

On the Kuhn Lattice, the Long-Wavelength Vertices of Regge's Action Are One Half of Einstein's Through Quartic Order

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Abstract

On the Kuhn lattice, the long-wavelength (two-derivative) quadratic, cubic and quartic vertices of Regge's action around a flat background are one half of those of the Einstein–Hilbert density $\sqrt{|g|}\,R$, for all polarizations and conserved momenta, in the metric sector, on the Euclidean lattice and on the Lorentzian one with flat metric $I - \mathbf{1}\mathbf{1}^{\top}$. The $\frac{1}{2}$ is one normalization at all three orders, not a changed coupling. On the Euclidean lattice these identities also follow from Cheeger, Müller and Schrader's convergence theorem; here they come with explicit tables and a finite, exact proof. The quartic table, rational in the background parameter κ, is Regge's own fourth derivative at every Euclidean κ; its contact vertex is half Einstein's at every κ off three poles. For every $\kappa > \frac{1}{2}$, beyond that theorem's reach, the Lorentzian action's derivatives are $\sqrt{4\kappa - 1}$ times the same tables. At $\kappa = 1$ the five non-metric edge modes decouple from these vertices at tree level; at $\kappa = 0$ the body diagonal is a non-gauge zero mode, and the results describe the metric truncation. The tree four-graviton amplitude from the tables is gauge invariant, with general relativity's helicity selection and MHV amplitude. No hinge shape cost, linear in the deficit with a coefficient set by the hinge's class and three squared lengths, completes a finite change of the coupling at quartic order (metric sector; $\kappa = 0$, $\frac{1}{10}$, 1; one stated inference). Every theorem is proved in the paper or by exact computation in its supplement.

1 Regge's action through quartic order

On the Kuhn lattice, the long-wavelength vertices of Regge's action reproduce Einstein's through fourth order in the field. Take the lattice, give every edge a squared length close to that of a flat lattice, and expand the action in plane waves of the metric. The terms with two derivatives, the order of Einstein's theory, organize into a quadratic form (the graviton's propagator), a cubic vertex and a quartic vertex. This paper proves that each of the three is one half of the corresponding vertex of $\sqrt{|g|}\,R$, for every polarization and every set of momenta summing to zero, on the Euclidean member of the lattice family and on the Lorentzian lattice

whose flat metric is $g_1 = I - \mathbf{1}\mathbf{1}^\top$ (Theorem 3.1). The proof is finite and exact. It runs through explicit per-simplex tables, polynomials in the edge perturbations whose coefficients are rational in the background parameter κ, and every identity it uses is checked by exact computation in the supplement [8].

The factor $\frac{1}{2}$ is not a halving of gravity. It is the classical normalization of Regge calculus, in which $\sum_h A_h\delta_h$ corresponds to $\frac{1}{2}\int\sqrt{|g|}\,R$ [19, 22], and it is the same at orders two, three and four. A common factor in front of an action fixes its units. A different factor at different orders would change the ratio of the cubic to the quadratic coupling, which is the gravitational coupling itself; that does not happen. With an action scale s_{p}, $s_{\mathrm{p}}\sum_h A_h\delta_h = \frac{s_{\mathrm{p}}}{2}\int\sqrt{|g|}\,R = (16\pi G)^{-1}\int\sqrt{|g|}\,R$ at $G = 1/(8\pi s_{\mathrm{p}})$. On the Lorentzian lattice this is also the coupling of the lattice's linear field and of Newton's law [3, Sec. 3.5 and Thms. 4.6, 4.10]. Reading the lattice action as gravity's action is an IDENTIFICATION. On the $\kappa = 1$ lattice the factor is one factor $\frac{1}{2}\sqrt{|\det g_1|}$, as in $\frac{1}{2}\sqrt{|g|}\,R$: the second, third and fourth derivatives of the Lorentzian simplex function are $\sqrt{3} = \sqrt{|\det g_1|}$ times the tables (Proposition 4.2(d)).

Relation to earlier work. Much of the Euclidean side is classical. At $\kappa = 0$ the Kuhn lattice is the triangulated hypercubic lattice, each unit cube cut into the 24 simplices of its chains from 0 to **1**, with fifteen edges per vertex. On it Roček and Williams began linearized Regge calculus and found the continuum linearized Einstein action in the weak-field, long-wavelength limit [11, 12]. Hamber and Williams wrote that expansion out on this lattice [13, Sec. 2], and Dittrich recomputed it [14] as the case $s = 0$ of a tilted lattice, the member $\kappa = -(2s + 4s^2)$ of the family used here. The Euclidean quadratic identity of Theorem 3.1(a) is therefore classical in substance. Their rules relating edge lengths to the metric are based at the lattice vertices, and the non-metric combinations of edges are eliminated through their equations of motion; here the metric is sampled at edge midpoints, (4), and the vertices are those of the metric sector. The same works record the cube's body diagonal, the edge class 1111 (their hyperbody diagonal), as a fifth zero mode of the quadratic action, besides the four periodic translations of the lattice points [17, 18, 13], and [17] counts all five as gauge fluctuations; Section 8 shows that the body diagonal is not one and enters the action at cubic order. Dittrich, Freidel and Speziale studied the linearized dynamics of the single four-simplex Regge action [15].

Beyond quadratic order, convergence theorems already contain the Euclidean identities. Cheeger, Müller and Schrader proved that for piecewise flat spaces obtained from a smooth triangulation of a Riemannian manifold, with geodesic edge lengths [19, Thm. 5.1], or secant ones [19, Sect. 1], and fatness bounded below, $\sum_h A_h\delta_h$ converges to $\frac{1}{2}\int\sqrt{g}\,R$ as the mesh goes to zero; Barrett proved a convergence result for linearized Regge calculus [20]. Feinberg, Friedberg, Lee and Ren state that lattice gravity approaches its continuum limit as the link length goes to zero, on any lattice and under general boundary conditions, with corrections in powers of the squared link length [21]. Applied to the plane waves used here, the theorem of [19] gives $V_n[S_{\mathrm{R}}] = \frac{1}{2}V_n[\sqrt{g}\,R]$ at every order n and every Euclidean κ (Remark 3.5); with Proposition 4.2 it gives the table identities of Theorems 5.1 and 6.1 as identities of rational functions of κ. It does not reach the Lorentzian lattice. There the deposited preprints [4, Thm. 3.3] and [2, Thm. 4.2] state, for every $\kappa > \frac{1}{2}$, that the metric part of the two-derivative coefficient of the lattice field equations on sampled metric jets is $-\frac{1}{2}\sqrt{|\det g|}\,G^{\mu\nu}$, with every nonlinear term; expanded about the flat metric, that statement gives the same metric-sector

vertices. Theorem 7.1 is the classical total-derivative identity between $\sqrt{|g|}\,R$ and the ΓΓ Lagrangian [25, §93], written for vertices. At $\kappa = 0$ the amplitude of Section 9 is general relativity's tree four-graviton amplitude [26, 27] at one half of Einstein's normalization, so its gauge invariance, helicity selection and MHV value are consistency checks.

What is new. Measured against that work, the paper adds four things.

1. *The quartic table and an exact, finite proof* (Theorem 6.1). The lattice's quartic table is an explicit per-simplex polynomial of degree four in the ten edge perturbations, with 691 coefficients rational in κ, and it is Regge's own fourth derivative at every Euclidean κ (Proposition 4.2). Its long-wave contact vertex is one half of Einstein's quartic vertex on the background $g_\kappa = I - \kappa\mathbf{1}\mathbf{1}^\top$ for every κ with $(2\kappa-1)(3\kappa-1)(4\kappa-1) \neq 0$, over every field of characteristic zero. The proof is 53 polynomial identities, checked exactly, and uses no convergence theorem. The three excluded values are those at which a class of the lattice's triangles degenerates.
2. *The Lorentzian action through quartic order, with explicit tables* (Proposition 4.2(d), Theorem 3.1(b)). For every $\kappa > \frac{1}{2}$ the second, third and fourth derivatives of the Lorentzian simplex function are $\sqrt{4\kappa-1}$ times the continued Euclidean tables. So the Lorentzian contact vertices through quartic order are given by explicit tables, and they are one half of those of $\sqrt{-\det g}\,R$ by exact computation.
3. *The other five edge modes* (Section 8). At $\kappa = 1$, and at $\kappa = \frac{1}{10}$, the five non-metric combinations of the fifteen edge classes are nondegenerate at zero momentum, so they can be eliminated at tree level, and eliminating them leaves every two-derivative vertex unchanged (Lemma 8.4): there the metric sector carries the whole lattice at this order. At $\kappa = 0$ the body diagonal is a non-gauge zero mode at every momentum that enters the action at cubic order (Proposition 8.3), and the statements there concern the metric truncation.
4. *A changed coupling* (Theorem 10.3). Within shape costs, one smooth function of a hinge's three squared edge lengths for each of the lattice's six classes of hinges, changed through their sixty third-derivative directions, no finite change of the gravitational coupling is completed at quartic order, at $\kappa = 0$, $\frac{1}{10}$ and 1, with the non-metric modes included at $\frac{1}{10}$ and 1. The reading uses one stated INFERENCE (Remark 10.4); in the metric sector it covers every shape cost whose quadratic and cubic parts have no long-wave vertex.

Why the vertices matter. The vertices are what the action contributes to scattering. Once the quadratic form, the cubic vertex and the quartic vertex are fixed, the tree-level four-graviton amplitude is fixed. At $\kappa = 1$, where the non-metric modes decouple from these vertices at tree level, the identity through quartic order therefore says that the lattice's gravitons scatter at long wavelength and at tree level as general relativity's do, at one half of Einstein's normalization in the lattice's units; at $\kappa = 0$ it says the same of the metric truncation. Section 9 checks this exactly for the amplitude built from the lattice's own tables: it is gauge invariant, and it has general relativity's helicity selection and maximally helicity violating (MHV) amplitude, at $\kappa = 0$ and at $\kappa = 1$.

The lattice. The flat members $g_\kappa = I - \kappa \mathbf{1}\mathbf{1}^\top$ agree on the hyperplane orthogonal to $\mathbf{1}$ and differ only in the squared length $4(1-4\kappa)$ of the body diagonal $\mathbf{1}$; at $\kappa = 1$ every single step is null. The recognition program of [1, 2] takes $\kappa = 1$ as the physical lattice. That is an IDENTIFICATION, which no theorem below uses; Section 11 says what it rests on. The $\kappa = 1$ lattice has two known artifacts at the lattice scale. Among the members with $\kappa > \frac{1}{2}$ it alone has massless corner doublers, at the four axis corners of the momentum cell [2, Thm. 2.1], and through them the zero-momentum anisotropy of the one-loop graviton self-energy in the diagram model of [3] has no ordered limit [3, Thm. 6.8]. The tree-level, long-wave vertices of this paper do not see them.

Scope. Everything here is tree level and long wavelength (the terms with two derivatives), on the Kuhn lattice. Every configuration of the main theorem is a sampled metric, ten numbers per point for fifteen edge classes: the metric sector. At $\kappa = 1$ the other five combinations decouple from the two-derivative vertices at tree level (Lemma 8.4); at $\kappa = 0$ the statements concern the metric truncation (Proposition 8.3); Theorem 10.3(b) includes the five modes. The one-loop graviton of the lattice is treated in [3, 2], and Newton's constant is not derived here. Section 11 lists what the paper does not say.

How to read the status of a claim. Every theorem, proposition, lemma and corollary is proved in this paper: in the text, or, where a proof ends in finitely many identities between explicit polynomials or in an exact certificate, by an exact computation in the supplement [8]. The supplement has eight parts, S1 to S8, listed in the last section; each proof that ends in such a computation names its part, so a reader can check every claim with the paper and the supplement alone. Results of the cited preprints are used only for comparison, never as steps of a proof. Four words mark what is not a theorem: IDENTIFICATION, a physical reading that the definitions do not force; INFERENCE, a reasoned step that is not one of the theorems; NUMERICAL, a computation that is not part of a proof; and OPEN, a precise statement that is neither proved nor refuted.

Map. Section 2 defines the lattice, the actions and the vertices. Section 3 states the main theorem and shows how it is assembled. Section 4 identifies the lattice's tables with the derivatives of the lattice actions, Section 5 proves the quadratic and cubic identities, Section 6 the quartic one, and Section 7 passes from the first-order Lagrangian to $\sqrt{|g|}\,R$. Section 8 treats the five non-metric edge modes, Section 9 the four-graviton amplitude, and Section 10 hinge shape costs and a changed coupling; Section 11 closes with what is not claimed.

2 The lattice, the actions and their vertices

2.1 The Kuhn lattice and its flat members

The vertices are the points $x \in \mathbb{Z}^4$, or of the torus $(\mathbb{Z}/N)^4$. Each unit cube $x + [0,1]^4$ is cut into 24 simplices, one for each order $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ of the four axes: the simplex σ_π at x has the five vertices

$$v_0 = x, \qquad v_k = v_{k-1} + e_{\pi_k} \quad (k = 1, \dots, 4),$$

so it climbs from x to $x+\mathbf{1}$, $\mathbf{1}=(1,1,1,1)$, one axis at a time [23, 24]. Its ten edges, the *slots* $f=(jk)$ with $0\le j<k\le 4$, have the *steps* $d_f=v_k-v_j\in\{0,1\}^4\setminus\{0\}$ and the *midpoints* $m_f=\frac{1}{2}(v_j+v_k)-x$, measured from the cell's corner. Every edge of the lattice joins y to $y+d$ for one of fifteen edge classes d. The triangles are the *hinges*. The graph is fixed throughout; we never sum over triangulations.

A *configuration* ℓ gives every edge a squared length. The flat lattice of parameter κ is

$$\ell_\kappa(y,d)=d^{\top}g_\kappa d=|d|\,(1-\kappa|d|),\qquad g_\kappa=I-\kappa\,\mathbf{1}\mathbf{1}^{\top},$$

with $|d|$ the number of ones in d, $\det g_\kappa=1-4\kappa$ and, for $\kappa\neq\frac{1}{4}$, $g_\kappa^{-1}=I+\frac{\kappa}{1-4\kappa}\,\mathbf{1}\mathbf{1}^{\top}$. For $\kappa<\frac{1}{4}$ the metric is Euclidean, and $\kappa=0$ is the standard Kuhn lattice. For $\kappa>\frac{1}{4}$ it is Lorentzian with time along $\mathbf{1}$. A hinge has vertices y, $y+a$, $y+a+b$ with a, b disjoint nonempty $0/1$ steps, and on ℓ_κ Heron's form $Q=2(xy+yz+zx)-x^2-y^2-z^2$ of its squared sides x,y,z is

$$Q_h=4|a||b|\big(1-\kappa(|a|+|b|)\big),\qquad |a|+|b|\in\{2,3,4\}.$$

So for $\kappa<\frac{1}{4}$ every hinge is a nondegenerate Euclidean triangle, $Q_h>0$; for $\kappa>\frac{1}{2}$ every hinge is timelike, $Q_h<0$; and $\kappa=\frac{1}{2},\frac{1}{3},\frac{1}{4}$ are the values at which the hinges with $|a|+|b|=2,3,4$ degenerate. At $\kappa=1$ every single step is null, and $g_1^{-1}=I-\frac{1}{3}\mathbf{1}\mathbf{1}^{\top}$. A permutation of the four axes permutes the 24 simplices of a cell among themselves and fixes g_κ, so every simplex of a flat lattice is congruent to the *canonical simplex* σ_{id}, slot by slot.

2.2 Regge's action and the simplex function

A hinge h has an area $A_h(\ell)$ and a deficit angle $\delta_h(\ell)=2\pi-\sum_{\sigma\ni h}\theta_{h,\sigma}(\ell)$, where $\theta_{h,\sigma}$ is the dihedral angle of the simplex σ at h. Regge's action is [9]

$$S_{\mathrm{R}}(\ell)=\sum_h A_h(\ell)\,\delta_h(\ell).$$

Every flat lattice has $\delta_h=0$, so near ℓ_κ the deficit is minus the sum of the angle changes, and the action splits into simplex shares:

$$S_{\mathrm{R}}(\ell)=-\sum_\sigma\Psi_\sigma(\ell),\qquad \Psi_\sigma(\ell)=\sum_{h\subset\sigma}A_h(\ell)\big(\theta_{h,\sigma}(\ell)-\theta_{h,\sigma}(\ell_\kappa)\big),\tag{1}$$

the sum running over the ten triangles h of σ. We call $\Psi=\Psi_{\sigma_{\mathrm{id}}}$, a function of the canonical simplex's ten squared lengths, the *simplex function*. By the congruence above, every Ψ_σ is Ψ read in σ's slot labels.

On the Lorentzian members with $\kappa>\frac{1}{2}$ every hinge is timelike, and the Lorentzian lattice action S_{L} is Regge's sum with Lorentzian areas and angles: $A_h=\frac{1}{4}\sqrt{-Q_h}$; the plane normal to a timelike hinge is spacelike, and $\theta_{h,\sigma}$ is the ordinary angle in that plane between the two tetrahedra of σ that contain h; $\delta_h=2\pi-\sum_\sigma\theta_{h,\sigma}$, and $S_{\mathrm{L}}(\ell)=\sum_h A_h(\ell)\,\delta_h(\ell)$. These are the conventions of [3, Sec. 3.1] and [6, Sec. 6.1]. The action S_{L} is real, every flat member has $\delta_h=0$, and S_{L} splits into simplex shares $-\Psi_{\mathrm{L},\sigma}$ exactly as in (1); Ψ_{L} is the Lorentzian simplex function.

Schläfli's identity $\sum_h A_h\, d\theta_h = 0$ [10], as used in [9], makes $\Psi = O(r^2)$ at ℓ_κ: along a line $\ell_\kappa + r\lambda$, with primes for d/dr, $\Psi' = \sum_h A'_h(\theta_h - \theta^0_h)$, so $\Psi(\ell_\kappa) = \Psi'(\ell_\kappa) = 0$. The same identity, differentiated, gives the derivatives at $r = 0$ in closed form:

$$\Psi'' = \sum_h A'_h\theta'_h, \qquad \Psi''' = \sum_h \big(2A''_h\theta'_h + A'_h\theta''_h\big), \qquad \Psi^{(4)} = \sum_h \big(3A'''_h\theta'_h + 3A''_h\theta''_h + A'_h\theta'''_h\big). \tag{2}$$

The identity holds on the Lorentzian simplices used here as well. In the notation of the proof of Proposition 4.2, write the derivative of each dihedral angle as $\theta'_h = \rho_h/\sqrt{D_h}$, where $D_h = b_{ll}b_{mm} - b_{lm}^2$ is the Gram determinant of the two dual normals spanning the plane normal to h and ρ_h is a rational function of the squared lengths and λ. Jacobi's identity gives $D_h = Q_h/(4\det G)$, with G the Gram matrix of the edge vectors from v_0, so $A_h/\sqrt{D_h} = \frac{1}{2}\sqrt{|\det G|}$ for every hinge, on both sides of the light cone. Hence $\sum_h A_h\theta'_h = \frac{1}{2}\sqrt{|\det G|}\sum_h \rho_h$. The Euclidean identity makes the rational function $\sum_h \rho_h$ vanish on an open set, hence everywhere, and with it the Lorentzian sum. Supplement S2 also checks the identity along every line through ℓ_κ, $\kappa > \frac{1}{2}$, to third order in r, the order (2) uses.

2.3 Tables

A *table* is a homogeneous polynomial $T(\lambda)$ in the ten slot variables $\lambda = (\lambda_f)$ of one simplex. We write $T(\lambda) = \sum_{f_1,\dots,f_n} \hat{T}_{f_1\dots f_n}\lambda_{f_1}\cdots\lambda_{f_n}$ with a symmetric coefficient tensor $\hat{T}$. On the Euclidean side the n-th derivative of the simplex function is $\sqrt{1-4\kappa} = \sqrt{\det g_\kappa}$ times a table with coefficients rational in κ:

$$\frac{d^n}{dr^n}\Psi(\ell_\kappa + r\lambda)\Big|_{r=0} = \sqrt{1-4\kappa}\; T_n(\kappa;\lambda) \qquad (n = 2,3,4,\ \kappa < \tfrac{1}{4}), \tag{3}$$

by Proposition 4.2. The quadratic table is $T_2(\kappa;\lambda) = -\lambda^\top\Omega(\kappa)\lambda$, with $\Omega(\kappa)$ a symmetric 10×10 matrix of rational functions of κ; the cubic table $T_3(\kappa)$ has 220 coefficients, rational in κ; the quartic table $T_4(\kappa)$ is given in Definition 4.1. Their values at $\kappa = 1$ are defined by continuing these rational functions, and Proposition 4.2(d) shows that they are the Lorentzian action's tables, up to the factor $\sqrt{|\det g_1|} = \sqrt{3}$.

2.4 Plane waves and long-wave vertices

A plane wave of the metric has a symmetric 4×4 polarization H and a momentum $K \in \mathbb{R}^4$, or in $\mathbb{C}^4$. Sampled at edge midpoints, it changes the squared length of the edge (y,d) with midpoint $y + d/2$ by

$$h_{H,K}(y,d) = (d^\top H d)\, e^{iK\cdot(y+d/2)}. \tag{4}$$

From a metric to edge lengths (IDENTIFICATION). Equation (4) reads a metric $g(x) = g_\kappa + \varepsilon H e^{iK\cdot x}$ on the lattice by giving the edge (y,d) the squared length $d^\top g(y+d/2)\,d$, the metric taken at the edge's midpoint. This relation between the lattice's variables and a continuum field is an IDENTIFICATION: the definitions do not force it. The two-derivative vertices do not depend on it at second order in the momenta: a linear rule that changes the amplitude of every edge only at second order in the momenta gives the same long-wave vertices. Two examples are the average of $d^\top g\, d$ over the segment from y to $y + d$, which multiplies the

amplitude of the wave by $\sin(K\cdot d/2)/(K\cdot d/2) = 1-(K\cdot d)^2/24 + O(K^4)$, and the mean of its values at the two ends, which multiplies it by $\cos(K\cdot d/2) = 1-(K\cdot d)^2/8 + O(K^4)$. Such a rule adds to each leg a change of second order in that leg's momentum, and a cell vertex whose legs are all metric waves but one is itself of second order in the momenta (Lemma 8.2); so every vertex changes only at fourth order. A rule that differs at first order can give different vertices: with the metric sampled at one end of each edge, $d^{\top} g(y)\, d$, the quadratic vertex at $\kappa = 0$ of the transverse-traceless strain $X = X' = E_{12}$ at $k = e_3$ is -1, against $-\frac{1}{2} = -\frac{1}{4}\mathrm{FP}_I(X, X'; k)$ for midpoint sampling, and the two rules differ by no common factor (supplement S3). So Theorem 3.1 rests on the identification (4) at first order in the momenta; midpoint sampling is the rule that agrees with the geodesic lengths of the metric to relative order N^{-2} (Remark 3.5(ii)).

For n waves (H_i, K_i) with $K_1 + \cdots + K_n = 0$ and an action S that is a sum of simplex shares, consider $S(\ell_\kappa + \sum_i \varepsilon_i h_{H_i,K_i})$. Its coefficient of $\varepsilon_1 \cdots \varepsilon_n$ is the same in every cell, because the phases of the cell corners multiply to $e^{i(K_1+\cdots+K_n)\cdot y} = 1$. On a simplex the coefficient comes from the share's n-th order part, a table, and the vertex is the part of it of second order in the momenta. We define it through the table.

Definition 2.1 (Long-wave vertex of a table)**.** For a table T of degree n with coefficient tensor $\hat{T}$ and n waves (H_i, K_i), the *long-wave vertex* of T is

$$\mathscr{V}[T](H;K) = \frac{1}{2}\sum_{\sigma}\sum_{f_1,\dots,f_n} \hat{T}_{f_1\dots f_n} \prod_{i=1}^{n} \big(d_{\sigma f_i}^{\top} H_i d_{\sigma f_i}\big)\Big(\sum_{i=1}^{n} K_i \cdot m_{\sigma f_i}\Big)^2, \tag{5}$$

the outer sum over the 24 simplices of a cell, with $d_{\sigma f}$ and $m_{\sigma f}$ the step and midpoint of slot f of σ.

Expanding $\frac{1}{2}(\sum_i z_i)^2 = \sum_{m_1+\cdots+m_n=2} \prod_i z_i^{m_i}/m_i!$ shows that (5) is the sum over the ten (for $n = 4$) ways of distributing two momentum factors over the legs, which is how the vertex is usually computed.

Definition 2.2 (Long-wave vertices of an action)**.** Let S be a lattice action whose simplex shares are smooth near ℓ_κ and congruent slot by slot, and write the n-th order part of each share as $-\frac{1}{n!}Q_n(\lambda)$ for a table Q_n. The *long-wave vertex* of S is $V_n[S](H;K) = \mathscr{V}[Q_n](H;K)$. For a continuum density $\mathscr{L}(g, \partial g, \partial\partial g)$ around a constant metric g_0, $V_n[\mathscr{L}](H;K)$ is the coefficient of $\varepsilon_1 \cdots \varepsilon_n$ in $\mathscr{L}$ at $g = g_0 + \sum_i \varepsilon_i H_i e^{iK_i\cdot x}$, which is constant in x when the momenta sum to zero.

Remark 2.3 (What the vertices are)**.** Both vertices are actual coefficients of the actions, with the factors i of the derivatives included, and one cell of the lattice and one unit of coordinate volume are the same region; so $V_n[S]$ and $V_n[\mathscr{L}]$ are comparable directly. For the lattice: the $\varepsilon_1 \cdots \varepsilon_n$ coefficient of $-\frac{1}{n!}Q_n(\sum_i \varepsilon_i h_i)$ is $-\sum_f \hat{Q}_f \prod_i h_i(f_i)$, by the symmetry of the coefficient tensor $\hat{Q}$. On the slot f_i of σ the wave has the amplitude $d_{\sigma f_i}^{\top} H_i d_{\sigma f_i}$ and the phase e^{iz_i}, $z_i = K_i \cdot m_{\sigma f_i}$, apart from the corner phase, and the part of second order in the momenta of $\prod_i e^{iz_i}$ is $\frac{1}{2}(i\sum_i z_i)^2 = -\frac{1}{2}(\sum_i z_i)^2$; so the part of second order of the coefficient is $\mathscr{V}[Q_n]$. For the density: it has two derivatives, so its coefficient is of second order in the momenta.

By (1) and (3), the n-th order part of a Euclidean simplex share of S_{R} is $-\Psi_n = -\frac{1}{n!}\sqrt{1-4\kappa}\, T_n(\kappa)$, so for $n = 2, 3, 4$ and every $\kappa < \frac{1}{4}$

$$V_n[S_{\mathrm{R}}] = \sqrt{\det g_\kappa}\; \mathscr{V}[T_n(\kappa)]. \tag{6}$$

2.5 Einstein's side

Write $\Gamma_{e,ab} = \frac{1}{2}(\partial_a g_{eb} + \partial_b g_{ea} - \partial_e g_{ab})$ for the Christoffel symbols with all indices down, and $\Gamma^c_{ab} = g^{ce}\Gamma_{e,ab}$. Einstein's first-order Lagrangian is [25]

$$\mathscr{L}_{\Gamma\Gamma} = \sqrt{|\det g|}\; g^{ab}\big(\Gamma^c_{ad}\Gamma^d_{bc} - \Gamma^c_{ab}\Gamma^d_{cd}\big) = \sqrt{|\det g|}\; g^{ab}g^{ce}g^{df}\big(\Gamma_{e,ad}\Gamma_{f,bc} - \Gamma_{e,ab}\Gamma_{f,cd}\big). \tag{7}$$

It differs from $\sqrt{|\det g|}\,R$ by a total derivative (Section 7). On plane waves, $2\Gamma_{e,ab}$ of the wave (H,K) is $i\,\hat\Gamma_{e,ab}(H,K)e^{iK\cdot x}$ with

$$\hat\Gamma_{e,ab}(H,K) = K_a H_{eb} + K_b H_{ea} - K_e H_{ab}.$$

For three symmetric matrices M, M', M'' and two waves, put

$$\begin{aligned}&\Xi(M,M',M'';\,H,K;\,H',K')\\ &\qquad = \sum M^{ab}M'^{ce}M''^{df}\big(\hat\Gamma_{e,ad}(H,K)\,\hat\Gamma_{f,bc}(H',K') - \hat\Gamma_{e,ab}(H,K)\,\hat\Gamma_{f,cd}(H',K')\big).\end{aligned} \tag{8}$$

Definition 2.4 (Einstein's vertices)**.** For a symmetric inverse background metric $\gamma = g_0^{-1}$, let $W_n(\gamma;H;K) = V_n[\mathscr{L}_{\Gamma\Gamma}]/\sqrt{|\det g_0|}$: the vertex of the first-order Lagrangian with the background volume factor removed. Expanding (7), with $(i/2)^2 = -\frac{1}{4}$ from the two Christoffel symbols,

$$\begin{aligned}W_2 &= -\tfrac{1}{4}\sum_{(j,k)}\Xi(\gamma,\gamma,\gamma;H_j,K_j;H_k,K_k),\\ W_3 &= -\tfrac{1}{4}\sum_{i}\sum_{(j,k)}M_1(H_i)\cdot\Xi(\,\cdot\,;H_j,K_j;H_k,K_k),\\ W_4 &= -\tfrac{1}{4}\sum_{\{i,j\}}\sum_{(k,l)}M_2(H_i,H_j)\cdot\Xi(\,\cdot\,;H_k,K_k;H_l,K_l).\end{aligned}$$

In W_2 the sum runs over the two orders of the two legs; in W_3 over the leg i in the metric factor and the two orders (j,k) of the other two, six terms; in W_4 over the six pairs $\{i,j\}$ in the metric factor and the two orders (k,l) of the other two, twelve terms. Here $M_1(h)$ and $M_2(h,h')$ are the coefficients of ε and of $\varepsilon\varepsilon'$ in $\sqrt{|\det g|}\,g^{ab}g^{ce}g^{df}/\sqrt{|\det g_0|}$ at $g = g_0 + \varepsilon h + \varepsilon' h'$; each is a sum of products of three symmetric matrices, and each product is contracted into the three slots of Ξ. Explicitly, with $\gamma h\gamma$ written $\hat h$,

$$\begin{aligned}M_1(h)\cdot\Xi &= \tfrac{1}{2}\operatorname{tr}(\gamma h)\,\Xi(\gamma,\gamma,\gamma) - \Xi(\hat h,\gamma,\gamma) - \Xi(\gamma,\hat h,\gamma) - \Xi(\gamma,\gamma,\hat h),\\ M_2(h,h')\cdot\Xi &= \big(\tfrac{1}{4}\operatorname{tr}(\gamma h)\operatorname{tr}(\gamma h') - \tfrac{1}{2}\operatorname{tr}(\gamma h\gamma h')\big)\Xi(\gamma,\gamma,\gamma) - \tfrac{1}{2}\operatorname{tr}(\gamma h)\,\Sigma_1(\hat h') - \tfrac{1}{2}\operatorname{tr}(\gamma h')\,\Sigma_1(\hat h)\\ &\quad + \Sigma_1\big(\gamma(h\gamma h' + h'\gamma h)\gamma\big) + \Sigma_2(\hat h,\hat h'),\end{aligned}$$

where $\Sigma_1(A)$ is the sum of the three terms with A in one slot and γ in the other two, and $\Sigma_2(A,B)$ the sum of the six terms with A and B in two different slots and γ in the third. They come from the volume factor $1 + \frac{1}{2}\tau_1 + \frac{1}{8}\tau_1^2 - \frac{1}{4}\tau_2 + \cdots$, $\tau_m = \operatorname{tr}\big((\gamma h)^m\big)$, and the inverse metric $\gamma - \gamma h\gamma + \gamma h\gamma h\gamma - \cdots$.

Every term of W_4 carries exactly five factors of γ, every term of W_3 four and every term of W_2 three: W_n is homogeneous of degree $n+1$ in γ. The quadratic vertex is the Fierz–Pauli form, $W_2(\gamma;X,X';k,-k) = -\frac{1}{2}\mathrm{FP}_\gamma(X,X';k)$ (Theorem 7.1), with

$$\begin{aligned}\mathrm{FP}_\gamma(X,X';k) &= (k\gamma k)\operatorname{tr}(\gamma X\gamma X') - 2\,k\gamma X\gamma X'\gamma k\\ &\quad + \operatorname{tr}(\gamma X')\,k\gamma X\gamma k + \operatorname{tr}(\gamma X)\,k\gamma X'\gamma k - (k\gamma k)\operatorname{tr}(\gamma X)\operatorname{tr}(\gamma X').\end{aligned} \tag{9}$$

On transverse-traceless strains, FP_γ is $(k\gamma k)\operatorname{tr}(\gamma X\gamma X')$, the graviton's dispersion.

3 The main theorem

Theorem 3.1 (Regge's action through quartic order)**.** *Let $n \in \{2,3,4\}$, let $H_1, \dots, H_n$ be real symmetric 4×4 matrices and $K_1, \dots, K_n \in \mathbb{R}^4$ with $K_1 + \cdots + K_n = 0$.*

(a) On the Euclidean Kuhn lattice $\kappa = 0$, around $g_0 = I$: $V_n[S_{\mathrm{R}}](H;K) = \frac{1}{2} V_n\big[\sqrt{\det g}\, R\big](H;K)$.

(b) On the Lorentzian Kuhn lattice $\kappa = 1$, around $g_0 = g_1 = I - \mathbf{1}\mathbf{1}^{\top}$: $V_n[S_{\mathrm{L}}](H;K) = \frac{1}{2} V_n\big[\sqrt{-\det g}\, R\big](H;K)$.

Corollary 3.2 (One half of Einstein–Hilbert)**.** *At long wavelength and through quartic order in the metric perturbation, the metric sector of Regge's action on the $\kappa = 0$ and on the $\kappa = 1$ Kuhn lattice is $\frac{1}{2}\int \sqrt{|g|}\, R\, d^4x$ in lattice units, with $\sqrt{|g|}\, R$ itself at each of the three orders: the parts of order zero and one in the momenta vanish (Lemma 4.3), and the parts of order two are those of $\frac{1}{2}\sqrt{|g|}\, R$. At $\kappa = 1$ the same holds for the whole fifteen-class lattice, the non-metric modes eliminated at tree level (Lemma 8.4); at $\kappa = 0$ it is a statement about the metric truncation (Proposition 8.3).*

Proof of Theorem 3.1. The proof has three links, each a theorem below. First, the lattice's vertices are those of its tables, with the volume factor $c = \sqrt{|\det g_\kappa|}$ in front: $V_n[S] = c\,\mathscr{V}[T_n(\kappa)]$ (Proposition 4.2, with Definition 2.2). Second, the tables' vertices are one half of Einstein's first-order vertices: $\mathscr{V}[T_n(\kappa)] = \frac{1}{2} W_n(g_\kappa^{-1})$ (Theorem 5.1 for $n = 2, 3$, Theorem 6.1 for $n = 4$). Third, the first-order vertices are those of $\sqrt{|g|}\, R$: $V_n[\sqrt{|g|}\, R] = c\, W_n(g_\kappa^{-1})$ (Theorem 7.1). Chaining them, $V_n[S] = \frac{c}{2} W_n = \frac{1}{2} V_n[\sqrt{|g|}\, R]$. □

Figure 1 draws the three links.

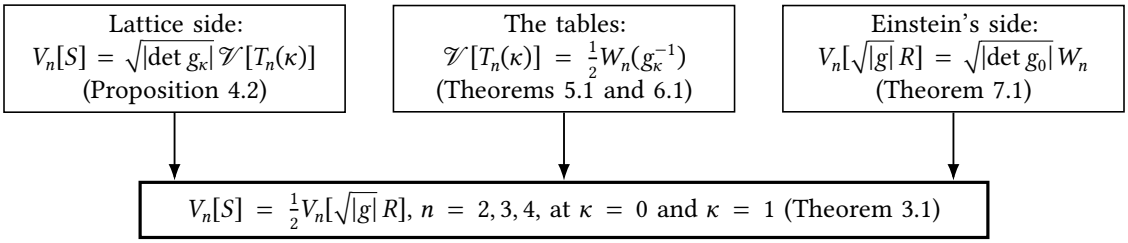

Figure 1. How Theorem 3.1 is assembled: the three equalities in the upper boxes, each proved in the section it names, chain into the main theorem. The quartic case of the middle box, with its explicit table and exact proof (Section 6), is what this paper adds.

Remark 3.3 (The factor $\frac{1}{2}$)**.** Scaling the three orders together is a normalization of the action (Section 1). Changing their ratio would change the coupling: the action $v^{-2} S_{\mathrm{R}}(\ell_\kappa + v\lambda)$ has the quadratic part of S_{R}, v times its cubic part and v^2 times its quartic part, so it is an action whose coupling is v times Regge's. Section 10 shows that, within the sixty third-derivative directions of hinge shape costs, such a change cannot be completed at quartic order (Theorem 10.3, read through the INFERENCE of Remark 10.4).

Remark 3.4 (Every flat member)**.** The three links hold on more members than the two of Theorem 3.1. Proposition 4.2 identifies the tables with the derivatives of the simplex function at every $\kappa < \frac{1}{4}$ and of the Lorentzian simplex function at every $\kappa > \frac{1}{2}$, and Theorems 5.1

and 6.1 hold at every κ off the three poles. So $V_n[S_{\mathrm{R}}] = \frac{1}{2}V_n[\sqrt{\det g}\,R]$ around every Euclidean member and $V_n[S_{\mathrm{L}}] = \frac{1}{2}V_n[\sqrt{-\det g}\,R]$ around every member with $\kappa > \frac{1}{2}$, for $n = 2, 3, 4$. The paper states the main theorem at $\kappa = 0$ and $\kappa = 1$, the two members it is about.

Remark 3.5 (The Euclidean identities from convergence)**.** On the Euclidean members the identities also follow from the convergence theorem of [19], by an argument the exact proofs of Sections 5 and 6 do not use. Fix $\kappa < \frac{1}{4}$, put the metric $g = g_\kappa + \sum_i \varepsilon_i H_i \cos(2\pi m_i \cdot x)$, $m_i \in \mathbb{Z}^4$, on the torus $\mathbb{R}^4/\mathbb{Z}^4$, and triangulate it by the Kuhn lattice of spacing $1/N$. (i) The simplices are congruent, so their fatness is bounded below uniformly in N for small ε, and [19, Thm. 5.1] applies: with geodesic edge lengths, $\sum_h A_h\delta_h \to \frac{1}{2}\int \sqrt{g}\,R$. (ii) Midpoint sampling, (4), has relative error $O(N^{-2})$ against geodesic lengths, because the squared geodesic distance between $x - v/2$ and $x + v/2$ is even in v; along the segment between the two configurations Schläfli's identity gives $dS = \sum_h \delta_h\, dA_h$ with $\delta_h = O(N^{-2})$ and $dA_h = O(N^{-4})$ on $O(N^4)$ hinges, so the action moves by $O(N^{-2})$; this is why (4) samples at the midpoint. (iii) The midpoint action is $N^{-2}\sum_y F(y/N; \varepsilon, 1/N)$, where the cell function $F(x; \varepsilon, \varrho)$ is analytic in (ε, ϱ) and $F(x; \varepsilon, 0) = 0$, since a constant metric is flat; hence it equals $NA_1(\varepsilon) + A_2(\varepsilon) + O(1/N)$. (iv) Convergence forces $A_1 = 0$ and $A_2 = \frac{1}{2}\int \sqrt{g}\,R$ on an interval of real ε, so all their Taylor coefficients agree. (v) Choose the m_i with $m_1 + \cdots + m_n = 0$ and with no other signed sum $\pm m_1 \pm \cdots \pm m_n$ equal to zero; such integer configurations are Zariski-dense in the hyperplane $\sum_i K_i = 0$. Then only the two sign patterns $\pm(1, \ldots, 1)$ contribute to the $\varepsilon_1 \cdots \varepsilon_n$ coefficient of A_2, and they contribute equally, since a vertex is quadratic in the momenta. So $V_n[S_{\mathrm{R}}] = \frac{1}{2}V_n[\sqrt{g}\,R]$ at the momenta $2\pi m_i$, hence, both sides being polynomials in the momenta on $\sum_i K_i = 0$, for all real and complex momenta, at every order n; and the parts of order zero and one vanish. With Proposition 4.2 at every $\kappa < \frac{1}{4}$, the cleared identities of Sections 5 and 6 then hold for infinitely many κ, hence as polynomials. The argument works one κ at a time, and a rule with an error of relative order N^{-1}, such as sampling at an edge's endpoint, is not covered. The Lorentzian action is outside [19]; there Proposition 4.2(d) is needed.

4 The tables as derivatives of the lattice actions

The lattice side of Theorem 3.1 rests on the tables of (3). This section says what they are and proves that they are the derivatives of Regge's action on the Euclidean members and, continued, those of the Lorentzian lattice action on the members with $\kappa > \frac{1}{2}$. The quartic table is new and is written out in supplement S1.

Definition 4.1 (The quartic table)**.** The quartic table $T_4(\kappa; \lambda)$ is the homogeneous quartic polynomial in the ten slot variables whose coefficient of the monomial μ is $N_\mu(\kappa)/D(\kappa)$, with

$$D(\kappa) = 1536\,(2\kappa - 1)^3(3\kappa - 1)^3(4\kappa - 1)^3$$

and integer polynomials N_μ of degree at most 9, listed in supplement S1. Of the 715 quartic monomials in ten variables, 691 have a coefficient that is not identically zero; at $\kappa = 0$, $\frac{1}{10}$ and 1 the numbers of nonzero coefficients are 500, 691 and 629.

The table is the outcome of an exact symbolic computation over $\mathbb{Q}(\kappa)$ of the fourth derivative (2) of the canonical simplex's function Ψ at ℓ_κ, with the factor $\sqrt{1 - 4\kappa}$ removed; Proposition 4.2(c)

states what that computation proves. As a definition it is just data, and the theorems below are about this explicit polynomial. Every coefficient has poles of order at most three at $\kappa = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and nowhere else, the values at which a class of hinges degenerates.

Proposition 4.2 (The tables). *(a)* Quadratic. *For every $\kappa < \frac{1}{4}$ the Hessian of Ψ at ℓ_κ is $-\sqrt{1-4\kappa}\,\Omega(\kappa)$, where $\Omega(\kappa) = B_0 + B_1/(\kappa - \frac{1}{4}) + B_2/(\kappa - \frac{1}{3}) + B_3/(\kappa - \frac{1}{2})$ for four rational 10×10 matrices B_j; $48\,\Omega(0)$ and $48\,\Omega(1)$ are integer matrices.*

(b) Cubic. *For every $\kappa < \frac{1}{4}$ and every λ, $\Psi''' = \sqrt{1-4\kappa}\,T_3(\kappa;\lambda)$, where the 220 coefficients of $T_3(\kappa)$ are rational functions of κ with poles only at $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$.*

(c) Quartic. *For every $\kappa < \frac{1}{4}$ and every λ, $\Psi^{(4)} = \sqrt{1-4\kappa}\,T_4(\kappa;\lambda)$. At $\kappa = \frac{1}{10}$ the factor is $\sqrt{3/5}$.*

(d) The Lorentzian action. *For every $\kappa > \frac{1}{2}$, every λ and $n = 2, 3, 4$, the n-th derivative of the Lorentzian simplex function Ψ_{L} at ℓ_κ along λ is $\sqrt{4\kappa-1}\,T_n(\kappa;\lambda)$, with $T_2(\kappa;\lambda) = -\lambda^{\top}\Omega(\kappa)\lambda$. At $\kappa = 1$ a simplex share $-\Psi_{\mathrm{L}}$ of S_{L} has the flat Hessian $\sqrt{3}\,\Omega(1) = (16\sqrt{3})^{-1} \cdot 48\,\Omega(1)$ and the third and fourth derivatives $-\sqrt{3}\,T_3(1;\lambda)$ and $-\sqrt{3}\,T_4(1;\lambda)$.*

Consequently, for $n = 2, 3, 4$, $V_n[S_{\mathrm{R}}] = \sqrt{\det g_\kappa}\,\mathscr{V}[T_n(\kappa)]$ at every $\kappa < \frac{1}{4}$ and $V_n[S_{\mathrm{L}}] = \sqrt{|\det g_\kappa|}\,\mathscr{V}[T_n(\kappa)]$ at every $\kappa > \frac{1}{2}$; at $\kappa = 1$ the factor is $\sqrt{3}$.

Proof. The derivatives (2) need the first three derivatives of the areas and dihedral angles along a line of squared lengths, and all of them have closed forms. A triangle's area is $\frac{1}{4}\sqrt{\pm Q_h}$, with Q_h Heron's quadratic in its squared lengths. Let b_{lm} be the inner product of the gradients of the barycentric coordinates of the vertices v_l and v_m: with G the Gram matrix of the edge vectors from v_0, $(b_{lm}) = NG^{-1}N^{\top}$ for a fixed integer 5×4 matrix N, and along the line $db/dr = \frac{1}{2}\,b\Lambda b$, with Λ the symmetric matrix of λ. The two gradients of v_l and v_m are normal to the triangle opposite l and m, so its dihedral angle has $\cos\theta_h = -b_{lm}/\sqrt{b_{ll}b_{mm}}$. At the flat simplex $b = (1-4\kappa)^{-1}\hat{b}$ with $\hat{b}$ of degree one in κ, and Jacobi's identity for complementary minors gives $\hat{b}_{ll}\hat{b}_{mm} - \hat{b}_{lm}^2 = (1-4\kappa)Q_h/4$ for the triangle opposite l, m. So the sine of each dihedral angle carries $\sqrt{\pm(1-4\kappa)}\sqrt{\pm Q_h}$, the $\sqrt{\pm Q_h}$ cancels against the area's, and each triangle's contribution to $\Psi^{(n)}$ is $\sqrt{1-4\kappa}$, or $\sqrt{4\kappa-1}$ on the Lorentzian side, times a polynomial in λ with coefficients rational in κ. On the Lorentzian side the same formulas hold in the spacelike plane normal to the hinge, where the two dual normals have a positive definite Gram matrix. Adding the ten triangles gives, for each monomial of λ, a rational function of κ, and the proposition says that these are the coefficients of $T_n(\kappa)$: 55, 220 and 715 identities of rational functions of κ for $n = 2, 3, 4$, the same on both sides of the light cone. Supplement S2 carries out this computation in exact arithmetic with κ kept as a variable, checks Schläfli's identity along the line to the order used, and finds every identity true; at $\kappa = 1$ it repeats the Lorentzian computation in exact arithmetic with $\sqrt{3}$. The only poles, $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$, lie outside both ranges, so the identities hold at every $\kappa < \frac{1}{4}$ and every $\kappa > \frac{1}{2}$. The last sentence is Definition 2.2: by (1) the n-th order part of a simplex share is $-\frac{1}{n!}\sqrt{1-4\kappa}\,T_n(\kappa)$ on the Euclidean side and, by (d), $-\frac{1}{n!}\sqrt{4\kappa-1}\,T_n(\kappa)$ on the Lorentzian side. □

The flat Hessian of (d) is the matrix printed in [3, App. B], and the long-wave quadratic and cubic vertices it gives agree with those reported in [3, Secs. 4.4, 4.10], [5, Thm. 3.1] and [6, Thm. 6.1]; these are comparisons, not steps of the proof.

The long-wave vertex of Definition 2.2 is the part of second order in the momenta of a cell's coefficient. Reading the vertex identities as statements about the action (Corollary 3.2) needs the parts of order zero and one to vanish as well; otherwise the long-wave action would contain terms with fewer than two derivatives. They do vanish, for all three tables and every κ. For a table T of degree n with coefficient tensor $\hat{T}$ and n waves (H_i, K_i), write

$$\mathscr{X}[T](H;K) = \sum_{\sigma} \sum_{f_1,\dots,f_n} \hat{T}_{f_1\dots f_n} \prod_{i=1}^{n} \big(d_{\sigma f_i}^{\top} H_i d_{\sigma f_i}\big)\, e^{iK_i\cdot m_{\sigma f_i}} \tag{10}$$

for the coefficient of $\varepsilon_1 \cdots \varepsilon_n$ in the table's sum over the 24 simplices of a cell, without the corner phase, and $\mathscr{X}^{(j)}[T]$ for its part of order j in the momenta. By Remark 2.3, $\mathscr{X}^{(2)}[T] = -\mathscr{V}[T]$.

Lemma 4.3 (The lower orders vanish)**.** *Let $n \in \{2,3,4\}$ and $(2\kappa-1)(3\kappa-1)(4\kappa-1) \neq 0$. For all symmetric $H_1,\dots,H_n$ and all $K_1,\dots,K_n \in \mathbb{C}^4$,*

$$\mathscr{X}^{(0)}[T_n(\kappa)](H) = 0 \qquad \text{and} \qquad \mathscr{X}^{(1)}[T_n(\kappa)](H;K) = 0.$$

So the quadratic, cubic and quartic terms of the long-wave expansion of S_{R} ($\kappa < \frac{1}{4}$) and of S_{L} ($\kappa > \frac{1}{2}$) start at two derivatives, as Einstein's do.

Proof. Order zero: constant strains are flat. The coefficient of $\varepsilon_1 \cdots \varepsilon_n$ in the cell sum $\sum_\sigma T(\lambda_\sigma)$ at the constant strain $\lambda_{\sigma f} = d_{\sigma f}^{\top} X d_{\sigma f}$, $X = \sum_i \varepsilon_i H_i$, is $n!\,\mathscr{X}^{(0)}[T](H)$; so it suffices that the cell sum vanishes on every constant strain. Let first $\kappa < \frac{1}{4}$. For a small symmetric X the configuration $\ell_X(y,d) = d^{\top}(g_\kappa + X)d$ is the Kuhn lattice of the constant metric $g_\kappa + X$: its simplices are flat simplices of that metric, and the simplices around each hinge fill a neighbourhood of it, so every deficit vanishes. On configurations that are the same in every cell, the action of one cell is $-\sum_\sigma \Psi_\sigma = \sum_h A_h \delta_h$, the second sum running over one hinge of each translation class; this is (1) regrouped by hinges. So the action of a cell vanishes at every ℓ_X. By Proposition 4.2 its part of order n in X is $-\frac{1}{n!}\sqrt{1-4\kappa}\sum_\sigma T_n(\kappa;\lambda_\sigma)$, which therefore vanishes for every X. For the other values of κ, $\mathscr{X}^{(0)}[T_n(\kappa)](H)$ is a rational function of κ that vanishes for every $\kappa < \frac{1}{4}$, so it vanishes identically.

Order one: the central inversion. The map $x \mapsto \mathbf{1} - x$ sends the unit cell to itself and is an isometry of every g_κ. It sends the simplex of axis order $(\pi_1,\pi_2,\pi_3,\pi_4)$ to the simplex of order $(\pi_4,\pi_3,\pi_2,\pi_1)$ and the vertex v_j of the first to the vertex v_{4-j} of the second, so it sends slot jk to slot $\iota(jk) = (4-k)(4-j)$: the permutation ι exchanges $01 \leftrightarrow 34$, $02 \leftrightarrow 24$, $03 \leftrightarrow 14$, $12 \leftrightarrow 23$ and fixes 04 and 13. Each step is kept, and each midpoint m goes to $\mathbf{1} - m$. Renumbering the vertices of a simplex by $j \mapsto 4-j$ does not change its geometry, and ℓ_κ on slot jk depends only on $k-j$, which ι keeps; so $\Psi(\ell\circ\iota) = \Psi(\ell)$, and every table is invariant, $\hat{T}_{\iota(f_1)\dots\iota(f_n)} = \hat{T}_{f_1\dots f_n}$, for $\kappa < \frac{1}{4}$ and then for every κ off the poles by rationality. Summing (10) over the images of the simplices and slots instead of the simplices and slots themselves gives

$$\mathscr{X}[T](H;K) = e^{i(K_1+\cdots+K_n)\cdot\mathbf{1}}\,\mathscr{X}[T](H;-K),$$

and the parts of order one of the two sides give $2\,\mathscr{X}^{(1)}[T](H;K) = i\,(K_1+\cdots+K_n)\cdot\mathbf{1}\;\mathscr{X}^{(0)}[T](H)$, which vanishes by the first part. □

Supplement S3 checks both statements exactly for T_2, T_3 and T_4 on all basis polarizations, at $\kappa = 0, \frac{1}{10}, 1$ and, with κ kept as a variable, at every κ off the poles, together with the invariance of the tables under ι.

5 Quadratic and cubic order

Theorem 5.1 (Quadratic and cubic vertices of the tables)**.** *Let $\mathbb{F}$ be a field of characteristic zero and $\kappa \in \mathbb{F}$ with $(2\kappa-1)(3\kappa-1)(4\kappa-1) \neq 0$, and put $\gamma = g_\kappa^{-1}$. Let $X, X' \in \mathbb{F}^{4\times 4}$ be symmetric and $k \in \mathbb{F}^4$; let $H_1, H_2, H_3 \in \mathbb{F}^{4\times 4}$ be symmetric and $K_1, K_2, K_3 \in \mathbb{F}^4$ with $K_1 + K_2 + K_3 = 0$. Then*

$$\mathscr{V}[T_2(\kappa)](X, X'; k, -k) = -\tfrac{1}{4}\mathrm{FP}_\gamma(X, X'; k) = \tfrac{1}{2}W_2(\gamma; X, X'; k, -k),$$
$$\mathscr{V}[T_3(\kappa)](H; K) = \tfrac{1}{2}W_3(\gamma; H; K).$$

In particular this holds at $\kappa = 0$, with $\gamma = I$, and at $\kappa = 1$, with $\gamma = \gamma_1 = g_1^{-1} = I - \frac{1}{3}\mathbf{1}\mathbf{1}^\top$, over $\mathbb{R}$ and over $\mathbb{C}$.

In the normalization of Definition 2.2 the quadratic statement at $\kappa = 1$ reads

$$V_2[S_{\mathrm{L}}](X, X'; k, -k) = -\tfrac{\sqrt{3}}{4}\mathrm{FP}_{\gamma_1}(X, X'; k).$$

For cosine waves of integer wave m on the torus of side N, so that $k = 2\pi m/N$, this is the limit $-\frac{\sqrt{3}\pi^2}{2}\mathrm{FP}_{\gamma_1}(X, X'; m)$ of N^{-2} times the flat Hessian reported in [5, Thm. 3.1(ii)] and [6, Thm. 6.1], where FP_{g_1} stands for our FP_{γ_1}; the cubic statement agrees with the direct computation on the Lorentzian simplex in [5, Thm. 3.1(iii)]. At $\kappa = 0$ the quadratic statement is the classical one of [11, 12], here with midpoint sampling and in the metric sector.

Proof. Quadratic. By (5) the left side is $-\frac{1}{2}\sum_\sigma \sum_{f,g} \Omega(\kappa)_{fg}\,(d_{\sigma f}^\top X d_{\sigma f})(d_{\sigma g}^\top X' d_{\sigma g})\big(k\cdot(m_{\sigma f} - m_{\sigma g})\big)^2$. Both sides are bilinear in X and X', so it suffices to take each among the ten symmetric basis matrices E_{ab}: E_{aa} has a single 1 at (a,a), and E_{ab} for $a<b$ has 1 at (a,b) and (b,a). With $D_2(\kappa) = (1-4\kappa)(1-3\kappa)(1-2\kappa)$ the matrix $D_2\Omega$ is polynomial in κ, and FP_γ is homogeneous of degree three in γ, so $(1-4\kappa)^3\mathrm{FP}_{g_\kappa^{-1}} = \mathrm{FP}_{(1-4\kappa)I+\kappa\mathbf{1}\mathbf{1}^\top}$. On the pair (E_p, E_q) the first equality is therefore

$$-4(1-4\kappa)^3\,\mathscr{V}[D_2T_2(\kappa)](E_p, E_q; k, -k) = D_2(\kappa)\,\mathrm{FP}_{(1-4\kappa)I+\kappa\mathbf{1}\mathbf{1}^\top}(E_p, E_q; k),$$

a polynomial identity in κ and the four components of k with rational coefficients. Supplement S3 verifies it by exact expansion on all 100 ordered pairs. The second equality is the Fierz–Pauli form of W_2 (Theorem 7.1).

Cubic. The difference $\Delta = \mathscr{V}[T_3(\kappa)] - \frac{1}{2}W_3(\gamma)$ is linear in each leg, unchanged when the three legs are permuted together with their momenta, and invariant under the 24 relabellings of the axes, because a relabelling permutes the simplices of a cell slot by slot and fixes g_κ and so γ. These 144 symmetries split the 1000 ordered triples of basis matrices into 20 orbits, and Δ vanishes for all symmetric legs once it vanishes, for all conserved momenta, on one triple of each orbit. On a representative, put $K_3 = -K_1 - K_2$, so that the eight components y of K_1, K_2 are free. With $D_3(\kappa) = (1-4\kappa)^2(1-3\kappa)^2(1-2\kappa)^2$ the table D_3T_3 is polynomial in κ, W_3 is homogeneous of degree four in γ, and $\Delta = 0$ multiplied by $2(1-4\kappa)^4 D_3(\kappa)$ is

$$2(1-4\kappa)^4\,\mathscr{V}[D_3T_3(\kappa)] = D_3(\kappa)\,W_3\big((1-4\kappa)I + \kappa\mathbf{1}\mathbf{1}^\top\big),$$

a polynomial identity in κ and y with rational coefficients. Supplement S3 verifies the twenty identities by exact expansion, and, as a check that uses neither symmetry, the identity on all 1000 ordered triples at $\kappa = 0$ and at $\kappa = 1$. Polynomial identities with rational coefficients hold over every field of characteristic zero. □

6 The quartic identity

Theorem 6.1 (The quartic vertex at every κ)**.** *Let $\mathbb{F}$ be a field of characteristic zero and $\kappa \in \mathbb{F}$ with $(2\kappa-1)(3\kappa-1)(4\kappa-1) \neq 0$. For all symmetric $H_1, \dots, H_4 \in \mathbb{F}^{4\times 4}$ and all $K_1, \dots, K_4 \in \mathbb{F}^4$ with $K_1 + K_2 + K_3 + K_4 = 0$,*

$$\mathscr{V}[T_4(\kappa)](H;K) = \tfrac{1}{2}\, W_4\big(g_\kappa^{-1}; H; K\big), \qquad g_\kappa^{-1} = I + \tfrac{\kappa}{1-4\kappa}\, \mathbf{1}\mathbf{1}^\top.$$

Both sides are defined over any field: $\mathscr{V}$ by (5) and W_4 by Definition 2.4, which are polynomial formulas. Over $\mathbb{R}$ the theorem covers the Euclidean backgrounds $\kappa < \frac{1}{4}$ and the Lorentzian ones $\kappa > \frac{1}{4}$ off $\frac{1}{3}, \frac{1}{2}$; over $\mathbb{C}$ it covers complex momenta, which the gauge tests of Section 9 need. The identity fails in general when the momenta do not sum to zero: at $\kappa = 0$, with the legs $(E_{11}, E_{11}, E_{11}, E_{23})$ and $K_1 = e_2$, $K_2 = e_3$, $K_3 = K_4 = 0$, $\mathscr{V}[T_4(0)] = -\frac{3}{8}$ while $\frac{1}{2}W_4(I) = 0$ (supplement S3). It is a statement about a contact vertex, not about the two Lagrangians pointwise.

Corollary 6.2. *(a) At $\kappa = 0$ and $\kappa = 1$, over $\mathbb{R}$ and over $\mathbb{C}$: $\mathscr{V}[T_4(0)] = \frac{1}{2}W_4(I)$ and $\mathscr{V}[T_4(1)] = \frac{1}{2}W_4(\gamma_1)$.*

(b) For every real $\kappa < \frac{1}{4}$, $V_4[S_{\mathrm{R}}] = \frac{1}{2}V_4[\sqrt{\det g}\, R]$ around g_κ, and for every real $\kappa > \frac{1}{2}$, $V_4[S_{\mathrm{L}}] = \frac{1}{2}V_4[\sqrt{-\det g}\, R]$ around g_κ.

Part (b) is Theorem 6.1 with Proposition 4.2 and Theorem 7.1.

Proof of Theorem 6.1. Write $\Delta(H;K) = \mathscr{V}[T_4(\kappa)](H;K) - \frac{1}{2}W_4(g_\kappa^{-1}; H; K)$.

Step 1: symmetries. (i) Δ is linear in each polarization: (5) is a product of one factor $d^\top H_i d$ per leg, and W_4 is multilinear by definition. (ii) Δ is unchanged when the four legs are permuted, together with their momenta: (5) because the coefficient tensor is symmetric, W_4 because it is a coefficient of $\varepsilon_1 \cdots \varepsilon_4$. (iii) Δ is unchanged by a relabelling of the axes. A permutation ρ of the axes, with permutation matrix P, sends the simplex of axis order π to that of order $\rho \circ \pi$, slot by slot, with steps and midpoints Pd and Pm; so $\mathscr{V}[T_4](PH_iP^\top; PK_i) = \mathscr{V}[T_4](H_i; K_i)$. On Einstein's side $W_4(\gamma; PHP^\top; PK) = W_4(P^\top\gamma P; H; K)$, and $P^\top g_\kappa^{-1} P = g_\kappa^{-1}$.

Step 2: fifty-three representatives. By (i), Δ vanishes for all symmetric polarizations once it vanishes whenever every H_i is a basis matrix E_{ab}. Leg permutations and the 24 axis relabellings act on the 10^4 ordered quadruples of basis matrices, and every quadruple lies in the orbit of one of 53 representatives (Appendix A, Table 1). Because (ii) and (iii) move the momenta along with the legs and the claim is for all conserved momenta, it suffices to prove $\Delta = 0$ on the 53 representatives, for all K with $K_1 + \cdots + K_4 = 0$.

Step 3: one polynomial identity per representative. Fix a representative and put $K_4 = -K_1 - K_2 - K_3$; the twelve components y of K_1, K_2, K_3 are free. By (5) and Definition 4.1, $D(\kappa)\,\mathscr{V}[T_4(\kappa)]$ is a polynomial in κ of degree at most 9 and a quadratic form in y, with rational coefficients. On Einstein's side, W_4 is homogeneous of degree five in γ, so

$$(1-4\kappa)^5\, W_4\big(g_\kappa^{-1}\big) = W_4\big((1-4\kappa)I + \kappa\mathbf{1}\mathbf{1}^{\top}\big),$$

a polynomial in κ of degree at most 5, again quadratic in y. Multiplying Δ by $2(1-4\kappa)^5 D(\kappa)$, which is nonzero under the hypothesis, $\Delta = 0$ on the representative is equivalent to

$$2(1-4\kappa)^5 \cdot D(\kappa)\mathscr{V}[T_4(\kappa)] = D(\kappa)\cdot W_4\big((1-4\kappa)I + \kappa\mathbf{1}\mathbf{1}^{\top}\big) \tag{11}$$

as polynomials in κ and y: an identity between two polynomials with rational coefficients, of degree at most 14 in κ and two in y. If it holds over $\mathbb{Q}$ it holds over every field of characteristic zero.

Step 4: the check. For each of the 53 representatives, both sides of (11) are computed from their definitions as polynomials in exact rational arithmetic, the left from the 24 simplices of a cell, the table's integer numerators and the slots' steps and midpoints, the right from the six pairs of legs that carry the metric factor and the two orders of the Christoffel legs; supplement S4 does this, and finds the two sides equal on all 53. It also checks (11) on all 715 unordered quadruples of basis matrices, which does not use the relabelling symmetry. □

7 From the first-order Lagrangian to $\sqrt{|g|}\,R$

Einstein's first-order Lagrangian (7) is the form in which the lattice identities are proved, because it has only first derivatives of the metric. It is classical that it differs from $\sqrt{|g|}\,R$ by a total derivative [25], and the vertex statement below follows from that in a few lines. We include it because the chain of Theorem 3.1 uses it in this form, at orders two, three and four, on every background.

Theorem 7.1 (The vertices of $\sqrt{|g|}\,R$)**.** *Let $\chi \in \{1,-1\}$ and let g_0 be a real symmetric 4×4 matrix with $\chi \det g_0 > 0$; put $\gamma = g_0^{-1}$. Let $n \in \{2,3,4\}$, let $H_1,\dots,H_n$ be real symmetric and $K_1,\dots,K_n \in \mathbb{R}^4$ with $K_1+\cdots+K_n = 0$. Then*

$$V_n\big[\sqrt{\chi\det g}\;R\big](H;K) = \sqrt{\chi\det g_0}\;W_n(\gamma;H;K),$$

where R is the scalar curvature of g written by its coordinate formula. For $n=2$ the vertex is the Fierz–Pauli form:

$$W_2(\gamma;X,X';k,-k) = -\tfrac{1}{2}\mathrm{FP}_\gamma(X,X';k).$$

With $\chi = 1$ this is the Euclidean density $\sqrt{\det g}\,R$, and with $\chi = -1$ the Lorentzian density $\sqrt{-\det g}\,R$. On the backgrounds of the lattice family, $\det g_\kappa = 1-4\kappa$.

Proof. Step 1: the total derivative, in any commutative ring. Let A be a commutative ring containing $\frac{1}{2}$, with four derivations ∂_c, and let g^{ab} (symmetric), $\partial_c g_{ab}$ (symmetric in a,b), $\partial_c\partial_d g_{ab}$ and v be elements of A with

$$\partial_c g^{de} = -g^{da}(\partial_c g_{ab})g^{be}, \qquad \partial_c(\partial_d g_{ab}) = \partial_c\partial_d g_{ab}, \qquad \partial_c v = v\Gamma^{e}_{ec},$$

where $\Gamma^c_{ab} = \frac{1}{2}g^{ce}(\partial_a g_{eb} + \partial_b g_{ea} - \partial_e g_{ab})$. Then

$$\nu R = \nu g^{ab}\big(\Gamma^c_{ad}\Gamma^d_{bc} - \Gamma^c_{ab}\Gamma^d_{cd}\big) + \sum_c \partial_c j^c, \qquad j^c = \nu\big(g^{bd}\Gamma^c_{bd} - g^{bc}\Gamma^a_{ba}\big),$$

with $R = g^{ab}(\partial_c\Gamma^c_{ab} - \partial_b\Gamma^c_{ac} + \Gamma^c_{cd}\Gamma^d_{ab} - \Gamma^c_{bd}\Gamma^d_{ac})$ and every $\partial\Gamma$ expanded by the rules above. This is the classical identity [25, §93], here as an identity in the ring: expand $\partial_c j^c$ by the Leibniz rule and the three rules, and the terms with second derivatives of g and the $\Gamma\Gamma$ terms match.

Step 2: plane waves as jets. Let $J = \mathbb{R}[\varepsilon_1, \dots, \varepsilon_4]/(\varepsilon_1^2, \dots, \varepsilon_4^2)$; write $\mathrm{top}(x)$ for the coefficient of $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4$. The metric $g = g_0 + \sum_k \varepsilon_k H_k e^{iK_k\cdot x}$ and its first and second derivatives at $x = 0$ are the jets $g_0 + \sum_k \varepsilon_k H_k$, $\sum_k \varepsilon_k (iK_{kc})H_k$ and $\sum_k \varepsilon_k (iK_{kc})(iK_{kd})H_k$ (complexify J for the factors i). Taking the multilinear part of the Taylor expansion in $\varepsilon_1, \dots, \varepsilon_4$ is a ring homomorphism from functions smooth near $\varepsilon = 0$ into J; so the jets of $\sqrt{\chi \det g}$ and of g^{-1} are the unique $\nu \in J$ with $\nu^2 = \chi \det g$ and constant term $\sqrt{\chi \det g_0}$, and the inverse of the jet metric. Differentiation in x acts on these jets as the derivation $\partial_c \varepsilon_k = iK_{kc}\varepsilon_k$ of J. It satisfies the three rules of step 1: the first because $g\,g^{-1} = 1$, and the third by Jacobi's formula for a derivation, $\partial_c \det g = \det g\ \mathrm{tr}(g^{-1}\partial_c g)$, since ν is a unit of J. So $V_4[\sqrt{\chi \det g}\, R] = \mathrm{top}(\nu R)$, and by step 1

$$\mathrm{top}(\nu R) = \mathrm{top}\big(\nu g^{ab}(\Gamma\Gamma - \Gamma\Gamma)\big) + \sum_c \mathrm{top}(\partial_c j^c).$$

The derivation multiplies every monomial ε_S by $i\sum_{k\in S} K_{kc}$, so $\mathrm{top}(\partial_c j^c) = i(K_{1c} + K_{2c} + K_{3c} + K_{4c})\,\mathrm{top}(j^c) = 0$ when the momenta are conserved.

Step 3: the first-order Lagrangian. The remaining term is $V_4[\mathscr{L}_{\Gamma\Gamma}] = \sqrt{\chi \det g_0}\, W_4(\gamma)$ by Definition 2.4: each Γ is of first order in the ε's, so the top coefficient takes one ε from each of the two Christoffel factors and the other two from the polarized second-order part of $\nu g^{ab} g^{ce} g^{df}$. For $n = 3$ and $n = 2$, set the unused polarizations to zero and read the coefficients of $\varepsilon_1\varepsilon_2\varepsilon_3$ and $\varepsilon_1\varepsilon_2$; the argument is the same, with top replaced by these coefficients. Finally, $W_2(\gamma; X, X'; k, -k) = -\frac{1}{2}\mathrm{FP}_\gamma(X, X'; k)$ for every symmetric γ. Both sides are polynomials in the entries of γ, so it suffices to take γ invertible over $\mathbb{C}$. Both are full contractions with γ, hence covariant: $W_2(\gamma; EXE^\top, EX'E^\top; Ek, -Ek) = W_2(E^\top\gamma E; X, X'; k, -k)$, and likewise for FP. An invertible complex symmetric γ has $E^\top\gamma E = I$ for some invertible E, and at $\gamma = I$ the identity is checked by exact expansion in supplement S3. □

The proof uses no tables and no case analysis: the identity holds on every background at once, and the background enters only through $\chi \det g_0 > 0$, which makes the square root and the inverse exist. Supplement S5 checks the total-derivative step in the jet algebra on random rational backgrounds of both signatures, and S3 the Fierz–Pauli form at $\gamma = g_\kappa^{-1}$ for every κ.

8 The other five edge modes

The lattice has fifteen edge classes, so a plane wave on it has fifteen amplitudes, one for each class, and the metric waves (4) form a ten-dimensional family of them. The vertices of Theorem 3.1 are those of this family, the metric sector. This section asks what the other five amplitudes do. At $\kappa = 1$ and $\kappa = \frac{1}{10}$ they are nondegenerate at zero momentum, so they can be eliminated at tree level, and eliminating them changes no vertex at two derivatives

(Lemma 8.4): there the vertices of the metric sector are those of the whole lattice. At $\kappa = 0$ one of them, the body diagonal of the cube, has no quadratic term at any momentum and enters the action at cubic order (Proposition 8.3); there the theorems describe the metric truncation.

Definition 8.1 (Edge fields and the cell form)**.** An *edge field* $w = (w_d)$ gives each of the fifteen classes $d \in \{0,1\}^4 \setminus \{0\}$ an amplitude $w_d \in \mathbb{C}$; its wave of momentum K changes the squared length of the edge (y,d) by $w_d e^{iK\cdot(y+d/2)}$. The *metric field* of a symmetric matrix H is $\Phi(H)_d = d^\top H d$, so the wave (4) is the edge field $\Phi(H)$. For a table T of degree n with coefficient tensor $\hat{T}$ and n edge fields w_i with momenta K_i, the *cell vertex* is

$$\mathcal{X}[T](w;K) = \sum_\sigma \sum_{f_1,\dots,f_n} \hat{T}_{f_1\dots f_n} \prod_{i=1}^{n} (w_i)_{d_{\sigma f_i}}\, e^{iK_i\cdot m_{\sigma f_i}},$$

with $\mathcal{X}^{(j)}[T]$ its part of order j in the momenta; on metric fields it is (10), $\mathcal{X}[T](\Phi(H);K) = \mathcal{X}[T](H;K)$. The *cell form* at momentum K is the 15×15 matrix

$$M(K)_{dd'} = \sum_\sigma \sum_{f,g} \Omega(\kappa)_{fg}\,[d_{\sigma f} = d]\,[d_{\sigma g} = d']\, e^{iK\cdot(m_{\sigma f} - m_{\sigma g})},$$

so that $w^\top M(K)\, w' = -\mathcal{X}[T_2(\kappa)](w,w';K,-K)$. We write $M^{(2)}(K)$ for its part of order two in K, and $M^\perp(\kappa)$ for the restriction of $M(0)$ to the five classes with three or four steps, 0111, 1011, 1101, 1110 and 1111.

Two facts fix what these objects mean. First, for every $\kappa < \frac{1}{4}$ (Regge's action) and every $\kappa > \frac{1}{2}$ (S_{L}), and for $n = 2,3,4$, the coefficient of $\varepsilon_1 \cdots \varepsilon_n$ in the action of one cell when the edge (y,d) is changed by $\sum_i \varepsilon_i (w_i)_d e^{iK_i\cdot(y+d/2)}$, with $K_1 + \cdots + K_n = 0$, is $-\sqrt{|1-4\kappa|}\,\mathcal{X}[T_n(\kappa)](w;K)$, because the part of order n of a simplex share is $-\frac{1}{n!}\sqrt{|1-4\kappa|}\,T_n(\kappa)$ (Proposition 4.2). In particular the coefficient of $\varepsilon\varepsilon'$ for the waves (w,K) and $(w',-K)$ is $\sqrt{|1-4\kappa|}\,w^\top M(K)\,w'$: up to the positive factor $\sqrt{|1-4\kappa|}$, $M(K)$ is the matrix of the action's quadratic form on waves of momentum K, and for real K it has that form's inertia. Second, Φ is injective, and on the ten classes with one or two steps it is invertible ($\Phi(H)_{e_a} = H_{aa}$, $\Phi(H)_{e_a+e_b} = H_{aa} + H_{bb} + 2H_{ab}$); so every edge field is $\Phi(H) + w^\perp$ for exactly one H and one $w^\perp$ supported on the five classes with three or four steps. We call $w^\perp$ its *non-metric part*. Since $M(0)$ annihilates the metric fields (Lemma 8.2), $(\Phi(H)+w^\perp)^\top M(0)\,(\Phi(H')+w'^\perp) = w^{\perp\top} M^\perp(\kappa)\, w'^\perp$: $M^\perp(\kappa)$ is the form of the five non-metric modes at zero momentum.

The lattice's gauge modes are edge fields too. Moving every lattice point x by $\hat{\zeta} e^{iK\cdot x}$, with $\hat{\zeta} = \gamma\zeta$ and $\gamma = g_\kappa^{-1}$, changes the squared length of the edge (y,d), to first order, by $4i\,(d\cdot\zeta)\sin(K\cdot d/2)\,e^{iK\cdot(y+d/2)}$. We call these four edge fields, $\zeta \in \mathbb{C}^4$, the *displacement modes*. They are null vectors of $M(K)$ at every K, because displacing the points of a flat lattice keeps it flat; supplement S6 checks this exactly at $\kappa = 0$, $\frac{1}{10}$ and 1, as identities in the phases. To first order in K the displacement mode is $i\,\Phi(K\otimes\zeta + \zeta\otimes K)$, i times the metric field of the pure-gauge polarization of Section 9.

Lemma 8.2 (Parity and flatness)**.** *Let* $(2\kappa-1)(3\kappa-1)(4\kappa-1) \neq 0$ *and* $n \in \{2,3,4\}$.

(a) Parity. *For every* $K \in \mathbb{C}^4$, $M(K)$ *is symmetric and* $M(-K) = M(K)$*; for real* K *it is real. For all edge fields* $w_1,\dots,w_n$ *and all momenta with* $K_1 + \cdots + K_n = 0$, $\mathcal{X}[T_n(\kappa)](w;-K) = \mathcal{X}[T_n(\kappa)](w;K)$*; in particular* $\mathcal{X}^{(1)}[T_n(\kappa)](w;K) = 0$.

(b) Flatness. *For all symmetric $H_1, \dots, H_{n-1}$ and every edge field w,*

$$\mathscr{X}^{(0)}[T_n(\kappa)]\big(\Phi(H_1), \dots, \Phi(H_{n-1}), w\big) = 0.$$

For $n = 2$ this says $M(0)\Phi(H) = 0$ for every symmetric H.

Consequently $M(K)\Phi(H)$ is of second order in K, and a cubic or quartic cell vertex whose legs are all metric fields but one is of second order in the momenta, whatever the last leg.

Proof. (a) This is the inversion argument of Lemma 4.3, which nowhere uses that the legs are metric: since the table is invariant under ι, $\mathscr{X}[T_n](w; K) = e^{i(K_1+\cdots+K_n)\cdot\mathbf{1}}\,\mathscr{X}[T_n](w; -K)$ for all edge fields. For $n = 2$ and the momenta $K, -K$ this gives $M(-K) = M(K)$. Exchanging f and g in the definition gives $M(K)^\top = M(-K)$, and complex conjugation gives $\overline{M(K)} = M(-\bar{K})$; so $M(K)$ is symmetric, and real for real K.

(b) This is the flatness argument of Lemma 4.3, with one leg free. Let first $\kappa < \frac{1}{4}$, and let w change the squared lengths in the same way in every cell. At every ℓ_X the derivative of the action of a cell along w is $\sum_h \delta_h\, \partial_w A_h$, by Schläfli's identity, and it vanishes because every $\delta_h(\ell_X)$ does. By Proposition 4.2 the parts of this derivative of degree one, two and three in X are $-\sqrt{1-4\kappa}\,\mathscr{X}^{(0)}[T_k(\kappa)]\big(\Phi(X), \dots, \Phi(X), w\big)/(k-1)!$ for $k = 2, 3, 4$; each vanishes, and polarizing in X gives (b) for $\kappa < \frac{1}{4}$. Rationality in κ gives it at every κ off the poles. For $n = 2$, $\mathscr{X}^{(0)}[T_2](\Phi(H), w) = -\Phi(H)^\top M(0)\, w$.

Finally, by (a) and (b), $M(K)\Phi(H) = M(0)\Phi(H) + O(K^2) = O(K^2)$, and a cell vertex whose legs are all metric but one has vanishing parts of orders zero and one. □

Supplement S6 checks (a) and (b) exactly at $\kappa = 0$, $\frac{1}{10}$ and 1: the invariance of T_2, T_3 and T_4 under ι, $M(0)\Phi(E_{ab}) = 0$ for the ten basis strains, the vanishing of the order-zero vertices with all legs but one metric (1500 cubic and 15000 quartic values at each κ), and the vanishing of the order-one vertices on all tuples of edge classes.

Proposition 8.3 ($\kappa = 0$: the body diagonal)**.** *Let $\kappa = 0$, and let w^{body} be the edge field with amplitude one on the class* 1111 *and zero on the others.*

(a) The row and the column 04 *of $\Omega(0)$ vanish.*

(b) $M(K)\, w^{\mathrm{body}} = 0$ for every $K \in \mathbb{C}^4$. The field w^{body} is not a metric field, and at every momentum none of whose components is a multiple of 2π it is not a displacement mode. $M(0)$ has rank 4, and $M^\perp(0) = \mathrm{diag}(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 0)$.

(c) Changing the squared length of one body diagonal by β, and no other edge, changes Regge's action by

$$S(\beta) = \tfrac{1}{8}\beta^3 + \tfrac{1}{32}\beta^4 + O(\beta^5).$$

Proof. (a) The triangles of a simplex that contain slot 04 are $0j4$, $j = 1, 2, 3$. At $\kappa = 0$ their squared sides are $\ell_{0j} = j$, $\ell_{j4} = 4-j$ and $\ell_{04} = 4$, with $1{+}3 = 2{+}2 = 3{+}1 = 4$: each is right-angled at v_j, where the steps $v_j - v_0$ and $v_4 - v_j$, having disjoint supports, are orthogonal. A triangle with squared sides x, y, z has $16A^2 = 2(xy + yz + zx) - x^2 - y^2 - z^2$, so $\partial A/\partial z = (x + y - z)/(16A)$, which vanishes when $z = x + y$. So the areas of 014, 024 and 034 are stationary in ℓ_{04} at ℓ_0,

and no other triangle contains slot 04. By Schläfli's identity $\partial_g \Psi = \sum_h \partial_g A_h \, (\theta_h - \theta_h^0)$, so the Hessian at ℓ_0 is $\partial_f \partial_g \Psi = \sum_h \partial_g A_h \, \partial_f \theta_h$, and for $g = 04$ every term vanishes. So the column 04 of the Hessian $-\Omega(0)$ vanishes, and by symmetry so does the row.

(b) The body diagonal is slot 04 of every simplex and no other slot, since the other slots have at most three steps; so the column 1111 of $M(K)$ involves only $\Omega(0)_{f,04} = 0$, and it vanishes, and by symmetry so does the row. A metric field that vanishes on the single and double steps has $H_{aa} = 0$ and $H_{aa} + H_{bb} + 2H_{ab} = 0$, hence $H = 0$, so w^{body} is not one. A displacement mode that vanishes on the four single steps has $\zeta_a \sin(K_a/2) = 0$ for every a, so it is zero when no K_a is a multiple of 2π. The matrix $M^{\perp}(0)$ is computed exactly from $\Omega(0)$ (supplement S6); since $M(0)$ annihilates the metric fields (Lemma 8.2), its kernel is spanned by the ten metric fields and w^{body}, and its rank is 4.

(c) The body diagonal of the cell at y, the edge $(y, 1111)$, is slot 04 of the 24 simplices of that cell and of no other simplex. So $S(\beta) = -24\,\Psi(\ell_0 + \beta\lambda)$, with $\lambda_{04} = 1$ and $\lambda_f = 0$ for $f \neq 04$. By Proposition 4.2 at $\kappa = 0$, $\Psi(\ell_0 + \beta\lambda) = -\frac{1}{2}\Omega(0)_{04,04}\beta^2 + \frac{1}{6}T_3(0;\lambda)\beta^3 + \frac{1}{24}T_4(0;\lambda)\beta^4 + O(\beta^5)$. Here $\Omega(0)_{04,04} = 0$ by (a), and $T_3(0;\lambda)$ and $T_4(0;\lambda)$ are the coefficients of λ_{04}^3 in $T_3(0)$ and of λ_{04}^4 in $T_4(0)$, both $-\frac{1}{32}$. So $S(\beta) = -24\big(-\frac{1}{192}\beta^3 - \frac{1}{768}\beta^4\big) + O(\beta^5)$. □

So the flat $\kappa = 0$ lattice is a degenerate critical point of Regge's action on the fifteen-class configuration space. Besides the four displacement modes, its Hessian annihilates the body diagonal at every momentum, and along the body diagonal the action begins at cubic order. The body diagonal therefore has no propagator, and tree-level perturbation theory about this background is not defined for it. The vertices of Theorem 3.1(a), and the $\kappa = 0$ amplitude of Section 9, are those of the *metric truncation*, in which the body diagonal of every cube takes the value $\Phi(H)_{1111} = \mathbf{1}^{\top} H \mathbf{1}$ of the metric field. Whether the body diagonal changes gauge-invariant amplitudes at two derivatives is OPEN. The linear zero mode is known in Regge calculus on this lattice [13, 17, 18], and the cubic onset for a uniform stretch of every body diagonal is in [1, Cor. 5.2]. [17] reads the linear zero mode as a gauge fluctuation; (b) shows that it is not a displacement mode, and (c) that it is not a symmetry beyond linear order. The displacement symmetries of Regge calculus are themselves broken away from flat configurations [16]. At $\kappa = \frac{1}{10}$, for comparison, $\Omega(\frac{1}{10})_{04,04} = -\frac{1}{48}$ and $S(\beta) = -\frac{1}{4}\sqrt{3/5}\,\beta^2 + O(\beta^3)$. Supplement S6 also evaluates $S(\beta)$ at $\kappa = 0$ directly from the simplex geometry (NUMERICAL, 60 digits), in agreement with (c).

Lemma 8.4 (Decoupling at $\kappa = 1$ and $\kappa = \frac{1}{10}$)**.** *(a) For every κ with $(2\kappa-1)(3\kappa-1)(4\kappa-1) \neq 0$,*

$$\det M^{\perp}(\kappa) = \frac{3\kappa\,(2\kappa-1)^3}{16\,(4\kappa-1)^4}.$$

At $\kappa = 1$, in the class order 0111, 1011, 1101, 1110, 1111*,*

$$48\,M^{\perp}(1) = \begin{pmatrix} 0 & 8 & 8 & 8 & -24 \\ 8 & 0 & 8 & 8 & -24 \\ 8 & 8 & 0 & 8 & -24 \\ 8 & 8 & 8 & 0 & -24 \\ -24 & -24 & -24 & -24 & 48 \end{pmatrix},$$

and the eigenvalues of $M^{\perp}(1)$ are $-\frac{1}{6}$ (three times) and $\frac{1}{4}(3 \pm \sqrt{17})$: its inertia is $(1, 4)$, one positive and four negative directions. At $\kappa = \frac{1}{10}$ the eigenvalues of $M^{\perp}$ are $-\frac{2}{3}$ (three times) and $\frac{1}{4}(-3 \pm \sqrt{5})$, so $M^{\perp}(\frac{1}{10})$ is negative definite. At both values the kernel of $M(0)$ is exactly the space of metric fields: the five non-metric modes are nondegenerate at zero momentum.

(b) At $\kappa = 1$ and at $\kappa = \frac{1}{10}$, eliminating the five non-metric modes at tree level changes the quadratic, cubic and quartic terms of the action in the metric fields only at fourth order in the momenta. So the two-derivative vertices of the whole fifteen-class lattice are those of its metric sector; at $\kappa = 1$ they are the vertices of Theorem 3.1(b).

Proof. (a) Both sides of the determinant formula are rational in κ. Multiplied by $\big((\kappa - \frac{1}{4})(\kappa - \frac{1}{3})(\kappa - \frac{1}{2})\big)^5$ they become polynomials of degree at most 15, and they agree at twenty values of κ (exact computation, supplement S6), so they are equal. The matrix $48M^{\perp}(1)$ is computed exactly from $\Omega(1)$. Let w^{three} be the edge field with amplitude one on each of the four classes with three steps and zero on the others. On fields supported on the four classes with three steps whose amplitudes sum to zero, $48M^{\perp}(1)$ acts as -8; on the span of w^{three} and w^{body} it maps $w^{\text{three}} \mapsto 24w^{\text{three}} - 96w^{\text{body}}$ and $w^{\text{body}} \mapsto -24w^{\text{three}} + 48w^{\text{body}}$, with eigenvalues $36 \pm 12\sqrt{17}$. At $\kappa = \frac{1}{10}$, $48M^{\perp}$ has the entries -36 on the diagonal and -4 off it among the four classes with three steps, 12 between these and the body diagonal, and -24 on the body diagonal; it acts as -32 on the first kind of field and maps $w^{\text{three}} \mapsto -48w^{\text{three}} + 48w^{\text{body}}$ and $w^{\text{body}} \mapsto 12w^{\text{three}} - 24w^{\text{body}}$, with eigenvalues $-36 \pm 12\sqrt{5} < 0$. Finally, $M(0)$ annihilates the metric fields (Lemma 8.2); for a field $w^{\perp}$ supported on the five classes, $M(0)w^{\perp} = 0$ exactly when $w^{\perp\top} M(0)\, w'^{\perp} = 0$ for every such $w'^{\perp}$, that is when $M^{\perp}w^{\perp} = 0$, which forces $w^{\perp} = 0$.

(b) Fix $\kappa \in \{1, \frac{1}{10}\}$, write every edge field as $\Phi(H) + w^{\perp}$ with $w^{\perp}$ its non-metric part, and expand the action to fourth order in the fields, in waves as in Definition 2.2, with all momenta of the form ηK; "of order η^k" refers to $\eta \to 0$. The terms that contain $w^{\perp}$ are of three kinds.

(i) The quadratic terms, $\sqrt{|1 - 4\kappa|}$ times $\Phi(H)^{\top} M(P)\, w^{\perp}$ or $w^{\perp\top} M(P)\, w'^{\perp}$ at each momentum P. By Lemma 8.2, $M(P)\Phi(H)$ is of order η^2; the block of $M(P)$ on the non-metric parts is $M^{\perp}(\kappa) + O(\eta^2)$, invertible for small η by (a).

(ii) The cubic and quartic terms with one non-metric leg and all other legs metric. By Lemma 8.2 their parts of orders zero and one vanish, so they are of order η^2.

(iii) The terms with two or more non-metric legs.

At tree level $w^{\perp}$ is eliminated by its field equation, the vanishing of the derivative of the action with respect to $w^{\perp}$. Its linear part is $M^{\perp}(\kappa) + O(\eta^2)$ applied to $w^{\perp}$, up to a nonzero factor. Its source, the derivative of the part of the action linear in $w^{\perp}$, comes from the terms of kind (i) with one non-metric leg and of kind (ii), so it is of order η^2 at each order in the metric fields. Solving the equation order by order in the metric fields therefore gives $w^{\perp}$ of order η^2 at each order up to three. Substituting back, every term with exactly one non-metric leg is a source term times $w^{\perp}$, of order $\eta^2 \cdot \eta^2$, and every term with two or more is of order η^4. So, through quartic order in the metric fields, the action at the stationary $w^{\perp}$ differs from the action at $w^{\perp} = 0$, which is the metric sector, by terms of order η^4, and their parts of order η^2, the long-wave vertices, are equal. At $\kappa = 1$ the action is S_{L}, whose second, third and fourth

derivatives are $\sqrt{3}$ times the tables (Proposition 4.2(d)); so its metric-sector vertices are those of Theorem 3.1(b). □

The lemma says nothing about the physical character of these modes. At $\kappa = 1$ the form $M^{\perp}(1)$ has one direction of the opposite sign to the other four, and whether the modes are healthy, ghost-like or tachyonic depends on their two-derivative block, which this paper does not compute. The exact computations behind the lemma are in supplement S6: the determinant formula, $48M^{\perp}(1)$ and the inertia at $\kappa = 1$ and $\frac{1}{10}$, and, at $\kappa = 0$, $\frac{1}{10}$ and 1, the facts of Lemma 8.2 used in (i) and (ii). The elimination in (b) is the argument above, not a computation. At $\kappa = 0$ the argument stops at (a): $M^{\perp}(0)$ is singular, and the body diagonal cannot be eliminated (Proposition 8.3).

9 Four gravitons at tree level

The vertices of Theorem 3.1 determine the tree-level four-graviton amplitude. This section builds it from the lattice's own tables and propagator and proves three properties: pure-gauge gravitons decouple, the amplitude does not depend on the gauge parameter, and it has general relativity's helicity selection and MHV amplitude, at $\kappa = 0$ and at $\kappa = 1$. At $\kappa = 0$ the amplitude is general relativity's tree amplitude at one half of Einstein's normalization, so these are consistency checks of Theorem 3.1; what the section adds is that they hold exactly for the amplitude built from the explicit tables, and on the Lorentzian lattice through a complex change of coordinates. The amplitude is that of the metric sector: at $\kappa = 1$ the non-metric modes decouple (Lemma 8.4), and at $\kappa = 0$ it is the amplitude of the metric truncation (Proposition 8.3).

9.1 The amplitude

Fix $\kappa \in \{0, 1\}$ and write $\gamma = g_\kappa^{-1}$ and $g = g_\kappa$, so $\gamma = I$ or $\gamma = \gamma_1 = I - \frac{1}{3}\mathbf{1}\mathbf{1}^{\top}$, and $P^2 = P^{\top}\gamma P$. All quantities are complex.

Definition 9.1 (The tree four-graviton amplitude)**.** For four legs (H_i, K_i) with $K_1 + \cdots + K_4 = 0$:

- the *contact term* is the quartic vertex $\mathscr{V}[T_4(\kappa)](H; K)$;
- the *current* of legs i, j is the matrix J_{ij} with entries $J_{ij}^{ab} = \mathscr{V}[T_3(\kappa)]\big(H_i, H_j, \frac{1}{2}(e_{ab} + e_{ba}); K_i, K_j, -K_i - K_j\big)$, e_{ab} the matrix with a single 1 at (a, b), so that $\mathscr{V}[T_3](H_i, H_j, Z) = \sum_{ab} J_{ij}^{ab} Z_{ab}$ for every symmetric Z;
- the *quadratic form* of a channel with momentum P is the lattice's quadratic vertex $B(X, X'; P) = \mathscr{V}[T_2(\kappa)](X, X'; P, -P)$, which is $-\frac{1}{4}\mathrm{FP}_\gamma(X, X'; P)$ by Theorem 5.1; de Donder gauge fixing with parameter $\frac{1}{2}$ subtracts $\frac{1}{2}\sum_{b,d}\gamma^{bd}F_b(X)F_d(X')$, with $F_b(X) = \gamma^{ac}P_c X_{ab} - \frac{1}{2}P_b\,\gamma^{cd}X_{cd}$;
- the *propagator* of a current J is $\Pi(J; P) = \frac{4}{P^2}\big(gJg - \frac{1}{2}(g\colon J)\,g\big)$, with $g\colon J = \sum_{ab} g_{ab}J^{ab}$, and the *exchange* of two currents is $E(J, J'; P) = \sum_{ab} J^{ab}\,\Pi(J'; P)_{ab}$.

The amplitude is the contact term plus the three exchanges,

$$\mathscr{A}(H;K) = \mathscr{V}[T_4(\kappa)](H;K) + E\big(J_{12},J_{34};K_1+K_2\big) + E\big(J_{14},J_{23};K_1+K_4\big) + E\big(J_{13},J_{24};K_1+K_3\big),$$

defined when the channel invariants $s=(K_1+K_2)^2$, $t=(K_1+K_4)^2$ and $u=(K_1+K_3)^2$ are nonzero.

The propagator solves the gauge-fixed field equation: the gauge-fixed form of Z and $\Pi(J;P)$ is $-\sum_{ab} J^{ab}Z_{ab}$ for every symmetric Z. The exchange is the stationary value of the channel's gauge-fixed quadratic-plus-cubic action in the two internal waves $e^{\pm iP\cdot x}$, with their two amplitudes varied independently, so contact and exchange enter with relative weight one; the external legs are annihilated by the gauge-fixed form (for the gauge leg this uses that K_4 is null, $F(H_4)=(K_4^{\top}\gamma K_4)\,\xi=0$), and $\mathscr{A}$ is the coefficient of $\varepsilon_1\cdots\varepsilon_4$ in the tree-level effective action of one cell of the metric sector, at second order in the momenta. By Theorem 3.1 the three vertices are those of $\frac{1}{2}\int\sqrt{|g|}\,R$, so at $\kappa=0$ the amplitude is the tree amplitude of $\frac{1}{2}\int\sqrt{|g|}\,R$ in this gauge.

A *physical configuration* has three gravitons and one pure-gauge leg: for $i=1,2,3$, K_i is null ($K_i^{\top}\gamma K_i=0$) and H_i is symmetric, transverse ($H_i\gamma K_i=0$) and traceless ($\operatorname{tr}\gamma H_i=0$); $K_4=-K_1-K_2-K_3$ is null; and $H_4=K_4\otimes\xi+\xi\otimes K_4$ for some ξ.

9.2 Noether identities and gauge invariance

The invariance of $\sqrt{|g|}\,R$ under changes of coordinates gives, order by order, identities among its vertices. Let $\xi=\zeta e^{iQ\cdot x}$ be a gauge parameter. The Lie derivative of $g=g_0+h$ along ξ has a part independent of h, the pure-gauge polarization $\delta_0=Q\otimes\zeta+\zeta\otimes Q$ at momentum Q, and a part linear in h: for $h=He^{iK\cdot x}$ it is $\delta_1(H,K)=(\hat{\zeta}\cdot K)H+Q\otimes(H\hat{\zeta})+(H\hat{\zeta})\otimes Q$ at momentum $K+Q$, with $\hat{\zeta}=\gamma\zeta$ (a common factor i dropped).

Theorem 9.2 (Noether identities of Einstein's vertices)**.** *Let γ be an invertible complex symmetric 4×4 matrix, for example $\gamma=I$ or $\gamma=\gamma_1$, and let $\zeta\in\mathbb{C}^4$. For complex symmetric legs and complex momenta, $W_2(\gamma;\delta_0,H;Q,K)=0$ when $K+Q=0$; and, when $K_A+K_B+Q=0$, respectively $K_1+K_2+K_3+Q=0$,*

$$\begin{aligned}
&W_3(\gamma;A,B,\delta_0;K_A,K_B,Q)+W_2(\gamma;\delta_1(A,K_A),B;K_A+Q,K_B)\\
&\qquad\qquad+W_2(\gamma;\delta_1(B,K_B),A;K_B+Q,K_A)=0,\\
&W_4(\gamma;H_1,H_2,H_3,\delta_0;K_1,K_2,K_3,Q)\\
&\qquad\qquad+\textstyle\sum_{m=1}^{3}W_3\big(\gamma;H_k,H_l,\delta_1(H_m,K_m);K_k,K_l,K_m+Q\big)=0,
\end{aligned}$$

with $\{k,l\}$ the two legs other than m.

Proof. At $\gamma=I$. Each identity is linear in every polarization and in ζ, and polynomial in the momenta, so it holds for all legs once it holds when every polarization is one of the ten basis matrices E_{ab} and ζ is one of the four unit vectors, with the last momentum eliminated by conservation. The quadratic identity becomes 40 polynomial identities in the four components of Q, the cubic one 400 in eight components, and the quartic one 4000 (1000 ordered triples of basis matrices and 4 unit vectors) in twelve components, each of degree three. Supplement S7

verifies every one by expanding both sides in exact rational or Gaussian-rational arithmetic, at $\gamma = I$, at $\gamma = \gamma_1$ directly, and at a random complex symmetric γ.

Every invertible symmetric γ. Over $\mathbb{C}$ there is an invertible E with $E^{\top}\gamma E = I$. Einstein's vertices are full contractions of γ with the legs and momenta (Definition 2.4), so they are covariant under linear changes of coordinates, $W_n(\gamma; EHE^{\top}; EK) = W_n(E^{\top}\gamma E; H; K)$. So are δ_0 and δ_1: with $\zeta = E\zeta'$ and $Q = EQ'$ one has $\hat{\zeta} = \gamma\zeta = E^{-\top}(E^{\top}\gamma E)\zeta'$, hence $\hat{\zeta}\cdot K = \hat{\zeta}'\cdot K'$ for $K = EK'$, and $\delta_0 = E\,\delta_0'\,E^{\top}$ and $\delta_1(EH'E^{\top}, EK') = E\,\delta_1'(H', K')\,E^{\top}$, the primed quantities being formed with $E^{\top}\gamma E$ in place of γ. So each identity at γ is the same identity at $E^{\top}\gamma E = I$. For $\gamma = \gamma_1$ one can take

$$E_1 = I + \tfrac{1}{2}\omega\,\mathbf{1}\mathbf{1}^{\top}, \qquad \omega = e^{2\pi i/3},$$

a complex change of coordinates: for $E = I + c\,\mathbf{1}\mathbf{1}^{\top}$ one has $E^{\top}\gamma_1 E = I - \frac{1}{3}(4c^2 + 2c + 1)\mathbf{1}\mathbf{1}^{\top}$, and $c = \frac{1}{2}\omega$ is a root of $4c^2 + 2c + 1$. □

Theorem 9.3 (Gauge invariance)**.** *At every physical configuration whose channel invariants s, t, u are nonzero, the amplitude of Definition 9.1 vanishes, at $\kappa = 0$ and at $\kappa = 1$.*

Proof. By Theorem 5.1 and Corollary 6.2, over $\mathbb{C}$, the contact term is $\frac{1}{2}W_4$, the currents are those of $\frac{1}{2}W_3$ and the quadratic form is $\frac{1}{2}W_2$, all on γ. The cubic identity of Theorem 9.2 gives two facts: the current of two on-shell gravitons is conserved, $\sum_a P_a J^{ab} = 0$, and the current of a graviton and the gauge leg is minus the quadratic form applied to the δ_1 leg. On-shell legs annihilate the quadratic form. So in each of the three channels the de Donder term drops on the conserved current, the propagator inverts the quadratic form, and the exchange collapses to the cubic vertex with the δ_1 leg inserted. The quartic identity of Theorem 9.2 cancels the three collapsed exchanges against the contact term. □

The proof does not use that K_4 is null; that condition makes the gauge leg an external state. Supplement S7 evaluates the amplitude exactly, built from the tables themselves, at three physical configurations at $\kappa = 0$ (Gaussian-integer null momenta) and three at $\kappa = 1$ (real integer null momenta), and finds it zero; with a graviton in place of the gauge leg it is nonzero.

Corollary 9.4 (The Lorentzian action's amplitude)**.** *At $\kappa = 1$ the amplitude built in the same way from the vertices of S_{L} itself is $\sqrt{3}\,\mathscr{A}$, with $\sqrt{3} = \sqrt{|\det g_1|}$; so Theorem 9.3 and Theorem 9.6 below hold for it, with the MHV value of Theorem 9.6(b) and (c) multiplied by $\sqrt{3}$.*

Proof. By Proposition 4.2(d) the vertices of S_{L} are $\sqrt{3}$ times those of the tables at each of the three orders. A common factor c multiplies the contact term and each current by c and the inverse of the quadratic form, the propagator, by c^{-1}, so each exchange and the amplitude by c; the gauge-fixing term may be scaled with the quadratic form, since by Lemma 9.5 below its weight does not change the amplitude. □

The gauge fixing of Definition 9.1 has a parameter, $\frac{1}{2}$ there. The amplitude does not depend on it.

Lemma 9.5 (The gauge parameter)**.** *Let γ be an invertible complex symmetric 4×4 matrix, $g = \gamma^{-1}$, $P \in \mathbb{C}^4$ with $P^2 = P^{\top}\gamma P \neq 0$, and $\alpha \in \mathbb{C}$ with $\alpha \neq 0$. For symmetric X, X' put*

$$\mathscr{B}_{\alpha}(X, X'; P) = -\tfrac{1}{4}\mathrm{FP}_{\gamma}(X, X'; P) - \alpha \sum_{b,d} \gamma^{bd} F_b(X) F_d(X'),$$

with F_b as in Definition 9.1, so that $\mathscr{B}_{1/2}$ is the gauge-fixed form there.

(a) $\mathscr{B}_\alpha$ is nondegenerate on symmetric matrices. For every symmetric J (indices up) the unique symmetric $\Pi_\alpha(J;P)$ with $\mathscr{B}_\alpha\big(Z,\Pi_\alpha(J;P);P\big) = -\sum_{a,b} J^{ab}Z_{ab}$ for every symmetric Z is

$$\Pi_\alpha(J;P) = \Pi(J;P) + \frac{2(1-2\alpha)}{\alpha(P^2)^2}\big(P\otimes gJP + gJP\otimes P\big), \qquad (gJP)_b = g_{bc}J^{cd}P_d.$$

(b) For all symmetric J and J',

$$\sum_{a,b} J'^{ab}\,\Pi_\alpha(J;P)_{ab} - E(J',J;P) = \frac{4(1-2\alpha)}{\alpha(P^2)^2}\,(J'P)^{\top} g\,(JP).$$

If J is conserved, $\sum_a P_a J^{ab} = 0$ for every b, then $\Pi_\alpha(J;P) = \Pi(J;P)$; and an exchange in which one of the two currents is conserved does not depend on α.

The contraction $\sum_a P_a J^{ab}$ pairs the lower index of P with an upper index of the current; no metric enters.

Proof. Three facts follow from the definitions by expansion. For every vector ω' the pure-gauge field $P\otimes\omega'+\omega'\otimes P$ is annihilated by the Fierz–Pauli form, $\mathrm{FP}_\gamma(X, P\otimes\omega'+\omega'\otimes P;P) = 0$, and has $F(P\otimes\omega'+\omega'\otimes P) = P^2\omega'$. The form at $\alpha = \frac12$ is $\mathscr{B}_{1/2}(X,X';P) = -\frac14 P^2\big(\mathrm{tr}(\gamma X\gamma X') - \frac12\,\mathrm{tr}(\gamma X)\,\mathrm{tr}(\gamma X')\big)$. With it, $\mathscr{B}_{1/2}(Z,\Pi(J;P);P) = -\sum_{ab} J^{ab}Z_{ab}$, which is the statement after Definition 9.1, and $F(\Pi(J;P)) = (4/P^2)\,gJP$.

(a) Write $\mathscr{B}_\alpha = \mathscr{B}_{1/2} + (\frac12-\alpha)F^{\top}\gamma F$. By the three facts, and since $\gamma\,gJP = JP$, the correction term of Π_α contributes $-\alpha P^2\cdot\frac{2(1-2\alpha)}{\alpha(P^2)^2}\,F(Z)^{\top}JP$ to $\mathscr{B}_\alpha(Z,\Pi_\alpha(J;P);P)$, and $\Pi(J;P)$ contributes $-\sum_{ab}J^{ab}Z_{ab} + (\frac12-\alpha)\frac{4}{P^2}F(Z)^{\top}JP$; the two terms in $F(Z)^{\top}JP$ cancel. For uniqueness, let $\mathscr{B}_\alpha(Z,X;P) = 0$ for every symmetric Z. With $Z = P\otimes\omega'+\omega'\otimes P$ this reads $-\alpha P^2\,\omega'^{\top}\gamma F(X) = 0$ for every ω', so $F(X) = 0$. Then $\mathscr{B}_{1/2}(Z,X;P) = 0$ for every Z, that is $\mathrm{tr}\big(\gamma Z\gamma(X - \frac12\,\mathrm{tr}(\gamma X)\,g)\big) = 0$ for every symmetric Z, so $X = \frac12\,\mathrm{tr}(\gamma X)\,g$; taking the trace with γ gives $\mathrm{tr}(\gamma X) = 0$, and $X = 0$.

(b) The difference is the contraction of J' with the correction term of Π_α, which is $\frac{2(1-2\alpha)}{\alpha(P^2)^2}\cdot 2\,(J'P)^{\top}g\,(JP)$. It vanishes if $JP = 0$ or $J'P = 0$. □

In the amplitude at a physical configuration, and at a configuration of four gravitons, each of the three exchanges pairs two currents at least one of which is the current of two on-shell gravitons. That current is conserved, by the cubic identity of Theorem 9.2, as in the proof of Theorem 9.3. So, built with $\mathscr{B}_\alpha$ and Π_α in place of $\mathscr{B}_{1/2}$ and Π, these amplitudes are the same for every $\alpha \neq 0$; in particular Theorems 9.3 and 9.6 hold for every $\alpha \neq 0$. Supplement S7 checks the lemma exactly at $\gamma = I$ and $\gamma = \gamma_1$, including the determinant of $\mathscr{B}_\alpha$ on the ten basis matrices, $-(\det\gamma)^5\alpha^4(P^2)^{10}/1024$.

9.3 Helicity selection and the MHV amplitude

On the Euclidean lattice the null momenta are complex, and spinors describe them. For $\lambda, \tilde{\lambda} \in \mathbb{C}^2$ let $v(\lambda, \tilde{\lambda}) \in \mathbb{C}^4$ be the vector y with

$$\begin{pmatrix} y_4 + iy_3 & y_2 + iy_1 \\ iy_1 - y_2 & y_4 - iy_3 \end{pmatrix} = \lambda\tilde{\lambda}^\top.$$

The determinant of this matrix is $y \cdot y$, so $v(\lambda, \tilde{\lambda}) \cdot v(\lambda, \tilde{\lambda}) = 0$ and $v(\lambda, \tilde{\lambda}) \cdot v(\mu, \tilde{\mu}) = \frac{1}{2}\langle\lambda\mu\rangle[\tilde{\lambda}\tilde{\mu}]$, where $\langle\lambda\mu\rangle$ is the determinant of the 2×2 matrix with columns λ and μ, and $[\tilde{\lambda}\tilde{\mu}]$ the same bracket. Leg i has the momentum $K_i = v(\lambda_i, \tilde{\lambda}_i)$. With reference spinors $\mu_i, \tilde{\mu}_i$, its polarization vector is $\epsilon_i^+ = v(\mu_i, \tilde{\lambda}_i)/\langle\mu_i\lambda_i\rangle$ or $\epsilon_i^- = v(\lambda_i, \tilde{\mu}_i)/[\tilde{\lambda}_i\tilde{\mu}_i]$, and the graviton's polarization is $H_i = \epsilon_i \otimes \epsilon_i$. We write $\langle ij\rangle = \langle\lambda_i\lambda_j\rangle$ and $[ij] = [\tilde{\lambda}_i\tilde{\lambda}_j]$; the channel invariants are $s = \langle 12\rangle[12]$, $t = \langle 14\rangle[14]$ and $u = \langle 13\rangle[13]$. The data are *generic* when $\sum_i \lambda_i\tilde{\lambda}_i^\top = 0$ (momentum conservation), $s, t, u \neq 0$, and every $\langle\mu_i\lambda_i\rangle$ and $[\tilde{\lambda}_i\tilde{\mu}_i]$ is nonzero.

Theorem 9.6 (Helicity selection and the MHV amplitude)**.** *For generic spinor data:*

(a) at $\kappa = 0$ the amplitude vanishes unless exactly two of the four helicities are positive;

(b) at $\kappa = 0$, if legs m, m' are negative and p, p' positive,

$$\mathscr{A} = -\frac{1}{256}\,\frac{\langle\lambda_m\lambda_{m'}\rangle^4\,[\tilde{\lambda}_p\tilde{\lambda}_{p'}]^4}{s\,t\,u};$$

(c) at $\kappa = 1$ the same two statements hold for the amplitude at the momenta E_1K_i and the polarizations $E_1H_iE_1^\top$, with $E_1 = I + \frac{1}{2}\omega\,\mathbf{1}\mathbf{1}^\top$, $\omega = e^{2\pi i/3}$, the complex change of coordinates of the proof of Theorem 9.2, $E_1^\top\gamma_1E_1 = I$. Its channel invariants are the same s, t, u, since $(E_1P)^\top\gamma_1(E_1P) = P^\top P$.

At $\kappa = 1$ the legs of (c) reach every null momentum: E_1 maps the null cone of I onto that of γ_1, so every γ_1-null momentum, the real Lorentzian ones included, is $E_1v(\lambda, \tilde{\lambda})$ for some spinors. And $E_1HE_1^\top$ is transverse and traceless for γ_1 exactly when H is for I. The dependence on the spinors in (b) is that of general relativity's four-graviton MHV amplitude [26, 27]; the constant is the one that $\frac{1}{2}\int\sqrt{g}\,R$ gives in these conventions, since the amplitude is that action's.

Proof. Five steps prove (a) and (b). (1) *Reference independence.* Changing a reference spinor shifts ϵ_i by a multiple of K_i (Schouten's identity), so $\epsilon_i \otimes \epsilon_i$ changes by a pure-gauge polarization; the amplitude is additive in each leg and symmetric under permutations of the legs, so Theorem 9.3 removes the change, one reference at a time. (2) *Spinor coordinates.* In coordinates where momenta and polarizations are the 2×2 matrices $\lambda\tilde{\lambda}^\top$, the background becomes an integer matrix and the amplitude becomes Einstein's tree amplitude on that background, times the exact factor $\frac{1}{32}$; this uses the covariance of Theorem 9.2. (3) *Lorentz invariance.* The maps $\lambda\tilde{\lambda}^\top \mapsto U\lambda\tilde{\lambda}^\top U'^\top$ with $\det U = \det U' = 1$ preserve that background, so the amplitude is invariant under $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. (4) *Isotropic vanishing.* Every term of every diagram carries eight polarization vectors and at most four momenta (two per vertex; the propagator carries none), so it contains a contraction of two polarizations; if every polarization vector

lies in one isotropic plane, every term vanishes; a weight count for a torus of $SL(2,\mathbb{C})$ proves it with no computation. Choosing the references by helicity pattern puts all polarizations of an all-equal or one-different pattern in one isotropic plane, which gives (a).

(5) *The canonical value.* By the symmetry under permutations of the legs we may take $m, m' = 1, 2$ and $p, p' = 3, 4$. Since $s = \langle 12\rangle[12] = \langle 34\rangle[34] \neq 0$, both $\langle 12\rangle$ and $[34]$ are nonzero, and step (3) brings $\lambda_1, \lambda_2, \tilde\lambda_3, \tilde\lambda_4$ to $(1,0), (0,p), (1,0), (0,q)$, with $p = \langle 12\rangle$ and $q = [34]$. Write $\lambda_3 = -a\lambda_1 - c\lambda_2$ and $\lambda_4 = -b\lambda_1 - d\lambda_2$; momentum conservation then gives $\tilde\lambda_1 = a\tilde\lambda_3 + b\tilde\lambda_4$ and $\tilde\lambda_2 = c\tilde\lambda_3 + d\tilde\lambda_4$. So

$$s = pq\,(ad - bc), \qquad t = -pq\,ad, \qquad u = pq\,bc,$$

and genericity means $a, b, c, d, p, q \neq 0$ and $ad \neq bc$. By step (1) we may take the reference spinors $\mu_3 = \mu_4 = \lambda_1$ and $\tilde\mu_1 = \tilde\mu_2 = \tilde\lambda_3$. Then the contact term vanishes, and so does the current J_{13}, so the u-channel exchange is zero. The currents J_{12}, J_{34}, J_{14} and J_{23} are traceless and of rank one, so each remaining exchange is a single product:

$$E_s = E(J_{12}, J_{34}; K_1 + K_2) = \frac{pq}{256\, b^2c^2(ad - bc)} = \frac{\langle 12\rangle^4[34]^4}{256\, s\,u^2},$$
$$E_t = E(J_{14}, J_{23}; K_1 + K_4) = -\frac{pq}{256\, ab^2c^2d} = \frac{\langle 12\rangle^4[34]^4}{256\, t\,u^2}.$$

Their sum is $\langle 12\rangle^4[34]^4(s+t)/(256\,stu^2) = -\frac{1}{256}\langle 12\rangle^4[34]^4/(stu)$, since $s + t + u = 0$; this is (b). These values hold at every value of the parameters, by exact computation (supplement S7): with the polarization vectors written without their normalizing factors, the momenta are linear in (a, bq, cp, dpq), so the contact term and every entry of every current are quadratic forms in these four numbers, fixed by finitely many exact evaluations.

(c) By Theorem 5.1 and Corollary 6.2(a), over $\mathbb{C}$, the $\kappa = 1$ amplitude is the one built from $\frac{1}{2}W_n(\gamma_1)$: contact term $\frac{1}{2}W_4(\gamma_1)$, currents of $\frac{1}{2}W_3(\gamma_1)$, quadratic form $\frac{1}{2}W_2(\gamma_1) = -\frac{1}{4}\mathrm{FP}_{\gamma_1}$, and the propagator Π on γ_1. Write $\mathscr{A}_\gamma(H;K)$ for the amplitude built in this way on an invertible complex symmetric γ; the $\kappa = 0$ amplitude is $\mathscr{A}_I$. We show that

$$\mathscr{A}_\gamma\big(EHE^\top; EK\big) = \mathscr{A}_{E^\top\gamma E}(H;K) \qquad \text{for every invertible complex } E,$$

so that $\mathscr{A}_{\gamma_1}(E_1HE_1^\top; E_1K) = \mathscr{A}_I(H;K)$, and (a) and (b) give (c). Put $\gamma' = E^\top\gamma E$ and $g' = \gamma'^{-1} = E^{-1}gE^{-\top}$. The contact term is covariant, $W_4(\gamma; EHE^\top; EK) = W_4(\gamma'; H; K)$, as in the proof of Theorem 9.2. The current of legs i, j at the carried legs $EH_iE^\top$, $EH_jE^\top$ with momenta EK_i, EK_j is $E^{-\top}J'E^{-1}$, with J' the current of H_i, H_j with momenta K_i, K_j on γ', because for every symmetric Z

$$\tfrac{1}{2}W_3\big(\gamma; EH_iE^\top, EH_jE^\top, EZE^\top; EK\big) = \tfrac{1}{2}W_3(\gamma'; H_i, H_j, Z; K),$$
$$\textstyle\sum_{ab}(E^{-\top}J'E^{-1})^{ab}(EZE^\top)_{ab} = \sum_{ab}J'^{ab}Z_{ab}.$$

The propagator transforms with it, $\Pi\big(E^{-\top}J'E^{-1}; EP\big) = E\Pi'(J'; P)\,E^\top$, the prime meaning that Π' is formed with γ': indeed $(EP)^\top\gamma(EP) = P^\top\gamma' P$, $gE^{-\top}J'E^{-1}g = Eg'J'g'E^\top$ and $g : (E^{-\top}J'E^{-1}) = g' : J'$. So every exchange, the contraction of one current with the propagated other, has the same value at the two configurations. The gauge fixing is consistent with this because every ingredient of the gauge-fixed form is a contraction with the same γ: FP_γ is, and

the de Donder function transforms as a covector, $F_\gamma(EXE^\top; EP) = E\,F_{\gamma'}(X;P)$, so that $F^\top \gamma F$ is invariant. Hence the gauge-fixed form is invariant, its inverse, the propagator, transforms as a field, and the exchange, the stationary value of the channel's quadratic-plus-cubic action, is invariant. □

Supplement S7 checks (a) and (b) exactly at three generic spinor configurations, in all sixteen helicity patterns with two sets of reference spinors, and the canonical configuration of step (5) at every value of its parameters. For (c) it checks, with the tables $T_n(1)$ themselves and exact arithmetic in $\mathbb{Q}(i,\sqrt{3})$, that the $\kappa = 1$ amplitude at $(E_1 H_i E_1^\top, E_1 K_i)$ equals the $\kappa = 0$ amplitude diagram by diagram in all 96 cases, with the currents carried as $E_1^{-\top} J E_1^{-1}$.

Remark 9.7 (Every member of the family)**.** The amplitude of Definition 9.1 is defined at every κ off the three poles, with $\gamma = g_\kappa^{-1}$ and the tables $T_n(\kappa)$. By Theorems 5.1 and 6.1, over $\mathbb{C}$, it is the amplitude $\mathscr{A}_\gamma$ built from $\frac{1}{2}W_n(\gamma)$, and the proof of Theorem 9.3 applies unchanged, with Theorem 9.2 at $\gamma = g_\kappa^{-1}$: the amplitude vanishes at every physical configuration whose channel invariants are nonzero. With an invertible complex E_κ such that $E_\kappa^\top g_\kappa^{-1} E_\kappa = I$, the covariance in the proof of (c) gives $\mathscr{A}_\gamma(E_\kappa H E_\kappa^\top; E_\kappa K) = \mathscr{A}_I(H;K)$, so helicity selection and the MHV amplitude hold at every such κ in that frame. On the Euclidean members and on the Lorentzian members with $\kappa > \frac{1}{2}$ the vertices of the lattice action are $\sqrt{|\det g_\kappa|}$ times those of the tables (Proposition 4.2), so, as in Corollary 9.4, these are the statements for the lattice actions themselves, up to that factor.

10 Hinge shape costs cannot complete a changed coupling at quartic order

Regge's action has one gravitational coupling, the ratio of its cubic to its quadratic vertex. The lattice can carry ν times it for any ν: rescaling every departure from the flat configuration by ν and the action by ν^{-2} gives $\nu^{-2} S_{\mathrm{R}}(\ell_\kappa + \nu\lambda)$, which has the quadratic part of S_{R}, ν times its cubic part and ν^2 times its quartic part, and so ν times Regge's coupling at every order. The question of this section is narrower. A *hinge shape cost*, or *shape cost*, is a cost linear in each hinge's deficit, $\sum_h f_c(L_h)\,\delta_h$, whose coefficient depends only on the hinge's class c (Section 10.2) and its own three squared edge lengths L_h. Can costs of that form change the coupling? To first order in the change they can: the derivative of the rescaled action at $\nu = 1$ is $-Y$ for a shape cost Y with no quadratic part and minus Regge's cubic part (Proposition 10.1), so $S_{\mathrm{R}} + (1-\nu)Y$ keeps Regge's quadratic action and has coupling ν through cubic order. At quartic order it falls short by $(1-\nu)^2 S_4$. The rescaled action supplies the shortfall exactly: $U_\nu = \nu^{-2} S_{\mathrm{R}}(\ell_\kappa + \nu\lambda) - S_{\mathrm{R}} - (1-\nu)Y$ has no quadratic or cubic part and the quartic part $(1-\nu)^2 S_4$. But U_ν is built from the deficits of the rescaled geometry, not from the lattice's own deficits with coefficients set hinge by hinge. The question is whether a shape cost can do what U_ν does at quartic order.

Quartic order is where such questions are first decided. A hinge charge $\varphi(A)\,\delta$, one function of the area for every hinge, enters the quadratic and cubic actions only through φ, φ' and φ'' at the flat areas [5, Sec. 4.3]: along a line $\ell_\kappa + r\lambda$, with $A = A^0 + rA' + \frac{1}{2}r^2 A'' + \cdots$ and

$\delta = r\delta' + \frac{1}{2}r^2\delta'' + \frac{1}{6}r^3\delta''' + \cdots$,

$$\frac{d^2}{dr^2}\big[\varphi(A)\,\delta\big]_0 = \varphi(A^0)\,\delta'' + 2\varphi'(A^0)\,A'\delta',$$

$$\frac{d^3}{dr^3}\big[\varphi(A)\,\delta\big]_0 = \varphi(A^0)\,\delta''' + 3\varphi'(A^0)\big(A'\delta'' + A''\delta'\big) + 3\varphi''(A^0)\,A'^2\delta'.$$

So the charge $\big[A+(A-A^0)^3/(A^0)^2\big]\delta$, with A^0 the flat area of each hinge, has Regge's quadratic and cubic actions without being Regge's charge.

For shape costs the answer is no. For the shape costs $f_c(L_h)\,\delta_h$, one smooth function of a hinge's three squared edge lengths for each of the lattice's six classes of hinges, changed through their sixty third-derivative directions, no finite change of the coupling is completed at quartic order, at $\kappa = 0$, $\frac{1}{10}$ and 1 (Theorem 10.3), and the lattice's non-metric edge modes do not change this at $\kappa = \frac{1}{10}$ and 1. The statement rests on one INFERENCE, that completing a changed coupling requires exactly the quartic shortfall in the gauge variation (Remark 10.4). In the metric sector the lower jets of the shape costs do not change it: a change of the values, gradients and Hessians of the f_c that adds no long-wave quadratic or cubic vertex adds no long-wave quartic vertex either. Costs that are nonlinear in the deficit, or that depend on more than a hinge's three edge lengths, are not covered.

10.1 The coupling direction

Rescale every departure from a reference simplex L_0 by ν and the action by ν^{-2}. The resulting family has, around L_0, the quadratic part of Regge's action, ν times its cubic part and ν^2 times its quartic part: it is Regge's action with ν times the original coupling. Its derivative at $\nu = 1$ is a shape cost.

Proposition 10.1 (The coupling direction)**.** *For a reference simplex L_0, a nondegenerate simplex L and $\ell = L - L_0$, write $\Psi_{L_0}(L) = \sum_h A_h(L)\big(\theta_h(L) - \theta_h(L_0)\big)$ and $\Psi_{\mathrm{Y}}(L) = \sum_h \big(A_h(L) + L_0 \cdot \nabla A_h(L)\big)\big(\theta_h(L) - \theta_h(L_0)\big)$.*

(a) $\dfrac{d}{d\nu}\Big[\nu^{-2}\,\Psi_{L_0}(L_0 + \nu\ell)\Big]_{\nu=1} = -\Psi_{\mathrm{Y}}(L)$.

(b) Let $\kappa < \frac{1}{4}$, let $S_{\mathrm{R}} = S_2 + S_3 + S_4 + \cdots$ be Regge's action around ℓ_κ split into its parts of each order, and let $Y = \sum_h (A_h + L_0 \cdot \nabla A_h)\,\delta_h$ be the shape cost with L_0 the flat squared lengths of each hinge's class, with parts Y_n. Then $Y_n = -(n-2)\,S_n$ for every $n \geq 2$: the quadratic part of Y vanishes identically, its cubic part is $-S_3$, so its long-wave cubic vertex is minus Regge's, and its quartic part is $-2S_4$. The same holds for S_{L} at every $\kappa > \frac{1}{2}$.

Proof. (a) Put $F(\nu) = \Psi_{L_0}(L_0 + \nu\ell)$; then $\frac{d}{d\nu}[\nu^{-2}F]_{\nu=1} = F'(1) - 2F(1)$, and with primes for the derivative along ℓ at L, $F'(1) = \sum_h \big[A'_h(\theta_h - \theta_h(L_0)) + A_h\theta'_h\big]$. Schläfli's identity removes $\sum_h A_h\theta'_h$. Heron's area is homogeneous of degree one in the squared lengths, so $L \cdot \nabla A_h = A_h$ by Euler's identity, and $A'_h = \ell \cdot \nabla A_h = A_h - L_0 \cdot \nabla A_h$. So $F'(1) - 2F(1) = \sum_h (-A_h - L_0 \cdot \nabla A_h)(\theta_h - \theta_h(L_0)) = -\Psi_{\mathrm{Y}}(L)$.

(b) Take $L_0 = \ell_\kappa$ in every simplex and sum (a) over the simplices. Near ℓ_κ every flat deficit vanishes, so $S_{\mathrm{R}} = -\sum_\sigma \Psi_{\ell_\kappa,\sigma}$ and $Y = -\sum_\sigma \Psi_{\mathrm{Y},\sigma}$; hence the function $\nu \mapsto \nu^{-2} S_{\mathrm{R}}(\ell_\kappa + \nu\lambda)$ has

the derivative $-Y(\ell_\kappa + \lambda)$ at $\nu = 1$. The same function is $\sum_n \nu^{n-2} S_n(\lambda)$, with the derivative $\sum_n (n-2) S_n(\lambda)$ at $\nu = 1$. Comparing the parts of degree n in λ gives $Y_n = -(n-2)S_n$. The same computation holds simplex by simplex, so at orders two, three and four the per-simplex tables of Y are 0, -1 and -2 times those of S_{R}, and the long-wave cubic vertex of Y is $-V_3[S_{\mathrm{R}}]$. For S_{L} the proof of (a) uses two facts: the Lorentzian area $\frac{1}{4}\sqrt{-Q_h}$ is again homogeneous of degree one, and Schläfli's identity holds for the Lorentzian simplices used here (Section 2). □

So $Y = -S_3 - 2S_4 - \cdots$, and

$$S_{\mathrm{R}} + (1-\nu)Y = S_2 + \nu S_3 + (2\nu - 1)\, S_4 + \cdots .$$

Adding $(1-\nu)Y$ gives Regge's action with coupling ν through cubic order, and falls short at quartic order by $(1-\nu)^2 S_4$. At $\nu = \frac{1}{2}$ the shortfall is one quarter of Regge's quartic part. A completion is a further shape cost with no quadratic or cubic part whose quartic vertex supplies that shortfall where it matters: in its gauge variation, on physical configurations. The U_ν of the opening supplies the shortfall exactly, but it is a cost of the rescaled geometry; at $\kappa = 0$, $\frac{1}{10}$ and 1, Theorem 10.3(a) shows that no shape cost supplies it.

10.2 The obstruction

A shape cost gives each hinge h of class c the cost $f_c(L_h)\,\delta_h$, with L_h the hinge's three squared edge lengths and f_c one smooth function for each class. The classes are taken up to translation and permutation of the axes: a hinge with vertices y, $y + a$, $y + a + b$, where a and b are disjoint nonempty 0/1 steps, has class $(|a|, |b|)$, and there are six, $(1,1)$, $(1,2)$, $(1,3)$, $(2,1)$, $(2,2)$ and $(3,1)$. So the costs respect the lattice's symmetry under permutations of the axes, and every simplex has the same per-simplex table, read in its slot labels; costs that depend on the orientation of a hinge would need fifty classes. The value, gradient and Hessian of f_c at the flat hinge enter the quadratic and cubic actions; its third derivatives first enter at quartic order. The *third-derivative direction* (c, j) is the shape cost with

$$f_c = \frac{\mathrm{mult}_j}{6}\,\frac{\mu_j}{A_c^0}$$

on the class c and no cost on the other five classes. Here $\mu_1, \ldots, \mu_{10}$ are the ten cubic monomials in the changes of the hinge's three squared lengths, $\mathrm{mult}_j \in \{1, 3, 6\}$ is the number of distinct orderings of the factors of μ_j, and A_c^0 is the flat area of a hinge of class c; equivalently, the third-derivative tensor of f_c is $1/A_c^0$ on the index triple of μ_j and zero elsewhere. These sixty directions have no quadratic or cubic part. Write $F_{c,j}(\kappa)$ for their per-simplex quartic tables divided by $\frac{1}{2}\sqrt{|1-4\kappa|}$, the normalization in which Regge's quartic part per simplex is $-\frac{1}{12}T_4(\kappa)$. The factor $1/A_c^0$ makes the tables rational: the first-order change of a dihedral angle carries $1/\sqrt{|Q_h|}$, which cancels against $A_c^0 = \frac{1}{4}\sqrt{|Q_h|}$. So $F_{c,j}(\kappa)$ is a rational function of κ, with poles of order at most one at $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$, and for $\kappa < \frac{1}{4}$ it is the table of the Euclidean shape cost. At $\kappa = 1$ the Lorentzian shape cost (c, j) has, in the same normalization, the table $-F_{c,j}(1)$, while Regge's quartic part keeps its sign, $-\frac{1}{12}T_4(1)$; both are exact computations from the geometry in supplement S2. A sign common to the sixty tables changes neither their span nor any certificate below, so the theorem at $\kappa = 1$ is a statement about the Lorentzian shape costs themselves. The lower jets cannot supply the shortfall either. The values, gradients

and Hessians of the f_c at the flat hinges, $6 \times (1 + 3 + 6) = 60$ numbers, are all that enters the quadratic and cubic parts of a shape cost, and their sixty per-simplex pairs of quadratic and cubic tables are linearly independent; so, through quartic order, the shape costs with no quadratic or cubic part are exactly the combinations of the sixty directions. More generally, a change of the lower jets whose quadratic and cubic parts have no long-wave vertex is odd under the central inversion of the cell: at $\kappa = 0$, $\frac{1}{10}$ and 1 these changes form exactly the 26-dimensional space of inversion-odd combinations. Both statements are exact computations in supplement S2. A table T that is odd under the inversion has no long-wave vertex at any order, because the argument of Lemma 4.3 gives $\mathscr{X}[T](H;K) = -\mathscr{X}[T](H;-K)$ when $K_1 + \cdots + K_n = 0$, so the parts of even order in the momenta vanish. So the quartic part of such a change has no long-wave vertex, and in the metric sector part (a) below covers every shape cost whose quadratic and cubic parts have no long-wave vertex.

Part (b) of the theorem below includes the lattice's non-metric edge modes (Section 8). They enter the gauge variation at its own order, three in the momenta, through two corrections, defined next.

Definition 10.2 (Admixtures and the corrected variation)**.** Let $\kappa \in \{\frac{1}{10}, 1\}$ and fix a physical configuration on $\gamma = g_\kappa^{-1}$ (Section 9): gravitons (H_i, K_i), $i = 1, 2, 3$, and the gauge leg $H_4 = K_4 \otimes \xi + \xi \otimes K_4$ at $K_4 = -K_1 - K_2 - K_3$.

- An *admixture* of graviton i is an edge field $w_i^\perp$ with

$$M(0)\, w_i^\perp + M^{(2)}(K_i)\, \Phi(H_i) = 0.$$

- The gauge leg is the displacement mode with $\zeta = \xi$, whose expansion is $4i(d \cdot \xi) \sin(K_4 \cdot d/2) = i\,\Phi(H_4)_d - \frac{i}{12}(d \cdot \xi)(K_4 \cdot d)^3 + O(K_4^5)$. A *non-metric part* of the gauge leg is an edge field $w_4^\perp$ that differs from $d \mapsto -\frac{1}{12}(d \cdot \xi)(K_4 \cdot d)^3$ by a metric field.
- For a quartic table T, the *corrected variation* is

$$\mathscr{C}[T] = -\mathscr{V}[T](H_1, H_2, H_3, H_4; K) + \sum_{i=1}^{3} \mathscr{X}^{(0)}[T]\big(\dots, w_i^\perp, \dots, \Phi(H_4)\big) + \mathscr{X}^{(0)}[T]\big(\Phi(H_1), \Phi(H_2), \Phi(H_3), w_4^\perp\big),$$

  where in the i-th term of the sum $w_i^\perp$ takes the place of $\Phi(H_i)$, and the other two graviton legs are metric.

The admixture equation is the part of second order in the momenta of the lattice field equation $M(K_i)(\Phi(H_i) + w_i^\perp) = 0$ of the corrected leg: $M(K_i)\Phi(H_i) = M^{(2)}(K_i)\Phi(H_i) + O(K_i^4)$ by Lemma 8.2, and $w_i^\perp$ is of second order in K_i. It has a solution, unique up to a metric field and unique when supported on the five classes with three or four steps. Indeed, by Lemma 8.4(a) the kernel of the symmetric matrix $M(0)$ is the space of metric fields, so the equation is solvable exactly when its second term is orthogonal to every metric field, $\Phi(Z)^\top M^{(2)}(K_i)\Phi(H_i) = 0$ for every symmetric Z. That is the lattice's quadratic vertex $\mathscr{V}[T_2(\kappa)](Z, H_i; K_i, -K_i) = -\frac{1}{4}\mathrm{FP}_\gamma(Z, H_i; K_i)$ (Theorem 5.1), which vanishes because H_i is transverse and traceless and K_i is null. The gauge leg's non-metric part is unique when

supported on those five classes. The first term of $\mathscr{C}[T]$ is minus the gauge variation of part (a), of order three in the momenta: two from $\mathscr{V}$ and one from H_4. The other four terms are the contributions of the non-metric modes at the same order, each at vertex order zero: an admixture, of order two, with the gauge leg's part $\Phi(H_4)$, of order one; and the gauge leg's non-metric part, of order three.

Theorem 10.3 (The quartic obstruction). *Let $\kappa \in \{0, \frac{1}{10}, 1\}$ and use physical configurations on $\gamma = g_\kappa^{-1}$ (Section 9).*

(a) There are no complex numbers $z_{c,j}$ such that, on every physical configuration,

$$\sum_{c,j} z_{c,j}\, \mathscr{V}[F_{c,j}(\kappa)](H;K) = \mathscr{V}\big[-\tfrac{1}{48}T_4(\kappa)\big](H;K),$$

the right side being the long-wave quartic vertex of one quarter of Regge's quartic part.

(b) At $\kappa = 1$ and at $\kappa = \frac{1}{10}$ the same holds with the lattice's non-metric edge modes included: there are no complex numbers $z_{c,j}$ such that $\sum_{c,j} z_{c,j}\, \mathscr{C}[F_{c,j}(\kappa)] = \mathscr{C}\big[-\frac{1}{48}T_4(\kappa)\big]$ on every physical configuration with its admixtures and gauge part (Definition 10.2). The corrected variation does not depend on how the admixtures and the gauge part are chosen. Along the line of momenta $\eta K_1, \dots, \eta K_4$, the cell vertex of T with the legs $\Phi(H_i) + \eta^2 w_i^\perp$ ($i = 1, 2, 3$) and the exact displacement mode $4i(d \cdot \xi)\sin(\eta K_4 \cdot d/2)$ is $i\,\mathscr{C}[T]\,\eta^3 + O(\eta^4)$. At $\kappa = 1$, on real configurations, $\mathscr{C}[T]$ is real, and the imaginary unit times it is the leading Taylor coefficient, of third order in the momenta, of the exact lattice vertex.

The space of the sixty variations is linear, so no nonzero multiple of Regge's quartic variation lies in it either: no change of the coupling, $\nu \neq 1$, is completed. One quarter is the multiple that $\nu = \frac{1}{2}$ needs.

Proof. Each claim is proved by an exact certificate. Choose finitely many physical configurations and evaluate the sixty-one vertices on them exactly: a matrix M with one column per direction, and the target column t. An integer vector y with $y^\top \mathsf{M} = 0$ and $y^\top \mathsf{t} \neq 0$ excludes every combination with real coefficients: applying y to the supposed equalities gives $0 = y^\top \mathsf{t}$.

(a) At $\kappa = 0$ and $\kappa = \frac{1}{10}$ the metric is Euclidean, the null momenta are complex, and the configurations have Gaussian-integer entries; 17 configurations suffice at each value, and M has two real rows for each, the real and the imaginary parts of its sixty-one values, 34 rows in all. A complex combination is excluded through a real one: the complex conjugate of a physical configuration is physical, and a table with rational coefficients takes conjugate values there, so a complex solution would give a real one, its real part, which the certificate excludes. At $\kappa = 1$ real null momenta exist, 33 real integer configurations are used, every value is real, and y excludes complex coefficients directly.

(b) At $\kappa = 1$ the first 24 configurations of (a) are used. Their admixtures and gauge parts, taken on the five classes, are real, because $M(0)$ and $M^{(2)}(K)$ are real for real K; so every corrected variation is real. On these 24 rows the sixty columns have rank 22 and the target column raises it to 23; the certificate has 24 integer entries, the largest of 64 digits, and excludes complex coefficients directly. At $\kappa = \frac{1}{10}$ the first 14 configurations of (a) give 28 real rows, on which the ranks are 26 and 27; the certificate has 28 integer entries, the largest of 246 digits. Complex

coefficients are excluded through real ones as in (a): the conjugate of a physical configuration is physical; the conjugates of its admixtures and gauge part are admixtures and a gauge part of the conjugate configuration, because $M(0)$ is real and $M^{(2)}(K)$ and the expansion of the displacement mode have real coefficients; and every corrected variation, a polynomial with rational coefficients in these data, takes the conjugate value.

The choices. Replacing $w_i^\perp$ by $w_i^\perp + \Phi(Z)$, or $w_4^\perp$ by $w_4^\perp + \Phi(Z)$, changes $\mathscr{C}[T]$ by an order-zero cell vertex with four metric legs. That vanishes for every one of the sixty-one tables: for Regge's by Lemma 4.3, and for a shape cost because the cost vanishes on the flat family, where every deficit is zero, so that the argument of Lemma 4.3 applies. For the target the four correction terms vanish, because its order-zero vertex vanishes on three metric legs and any edge leg (Lemma 8.2(b)); so $\mathscr{C}[-\frac{1}{48}T_4] = -\mathscr{V}[-\frac{1}{48}T_4]$, its metric-sector value.

The Taylor statement. Expand the cell vertex of T with the stated legs in powers of η. The displacement mode is $i\eta\,\Phi(H_4) - \frac{i}{12}\eta^3(d\cdot\xi)(K_4\cdot d)^3 + O(\eta^5)$, the graviton legs have no term of order one, and each phase is $1 + i\eta K_i\cdot m + \cdots$. The term of order η^0 vanishes because the gauge leg starts at order η. The term of order η is $i\,\mathscr{X}^{(0)}[T](\Phi(H_1),\Phi(H_2),\Phi(H_3),\Phi(H_4))$, an order-zero vertex on four metric legs, which vanishes as above. The term of order η^2 is $i\,\mathscr{X}^{(1)}[T](\Phi(H_1),\dots,\Phi(H_4);K)$. It vanishes because the part of first order in the momenta of each of the sixty-one cell vertices vanishes on four metric legs whose momenta sum to zero: for Regge's table by Lemma 4.3; for a shape cost because its quartic part at each hinge is a cubic monomial in the hinge's squared lengths times the first-order change $\delta_h^{(1)}$ of the deficit, and $\delta_h^{(1)}$ of a metric wave is of second order in its momentum. Near the hinge, to first order in K, the wave is a constant strain plus a strain $x^c\Lambda_{c,ab}$ linear in x; that strain is $\partial_a\varpi_b + \partial_b\varpi_a$ for the quadratic displacement $\varpi_b = \frac{1}{4}x^a x^c(\Lambda_{a,bc} + \Lambda_{c,ab} - \Lambda_{b,ac})$, and for a quadratic displacement, sampling its strain at the edge midpoints as in (4) gives the exact first-order change of the squared lengths. A constant strain and a displacement both keep the lattice flat, so $\delta_h^{(1)}$ vanishes on both. The term of order η^3 collects $-i\,\mathscr{V}[T](H_1,H_2,H_3,H_4;K)$ from the phases of second order, $i\,\mathscr{X}^{(0)}[T]$ with one admixture and $\Phi(H_4)$, and $i\,\mathscr{X}^{(0)}[T]$ with the third-order part of the displacement mode, which may be replaced by $w_4^\perp$ because the two differ by a metric field; it is $i\,\mathscr{C}[T]$. At $\kappa = 1$ every quantity in $\mathscr{C}[T]$ is real on real configurations.

The configurations, the admixtures, the gauge parts and the certificates have integer, Gaussian-integer or exact rational entries. Supplement S8 rechecks the certificates of (a), and S6 those of (b), configuration by configuration, together with the admixtures and the gauge parts, the independence of the choices, and the Taylor coefficients of orders one, two and three at every configuration of (b), at $\kappa = \frac{1}{10}$ also at the conjugate configurations. □

At $\kappa = 0$ part (b) is not stated: the admixture equation is singular there, because of the body diagonal (Proposition 8.3). At $\kappa = \frac{1}{10}$ the null momenta are complex, so the Taylor statement of (b) concerns the lattice vertex continued to complex momenta; the expansion $i\,\mathscr{C}[T]\,\eta^3 + O(\eta^4)$ holds there as it stands.

Remark 10.4 (What the obstruction says, and what it does not)**.** One step of the reading is an INFERENCE: that continuing a changed coupling ν to quartic order requires exactly the shortfall $(1-\nu)^2 S_4$ in the gauge variation. The argument is the quartic Noether identity at coupling ν, an identity of the long-wave vertices; it is not one of the theorems, and it does

not follow from an exact symmetry of the lattice action, whose displacement symmetry is broken away from flat configurations [16]. Granting it, the theorem says that within shape costs, changed through their sixty third-derivative directions, the coupling direction Y is an infinitesimal freedom only: no finite change of the coupling is completed at quartic order, at $\kappa = 0$, $\frac{1}{10}$ and 1, and at $\frac{1}{10}$ and 1 also with the non-metric modes. In the metric sector the lower jets add nothing, by the exact statement above; with the non-metric modes, part (b) concerns the sixty directions, since some inversion-odd changes of the lower jets couple a graviton to the non-metric modes at first order in the momenta, which the admixtures of Definition 10.2 do not include. Costs of other kinds are not covered: among them the rescaled action of the opening, which realizes every coupling exactly, and costs nonlinear in the deficit or depending on more than a hinge's class and three squared lengths. The obstruction is about a changed coupling. It does not touch Regge's own action, whose quartic vertex is one half of Einstein's (Theorem 6.1) and satisfies the quartic Noether identity (Theorems 9.2 and 9.3).

11 What this paper does not say

- *Tree level.* The vertices and the amplitude are classical. The one-loop graviton of the lattice is treated in [3, 2].
- *Long wavelength.* Only the terms with two derivatives are compared. The lattice action has higher-derivative terms, which these theorems do not address.
- *Metric sector.* The theorems use the ten metric combinations of the fifteen edge classes. At $\kappa = 1$ and $\kappa = \frac{1}{10}$ the other five are nondegenerate at zero momentum, so they can be eliminated at tree level, and eliminating them changes no two-derivative vertex (Lemma 8.4). The paper makes no claim about their physical character, which depends on their two-derivative block, not computed here; at $\kappa = 1$ their zero-momentum form has inertia $(1, 4)$. At $\kappa = 0$ the body diagonal is a zero mode, the results describe the metric truncation, and the mode's effect on gauge-invariant amplitudes is open (Proposition 8.3). Theorem 10.3(b) includes the five modes.
- *Two named members.* The main theorem is stated at $\kappa = 0$ and $\kappa = 1$, and Remark 3.4 extends its chain to every $\kappa < \frac{1}{4}$ and every $\kappa > \frac{1}{2}$. On the Euclidean members the identities also follow from the convergence theorem of Cheeger, Müller and Schrader (Remark 3.5); what is proved here for them is the exact form with explicit tables.
- *The lattice in the recognition program.* The program of [1, 2] reads a unit cell of the lattice as the records of four accounts, each posted at most once: the vertex $x + \sum_{a \in B} e_a$ is the record in which the accounts in B hold a posting, a single step is one *posting*, and the cell's 24 simplices are the 24 orders of its four postings [1, Sec. 2]. The program takes $\kappa = 1$ as the physical lattice: among the members with $\kappa > \frac{1}{2}$, with a fixed positive time advanced by each posting, it is the one at which the lattice interval reads the order of the postings on some pair of positive duration [2, Thm. 2.2(ii)]. Each reading is an IDENTIFICATION, and no theorem here uses either; the program's premises also admit other values of κ [2, Thm. 2.3].
- *Newton's constant.* The theorems fix the action up to its scale s_{p}, the action per unit area per unit deficit of the charge $s_{\mathrm{p}} A\delta$, and reading $s_{\mathrm{p}} S_{\mathrm{R}}$ as Einstein–Hilbert's action gives

$G = 1/(8\pi s_{\mathrm{p}})$ (Section 1). The scale is not derived here. In [6] it is tied to the lattice's unit of length λ_{rec} by the *posting identification* [6, Thm. 7.2], an IDENTIFICATION: a posting on a timelike hinge is a deficit of $\pi/6$, its action adds over area, and a full turn of twelve postings on a face of area λ_{rec}^2 carries the action $\hbar \ln 2$ of one bit. Then $s_{\mathrm{p}} = \hbar \ln 2/(2\pi\lambda_{\mathrm{rec}}^2)$ and $G\hbar = \lambda_{\mathrm{rec}}^2/(4 \ln 2)$, which fixes λ_{rec} in terms of G rather than predicting G [6, 7].

- *Which action.* Why the lattice carries Regge's action is not addressed here. A *hinge charge* is a cost $C(A_h, \delta_h)$ of each hinge's area and deficit, with $C(A, 0) = 0$, summed over the hinges. In the deposited preprints, flat-space stationarity at every stretched flat lattice fixes a hinge charge's slope in the deficit, and with it the long-wave quadratic and cubic vertices around the $\kappa = 1$ lattice [5, Thm. 3.1]. Two premises fix the charge itself [6, Prop. 7.3]: (B), that it depends on area and deficit only through $A\delta$, and (C), the *record clause*, that one posting on a face of area A carries the posting action $s_{\mathrm{p}} A\pi/6$ and a credit, a deficit of $-\pi/6$, carries minus it. Together they give $C(A, \delta) = s_{\mathrm{p}} A\delta$. Section 10 shows that, at $\kappa = 0$, $\frac{1}{10}$ and 1 and with one stated INFERENCE, hinge shape costs cannot complete a changed coupling at quartic order; whether conditions at quartic order single out Regge's charge in a wider class is not settled here.
- *Solutions.* The theorems concern vertices. They do not construct lattice solutions or prove their convergence to solutions of Einstein's equations; for $\kappa > \frac{1}{2}$ the lattice field equation is the subject of [4].

A Conventions and the fifty-three representatives

The canonical simplex. Its vertices are $v_0 = 0$, $v_1 = e_1$, $v_2 = e_1 + e_2$, $v_3 = e_1 + e_2 + e_3$, $v_4 = \mathbf{1}$. The ten slots, in the order used by every table, with their steps d_f and doubled midpoints $2m_f = v_j + v_k$:

slot	step d_f	$2m_f$	slot	step d_f	$2m_f$
01	$(1,0,0,0)$	$(1,0,0,0)$	13	$(0,1,1,0)$	$(2,1,1,0)$
02	$(1,1,0,0)$	$(1,1,0,0)$	14	$(0,1,1,1)$	$(2,1,1,1)$
03	$(1,1,1,0)$	$(1,1,1,0)$	23	$(0,0,1,0)$	$(2,2,1,0)$
04	$(1,1,1,1)$	$(1,1,1,1)$	24	$(0,0,1,1)$	$(2,2,1,1)$
12	$(0,1,0,0)$	$(2,1,0,0)$	34	$(0,0,0,1)$	$(2,2,2,1)$

The simplex of axis order π has the steps and midpoints of the canonical one with coordinate a replaced by coordinate π_a.

The representatives. Table 1 lists one representative of each orbit of ordered quadruples of symmetric basis matrices under leg permutations and the 24 relabellings of the axes. The entry (ab) stands for E_{ab}, $a \le b$.

Table 1. The 53 representatives of Theorem 6.1's proof, as quadruples of basis matrices E_{ab}.

1	(11, 11, 11, 11)	15	(11, 11, 23, 23)	29	(11, 12, 22, 34)	43	(12, 12, 12, 12)
2	(11, 11, 11, 12)	16	(11, 11, 23, 24)	30	(11, 12, 23, 23)	44	(12, 12, 12, 13)
3	(11, 11, 11, 22)	17	(11, 12, 12, 12)	31	(11, 12, 23, 24)	45	(12, 12, 12, 34)
4	(11, 11, 11, 23)	18	(11, 12, 12, 13)	32	(11, 12, 23, 33)	46	(12, 12, 13, 13)
5	(11, 11, 12, 12)	19	(11, 12, 12, 22)	33	(11, 12, 23, 34)	47	(12, 12, 13, 14)
6	(11, 11, 12, 13)	20	(11, 12, 12, 23)	34	(11, 12, 23, 44)	48	(12, 12, 13, 23)
7	(11, 11, 12, 22)	21	(11, 12, 12, 33)	35	(11, 12, 33, 34)	49	(12, 12, 13, 24)
8	(11, 11, 12, 23)	22	(11, 12, 12, 34)	36	(11, 12, 33, 44)	50	(12, 12, 13, 34)
9	(11, 11, 12, 33)	23	(11, 12, 13, 14)	37	(11, 12, 34, 34)	51	(12, 12, 34, 34)
10	(11, 11, 12, 34)	24	(11, 12, 13, 22)	38	(11, 22, 33, 44)	52	(12, 13, 14, 23)
11	(11, 11, 22, 22)	25	(11, 12, 13, 23)	39	(11, 22, 34, 34)	53	(12, 13, 24, 34)
12	(11, 11, 22, 23)	26	(11, 12, 13, 24)	40	(11, 23, 23, 23)		
13	(11, 11, 22, 33)	27	(11, 12, 13, 44)	41	(11, 23, 23, 24)		
14	(11, 11, 22, 34)	28	(11, 12, 22, 33)	42	(11, 23, 24, 34)		

Verification and data availability

Every theorem, proposition, lemma and corollary of the paper is proved in the text or by an exact computation in the supplement [8], which is archived with the paper. The supplement holds exact data and short scripts. Each of its eight parts describes the statements it checks, the data format, how to run the checks and the output observed, and its index lists, for each result of the paper, the checks that carry it out. Every exact check uses integer, rational, Gaussian-rational or exact algebraic arithmetic; the few floating-point cross-checks are labelled NUMERICAL and are part of no proof. The parts are:

S1 the tables T_2, T_3 and T_4 as rational functions of κ, with integer numerators, and their values at $\kappa = 0$, $\frac{1}{10}$ and 1;

S2 Proposition 4.2: the derivatives of the Euclidean and Lorentzian simplex functions against the tables, with κ kept as a variable, and at $\kappa = 1$ in exact arithmetic with $\sqrt{3}$; Schläfli's identity along every line through ℓ_κ to the order used; the sixty shape-cost tables, their normalization and their Lorentzian sign; and the lower jets of the shape costs (Section 10): the independence of their quadratic and cubic tables, and the inversion-odd space of their changes with no long-wave quadratic or cubic vertex;

S3 Theorem 5.1 on all ordered pairs and on the twenty cubic representatives with κ kept as a variable, the cubic identity on all 1000 ordered triples at $\kappa = 0$ and 1, the Fierz–Pauli form at $\gamma = I$ on all 100 ordered pairs of basis matrices, Lemma 4.3, and the two examples of Sections 2 and 6: sampling at one end of each edge, and momenta that do not sum to zero;

S4 Theorem 6.1: the 53 identities (11), the identity on all 715 unordered quadruples, and the orbit decomposition of Table 1;

S5 the total-derivative step of Theorem 7.1 in the jet algebra;

S6 Section 8: the fifteen-class form, Lemmas 8.2 and 8.4 and Proposition 8.3; and Theorem 10.3(b) at $\kappa = 1$ and $\frac{1}{10}$: the configurations, admixtures, gauge parts, corrected

variations, certificates, the independence of the choices and the Taylor coefficients;

S7 Section 9: Theorem 9.2 on all basis cases at three backgrounds, Lemma 9.5, and Theorems 9.3 and 9.6 with the amplitude built from the tables, at $\kappa = 1$ through the frame E_1;

S8 Theorem 10.3(a): the certificates at $\kappa = 0$, $\frac{1}{10}$ and 1, with a verifier that recomputes every vertex from the tables.

The supplement's checksum list gives the SHA-256 hash of each of its files; the SHA-256 hash of that list, for version 1.0, is

`9f80897365ce5f8e03ed3f9d250f55e2c7975cf312f670c2ead28808318d83b5`

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