (Lemma 21.3(a)). The same two records witness step-passability, and a cut holding neither $\circ$ nor an observed account receives no post under pad. (d) Combine (b) with Theorem 3.20(b), by which C is post-passable under phys exactly when $2 \in C$, or $1 \in C$ and $3 \in C$. □

Part (c) is why the field-current window does not wait on the padding question: on the native-step clock a cut without $\circ$ reads the same ticks under every rule in the membership

bracket, since padding posts land only on ∘. On the one-post clock the question matters, for every rule that writes padding.

Theorem 21.11 (Every rule that writes padding)**.** *Let W be complete, let padding be a write under W, and let $|W(j)| \le 8$ for every label j.*

(a) If C is post-passable under W, then $\circ \in C$ or $\{1,2,3\} \subseteq C$. If moreover $W(j) \subseteq \mathrm{pad}(j)$ as sets for every j, the converse holds.
(b) Consequently, if the field-current cut $\{2\}$ is post-passable under a complete rule with at most eight entries per posting, that rule does not book a padding post at every label with a silent axis.
(c) If $W(j) \subseteq \mathrm{pad}(j)$ as sets for every j, every cut holding $\circ$ passes on the record of eight postings $(0,0) \to (0,0)$, which has no physical write at all.
(d) The bound of eight entries cannot be raised. The rule that books $p_\circ$ first at every label with a silent axis, then the physical writes, and, at labels with $j_1 = j_3 = 0$, seven further copies of each physical write, lies in the membership bracket, writes padding, has at most nine entries per posting, and post-passes $\{2\}$.

Proof. (a) The last eight posts end with a whole posting, and with at most eight entries that posting lies inside them. If its label has a silent axis, its padding post lands on C; otherwise it books the three physical posts, on 1, 2 and 3. For the converse use the two records of Theorem 21.10(b): at $(0,0,0)$ the rule books only padding posts, at least one, and at $(1,1,-1)$ it books only posts on observed accounts. (b) is (a) with $C = \{2\}$. (c) Each posting books at least one entry, all equal to $p_\circ$. (d) On the oscillation $(1,0) \leftrightarrow (0,1)$ the labels are $(0,\pm1,0)$; each posting books $p_\circ$ and then eight posts on account 2, so the last eight posts land on $\{2\}$. The rule's entries are padding posts and physical writes, and it contains every physical write. □

So the two open choices of a verdict, which clock and whether padding writes, constrain each other. Part (d) also shows that membership alone cannot exclude a rule like this: it books the encoding's entries and nothing else, and only the number of times it books them is wrong.

21.5 Axis locality

Suppose each axis decides on its own whether to book its entry of the encoding. Then the rules are eight, one per set of padded axes, and two passes decide which.

Definition 21.12 (Axis locality)**.** A rule W is *axis-local* if there is a choice $\kappa(i,c) \in \{\mathrm{keep}, \mathrm{drop}\}$ for each axis i and each value $c \in \{-1,0,1\}$ such that $W(j)$ is obtained from $\mathrm{pad}(j) = [e_1(j), e_2(j), e_3(j)]$ by keeping $e_i(j)$ exactly when $\kappa(i, j_i) = \mathrm{keep}$. For $S \subseteq \{1,2,3\}$ the rule W_S keeps every physical post and the padding post of every silent axis in S:

$$W_S(j) = \big[\, e_i(j) \ :\ j_i \neq 0 \text{ or } i \in S \,\big].$$

Axis locality is a premise we supply. It says that whether an axis books its entry depends on that axis's own current component and on nothing else in the posting. Nothing in the native step derives it.

Lemma 21.13 (The local rules)**.**

(a) A complete rule is axis-local if and only if it equals W_S for some S.
(b) $\mathrm{phys}(j)$ *is a sublist of* $W_S(j)$*, which is a sublist of* $\mathrm{pad}(j)$*;* $W_\varnothing = \mathrm{phys}$ *and* $W_{\{1,2,3\}} = \mathrm{pad}$.
(c) Padding is not a write under W_S if and only if $S = \varnothing$.

Proof. (a) Completeness forces $\kappa(i, \pm 1) = \mathrm{keep}$; put $S = \{i : \kappa(i, 0) = \mathrm{keep}\}$. (b) is the definition. (c) The zero label is silent on every axis, so $W_S(0, 0, 0)$ holds $|S|$ padding posts. □

Theorem 21.14 (Two passes decide the padded axes)**.**

(a) The field-current cut $\{2\}$ is post-passable under W_S if and only if $1 \notin S$ and $3 \notin S$.
(b) The outer cut $\{1, 3\}$ is post-passable under W_S if and only if $2 \notin S$.
(c) Both hold if and only if $S = \varnothing$, that is, if and only if padding is not a write under W_S. On every field carrier with $\mathscr{Y} \neq 0$ this is equivalent to $\mathscr{X}_{W_S,k}(v) = \mathscr{A}_k(v)$ for all k and v. The same three conditions are equivalent for every complete axis-local rule.
(d) One pass is not enough: $\{2\}$ is post-passable under $W_{\{2\}}$, padding is written at the zero label, and on every field carrier with $\mathscr{Y} \neq 0$, $\mathscr{X}_{W_{\{2\}},k}(v) \neq \mathscr{A}_k(v)$ for some k and v.
(e) Axis locality cannot be dropped. The zero-label rule zpad*, with* $\mathrm{zpad}(j) = \mathrm{phys}(j)$ *for* $j \neq (0, 0, 0)$ *and* $\mathrm{zpad}(0, 0, 0) = [p_\circ]$*, lies in the multiset bracket, post-passes both $\{2\}$ and $\{1, 3\}$, is not axis-local, writes a padding post, and on every field carrier with $\mathscr{Y} \neq 0$ its executed read differs from $\mathscr{A}_k$ for some k and v.*

Proof. (a), (b) An axis $i \in S$ books an entry at every posting: the physical post on i when $j_i \neq 0$, the padding post when $j_i = 0$. So when 1 or 3 lies in S, the newest posting books an entry off $\{2\}$, and when $2 \in S$ it books one off $\{1, 3\}$. Conversely, the oscillation $(1, 0) \leftrightarrow (0, 1)$ writes $(0, \pm 1, 0)$; axis 2 carries current at every posting, so under W_S with $1, 3 \notin S$ it posts only on account 2. The balanced oscillation $(0, 0) \leftrightarrow (1, 1)$ writes $(1, 0, 1)$ and $(-1, 0, -1)$; under W_S with $2 \notin S$ it posts only on 1 and 3. (c) From (a), (b), Lemma 21.13(c) and Theorem 21.8(c), since W_S lies in the multiset bracket; for a complete axis-local rule use Lemma 21.13(a). (d) By (a), and Theorem 21.8(c). (e) The two oscillations never write the zero label, so zpad posts as phys on them. If zpad were axis-local, some axis i would have $\kappa(i, 0) = \mathrm{keep}$, and then zpad would pad the nonzero labels silent on i, such as $(0, 1, -1)$, $(1, 0, 1)$ and $(1, 1, 0)$ for $i = 1, 2, 3$; it does not. The read differs by Theorem 21.8(c). □

The full classification of passable cuts under every local rule follows the same lines, with one twist that comes from the step itself.

Theorem 21.15 (Every cut under every local rule)**.** *C is post-passable under W_S if and only if either $S \neq \varnothing$ and $\circ \in C$, or all of the following hold:*

(i) $2 \in C$, or $1 \in C$ and $3 \in C$;
(ii) every axis in S is an account of C;
(iii) not both $\{1, 2\} \subseteq S$ and $3 \notin C$;
(iv) not both $\{2, 3\} \subseteq S$ and $1 \notin C$.

For $S = \varnothing$ this is Theorem 3.20(b), and for $S = \{1, 2, 3\}$ it is Theorem 21.10(b).

Proof outline. Sufficiency uses six records written by occupation paths: eight postings $(0, 0) \to (0, 0)$ (for $S \neq \varnothing$ and $\circ \in C$), eight postings $(1, 0) \to (1, 0)$, the two oscillations, and the two four-cycles $(0, 0) \to (1, 0) \to (1, 1) \to (0, 1) \to (0, 0)$ and $(0, 0) \to (0, 1) \to (1, 1) \to (1, 0) \to (0, 0)$,

each run twice. The first four-cycle writes $(1,0,0),(1,1,0),(-1,0,0),(-1,-1,0)$, with current on axis 1 at every posting and never on axis 3; the second is its mirror. On each witness the padded axes that matter carry current at every posting, so W_S books the physical posts there and the physical verdict transfers.

Necessity, for $S \neq \emptyset$ and $\circ \notin C$. Each posting books one to three entries, so the last eight posts contain the two newest postings whole, and a record written by an occupation path ends with two transitions $x \to y \to z$. No padding post may occur in them, so every axis of S carries current at both, and its physical post lands on C: this is (ii). Suppose $\{1,2\} \subseteq S$ and $3 \notin C$. Then the transition $x \to y$ has current on axes 1 and 2 and none on axis 3. Current on axis 2 means x is mixed, and from a mixed input, current on exactly one outer axis ends at a balanced occupation: from $x = (1,0)$ the label $(y_1, 1, y_2 - 1)$ has $y_1 \neq 0$ and $y_2 = 1$, so $y = (1,1)$, and from $x = (0,1)$ likewise $y = (0,0)$. Then $y \to z$ has no current on axis 2, a contradiction. This gives (iii), and (iv) is its mirror. Condition (i) and the remaining cases are decided by checking the sixteen transitions in the same way. □

21.6 What a count and two passes can see

The executed count reads how many axes a local rule pads. The two passes add whether the padded axes are outer or middle. What is left over is a symmetry of the native step.

Theorem 21.16 (The count reads how many)**.** *Under the uniform cell, $\overline{\mathbb{E}}_k[N^\circ_{W_S}] = \frac{1}{2}|S|\,k$ and $\overline{\mathbb{E}}_k[X_{W_S}] = k\big(\frac{3}{2} + \frac{1}{2}|S|\big)$. Two local rules have the same expected executed count for every k if and only if $|S| = |S'|$.*

Proof. $W_S(j)$ holds one padding post for each silent axis of j in S, and each axis is silent with probability $\frac{1}{2}$ at every posting (Theorem 21.9(b)). Add $\overline{\mathbb{E}}_k[N] = \frac{3}{2}k$ by Theorem 21.6(c). The map $|S| \mapsto \frac{3}{2} + \frac{1}{2}|S|$ is injective. □

Let τ exchange the axes 1 and 3, and also the accounts 1 and 3.

Theorem 21.17 (Passes and count decide up to the outer exchange)**.** *For $S, S' \subseteq \{1,2,3\}$, the rules W_S and $W_{S'}$ agree on the post-passability of $\{2\}$, on that of $\{1,3\}$, and on the expected executed count for every k, if and only if $S' = S$ or $S' = \tau S$.*

Proof. Theorems 21.14(a),(b) and 21.16 give each S the data in Table 21.1. Two sets have the same data exactly when they are equal or form one of the pairs $\{1\},\{3\}$ and $\{1,2\},\{2,3\}$, which τ exchanges; the other four sets are fixed by τ. □

The leftover pair is not an accident of the observables. It is the native swap of the two occupation bits, under which counts, weights and passes all transform together.

Theorem 21.18 (The residual is the occupation swap)**.** *Let $g_{\mathrm{s}}(x_1,x_2) = (x_2,x_1)$, the swap of the two channels (Chapter 4). The transition $g_{\mathrm{s}}x \to g_{\mathrm{s}}y$ has label $j^\tau = (j_3, -j_2, j_1)$ when $x \to y$ has label j; write h^τ for the record with every label replaced in this way.*

(a) $X_{W_{\tau S}}(h^\tau) = X_{W_S}(h)$ and $N^\circ_{W_{\tau S}}(h^\tau) = N^\circ_{W_S}(h)$ for every record h.

padded axes S	$\{2\}$ post-passable	$\{1,3\}$ post-passable	expected posts per posting
$\varnothing$	yes	yes	$3/2$
$\{2\}$	yes	no	2
$\{1\}$, $\{3\}$	no	yes	2
$\{1,3\}$	no	yes	$5/2$
$\{1,2\}$, $\{2,3\}$	no	no	$5/2$
$\{1,2,3\}$	no	no	3

Table 21.1. What the two passes and the expected executed count read of a local padding rule W_S (uniform cell). Rows with two sets are the pairs exchanged by τ.

(b) $w(j^\tau) = w(j)$. *Hence, under the uniform cell on every field carrier, every additive count* $\sum_t f(j^{(t)})$ *has the same expectation as* $\sum_t f\big((j^{(t)})^\tau\big)$.
(c) C *is post-passable under* W_S *if and only if* τC *is post-passable under* $W_{\tau S}$. *So no cut with* $\tau C = C$ *separates* W_S *from* $W_{\tau S}$.
(d) $W_{\{1\}}$ *post-passes* C *if and only if* $\circ \in C$, *or* $1 \in C$ *together with* $2 \in C$ *or* $3 \in C$. *A cut is post-passable under* $W_{\{1\}}$ *and not under* $W_{\{3\}}$ *if and only if* $\circ \notin C$, $1 \in C$, $2 \in C$ *and* $3 \notin C$. *So* $\{1,2\}$ *separates them, and every separating cut contains it.*
(e) $\{1,2\}$ *is post-passable under* W_S *if and only if* $3 \notin S$ *and* $\{1,2\} \not\subseteq S$.
(f) $W_{\{1,2\}}$ *and* $W_{\{2,3\}}$ *pass exactly the same cuts: those holding* $\circ$ *or all three observed accounts.*
(g) W_S *and* $W_{S'}$ *agree on the post-passability of every cut and on the expected executed count for every* k *if and only if* $S' = S$, *or* $2 \in S$ *and* $S' = \tau S$.

Proof. The label of $g_s x \to g_s y$ is $\big(y_2 - x_1,\ x_2 - x_1,\ y_1 - x_2\big) = (j_3, -j_2, j_1)$. (a) The silent axes of j^τ are the τ-images of those of j, and $|\operatorname{phys}(j^\tau)| = |\operatorname{phys}(j)|$. (b) w is the uniform average over the sixteen transitions (Theorem 21.9(a)), and g_s permutes them; then apply Theorem 21.9(a) to f and to $f \circ \tau$. (c) The conditions of Theorem 21.15 are invariant under exchanging 1 and 3 in S and in C simultaneously. (d), (e), (f) are Theorem 21.15 with $S = \{1\}$, $S = \{3\}$, $C = \{1,2\}$, and $S = \{1,2\}, \{2,3\}$. (g) If $2 \notin S$, then $S \in \{\varnothing, \{1\}, \{3\}, \{1,3\}\}$, the count and the two passes leave only $\{1\}$ against $\{3\}$, and (d) separates them. If $2 \in S$, then $\{2\}$ and $\{1,2,3\}$ are fixed by τ, and (f) shows that $\{1,2\}$ and $\{2,3\}$ are not separated. □

21.7 Multiplicity: what soundness must count

Soundness and completeness as defined fix which posts a posting books, not how often it books them. The native-step clock reads only which accounts a posting touches, so it cannot tell the difference. Counts and the one-post clock can.

Proposition 21.19 (Booking every physical write twice)**.** *Let* $\mathrm{dbl}(j) = \operatorname{phys}(j) \operatorname{phys}(j)$.

(a) dbl *is sound and complete. Each posting puts twice as many entries on a cut under* dbl *as under* phys, *and every native-step verdict of* dbl, *for every cut and every eight labels, is that of* phys.
(b) $X_{\mathrm{dbl}} = 2N$, *so the executed count is not the net-current count; under the uniform cell* $\overline{\mathbb{E}}_k[X_{\mathrm{dbl}}] = 3k = 2\,\overline{\mathbb{E}}_k[N]$.

(c) The record of four postings with label $(1,0,0)$ *has eight posts on account* 1 *under* dbl*, on which* $\{1\}$ *passes the one-post clock. Under* phys *the same record has four posts and no eight consecutive posts at all.*
(d) A rule that is count-sound and complete books, at every label, a rearrangement of phys(j)*; its executed count is N.*

Proof. (a) dbl(j) and phys(j) have the same entries; use Theorem 21.4(c). (b) Every posting books $2|\,\mathrm{phys}(j)|$ posts; use Theorem 21.9(c). (c) $\mathrm{phys}(1,0,0) = [(1,+)]$. (d) phys$(j)$ has no repeated entry, so a sub-multiset of it that contains all its entries is a rearrangement of it. □

The record in (c) is not written by an occupation path, since $(0,0) \to (1,0)$ cannot follow itself; it shows the arithmetic of the clock. The same arithmetic applies on paths: whatever a posting books once, dbl books twice, so the one-post clock fills eight posts in half the postings. Wherever counts or serialized clocks are read, the soundness to require is count-soundness, and by (d), together with completeness, it pins the rule to the physical writes.

The gap matters most for a ledger that also keeps a record of its own postings. Add to the six ledger accounts a *record arm r* and an *outside o*. An *entry* is an ordered pair (d,c) of accounts, a debit on d and a credit on c. A signed post becomes an entry, $(a,+) \mapsto (a,o)$ and $(a,-) \mapsto (o,a)$, which changes the potential of every ledger account exactly as the signed post does. At every posting a *writer account s* also writes the record arm, with the entry (s,r). The *native record ledger* of a record h is

$$\mathscr{L}_s(h) = \widehat{\mathrm{phys}}(j^{(1)})\,(s,r)\,\widehat{\mathrm{phys}}(j^{(2)})\,(s,r)\,\cdots\,\widehat{\mathrm{phys}}(j^{(k)})\,(s,r),$$

where $\widehat{\mathrm{phys}}(j)$ is the list of entries of the physical writes. An entry *crosses* a set B of accounts when exactly one of its two accounts lies in B. On the *crossing clock* each crossing entry is a tick of B, and B passes on a list of entries when at least eight of them cross B.

A *reading* R assigns a list $R(h)$ of entries to every record. It is *sound* (for the writer s) if every entry of $R(h)$ is an entry of $\mathscr{L}_s(h)$; *record-complete* if $R(h)$ has at least k entries crossing $\{r\}$ when h has length k; *ledger-complete* if the entry of every physical write of h lies in $R(h)$; and *count-sound* if $R(h)$ is a sub-multiset of $\mathscr{L}_s(h)$.

Two things here are supplied. That every posting writes the record arm once is a premise: the native step writes its label into the record (Chapter 2), and record writes are active operations (Chapter 20), but that the ledger the clock reads books exactly one record write per posting is not derived. Which account writes it, s, is left free. Reading the record arm as the place where a perspective keeps its own record is an identification, and nothing below depends on it.

Theorem 21.20 (The native ledger with a record arm)**.** *Let* $s \neq r$.

(a) $\mathscr{L}_s$ *is sound, record-complete and ledger-complete; each of its entries joins two different accounts, and* $\mathscr{L}_s(h)$ *is an initial segment of* $\mathscr{L}_s(h \cdot j)$*. The physical entries alone are sound and ledger-complete but not record-complete. The padded entries are not sound.*
(b) Every reading that is sound, record-complete and ledger-complete has exactly the entries of $\mathscr{L}_s(h)$*, for every record* h*.*
(c) If R *is count-sound and record-complete, in particular if each* $R(h)$ *is a rearrangement of* $\mathscr{L}_s(h)$*, then* $\{r\}$ *passes on* $R(h)$ *if and only if* h *has at least eight postings, whatever their labels.*

(d) $\mathscr{L}_s$ booked twice is sound, record-complete and ledger-complete, and on four zero-label postings $\{r\}$ passes on it, while on $\mathscr{L}_s$ it fails.

(e) For a cut C of ledger accounts, let $n_C(j)$ be the number of physical posts of j on C. The number of entries of $\mathscr{L}_s(h)$ crossing C is $\sum_{t=1}^{k} n_C(j^{(t)}) + k\,[s \in C]$. If $s \notin C$, a zero-label posting adds no crossing, and on eight zero-label postings $\{r\}$ passes while C fails. The condition $s \notin C$ is needed: for $s = 1$, $C = \{1\}$ and one zero-label posting there is one crossing and no physical post.

(f) Let W assign to each record a list of entries within $\mathscr{L}_s(h)$. Some reading whose entries lie in $W(h)$ is record-complete if and only if $W(h)$ contains the record write for every record of positive length. In particular no reading made of physical entries alone is record-complete.

(g) No sound and ledger-complete reading is unchanged under every relabelling of the fifteen labels.

(h) The padding post is neither a physical write nor a record write: it occurs no times in $\mathrm{pad}(1, 1, -1)$ *and three times in* $\mathrm{pad}(0, 0, 0)$*, its entry $(\circ, o)$ does not involve r, and the padded entries never cross $\{r\}$.*

Proof. The entries touching r in $\mathscr{L}_s(h)$ are the k record writes, each crossing $\{r\}$ because $s \neq r$; physical entries join a ledger account to o. (a) follows, with the one-posting record $(0, 0, 0)$, which has no physical entry, against the physical entries alone, and the entry $(\circ, o)$, not in $\mathscr{L}_s((0, 0, 0)) = [(s, r)]$, against the padded entries. (b) Soundness gives one inclusion and ledger-completeness the physical entries. For $k \geq 1$ record-completeness gives an entry crossing $\{r\}$; by soundness it is an entry of $\mathscr{L}_s(h)$ touching r, so it is (s, r). This part does not use $s \neq r$. (c) A count-sound reading has at most the k record writes as crossings of $\{r\}$, and a record-complete one at least k. (d) Booking twice doubles the crossings. (e) A physical entry crosses C exactly when its account lies in C, since $o \notin C$; the record write crosses C exactly when $s \in C$, since $r \notin C$. (f) If $W(h)$ contains (s, r), the reading that repeats it k times is record-complete; conversely a crossing of $\{r\}$ inside $\mathscr{L}_s(h)$ is a record write. (g) Exchange $(0, 0, 0)$ with $(1, 1, -1)$. A sound reading has no physical entry at the one-posting record $(0, 0, 0)$, and a ledger-complete reading has three at $(1, 1, -1)$. (h) is Lemma 21.3(b) and the form of the entry. □

The record arm counts occurrences, not content: with a count-sound reading, $\{r\}$ passes after eight native postings whatever they wrote, while every ledger cut without the writer keeps the silence of Theorem 21.4(b), for every C when the writer is the outside, $s = o$. The window verdicts of this book concern ledger cuts on the native-step clock and survive duplication (Proposition 21.19(a)). Executed counts, ensemble counts and one-post windows depend on multiplicity and need a count-sound attachment.

21.8 Least-cost booking decides

The cost of Proposition 2.1 prices each unit post at $\cosh 1 - 1$. Suppose a posting is booked as a list of unit posts, priced post by post, and the cheapest list that carries its current is chosen. That is a premise about whole postings; recognition dynamics prices single posts. Granted it, the padding question has an answer.

Definition 21.21 (Least-cost booking)**.** A list of signed posts *realizes* a label j when its net change of the potential of each observed account i is j_i. Booked post by post, a list of ℓ posts costs $\ell(\cosh 1 - 1)$. A write rule W is *booked at least cost* when every $W(j)$ realizes j and no list that realizes j costs less. A *tick floor* asks instead that every $W(j)$ be nonempty, realize j and cost no more than any nonempty list that realizes j.

Theorem 21.22 (Least-cost booking and the native ledger data)**.**

(a) A list that realizes j has at least $|\mathrm{phys}(j)|$ posts, and a realizing list of that length is a rearrangement of $\mathrm{phys}(j)$*. Hence W is booked at least cost if and only if every $W(j)$ is a rearrangement of* $\mathrm{phys}(j)$*.*

(b) If W is booked at least cost, padding is not a write under W, W is sound and complete, and $X_W = N$ on every record; and the load-weighted and ensemble counts of Section 21.3 agree with their net-current forms. No bracket, no axis locality and no condition on repeated entries is needed.

(c) For W in the multiset bracket, W is booked at least cost if and only if padding is not a write under W.

(d) pad*, the doubled rule* dbl *and the zero-label rule* zpad *are not booked at least cost, and W_S is booked at least cost if and only if $S = \emptyset$.*

(e) The ledger facts listed here do not decide. The rules phys*,* pad*,* zpad *and every W_S realize every label, lie in the multiset bracket, book only posts each of which is a unit posting, a cheapest nontrivial monotone extension (Proposition 2.1), and, like every ledger, conserve in the sense of recognition dynamics. Padding is not a write under* phys*, is a write under* pad*, and is neither under* zpad*.*

(f) If W lies in the multiset bracket and every $W(j)$ has the same length, padding is a write under W and $X_W \neq N$. No rule booked at least cost has this property.

(g) Under a tick floor, $W(0,0,0)$ is a single post on an account that is not observed, and if every entry of every $W(j)$ is an entry of $\mathrm{pad}(j)$*, then every $W(j)$ is a rearrangement of* $\mathrm{zpad}(j)$*: padding is neither a write nor not a write, and $X_W \neq N$. If instead the mandatory post is the posting's record write, booked on an account of its own, the shortest lists that realize the current together with one record write are the rearrangements of the physical writes together with the record write, and none contains $p_\circ$.*

(h) Booked as one ledger change, the padded posting costs the physical posting's cost (2.1) *plus* $\cosh \mathrm{sil}(j) - 1$*, strictly more whenever j has a silent axis.*

Proof. (a) An observed axis with $j_i = \pm 1$ needs at least one post of that sign on account i, so a realizing list has at least as many posts as j has nonzero components, which is $|\mathrm{phys}(j)|$. At that length each nonzero axis carries exactly one post, of the right sign, and nothing else is booked, since any further post, on a silent axis or on another account, adds length. Cost is proportional to length. (b) follows from (a): a rearrangement of $\mathrm{phys}(j)$ has its entries and its length. (c) In the multiset bracket $W(j)$ is $\mathrm{phys}(j)$ plus some copies of $p_\circ$ (proof of Theorem 21.6(c)); by (a) it is cheapest exactly when there are none. (d) $\mathrm{pad}(0,0,0)$ has three posts, $\mathrm{dbl}(j)$ twice the physical ones at every nonzero label, and zpad one post at the zero label; $W_S(0,0,0)$ holds $|S|$ padding posts. (e) Each listed rule changes observed account i by j_i (Lemma 21.3(c)) and lies between phys and pad; each of its posts is a unit posting, which is a cheapest nontrivial monotone extension by Proposition 2.1; the readings of padding are read off the zero label and the definitions. (f) The labels $(1,1,-1)$ and $(-1,-1,1)$ book their

three physical posts in the bracket, so the common length is 3, and $W(j)$ is phys(j) plus sil(j) copies of $p_\circ$; then $X_W(h) = 3k$, which differs from N on the zero label. A least-cost rule books nothing at the zero label and three posts at $(1, 1, -1)$. (g) The zero label is realized by the empty list, which the floor excludes; a single post realizes it exactly when it is not on an observed account, and one post is cheapest among nonempty lists. At a nonzero label the physical posts already form a nonempty realizing list, so they are cheapest. Within the entries of pad the unobserved post at the zero label is $p_\circ$, which gives zpad; it pads at $(0, 0, 0)$ and at no other silent label, and it books one post where N counts none. With a record write on its own account, the record account needs one post and the observed accounts need the physical ones, and nothing else shortens or completes the list. (h) In one ledger change the observed counts rise as under phys, and the debit count of $\circ$ rises by sil(j), which adds $\cosh \text{sil}(j) - 1$ to (2.1). □

Granted least-cost booking, the attachment follows without the bracket, axis locality or a condition on repeated entries, by (b). Part (e) is why it is a premise and not a consequence: each single post of every rule considered is a cheapest ledger extension, and recognition dynamics as developed here prices ticks, not the set of posts a step books. Least-cost booking is a step-level premise, and it is minimal: on the multiset bracket it is equivalent to the silence reading, by (c). It is not derived. The opposite reading contradicts least cost already by (c). A fixed number of posts per posting is one way to force it (f), and no native source of such a rate is known; it is not the only way: booking one padding post at every label with a silent axis lies in the multiset bracket, writes padding, and books one, two or three posts. The direct question, whether a posting with zero current must book a post and where, has a definite answer in each case: under least cost, nothing; under a tick floor, one post on an unobserved account, the padding account if the post is one of the encoding's; with the record write as the mandatory post, only the record write. A conserved ledger quantity cannot decide instead: every ledger conserves in the sense of recognition dynamics, and within the encoding the padding account's balance stays fixed exactly when padding is not a write, which restates the question.

Remark 21.23 (Open: least-cost booking). Derive least-cost booking of each posting from recognition dynamics, for example from a least-action principle on whole histories of the ledger; under a tick floor, decide whether the mandatory post of a zero-current posting is its record write or a bare ledger post; and derive that each posting writes the record arm exactly once, and by which account.

What a pass is taken to mean, and what kind of thing it is a pass of, is the question of the next chapter.

Chapter 22

What a perspective is

The native step gives a perspective three layers, and each is a different kind of fact. Put together they define a perspective. The theorems give evidence for reading that definition experientially, and the one step that stays open is an identity, which we state exactly.

22.1 Three layers, three kinds of fact

A *boundary* is an exact superselection sector of the ledger's own history, drawn by no observer and produced by no environment, and which boundary a history occupies is fixed at its beginning (Chapters 4 and 5). A *participant* is a state that the postings keep as itself: four of the cell's 64 dimensions, kept through every record of every length, and on a network the parity law (Theorems 7.11 and 13.5). A *record* is what decides: it settles membership, purifies the present and carries the arrow of the present's long-run motion (Chapter 8, Corollary 10.12 and Theorem 11.4).

Definition 22.1 (Perspective)**.** A *perspective* is a participant inside a boundary together with its record. On one cell: a unit genesis state v with $w_{\mathrm{al}}(\rho_v) = 1$, the boundary formed by the join of the lineage atoms $\Lambda(x, n)$ at which v has weight, and the records h with their law $\mathbb{P}_v$. On a network the kept class $\mathscr{N}(E)$ replaces the aligned sector.

Each part of the definition is then fixed by the native step, its records and the stated choices: the event algebra that defines boundaries, the field-current window, and on networks the placement of contacts (derived on the ledger's torus from locality, the symmetry of the contact rule and the existence of a contact, Proposition 12.4) and their timing (Chapter 14). The definition is a proposal for what the word should name. The theorems characterize what it names.

22.2 The window criterion and the dynamics agree

Recognition Science reads each recognition event as having an observable outside and an experiential inside [4]. On that reading a complete experience spans an eight-tick cycle with no silent tick. We read this criterion on the native cell as the field-current window of Chapter 3: eight consecutive postings, one tick per posting, each writing the loop edge, which happens exactly when each starts from a mixed occupation. The account, the clock and the routing of the exchange current through the loop edge are supplied with the step (Chapters 2 and 3). A sure subject is a state for which every window passes with probability one. Reading a sure subject as a perspective that experiences is an identification, and this is where it enters.

The theorems do not prove that a sure subject experiences. They establish something that bears directly on whether the identification is reasonable. An identification of experience

with a dynamical class could have named an arbitrary set of states, or a set that the dynamics does not respect. This one does neither:

(1) The class the window names is exactly the class the postings keep (Theorem 7.16). The neutral sector is itself a loop-edge quantity, so the substance of the agreement is that the aligned sector never leaks and that three postings expose every other state.
(2) The class is spanned by physically distinct motions, two persistent currents and an oscillation, and their currents and field energies are explicit (Section 7.7).
(3) Among cuts that can pass, only the loop edge gives a first-window verdict that commutes with everything the record effects generate (Theorem 3.11). The criterion picks the one account on which the verdict does not depend on coherences.
(4) A participant's own record settles membership: the undecided part falls at the explicit rate $2^{-(n-1)}$ and every infinite record decides, although no finite record event decides for every state (Theorems 8.4, 8.5 and 8.6).
(5) On networks the class becomes the parity law, and on a connected network one participant's own test and the joint test keep the same states (Theorem 13.13).
(6) Composition creates no kept weight: on every network in its round order the kept projection commutes with every round word (Theorem 13.9, Section 22.4).

If experience is the inside of recognition activity, these are the properties a persisting experience should have. It should be something the ledger maintains, present at every tick, settled by its own history and not manufactured by combining others. That inference is the main evidence the book adds for the experiential reading. The next section adds a theorem of a different kind: under a faithful report, the participant's own verdict channel carries exactly what the window's inputs index, and exactly on the passing windows.

22.3 What the verdict channel carries

Take an occupation path $x^{(0)}, \dots, x^{(k+8)}$ and its record $j^{(t)} = j(x^{(t)}, x^{(t-1)})$, $t = 1, \dots, k+8$. Window k consists of the postings $k{+}1, \dots, k{+}8$. Its *inputs* are $x^{(k)}, \dots, x^{(k+7)}$ and its *central word* is $(j_2^{(k+1)}, \dots, j_2^{(k+8)})$, the loop-edge components of its labels. Every sequence of occupations is an occupation path, since every transition writes a label (Proposition 2.7).

Lemma 22.2 (The central value reads the input)**.** *$j(y,x)_2 = x_1 - x_2$ depends on the input alone. It is $+1$ at $(1,0)$, -1 at $(0,1)$ and 0 at both balanced inputs. Two paths have the same central word in window k exactly when, at each posting of the window, their inputs agree or are both balanced.*

Proof. The first two statements are the definition of the label. A nonzero value fixes the input by its sign; the value 0 is shared by $(0,0)$ and $(1,1)$ and by nothing else. □

Theorem 22.3 (The window passes exactly when its central word decodes its inputs)**.** *For every $k \geq 0$ and every occupation path x of length $k+8$, window k passes on the record of x if and only if every path x' with the same central word in window k has the same window inputs. If the window fails, there is such an x' whose inputs differ from those of x.*

Proof. If the window passes, every input is mixed, and Lemma 22.2 recovers each from the sign of its central value. If it fails at posting t, the input $x^{(t-1)}$ is balanced. Replace it by the

other balanced occupation and keep every other occupation. The result is a path; each central value reads only its own input, so the central word is unchanged, and the inputs differ at t. □

To connect the central word with felt content we need the content as an object, and we need a premise.

Definition 22.4 (Episodes, faithful reports and decoders)**.** Fix a set $\mathscr{E}$ of contents. An *episode* is a pair (x, e) of an occupation path and a content $e \in \mathscr{E}$. A *report* is a function r from paths to $\mathscr{E}$; a report *for window k* depends on the path only through the window's eight inputs. A report is *faithful* on the episode (x, e) when $r(x) = e$. For a family of episodes $(x_i, e_i)_{i \in I}$ and a read R of paths, a *decoder* of R is a function d with $d(R(x_i)) = e_i$ for every i.

The contents and the faithfulness of the report are supplied: the premise says that the content of each designated episode is a function of the window's inputs and that the report states it. A decoder is a mathematical lookup on the designated episodes, not an instrument.

Theorem 22.5 (Under a faithful report, the verdict channel decodes exactly on passing windows)**.** *Fix k, a set $\mathscr{E}$ and a report r for window k.*

(a) Let $(x_i, e_i)_{i \in I}$ be episodes on each of which r is faithful and window k passes. Then the central word decodes the content, by the explicit decoder $d(c) = r(\hat{x}(c))$, where $\hat{x}$ reads $(1, 0)$ from $+1$ and $(0, 1)$ from -1 at each posting. In particular $e_i = e_j$ whenever x_i and x_j have the same central word.

(b) If window k fails on x, take x' from Theorem 22.3. For every family of episodes that contains (x, e) and (x', e') with $e \neq e'$, the central word has no decoder.

(c) Window k passes on x if and only if, for every set $\mathscr{E}$, every report r and every pair of episodes (x, e), (x', e') on which r is faithful, the central word decodes the pair.

Proof. (a) On a passing window $\hat{x}$ returns the inputs (Theorem 22.3), and faithfulness gives $r(\hat{x}(c_i)) = e_i$, where r is read as a function of the inputs and c_i is the central word of x_i. (b) Equal central words with different contents defeat every function of the central word. (c) If the window passes, apply (a) to the pair. If it fails, take x' from Theorem 22.3, let $\mathscr{E}$ be the set of input octuples, r the identity, and e, e' the two input octuples; both episodes are faithful, $e \neq e'$, and (b) applies. □

Two criteria that could have come apart coincide here. The window criterion asks whether every tick of the window writes the loop edge. The access criterion asks whether the participant's own loop-edge record determines the window's inputs. Theorem 22.3 says they are the same condition, and Theorem 22.5 says that, for content indexed by the inputs, the passing windows are exactly the ones on which the verdict channel can carry it. On a failing window the channel cannot carry any difference indexed by the balanced bit, because $(0, 0)$ and $(1, 1)$ post the same central value at every output.

The whole record carries more than the central word.

Theorem 22.6 (What the whole record carries)**.** *Let the records have length $n \geq 1$.*

(a) All occupation paths writing a record h coincide if and only if h contains a nonzero label. The silent record is written exactly by the two constant balanced paths.

(b) Let r be any function of the path and let (x_i, e_i) be episodes on which r is faithful, none of them a constant balanced path. Then the record decodes the content, by reading back the path that writes it and reporting.
(c) If the two constant balanced paths are episodes with different contents, the record has no decoder.

Proof. (a) is Proposition 9.10. (b) follows from (a), and (c) from the collision in (a). □

The bound $n \geq 1$ is needed: the empty record is written by every one-point path, among them the two distinct mixed paths $(1, 0)$ and $(0, 1)$.

So content indexed by outputs, or by the balanced bit of a failing posting, is in the record but not in the verdict channel. That is a precise place where "in the record" and "available to the verdict" separate. In probability the two channels behave differently as well.

Theorem 22.7 (How often each channel decodes)**.** *Let ν be a unit genesis state.*

(a) For $n \geq 2$, the probability that the central word of the first n postings decodes all n inputs is

$$\mathbb{P}_\nu(\mathscr{J}_*^n) = w_{\mathrm{al}}(\rho_\nu) + 2^{-(n-1)}\, w_{\mathrm{ne}}(\rho_\nu),$$

which tends to the aligned weight.
(b) The probability that the central word of window k decodes its inputs is the pass probability $\mathrm{tr}(\rho_\nu W_k)$.
(c) The silent record of length $2m$ has probability at most 4^{-m}, and a silent posting never raises the silent weight. Hence the whole record decodes the occupation path with probability $1 - \mathbb{P}_\nu(\mathit{silent}) \geq 1 - 2 \cdot 2^{-n}$ at length n, which tends to 1.

Proof. (a) By Lemma 22.2 the central word decodes every input exactly when every label writes the loop edge, the event $\mathscr{J}_*^n$. Theorem 8.4(a) gives its probability: the aligned part contributes w_{al} and the rest $2^{-(n-1)} w_{\mathrm{ne}}$. (b) is Theorem 22.3 and Proposition 3.2(a). (c) The zero label is written only by $00 \to 00$ and $11 \to 11$; two consecutive silent postings keep at most a quarter of the weight, and one keeps at most all of it (a direct exact computation on the two balanced blocks). The decoding event is the complement of the silent record by Theorem 22.6(a). □

The verdict channel decodes everything forever only on the aligned weight. The full record decodes almost surely in the long run, for every genesis state. Nothing in these theorems identifies a current with feeling: they say what the channel can carry if the faithful-report premise holds.

22.4 Composition and the whole

Whether a whole made of participants is itself one perspective needs a composition law, and the network results constrain any such law in four ways.

Composition creates no kept weight. On every network in its round order the kept projection commutes with every round word, so a whole holds no kept weight its beginning lacked (Theorem 13.9 and Corollary 13.10).

Participation is joint. On a connected network one participant's own test keeps exactly the states the whole network keeps (Theorem 13.13).

A composite reads only crossed products. Under the product law, a composite record reads at two sites, beyond its members' two-site reads, at most the crossed products of what the members read at each site, and exactly these when the composite obeys the product law too (Theorem 16.40). Composition adds correlations of the members' reads, not new single-site content.

A participant can outlast all of its partners. A site keeps its awake configuration while one partner is exchanged for a fresh site exactly when the old partner carries a persistent current or that contact was its only one, and the new partner must take the old partner's clock and occupation (Proposition 12.19). An oscillating centre can replace all its partners in turn and stay awake at every stage (Chapter 12).

These are constraints, not a composition law. Whether a connected network of participants is itself one subject, and whether the whole ledger is one experiencing subject, remain open (Chapter 23). Any proposed law has to reproduce the four facts above: a whole that came out of composition with new kept weight, or with single-site content its members do not read, would contradict them.

22.5 The positive case and the open identity

The case for the experiential reading rests on three points.

Experience belongs to the reality being explained. Our experience is itself an occurrence within reality, so an adequate account has to place it in the same world as physical interaction. In Recognition Science, recognition is the common structure of distinctions, comparisons, interactions and retained records [1]. A perspective, if it is anything physical, is structure of that kind.

The reading gives a unified account. On the dual-aspect reading, experience is intrinsic to the process already doing the physical work [4]. It is not something added to the ledger, and not a label attached to it afterwards. The book shows what that reading requires of the process on the native cell, and the process supplies it: the window criterion names the class the postings keep, a class spanned by definite motions, settled by its own record, joint on networks and not produced by composition.

The agreement is not built in. The length of the window and its no-silent-tick condition come from the completeness condition on an eight-tick cycle [4], not from an analysis of the postings. Its class could have been empty, could have been a set the native step mixes with its complement, with no description independent of the window, or could have depended on coherences no record reads. It is none of these. Under a faithful report, the passing windows are exactly the windows on which the participant's own verdict channel carries everything its inputs index. An alternative reading of the same mathematics has to explain why the class the window names is the sector fixed by the pole product, four definite motions that three postings cut out and that only the loop-edge cut reads centrally.

If experience is the inside of indispensable recognition activity, removing it would remove

an aspect of the activity through which the ledger operates. On that reading experience is constitutive of reality, and the perspectives are the places where the ledger's own dynamics keeps that activity going without loss. We think the agreement proved here makes this the most economical reading of the results. The judgment is an inference from the theorems, not one of them.

The precise unresolved step is the identity itself: that the inside of the activity which keeps a state in the aligned sector is felt. The theorems are neutral about that identity, and they are what any argument for it now has to explain. Two further steps are separate from it. The faithful-report premise of Section 22.3 would become a theorem with a derivation of the participant's report channel as an instrument of the native dynamics, with the content it reports. The single-subject claim needs the composition law of Section 22.4.

22.6 Relation to familiar structures

Several results have recognizable counterparts, and the differences are informative. Superselection rules usually come from a symmetry of the theory [43] or emerge approximately when an environment records a preferred basis [45]. Here they are exact and come from the event algebra alone (Chapter 4). The kept subspace resembles a noiseless subspace of a quantum operation [25], with one difference: it is defined against a specific sector of leaking states rather than against all disturbance, and it is computed as a common kernel of the record operators (Chapter 7). The decision theorem has the structure of repeated nondemolition measurement of a conserved sector, in which conditional weights converge to 0 or 1 [7]; here the rate $2^{-(n-1)}$ is explicit (Chapter 8). Purification of the whole present (Theorems 10.5 and 10.11) is the Maassen–Kümmerer statement [30], with an explicit though not sharp rate. The arrow of Chapter 11 arises without a thermal bath or an imposed boundary condition: it is a circulation of the record chain itself.

What the book leaves open, and what would settle each item, is the subject of the last chapter.

Chapter 23

Open problems

The book keeps three things apart: what is proved, what is supplied, and what is open. This chapter lists the second and third. For each premise we say what is already known about removing it and what exact step remains. For each open proposition we give its content as the chapters state it, what is known, and what would settle it. Where a question is narrowed, its proved part is a theorem of the cited chapter and only the residue appears here. Refuted posed laws are results, stated with their counterexamples in the chapters, and are not repeated.

23.1 The premises and what would remove them

The step. The native operation is taken from recognition dynamics [3]; each channel has 2^3 states and unit postings are the cheapest ledger extensions (Proposition 2.1). The routing of the exchange's transfer through the loop edge is part of the step: the occupations fix the current only up to a circulation $m(1, 1, -1)$, and the tree-routed current never writes the loop edge (Proposition 2.8). Open: which physical interactions enact the exchange and the clock, and whether the loop-edge routing can be derived.

The window's account and clock. The eight-tick criterion fixes the window's length and that no tick is silent. Reading it on the field-current account with one tick per posting is supplied; Chapter 3 shows what other cuts and clocks give.

Locality of contacts. On the bare network carrier no proper placement is named by the native postings. On the ledger's torus the contacts are exactly the nearest-neighbour pairs, given three premises and the identification of the network's sites with the torus sites: a contact joins only sites one post can join (locality), the contact rule is computed from the native data alone and so carries their translations, axis permutations and reflection, and at least one contact exists (Proposition 12.4). One contact is taken per neighbour pair, a reading: keeping both orientations would make every pair a doubled contact. Deriving locality from the ledger dynamics would make placement native. A concrete next computation is the kept class of the derived network itself, the parity law on the torus graph with one contact per neighbour pair.

Orientation. No symmetric rule orients every neighbour pair one way, so orientation is supplied. It changes the written labels and not what is kept.

Posting order. No order is forced: a timing rule computed from the native data is unchanged by the one-tick shift and by relabellings that move every contact, so it picks out no fair schedule (Chapter 14). The order is physical, since it decides what is kept (Chapter 12). It therefore has to come from structure beyond the native posting family, and which structure supplies it is open.

The gate. The gated posting is the native posting on awake labels and silence otherwise. On a balanced pair the native instrument does post, with zero loop-edge current (a deterministic dressing when both clocks are undressed, a branching otherwise), so the

gate is a restriction of the native instrument and not a consequence of it. Deciding it is the question whether a posting that carries no loop-edge current is executed, which is the attachment question below in another form.

Attempts. An attempt law computed from the native data alone, a chance for each contact unchanged by their symmetries, gives every one of m contacts the chance $1/m$, so positivity, the only property of the chances the zero-one law uses, follows from that premise. The independence of the attempts is supplied. Open: derive independence, or some floor on every chance, from the native dynamics.

Event attachment. The ledger facts established so far cannot decide whether a padding post is a physical write: the physical writes, the padded encoding, the rule that pads only the zero label and every per-axis rule all satisfy the field tie, the encoding bracket, per-tick minimum cost and conservation (Chapter 21). The minimal extra premise is least-cost booking: each native step executes a cheapest list of unit posts that carries its current on the observed accounts. Under it padding is not a write and every count agrees. To make attachment native, derive least-cost booking of each step from recognition dynamics, for example from a history-level least-action principle on the ledger; recognition dynamics as developed so far gives minimality only per tick. If instead every step must book at least one post, the remaining question is whether that post is the step's record write (then padding is again not a write) or a bare ledger post (then the zero-label rule, under which neither reading holds). That the physical writes of a posting are the unit changes of its own current is an identification; given it, soundness and completeness force the attachment to the physical writes. Axis locality, used where least-cost booking is not assumed, and the premise that every posting writes the record arm exactly once, by a writer account outside the cut, are supplied in the same chapter.

Capture and control. The capture permutation, its tag chart and blank, the joined carrier with its holding, and every control, schedule, strength, blank, donor slot and recorded branch of Chapter 20 are supplied. Reading a non-passive write as the activity of a boundary that registers is an identification.

The relay. The protected-inside results of Chapters 18 and 19 use the relay. Its photon-free clock and reciprocal terms are eighth-turn logarithms of the posting's two factors, for fixed branches of the logarithm. Alternated at a common rate they give its photon-free evolution at $\phi = 0$, $\Delta\nu = 0$ as the limit of fractional native postings, and the clock terms' photon factors are the posting's own current incidence (Proposition 18.1). The relay equals that native generator exactly when $\Delta\nu = 0$ and $\phi \in 2\pi\mathbb{Z}$. In Chapter 19 the protocols that interleave relay intervals with postings, the fast-posting limit, and the half-turn kicks, applied instantaneously at supplied rates and in supplied words, are supplied as well; the joint half-turn is a native symmetry, but no native process applies it on a schedule and it is not a polynomial in the step (Proposition 19.18). Open: derive the branches and the common rate; derive the Coulomb difference, for example as a ledger cost of the doubly occupied cell; derive the momentum phase and the reciprocal's translated profile and recoil, for example as Bloch data of a total momentum on a translation-covariant chain of native cells with the posting's field incidence. Whether the reciprocal's photon factor is the native incidence with no recoil is a named open question; at $z = 0$ it would make the recoil act as the identity on every photon payload, which is not expected. Reading fractional postings as the dynamics between postings is an identification.

The read algebra. The retained read R and the grammar of placed reads are taken from

recognition dynamics [3], not derived from postings (Chapter 5). Which algebra describes the operations available inside the ledger, the closed event algebra or the larger algebra of native reads, is open (Chapters 5 and 6). So is the corner convention: whether a read placed at an earlier stage can act on a history already carrying newer records, which decides whether a stored record is classical for every available read (Chapters 6 and 9). Which physical process, if any, performs the retained read, the dwell read, the circulation projection or the read $RE_{(0,0,0),(0,0,0)}R$ as one operation is open.

The record law as frequencies. That the record law gives the frequencies with which records occur in repeated trials is an apparatus statement, separate from its construction (Section 2.7).

The present. Reading the conditional state $\rho_{v,h}$ as the physical present is an identification, not a collapse rule and not a derivation that individual occurrences sample the record law.

Genesis. Which lineages are occupied is an initial condition (Chapter 5). The physical status of orphan coordinates, which no beginning populates but native reads reach, is open.

The photon dictionary. Reading the axis as a plaquette of the photon field, the Gauss energy as electric field energy, and its scale $\eta > 0$, are identifications and supplied data (Section 7.7).

Experience. Reading a sure subject as a perspective that experiences is an identification; Chapter 22 gives the case for it and Section 23.7 states the open identity.

23.2 One cell

The oscillation's field. Whether the field displacement posted by the oscillation, a single open edge current, has a nonzero transverse profile, and what its coherent energy is beside that of the circulating loop; which sign an actual field displacement carries (Section 7.7).

Field energy with interactions. Whether the split of Theorem 7.29, bounded field energy exactly on the pure oscillation, survives when the interaction and control energies of the step are added, and for a general loop distribution, where a summability condition is needed for the expected energy to be finite.

Two cells. On two native cells, are the joint sure subjects of both field-current windows with bounded joint field energy exactly the products of two oscillations, or do entangled pairs also qualify?

The one-zero rate. Whether the decay $7/8$ per four postings of Theorem 11.2 is sharp; it is settled by the spectral radius of the transfer operator on records without a bad pair.

Instruments for the reversed chain. Whether an instrument built from native postings realizes the reversed kernel on the circulating classes 3 and 4, and which native process prepares a genesis state in those classes (Chapter 11).

Unsharp archive reads. How well an unsharp read recovers a record after an unknown real duration of the archive (Chapter 11).

A conjugation reference in the record law. Whether the couplings between a cell and the photon field carry a conjugation reference into the cell's record law, so that a real genesis acquires a circulation; and whether one real read serves every duration beside a non-real reference (Proposition 11.15).

Sharp constants. The entry constants of Theorem 10.11 are a proved envelope, far from sharp; the sharp purification rate is open.

23.3 Networks, timing and reading one's world

Leak constants. The optimal leak constants for the path and the triangle (Proposition 12.10), the sharp rates in Theorems 13.15 and 13.16, the least depth of an own test as a function of the network and the tested contact's position, and leak rates on general networks.

Order-freeness from the graph. Theorem 12.22 characterizes the networks whose awake patterns are the same in every order through the awake patterns of all orders. A test that reads order-freeness off the graph alone is not known.

Identity across turnover. Whether the oscillating centre of the partner-exchange example of Chapter 12 is the same participant across its stages is a question of identification that the exchange law does not settle.

Loops raise admission with any sharing. Does the inequality

$$m!\, N_{\mathrm{flow}} \le N_{\mathrm{adm}} \prod_{c} \binom{\deg c - 1}{\deg c / 2}$$

of Theorem 14.45(d) hold for every walled cluster with one contact per pair of sites and every clock of even degree, whatever the sharing pattern? It is proved when every many-contact clock shares at most two postings. It is open even when every many-contact clock has degree four, where it reads $m!\, N_{\mathrm{flow}} \le N_{\mathrm{adm}}\, 3^{\#}$ with $\#$ the number of many-contact clocks; that case asks whether sharing three or more contacts can lower the admission chance. Termwise positivity fails: on $K_{2,3}$ of four-contact clocks one term of the expansion of $\prod_c \alpha_c$ is $-\frac{1}{467775}$. Four postings alternate exactly when the linking number of the chord of the outgoing pair with the chord of the incoming pair is ± 1, so the alternation indicator is the square of a sum of sawtooth functions.

How far a cut count must reach. Is there a fixed k such that, on every tree cluster with balanced walls, one contact per pair of sites and even degrees at most four, cut counts on all sets of at most $k(d+1)$ clocks imply a token flow, d the diameter of the cluster? The expected answer, that no such k exists, is also unproved (Theorem 14.41(c) shows that no bound independent of the cluster suffices); a proposed refutation takes two copies of a root carrying three full binary trees of depth D.

The star with three unread contacts. What lasting content a star keeps when its centre meets three unread contacts. The law posed by analogy with the path (Theorem 14.47) fails in three ways, and strict-round powers of the round map do not converge; this is computational evidence, and no statement is proved.

A three-site whole with one recorded contact. What a three-site whole keeps when exactly one of its two contacts is recorded.

The copy clique. On every network and for every contact e, is the own copy number attained by a family of pairwise conflicting exit tests at e's sites, so that the copy number is the largest exit clique? It holds on the small networks checked and is undecided on the pentagon, where it asks whether four copies make a copy reader (five suffice; the exit clique has four tests).

Joint site cover. On every connected network and for every contact e, is the least joint cover number, the pooled copy number, equal to the site cover number, the largest over sites of the least site cover? In particular, is the pooled copy number of the contact $(2, 8)$ on G_{13} equal to 22 (Theorem 15.39)?

Single-anchor König property. On every connected network, for every anchor site s and

token site p with no free exits, is the least cover by search orders of the family of tests at p whose region reaches s equal to its largest pairwise conflicting subfamily? Proved for $s = p$ and for s adjacent to p; connected anchor sets need not have the property (Proposition 15.36), and the per-site law fails on G_{10} (Theorem 15.37).

Block rounding. At every site of every connected network, is the least site cover attained by an order-separated block family, equal to the largest sum over blocks, with no order serving tests of two blocks, of the rounded-up fractional cover numbers? Proved at every site whose least cover is its rounded fractional cover number and at the far site of $K_{2,r}$ glued at such a cut site (Proposition 15.40 and Section 15.6).

One-token content. Outside the case $(4, 2)$, do equal joint records of connected families of $K \geq 2$ members with no repeated clock pair, with positive weights summing to one, at any token numbers, give a labelling with equal member weights, equal cluster sizes and equal one-token walks P^s on the family's clocks (Chapter 15)?

Disjoint members and the exceptional pairs. Do two members sharing no clock ever give the two wholes of an exceptional pair ($2k + 1, k$ or $k + 1$) against $(2k + 2, k + 1)$, in either order, neither the case $(4, 2)$, with positive weights summing to one, the same joint record? Settled at $k = 2$ (Theorem 15.64); open for $k \geq 3$. With the result recorded for sharing members in Chapter 15, this is all that size reading by families with a disjoint pair or a star needs.

Sharing members read c_4. For two members sharing exactly one clock, under the hypotheses of the size-reading statement, do equal joint records give equal c_4? Proved when the members are alone on their clock pairs and $W_s \neq W_x$ (Proposition 15.60); open in the symmetric case $W_s = W_x$ and for silent copies of a member on its own clock pair.

23.4 Pooled records

Relative circulation through three erasures. On the path, with sites 1 and 2 at bit 0 carrying γ, γ against $\gamma, \bar{\gamma}$, and for every real state of the shared site, every list of at most three erased postings of the two contacts and every record of both contacts after them: do the two states have one law? Four erased postings read the difference (Proposition 16.7); two or fewer cannot.

The path criterion. Is every law-private instrument free of live path pairs: is law privacy exactly the absence of a pair of unrecorded paths from distinct inputs, with equal written values, equal end and equal parity mark, whose cross term is nonzero (Remark 16.25)? The summed criterion is proved (Theorem 16.23). A counterexample must be a record blind to the sum of two distinct live path pairs of one class, of length at least four.

How deep a privacy test must look. Is there a depth N near five such that, for every instrument, law privacy is equivalent to darkness for histories of at most N postings followed by native words of at most three labels? $N \geq 5$ is needed and $N = 1956$ suffices (Theorem 16.28). Is there any N for which privacy is equivalent to darkness for histories of at most N postings followed by one-label words?

The old record alone. Does the old participant's record, its neighbour unread, leave a dark subspace on the three path sites: is there a projection Q of rank at least two with $QE_w^{\mathrm{old}}Q \in \mathbb{C}Q$ for every word w? There is none if each of the 512 pole-line projections of the three sites lies in the span of the old effects, and that sufficient condition is open as well.

Exact pools and the cut. (i) Does every exact pool reach the cut? By Theorem 17.20 this

would give: a pool is exact if and only if the flip rule holds, on every network. (ii) The weaker statement that no exact pool contains a parallel pair is equivalent to that equality (Remark 17.21). Corollary 17.25 reduces (i) to maximal non-reaching pools with bracketed events.

The two-unread rule. On every network in which no site touches three distinct unread postings, is a pool exact if and only if it is joined? Joined implies exact everywhere. The rule follows from the exposure statement, that every unjoined pool of such a network has an exposing flip. Proved classes: one split posting per site, the two-factor table class, even gaps, clustered composites and sided final components, repair plans (Chapter 17); in exhaustive searches over seven families of contacts (about 43 million unjoined networks of up to twelve postings) and in random searches, every unjoined pool of such a network had an exposing flip.

Pair exposure, and all pairs. Does every maximal unjoined two-unread bracketed pool have a read component that splits an unread posting, holds a bit at the cut, and carries a crossing-assigned pair vanishing off the bits linked to it? The stronger statement for all pairs: on every two-unread network, every read component that splits an unread posting carries such a pair. Separately: with three or more common components, does a colouring always expose? All pairs implies pair exposure, which implies the exposure statement and the two-unread rule (Remark 17.30).

The clause statement. On every two-unread network with bracketed events, does every read-closed set have a satisfiable clause system (Remark 17.38)? It implies pair exposure. Two refinements: some candidate meets no propagation conflict on every read-closed set, and after propagation the two-bit shadows of the sites decide satisfiability. A proof must choose the assignment against cascades of three or more sites and against parity cycles: pair conditions do not decide a candidate, and propagation does not either.

The second contact's rule. Does the second contact keep lasting joint content beyond products exactly when the pattern has rule shape and reads some label (Remark 17.48)? It follows from two statements: every path pattern is read-saturating, and every pattern with lasting content at the second contact has rule shape and reads a label. Both reduce to single-read patterns. All two-contact patterns of length six with a read, and all single-read patterns of lengths seven and eight, agree.

Transience between untouching sites. Does every path pattern obey the product law at (a, b) from some depth on? Open only for patterns that use both contacts, read a label and do not open twice; all 1302 path patterns of length at most five with a read stabilize with the product law (Remark 17.52).

Pair saturation after every run. For every $k \geq 3$, does the second contact's pair of π_k read all 144 products at some depth? Also open is leading-run invariance: a pattern's sites and second pair saturate after $k \geq 3$ leading unread postings whenever they do after three. Rank 144 at depth 2 holds for every k from 3 to 15 modulo a large prime, as evidence. A proof cannot pass through the long-run limit, which reads at most 60 products (Theorem 17.53); it must control exponentially small terms, a non-vanishing problem of Skolem type.

23.5 Protected insides

The odd gap. For $z \neq 0$, is $W_{\mathrm{odd}} = W(z) - W(-z) - W(\tau_{-1}z) + W(-\tau_{-1}z)$ injective on the photon space? For $\varepsilon \neq 0$ this is equivalent to every nonzero sure subject having a nonzero pinched departure (Remark 18.13). It holds on the vacuum ray; a nonzero h with $W_{\mathrm{odd}}h = 0$ would give $\psi_\circ \otimes h$ kept to first order by unrecorded postings.

The frequent-posting limit at all times. Does the limit of Theorem 18.14 hold for every real T, negative T included? Proved for $T \geq 0$. At $z = 0$, is the persistent loss cubic: for every sure subject x are there $C, \delta > 0$ with $\left|\|\Pi^{\perp} e^{-iT(H_0+\bar{V})}x\|^2 - T^2(r_+(g_+)+r_-(g_-))\right| \leq CT^3$ for $0 \leq T \leq \delta$? Proved with $o(T^2)$ in place of CT^3 (Remark 18.16).

The persistent mean losses. At $\Delta v = 0$ and $\varepsilon \neq 0$, is $0.726 < m_+ < 0.728$ and $0.316 < m_- < 0.317$? Floating-point evidence gives $m_+ \approx 0.72700$ and $m_- \approx 0.31646$; an enclosure of the spectral formula of Theorem 18.17(e) would settle it (Remark 18.18).

Averaged delivery and the best preparation. Does the posting-averaged relay $\bar{H}$ deliver its circulation: for $\varepsilon \neq 0$ or $\Delta v \neq 0$, are there a record word w and a time T with $g_w(e^{-iT\bar{H}}, f_\gamma) \neq 0$? And what is the supremum of $|\langle f_\gamma, e^{-iTH}u\rangle|^2$ over real unit u and times T, known only to exceed $\frac{1}{2}$ (Remark 19.7)?

Every clock coupling reads. For every ϕ, v_0, Δv and every $\varepsilon \neq 0$, is no circulating kept state of Fix_1 or Fix_{-1} sealed by $\bar{H}(\phi, \varepsilon, v_0, \Delta v)$? Proved at $\Delta v = 0$ (Theorem 19.14). Under it a sealed kept circulation exists exactly when $\varepsilon = 0$; the minors computed exactly point to $\Delta v/\varepsilon = 0$ as the only candidate exception (Remark 19.16).

Coulomb-tuned seals. (i) For every t with $|t| \leq \frac{1}{20}$, does no admissible kick word seal a parity-split circulating kept state of $\mathrm{Fix}_{\pm 1}$ under $\bar{H}(\phi, \varepsilon, v_0, t\varepsilon)$, for any ϕ, v_0 and $\varepsilon \neq 0$? It holds outside a finite set of ratios and is equivalent to its restriction to that set. (ii) The same at every real t. Also undecided: the equivalence of Theorem 19.19(b) at the finitely many real roots of its two degree-10 polynomials (Remark 19.33).

Symmetric reading at every ratio. Theorem 19.38(a) and (b) at every $t \neq 0$, each equivalent to its restriction to finitely many ratios or to $|t| > \frac{1}{20}$; and whether every mirror-fixed record word with nonzero effect has all its letters mirror-fixed, which an exact computation over the 225 two-letter words supports (Remark 19.39).

One symmetric record near the symmetric point. Does some single mirror-fixed label read, under the plain word, every mirror-real circulating kept state of $\mathrm{Fix}_{\pm 1}$ at every ratio $0 < |t| \leq \frac{1}{20}$? It fails at $t = 0$ (Remark 19.41).

The rigid tie core. For each of the eleven readers, each $s = \pm 1$, each parity-split circulating kept state ψ of Fix_s that is not mirror-real, satisfies $Y\psi \neq \pm\psi$ and is not led by an extreme entry, and each toggling-frame list with a first block of tie fraction and a rigid rest of at least two positive intervals alternating in frame: is the reader never blind to ψ along the list (Remark 19.52)? Kicks add no privacy against one record exactly when this holds. The first open case is a rest of exactly two intervals: do two half-turns ever hide an inside?

23.6 Activity and attachment

Actuators. Derive the controls, strengths and schedules of capture from native interactions with spatial admission and bounded strength, and decide whether the static relay or the controlled preparation contact moves the source pair charge (Chapter 20).

Blanks and work. Prepare and fund the blanks at slots $4, 6, 7, 11$ inside the paid grammar, and supply the work from a constructed source with an autonomous clock.

Uniform resources. Find bounds uniform in the flux m on the errors, peak energy, duration and word length of the paid constructions; every present bound scales with the norm of the holding.

Capture as one operation. Enact write-after-exchange with coherent references by one composite paid operation or one fixed interaction, with the full payload, and construct the blank, the latch and the history.

The control-grammar invariant. Find the invariant of the control grammar beyond the total charge at every instant and the local commutation of the source algebra with charge.

Matter-side sectors. Does the holding move the sector label on the matter slots? The chapter proves motion on record pairs only.

Distance from passive flows. Is there a uniform positive lower bound on the distance of a record write from every passive flow, and is it smaller in the five degenerate sectors?

The two-charge receiver. On the two-charge carrier with supplied species, fixed centres and a shared transverse field: is there a finite-time receiver contrast caused by the source, or an exact cancellation? Then its mechanism, capture and retained records.

Least-cost booking. Derive least-cost booking of each posting from recognition dynamics, for example from a least-action principle on whole histories of the ledger (Section 23.1); under a tick floor, decide whether the mandatory post of a zero-current posting is its record write or a bare ledger post; and derive that each posting writes the record arm exactly once, and by which account.

23.7 Perspective

The experiential identity. That the inside of the activity which keeps a state in the aligned sector is felt. The theorems of Chapter 22 are neutral about it and are what an argument for it has to explain.

The faithful report. Derive the participant's report channel as an instrument of the native dynamics, with the content it reports, so that Theorem 22.5 holds without its premise.

Composition into one subject. A composition law deciding whether a connected network of participants is itself one subject, and whether the whole ledger is one experiencing subject. Any such law must respect the four constraints of Section 22.4: no kept weight from composition, joint participation, crossed products only, and persistence through exchange of partners.

Appendix A

Tables and finite certificates

A.1 The native step in coordinates

The clock maps. In the basis $e_0, \dots, e_3$, with $(Lv)_i = v_{i+1}$,

$$\text{stay} = \tfrac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{flip} = \tfrac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

On a mixed input each cell is dressed first. Table A.1 gives the dressed clock maps on the pole basis; it follows from Table 2.2.

	p	a	P	A
stay D	$\frac{1}{2}(p+a)$	$\frac{1}{2}(a-p)$	0	A
flip D	$\frac{1}{2}(p-a)$	$\frac{1}{2}(p+a)$	P	0

Table A.1. The dressed clock maps on the pole basis. On the dressed pole lines a cell moves surely (P flips to P, A stays to A); on p and a it stays or flips with weight $\frac{1}{2}$.

The sixteen blocks. Table A.2 lists $M_{yx} = \sigma(x)\, F_1 \otimes F_2$ with $F_1 = K(y_1, x_2)C_x$ acting on clock cell 1 and $F_2 = K(y_2, x_1)C_x$ on clock cell 2, together with the label $j(y, x)$.

input x	$\sigma(x)$	output y	F_1	F_2	label $j(y,x)$
00	-1	00	stay	stay	$(0,0,0)$
		10	flip	stay	$(1,0,0)$
		01	stay	flip	$(0,0,1)$
		11	flip	flip	$(1,0,1)$
10	-1	00	stay D	$-$flip D	$(0,1,-1)$
		10	flip D	$-$flip D	$(1,1,-1)$
		01	stay D	$-$stay D	$(0,1,0)$
		11	flip D	$-$stay D	$(1,1,0)$
01	$+1$	00	$-$flip D	stay D	$(-1,-1,0)$
		10	$-$stay D	stay D	$(0,-1,0)$
		01	$-$flip D	flip D	$(-1,-1,1)$
		11	$-$stay D	flip D	$(0,-1,1)$
11	-1	00	$-$flip	$-$flip	$(-1,0,-1)$
		10	$-$stay	$-$flip	$(0,0,-1)$
		01	$-$flip	$-$stay	$(-1,0,0)$
		11	$-$stay	$-$stay	$(0,0,0)$

Table A.2. The sixteen step blocks $M_{yx} = \sigma(x) F_1 \otimes F_2$ of the native step and their labels. Clock cell 1 reads x_2 and writes y_1; clock cell 2 reads x_1 and writes y_2. Mixed inputs are dressed by D on both cells.

A.2 Window tables for Chapter 3

The tables below carry the finite computations behind the later windows of the field-current cut (Section 3.5) and the long-run verdict (Section 3.6). Every entry is exact.

The eight modes. Each mode is $M_m = \sum_x |x\rangle\langle x| \otimes B_m(x)$, with the blocks of Table A.3; modes 1 to 6 are pole tables and modes 7, 8 carry coherence. One free posting acts on the weights by $\theta \mapsto S\theta$:

$$S = \begin{pmatrix} 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

That is, $\Phi\big(\sum_m \theta_m M_m\big) = \sum_m (S\theta)_m M_m$, a direct exact computation from the one-cell pullbacks of Table 9.1, applied to the two cells independently.

m	$B_m(x)$, x balanced	$B_m(x)$, x mixed	reading
1	$c_x^2 \otimes c_x^2$	0	both poles nonzero, balanced
2	$c_x \otimes c_x$	0	joint pole sign, balanced
3	0	$c_x^2 \otimes c_x^2$	both poles nonzero, mixed
4	0	$c_x \otimes c_x$	joint pole sign, mixed
5	$\Pi_0^x \otimes \Pi_0^x$	$\Pi_0^x \otimes \Pi_0^x$	both poles zero
6	$c_x^2 \otimes \Pi_0^x + \Pi_0^x \otimes c_x^2$	the same	exactly one pole zero
7	$Z_x \otimes Z_x$	$-Z_x \otimes Z_x$	coherence in the joint zero plane
8	$X_x \otimes X_x$	$-X_x \otimes X_x$	coherence in the joint zero plane

Table A.3. The eight modes of the later field-current windows. $W_k = \sum_m \theta_m(k) M_m$ for $k \geq 1$, with $\theta(1) = (0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{256}, \frac{1}{256}, 0, 0)$. In these terms $I = M_1 + M_3 + M_5 + M_6$, $Q_{\mathrm{al}} = \frac{1}{2}(M_3 + M_4)$ and $Q_{\mathrm{oz}} = M_6$.

The integer sequence. Table A.4 lists $v(n) = (z_n, g_n, d_n, u_n, q_n, e_n)$ from Proposition 3.13. The window $k = n+1$ has coherence $\theta_7 = 2^{-(n+8)} q_n$; the chance state passes it with probability $\frac{9}{25} + \frac{16}{25} 2^{-(n+8)} u_n$ and the ready vector with probability $2^{-(n+7)} d_n$.

Sector values. Every pole table is a combination of the 36 sector projections $\Pi(x, s, t)$. Table A.5 lists the values of the first three field-current windows and of the long-run window on the four kinds of joint pole sector. For a mode combination the values are $\theta_1 \pm \theta_2$ (balanced) or $\theta_3 \pm \theta_4$ (mixed) on nonzero poles of equal or opposite sign, θ_6 where exactly one pole is zero, and θ_5 where both are.

n	z_n	g_n	d_n	u_n	q_n	e_n
0	1	0	0	0	0	0
1	1	1	0	2	0	0
2	2	1	2	2	1	0
3	5	2	2	4	−1	1
4	9	6	4	8	1	−1
5	19	8	8	20	1	1
6	35	20	20	36	1	1
7	75	36	36	68	1	1
8	147	76	68	148	1	1
9	291	148	148	292	9	1

Table A.4. The integer coordinates of the later windows. For $n \geq 3$ the parities of $(z_n, g_n, d_n, u_n, q_n, e_n)$ are (odd, even, even, even, odd, odd); in particular q_n is odd for every $n \geq 2$.

occupation	joint pole sector	W_0	W_1	W_2	W_∞
balanced	equal signs	0	0	$\frac{1}{256}$	$\frac{1}{448}$
	opposite signs	0	0	0	$\frac{1}{448}$
	one pole zero	0	$\frac{1}{256}$	$\frac{1}{256}$	$\frac{1}{256}$
	both poles zero	0	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{448}$
mixed	equal signs (aligned)	1	1	1	1
	opposite signs (opposed)	0	0	$\frac{1}{256}$	$\frac{1}{448}$
	one pole zero	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{256}$	$\frac{1}{256}$
	both poles zero	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{448}$

Table A.5. Values of the central field-current windows on the joint pole sectors. $W_0 = Q_{\mathrm{al}} + \frac{1}{128} Q_{\mathrm{ne}}$ and $W_\infty = \frac{1}{448} I + \frac{447}{448} Q_{\mathrm{al}} + \frac{3}{1792} Q_{\mathrm{oz}}$. The windows W_k with $k \geq 3$ are not pole tables.

A.3 Tables for the selection theorem of Chapter 7

These tables let a reader recompute every finite step of Chapter 7 by hand. Work in the product pole basis $e \otimes f$, with e (first clock cell, rows) and f (second clock cell, columns) running over p, a, P, A.

	p	a	P	A
$\mathrm{stay}^{\top}$	$\frac{1}{2}(P - A)$	$\frac{1}{2}(P + A)$	0	a
$\mathrm{flip}^{\top}$	$\frac{1}{2}(P + A)$	$\frac{1}{2}(A - P)$	p	0

Table A.6. Transposed clock maps on the pole basis. With D from Table 2.2 they give every $M_{yx}^{\top}(e \otimes f)$, hence every pull of a rank-one product projection. Signs of blocks drop out of every pull.

Every block of $\Phi^k(Q_{\mathrm{ne}})$, $0 \le k \le 6$, has the form

$$\mathrm{diag}(d_k) + \gamma_k S_{pa} \otimes S_{pa} + \delta_k S_{PA} \otimes S_{PA},$$

where d_k is a diagonal weight table, $S_{pa} = |p\rangle\langle a| + |a\rangle\langle p|$ and $S_{PA} = |P\rangle\langle A| + |A\rangle\langle P|$. Thus $S_{pa} \otimes S_{pa}$ couples $p \otimes p$ with $a \otimes a$ and $p \otimes a$ with $a \otimes p$, and $S_{PA} \otimes S_{PA}$ does the same for P, A. On mixed blocks $\delta_k = 0$; on balanced blocks $\gamma_k = 0$. The two mixed occupations share one block, as do the two balanced occupations. Table A.7 lists $64\, d_k$ and the coherence for $k \le 6$ (direct exact computation from Table A.6). The rows $k \le 3$ carry the certificate of Lemma 7.14; the rows $k = 1, 2, 3, 5, 6$ carry the identity (7.3).

Checking the quintic identity. Equation (7.3) holds entry by entry in Table A.7. On the diagonal, 7Λ is $\frac{7}{2}$ at the one-zero products ($p \otimes P$, $P \otimes p$ and their relatives, on either kind of block), 0 at the aligned products $P \otimes P$, $A \otimes A$ of mixed blocks, and 2 at every other product. For example, at $P \otimes A$ on a mixed block the left side is $8 \cdot \frac{20}{64} + 4 \cdot \frac{16}{64} - 2 \cdot \frac{16}{64} - 2 \cdot \frac{32}{64} - 0 = 2$, and at $p \otimes P$ on a balanced block it is $8 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} - 2 \cdot \frac{1}{2} - 2 \cdot \frac{1}{2} - \frac{1}{2} = \frac{7}{2}$. The coherences cancel: $8 \cdot (-\frac{1}{64}) + 4 \cdot (-\frac{1}{32}) - 2 \cdot (-\frac{1}{8}) = 0$ on mixed blocks and $8 \cdot \frac{1}{64} + 4 \cdot \frac{1}{32} - 2 \cdot \frac{1}{8} = 0$ on balanced blocks.

k	mixed: $64\,d_k,\ \gamma_k$	balanced: $64\,d_k,\ \delta_k$
0	$\begin{pmatrix} 64 & 64 & 64 & 64 \\ 64 & 64 & 64 & 64 \\ 64 & 64 & 0 & 0 \\ 64 & 64 & 0 & 0 \end{pmatrix}, 0$	$0, 0$
1	$\begin{pmatrix} 32 & 32 & 32 & 32 \\ 32 & 32 & 32 & 32 \\ 32 & 32 & 0 & 0 \\ 32 & 32 & 0 & 0 \end{pmatrix}, 0$	$\begin{pmatrix} 0 & 0 & 32 & 32 \\ 0 & 0 & 32 & 32 \\ 32 & 32 & 32 & 32 \\ 32 & 32 & 32 & 32 \end{pmatrix}, 0$
2	$\begin{pmatrix} 16 & 16 & 32 & 32 \\ 16 & 16 & 32 & 32 \\ 32 & 32 & 0 & 32 \\ 32 & 32 & 32 & 0 \end{pmatrix}, 0$	$\begin{pmatrix} 32 & 0 & 32 & 32 \\ 0 & 32 & 32 & 32 \\ 32 & 32 & 16 & 16 \\ 32 & 32 & 16 & 16 \end{pmatrix}, 0$
3	$\begin{pmatrix} 16 & 16 & 32 & 32 \\ 16 & 16 & 32 & 32 \\ 32 & 32 & 0 & 16 \\ 32 & 32 & 16 & 0 \end{pmatrix}, -\frac{1}{8}$	$\begin{pmatrix} 16 & 32 & 32 & 32 \\ 32 & 16 & 32 & 32 \\ 32 & 32 & 16 & 16 \\ 32 & 32 & 16 & 16 \end{pmatrix}, +\frac{1}{8}$
4	$\begin{pmatrix} 16 & 24 & 32 & 32 \\ 24 & 16 & 32 & 32 \\ 32 & 32 & 0 & 16 \\ 32 & 32 & 16 & 0 \end{pmatrix}, +\frac{1}{16}$	$\begin{pmatrix} 16 & 16 & 32 & 32 \\ 16 & 16 & 32 & 32 \\ 32 & 32 & 24 & 16 \\ 32 & 32 & 16 & 24 \end{pmatrix}, -\frac{1}{16}$
5	$\begin{pmatrix} 20 & 16 & 32 & 32 \\ 16 & 20 & 32 & 32 \\ 32 & 32 & 0 & 16 \\ 32 & 32 & 16 & 0 \end{pmatrix}, -\frac{1}{32}$	$\begin{pmatrix} 24 & 16 & 32 & 32 \\ 16 & 24 & 32 & 32 \\ 32 & 32 & 16 & 20 \\ 32 & 32 & 20 & 16 \end{pmatrix}, +\frac{1}{32}$
6	$\begin{pmatrix} 18 & 20 & 32 & 32 \\ 20 & 18 & 32 & 32 \\ 32 & 32 & 0 & 20 \\ 32 & 32 & 20 & 0 \end{pmatrix}, -\frac{1}{64}$	$\begin{pmatrix} 16 & 16 & 32 & 32 \\ 16 & 16 & 32 & 32 \\ 32 & 32 & 20 & 18 \\ 32 & 32 & 18 & 20 \end{pmatrix}, +\frac{1}{64}$

Table A.7. The pulled neutral sector $\Phi^k(Q_{\mathrm{ne}})$ for $k = 0, \dots, 6$. Rows and columns of each 4×4 table run over p, a, P, A. The diagonal entries at $P \otimes P$ and $A \otimes A$ on mixed blocks vanish for every k: these are the aligned directions. The balanced entry at $p \otimes a$ is $0, 0, 0, 32$ for $k \le 3$: this is the ready vector of Theorem 7.11.

mixed	balanced
$\begin{pmatrix} \frac{3}{2} & \frac{3}{2} & 2 & 2 \\ \frac{3}{2} & \frac{3}{2} & 2 & 2 \\ 2 & 2 & 0 & \frac{1}{4} \\ 2 & 2 & \frac{1}{4} & 0 \end{pmatrix}$, coherence $-\frac{1}{8}$	$\begin{pmatrix} \frac{1}{4} & 0 & 1 & 1 \\ 0 & \frac{1}{4} & 1 & 1 \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, coherence $+\frac{1}{8}$

Table A.8. The certificate $\sum_{k \le 3} \Phi^k(Q_{\mathrm{ne}}) - \frac{1}{2}(I - Q_{\mathrm{al}})$ of Lemma 7.14: its diagonal in the product pole basis, and its only off-diagonal entries. All diagonal entries are nonnegative, and each coherence joins two entries equal to $\frac{3}{2}$ (mixed) or $\frac{1}{2}$ (balanced).

A.4 The pointer chain of Chapter 10

A.4.1 Stationarity of the one-zero and rest classes by hand

The lines of classes 3 and 4 fall into a few types, and Table 10.1 moves whole types. Two facts do the counting. On a balanced input a cell map exchanges the planes $\mathcal{U}$ and $D\mathcal{U}$; on a mixed input it keeps them. And given the output occupation y and the two moves, the input is fixed ($x_2 = y_1$ if cell 1 stays and $x_2 = 1 - y_1$ if it flips, and likewise x_1 from y_2), so whether a candidate source is balanced or mixed is a parity condition on the two moves.

Class 4. Write UP, UD, DP for lines with both cells on $\{p, a\}$, on $\{\omega_+, \omega_-\}$, on $\{P, A\}$, with a subscript B or M for a balanced or mixed occupation; π_4 is $\frac{1}{84}, \frac{1}{42}, \frac{1}{28}$ on UP, UD, DP. The moves are: $\mathrm{UP_B} \to \mathrm{DP}$ with weight 1 (p flips to P, a stays to A; the output is mixed exactly when the two cells differ, and then they land on opposite poles); $\mathrm{UP_M} \to \mathrm{UD}$, $\mathrm{UD_B} \to \mathrm{DP}$, $\mathrm{UD_M} \to \mathrm{UP}$ and $\mathrm{DP_B} \to \mathrm{UD}$, each to four lines with weight $\frac{1}{4}$; and $\mathrm{DP_M} \to \mathrm{DP_B}$ with weight 1. Inflows: a $\mathrm{DP_B}$ line with equal cells receives $\frac{1}{84}$ from one $\mathrm{UP_B}$ line and $4 \cdot \frac{1}{42} \cdot \frac{1}{4}$ from four $\mathrm{UD_B}$ lines, in total $\frac{1}{28}$; one with unequal cells receives $\frac{1}{28}$ from one $\mathrm{DP_M}$ line; a $\mathrm{DP_M}$ line receives $\frac{1}{84} + \frac{1}{42} = \frac{1}{28}$ in the same way as an equal-cell $\mathrm{DP_B}$ line; a UD line receives $2 \cdot \frac{1}{84} \cdot \frac{1}{4} + 2 \cdot \frac{1}{28} \cdot \frac{1}{4} = \frac{1}{42}$ from two $\mathrm{UP_M}$ and two $\mathrm{DP_B}$ lines; a UP line receives $2 \cdot \frac{1}{42} \cdot \frac{1}{4} = \frac{1}{84}$ from two $\mathrm{UD_M}$ lines.

Class 3. One cell lies in $\mathcal{U}$ (on a pole p, a or a wave $\omega_\pm$) and the other on a pole of $D\mathcal{U}$; π_3 is $\frac{1}{96}$ or $\frac{1}{48}$ as the $\mathcal{U}$-cell is on a pole or a wave. On a balanced occupation the roles exchange: the $\mathcal{U}$-cell goes to a pole of $D\mathcal{U}$ (weight 1 from a pole, $\frac{1}{2}$ from a wave) and the other cell to a wave of $\mathcal{U}$ (weight $\frac{1}{2}$). On a mixed occupation the $D\mathcal{U}$-cell keeps its pole with weight 1 and the $\mathcal{U}$-cell moves between poles and waves with weight $\frac{1}{2}$. A line whose $\mathcal{U}$-cell is on a pole is entered only from mixed wave lines, one of the two candidates having the right parity: inflow $\frac{1}{48} \cdot \frac{1}{2} = \frac{1}{96}$. A line whose $\mathcal{U}$-cell is on a wave is entered from one balanced pole line ($\frac{1}{96} \cdot \frac{1}{2}$), two balanced wave lines ($2 \cdot \frac{1}{48} \cdot \frac{1}{4}$) and one mixed pole line ($\frac{1}{96} \cdot \frac{1}{2}$): in total $\frac{1}{48}$.

The same type moves show that classes 3 and 4 are closed. Their irreducibility (closed walks of positive weight through every line, of lengths 112 and 97) and the transient bound of Theorem 10.14(b) are direct exact computations over the 256 lines with the weights of Table 10.1.

A.4.2 The two-step pair certificate

Theorem 10.11 rests on one finite fact: for every two distinct pointer lines $\ell \neq \ell'$,

$$\sum_{u \in \mathcal{J}^2} \sqrt{w_u(\ell)\, w_u(\ell')}\, [\ell_u \neq \ell'_u] \leq \frac{41}{48}.$$

Lines at different occupations share only the zero label, which is written by both balanced occupations, and two consecutive zero-label postings weigh at most $\frac{1}{4}$; so for them the sum is at most $\frac{1}{4}$.

For lines at one occupation the two postings act on the two clock cells separately, through a stay or flip move of each cell at each posting. The intermediate occupation, and so the dressing at the second posting, is fixed by the input dressing and by whether the two cells

made the same first move. The pair ends apart unless both cells' image vectors coincide. So the two-step sum is a sum over the two first moves of products of per-cell quantities, minus the products over the cases where both cells' images coincide. For a pair (a, b) of cell vectors we record its *profile*: whether $a = b$, and, for each first move, an upper bound for $\sqrt{w_a w_b}$ and, for each dressing of the second posting, the sum over the second move of the same bound for the image pair, and the part of that sum on which the two images coincide. Every pair of cell vectors, at either input dressing, has one of 18 profiles. The bound $\frac{41}{48}$ is checked exactly over all $648 = 18 \cdot 18 \cdot 2$ choices of two profiles and an input dressing, other than two coincident pairs, with the square root of $\frac{1}{2}$ bounded by $\frac{17}{24} \geq 1/\sqrt{2}$ and all other weights exact. This is a direct exact computation in rational arithmetic.

A.5 Tables for Chapter 11

The free pull on the record content. In Theorem 11.6, M is the 92×92 matrix of Φ in the basis of the record reads R_q, each entry a product of two one-cell pull coefficients from Table 9.1. With Per the projection onto the span of $\Lambda_0, \dots, \Lambda_4, \Psi_\circ$ and e_w the coefficient vector of E_w,

$$C_v = \sum_q \left|\operatorname{tr}(\rho_v R_q)\right|, \qquad C_w = \sum_{s=0}^{7} \left\| M^s (e_w - \operatorname{Per} e_w) \right\|_\infty .$$

The record reads of each class form one diagonal block of M, and Table A.9 lists an annihilating polynomial of $Y = 16M$ on each block. Substituting $Y = 16x$ in the quintic factor and dividing by 2^{17} gives $8x^5 + 4x^4 - 2x^2 - 2x - 1$.

class	annihilating polynomial of $Y = 16M$ on the block	reads
0, 1	$Y - 16$	1 each
2	$Y^2 - 256$	2
3	$(Y - 16)\,Y^3\,(Y^2 - 64)(Y^2 + 64)(Y^2 - 8Y + 64)(Y^2 + 128)$	48
4	$(Y - 16)\,Y^3\,(Y^2 + 64)(Y^4 + 64Y^2 + 16384)$ $\times (Y^5 + 8Y^4 - 1024Y^2 - 16384Y - 131072)$	40

Table A.9. Annihilating polynomials of the free pull on the record reads of each class. Apart from $Y = \pm 16$ and the quintic factor, every root has $|Y| \leq 8\sqrt{2}$.

Archive codes. Table A.10 gives the code $(j_1 + 1, j_2 + 1, j_3 + 1)$ of each label and the ten merge classes of records of length one (Theorem 11.13). Two labels share a class when one common shift modulo 4 carries one code to the other.

labels	codes	shift
$(1,1,-1)$, $(-1,-1,1)$	$(2,2,0)$, $(0,0,2)$	2
$(0,1,0)$, $(-1,0,-1)$	$(1,2,1)$, $(0,1,0)$	3
$(0,-1,0)$, $(1,0,1)$	$(1,0,1)$, $(2,1,2)$	1
$(0,0,1)$, $(-1,-1,0)$	$(1,1,2)$, $(0,0,1)$	3
$(1,1,0)$, $(0,0,-1)$	$(2,2,1)$, $(1,1,0)$	3
$(0,0,0)$; $(1,0,0)$; $(-1,0,0)$; $(0,1,-1)$; $(0,-1,1)$	each alone	

Table A.10. Merge classes of the fifteen labels in the clock archive.

A.6 The triangle in its six fixed orders

Table A.11 lists, for each of the six fixed orders of the triangle's contacts $(0,1)$, $(0,2)$, $(1,2)$, the awake configurations of Theorem 12.16. Every order admits exactly the four odd flip patterns, each at two occupations, x and its complement; the table gives the one with $x_0 = 0$. By Theorem 12.28(a) the class each order keeps is spanned by the corresponding eight product states with lines $\ell_z = P$ for $f_z = 0$ and $\ell_z = A$ for $f_z = 1$. The entries follow from condition (ii) of Theorem 12.16, every degree being 2; we computed them directly for all 64 dressed configurations and every order.

	posting order					
flips $f_0 f_1 f_2$	01, 02, 12	01, 12, 02	02, 01, 12	02, 12, 01	12, 01, 02	12, 02, 01
100	010	010	001	001	010	001
010	011	011	011	001	001	001
001	011	010	011	011	010	010
111	010	011	001	011	001	010

Table A.11. Awake occupations $x_0x_1x_2$ (with $x_0 = 0$; the complement is awake too) of the triangle in each fixed order, for each odd flip pattern. Two orders share a kept state exactly where their entries in a row agree. Orders that differ by exchanging two consecutive contacts agree in two rows (a 4-dimensional common class); all other pairs agree in one row (2 dimensions). No row has the same entry in four columns, and the three cyclic phases 01, 02, 12, then 02, 12, 01, then 12, 01, 02 agree in no row.

A.7 Witness networks for timing and births

The networks below are the explicit witnesses of Chapter 14. Sites are numbered from 0; a contact (u, v) puts u in the first position of the native pair; a state lists bits and lines site by site. Where the property concerns some round order or all round orders, the listed order of the contacts does not matter. Each property is a direct exact computation from the definitions, or follows from the theorem cited.

W1. *Three-leaf star* (Proposition 14.8). Contacts $(0,1)$, $(0,2)$, $(0,3)$. Beginning: bits

$(0,0,0,0)$, lines (a, p, P, P). Fairly kept by the sequence $(0,1)$, $(0,1)$, $(0,2)$, $(0,1)$, $(0,3)$, $(0,1)$, $(0,2)$, ...; not born, since its dressed centre is A of degree 3 beside P leaves. No beginning of this star is born (Table 14.1).

W2. *Four-leaf star* (Proposition 14.32(a)). Contacts $(0,1)$, $(0,2)$, $(0,3)$, $(0,4)$. Beginning: bits $(0,0,0,0,0)$, lines (a, p, P, P, P); dressing: bits $(0,1,0,0,0)$, lines (A, P, P, P, P). A degree-balanced fair birth that fails the wall law at the centre (walls $1,0,0,0$, unbalanced), born in no order. No beginning of this star is born in the listed order.

W3. *Six-cycle* (Proposition 14.32(b)). Contacts $(0,1)$, $(1,2)$, $(2,3)$, $(3,4)$, $(4,5)$, $(5,0)$. Lines (a, a, A, A, A, A). Bits $(0,0,0,1,1,0)$: a degree-balanced fair birth with two tokens, not born. Bits $(0,0,0,1,0,0)$: born. All-A first-round solutions: $(1,0,0,0,0,0)$ and $(0,1,1,1,1,1)$.

W4. *Seven sites, half charge* (Theorem 14.39). Contacts: the six pairs of $\{0,1,2,3\}$, and $(4,5), (5,6)$. Beginning: bits $(1,0,0,0,0,0,0)$, lines (A, A, A, A, p, a, P). A degree-balanced fair birth; the all-A cluster $\{0,1,2,3\}$ has odd degrees, no walls and one token, so it fails half charge, and the beginning is born in no order.

W5. *Nine sites, no token flow* (Theorems 14.38(c) and 14.39). Contacts $(3,0)$, $(4,0)$, $(5,0)$, $(0,1)$, $(1,2)$, $(2,6)$, $(2,7)$, $(2,8)$. Beginning: bits $(1,1,1,1,1,1,0,0,0)$, lines $(a, a, A, P, P, P, P, P, P)$. The dressed cluster is the path $0,1,2$ of A clocks with three source walls at 0 and three sink walls at 2; walls balance and every clock has even degree, but clock 0 has $\mathrm{src}(0) = 3 > \deg 0/2 = 2$. A degree-balanced fair birth with half charge and balanced walls, born in no order.

W6. *Eight sites, single-clock bounds insufficient* (Theorem 14.38(c)). Contacts $(0,1)$, $(1,3)$, $(3,2)$, $(2,0)$, $(4,0)$, $(4,1)$, $(5,0)$, $(5,1)$, $(2,6)$, $(3,6)$, $(2,7)$, $(3,7)$. Dressed state: bits $(1,1,1,1,1,1,0,0)$, lines (A, A, A, A, P, P, P, P); the beginning has lines (a, a, A, A, P, P, P, P) with the same bits. One contact per pair of sites, maximum degree 4, even degrees, balanced walls ($4 = 4$) and both single-clock bounds at every clock; the set $\{0,1\}$ has src $= 4$, snk $= 0$, cut $= 2$ and fails. Born in no order.

W7. *Ring family*, $k \geq 0$ (Theorem 14.41(c)). Sites $0, \ldots, 4k+7$. Contacts $(j, j+1)$ for $j = 0, \ldots, 4k+3$ (indices modulo $4k+4$), the source walls $4k+4$ and $4k+5$ each joined to clocks 0 and $k+1$, the sink walls $4k+6$ and $4k+7$ each joined to clocks $2k+2$ and $3k+3$. Dressed state: every clock A of bit 1, walls P with bits $1,1,0,0$; the beginning has clocks 0 and 1 on a. Surpluses $+2, +2, -2, -2$ at clocks $0, k+1, 2k+2, 3k+3$, so $P_{k+1} = 4$ and there is no flow, while every set of at most k clocks passes the cut count. For $k = 0$ this is W6.

W8. *Path family*, $k \geq 0$ (Theorem 14.41(c)). Sites $0, \ldots, 2k+9$. Contacts $(j, j+1)$ for $j = 0, \ldots, 2k+2$; walls $2k+4, \ldots, 2k+9$, of bits $1,1,1,0,0,0$, joined to clocks 0, $k+1$, $k+1$, $k+2$, $k+2$, $2k+3$. Dressed state: every clock A of bit 1; the beginning has clocks 0 and 1 on a. Surpluses $+1, +2, -2, -1$; the side $\{0, \ldots, k+1\}$ of the middle contact has surplus 3. Every set of at most $k+1$ clocks passes the cut count; the side, of $k+2$ clocks, fails.

W9. *Triangle of four-contact clocks* (Theorems 14.43(d) and 14.45). Contacts $(0,1)$, $(1,2)$, $(2,0)$, $(3,0)$, $(3,1)$, $(3,2)$, $(0,4)$, $(1,4)$, $(2,4)$. State: clocks $0,1,2$ on A with bit 0, walls 3 (P, bit 1) and 4 (P, bit 0). Two flows (the circulations), each realized by 13536 of the 9! orders; 27072 admitting orders.

W10. *Triangle of six-contact clocks* (Theorem 14.45). Contacts $(0,1)$, $(1,2)$, $(2,0)$, $(3,c)$ and $(4,c)$ for $c = 0,1,2$, $(c,5)$ and $(c,6)$ for $c = 0,1,2$. State: clocks $0,1,2$ on A with bit

0, walls $3, 4$ of bit 1 and $5, 6$ of bit 0. Two flows, loop factor $1 + \frac{9344}{1702701}$, each flow alternated with chance $\frac{342409}{340540200}$; 2629701120 admitting orders of the 15!.

A.8 Tables for Chapter 16

Four writes. Table A.12 lists the writes used in Proposition 16.26 and Theorem 16.28. Each entry is the value $f(y, x)$ written by the transition from the input x (column) to the output y (row). All four are instruments by Theorem 16.15 and separate up to the awake bit; a reader can check both properties from the table alone.

d	00	01	10	11	π	00	01	10	11
00	1	0	2	0	00	3	6	8	6
01	3	0	4	0	01	1	5	4	5
10	3	2	0	1	10	5	8	5	1
11	1	4	2	3	11	7	0	7	2
d_3	00	01	10	11	d_5	00	01	10	11
00	2	3	2	0	00	5	2	5	3
01	1	0	1	3	01	4	2	4	1
10	1	3	1	0	10	3	0	4	1
11	3	0	3	1	11	1	4	2	3

Table A.12. The writes d (a dead path pair), π (law-private with path pairs), d_3 (first read after three postings) and d_5 (first read after five). Rows: output y; columns: input x. The path pair of d is $(1,0) \to (1,0) \to (0,1) \to (0,0)$ and $(1,1) \to (0,1) \to (1,1) \to (0,0)$, written $(0, 4, 0)$; that of π is $(0,0) \to (1,0) \to (0,0) \to (1,0)$ and $(0,1) \to (0,1) \to (1,0) \to (1,0)$, written $(5, 8, 5)$; d_5 has the path pair $(0,0) \to (0,1) \to (0,0) \to (1,0) \to (0,1)$ and $(1,0) \to (1,0) \to (1,1) \to (0,0) \to (0,1)$, written $(4, 2, 3, 4)$.

Label identities on the triangle. Table A.13 gives the three identities behind Theorems 16.51 and 16.52. In each, one contact's label is an integer-linear function of the labels the other two contacts write around it (o, n, t for the old, new and third contact, primed in the later round), valid for all values of the occupation bits involved. Labels are written $j = (j_1, j_2, j_3) = (y_1 - x_2,\ x_1 - x_2,\ y_2 - x_1)$ for the contact's own pair, first site first: the old contact on $(0, 1)$, the new contact on $(0, 2)$ and the third contact on $(1, 2)$.

For the first identity, let the new contact take (y_0, x_2) to (z_0, z_2) and the third contact (y_1, z_2) to (w_1, w_2) in round $r - 1$, and in round r let the old contact take (z_0, w_1) to (u_0, u_1), the new contact (u_0, w_2) to (v_0, v_2) and the third contact (u_1, v_2) to (s_1, s_2). The old label is $(u_0 - w_1,\ z_0 - w_1,\ u_1 - z_0)$, and for instance

$$n_2' + t_3 + t_2 - t_1 = (u_0 - w_2) + (w_2 - y_1) + (y_1 - z_2) - (w_1 - z_2) = u_0 - w_1.$$

The other components, and the other two identities, telescope in the same way.

postings	identity
round $r-1$: new writes n, third writes t; round r: new writes n', third writes t'	old label of round $r = \big(n'_2 + t_3 + t_2 - t_1,\ n_1 - n_2 - n_3 - t_1,\ t'_2 + n'_3 + n'_2 + t_3 + t_2 + n_3 + n_2 - n_1\big)$
round r: old writes o, new writes n; round $r+1$: old writes o', new writes n'	third label of round $r = \big(n_1 - n_2 - n_3 - o'_2,\ o_3 - o_1 + o_2 - n_3,\ o'_1 - o'_2 - n'_2 + n_1 - n_2 - o_3 + o_1 - o_2\big)$
round $r-1$: third writes t; round r: old writes o, third writes t'; round $r+1$: old writes o'	new label of round $r = \big(o'_2 + t'_1 - t'_2 + o_3 + o_2 + t_1 - t_2 - t_3,\ o_1 + t_1 - t_2 - t_3,\ o_3 + o_2 - o_1 - t'_2\big)$

Table A.13. Current-conservation identities on the triangle. Each follows from Lemma 16.47 by substituting the definition of the labels; every component telescopes along the shared bits.

A.9 Tables for Chapter 17

Dying type sequences. Table A.14 lists the sequences of Theorem 17.10 found by running the line automaton of Table 17.1 on all 4^4 sequences of length four and all 4^5 of length five. No sequence of length three dies, so every dying sequence of length four is minimal. Types are listed in posting order.

length four (all dying sequences)			
bs bs mf bf	bs bf ms bf	bs mf bs mf	bs mf bf ms
bf bs mf bs	bf bf ms bs	bf ms bs mf	bf ms bf ms
ms bs mf bf	ms bf ms bf	ms mf bs mf	ms mf bf ms
mf bs mf bs	mf bf ms bs	mf ms bs mf	mf ms bf ms
length five (minimal dying sequences)			
bs mf bs ms mf	bs mf bf mf ms	bs mf mf bs mf	bs mf mf bf ms
bf ms bs ms mf	bf ms bf mf ms	bf ms ms bs mf	bf ms ms bf ms
ms mf bs ms mf	ms mf bf mf ms	ms mf mf bs mf	ms mf mf bf ms
mf ms bs ms mf	mf ms bf mf ms	mf ms ms bs mf	mf ms ms bf ms

Table A.14. The sixteen dying type sequences of length four and the sixteen minimal dying sequences of length five. A sequence is minimal if neither of its stretches without the first or the last factor dies.

Two factors among balanced stays. For $(\mathrm{bs})^a\, x\, (\mathrm{bs})^b\, y\, (\mathrm{bs})^c$ with arbitrary types x, y, death is decided by Table A.15. The table was obtained by reducing a to $\min(a, 2)$, c to $\min(c, 1)$ and b to itself if $b < 4$ and to $2 + (b \bmod 2)$ otherwise, and evaluating the automaton on the representatives.

x	y	dies exactly when	description
bf	mf	$b = 1, c \geq 1$	one stay apart, a stay after
ms	ms	$a \geq 2$, b odd, $c \geq 1$	odd gap after two stays, a stay after
ms	mf	$a \geq 2, b \geq 2$ even	even gap of at least two, after two stays
mf	bf	$a \geq 2, b = 0$	adjacent, after two stays
mf	ms	$a \geq 1, b \geq 2$ even, $c \geq 1$	even gap of at least two, a stay before and after
mf	mf	$a \geq 1$ and b odd, or $a = 0, b = 1$, $c \geq 1$	odd gap after a stay, or the second case

Table A.15. The six dying families of two factors off balanced stay. All other pairs (x, y) and parameters survive.

Three mixed factors among balanced stays. For $(\mathrm{bs})^a \mu_1 (\mathrm{bs})^b \mu_2 (\mathrm{bs})^c \mu_3 (\mathrm{bs})^d$ with μ_i mixed and $f_i = [\mu_i \text{ flips}]$, the sequence dies exactly when the truth-value expression of Table A.16 is true. The rule reads the window rule group by group: adjacent mixed factors act as one group on the odd run and rotate the even run.

case	dies exactly when
$b \geq 1, c \geq 1$	$(o_1 \vee f_2 r_2 \vee v_2 \bar{r}_2 \vee f_3 r_3 \vee v_3 \bar{r}_3) \wedge (v_1 \vee f_2 \bar{r}_2 \vee v_2 r_2 \vee f_3 \bar{r}_3 \vee v_3 r_3)$, where $o_1 = [a \geq 1] f_1$, $v_1 = [a \geq 2] \bar{f}_1$, $v_2 = \bar{f}_2$ if $b \geq 2$ and $[f_1 = f_2]$ if $b = 1$, $v_3 = [d \geq 1]$ times ($\bar{f}_3$ if $c \geq 2$, $[f_2 = f_3]$ if $c = 1$), $r_2 = [b \text{ even}]$, $r_3 = r_2$ if c is even and $\bar{r}_2$ otherwise
$b = 0, c \geq 1$	$(o_1 \vee f_3 r \vee v\bar{r}) \wedge (f_3 \bar{r} \vee vr)$, where $o_1 = [a \geq 1](f_1 \vee f_2) \vee [a = 0][f_1 \neq f_2]$, $v = [d \geq 1]$ times ($\bar{f}_3$ if $c \geq 2$, $[f_2 = f_3]$ if $c = 1$), $r = [c \text{ even}]$
$b \geq 1, c = 0$	$(o_1 \vee (f_2 \vee f_3) r) \wedge (v_1 \vee (f_2 \vee f_3) \bar{r})$, where $o_1 = [a \geq 1] f_1$, $v_1 = [a \geq 2] \bar{f}_1$, $r = [b \text{ even}]$
$b = c = 0$	$\big([a \geq 1](f_1 \vee f_2 \vee f_3) \vee [a = 0]([f_1 \neq f_2] \vee [f_1 \neq f_3])\big) \wedge [a \geq 2][d \geq 1](f_1 \oplus f_2 \oplus f_3)$

Table A.16. The three-mixed rule. A bar denotes negation and juxtaposition conjunction.

Fragility of the three-crossing pool. Table A.17 gives the witnesses of Proposition 17.18(2),(3) when one separating read posting of E_3 is doubled or left unread. A crossed pair consists of two histories that agree except that one read component is empty in the first and filled in the second; times in a doubled network count its twelve postings. Each history is a history of its network, the two histories of a pair write equal read labels and end differently, each colouring writes the zero label at every read posting and a nonzero label at a crossing, and every history is alive at every site by the automaton of Table 17.1.

The split rule. Theorem 17.29 decides survival at a site with two unread postings $u_1 < u_2$ by a finite condition. On networks with bracketed events the structural rule of Table A.18 implies it. Here a and b count the site's postings before u_1 and between u_1 and u_2; $\beta_1, \beta_2, \beta_3$

change to E_3	witness	bits equal to 1, all other bits 0
posting 4 doubled	crossed pair	sites 0, 1 throughout in the second history; in both, site 2 at times 7 to 9 and site 3 at times 6 to 8
posting 5 doubled	crossed pair	sites 2, 3 throughout in the second history; in both, site 0 at times 7 to 9 and site 1 at times 4 to 7
posting 7 doubled	crossed pair	sites 0, 1 throughout in the second history; in both, site 2 at times 5, 10, 11
posting 8 doubled	crossed pair	sites 2, 3 throughout in the second history; in both, site 0 at times 4, 5, 10
posting 4 unread	colouring	site 0 at times 0 to 2, site 1 at times 0 to 3
posting 5 unread	colouring	site 2 at times 0 to 2, site 3 at times 0 to 4
posting 7 unread	colouring	site 0 at time 6
posting 8 unread	colouring	sites 0, 1 throughout, site 2 at times 0 to 5, site 3 at times 0 to 7

Table A.17. Witnesses that one doubled or unread separating posting makes the pool of E_3 inexact.

record whether the site's bits entering u_1, entering u_2 and at the end lie in S; and σ_i records whether u_i, when it is a crossing, is assigned to h^{κ}_{S}, that is, mixed there.

crossings	the rule
both u_1 and u_2	b even: $b \geq 2$ and $(\sigma_1 \vee \sigma_2)$; $b \geq 3$ odd: $\sigma_1 \neq \sigma_2$; $b = 1$: $\sigma_1 \neq \sigma_2$, and if σ_1 then $\beta_2 \vee \beta_3$, otherwise not $(\bar{\beta}_1 \wedge \bar{\beta}_2 \wedge a \geq 2)$
only u_2	σ_2, or not $(b = 1 \wedge \beta_1 \neq \beta_2 \wedge \beta_2 = \beta_3)$
only u_1, or neither	always

Table A.18. The split rule, a sufficient condition with no clock in it for a site with two unread postings to survive in both shared-out marked histories.

Forced repair bits. Table A.19 lists the configurations of Theorem 17.41(2). A site meets crossings $j < k$ and is kept in one history with frame value $\kappa = \kappa(j) = \kappa(k)$; the sides are those of its bits entering j, entering k and leaving k. The last row is a site with a single crossing.

The example networks. Table A.20 gives every network named in the chapter, with its numbers n of sites, m of postings and u of unread postings. A posting uv is the contact (u, v); a star marks an unread posting; posting i takes the network from time $i - 1$ to time i.

Openings of the second contact's rule. Table A.21 lists the openings of Theorem 17.47(2) and (3), each a list of postings in posting order, with aR, aU a read and an unread posting of (c, a) and bR, bU those of (c, b). One round of each opening in the first column reads all of $\mathscr{K}$ at each of the three sites; one round of each opening in the second column reads all 144 products at (c, b).

sides at j, k, after k	further condition	the site survives exactly when	rule reduces to
in, in, out	odd gap	$y_{C(k)} = 1 - \kappa$	e_3 (e_1 true, e_2 false)
in, in, out	even gap	value at $D(k) = \kappa$	e_4 (e_1 true, e_2 false)
out, in, in	odd gap	$y_{B(j)} = 1 - \kappa$	e_1 (e_3 true, e_4 false)
out, in, in	even gap, $A(j)$ exists	$y_{A(j)} = \kappa$	e_2 (e_3 true, e_4 false)
in, out, in	one posting x between, fewer than two before j	$y_x = 1 - \kappa$	e_3 (e_2 false, e_4 true)
in, out, out	one posting x between, two before j	$y_x = 1 - \kappa$ and value at $D(k) = 1 - \kappa$	Theorem 17.40
free head at c		$y_{A(c)} = \kappa(c)$	unit clause

Table A.19. Repair bits forced by single sites. A, B, C, D are the site's slots two before, one before, one after and two after a crossing; the value at $D(k)$ is a repair bit or the final value.

network	$n/m/u$	postings	read-closed set S	shows
E_3	4/11/3	23 01 02* 01 23 02* 01 23 02* 01 23	components: sites 0, 1; sites 2, 3	clocks make an unjoined pool exact
E_2	4/8/2	23 01 02* 01 23 02* 01 23	as for E_3	two crossings do not suffice
E_{W}	3/7/2	12 01 12* 12 01* 12 01	site 0 from time 5; sites 1, 2 from time 3	even gaps, outside the table class
E_{M}	4/8/2	23 01 02* 23 01 02* 23 01	sites 0, 1	outside both; an explicit pair
$E_{\mathrm{M}'}$	4/10/2	01 01 23 23 02* 23 01 02* 23 01	sites 0, 1	no assignment at rest
E_{R}	4/14/3	01 01 23 23 02* 23 01 02* 23 01 13* 23 01 01	sites 0, 1	split plan, no private repair
E_{O}	5/19/3	03 03 13 24 24 20* 04 13 23 23 01* 03 03 14 23 12* 03 13 24	site 0 at times ≤ 5 or ≥ 11; site 1 at ≤ 10 or ≥ 16; site 2 at 6 to 15; site 3	targeted plan, no split
E_{F}	5/12/2	03 03 23 14 14 01* 01 01 20* 03 23 14	site 0 at times ≤ 5 or ≥ 9; sites 2, 3	clauses beyond plans
E_{D}	5/22/4	31 41 31 41 04 41 04* 24 41* 13 24 40 24 10* 02 23* 02 40 03 20 41 01	sites 0, 4 from time 7; site 1 from 14; site 2; site 3 from 16	two free heads share a slot
E_{K}	4/10/2	20 13 20 13 03* 31 02 30* 02 13	components: sites 0, 2; sites 1, 3	no constant repair
E_{G}	4/11/3	02 23 32* 20 31 10* 31 12* 02 21 20	site 1 at times ≤ 7; site 3 at times ≥ 3	repairs interfere
E_{C}	8/24/5	27 10 27 12* 10 27 27 12* 27 30 47 34* 30 47 47 34* 47 50 50 67 56* 67 35 15	site 0; site 1 at ≤ 7; site 3 at ≤ 15; site 5 at ≤ 20	pairs do not decide
E_{P}	5/14/4	23 04 04 03* 02 42* 41 34 20 30* 40 21* 23 12	sites 1, 4; site 0 at ≤ 3 or ≥ 10; site 2 from 12; site 3 from 4	propagation does not decide
E_{B}	4/13/4	30 21 12 23* 20 10* 10 12* 32 30* 02 03 21	site 0 at times 6 to 9; site 1 at ≤ 7; site 2 at ≤ 3	two sites fail beyond units

Table A.20. The example networks. Where the chapter speaks of a component or set, its bits are listed: a site with no times given contributes all its bits.

fifteen read-saturating openings			seven product openings
aU bU bR aR	aU aR bU bU bR	aU bU bU aR bR	aU bR aU bR
aU bU aR bR	aU bR bU bU aR	aU aU aU bU bR	aU aR bU bR
aU aR bU bR	aU bU aU bR aR	aU bU aR bU bR	aU aU bR bR aR
aU aU bU bR aR	aU bU bR bU aR	aU bU bR aU aR	aU aU bR aU bR
aU aU bU aU bR	aU bU aU aU bR	aU bU bU bR aR	aU bR aU aU bR
			aU aR aU bU bR
			aU aU aR bU bR

Table A.21. The openings of the second contact's rule.

The position-two functionals. The space $\mathscr{W}_{\mathrm{II}}$ of Theorem 17.45 is the common kernel of the twenty functionals of Table A.22. Here $x_{\beta p|\beta' q} = B[(\beta, p), (\beta', q)]$ is the coordinate of a pair operator on (c, b) at c's bit β and coordinate p and b's bit β' and coordinate q. Each functional has a distinguished coordinate at which no other has weight, so the twenty are independent.

φ_{1}	$-x_{01\|00} + x_{01\|01}$
φ_{2}	$-x_{01\|10} + x_{01\|11}$
φ_{3}	$x_{00\|00} - x_{00\|01} + x_{00\|02} - x_{00\|03} + x_{00\|10} - x_{00\|11} + x_{00\|12} - x_{00\|13} - x_{01\|02} + x_{01\|03} - x_{01\|12} + x_{01\|13}$
φ_{4}	$-x_{03\|02} + x_{03\|03}$
φ_{5}	$x_{02\|00} - x_{02\|01} + x_{02\|02} - x_{02\|03} + x_{02\|10} - x_{02\|11} + x_{02\|12} - x_{02\|13} - x_{03\|00} + x_{03\|01} - x_{03\|10} + x_{03\|11}$
φ_{6}	$-x_{03\|12} + x_{03\|13}$
φ_{7}	$-x_{00\|00} + x_{00\|01} - x_{00\|02} + x_{00\|03} - x_{00\|10} + x_{00\|11} - x_{00\|12} + x_{00\|13} + 2x_{01\|02} - 2x_{01\|03} - 2x_{04\|00} + 2x_{04\|01} - 2x_{04\|10} + 2x_{04\|11}$
φ_{8}	$-x_{00\|00} + x_{00\|01} - x_{00\|02} + x_{00\|03} - x_{00\|10} + x_{00\|11} - x_{00\|12} + x_{00\|13} + 2x_{01\|02} - 2x_{01\|03} - 4x_{04\|00} + 4x_{04\|01} + 2x_{04\|02} - 2x_{04\|03} - 2x_{04\|12} + 2x_{04\|13}$
φ_{9}	$-x_{02\|00} + x_{02\|01} - x_{02\|02} + x_{02\|03} - x_{02\|10} + x_{02\|11} - x_{02\|12} + x_{02\|13} + 2x_{03\|00} - 2x_{03\|01} + 2x_{05\|00} - 2x_{05\|01} - 4x_{05\|02} + 4x_{05\|03} - 2x_{05\|10} + 2x_{05\|11}$
φ_{10}	$-x_{02\|00} + x_{02\|01} - x_{02\|02} + x_{02\|03} - x_{02\|10} + x_{02\|11} - x_{02\|12} + x_{02\|13} + 2x_{03\|00} - 2x_{03\|01} - 2x_{05\|02} + 2x_{05\|03} - 2x_{05\|12} + 2x_{05\|13}$
φ_{11}	$-x_{11\|00} + x_{11\|01}$
φ_{12}	$-x_{11\|10} + x_{11\|11}$
φ_{13}	$x_{10\|00} - x_{10\|01} + x_{10\|02} - x_{10\|03} + x_{10\|10} - x_{10\|11} + x_{10\|12} - x_{10\|13} - x_{11\|02} + x_{11\|03} - x_{11\|12} + x_{11\|13}$
φ_{14}	$-x_{13\|02} + x_{13\|03}$
φ_{15}	$x_{12\|00} - x_{12\|01} + x_{12\|02} - x_{12\|03} + x_{12\|10} - x_{12\|11} + x_{12\|12} - x_{12\|13} - x_{13\|00} + x_{13\|01} - x_{13\|10} + x_{13\|11}$
φ_{16}	$-x_{13\|12} + x_{13\|13}$
φ_{17}	$x_{10\|00} - x_{10\|01} + x_{10\|02} - x_{10\|03} + x_{10\|10} - x_{10\|11} + x_{10\|12} - x_{10\|13} - 2x_{11\|02} + 2x_{11\|03} - 2x_{14\|00} + 2x_{14\|01} - 2x_{14\|10} + 2x_{14\|11}$
φ_{18}	$x_{10\|00} - x_{10\|01} + x_{10\|02} - x_{10\|03} + x_{10\|10} - x_{10\|11} + x_{10\|12} - x_{10\|13} - 2x_{11\|02} + 2x_{11\|03} - 4x_{14\|00} + 4x_{14\|01} + 2x_{14\|02} - 2x_{14\|03} - 2x_{14\|12} + 2x_{14\|13}$
φ_{19}	$x_{12\|00} - x_{12\|01} + x_{12\|02} - x_{12\|03} + x_{12\|10} - x_{12\|11} + x_{12\|12} - x_{12\|13} - 2x_{13\|00} + 2x_{13\|01} + 2x_{15\|00} - 2x_{15\|01} - 4x_{15\|02} + 4x_{15\|03} - 2x_{15\|10} + 2x_{15\|11}$
φ_{20}	$x_{12\|00} - x_{12\|01} + x_{12\|02} - x_{12\|03} + x_{12\|10} - x_{12\|11} + x_{12\|12} - x_{12\|13} - 2x_{13\|00} + 2x_{13\|01} - 2x_{15\|02} + 2x_{15\|03} - 2x_{15\|12} + 2x_{15\|13}$

Table A.22. The twenty integer functionals cutting out the position-two write space.

A.10 Clock actions and resonance tables of the relay

This section lists the finite data used in Chapter 18. Vectors of the clock cell are written in the basis $e_0, \dots, e_3$, so that $P = \frac{1}{2}(1,-1,-1,1)$ and $A = \frac{1}{2}(1,1,-1,-1)$. The clock blocks $B(b',b)$ are the blocks of the eight-state clock Ω in the basis f_j^b of Section 18.1. Each entry is a direct exact computation over $\mathbb{Q}(\sqrt{2}, i)$.

vector	coordinates
$B(0,0)A$	$\left(\frac{3}{2} - \frac{i}{4},\ \frac{5}{2} + \frac{\sqrt{2}}{2} + \frac{i}{4},\ -\frac{5}{2} + \frac{i}{4},\ -\frac{3}{2} + \frac{\sqrt{2}}{2} - \frac{i}{4}\right)$
$B(1,1)A$	$\left(\frac{5}{2} - \frac{i}{4},\ \frac{3}{2} - \frac{\sqrt{2}}{2} + \frac{i}{4},\ -\frac{3}{2} + \frac{i}{4},\ -\frac{5}{2} - \frac{\sqrt{2}}{2} - \frac{i}{4}\right)$
$B(1,0)A$	$\left(-\frac{1}{2} - \frac{i}{4},\ \frac{1}{2} - \frac{\sqrt{2}}{2} + \frac{i}{4},\ -\frac{1}{2} + \frac{i}{4},\ \frac{1}{2} - \frac{\sqrt{2}}{2} - \frac{i}{4}\right)$
$B(0,1)A$	$\left(\frac{1}{2} - \frac{i}{4},\ -\frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{i}{4},\ \frac{1}{2} + \frac{i}{4},\ -\frac{1}{2} + \frac{\sqrt{2}}{2} - \frac{i}{4}\right)$
$B(0,0)P$	$\left(\frac{3}{2} - \frac{\sqrt{2}}{2} + \frac{3i}{4},\ -\frac{3}{2} + \frac{3i}{4},\ -\frac{3}{2} - \frac{\sqrt{2}}{2} + \frac{i}{4},\ \frac{3}{2} + \frac{i}{4}\right)$
$B(1,1)P$	$\left(\frac{3}{2} + \frac{\sqrt{2}}{2} - \frac{i}{4},\ -\frac{3}{2} - \frac{i}{4},\ -\frac{3}{2} + \frac{\sqrt{2}}{2} - \frac{3i}{4},\ \frac{3}{2} - \frac{3i}{4}\right)$
$B(0,1)P$	$\left(\frac{1}{2} - \frac{\sqrt{2}}{2} - \frac{i}{4},\ -\frac{1}{2} - \frac{i}{4},\ -\frac{1}{2} - \frac{\sqrt{2}}{2} - \frac{3i}{4},\ \frac{1}{2} - \frac{3i}{4}\right)$
$B(1,0)P$	$\left(\frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{3i}{4},\ -\frac{1}{2} + \frac{3i}{4},\ -\frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{i}{4},\ \frac{1}{2} + \frac{i}{4}\right)$

for $b \in \{0,1\}$	stay $B(b,b)$	transfer $B(1-b,b)$
$\langle A, B\,A\rangle$	4	0
$\lVert B\,A - \langle A, B\,A\rangle A\rVert^2$	$\frac{9}{4} + \sqrt{2}$	$\frac{9}{4} - \sqrt{2}$
$\langle P, B\,P\rangle$	3	1
$\lVert B\,P - \langle P, B\,P\rangle P\rVert^2$	$\frac{9}{4}$	$\frac{9}{4}$
$\lVert B\,P\rVert^2$	$\frac{45}{4}$	$\frac{13}{4}$
$\langle P, B\,A\rangle$	$-\frac{i}{2}$	$-\frac{i}{2}$
$\langle A, B\,P\rangle$	$\frac{i}{2}$	$\frac{i}{2}$

Table A.23. The clock blocks of the relay on the dressed pole vectors A and P. The stays $B(b,b)$ keep a mean of 4 on A and 3 on P, which with the diagonal $2\varepsilon + \nu_0$ gives the mode energies $10\varepsilon + \nu_0$ and $8\varepsilon + \nu_0$ of Proposition 18.5; their centred parts and the transfers $B(1-b,b)$ are the departures of Theorems 18.3 and 18.6.

The second table concerns the resonant parts used in Theorem 18.11. A persistent mode $\psi_\pm \otimes g$ departs through two balanced emissions, carrying the payloads $W(-z)g, W(-\tilde{z})g$ for ψ_+ and $W(\tilde{z})g, W(z)g$ for ψ_-, through its in-place clock spread, carrying g itself, and through the recoil. Let u_1, u_2 be the parts of the two emission vectors in the $+1$ eigenspace of U_{nat} (for ψ_+) or the -1 eigenspace (for ψ_-), and u_3 the part of the in-place spread. The recoil has no part in either eigenspace. The vectors u_k have entries in $\mathbb{Q}(\sqrt{2}, i)$, and each is certified by an explicit vector c_k with entries in $\mathbb{Q}(\sqrt{2}, i)$ and $(U_{\mathrm{nat}} \mp 1)c_k = y_k - u_k$, y_k the departure vector, so

that u_k is the full spectral projection of y_k. Then

$$r_+(g) = \varepsilon^2 \left\| u_1 \otimes W(-z)g + u_2 \otimes W(-\tilde{z})g + u_3 \otimes g \right\|^2,$$
$$r_-(g) = \varepsilon^2 \left\| u_1 \otimes W(\tilde{z})g + u_2 \otimes W(z)g + u_3 \otimes g \right\|^2,$$

which expand into the forms of Theorem 18.11(a) with the Gram entries below.

	ψ_+ (eigenvalue $+1$)	ψ_- (eigenvalue -1)
$\langle u_1, u_1 \rangle = \langle u_2, u_2 \rangle$	$\frac{5711}{9744}$	$\frac{5711}{9744}$
$\langle u_1, u_2 \rangle$	$\frac{4381}{9744}$	$-\frac{2003}{9744}$
$\langle u_1, u_3 \rangle = \langle u_2, u_3 \rangle$	$\frac{1}{4} + \frac{\sqrt{2}}{2} i$	$-\frac{11}{116}$
$\langle u_3, u_3 \rangle$	$\frac{3}{2}$	$\frac{11}{58}$
squared length of each emission	$\frac{13}{4}$	$\frac{13}{4}$
squared length of the spread	$\frac{9}{2}$	$\frac{9}{2}$
$\|f\|^2$	$\frac{12}{29}$	$\frac{44}{309}$

Table A.24. Gram tables of the resonant parts of the persistent departures. The last three rows give the squared lengths of the full departure vectors, for comparison, and of the witness. The vector f is an explicit witness orthogonal to u_1 and u_2 with $\langle f, u_3 \rangle = \|f\|^2$; $\|f\|^2$ is the squared distance of u_3 from $\operatorname{span}\{u_1, u_2\}$, and it gives the payload-independent lower bounds $r_+(g) \geq \frac{12}{29}\varepsilon^2\|g\|^2$ and $r_-(g) \geq \frac{44}{309}\varepsilon^2\|g\|^2$.

A.11 Finite certificates for Chapter 19

Several proofs of Chapter 19 end in an exact finite computation. The certificates themselves (rank certificates with integers of up to about 190 digits, and interval enclosures of eigen-data) are too long to print. Table A.25 says what each one computes, on which data, and in which arithmetic, so that it can be reproduced. "Rows" are the linear conditions of Lemma 19.13 or its first-order analogue on Plücker coordinates in a rational basis of a kept eigenspace; a certificate is an exact matrix identity $NM = dI$ with $d \neq 0$ on the coordinates in question. Interval arithmetic is over $\mathbb{Q}$ with outward rounding.

result	what is computed	data and arithmetic
Thm. 19.3, 19.5	real parts of the clock blocks; the four effects E_{w_y} on f_{r}, f_{i}; all 15 single labels	4×4 tables over $\mathbb{Q}(\sqrt{2})$; $E_{w_{(1,1)}} f_{\mathrm{r}} = 0$, $E_{w_{(1,1)}} f_{\mathrm{i}} = \frac{1}{4} f_{\mathrm{i}}$
Thm. 19.10	rational bases of $\mathrm{Fix}_{\pm 1}$ with Gram inverses; compressed commutators on two kept pairs	Grams $(8, \frac{1408}{9}, 0)$, $(\frac{88}{3}, 15, 0)$; coefficients $(\frac{53}{21}, -\frac{1}{14})$, $(0, -\frac{1}{14})$, $(\frac{9809}{24360}, -\frac{11297}{24360}, \frac{613}{2030})$
Thm. 19.14	28 rows on 28 Plücker coordinates	Fix_1: levels 1–4, 12 words; Fix_{-1}: levels $1, 3, 5$, 13 words; $\mathbb{Z}[\sqrt{2}]$
Thm. 19.19(b), (c)	12 rows on the parity-diagonal coordinates; the same as polynomials in t	10 rows at level 1, 2 at level 3; integer; two determinants of degree 10 in t
Thm. 19.23(a)	16 rows on the cross-parity coordinates	Fix_1: levels 1 (10 words), 3 (6); Fix_{-1}: levels 2, 4; d of about 71 and 190 digits
Thm. 19.24(c)	the same with G_{Ξ} in place of M_{-}	rank 16 from levels $2, 4$ (Fix_1) and $1, 3$ (Fix_{-1})
Thm. 19.27	eigenvalues of each block of the relay on $\mathrm{Fix}_{\pm 1}$; eigenvector coordinates; independence of four records' top–bottom elements	characteristic polynomials over $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2})$; isolating intervals of width 2^{-80}; coordinates to 2^{-96}; approximate inverse with a Neumann bound
Thm. 19.30	one-block eigen-data of the relay at $t = 2$	isolating intervals of width about 2^{-55}; all 64 parity overlaps and 64 combined record forms bounded away from 0
Thm. 19.32	eigenvalue branches of the relay on Fix_{-1} over $t \in [-\frac{193}{5000}, -\frac{77}{2000}]$	centred bivariate tables at $t_c = -\frac{771}{20000}$; boxes of half-width $\frac{1}{5000}$; sign change of the gap difference
Thm. 19.37	rows of the five Y-fixed labels on the $3 + 10$ coordinates inside the eigenspaces of Y	levels to 4 (Fix_1) and 5 (Fix_{-1}); $\mathbb{Z}[\sqrt{2}]$
Thm. 19.38(a)	first-order rows on the 15 cross coordinates; rows of every order in t	levels to 4 and 7; centre certificates with positive weights on $\lvert t \rvert \le \frac{1}{20}$
Thm. 19.40	each single record's matrix elements between relay eigenvectors at $t = 2$	40 interval checks, 64 forms each
Thm. 19.44	readers on all 28 pairs of distinct modes; kept-Y-fixed records on the 13 pairs inside the eigenspaces of Y	interval forms with the eigenvector enclosures of Thm. 19.27
Prop. 19.46	all tie fractions in $(0, 1)$	854 and 1062 ordered pairs of pairs; enclosures of width below 10^{-15}
Thm. 19.48	brackets, windows and step orders of the second-level probe	gap separation $\frac{21}{250}$ on Fix_1; pairwise windows on Fix_{-1}
Thm. 19.49(a)	on Fix_{-1} with $\Xi\psi = Y\psi$: the five tie fractions left by the support analysis, and how each closes	$F \approx 0.7243, 0.4908, 0.2729, 0.4378, 0.5484$; four by the second-level probe with (c) for one rest interval (one of them after an exceptional two-interval timing is settled by a probe on the longer interval),

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