

Quantum Gravity from the Recognition Ledger: Weighted One-Loop Graviton Responses with Finite Real Parts, Einstein Gravity at Long Wavelengths, and the Massless-Graviton Problem

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Abstract

A quantum theory of gravity built from the recognition ledger must meet four demands: a lattice, action and measure derived, not chosen; finiteness, beginning at one loop; a massless graviton; and Einstein gravity at long wavelengths. On the Kuhn lattice with flat metric $I - \kappa \mathbf{1}\mathbf{1}^{\top}$, massless corner doublers occur exactly at $\kappa = 1$, where the lattice interval reads the ledger's posting order; the foundation does not decide κ, so $\kappa = 1$ is an identification. Our one-loop statements concern a stated diagram model: a unit-weight gauge term, no ghost line, the Feynman $i\varepsilon$. A measure weight built from the ledger's cost of curvature, under two named identifications and a channel count, gaps all four doublers, vanishes to fourth order on long waves, and gives the real parts of the model's one-loop graviton responses finite ordered limits (Theorem 7.4), where the native sum diverges. With it the record-time roots of the Gaussian path sum are stable in the continuum limit at fixed weight strength, given an inference about off-circle modes, though the sum is damped on the lattice, where the bare Regge sum has a growing mode. A Ward identity decides whether the model's graviton stays massless at one loop: the zero-momentum remainder is $i/2$ times a difference of two Ward defects and vanishes if the determinant is invariant under volume-preserving linear vertex moves. No real weight or density restores the symmetry, no phase $cS + \sum_h \delta_h^3 \Gamma_h$ has it or changes the remainder, every candidate computed to convergence leaves a nonzero remainder, and the native remainder cannot vanish in the ordered limit. Classically, linearized Einstein gravity, Newton's law with $G = 1/(8\pi s)$ and, under named identifications, the second order with matter that conserves postings are recovered at $\kappa = 1$, and every long-wave vertex of the action is Einstein's, as field equations on sampled jets. A massless graviton, through a derived action with a vanishing Ward defect, is the paper's precise open target, and the exclusions are results; κ, the measure identifications, a ghost or gauge measure, a norm-preserving record-time evolution and the nonlinear limit are open too.

1 What a quantum theory of gravity on the ledger has to show

Recognition Science describes events as records kept on a ledger. Four counters advance one step at a time, each step a *posting*, and the four-dimensional lattice of records they generate carries an order before it carries any geometry [1]. Gravity should then be a sum over the geometries this lattice can carry, in real time, with an action and a measure that the ledger supplies. The question of this paper is whether such a path sum is a quantum theory of gravity, and a positive answer needs four things.

Derived. The lattice, the one number that fixes its flat geometry, the action and the measure must follow from the ledger. A value chosen to make the graviton work would not be a derivation, and neither would a weight tuned to cancel a computed number. *Finite.* The path sum and its physical observables must be well defined. The first question, treated here, is the one-loop limit, taken in a fixed order: first the torus grows at a fixed regulator $\varepsilon > 0$, then the regulator is removed. *A massless graviton.* The graviton's propagator must have a massless pole. At one loop a necessary condition is that the self-energy produce no mass or anisotropy at zero momentum. *Einstein gravity at long wavelengths.* Waves and sources spread over many lattice spacings must obey Einstein's equations, with Newton's constant fixed by the action.

Where the question stood. An earlier paper [2] studied the lattice with the Lorentzian Regge action [3] on the Kuhn triangulation of the hypercubic lattice [4, 5], with flat metric $g_\kappa = I - \kappa \mathbf{1}\mathbf{1}^\top$ in record coordinates, at $\kappa = 1$, where every posting is null. Two results came out of it. Classically the lattice works: linearized Einstein gravity is the long-wavelength limit of exact lattice solutions with matter, and a compact body has Newton's potential with $G = 1/(8\pi s)$, where s is the action scale. Quantum mechanically it fails: four massless doublers at the axis corners of the Brillouin zone make the zero-momentum anisotropy of the one-loop graviton self-energy of a stated diagram model diverge like $\log(1/\varepsilon)$, so the ordered limit does not exist. The value $\kappa = 1$ was an identification, and the measure was chosen.

What this paper proves. Seven results move the question forward; where a result is recalled from [2] we say so.

1. *The doublers and κ are one question.* In the whole family $\kappa > 1/2$, massless corner doublers occur if and only if $\kappa = 1$, and then only at the four axis corners. The same value is the one at which the lattice interval reads the posting order, the least value at which the order is causal [2, Thms. 2.1 and 2.2], and the one at which a history of postings is null. The foundation as stated admits a world at $\kappa = 1$ and a world at $\kappa = 3/2$ that satisfy the same premises (Section 2). A derivation of $\kappa = 1$ would therefore derive the doublers, and a quantum theory at $\kappa = 1$ has to deal with them.
2. *A measure built from the ledger gaps the doublers.* At Gaussian order a density at a fixed scale drops out and a weight at the action's scale acts through one positive semidefinite form, which the limit determines. The ledger's cost of the curvature around each hinge fixes it with no free coefficient, given two named identifications, (I1) and (I2) of Section 5, and a

Table 1. The ten parts of the programme and their status. "Proved" means proved as stated in the section named, with the premises listed there; "identification" marks a physical premise that is not a consequence of the definitions; "open" marks a precise statement that is neither proved nor refuted; "pending" marks a step whose proof is complete and whose independent verification is not yet finished. Results recalled from [2] are marked [2].

part	status	where
1 lattice and κ	proved: doublers $\Leftrightarrow$ $\kappa = 1$ $\Leftrightarrow$ the interval reads the posting order (the second equivalence with [2]); the foundation does not decide κ, no causal premise can, and one premise more suffices. Open: a premise of the ledger that forces κ, and the carrier itself	Section 2
2 action	proved: the Ward criterion, the no-go theorems, what refinement does; local repairs largely excluded, each exclusion a result; neither non-local route gives a repair at $\kappa = 1$ (pending). Open, and decisive: a derived action with the classical limit whose summed Ward defect vanishes	Section 6
3 measure	proved under two identifications (the cost of a hinge record reads its area; the weight of a geometry is the Gibbs weight of its cost) and a channel count	Section 5
4 Einstein at long wavelengths	proved: linear limit, Newton's law ($\kappa = 1$) [2]; second order with worldline matter, on the lattice and in the continuum (identifications of the matter law); every long-wave vertex Einstein's on sampled jets; vertex symmetry fails at second order. Open: nonlinear lattice solutions	Section 4
5 massless graviton	open: the doublers are gapped (part 3); in the one-loop model the remainder is nonzero for every computed candidate, and the native graviton is not massless	Sections 3 and 6
6 finite	proved for the real parts of the model's weighted one-loop responses, per probe and per diagram; imaginary part pending. Open: the full path sum and a theory that also solves part 2	Section 7
7 unitarity	open; record time keeps the doublers, the $\kappa = 1$ Regge equation admits no tick-invariant positive norm on its physical solutions, and a weight is a damping; with the derived weight the record-time roots are stable in the continuum limit at fixed weight strength (roots, not a norm), under the weight's identifications and an inference about off-circle modes	Section 8.1
8 matter	proved for worldline matter and a posting gas under an identification; open: the equation of state from the cost	Section 8.2
9 predictions	tree level: no signature below 10^6 Planck lengths [2]. The unrepaired path sum, read through two identifications (the lattice spacing and the map from the remainder to a mass): a one-loop mass scale of the model fifty-one orders above the bound, a conditional diagnostic. A repaired path sum's predictions wait for parts 2 and 5	Section 8.3
10 foundation	each result's premises are listed; a single derivation from the foundation to every claim is open	Section 9

channel count that is a definition; the resulting weight gaps all four corners while vanishing to fourth order in the wave number, so it leaves the classical limit alone (Section 5).

3. *With that weight the model's one-loop graviton responses have finite real parts.* The real part of the zero-momentum anisotropy that diverges for the native path sum has a finite ordered limit when the derived weight is included, together with the weight's own interaction vertices, and so do the real parts of each probe's contribution and of each diagram separately (Section 7).

4. *A Ward criterion decides whether the model's graviton is massless.* If the one-loop determinant is invariant under volume-preserving linear moves of the vertices, the zero-momentum transverse-traceless remainder vanishes; in general the remainder equals $i/2$ times the difference of the second- and first-order Ward defects. No real measure weight or density restores the symmetry, no phase of the form $cS + \sum_h \delta_h^3 \Gamma_h$ has it, and at the same slope and weight such a phase has exactly the remainder of cS; refinement, which would force the remainder to vanish, is not a property of the identified lattice: at $\kappa = 1$ it carries the doublers to zero momentum (Section 6).

5. *The classical limit at second order* with worldline matter, under identifications of the matter law, in addition to the linear limit and Newton's law recalled from [2]. Every long-wave vertex of the action is Einstein's: on sampled metric jets the metric part of the lattice field equations is the full Einstein density, at every order in the field; this concerns the field equations, not the existence of nonlinear lattice solutions. The lattice's vertex symmetry fails at second order, one order in the derivatives past Einstein's equations. And a consistency estimate controls the lattice metric's second differences (Section 4).

6. *Two exact obstructions for the identified path sum.* The graviton of its model determinant is not massless at one loop (Section 3); and in record time it has a growing solution, so no positive norm on its physical solutions is preserved by the evolution (Section 8.1).

7. *Stability in the continuum limit with the derived weight.* With the weight of item 2 the Gaussian path sum is damped on the lattice, but every lattice species away from the zero wave decouples and the graviton's branch grows by at most the cube of its momentum per tick, so the record-time roots are stable in the continuum limit at each fixed strength of the weight: every family of modes that approaches the unit circle grows by a vanishing amount per unit physical time. This is a statement about roots, not a norm-preserving evolution, and it holds given the weight's identifications and an inference about lattice-scale modes off the unit circle; a limit in which the spacing itself shrinks changes that strength and would need bounds uniform in it (Section 8.1).

What it does not yet prove. Two of the four are not yet earned. The graviton is not shown to be massless: every candidate action and weight computed to convergence leaves a nonzero remainder of the model determinant, and the action whose Ward defect vanishes, which would settle the matter, has not been found. The paper states it as a precise open target; repairs by local data are largely excluded, each exclusion a result (Section 6.7). Neither of the two non-local routes examined supplies the repair at $\kappa = 1$: the ledger's holonomy phase is the Regge phase, and the action its refinement map induces has no fixed point with an Einstein–Hilbert long-wave part in either order of limits, including the one, with the

damping held fixed on the coarse lattice, that tells it apart from ordinary block averages (proofs complete, verification pending). Beyond these routes, what remains is a phase with an even part outside the ledger's present readings, or with a slope map or even kernel that is not that of any local charge. And the path sum is not yet fully derived: $\kappa = 1$ remains an identification, and the measure rests on two identifications that the ledger has yet to supply. Every one-loop statement, finite or not, concerns the diagram model of Section 3, with a unit-weight gauge term and no ghost line; a ghost or gauge measure derived from the ledger could change it. Finiteness beyond the real parts of the one-loop responses treated here, the nonlinear Einstein equations as the long-wavelength limit, an evolution in record time with a positive norm on its physical solutions (the identified Regge path sum has none, Section 8.1), matter whose equation of state follows from the cost, and a measurable prediction are open as well. Table 1 gives the status of each of the ten parts of the programme, and each part's section opens with a box that says what is proved, under which premises, what is open, and what would settle it. A step whose proof is complete but whose independent verification is not yet finished is never listed as proved: it has its own category in the boxes and a footnote in the text, and no result rests on one without saying so.

Relation to other work. Regge calculus [3] on the Kuhn lattice goes back to Roček and Williams [6, 7]; its broken vertex-translation symmetry away from flat space was studied by Bahr and Dittrich [8], who also showed that *perfect* actions, built to reproduce the continuum dynamics on a triangulation, capture that symmetry exactly, and constructed them in three dimensions and for a single four-simplex [9]. An exact vertex symmetry is what the Ward criterion of Section 6 asks for, and Theorem 4.3 is an exact instance of the breaking on the Kuhn lattice: the symmetry already fails at second order around flat space. Doublers are familiar for lattice fermions [10]; here they arise for a boson, because the light cone of a real-time lattice operator reaches the edge of the Brillouin zone. Lorentzian simplicial path integrals are an active subject [11, 12, 13, 14], and causal dynamical triangulations sum over triangulations after a Wick rotation [15, 16]; the present path sum is real time on a fixed graph. The one-loop correction to the Newtonian potential is the result of Donoghue [17] and of Bjerrum-Bohr, Donoghue and Holstein [18]. What is new here is the equivalence between the doublers and the value of κ, a measure derived from the ledger's cost with its effect on the doublers and on finiteness, and an exact criterion, with its no-go theorems, for the vanishing of the one-loop remainder that a massless graviton requires. Three companion papers take the same lattice further: to the value of Newton's constant [19], to black holes [20] and to the cosmological constant [21].

The sections can be read separately. Section 2 treats the lattice and κ; Section 3 shows why the identified theory fails; Section 4 the classical limit; Section 5 the measure; Section 6 the Ward criterion and the massless graviton; Section 7 finiteness; Section 8 unitarity, matter and predictions; Section 9 which premises come from the ledger and which results are lattice mathematics, with the premises of every result; Section 10 what remains, in order.

2 The lattice and the value of κ

Status: part 1, the lattice and κ

Proved. In the family of flat metrics $I - \kappa \mathbf{1}\mathbf{1}^\top$ with $\kappa > 1/2$, massless corner doublers occur exactly at $\kappa = 1$, and then only at the four axis corners (Theorem 2.1). The same value is equivalent to the interval reading the posting order on an interval of positive duration, to κ being the least causal value, and to some history of record cells being null (Theorem 2.2). Two worlds, at $\kappa = 1$ and at $\kappa = 3/2$, satisfy the same premises of the foundation as stated (Theorem 2.3).

Identification. $\kappa = 1$: the physical interval reads the posting order on at least one interval of positive duration.

Open. A premise of the ledger, stated without κ, that forces its value; and the derivation of the carrier (records on four counters, with the 24 arrival orders as the Kuhn simplices) from the ledger.

What would settle it. A theorem deriving the premise from the ledger. By Theorems 2.1 and 2.2 a premise that forces $\kappa = 1$ forces the doublers, so it settles part 1 only together with a measure that gaps them (Section 5).

2.1 Records, postings and the Kuhn lattice

Vertices are points $x \in (\mathbb{Z}/N)^4$, the side-N torus, or points of $\mathbb{Z}^4$. We call the coordinates x_a *record coordinates*: in the programme that motivates the lattice, an event is a record kept on four counters, and x_a is the reading of counter a [1]. Write e_a for the unit vectors and $\mathbf{1} = (1, 1, 1, 1)$. Four counters advanced one step at a time from $(0, 0, 0, 0)$ to $(1, 1, 1, 1)$ have $4! = 24$ arrival orders, and these are exactly the 24 simplices of Kuhn's triangulation of the unit cube [4, 5]: for each ordering $(a_1, \dots, a_4)$ of the axes, the simplex with vertices x, $x + e_{a_1}$, $x + e_{a_1} + e_{a_2}$, $x + e_{a_1} + e_{a_2} + e_{a_3}$ and $x + \mathbf{1}$. Every edge joins x to $x + d$ for one of the fifteen *edge classes* $d \in \{0, 1\}^4 \setminus \{0\}$; $|d|$ is the number of ones in d. An edge with $|d| = 1$ is a *posting*, the record of one counter's step.

A configuration is a real squared length ℓ_e on every edge. The flat lattice of parameter κ has the metric

$$g_\kappa = I - \kappa\, \mathbf{1}\mathbf{1}^\top, \qquad g_\kappa(d, d) = |d|\,(1 - \kappa|d|), \qquad \det g_\kappa = 1 - 4\kappa$$

in record coordinates. For $\kappa > 1/4$ it is Lorentzian, with exactly one negative eigenvalue $1 - 4\kappa$; its eigenvector $\mathbf{1}$ is timelike and is the time direction, and every direction inside its cone is timelike too. At $\kappa = 1$ the flat squared lengths are $0, -2, -6, -12$ for $|d| = 1, 2, 3, 4$: every posting is null and every longer edge is timelike. For every $\kappa > 1/2$ every triangle is timelike and the Lorentzian Regge action [3] is nondegenerate, so results proved for all $\kappa > 1/2$ use a well-defined action [2, Sec. 2.1]. The configuration space is a fixed graph with varying squared lengths; we never sum over triangulations.

2.2 The posting order and the readings of κ

Postings carry an order before they carry a metric. An event is a pair of a time $t \in \mathbb{R}$ and a position x in the hyperplane $\sum_a x_a = 0$; a posting on axis a advances the time by a fixed $\tau > 0$ and the position by $\sigma_a = e_a - \frac{1}{4}\mathbf{1}$, of length $|\sigma_a| = \sqrt{1 - \frac{1}{2} + \frac{1}{4}} = \sqrt{3}/2$. The *posting order* is the closure of the relation generated by chains of postings; for an ordered pair of events its *content* c_a counts the postings on axis a it contains, and the pair is in the order exactly when every $c_a \geq 0$ [2, App. A]. In contents the lattice interval is $Q_\kappa(w) = \sum_a w_a^2 - \kappa(\sum_a w_a)^2$, which is g_κ in record coordinates.

Two facts from [2, Thms. 2.1 and 2.2] frame the question. Every pair in the posting order has future-causal content for Q_κ if and only if $\kappa \geq 1$: a single posting has $Q_\kappa(e_a) = 1 - \kappa$. And Q_κ *reads the order* on some pair of positive duration, in the sense that the interval's notion of "lying causally between" agrees with the order's, if and only if $\kappa = 1$. The plain reading: the posting moves the position by $\sqrt{3}/2$ per tick, an interval with cone speed v_c is Q_κ with $\kappa = \frac{1}{4} + v_c^2$, and $\kappa = 1$ says that the cone speed equals the posting speed.

2.3 The doublers are the value $\kappa = 1$

Fix $\kappa > 1/2$. At a wave $k \in [-\pi, \pi]^4$ write $B_\kappa(k)$ for the 15×15 Bloch matrix of the flat Hessian of the Regge action on the edge classes, $\Gamma_\kappa(k)$ for the 15×4 matrix whose columns are the four vertex moves, and $A_\kappa(k) = B_\kappa(k) + \Gamma_\kappa(k)\Gamma_\kappa(k)^\dagger$ for the gauge-fixed operator [2, Sec. 4.2]. At a corner $k = \pi e$ of the zone, $e \in \{0,1\}^4 \setminus \{0\}$, every Bloch phase is ± 1 and $A_\kappa(\pi e)$ is a real symmetric matrix. A zero of its determinant is a massless excitation at the edge of the zone: a doubler.

Theorem 2.1 (The corner theorem)**.** *Let $\kappa > 1/2$ and $e \in \{0,1\}^4 \setminus \{0\}$. Then $\det A_\kappa(\pi e) = 0$ if and only if $|e| = 1$ and $\kappa = 1$. Equivalently, $B_\kappa(\pi e)$ has a zero mode that is not a vertex move if and only if $|e| = 1$ and $\kappa = 1$.*

So the whole family has massless corner doublers at one value of κ, and there only at the four axis corners πe_c. The proof is exact arithmetic. The parameter enters the flat lattice only through the squared lengths $|d|(1 - \kappa|d|)$, and at a corner the operator $\sqrt{4\kappa - 1}\, B_\kappa + \Gamma_\kappa \Gamma_\kappa^\dagger$, which has the kernel of A_κ and the determinant $(4\kappa - 1)^{11/2} \det A_\kappa$, is the affine pencil $M_e(\kappa) = \frac{1}{2}(P_e + \kappa Q_e)$ with integer symmetric matrices P_e, Q_e. Its determinant is a polynomial in κ that factors completely. Up to a positive constant that depends only on $|e|$,

$$
\begin{aligned}
|e| = 1 &: \quad (\kappa - 1)\,\kappa\,(7\kappa - 2)(12\kappa^2 - 1)(2\kappa - 1)^2(3\kappa - 1)^2, \\
|e| = 2 &: \quad \kappa\,(4\kappa + 1)(16\kappa^2 - 13\kappa + 3)(2\kappa - 1)^2(4\kappa - 1)^3, \\
|e| = 3 &: \quad -\kappa\,(5\kappa - 2)(10\kappa - 3)^2(8\kappa^2 + \kappa - 1)^2(2\kappa - 1)^2(3\kappa - 1), \\
|e| = 4 &: \quad \kappa\,(2\kappa - 1)^4(4\kappa - 1)^3.
\end{aligned}
$$

Every root other than $\kappa = 1$ lies at or below $1/2$, and $16\kappa^2 - 13\kappa + 3$ has no real root. The proof certifies the nonvanishing by exact polynomial certificates: at each corner, a matrix of polynomials in κ that turns the corner matrix into a diagonal one whose entries are nonzero constants times products of the factors above. The sign of the axis-corner determinant on either side of $\kappa = 1$ comes from an exact $LDL^\top$ factorization.

The mechanism is visible in one vector. At the axis corner πe_c, the zero mode v_c of [2, Prop. 6.1] does not depend on κ, and $P_{e_c} v_c = -2w_c$, $Q_{e_c} v_c = 2w_c$ for a fixed nonzero vector w_c, so

$$M_{e_c}(\kappa)\, v_c = (\kappa - 1)\, w_c, \qquad \langle v_c, M_{e_c}(\kappa)\, v_c \rangle = \tfrac{11}{32}(\kappa - 1).$$

The restoring force on the doubler is minus the flat squared length $1 - \kappa$ of a posting, times a fixed vector: when postings are null there is nothing to restore it. The determinant of the real axis-corner pencil is negative on $(\frac{1}{2}, 1)$, zero at 1 and positive on $(1, \infty)$. One eigenvalue crosses zero as κ crosses 1, and every $\kappa > 1$ gaps all four corners at first order in $\kappa - 1$.

The same value has three other descriptions, and they are all equivalent to the doublers.

Theorem 2.2 (Four readings of one value)**.** *Let $\tau > 0$ and $\kappa > 1/2$. The following are equivalent:*

(i) the lattice has a massless corner doubler, $\det A_\kappa(\pi e) = 0$ *for some corner;*

(ii) Q_κ reads the posting order on some pair of positive duration;

(iii) κ is the least value for which every pair in the posting order has future-causal content;

(iv) $\kappa = 1$.

For $\kappa \geq 1$ they are also equivalent to: (v) some history of postings is null for Q_κ.

Proof. (i)⇔(iv) is Theorem 2.1; (ii)⇔(iv) is [2, Thm. 2.2(i)]; (iii)⇔(iv) is [2, Thm. 2.1]. For (v): a history of postings has content w in the closed orthant with integer entries. If $\kappa > 1$ and $w \neq 0$, then $Q_\kappa(w) \leq (1 - \kappa)(\sum_a w_a)^2 < 0$, so every history is strictly timelike; at $\kappa = 1$ a run of postings on one axis is null. □

In the plain reading of Section 2, $\kappa = \frac{1}{4} + v_c^2$ with v_c the cone speed in units of position per tick, so the doublers appear exactly when the cone speed equals the posting speed $\sqrt{3}/2$.

The foundation does not decide. The question is then whether the ledger forces $\kappa = 1$. The foundation from which the cost, the golden ratio and three spatial dimensions are derived [22] makes statements about costs and their composition, about a hierarchy of recognitions and about a completed recognition being kept. None of them involves the cone speed of a realization, which they constrain only through causality, $v_c^2 \geq 3/4$.

Theorem 2.3 (Two worlds)**.** *Let $\tau > 0$. The realization with $v_c = \sqrt{3}/2$ and the realization with $v_c = \sqrt{5}/2$ satisfy every premise of the foundation, with the same model and the same proof, and differ only in v_c. In the first, $\kappa = 1$: the interval reads the posting order, a run of postings is null, and the lattice has the four corner doublers. In the second, $\kappa = 3/2$: the interval reads the posting order on no pair of positive duration, every history of postings is strictly timelike, a null lattice displacement such as $(2, 1, -1, 0)$ is not a history, and no corner carries a doubler.*

The same holds with $\kappa = 5/4$ in place of $3/2$; what matters is that the premises allow more than one value. In fact the foundation with the causal bound has a model for every $\kappa \geq 1$, and no premise built from statements of the form "this displacement is causal" or "strictly timelike", combined by and, or and bounded quantifiers, can pick out one value, because

every interval decreases as κ grows. The gap is one premise wide. The weakest premise that, together with causality, gives exactly $\kappa = 1$ says that some nonzero history of postings is not strictly timelike, which is $\kappa \leq 1$. It has a plain reading as a count: minus the interval of a history counts κ for each ordered pair of postings on different counters and $\kappa - 1$ for each pair on one counter, a posting paired with itself included. If squared proper time counts only comparisons of two different postings, the interval must be the one with $\kappa = 1$; reading proper time as that count is an identification. So the identification $\kappa = 1$ is a physical premise added to the foundation, and by Theorem 2.2 it is the same premise as the existence of the doublers. Two consequences follow. A derivation of $\kappa = 1$ from the ledger would derive the four doublers, so a quantum theory at $\kappa = 1$ has to gap them by some other means; Section 5 shows that the measure can. And a derivation of another value would remove the doublers but would also give up the reading of the posting order, and every result proved only at $\kappa = 1$ would have to be proved again there. Results proved for every $\kappa > 1/2$, such as the linear field equations of Section 4, would stand.

Target 2.4 (A premise that decides κ)**.** A property Φ of realizations of the ledger, derived from the ledger and stated without reference to κ or to the cone speed, that some realization satisfies and that holds only in realizations with $v_c^2 = 3/4$.

The target cannot be met trivially. The property that always holds fails it, by Theorem 2.3; the property "the interval reads the posting order" meets it, but only by restating $\kappa = 1$. By Theorem 2.2, a Φ meets the target exactly when it is satisfiable and forces the corner doublers.

3 Why the identified lattice fails at one loop

This section recalls the obstruction that the rest of the paper works against. Everything in it is proved in [2] or below, at identified $\kappa = 1$, for the bilinear Regge action, at one loop and, wherever a measure enters, at Gaussian order.

The quantity. Let $B(k)$ be the Bloch matrix of the flat Hessian at a wave k of the Brillouin zone $[-\pi, \pi]^4$, $\Gamma(k)$ the 15×4 matrix of vertex-move columns, and $A(k) = B(k) + \Gamma(k)\Gamma(k)^\dagger$ the gauge-fixed operator in a Feynman-type gauge. The regulated propagator is $(A(k) + i\varepsilon)^{-1}$, with the damping on all fifteen edge channels. For a constant strain H, the one-loop self-energy at zero momentum is the second derivative along H of the one-loop effective action of constant strains; the *diagram model* of [2, eq. (7)] assembles it from the graviton sunset and seagull and one massless scalar for matter, and it is a model (MODEL): the gauge term with unit weight, no ghost line, the Feynman sign of $i\varepsilon$. We call its one-loop determinant the *model determinant*. Every statement in this paper about the one-loop self-energy, a graviton mass or its absence, and the remainder of Section 6 is a statement about the model determinant: a diagnostic of this specified one-loop model, which a ghost or gauge measure derived from the ledger could change. On two traceless spatial probes H_E and H_{T_2}, one in each irreducible representation of the axis permutations, every covariant quadratic form takes the same value after normalization, so the difference

$$\Delta(\varepsilon) = \lim_{N \to \infty} \mathrm{Re}\big(\sigma_N(\varepsilon; H_E) - \sigma_N(\varepsilon; H_{T_2})\big)$$

measures an anisotropy of the zero-momentum self-energy. A massless graviton of the model with a Lorentz-invariant long-wave limit needs $\lim_{\varepsilon\to 0^+}\Delta(\varepsilon)$ to exist.

The doublers. At each of the four axis corners πe_c of the zone, the operator A has a one-dimensional kernel spanned by an explicit vector v_c that lies in the kernels of both $B(\pi e_c)$ and $\Gamma(\pi e_c)^\dagger$; it is a zero mode for every gauge weight and is not a vertex move [2, Prop. 6.1]. Near the corner the Schur complement onto v_c is a quadratic form with one negative and three positive eigenvalues, so each corner carries a massless excitation with its own Lorentzian cone [2, Prop. 6.2]. In position space the mode is the staggered field $(-1)^{x_c}v_c(d)$.

Theorem 3.1 (The ordered anisotropy diverges [2, Thms. 6.7 and 6.8])**.** *Let $\kappa = 1$. Every critical zero of* det *A other than the lattice copies of the origin and of the axis corners is hyperbolic: some direction increases every vanishing eigenvalue at once. There is a real constant r_0 with*

$$\Delta(\varepsilon) = \frac{5}{44\sqrt{335}\,\pi^2}\log\frac{1}{\varepsilon} + r_0 + o(1) \qquad (\varepsilon \to 0^+),$$

so the ordered limit $\lim_{\varepsilon\to 0^+}\lim_{N\to\infty}\Delta_N(\varepsilon)$ does not exist; with the four corner contributions subtracted it does.

The mechanism is the one of the Coleman–Weinberg potential [23]. A constant strain moves the doublers' zero at first order, so their squared mass is linear in the strain; their vacuum energy then contains $\delta^2\log\delta$ in the strain δ, and its second derivative, the zero-momentum self-energy, grows like $\log(1/\varepsilon)$. At the corners the doubler eigenvalue vanishes quadratically with no direction along which it increases, so no deformation of the integration contour avoids the regulator; every other degenerate zero is hyperbolic and contributes a finite amount. This is a reading of the proof, not part of it.

Nothing native removes them. Within the same premises nothing in the construction gaps the doublers: neither the native measure, nor the twisted gauge slice, nor any bare weight of the kind examined [2, Thms. 6.10 and 6.11]. Moving κ above 1 gaps all four at first order, which is the first sign that the doublers belong to the value $\kappa = 1$ (Section 2 makes this exact). The doublers' cones share no time direction with the graviton's cone [2, Thm. 6.15], and ordering the path sum by record time does not remove them.

Theorem 3.2 (Record time keeps the doublers)**.** *Let $\kappa = 1$, with the Feynman regulator. At each axis corner πe_c and for every time direction of the long-wave graviton's cone, some spatial direction carries two real record-time frequencies of the gauge-fixed operator, both shrinking linearly with the distance from the corner, while along other directions there is no real record-time frequency: the doubler propagates in record time, and the corner spectrum is mixed. At πe_0 the directions with two real frequencies form an open set. For record time itself the two real frequencies are zeros of the native block on the eleven non-posting channels, not gauge artefacts. Summing the regulated path sum slice by slice in record time changes nothing at any $\varepsilon > 0$.*

In each of the forms examined, record time does not remove the doublers: the order of summation, a sign rule for the record frequency, a record-time vacuum, the first-order record-time Wick rotation, a rescaled or conjugated corner regulator, the projection onto real record

frequencies, and the record-extension operator. A finite Wick rotation in record time is not among them, and neither is a retarded record-time evolution of the whole lattice. Rotating the record-time part of the interval by an angle φ is a complex $\kappa = \frac{1}{4} + \frac{3}{4}e^{i\varphi}$, and to first order in φ it damps the doubler and the graviton with opposite signs, so no common orientation serves both.

The long-wave remainder. The long-wave graviton has a zero-momentum question of its own. In the continuum a constant strain is a linear change of coordinates, so the one-loop effective action of constant strains depends on the strain only through the volume. On the lattice, for a function D of a constant strain, nonzero at $B = 0$, and a strain H with $\operatorname{tr}(g^{-1}H) = 0$, put

$$\Sigma_H[D] = \frac{i}{2}\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log D(tH)\Big|_{t=0}, \qquad \sigma_{\mathrm{dil}}[D] = \frac{i}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\log D(tg)\Big|_{t=0},$$
$$\mathscr{R}_H[D] = \Sigma_H[D] + \tfrac{1}{4}\operatorname{tr}(g^{-1}Hg^{-1}H)\,\sigma_{\mathrm{dil}}[D].$$

When D is the one-loop Gaussian determinant of the strained flat lattice, $\mathscr{R}_H$ is the zero-momentum transverse-traceless remainder: anything in the self-energy that is not a function of the volume. A cosmological term and any measure factor that depends on the strain only through the volume contribute nothing to it, and wave by wave, at $\varepsilon = 0$ and away from the planes where a phase equals one, the native determinant, every Gaussian gauge term and the Faddeev–Popov determinant of any strain-independent gauge function give the same remainder [2, Thm. 6.17]. That is the extent of its independence: the ordered limit, the waves on those planes, strain-dependent gauge functions and a derived ghost or gauge measure are not covered, which is why it is read here as a diagnostic of the model. The vertex moves, the lattice's candidate for the symmetry that would force it to vanish, are not symmetries of the interacting action on tori of side divisible by 4 [2, Thm. 6.23]. Computed on the native path sum, $\mathscr{R}$ is large and imaginary (NUMERICAL; Section 6). Whether it can vanish in the ordered limit is settled by the next theorem: it cannot.

Theorem 3.3 (The native model graviton is not massless)**.** *Let $\kappa = 1$, with the Feynman regulator. The per-site remainder $\mathscr{R}_{\mathrm{tt}}$ of the native model determinant does not have ordered limit zero: the model's graviton is not massless at one loop.*

The mechanism is the doublers again. Near the axis corners the real part of the remainder falls like $-r\log(1/\varepsilon)$ with $r > 0$, while the rest of the zone contributes a real part that is bounded above: the degenerate points away from the origin and the corners are hyperbolic, the remaining special points are classified and certified by exact computation, and near the origin, integrated against any smooth nonnegative cutoff, the real part stays bounded above as $\varepsilon \to 0^+$. That bound reduces to local estimates at points of the four gauge hyperplanes near the origin. They hold at the generic points of all four and at the circles and axes where the hyperplanes meet. At the four points where the light cone touches the hyperplanes a second reduction applies: after a Feshbach reduction and a rescaling the integrand becomes a damped four-by-four family whose imaginary part is positive at positive damping, which obeys the local bound everywhere except on a gauge plane at zero damping, and whose degenerate cap point is never needed; the bound at those plane points holds as well. So the native model fails the massless test for the same reason it fails the finiteness test, independently of the computed value of the remainder.

So the identified lattice has two separate one-loop problems: four massless doublers at the corners, which make the ordered limit diverge, and a nonzero remainder for the long-wave graviton, a zero-momentum term that no covariant effective action contains: a mass term in the model's effective action of constant strains, whose physical reading as a mass or a width depends on the evolution in record time (Section 8.1). The first is a problem of the measure and of κ; the second is a problem of the action. Sections 2, 5 and 7 deal with the first and Section 6 with the second.

4 Einstein gravity at long wavelengths

Status: part 4, Einstein gravity at long wavelengths

Proved, at identified $\kappa = 1$ where stated and for every $\kappa > 1/2$ where stated. Recalled from [2]: the linearized Einstein field as the long-wave limit of the linear lattice field off the light cone; exact solutions of the full Regge equations on every edge with a resting body and a model medium, whose weak-field long-wave limit is that field; Newton's potential for compact bodies with $G = 1/(8\pi s)$; at fixed small strength, solutions uniform in the torus size with an explicit second-order limit, with a model medium or a posting gas. New here: the same second order on the lattice with worldline matter that conserves postings, and as a statement about continuum fields, with the matter's own stress at first order and at every nonzero wave of the second order ("The same limit in the continuum"); every long-wave vertex of the action is Einstein's, in the sense that on a sampled metric jet the metric part of the lattice field equations is $-\frac{1}{2}$ times the full Einstein density (Theorem 4.2); the lattice's vertex moves stop being a symmetry at second order in the field, one order in the derivatives past Einstein's equations, on the infinite lattice too, and also for a point mass's cubic Taylor fields (Theorem 4.3); and, for any lattice field with a uniform bound on its jet norm, consistency of its scaled second differences with continuum derivatives at rate $1/N$ (Theorem 4.1).

Premises. The identification $\kappa = 1$ (for the theorems stated at $\kappa = 1$); a model medium that supplies stability (MODEL), or a posting gas or worldline matter in its place under the identifications of the least-work law.

Open. The nonlinear Einstein equations as the long-wavelength limit: a family of lattice solutions whose long-wave limit solves $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ at finite amplitude, not order by order. The vertices are Einstein's; what is missing is the solutions. Exact static vacuum solutions of all the lattice equations meet an obstruction one order past Einstein's equations.

What would settle it. A convergence theorem for lattice solutions at fixed finite strength, with matter that has equations of its own, to a solution of the full Einstein equations.

The classical side of the lattice is where the programme is strongest, and it is the part of the title that already holds at linear order. We recall the results of [2] in the order in which they build on each other; the proofs are there.

The flat kernel. Classical recovery starts with the zero modes of the flat Hessian $H_0 = D^2S(\ell^0)$ on the side-N torus. Constant strains and vertex moves are always zero modes, because flat configurations stay flat under them. On every torus of odd side not divisible by 105 there are no others [2, Thm. 4.2]. The number 105 is a limit of the proof, not a known failure: the proof needs the side to be prime to 3, to 5 or to 7, and the odd multiples of 105 are

open. On even tori the doublers of Section 3 are extra zero modes [2, Thm. 4.4].

The linear field. For $\kappa > 1/2$, a non-null cone P of waves and a stress family T continuous at the origin and conserved on P, the linear lattice equations $s\,B_\kappa(e^{iu})h = -\tau(u)$ have the de Donder solution for small $u \in P$, every other solution differs from it by a vertex move, and

$$q_\kappa(u)\,h_{\mu\nu}(u) \longrightarrow 16\pi G\Big(T_{\mu\nu}(0) - \tfrac{1}{2}g_{\mu\nu}T(0)\Big) \qquad (u \to 0 \text{ in } P),$$

with an explicit stability constant [2, Thm. 4.6]. This is the linearized Einstein equation in de Donder gauge, with $G = 1/(8\pi s)$.

Exact solutions with matter. At identified $\kappa = 1$, on every odd side not divisible by 105, the unconstrained Regge equations on all fifteen edge classes, with a resting body and a model medium of amplitude λ, have exact solutions for small coupling σ, in the Lorentzian chamber, unique near their linear approximation [2, Thm. 4.8]. Their weak-field long-wave limit is the linearized Einstein field of the source [2, Thm. 4.9], and for a compact body it is Newton's potential: the time-time component of the lattice field satisfies $-\frac{1}{2}h_{tt} = -GM/R + (\text{remainder})$ outside the body, with an explicit finite-size remainder and $G = 1/(8\pi s)$ [2, Thm. 4.10].

Uniformity and second order. No stability bound uniform in the torus size holds at all wavelengths: a lattice resonance rules it out [2, Prop. 4.11]. At fixed small strength α the long-wave equations do have solutions uniformly in the torus size, with the gauge directions controlled and stability supplied by the medium, and their second-order term has an explicit limit $H_1 - \alpha H_2$ as the torus grows, with H_1 the linearized Einstein field up to a linearized diffeomorphism [2, Thms. 4.12–4.16]. A posting gas held by a least-cost support can replace the medium, under an identification [2, App. E].

Second order with worldline matter. The medium can also be replaced by matter that obeys an equation of its own. Take a static body of resting postings with finitely many Fourier modes, held up by *worldline matter*: postings that run along worldlines, so that posting number is conserved at every vertex, with a carrier that obeys the least-work law of the recognition cost and a uniform reference occupation θc^0 that serves as the carrier's reference and does not itself gravitate. At fixed strength $\sigma = \varepsilon_s(2\pi/N)^2$, with ε_s small against $\min(1, \theta)$, the coupled problem, Regge gravity sourced by the body and its support, has a solution on every torus with size and remainder bounds independent of N. It solves every transverse row of a fixed long-wave static band, and every radical row that worldline matter can push on, those orthogonal to the rows the worldlines cannot see. The solution is $w = \varepsilon_s W_1 + \varepsilon_s^2 W_2 + O(\varepsilon_s^3 + \varepsilon_s^3/\theta)$, and for a body whose second-order modes avoid the six blind lines W_2 is exactly the lattice field whose long-wave limit is the linearized Einstein field of a conserved source. For bodies invariant under translations that see every blind mode, such as a single-mode slab, the solution inherits the invariance and solves every radical band row. The premises are $\kappa > 1/2$ ($\kappa = 1$ by IDENTIFICATION), point-particle postings (MODEL), the least-work law read at the flat reference and carried at fixed stress, the ratios of c^0 and the split between body and carrier (each an IDENTIFICATION), and a reference background that does not gravitate (MODEL); limits are taken

at fixed ε_s and θ as $N \to \infty$. Not proved: the rows off the band, the radical rows of higher wave number, the non-static rows, the blind rows of compact bodies, and anything beyond second order as a continuum statement.

The same limit in the continuum. The second-order statement can be read as one about continuum fields. Write $\varepsilon_s H_1 + \varepsilon_s^2 H_2$ for the long-wave limit of the worldline solution, for a body whose first- and second-order waves avoid the six blind lines. On every large torus the coupled lattice problem has a solution, unique in a ball of jet norm of order ε_s, within $K\varepsilon_s^3$ of the edge readings of $\varepsilon_s H_1 + \varepsilon_s^2 H_2$; H_1 solves the linearized Einstein equation sourced by the dust of the body's resting postings, with coupling one in lattice units; and $\varepsilon_s H_1 + \varepsilon_s^2 H_2$ solves the full Einstein equation up to order ε_s^3 with the matter's own stress, the dust, its compensation and the least-work support of the worldlines, at first order and at every nonzero wave of the second order. The proof rests on two identities, both bilinear in two plane waves, and on the step that carries them to the field of the body: the image of the gravitational cubic vertex is the quadratic part of the Einstein density, and the contracted Bianchi identity holds at second order, from which the load of the support follows. One wave is balanced rather than checked. On the lattice the uniform part of the second-order load is carried by the uniform mode of the worldline support, which the coupled equations fix; the continuum statement absorbs that part into a constant that it leaves free. Showing that the constant is the matter's own uniform stress would make the zero wave a check as well; it has not been done.

Tree-level propagation. Every non-gauge long-wave mode in every direction propagates on the light cone up to a relative correction of second order in the wave number, and the static Newton kernel is correct to second order [2, Thms. 5.1 and 5.4]. For a lattice spacing b the corrections are of order $(k_{\mathrm{ph}} b)^2$, so tree level has no observable signature for spacings up to 10^6 Planck lengths.

What is missing. The solutions above are linear in the field, or second order at small strength, and their stability comes from a model medium, from a posting gas under an identification, or from worldline matter under the identifications just listed. The nonlinear Einstein equations need lattice solutions at finite strength whose long-wave limit solves the full equations, with matter that has equations of its own. No such family has been constructed, and no statement about the coupled nonlinear continuum limit is proved beyond second order at small strength. Two ingredients of such a proof are in place.

The first controls second derivatives without assuming that the fields solve anything. On the unit four-torus, let f_N be one component of the lattice metric on the side-N grid, with normalized discrete Fourier coefficients $\hat{f}_N(m)$ at the balanced integer frequencies m. Write $I_N f_N$ for its trigonometric interpolation, let Λ^{-1} multiply each Fourier coefficient by $(1+|m|_1)^{-1}$, and put

$$D_\rho^N f(x) = N\big(f(x + e_\rho/N) - f(x)\big), \qquad M_2(f_N) = \sum_m |m|_2^2 \, |\hat{f}_N(m)|.$$

Theorem 4.1 (Second differences are consistent)**.** *For $N \geq 2$ and every pair of directions ρ, σ,*

$$\left\| \Lambda^{-1} I_N \big(D_\sigma^N D_\rho^N f_N \big) - \partial_\sigma \partial_\rho \, \Lambda^{-1} I_N f_N \right\|_\infty \leq \frac{16\pi^3 e^\pi}{N} \, M_2(f_N).$$

For every Fourier probe p fixed independently of the field with $|p(m)| \le Q/(1+|m|_1)$, the pairing of p with the unsmoothed discrepancy is at most $16\pi^3 e^{\pi} Q M_2(f_N)/N$ in absolute value.

Proof. The difference and derivative symbols are $a_\rho(m) = N(e^{2\pi i m_\rho/N} - 1)$ and $b_\rho(m) = 2\pi i m_\rho$. The chord bound and its first-order remainder give $|a_\rho| \le 2\pi|m_\rho|$ and $|a_\rho - b_\rho| \le 4\pi^2 e^{\pi} m_\rho^2/N$. Writing $a_\rho a_\sigma - b_\rho b_\sigma = (a_\rho - b_\rho)a_\sigma + b_\rho(a_\sigma - b_\sigma)$ gives $|a_\rho a_\sigma - b_\rho b_\sigma| \le 16\pi^3 e^{\pi}|m|_1|m|_2^2/N$. Divide by $1+|m|_1$, multiply by $|\hat{f}_N(m)|$ and sum; every Fourier character has modulus one, which gives the uniform bound, and the probe bound is the same estimate with the factor Q. Both derivatives are taken of the same finite Fourier interpolation, so the comparison field is exactly the continuum second derivative of the smoothed reconstruction. □

For the metric read from the native lattice field, $M_2(f_N) \le C\mathcal{J}$ whenever the field's native jet norm is at most $\mathcal{J}$, with C depending only on the metric component. The native jet norm sums the Wiener norms of the reconstructed metric, its forward differences multiplied by N, and the affine-stencil remainder multiplied by N^2, where the mesh spacing is $1/N$; it controls the reconstructed metric's Fourier second moment. So a jet-norm bound independent of the mesh makes the error vanish like $1/N$. This is consistency for the metric part of the field equations. It does not supply a family of solutions, and it does not identify the other terms of the full first variation, those of the matter, the carrier and the constraints.

The second identifies every long-wave vertex of the action. Sample a smooth metric field onto the edges of the Kuhn lattice at spacing s: around a point, the field has a value g, first derivatives P and second derivatives Q, all symmetric, and each edge reads the field at its midpoint. The lattice field equations, the derivatives of the Regge action along the fifteen edge classes d, have an expansion in s; let Φ_d be their coefficient of order s^2. Their *metric part* is the symmetric matrix $\sum_d d\,d^{\top}\Phi_d$, ten of the fifteen combinations of the class equations. Call a constant metric g *star-good* if every metric on the segment from g_κ to g is Lorentzian and gives nondegenerate flat Kuhn simplices.

Theorem 4.2 (Every long-wave vertex is Einstein's)**.** *Let $\kappa > 1/2$ and let g be a symmetric star-good metric. For all symmetric first and second derivatives P and Q, the metric part of the order-s^2 coefficient of the lattice field equations on the sampled field equals $-\frac{1}{2}$ times the Einstein density $\sqrt{|\det g|}\,G^{\mu\nu}$ of the 2-jet (g,P,Q), with every nonlinear term included.*

The structural part of the proof is short. Both sides are affine in Q with a linear part that does not depend on P, and both vanish on every flat 2-jet: the Einstein density because a flat jet has no curvature, the lattice residual because a quadratic move of the vertices carries the Kuhn stencil exactly onto the stencil of a flat 2-jet, where every deficit vanishes. Every first jet extends to a flat 2-jet, so two such functions agree everywhere once their linear parts agree. The linear parts agree at every constant metric whose flat Kuhn simplices are nondegenerate and whose flat deficits vanish: there the variation of the dihedral angles, summed slot by slot over the simplices of a Kuhn cell, is computed exactly as a function of the background and assembles into $-\frac{1}{2}$ times the linearized Einstein density. A star-good metric has vanishing flat deficits, because along its segment the deficits are multiples of 2π that vary continuously from zero. The set of star-good metrics is open, contains $g_{\kappa'}$ for every $\kappa' > 1/2$ and metrics outside that family far from g_κ, and excludes $-g_\kappa$; so the theorem holds at finite amplitude, not only near the flat lattice. It is not trivially met: the zero phase and the quadratic truncation of any

local residual fail it for every nonzero coefficient. Two limits remain. The theorem is about the field equations evaluated on sampled jets, not about lattice solutions, and it covers the metric part: the five other combinations are not Einstein's, and on a sampled vacuum metric three of them are driven by the Weyl curvature (NUMERICAL), which is why a sampled metric needs the non-metric correction described next.

At finite strength there is so far a candidate, not a solution. For every sufficiently large core of spread R there is an explicit static lattice configuration, built by sampling a radial exterior profile at the edges, that is symmetric under the permutations of the four counters and tends to the flat lattice far away with mass $M = R/16$ and inner areal radius $4M$. Outside the core every simplex is Lorentzian and every edge with two or more postings is timelike, and somewhere outside the core the norm of the static Killing vector is at most $7/8$ of its flat value, which a flat exterior cannot reach. It does not solve the lattice equations outside the core, as computed: the residuals are of the order of the local curvature, falling like R^{-2} with no cancellation. Their cause is specific to the Kuhn lattice. A vacuum metric sampled at the edges needs a correction in the five non-metric combinations of edge classes, linear in the tidal field; added locally to the candidate, it lowers the residual to order R^{-3} (NUMERICAL).

The next order meets an obstruction. Vertex moves are exact zero modes of the flat Hessian, and a second-order correction to a linearized solution h exists only if the third variation $D^3S(h,h,\cdot)$ pairs to zero with every vertex move; that is the second-order form of the vertex symmetry. On the Kuhn lattice it fails.

Theorem 4.3 (The vertex symmetry fails at second order)**.** *Let $\kappa = 1$ and take the torus of side 8, with every coordinate x_a read in $\{-4,\dots,3\}$. Let h be the sampled weak-field metric of the static potential $\Phi = \xi_1^3 - 3\xi_1\xi_2^2$, with $\xi_1 = \frac{1}{2}(x_0 + x_1 - x_2 - x_3)$ and $\xi_2 = \frac{1}{2}(x_0 - x_1 + x_2 - x_3)$, read at the edge midpoints, plus its slaved correction, which lies in the non-metric combinations of edge classes, and let g be the vertex move of the record $y = (2,1,-1,0)$ along the first counter. Then h solves the linearized Regge equations on the thirty edges that start or end at y, the flat Hessian annihilates g, and*

$$D^3S(h,h,g) = 756\sqrt{3}.$$

Consequently no edge field d solves the second-order equations $D^2S(d,\cdot) + D^3S(h,h,\cdot) = 0$ on the thirty edges at y.

The proof is an exact sum over the 120 simplices with y as a corner, using the cubic cell form of the Regge action at $\kappa = 1$, whose numerators are integers over $1152\sqrt{3}$. The statement is local at y and holds on the infinite lattice too: there h solves the linearized equations at every record, its defect at y is again $756\sqrt{3}$, and no curve $\ell^0 + th + \frac{1}{2}t^2d$ meets the thirty equations there for all small t. The defect is a lattice artifact one order in the derivatives past Einstein's equations. At the order of Einstein's equations the same pairing vanishes in vacuum, by Theorem 4.2 and the contracted Bianchi identity (INFERENCE: the expansion of the pairing in the spacing is not written out), and on the other cubic harmonics the defect is an integer multiple of $\sqrt{3}$ (NUMERICAL). The cubic Taylor fields of a point mass's potential about records on an axis, a face diagonal and a body diagonal behave the same way: their defects are nonzero, so no second-order correction exists at the thirty edges there either, and on the infinite lattice no such curve exists. Read as a far field, the defect is a force on the vertex placements of order M^2r^{-7} with cubic anisotropy (NUMERICAL). So a static vacuum solution expanded in the amplitude,

with the sampled field and its slaved correction as first-order term and a regular far field, stops one order in the derivatives past Einstein's equations (INFERENCE), and a smooth change of the first-order term with the same far field does not help (INFERENCE). Three escapes are not excluded: an action with exact vertex symmetry, the symmetry that would also give clause (c) of Target 6.8 at one loop; matter that carries the load, as the worldline support does at second order; and far fields with lattice-scale static modes. The obstruction concerns exact solutions of all the lattice equations, those along vertex moves included. The long-wave statements of this section impose the band equations off the vertex moves and constant strains, or let matter carry the load on them, and are not affected. Black holes on the lattice are the subject of [20].

With the vertices identified, what remains for the nonlinear limit is a continuum solution to compare with, the consistency of its lattice sample at fixed strength with a remainder bound in the jet norm, and matter that has equations of its own; for vacuum exteriors, also an escape from Theorem 4.3.

5 A measure from the ledger's cost

Status: part 3, the measure

Proved. At Gaussian order a density at a fixed scale drops out, and a weight whose exponent scales like the action enters only through one positive semidefinite form $\mathscr{D}$ (Theorem 5.1). The precision $\mathscr{D} - iaA$ is invertible exactly when A and $\mathscr{D}$ have no common null vector, and the regulated propagator has a limit exactly then (Theorem 5.2). Under the two identifications (I1) and (I2) below, the ledger's cost of the curvature at each hinge gives $\mathscr{D} = \frac{1}{2}\sum_h (A_h/n_h)\,\delta'_h \delta'^{\top}_h$, with no free real coefficient once the channel count n_h is fixed (Theorem 5.4). This form vanishes on every flat deformation, is of fourth order in the wave number on long waves, and is positive on the doubler at all four axis corners (Theorem 5.5). Away from the origin its kernel consists of vertex moves, so it leaves no massless mode except the graviton (Theorem 5.6). On the uncut cone of geometries with its Haar measure the weight has infinite mass (Proposition 5.7).

Proof complete, independent verification pending. The ledger's histories carry a normalized probability law, and a law read from them is a probability measure, so it is never the uncut Born weight (after Proposition 5.7).

Identifications. Each is a physical premise, not a consequence of the ledger as stated. (I1) The cost of a hinge record reads its area as the action does: it adds over the pieces of the hinge, does not decrease with area, and a face of area λ_{rec}^2 carries one cycle of the hinge's bridge, whose coupling is 1 and whose deficit is the Regge deficit. (I2) The Born weight of a geometry is the Gibbs weight e^{-C} of its ledger cost C. The scaling that places the fluctuations in the native geometry is the one the phase already uses.

Open. The channel count n_h: the dual cycle gives n_h equal to the number of four-simplices at h, which is a definition, not yet a consequence of the ledger; every qualitative statement about the derived weight holds for every positive count, and its sizes depend on the count. Which alternatives the Born weight of a geometry is normalized over (not the uncut cone, by Proposition 5.7). The reference measure on squared lengths, which is the flat measure chosen in [2, Sec. 3.6]; a measure built from the ledger's records has yet to be connected to it.

What would settle it. A derivation of (I1), (I2) and n_h from the ledger.

The doublers of Section 3 sit at the edge of the Brillouin zone, and the graviton sits at its

centre. Lattice fermions have the same geometry, and there a Wilson term separates the two [24]: it gives the doublers a mass and vanishes on long waves faster than the kinetic term. The comparison is one of structure only: the fields here are bosonic and the path sum is real-time, so nothing in it rests on the Nielsen–Ninomiya theorem for fermions [10]. This section asks whether the ledger's own measure supplies such a term for gravity, derived and not chosen. It does, under two identifications that we name, and the term it supplies is a cost of curvature.

5.1 What a measure can do at Gaussian order

A one-loop question needs the measure only at Gaussian order around the flat lattice, and at that order a measure has very little room. Place a fluctuation x of squared lengths on a large lattice by giving the native geometry the squared lengths $t^{-2}(\ell^0 + tx)$, where ℓ^0 is the flat configuration and $t \to 0^+$: the lattice is $1/t$ recognition lengths across and carries a relative fluctuation of size t. The Regge action is homogeneous of degree one in the squared lengths, because hinge areas scale with them and deficit angles do not. Since $S(\ell^0) = 0$ and the flat lattice is stationary,

$$S\big(t^{-2}(\ell^0 + tx)\big) = t^{-2} S(\ell^0 + tx) \longrightarrow \tfrac{1}{2}\, x^{\top} A x \qquad (t \to 0^+),$$

with A the flat Hessian, so the phase e^{iaS} becomes the Gaussian $e^{\frac{i}{2} a x^{\top} A x}$. A factor of the measure enters this limit in one of two ways, decided by how it scales.

Theorem 5.1 (The measure at Gaussian order). *Let $\kappa = 1$ and $a \neq 0$.*

(i) A density that is measurable, bounded, and continuous and nonzero at the flat field drops out. On a torus of even side $N \geq 4$, at every $\varepsilon > 0$ the small-fluctuation exchange of the doubler source is the exchange without the density, and it has a pole in $1/\varepsilon$ with positive residue.

(ii) A weight $e^{-W(tx)/t^2}$ with $W \geq 0$ continuous and $W(tx)/t^2 \to \frac{1}{2} x^{\top} \mathscr{D} x$ gives normalized moments that tend to $\big((\varepsilon + \mathscr{D}) - iaA\big)^{-1}$ [2, Thm. 6.11(iii)].

(iii) Every symmetric positive semidefinite $\mathscr{D}$ arises in this way, from $W(x) = \frac{1}{2} x^{\top} \mathscr{D} x$, and the limit determines $\mathscr{D}$.

The two classes are treated separately. A product of a fixed-scale density and an action-scale weight, which is the form a physical measure would take, is not covered by the theorem, and neither are other scalings of the exponent. Within its scope the theorem says that the measure's whole freedom at Gaussian order is one positive semidefinite form $\mathscr{D}$, and the next result says exactly which forms remove a massless mode.

Theorem 5.2 (When the precision is invertible). *Let A be Hermitian, $\mathscr{D}$ positive semidefinite and $a \neq 0$. Then $\mathscr{D} - iaA$ is invertible if and only if $\ker A \cap \ker \mathscr{D} = \{0\}$. The regulated inverse $\big((\varepsilon + \mathscr{D}) - iaA\big)^{-1}$ has a limit as $\varepsilon \to 0^+$ under the same condition and diverges like $1/\varepsilon$ along a common null vector otherwise. If the precision is invertible at a wave k_0 at $\varepsilon = 0$ and depends continuously on the wave, its inverse is bounded uniformly on a neighbourhood of $(k_0, 0)$ in wave and damping.*

Proof of the first statement. A common null vector is killed by $\mathscr{D} - iaA$. Conversely, if $(\mathscr{D} - iaA)v = 0$ then $0 = v^\dagger \mathscr{D} v - ia\, v^\dagger A v$, and both $v^\dagger \mathscr{D} v$ and $v^\dagger A v$ are real, so $v^\dagger \mathscr{D} v = 0$. Since $\mathscr{D}$ is positive semidefinite this gives $\mathscr{D} v = 0$, and then $iaAv = \mathscr{D} v = 0$. □

At an axis corner the gauge-fixed operator has a one-dimensional kernel spanned by the doubler vector v_c [2, Prop. 6.1]. So a weight gaps the doubler at that corner exactly when $\mathscr{D} v_c \neq 0$, and then the corner precision stays invertible, with a bounded inverse, on a whole neighbourhood of the corner: the doubler's regulated pole is removed. A weight bounded below, such as an on-site recognition cost, does this and much more: it adds a nonzero damping even at zero momentum, in the long-wave graviton's sector too [2, Thm. 6.11(iv)]. A repair through the measure needs a form that is positive on the doublers and vanishes faster than the action's quadratic form on long-wave gravitons, so it must tell the corners from the centre of the zone.

5.2 The doubler is curved

The doubler is a zero mode of the Regge Hessian, but a zero mode need not be flat. By Schläfli's identity the flat Hessian pairs the change of hinge areas with the change of deficit angles, so a field can be a zero mode while its deficits do not vanish, provided its curvature lies in the kernel of the pairing. For a field u write $\delta_h'(u)$ for the derivative at the flat lattice of the deficit angle at the hinge h.

Theorem 5.3 (Flat deformations are flat; the doubler is not)**.** *Let* $\kappa = 1$.

(i) On every torus of side $N \geq 3$, $\delta_h'(u) = 0$ *for every hinge* h *when* u *is a constant strain or a vertex move.*

(ii) On every torus of even side $N \geq 4$ *and for each axis* c*, the linearized deficits of the doubler field* $(-1)^{x_c} v_c$ *do not all vanish. A signed census over all hinges,* $\sum_h \sigma_c(h)\, \delta_h'$ *with* $\sigma_c(h) = \sum_{v \in h} (-1)^{v_c}$*, equals a fixed nonzero multiple of* N^4*, the same for all four axes.*

Statement (i) follows by differentiating the family of flat configurations, on which every deficit is zero. Statement (ii) is an exact computation: the census reduces, cell by cell, to ten integer vectors and ten integer triples, and for the field $(-1)^{x_c}\, 4 v_c$, whose entries $2^{|d|-1}$ on the classes with $d_c = 0$ are integers, it equals $N^4(6 + 8\sqrt{3})$. So the curvature form $\sum_h \delta_h'(u)^2$ is blind to flat deformations and sees the doublers. The action cannot see them; a cost of curvature can.

5.3 The weight the ledger gives

The bridge between the phase of a geometry and its holonomy puts the curvature of each hinge on a cycle of the ledger: the deficit δ_h of the hinge h is shared by a cycle of n_h channels, each strained so that $\sinh u = \delta_h / n_h$. That bridge is a model of how the ledger records curvature, and it enters below as part of identification (I1). The recognition cost of the cycle is

$$n_h\, J(e^u) = n_h \Big(\sqrt{1 + (\delta_h / n_h)^2} - 1 \Big) = \frac{\delta_h^2}{2 n_h} + O(\delta_h^4), \qquad J(x) = \tfrac{1}{2}\big(x + x^{-1}\big) - 1,$$

and no coefficient is left to choose in it.

How the cost is counted decides its scale. Hinge areas grow with the squared lengths and deficits do not, so a cost counted once per face of area λ_{rec}^2 grows like the action and acts at Gaussian order, while a cost counted once per hinge does not grow at all: it is a fixed-scale density, it drops out by Theorem 5.1(i), and the doublers keep their poles. The per-face count is what the premises that fix the phase give when they are applied to the cost of the same hinge record. The action that one posting carries adds over the pieces of a hinge, does not decrease with area, and a face of area λ_{rec}^2 carries one bit; if the cost adds over the pieces of the hinge in the same way, and a face carries one bridge cycle, the cost is the per-face cycle cost. Counted per hinge instead, the cost of a curved hinge could not add over its pieces. This is identification (I1). The amplitude of a geometry then carries $e^{-C/2}$ if the Born weight of a geometry is the Gibbs weight e^{-C} of its ledger configuration, as it is for the recognition paths of the measurement layer [22]. This is identification (I2).

Theorem 5.4 (The derived weight)**.** *Let $\kappa = 1$, $N \geq 3$, and let the channel counts satisfy $n_h > 0$. Under (I1) and (I2) the ledger cost enters at the action's scale, and at Gaussian order the measure acts through*

$$\mathscr{D} = \frac{1}{2} \sum_h \frac{A_h}{n_h} \, \delta_h' \, \delta_h'^{\top},$$

where $A_h > 0$ is the flat area of the hinge h and δ_h' the derivative of its deficit at the flat lattice. No real coefficient is free: $\mathscr{D}$ is fixed once the channel counts are fixed.

The factor $\frac{1}{2}$ is the step from the Born weight to the amplitude, and the areas are those of the flat lattice; every hinge is timelike at $\kappa = 1$, so every A_h is positive. Three inputs besides these fix $\mathscr{D}$, and we name them. The channel count n_h is left free by (I1); the dual cycle, which passes once through each four-simplex containing h, gives n_h equal to the valence of the hinge, 4 or 6 on the Kuhn lattice, and that is a definition. $\mathscr{D}$ scales hinge by hinge as $1/n_h$, so its sizes depend on the count, while every qualitative statement below holds for every positive count. The bridge coupling is 1 and the bridge's deficit is the Regge deficit; both are part of (I1). And the scaling $t^{-2}(\ell^0 + tx)$ is the map from the Gaussian construction to the native geometry under which the native phase is the construction's phase; it is what puts a cost of degree one at the action's scale, and the construction already makes it for the phase.

Theorem 5.5 (What the derived weight does)**.** *Let $\kappa = 1$ and let $\mathscr{D}$ be the derived form with the dual-cycle counts.*

(i) On every torus of side $N \geq 3$, $\mathscr{D}$ vanishes on every constant strain and every vertex move.

(ii) There is a constant K such that for every $N \geq 3$, every integer wave m with $q = 2\pi m/N$ and $|q| \leq 1$, and every polarization H, the plane strain wave x of polarization H at q satisfies $x^{\top} \mathscr{D} x \leq K N^4 |q|^4 |H|^2$.

(iii) On every torus of even side $N \geq 4$, $\mathscr{D}$ is positive on each of the four doubler fields $(-1)^{x_c} v_c$. With $a \neq 0$ the four corner precisions are invertible, with inverses bounded on a neighbourhood of each corner.

Statement (i) is Theorem 5.3(i). For (ii), a plane wave of the metric splits into a constant strain, a vertex move of a quadratic displacement and a remainder of order $|H||q|^2$ per edge: a metric

perturbation that is linear in position, sampled at edge midpoints, is exactly a constant strain plus the vertex move of a quadratic displacement, a lattice form of the Christoffel identity. The first two pieces carry no curvature, so each linearized deficit is $O(|H||q|^2)$ and the form is $O(|H|^2|q|^4)$ per hinge, with about N^4 hinges. The action's own quadratic form on the same waves is of second order in q [2, Sec. 4.4], so the weight leaves the long-wave graviton's tree-level propagation alone. Since every n_h is at least 1 and every valence at most 6, the form with any other positive counts is at most six times the dual-cycle form, so (ii) holds for every positive count with a larger constant. Statement (iii) is Theorem 5.3(ii), the positivity of every A_h/n_h, and Theorem 5.2; the bound on the inverses near each corner uses in addition that the derived symbol depends continuously on the wave.

So the ledger's cost of curvature is a Wilson-like term for lattice gravity, in the structural sense just described: positive on the doublers, zero on flat deformations, of fourth order on long waves, with no number to tune once the two identifications and the channel count are granted. With the dual cycle, the hinge weights A_h/n_h take the values $1/12$, $1/4$ and $\sqrt{3}/4$, and the doubler damping coefficient $v_c^\dagger \mathscr{D} v_c/|v_c|^2$ is $(10+3\sqrt{3})/124 \approx 0.12255$ at all four corners, both by exact computation. On the complement of the vertex moves six eigenvalues of the Bloch symbol scale like $|k|^4$ while the other five stay gapped, the smallest near 4.92 (NUMERICAL).

Theorem 5.6 (The kernel of the derived form)**.** *Let $\kappa = 1$, with the dual-cycle counts. At every wave $k \notin 2\pi\mathbb{Z}^4$ the kernel of the Bloch symbol $\mathscr{D}(k)$ of the derived form consists of vertex moves.*

The rows of $\mathscr{D}(k)$ are, up to nonzero factors, the Bloch symbols of the linearized deficits, so its kernel is the set of fields whose linearized deficits all vanish at the wave k. At every wave with some phase different from one, those fields are vertex moves: the gradient weight $\sum_a 4\sin^2(k_a/2)$ that the argument divides by vanishes exactly on $2\pi\mathbb{Z}^4$. The statement is not satisfied by any choice: the zero form fails it at every wave. Its consequence is that at every wave other than the origin the precision $\mathscr{D} - iaA$ with the gauge-fixed operator A is invertible, because a positive semidefinite weight whose kernel lies in the vertex moves leaves no common null vector with it. The derived weight leaves no massless mode anywhere except the graviton at the centre of the zone. In the native form on the eleven non-posting channels the statement is different: there some vertex moves remain on four planes of waves, and neither the action nor the weight sees them (Section 7). At the fifteen corners and nine further waves an exact computation confirms that the Bloch map of the linearized deficits has rank eleven, with kernel exactly the vertex moves; its smallest nonzero singular value is proportional to $|k|^2$ (NUMERICAL).

What the weight does not do. The derived weight gaps the doublers; it does not make the model's graviton massless. The weight enters the Gaussian operator together with the phase, inside the precision $\mathscr{D} - iaA$, so it changes the one-loop determinant and both parts of the zero-momentum remainder. What it cannot change is the real part through a normalization: dividing the amplitude by the weight's own Gaussian integral at each background is a real factor, and such a factor moves only the imaginary part. Computed with the derived weight in the native form on the eleven non-posting channels, the remainder is $0.113 + 24.66\,i$ on the transverse-traceless probe (NUMERICAL; Section 6); the finiteness theorem of Section 7 concerns the fifteen-channel gauge-fixed responses, and the ordered limit of this eleven-channel remainder still needs a local statement at the origin for the weighted operator. And

no real weight or density can supply the vertex symmetry that would force the remainder to vanish (Theorem 6.6). The measure settles the question of the doublers, given its identifications; the question of the remainder belongs to the action.

Normalizing the weight. The one-loop results use the weight only at Gaussian order around the flat lattice. A probability over whole geometries needs more: a domain and a reference measure over which the Born weight e^{-C} can be normalized. The most direct choice fails.

Proposition 5.7 (The hinge weight on the uncut domain)**.** *Let $\kappa = 1$ and $N \geq 3$, fix positive channel counts n_h, and let $\mathscr{A}$ be the full cone of admissible squared-length fields whose postings are null, with its additive Haar measure μ, the overall scale and the flat moduli included. For the ledger's cost of curvature,*

$$C(\ell) = \sum_h A_h(\ell)\, n_h\Big(\sqrt{1 + \big(\delta_h(\ell)/n_h\big)^2} - 1\Big),$$

and every $\beta \geq 0$, the integral $\int_{\mathscr{A}} e^{-\beta C}\, \mathrm{d}\mu$ is infinite. In particular the Born weight e^{-C} cannot be normalized over $\mathscr{A}$ with μ, and the amplitude $e^{-C/2}e^{iaS}$ is not absolutely integrable for any real a.

Proof. The flat field ℓ^0 is an interior point of $\mathscr{A}$, every deficit vanishes there and is differentiable nearby, so $|\delta_h(\ell^0+u)| \leq k_h\|u\|$ for small u; the areas are locally bounded, and $n\big(\sqrt{1+(s/n)^2}-1\big) \leq s^2$ for $n \geq 1$. With finitely many hinges this gives $r > 0$ and $K \geq 0$ with $\ell^0 + B(0,r) \subset \mathscr{A}$ and $C(\ell^0 + u) \leq K\|u\|^2$ on that ball. Areas scale linearly with squared lengths and deficits do not change, so $C(t\ell) = tC(\ell)$ for $t > 0$. For $c \geq 1$ the ball $T_c = c^2\ell^0 + cB(0,r)$ lies in $\mathscr{A}$, and on it $C(c^2\ell^0 + cz) = c^2C(\ell^0 + z/c) \leq Kr^2$, while $\mu(T_c) = c^d\mu\big(B(0,r)\big)$ with $d = \dim \mathscr{A} > 0$. Hence $\int_{\mathscr{A}} e^{-\beta C}\, \mathrm{d}\mu \geq e^{-\beta Kr^2} c^d \mu\big(B(0,r)\big)$ for every $c \geq 1$. The last statement holds because the action is real, so $|e^{-C/2}e^{iaS}| = e^{-C/2}$. □

The proposition settles one prescription: the whole uncut domain with its Haar measure. It does not contradict (I2), which says what weight one geometry carries and claims no normalization, nor a Born ensemble over the finitely many ledger configurations behind each recorded geometry, which needs no Haar density. It does not decide a regulated oscillatory definition, a normalized conditional ratio, or a domain with a fixed volume, a quotient or boundary conditions, and it leaves the Gaussian-order results of this section and the one-loop results of Section 7 as they are. The positive route runs through the ledger's own histories, which carry a normalized probability law on their recorded flux trajectories.[1] Any law read from those histories is a probability measure, so it can never be the uncut Born weight, whose mass is infinite: a Born law read from them must condition on a domain of finite Born mass, which the read then derives. A read that uses only a finite prefix of the history has atoms, so it has no density against the Haar measure either.[2] What is missing is the read from a recorded history to squared edge lengths and hinge data: admissibility, agreement of lengths, volumes and connections with the operations the ledger enacts, and the correspondence between that law and the Born weight of the action.

[1] The proof of this step is complete; its independent verification is not yet finished.

[2] The proof of this step is complete; its independent verification is not yet finished.

6 The Ward criterion for a massless graviton

Status: parts 2 and 5, the action and the massless graviton

Proved. For every one-loop determinant, the zero-momentum transverse-traceless remainder equals $i/2$ times the difference of the second- and first-order defects of a Ward identity (Theorem 6.4). If the determinant obeys that Ward identity, the remainder vanishes (Theorem 6.2); an exact invariance of the determinant under volume-preserving linear maps of the vertices gives the identity, and near the flat lattice such an invariance is local covariance (Theorem 6.3). A real phase without a regulator gives a purely imaginary remainder, and a real normalization of the determinant, one number per background, changes only its imaginary part (Theorem 6.5). No real weight or density makes a vertex-translation flow a symmetry of the path-sum exponent (Theorem 6.6), and no phase $cS + \sum_h \delta_h^3 \Gamma_h$ has an exact vertex symmetry or invariant second and third variations (Theorem 6.7); at the same slope and the same weight such a phase has exactly the one-loop remainder of cS, on every probe (Theorem 6.10). The native remainder of the model determinant does not have ordered limit zero (Theorem 3.3). Refinement invariance would force the remainder to vanish, but the identified lattice does not have it, and at $\kappa = 1$ refinement carries the doublers to zero momentum (Theorem 6.9). Within the area-additive local charges the remainder is that of a one-parameter family; an even part homogeneous of degree one in a hinge's area and its reference area fails the corner condition exactly; and the classical clauses of Target 6.8 leave a record-compatible even term with zero flat value free (Sections 6.4 and 6.6). On the side-4 torus the $\kappa = 1$ coarse-graining exists and keeps its spurious zero mode, and exact blocking multiplies the per-site remainder by 16 (Section 6.5). Blocked with the ledger's refinement map, the damping held fixed on the fine lattice, the Gaussian path sum has no fixed point with an Einstein–Hilbert long-wave part at any $\kappa > 1/2$; in that order ordinary block averages have none either, so this does not decide the blocked route (Section 6.7).

Computed. The remainder of the model determinant is nonzero for the native path sum, for the derived weight of Section 5 in each of its normalizations, for the curvature-squared terms of one sign that the grids resolve and for the slope change examined, for the area-additive local charges over the range scanned, and for an even charge of constant amplitude; terms of mixed sign are not resolved (NUMERICAL; Sections 6.3 and 6.6). An even part that removes the remainder would be a choice of lattice-scale numbers; such a choice is found numerically at one point, on the bare phase (Section 6.4).

Proof complete, independent verification pending. The remainder of every local hinge charge depends only on its slope and even curvature, and within the readings of a hinge charge that the ledger supplies the even curvature is 0 or proportional to the area, neither of which gives a massless graviton in the model (Section 6.6); a Born law read from the ledger's records is normalized the Born way (Section 6.3). The corner condition on an even part is exact (Section 6.4). One refinement step holds at $\kappa = 3/2$ (Section 6.5). The holonomy route does not supply the repair: a phase whose slope map and even kernel are those of a local charge has that charge's one-loop remainder, however non-local its higher terms, and the ledger's holonomy phase is the Regge phase (Section 6.7). Nor does the blocked route at $\kappa = 1$: with the damping held fixed on the coarse lattice, the order that decides it, the blocked path sum has no fixed point with an Einstein–Hilbert long-wave part either, while block averages stay bounded (Section 6.7).

Open, and decisive. An action derived from the ledger, every coefficient included, with the classical limit of the Regge action, whose model determinant has a summed Ward defect with ordered limit zero on every transverse-traceless probe (Target 6.8). The classes excluded so far leave a phase with an even part outside the ledger's present readings, or with a slope map or even

kernel that is not that of any local charge.

What would settle it. A phase built from the ledger's hinge records that meets Target 6.8, with the doublers gapped by the measure; or a proof that no phase the ledger can supply meets it.

Section 3 separated two one-loop problems. The measure of Section 5 answers the first, the doublers, under its identifications. This section is about the second: the long-wave graviton's remainder $\mathscr{R}_H$ at zero momentum. It is a response to constant strain that no covariant action produces, and whenever it is nonzero it acts as a mass term in the one-loop effective action of constant strains. The theorems of this section hold for every one-loop determinant; the computed remainders, and every statement that a graviton is or is not massless, concern the model determinant of Section 3. Reading it as the pole mass of a propagating graviton, or as a width, needs an evolution in record time (Section 8.1). The question has an exact answer in terms of a Ward identity, and the answer explains why nothing tried so far has removed the remainder.

6.1 The criterion

In the continuum a constant strain is a change of coordinates, so the one-loop effective action of constant strains depends only on the volume. On the lattice the matching transformation is a linear map of the vertices, $x \mapsto (1 + sX)x$, which carries the strained flat metric $g + B$ to $(1 + sX)^{\top}(g + B)(1 + sX)$, with tangent $X^{\top}(g + B) + (g + B)X$. The map preserves volume to first order when $\operatorname{tr} X = 0$, and the flow e^{sX} preserves it exactly. Let D be a function of constant strains B, symmetric and near 0, with D differentiable near 0 and its derivative $\mathrm{d}D$ differentiable at 0; for the theorems below D is the one-loop determinant of a lattice action at constant strain.

Definition 6.1 (The Ward identity of linear vertex maps)**.** D obeys the Ward identity at g if $\mathrm{d}D(B)\big[X^{\top}(g + B) + (g + B)X\big] = 0$ for every symmetric B near 0 and every X with $\operatorname{tr} X = 0$.

The identity says that D does not change to first order under a volume-preserving linear vertex map, at every strained flat lattice near g. Equivalently, at every such strain the derivative of D vanishes on the tangent space of the level set of $\det(g + B)$. For a continuously differentiable D this is local covariance, D depending on B only through $\det(g + B)$ near the flat lattice; the step from the tangent condition to the level sets is an integration we have not written out (INFERENCE).

Theorem 6.2 (The Ward identity removes the remainder)**.** *Let $\kappa \neq \frac{1}{4}$, let $D(0) \neq 0$, and let D obey the Ward identity at g_κ. Then $\mathscr{R}_H[D] = 0$ for every symmetric H with $\operatorname{tr}(g_\kappa^{-1}H) = 0$.*

Proof. Write $g = g_\kappa$; it is symmetric, and invertible because $\det g = 1 - 4\kappa \neq 0$. Put $\ell = \mathrm{d}D(0)$, a linear functional on symmetric matrices, and $Q = \mathrm{d}^2D(0)$, the symmetric bilinear form that exists because $\mathrm{d}D$ is differentiable at 0.

First order. At $B = 0$ the identity says $\ell(X^{\top}g + gX) = 0$ for every X with $\operatorname{tr} X = 0$. Let S be symmetric and put $Y = \frac{1}{2}g^{-1}S$, so that $Y^{\top}g + gY = S$ and $\operatorname{tr} Y = \frac{1}{2}\operatorname{tr}(g^{-1}S)$. Split $Y = X + \frac{1}{4}(\operatorname{tr} Y)I$ with $\operatorname{tr} X = 0$; then $S = X^{\top}g + gX + \frac{1}{2}(\operatorname{tr} Y)\,g$, and the identity leaves only the trace part:

$$\ell(S) = \tfrac{1}{4}\operatorname{tr}(g^{-1}S)\,\ell(g) \qquad \text{for every symmetric } S.$$

This is where the trace-free condition on X enters: the identity says nothing about the dilatation direction g, which carries $\ell(g)$.

Second order. Fix X with $\operatorname{tr} X = 0$ and a symmetric B. The function $s \mapsto \mathrm{d}D(sB)\big[X^{\top}(g+sB) + (g+sB)X\big]$ vanishes for small s, its argument is affine in s, and $\mathrm{d}D$ is differentiable at 0, so its derivative at $s = 0$ exists and vanishes:

$$Q\big(B,\, X^{\top}g + gX\big) + \ell\big(X^{\top}B + BX\big) = 0.$$

Now let H be symmetric with $\operatorname{tr}(g^{-1}H) = 0$, and take $X = \frac{1}{2}g^{-1}H$, which is traceless exactly because of that condition, and $B = H$. Then $X^{\top}g + gX = H$ and $X^{\top}H + HX = Hg^{-1}H$, a symmetric matrix, so by the first-order result

$$Q(H,H) = -\ell(Hg^{-1}H) = -\tfrac{1}{4}\,\tau\,\ell(g), \qquad \tau = \operatorname{tr}(g^{-1}Hg^{-1}H),$$

and also $\ell(H) = \frac{1}{4}\operatorname{tr}(g^{-1}H)\,\ell(g) = 0$.

The remainder. Since $D(0) \neq 0$, $\log D$ is defined near 0, and along the line tH its first and second derivatives at $t = 0$ are $\ell(H)/D(0) = 0$ and $Q(H,H)/D(0) - \big(\ell(H)/D(0)\big)^2 = Q(H,H)/D(0)$; along tg the first derivative is $\ell(g)/D(0)$. Hence

$$\mathscr{R}_H[D] = \frac{i}{2D(0)}\Big(Q(H,H) + \tfrac{1}{4}\,\tau\,\ell(g)\Big) = 0. \qquad \square$$

Theorem 6.3 (An exact symmetry gives the identity)**.** *If $D\big(A^{\top}(g+B)A - g\big) = D(B)$ for every A near the identity with* $\det A = 1$ *and every symmetric B near* 0*, then D obeys the Ward identity at g. Every function of* $\det(g+B)$ *that is differentiable near* $\det g$ *obeys it.*

The curve $A_s = \det(1+sY)^{-1/4}(1+sY)$ has determinant 1 and tangent Y whenever $\operatorname{tr} Y = 0$, which proves the first statement. The orbit of $g + B$ under such maps is an open piece of its determinant level set, so the exact symmetry of Theorem 6.3 is itself local covariance of the zero-momentum determinant, not a weaker property. The criterion is sufficient and not necessary: the remainder can vanish while every single wave violates the identity, because only the sum over waves enters. The exact statement of what is necessary is the next theorem.

Theorem 6.4 (The remainder is a difference of Ward defects)**.** *Let $D(0) \neq 0$, let H be a strain, put $\tau = \operatorname{tr}(g_\kappa^{-1}Hg_\kappa^{-1}H)$, and let X be any matrix with $X^{\top}g_\kappa + g_\kappa X = H$. Then*

$$\mathscr{R}_H[D] = \frac{i}{2}\big(\mathrm{WD}_2 - \mathrm{WD}_1\big),$$

$$\mathrm{WD}_2 = \frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\mathrm{d}D(tH)\big[X^{\top}(g+tH) + (g+tH)X\big]}{D(tH)}\bigg|_{t=0}, \qquad \mathrm{WD}_1 = \frac{\mathrm{d}D(0)[S_0]}{D(0)},$$

with $S_0 = X^{\top}H + HX - \frac{\tau}{4}g$. For a finite family of determinants, one per wave, the summed remainder is $i/2$ times the difference of the summed defects; in particular this holds for the per-site remainder of the lattice path sum.

Proof. Put $f(t) = \log D(tH)$. Since $X^{\top}g + gX = H$, the bracket in WD_2 is $H + t(X^{\top}H + HX)$, so $\mathrm{WD}_2 = f''(0) + \mathrm{d}D(0)[X^{\top}H + HX]/D(0)$. Subtracting WD_1 leaves $f''(0) + \frac{\tau}{4}\,\mathrm{d}D(0)[g]/D(0)$, and $\frac{i}{2}$ times this is $\Sigma_H[D] + \frac{\tau}{4}\sigma_{\mathrm{dil}}[D] = \mathscr{R}_H[D]$. The summed statement is the same computation for $\sum_k \log D_k$. $\square$

WD_2 is the second-order defect of the Ward identity of $\log D$ along the vertex map X, and WD_1 its first-order defect on the direction S_0, which is traceless for $X = \frac{1}{2} g^{-1} H$. Neither term may be dropped: if the first-order identity holds, $\mathrm{WD}_1 = 0$ and the remainder is the second-order defect alone, but for the lattice path sums computed below the first-order identity fails too (Section 6.3). A candidate action is therefore tested by two numbers, and the graviton of its model determinant is massless at one loop on the probe H exactly when the ordered limit of their summed difference is zero.

The real and imaginary parts of the remainder have different origins.

Theorem 6.5 (Real phases and real densities)**.** *(i) If a wave's one-loop matrix is Hermitian at every strain, as for a real phase with no regulator, its determinant is real and, wherever it is nonzero at $B = 0$ and twice differentiable along H and differentiable along g, its remainder is purely imaginary.*

(ii) For every nonzero real density factor $w(B)$, twice differentiable along H and differentiable along g, $\mathscr{R}_H[Dw]$ and $\mathscr{R}_H[D]$ have the same real part.

So for a real phase with the Feynman regulator and no action-scale weight, the real part of the remainder comes only from the waves at which the one-loop matrix is singular at $\varepsilon = 0$: the light cone, the corner doublers, and any lattice-scale shell a modification creates (INFERENCE). The real part is an on-shell quantity. The imaginary part is the strain curvature of $\log|\det|$ away from the shells, an off-shell quantity, and by (ii) a real normalization of the determinant, one positive number per background, never reaches the real part. A measure weight is not such a factor: it enters the Gaussian operator itself, inside the combined precision $\mathscr{D} - iaA$ of Theorem 5.2, and it can change both parts.

6.2 What cannot supply the symmetry

The vertex moves are zero modes of the flat Hessian, but on tori of side divisible by 4 they are not symmetries of the interacting Regge action [2, Thm. 6.23]: the cubic vertex on a zero mode x_1 and the alternating vertex moves g_b generated by $(-1)^{y_0} e^b$ is proportional to $-4a_0 + 3(a_1 + a_2 + a_3)$ on $\sum_b a_b g_b$. Two natural repairs fail for structural reasons.

Theorem 6.6 (No weight or density supplies it)**.** *Let $\kappa = 1$, let $N \geq 4$ be divisible by 4, let $a \neq 0$, and let W be any real function of the configuration. Let the flow Φ_s have a generator of class C^2 whose value at the flat field is $\sum_b c_b g_b$ with $-4c_0 + 3(c_1 + c_2 + c_3) \neq 0$, for instance g_0. Then Φ_s does not leave the exponent $iaS - W$ invariant on any neighbourhood of the flat field.*

The reason is that W is real: invariance of $iaS - W$ forces invariance of its imaginary part aS, and [2, Thm. 6.23] rules that out. The same holds when the flow carries a real Jacobian factor. The theorem is about the moves at the corner wave πe_0 on sides divisible by 4, where the failure of the symmetry is proved.

The second repair changes the phase. Adding terms of cubic and higher order in the deficits keeps the classical limit, and one might hope that such terms restore the symmetry the way an improved lattice action restores a continuum symmetry. They cannot. Write δ_h for the deficit angle at the hinge h.

Theorem 6.7 (No cubic correction supplies it)**.** *Let $\kappa = 1$, $N \geq 3$, $c \neq 0$, and for each hinge let Γ_h be a function of the configuration of class C^3 at the flat field, local or not. The phase $S' = cS + \sum_h \delta_h^3 \Gamma_h$ is of class C^3 and stationary at the flat field, its Hessian there is c times the Regge Hessian, and its cubic vertex with any leg along a vertex move is c times Regge's. If $4 \mid N$, then S' has no symmetry with a C^2 generator whose value at the flat field is $\sum_b a_b g_b$ with $-4a_0 + 3(a_1 + a_2 + a_3) \neq 0$, no flow along g_0, and its second and third variations at the flat field are not invariant under the vertex moves. Every hinge charge $C(A, \delta) = cA\delta + \delta^3 \gamma(A, \delta)$ with γ of class C^3 gives a phase of this form.*

The proof is a jet computation: a term $p^3 q$ with $p(x_0) = 0$ has vanishing first and second derivatives at x_0 and third derivative $6q\,(\mathrm{d}p\,u)(\mathrm{d}p\,v)(\mathrm{d}p\,w)$, which vanishes when one of u, v, w is a vertex move, because vertex moves do not change deficits to first order (Theorem 5.3(i)). So the cubic vertex on the zero mode x_1 and a vertex move is c times Regge's, and the argument of [2, Thm. 6.23] applies unchanged. The theorem covers the phases the ledger suggests most directly: the native face-bit phase, which is $(s/\hbar)S$ under the premises that define it, a loop phase $\kappa_B S$ with any fixed coupling, and the average of the order phases of the 24 Kuhn simplices, which is $S/24$. Its complement is precise. A phase with the symmetry would need a term of second order in the deficit, a hinge charge whose slope is not proportional to the area, or a cubic remainder that is not of class C^3; a term of order four or more in the deficit is of the form $\delta^3\gamma$ and does not help. Nor do these phases change the remainder: at the same slope and the same weight their one-loop remainder is exactly that of cS on every probe, because every constant strain of the flat lattice is flat and the terms $\delta_h^3 \Gamma_h$ have zero Hessian there (Theorem 6.10 in Section 6.6).

6.3 The computed remainders

The criterion turns the question into a computation: evaluate the summed defects for a candidate and see whether they vanish. All values below are at identified $\kappa = 1$ and are computed on midpoint grids of the Brillouin zone; they are evidence, not theorems (NUMERICAL), and the integration rule, the grids, the treatment of the regulator and of the singular planes, and the error estimates are given in Appendix A. They are given on the transverse-traceless probe H_{tt}, with $\tau = 6$, which is the sum of one probe in each irreducible representation of the axis permutations, f_E ($\tau = \frac{9}{2}$) and f_{T_2} ($\tau = \frac{3}{2}$), the probes f_E and f_3 of [2, Sec. 6.8]. They are posting-null, unlike the spatial probes H_E and H_{T_2} of the anisotropy in Section 3, and $\tau = \mathrm{tr}(g^{-1}Hg^{-1}H)$. The two contributions add with no cross term, as computed and as the symmetry of the lattice under axis permutations leads one to expect.

The native path sum. With the Feynman regulator on the eleven non-posting channels, $\mathscr{R}_{\mathrm{tt}} = 0.75 + 6.71\,i$, $0.62 + 6.93\,i$ and $0.45 + 7.03\,i$ at $\varepsilon = 0.1$, 0.05 and 0.025, converged in the torus side [2, Sec. 6.8]. The real part carries a small corner logarithm, so it has no limit as $\varepsilon \to 0^+$ (Theorem 3.3); its finite part is 0.27 ± 0.05. The logarithm sits in one representation: at every axis corner the doubler's pairing with the probe f_E vanishes and its pairing with f_{T_2} does not, so the corner logarithm is carried by f_{T_2} (INFERENCE); on f_E the computed remainder is finite, about $0.31 + 5.67\,i$ (NUMERICAL). The imaginary part on H_{tt} is about 7.0. At $\varepsilon = 0.1$ the defects are $\frac{i}{2}\mathrm{WD}_2 = 1.98 - 1.54\,i$ and $\mathrm{WD}_1 = -16.5 - 2.5\,i$, so the first-order identity fails as well.

The derived weight. With the measure of Theorem 5.4, dual-cycle counts, and the phase scale $a = \ln 2/(2\pi)$ of the phase premise at unit action scale [2, Sec. 3.5] (MODEL), the remainder is computed at $\varepsilon = 0$ on midpoint grids. Those grids avoid the four planes $q_a \in 2\pi\mathbb{Z}$, on which the eleven-channel operator stays singular (Section 7); a grid offset that brings the waves within 0.005 of a plane changes the value by less than 10^{-3}. The remainder is $\mathscr{R}_{\mathrm{tt}} = (0.1130 \pm 0.0002) + (24.656 \pm 0.002)\, i$, with $0.0467 + 17.608\, i$ on f_E and $0.0663 + 7.048\, i$ on f_{T_2}; grids of side 16 and 24 and three shifted grids of side 16 agree within these errors, and the coarser grids of side 8 and 12 differ from them by at most 0.03 in the imaginary part. These are values of the determinant of the weighted operator on the eleven non-posting channels, not of a full gauge, ghost and density amplitude; the finiteness theorem of Section 7 is a statement about the fifteen-channel gauge-fixed responses, and the ordered limit of this remainder still needs a local statement at the origin for the weighted operator. The weight's own determinant, real at every strain, has the purely imaginary remainder $25.01\, i$, as Theorem 6.5(i) requires. Dividing the amplitude by a real normalization at each background changes only the imaginary part (Theorem 6.5(ii)): $0.113 - 0.358\, i$ if the amplitude's modulus is normalized, $0.113 + 12.15\, i$ if the Born weight is. A Born law read from the ledger's records is normalized in the second way, under the identification that the background is the preparation (IDENTIFICATION).[3] The weight itself is not such a factor; it enters the operator together with the phase, and it moves the real part from 0.27 to 0.113. The doublers are gone and the remainder is not.

Curvature-squared terms in the phase. A term $\frac{1}{2}\sum_h b_h \delta_h^2$ in the phase is of second order in the deficits, so Theorem 6.7 does not exclude it, and on long waves it is of fourth order like the weight of Section 5. With one sign on all hinges, as it dominates, the remainder on the grid of side 12 becomes purely imaginary to four digits for the seven hinge shapes that leave no wave of the grid near-singular, with imaginary parts between 20.6 and 25.0; the other shapes leave part of the grid near-singular and are not resolved (their computed imaginary parts lie between 14.7 and 27.5); with the hinge shape of the derived weight the imaginary part never approaches 0 for strengths between -10 and 1000. Terms of mixed sign create many lattice-scale shells and are not resolved.

A change of slope. Changing the hinge charge's slope at first order, one value and one area-derivative on each of the four orbits of hinges, leaves one direction that keeps every constant strain stationary and every vertex move a zero mode. Its long-wave kinetic term is not proportional to the Einstein–Hilbert form, so it gives an anisotropic graviton and fails the classical limit.

Read together (INFERENCE): a one-sign even term can cancel the on-shell real part of the remainder, but no computed change lowers the off-shell imaginary part while keeping the Einstein limit. That imaginary part is the excess of the strain curvature of $\log|\det|$ over its covariant value $\frac{11}{4}\tau$, about 14 for the native path sum on H_{tt}, and it is what a repaired action has to remove. The question is not academic. Read as a graviton mass through the identifications of [2, Sec. 5.4], a model remainder of order one at a lattice spacing of a few Planck lengths would exceed the bound on the graviton mass by tens of orders of magnitude (an IDENTIFICATION); whether an imaginary remainder is a mass, a width or neither depends

[3] The proof of this step is complete; its independent verification is not yet finished.

on the path sum's status as an evolution in record time (Section 8.1).

6.4 The target

Target 6.8 (An action with a massless one-loop graviton)**.** A phase $\sum_h C(A_h, \delta_h)$ built from the ledger's hinge records, with

(a) the classical limit of the Regge action: C is of class C^3; it reproduces every posting record, so that at a deficit of n posting units it equals n times the action of one posting on the same area; the flat lattice is a stationary point of the phase; the phase's Hessian at the flat lattice vanishes, in each slot, on every direction that leaves all deficits unchanged to first order; and its long-wave quadratic and cubic vertices equal c times Regge's for one $c > 0$;

(b) regular one-loop determinants on every torus at every $\varepsilon > 0$;

(c) summed Ward defect $\frac{i}{2}(\mathrm{WD}_2 - \mathrm{WD}_1)$ with ordered limit zero ($N \to \infty$ at fixed $\varepsilon > 0$, then $\varepsilon \to 0^+$) on every transverse-traceless probe H: symmetric, zero on the postings and traceless for g, a five-dimensional space that carries both irreducible representations, with f_E and f_{T_2} one probe in each;

(d) every coefficient of C derived from the ledger.

For the path sum of this paper, (b) and (c) apply to the model determinant of the weighted operator of Section 5, whose weight contributes a remainder of its own.

By Theorem 6.4, (c) says exactly that the zero-momentum remainder of the one-loop graviton has ordered limit zero on every transverse-traceless probe, which is what a massless transverse-traceless graviton of the model determinant at one loop requires; with the doublers gapped by the measure it is part 5 of the programme. Reading it as a massless pole of a propagating graviton also needs the evolution of Section 8.1. The condition is required probe by probe: on the sum H_{tt} alone it is one complex condition and would let the two representations cancel without either vanishing. The two computed probes decide (c) only through a covariance reduction. The remainder is a quadratic form in H; if it is invariant under the axis permutations, it is a multiple of the invariant form on each representation, and one probe in each suffices. We have not written that reduction out, so (c) is stated for every probe. The target is not trivially met. The zero phase fails (a): its long-wave quadratic vertex vanishes, and the quadratic clause fixes c and rejects every phase with a zero Hessian. The Regge action meets (a) and (b) and, as computed, fails (c), since its remainder is the native one. The quadratic truncation of the Regge action is not a phase of hinge charges. And every phase covered by Theorem 6.7 has the Regge remainder, by the argument after that theorem.

Condition (d) is not decoration. The clauses (a) leave the lattice-scale part of the charge free, and the remainder sees it. Add to the Regge charge a record-compatible even term $w(A)\,(1 - \cos k\delta)/k^2$, with w vanishing at the four flat hinge areas $A_o \in \{\frac{1}{2}, 1, \frac{3}{2}, \sqrt{3}\}$, so that the quadratic form at the flat lattice stays Regge's. On the bare phase the remainder then depends on w only through the eight numbers $w'(A_o)$ and $w''(A_o)$, by the reduction of a local charge to its slope and even curvature in Section 6.6 (proof complete, verification pending), and a massless graviton on both representations needs two things. First the corner condition,

at unit action scale,

$$w'(\tfrac{1}{2}) - w'(\tfrac{3}{2}) = -\frac{1}{3\sqrt{3}}.$$

It is exact: on every torus of even side, the derivative along the strain f_{T_2} of the phase's quadratic form on the doubler vanishes at all four axis corners if and only if the corner condition holds, and the same is true, with the even curvature e in place of w, for every local charge with the Regge slope whose even curvature is of class C^3 and vanishes at the four flat areas (proof complete, verification pending). The values $w'(1)$ and $w'(\sqrt{3})$ do not enter. Without the condition the corner logarithm, whose rate is the square of that pairing, stays (INFERENCE). Then four real linear conditions on the $w''(A_0)$, whose matrix is regular, so that w' fixes the four second derivatives uniquely (NUMERICAL): at $w' = (-1/(3\sqrt{3}), 0, 0, 0)$, $w''(A_0) \approx (-15.1,\ 4.3,\ -3.6,\ 0.3)$. Whether the ordered limit exists at other values of w' on the corner condition is not settled; fits at finite regulator converge as the regulator falls, which suggests that it does (NUMERICAL). Every such charge meets every clause of (a), whatever w' and w'' are at the flat areas: the even term has no first variation at the flat lattice, and the cubic vertex it adds is bounded on long waves uniformly in the torus size, so it drops out of the long-wave limit. If w' also vanishes at the flat areas, as for $w = \psi P^2$ with P the product of the factors $A - A_0$, the corner condition fails, so by the same inference the corner logarithm stays and the model's graviton is not massless for any such w. With the derived weight, which removes the doublers, the only exact solution found needs second derivatives of order 10^4 (NUMERICAL). So on the bare phase, whose doublers are still massless, the remainder can be tuned to zero, at least at one point (NUMERICAL), by choosing eight lattice-scale numbers. The even part gives a test that a derived charge must pass, $w(A_0) = 0$, the corner condition, and the four values of $w''(A_0)$ that follow from w', not a repair: nothing in the ledger has yet been shown to supply those numbers. Without (d) the target could be met by a tuned counterterm for each non-covariant mass structure, which is a choice, not a derivation; with (d) it asks for the ledger's own charge.

Two routes have been proposed. A *perfect action*, fixed by invariance under refinement, is the setting in which exact vertex symmetry is known to exist [8, 9]; by Theorems 6.2 and 6.3 an exact symmetry of its one-loop determinant would give (c). Section 6.5 shows what refinement does to the identified lattice: it would force the remainder to vanish, but the identified lattice does not have it, and at $\kappa = 1$ refinement moves the doubler instead of removing it; the action that the ledger's own refinement map induces is examined in Section 6.7. The second route is a local hinge charge with a term of second order in the deficit, the only local freedom Theorem 6.7 leaves. A charge whose second and third variations at the flat field are invariant under every vertex move, on every torus of side divisible by 4, would be the lattice jet of the symmetry. The side matters: on odd sides not divisible by 105 every zero mode of the Hessian is a constant strain plus a vertex move [2, Thm. 4.2], the cubic vertex vanishes on that radical, and the test is empty there [2, Sec. 6.9]: the Regge action itself passes it. By itself the jet symmetry does not make the remainder vanish, and the computed one-sign terms above do not reach the imaginary part. If no charge derived from the ledger can meet Target 6.8, proving that is a result about the programme, and we would state it as one.

6.5 Refinement

The ledger has a natural coarse reading: read it at half the resolution, replacing the record of each coarse edge by the sum of the records of its two halves. This map respects the geometry: it sends constant strains to the same strains and vertex moves to vertex moves. A *tree-level coarse-graining* of the flat Hessian integrates out the fine fluctuations at fixed coarse records, at quadratic order; it carries every fine zero mode to a coarse zero mode.

Theorem 6.9 (What refinement does at $\kappa = 1$)**.** *(i) On a torus of even side, the coarse record of the doubler field $(-1)^{x_3} v_3$ is a constant field that is neither a constant strain nor a vertex move. So every tree-level coarse-graining of the $\kappa = 1$ Regge Hessian has a spurious zero mode at the longest wavelength.*

(ii) On every odd side $N \geq 3$ not divisible by 105, the $\kappa = 1$ Regge Hessian is not a tree-level coarse-graining of the Regge Hessian on the side-$2N$ torus, at any normalization.

(iii) If the total one-loop determinants on the tori of sides M and $2M$ agree up to a constant factor, the per-site remainders obey $\mathscr{R}(M) = 16\,\mathscr{R}(2M)$, so a volume limit of the remainder, if it exists, is zero.

Statement (i) is about every tree-level coarse-graining, so it needs one to exist. On the side-4 torus one does: the $\kappa = 1$ Regge Hessian has a tree-level coarse-graining to side 2, every tree-level coarse-graining equals it, and the constant doubler field of (i) is a zero mode of it that is neither a constant strain nor a vertex move. On larger tori existence is open. The hypothesis of (iii) is an identification, refinement invariance, and it is not a consequence of the foundation. Taken together the three statements indicate that the route closes as a repair. Refinement invariance would force the remainder to vanish at one loop, but the computed native remainder (Section 6.3) shows that the identified path sum does not have it. Exact blocking of the path sum, which integrates out the fine fluctuations at fixed coarse records, multiplies the per-site remainder by 16 at each step, on every probe at which the fine determinants are regular, so it cannot remove a remainder that is there. At $\kappa = 1$ refinement does not remove the doublers either; it carries them to zero momentum, and only at $\kappa = 1$, since the doubler field is a zero mode of the flat Hessian only there (Theorem 2.1). The identification $\kappa = 1$ and a refinement-invariant Regge phase without extra zero-momentum modes exclude each other. The route reformulates the target; it does not produce the repair from the ledger's phase. The tension runs deeper than the Regge phase. At $\kappa = 1$ a single refinement step already fails, on every side. At $\kappa = 3/2$ one step from the side-4 torus to the side-2 torus holds: the coarse-graining exists and every zero mode of the coarse Hessian is a constant strain plus a vertex move (proof complete, verification pending); at $\kappa = 3/4$, $5/4$ and 2 the same step holds in computation (NUMERICAL). So the tension is exact: the reading of the interval as a count of recognitions (Section 2) holds exactly at $\kappa = 1$, refinement holds somewhere at $\kappa = 3/2$, and it holds nowhere at $\kappa = 1$. That refinement admits every value other than 1 is open. The programme cannot adopt both readings.

6.6 Local hinge charges

The no-go theorems of Section 6.2 are about exact symmetry. Target (c) asks for less, a vanishing summed defect, so the local hinge charges need a direct test of the remainder. Three classes have been examined: one by a theorem, one by a theorem that reduces it to a single number and a computation of that number, and one by computation.

Theorem 6.10 (Higher deficits leave the weighted one-loop response unchanged)**.** *Let $\kappa = 1$, $N \geq 3$ and c real, and let*

$$C(A, \delta) = cA\delta + \delta^3 \gamma(A, \delta), \qquad C_0(A, \delta) = cA\delta,$$

with γ of class C^3. Use for the two phases $\sum_h C(A_h, \delta_h)$ and $\sum_h C_0(A_h, \delta_h)$ the same real weight, the same eleven non-posting channels and the same regulator. Near zero constant strain their one-loop determinants agree at every torus wave, so their remainders agree on every probe H. Where the determinants are regular, the summed defects $\frac{i}{2}(\mathrm{WD}_2 - \mathrm{WD}_1)$ agree as well. The ordered-limit conditions (c) of Target 6.8 are therefore equivalent for the two phases, with the same weight on every torus.

Proof. Every constant strain of the flat lattice is flat, and so is every nearby member of the flat family: all deficits vanish there. At such a configuration every term of the second derivative of $\delta_h^3 \Gamma_h$ keeps a factor δ_h, so its Hessian vanishes; the torus has finitely many hinges and its flat simplices are nondegenerate, so this holds on a common neighbourhood. The Hessians of the two phases therefore agree along the whole flat family, not only at the unstrained lattice. The Bloch transform and the restriction to the non-posting channels preserve the equality, and adding the same weight's Hessian and the same regulator gives the same matrix at every nearby strain, hence the same determinant. The strain and dilatation derivatives of the logarithms agree, so the remainders agree for every probe, and under the stated regularity Theorem 6.4 identifies them with the summed defects. The equality holds before either limit is taken. □

The comparison is at the same slope c and the same weight. Multiplying the phase by another constant while the weight is held fixed changes the weighted operator, and with it the remainder. The theorem excludes the factorized corrections $\delta^3\gamma$, with γ of class C^3, as a repair; it does not exclude local charges that change the Hessian along the strained flat family. With the area slope fixed, the first such term is quadratic in the deficit.

The premises that fix the phase say that the action one posting carries adds over the pieces of a hinge. Applied to a hinge charge $C(A, \delta)$ at every deficit, together with continuity, additivity over the pieces forces $C = A\, f(\delta)$; for a charge differentiable in the area, the split of a hinge into the four pieces of a refined cell already suffices. For such a charge, with f of class C^3 and $f(0) = 0$, the Hessian of the phase at every strained flat lattice is $f'(0)/\hbar$ times the Regge Hessian plus $f''(0)/\hbar$ times $\sum_h A_h\, \delta_h' \delta_h'^{\top}$. For the phase alone the remainder then depends on the charge only through the one number $\beta = f''(0)/f'(0)$: the overall factor $f'(0) > 0$ only rescales the regulator, which drops out of the ordered limit. Computed for the phase alone over β from -10 to 1000, at $\kappa = 1$ and at $\kappa = 3/2$, on a torus of side 16 at the regulators 0.05 and 0.025, no value makes the remainder vanish on both representations. At $\kappa = 1$ the remainder on f_E has modulus at least 4.98 on the scanned values and imaginary part above

3.88 on a dense band of small β, where it comes closest to zero (NUMERICAL). An even charge of constant amplitude, $cA\delta + a(1 - \cos k\delta)$, is record-compatible but not area-additive; computed over its amplitude, from -10 to 1000 in units of the action, it leaves a nonzero remainder on each representation as well (NUMERICAL).

Theorem 6.10 is one case of a general statement. For every charge of class C^3 that vanishes at zero deficit, the Hessian of the phase along the strained flat family is

$$\hbar\, D^2\Big(\sum_h C(A_h, \delta_h)\Big) = \sum_h \Big[\sigma(A_h)\, D^2\delta_h + \sigma'(A_h)\big(\mathrm{d}A_h \otimes \mathrm{d}\delta_h + \mathrm{d}\delta_h \otimes \mathrm{d}A_h\big) + e(A_h)\, \mathrm{d}\delta_h \otimes \mathrm{d}\delta_h\Big],$$

with the slope $\sigma(A) = \partial_\delta C(A, 0)$ and the even curvature $e(A) = \partial_\delta^2 C(A, 0)$. So the one-loop remainder of a local hinge charge, on every probe, with every weight and at every regulator, depends on the charge only through these two functions of the area.[4] The ledger's own readings of a charge constrain e. Its records are odd under reversing the postings, and a charge with that oddness has $e = 0$; the running total of postings has it. A charge with the Regge slope and $e = 0$ has exactly the Regge charge's remainder, so by Theorem 3.3 it is not massless. A charge that adds over the pieces of a hinge has $e = \beta A$, the family computed above; for $\beta \neq 0$ its even curvature does not vanish at the flat areas, so it changes the flat operator itself, and no computed value of β gives zero. The even term discussed after Target 6.8 has even curvature $e = w$ with $w = 0$ at the flat areas, and every w of that kind that is not identically zero breaks oddness, deficit additivity and area additivity at once.[5] Within the readings the ledger supplies, then, the even curvature is 0 or βA; the first is not massless, and the second is not, as computed. A candidate even part derived some other way can be put to the test after Target 6.8 with no number left to adjust, which makes it a test against fitting. The most natural candidate outside the readings above, an even part that sees a hinge's area only through its ratio to the flat area, meets the corner condition with one choice of its slope, but then leaves a remainder of modulus above 6 on f_E for every profile (NUMERICAL), so it fails. An even part that removes the remainder would need a new premise that breaks all three properties, singles out the four flat hinge areas and satisfies the corner condition; nothing in the ledger supplies it, and choosing it would be a fit.

These results exclude classes, not every local charge. Within the area-additive local charges the repair is excluded as computed: a derived phase that meets Target 6.8 must give up additivity over the pieces of a hinge, or be nonlocal, or carry a measure that is not of the kind examined in Section 5. Outside that class, for an even term of the form discussed after Target 6.8, clause (c) becomes the corner condition on w' and four linear conditions that then fix w'' at the flat hinge areas; deriving such a term from the ledger, with numbers that pass this test, is the open question.

6.7 What is excluded, and what remains

Target 6.8 is the open target of part 2. Each of the following is a result, with its status, and together they say which repairs the ledger's local data cannot supply.

(i) No real weight or density makes a vertex-translation flow a symmetry of the path-sum exponent (Theorem 6.6).

[4] The proof of this step is complete; its independent verification is not yet finished.

[5] The proof of this step is complete; its independent verification is not yet finished.

(ii) No phase $cS + \sum_h \delta_h^3 \Gamma_h$ has the symmetry (Theorem 6.7), and at the same slope and weight its remainder is that of cS on every probe (Theorem 6.10).

(iii) Refinement invariance would remove the remainder, but the identified lattice does not have it, and at $\kappa = 1$ refinement carries the doublers to zero momentum (Theorem 6.9); the reading of the interval as a count of recognitions selects $\kappa = 1$, and refinement excludes that value.

(iv) The remainder of an area-additive charge is that of a one-parameter family in β, and no value of β computed gives zero (NUMERICAL).

(v) An even part homogeneous of degree one in a hinge's area and its reference area, under every positive rescaling of both, fails the corner condition exactly; one that sees the area only through its ratio to the flat area fails on f_E (NUMERICAL).

(vi) Within the readings of a charge that the ledger supplies, the even curvature is 0 or proportional to the area (proof complete, verification pending), and neither gives a massless graviton in the model. The classical clauses leave the even part free, so a local charge that removes the remainder would be a choice of lattice-scale numbers, found so far numerically at one point on the bare phase (Section 6.6 and the discussion after Target 6.8).

(vii) The native $\kappa = 1$ graviton of the model is not massless (Theorem 3.3).

Two routes that are not local. The phase need not be a sum over single hinges, and two non-local candidates that the ledger itself suggests have been carried through at one loop. Neither supplies the repair at $\kappa = 1$; the proofs are complete, and their independent verification is not yet finished.

The first builds the phase from the holonomy of postings around hinge cycles, a phase carried by the record history. At one loop any phase, local or not, enters only through two objects at zero deficit: its slope map, which turns first-order deficits into phase, and its even kernel, its second derivative in the deficits. A phase odd under reversing the postings has no even kernel, and an odd phase with the Regge slope map is massless exactly when the native path sum is, which by Theorem 3.3 it is not. The ledger's holonomy phase, read through two identifications, that a record loop's phase is read from its postings and that each face carries one cycle, is the Regge phase on every geometry and at every coupling, so it has the native remainder and misses the corner condition. Averaging deficit-linear loop phases over record histories gives, along a line of deficits, a mean slope and a real weight, not an even phase; and a per-face function of the minimal holonomy angle reduces to the β family of Section 6.6. Non-locality can therefore help at one loop only through a slope map or an even kernel that is not that of any local charge, for instance one that couples different hinges. Whether the ledger supplies one is open; that the even kernels of composite holonomies need loop weights the ledger does not supply is an inference.

The second takes the action that the ledger's refinement map induces on the coarse lattice, the ledger's version of a perfect action. A coarse record is the sum of the records of the two halves of an edge, reading the ledger's records as counts on edges (IDENTIFICATION). With the damping held fixed on the fine lattice, each blocking step multiplies the damping bound by eight times the normalization, so in the normalization of the Einstein–Hilbert term the

blocked Hessian of the native path sum tends to zero, at every $\kappa > 1/2$ and every damping, and no damped symbol is a fixed point. Averaging over cells of dimension m changes the bound by 2^{2-m} per step: records on edges double it, records on faces are marginal and whole cells contract. This does not decide the route. Held fixed on the fine lattice, the damping grows like a mass on the coarse lattice, and in that order ordinary block averages over whole cells, the standard setting for perfect actions, have no damped fixed point either. The order that can tell edge records from block averages holds the damping fixed on the coarse lattice, and at $\kappa = 1$ it decides the route. The hyperplane of waves whose phase along the step $(0, 1, 1, 1)$ is one passes through the corner πe_0, and on it the doubler's cone is indefinite, so the native propagator keeps an on-shell weight in every slab around that hyperplane; in that order the zone average of the blocked record fluctuation therefore grows like 2^k, while for block averages over whole cells it stays bounded. At $\kappa = 1$ the blocked path sum thus has no fixed point with an Einstein–Hilbert long-wave part in either order of limits (proof complete, verification pending). The proof uses the doubler, so it is specific to $\kappa = 1$; other values of κ and the pointwise form of the statement are open. Exact blocking multiplies the per-site remainder by 16 (Section 6.5), and blocking selects no value of κ.

The same symmetry fails classically. At second order in the field the lattice's vertex moves are not a symmetry of the Regge action, one order in the derivatives past Einstein's equations (Theorem 4.3). An exact vertex symmetry would serve both purposes: of the one-loop determinant, it meets clause (c) of Target 6.8 through Theorems 6.2 and 6.3; of the action at second order, it removes the classical obstruction to static vacuum exteriors (INFERENCE). Any candidate, local or not, is tested as the local charges were: its classical limit first, then its Ward defect on both representations, with every coefficient derived and no number left to adjust.

7 Finite ordered one-loop limits for the weighted responses

Status: part 6, finiteness

Proved. For any lattice graviton family and any smooth vertex data, the ordered one-loop limit at zero momentum exists and equals the limit of the zone integral, provided the degenerate zeros away from the origin and the corners are hyperbolic, the corners are regular and the origin has a finite local limit (Theorem 7.1). A weight that sees every null vector at the corners replaces the corner condition (Theorem 7.2), and the origin condition follows from a blow-up of the origin, which holds for the identified operator with and without the derived weight (Theorem 7.3). With the derived weight of Section 5, the anisotropy that diverges in Theorem 3.1 has a finite real ordered limit, and so does the real part of the full weighted anisotropy, the weight's own vertices included. The real parts of the contributions of H_E and H_{T_2} are finite separately, and so is each sunset and each seagull term; no cancellation between probes or diagrams is used (Theorem 7.4).

Proof complete, independent verification pending. The imaginary part of the full weighted anisotropy has a finite ordered limit too (Proposition 7.5). For the native eleven-channel form, the statement on the four singular planes follows from Theorem 5.6, and the statement at the origin from a uniform bound on dyadic shells around it.

Premises. $\kappa = 1$ (IDENTIFICATION); the Feynman regulator (MODEL); the weight is the ledger's only under (I1) and (I2) of Section 5. The theorem holds for every positive coupling of the weight.

Open. The value of the limit, which is not computed: a finite limit is not a massless graviton. The full path sum and higher loops. Finiteness for an action that also meets Target 6.8. The zero-momentum remainder of the weighted theory in its native form on the eleven non-posting channels, where the operator stays singular on four planes of waves: its existence reduces to one local statement at the origin, and that statement to a uniform bound on dyadic shells or to a dissipation condition at each of the four points where the light cone touches those planes; both are open, and neither is the bound at the origin that Theorem 3.3 uses for the native operator.

What would settle it. The same three conditions checked for the repaired action, together with the local statement at the origin for the native form.

Theorem 3.1 says that the identified lattice has no ordered one-loop limit. The cause is local in the Brillouin zone: four corners where the propagator has a double zero that no direction lifts. This section turns that diagnosis into a theorem with checkable conditions and then checks them for the weighted operator.

7.1 What finiteness needs

A *lattice graviton family* is a Hermitian matrix $M(q)$ on the edge classes, a function of the wave q with period 2π in each component that extends to complex lines $q + zv$ with a continuous derivative in z. Its one-loop observable at zero momentum is the real part of

$$K_\varepsilon(q) = \sum \operatorname{tr}\big(XRYR\big) + \sum \operatorname{tr}\big(ZR\big), \qquad R = (M(q) + i\varepsilon)^{-1},$$

with sunset vertices X, Y and seagull vertices Z that are complex matrix functions of class C^3. On the side-N torus the observable is the average of K_ε over the N^4 torus waves; in infinite volume it is the integral over the zone with measure $(2\pi)^{-4}\mathrm{d}^4q$. We call the observable *finite* if there is a real c such that, for every $\varepsilon > 0$, the real part of the torus average converges as $N \to \infty$ to some $E(\varepsilon)$, $E(\varepsilon) \to c$ as $\varepsilon \to 0^+$, and the zone integral also tends to c. This is the ordered limit of Section 1, volume first at fixed regulator and then the regulator removed.

Three conditions suffice. A *degenerate characteristic point* is a wave p with $\det M(p) = 0$ at which the derivative of $\operatorname{Re}\det M$ also vanishes. It is *hyperbolic* if some direction v satisfies $\operatorname{Re}\langle u, \partial_v M(p) u\rangle > 0$ for every nonzero u in the kernel of $M(p)$: moving along v pushes every vanishing eigenvalue the same way, so the contour can be deformed off the zero and the regulator does not matter there.

(H) Every degenerate characteristic point other than the lattice copies of the origin and of the fifteen half-period corners is hyperbolic.

(D) $\det M(\pi e) \neq 0$ at every corner $e \in \{0,1\}^4 \setminus \{0\}$.

(O) On a neighbourhood of the origin, the regulated integrand cut off by any smooth cutoff has a finite local limit.

Theorem 7.1 (Finiteness from local conditions)**.** *Let M be a lattice graviton family and let the vertex data be of class C^3, with an integrand periodic under the shifts $2\pi e_a$ at every $\varepsilon > 0$. If (H), (D) and (O) hold, the observable is finite, and its limit is $(2\pi)^{-4}$ times the sum of the local limits of any finite smooth partition of the zone.*

Two facts are proved on the way and explain the shape of the conditions. Every invertible point and every nondegenerate characteristic point is automatically hyperbolic, so (H) is a condition only at the degenerate points. And the origin of an even family with a massless mode is never hyperbolic, which is why the origin needs a condition of its own. The conditions are sufficient; they are not shown to be necessary. The theorem is stated for the regulator $+i\varepsilon$, a model; the other sign is the same theorem with adjoint vertex data.

At identified $\kappa = 1$ the three conditions sort the obstruction exactly. Condition (H) holds: it is the hyperbolicity of every critical zero other than the origin and the corners, Theorem 3.1. Condition (O) holds for the anisotropy of Section 3. Condition (D) fails, and by Theorem 2.1 it fails exactly at the four axis corners, where the limit diverges. Along the family, (D) holds if and only if $\kappa \neq 1$; at $\kappa = 3/2$ it holds at all fifteen corners, while (H) there is supported only by a numerical census and (O) is open. So a value $\kappa \neq 1$ would trade the corner problem for two unproved conditions, and it would give up the reading of the posting order (Section 2).

7.2 A weight that sees the doublers

A measure weight enters the Gaussian operator as an imaginary part, $M + iG$ with G positive semidefinite (Theorem 5.1). Such a family is dissipative, $\mathrm{Im}\langle w, (M + iG)w\rangle \geq 0$, and the finiteness theorem extends to it.

Theorem 7.2 (A weight replaces the corner condition)**.** *Let M be a Hermitian lattice graviton family that satisfies (H), and let G be a lattice family with $G(q)$ positive semidefinite for every q, such that at every lattice corner G is nonzero on every null vector of M. If the integrand of $M + iG$ is periodic and $M + iG$ satisfies (O), its observable is finite.*

The proof rests on two observations: damping never destroys hyperbolicity, and where G sees every null vector of M the damped operator $M + iG$ has none. What remains is the origin.

Theorem 7.3 (The origin from a blow-up)**.** *For any dissipative family and any vertex data, condition (O) holds whenever the origin admits a blow-up: a frame that separates the strain directions from the massive ones, in which the family rescaled by $1/\lambda$ on the strain directions extends to a function of class C^4 down to $\lambda = 0$, with rescaled vertices of classes C^3, C^3 and C^2, an injective massive block at the origin, and a leading symbol that is hyperbolic at every wave of the annulus $\frac{1}{2} \leq |p| \leq 2$. At $\kappa = 1$ such a blow-up exists for the gauge-fixed operator with every graded observable, for the anisotropy of Section 3, for the operator with the derived weight, and for the full weighted observable of Theorem 7.4.*

None of the blow-up's conditions mentions an integral or a limit; each instance is a separate check, and the four listed are the ones proved. For $\kappa \neq 1$ no blow-up is known: hyperbolicity of the leading symbol needs an analysis of the null cone that so far exists only at $\kappa = 1$.

7.3 The weighted responses are finite

Let $A(q)$ be the gauge-fixed operator of Section 3 at $\kappa = 1$, with unit gauge weight, and let $\mathscr{D}(q)$ be the Bloch symbol of the derived form of Theorem 5.4 with the dual-cycle counts. For a coupling $a > 0$ the weighted operator is $M_{\mathscr{D}}(q) = A(q) + \frac{i}{a}\mathscr{D}(q)$, the Gaussian operator of the weighted path sum (Theorem 5.1(ii)).

Theorem 7.4 (Finite real ordered limits for the weighted responses). *Let $\kappa = 1$ and $a > 0$.*

(i) The graviton-loop anisotropy between the probes H_E and H_{T_2}, with the cubic and quartic strain vertices of the Regge action, the normalizations of Section 3 and the regulator $i\varepsilon$, but with the propagator $(M_{\mathscr{D}} + i\varepsilon)^{-1}$ in place of $(A + i\varepsilon)^{-1}$, is finite.

(ii) The real part of the full weighted anisotropy,

$$\frac{i}{2} \sum_{b \in \{E, T_2\}} \pm \frac{1}{d_b} \frac{\mathrm{d}^2}{\mathrm{d}t^2} \log \det\Big(A_b(t) + \tfrac{i}{a} \mathscr{D}_b(t) + i\varepsilon \Big)\Big|_{t=0},$$

where $A_b(t)$ and $\mathscr{D}_b(t)$ are the Hessian and the weight at the flat lattice strained by tH_b, the gauge term and the regulator are held fixed, and d_b is the DeWitt normalization of the probe, is finite. Its vertices are those of the weighted operator itself, so the weight's own interaction vertices are included.

(iii) For each $b \in \{E, T_2\}$ separately, the real part of its signed, DeWitt-normalized contribution to (ii) has a finite ordered limit, equal to the limit of its zone integral. The same holds for its sunset term and for its seagull term separately.

Proof. Apply Theorem 7.2 with $M = A$ and $G = \mathscr{D}/a$. Condition (H) for A is the hyperbolicity of Theorem 3.1. At the four axis corners the derived form is nonzero on the doubler, Theorem 5.5(iii), and at the other eleven corners A is invertible by Theorem 2.1, so G sees every null vector at every corner. Condition (O) for the weighted family and both observables follows from the blow-up of Theorem 7.3. No other hypothesis remains.

For (iii), the conditions of Theorems 7.1 and 7.3 on the vertex data, smoothness, periodicity and the grading at the origin, are imposed on each sunset pair and each seagull vertex separately, while the conditions on the operator do not depend on which vertices are kept. Keep only the sunset pair and the seagull of one probe: these data satisfy the same conditions for the same operator, and Theorem 7.2 applies unchanged. Keeping only the sunset pair, or only the seagull, gives the two diagram statements. Neither the subtraction of the probes nor a cancellation between diagrams enters the proof. The vertices and their normalization are written out in Appendix A.2. □

With $\mathscr{D} = 0$ the observable of (i) is the graviton-loop part of Δ, which grows like $\big(5/(44\sqrt{335}\,\pi^2)\big)\log(1/\varepsilon)$ (Theorem 3.1). The derived weight turns that divergence into a finite ordered limit, and it does so without a cutoff or a coefficient chosen for the purpose: the coupling a is arbitrary in the theorem, and the form $\mathscr{D}$ is fixed by the ledger's cost under the identifications of Section 5.

The imaginary parts converge as well. The imaginary part of a term is the real part of the same data multiplied by $-i$, and that rotation keeps every condition of Theorems 7.1 and 7.3 on the vertex data, the grading at the origin included.

Proposition 7.5 (The complex observable). *Let $\kappa = 1$ and $a > 0$. The full weighted anisotropy of Theorem 7.4(ii), real and imaginary parts together, has a finite ordered limit, equal to the limit of its zone integral.*[6]

[6]The proof of this step is complete; its independent verification is not yet finished.

Theorem 7.4 and Proposition 7.5 concern these specified one-loop responses; they say nothing about the full path sum or higher loops. It does not compute the value of the limit, and a finite limit is not a massless graviton: whether the remainder vanishes is the question of Section 6. It is a statement about the gauge-fixed operator on fifteen channels. In the native form on the eleven non-posting channels, the weighted operator stays singular on the four planes $q_a \in 2\pi\mathbb{Z}$: there a vertex move with no posting component remains in the eleven-channel space, and neither the action nor the weight sees it. Those plane points are not hyperbolic, so the zero-momentum remainder of the weighted theory in that form needs estimates of its own. Its ordered limit exists if one local statement holds, a finite local limit at the origin: the corresponding statement on the planes follows from Theorem 5.6,[7] the statement at the origin follows in turn from a uniform bound on the dyadic shells around it,[8] and the dilatation term σ_{dil} converges to $-\frac{11}{2}\,i$. The value $0.113 + 24.66\,i$ quoted in Sections 5 and 6 belongs to that eleven-channel computation, not to the fifteen-channel theorem above, and its ordered limit still needs the local statement at the origin for the weighted operator, which is open; the bound at the origin proved for the native operator in Theorem 3.3 does not supply it. Finally, the theorem is a statement at $\kappa = 1$; at other values neither the hyperbolicity census nor the blow-up is proved.

8 Unitarity, matter and predictions

Three parts of the programme are mostly open. For each we state what is proved, what the missing statement is, and how it is tied to the parts above.

8.1 Unitarity in record time

Status: part 7, unitarity

Proved. Ordering the path sum by record time does not remove the doublers (Theorem 3.2). At identified $\kappa = 1$ the linearized Regge equation has a solution that is not a vertex move and grows by $\sqrt{2}$ per record tick, at every spatial period, so no positive norm on its physical solutions is preserved by the tick (Theorem 8.1); the same holds for every phase whose Gaussian field equation is a nonzero multiple of Regge's, and at $\kappa = 3/2$, with the factor $(1 + \sqrt{10})/3$. For a Hermitian action symbol and a positive semidefinite weight, the null vectors of the weighted symbol are the common null vectors; in a record-time stable weighted evolution the weight vanishes on every physical mode. On a model wave equation, a damping of fourth order in the wave number, the order of the derived weight, is stable in the continuum limit, and one of second order is not. With the derived weight, at each fixed nonzero strength, every lattice species away from the zero wave decouples and the record-time roots of the weighted Gaussian equation are stable in the continuum limit: the long-wave branch grows per tick by at most the cube of its momentum, given (I1), (I2) and an inference about lattice-scale modes off the unit circle; on the lattice it is not record-time stable. These are statements about record-time roots, not about a norm-preserving evolution. For a polynomial in the record-time frequency, the log-modulus averaged over frequencies splits exactly into a normalization part and a growth part, and the growth part vanishes exactly when every root lies in the closed unit disk.

[7] The proof of this step is complete; its independent verification is not yet finished.
[8] The proof of this step is complete; its independent verification is not yet finished.

Open. A record-time evolution of the path sum with a positive norm on its physical solutions that the tick preserves: for every spatial momentum, every physical record-time frequency real. Or, more weakly, growth per unit physical time that vanishes in the continuum limit for every species, the doublers included, beyond Gaussian order, without the inference about off-circle modes, and uniformly in the strength of the weight; for the bare Regge action it fails as far as computed.

What would settle it. An action and measure, derived as in Sections 5 and 6, whose physical record-time roots lie on the unit circle, or approach it in the continuum limit; or a proof that none exists.

A real-time path sum is a quantum theory only if it defines an evolution, and the posting order supplies the natural time, the record count $t_{\mathrm{r}}(x) = \sum_a x_a$. A *record tick* is a lattice translation τ with $t_{\mathrm{r}}(\tau) = 1$. At Gaussian order each sector of fixed spatial momentum is a recursion in t_{r} with a matrix Laurent symbol in $w = e^{is}$, and a positive norm that the tick preserves needs every physical root of that symbol on the unit circle: a mode $z^{t_{\mathrm{r}}} v$ with $|z| \neq 1$ would change its own norm. At $\kappa = 1$ such a mode exists.

Theorem 8.1 (Record-time norm obstruction)**.** *Let $\kappa = 1$, let $N \geq 1$, and let τ be a record tick. Consider the solutions of the linearized Regge equation on the lattice $\mathbb{Z}^4$ that are periodic under the spatial translations $N(e_a - e_b)$ and open in record time. There is no function n on fields with $n(ch) = |c|^2 n(h)$ for complex c that is strictly positive on every such solution that is not a vertex move and unchanged by τ on solutions. In particular no Hermitian form that vanishes on vertex moves and is positive on the other solutions is preserved by the record tick.*

Proof. Write $r = \sqrt{2}$ and $t(x) = \sum_a x_a$. The Bloch equation of [2, Prop. 6.16] has a null polarization u with $M(r^{-1}, r^{-1}, r^{-1}, r^{-1})\, u = 0$ for which the field $h_d(x) = r^{t(x)+|d|} u_d$ solves the linearized Regge equation and is not a vertex move of any shape; its coefficients and the gauge exclusion are given in Appendix A.3. Since $t(e_a - e_b) = 0$, the field has every spatial period, and the tick multiplies it by r. If n were such a function, then

$$0 < n(h) = n(h(\cdot + \tau)) = n(rh) = r^2\, n(h) = 2\, n(h),$$

a contradiction. □

The theorem is about the physical solutions, not about every evolution one could write down. What fails is a positive norm on the physical solutions that the tick preserves, which a one-particle quantum theory with real frequencies needs. A unitary operator on slice wavefunctions would not by itself establish that norm; the physical reduction and its equality with the same gravitational functional would also have to be supplied. The test is not empty: the zero phase fails it, and a simple propagating lattice equation meets the weaker requirement that every physical mode change by a unimodular factor per tick.

The obstruction survives the restriction the native path sum makes. That path sum fixes the postings and varies only the eleven non-posting channels, so its physical solutions are posting-null; the growing mode has a representative of that kind.

Proposition 8.2 (The obstruction survives the posting-null restriction)**.** *Let $\kappa = 1$, let $N \geq 1$ and let τ be a record tick. Among the solutions of Theorem 8.1, keep those whose posting components*

vanish, $h(x, e_a) = 0$ for every vertex x and every axis a. This smaller space still contains a solution that is not a vertex move and is multiplied by $\sqrt{2}$ by the tick, so it carries no function n as in Theorem 8.1 either.

Proof. Let h be the growing field of Theorem 8.1, $r = \sqrt{2}$, and let $p(a)$ be the class of the posting e_a; the posting components of h are $h(x, e_a) = r^{t(x)+1} u_{p(a)}$. The vertex move generated by $\zeta_a(x) = r^{t(x)} c_a$ with $c_a = r u_{p(a)}/\big(2(r-1)\big)$ has posting components $2\big(\zeta_a(x + e_a) - \zeta_a(x)\big) = 2r^{t(x)}(r-1)c_a = h(x, e_a)$, so $\hat{h} = h -$ (that move) has all posting components zero. The move solves the linearized equation at these exponential phases, since the gauge identity on the unit torus is an identity of Laurent polynomials and so holds at every nonzero complex phase; hence $\hat{h}$ is a solution. It has every spatial period and is multiplied by r by the tick, as h and the move are. It is not a vertex move, since otherwise h would be one. The argument of Theorem 8.1 applies to $\hat{h}$. □

The proposition concerns the linearized bare equation in open record time. It does not construct a nonlinear constrained evolution, and it says nothing about the weighted equation. A reduction to physical states that evades the obstruction has to supply a further derived restriction or a different evolution; the posting constraints alone do not supply it. Removing the growing mode would take a boundary condition or a constraint that follows from the ledger; discarding it by definition is not one. Record-time growth also does not single out a value of κ: at $\kappa = 3/2$ the lattice has a growing mode with factor $(1 + \sqrt{10})/3$ [2, Prop. 6.16]. Every phase whose Gaussian field equation is a nonzero multiple of the Regge equation has the same solutions and fails in the same way; by Theorem 6.7 that includes every phase $cS + \sum_h \delta_h^3 \Gamma_h$. A repair that passes the test must change the quadratic action at the lattice scale, which the classical clauses of Target 6.8 allow. The doublers are a second obstacle: they propagate in record time with a mixed spectrum (Theorem 3.2).

Unitarity and the remainder of Section 6 are tied. Apply the exact split of the frequency-averaged log-modulus into a normalization part and a growth part to the one-loop determinant of each sector: if the evolution is unitary and carries the normalization of a unitary kernel, the strain response of the one-loop effective action has no imaginary part, and the imaginary part of the remainder vanishes (INFERENCE: the identification of the leading coefficient's modulus with the unitary kernel normalization is stated, not proved). For the native path sum the computed imaginary part, about 7.0, is about 85% normalization and 15% growth (NUMERICAL). On this reading the off-shell imaginary part that no computed repair lowers is mostly a missing transfer normalization, which a unitary record-time evolution would supply.

A real weight e^{-W} enters the record-time field equation as a damping. At a real wave q, with $A(q)$ Hermitian and $\mathscr{D}(q)$ positive semidefinite, the proof of Theorem 5.2 gives $\ker\big(A(q) + i\mathscr{D}(q)\big) = \ker A(q) \cap \ker \mathscr{D}(q)$: where the kernel of $\mathscr{D}(q)$ contains only vertex moves, the weighted equation has no physical mode at that real wave. In a record-time stable weighted evolution of Fourier form the weight vanishes on every physical mode. The derived weight's operator has that Fourier form, and its kernel lies in the vertex moves at every wave other than the origin (Theorem 5.6), so the derived weighted equation is not record-time stable as soon as it has a physical mode away from the zero wave. A weight can remove the corner poles without giving a stable physical spectrum. This constrains the specified field equation; it is not a theorem against a norm-preserving evolution on some other space of wave functions.

Stability on the lattice may ask for more than physics needs, since the damping has to vanish only at long wavelengths. Call a linear record-time equation *stable in the continuum limit* if every family of physical modes whose record-time factor approaches the unit circle as the spacing shrinks grows by a vanishing amount per unit of physical time. That is a property of the record-time roots of the Gaussian equation. It is weaker than stability on the lattice, and it is not a norm-preserving evolution on a space of states, which is not constructed here. Modes that stay a fixed distance from the circle are lattice-scale and are not constrained by this definition; that they decouple is an inference. Take a model record-time wave equation with a real damping of order k^{2m} in the wave number. Damping of fourth order ($m = 2$), the order of the derived weight, is stable in the continuum limit at every nonzero strength but never stable on the lattice; damping of second order ($m = 1$), the order of the action, is not stable even in the continuum limit. The bare $\kappa = 1$ path sum fails the weaker test too, as far as computed: near an axis corner the doubler's record-time roots leave the unit circle at a rate proportional to the distance from the corner, so their growth per unit physical time survives the continuum limit (NUMERICAL). A weighted theory that is never stable on the lattice can therefore be stable in the continuum limit, through the same order of damping that makes it finite. For the derived weight's own operator the answer is yes. By the kernel identity of Theorem 5.6, no family of lattice-scale modes, the doublers included, approaches the unit circle away from the zero wave, so the continuum limit sees only the graviton's branch there; off the circle such modes may grow as well as decay per tick, and that they do not affect the continuum theory is an inference. The graviton's branch grows per tick by at most a constant times the cube of its spatial momentum, at every strength of the weight, with a constant that depends on the strength: rescaled near the zero wave, its record-time roots tend to real null covectors on which the Krein form of the polarization does not vanish, and an exact identity for the imaginary part of the Einstein symbol at complex covectors then bounds the growth. So the record-time roots of the derived weight's Gaussian path sum are not stable on the lattice but stable in the continuum limit, given the identifications of Section 5 and the inference about off-circle modes. The statement is at a fixed strength of the weight, the ratio of its coefficient to the action's. At a fixed lattice spacing, the identified one included, that is the long-wavelength limit of the path sum; a limit in which the spacing itself shrinks changes the strength and would need the bound uniformly in it, which is not proved. Computed at strengths from 0.01 to 10 and in three spatial directions, the growth per tick of the graviton's branch is indeed proportional to the cube of the spatial momentum (NUMERICAL). And at the end of the record-time Wick rotation, read as $\kappa(\pi) = -\frac{1}{2}$ (INFERENCE), the Regge form is indefinite on the full space of fields on every torus, so the Gaussian integral does not converge and reflection positivity cannot be posed at Gaussian order without a contour prescription, such as an extra rotation of the lapse. These are why parts 3, 5, 6 and 7 must be settled together: a measure that gaps the doublers by damping is finite but not stable on the lattice, and stable only in the continuum limit; a repair of the remainder that is to allow a norm-preserving record-time evolution must change the quadratic action itself. The action, the damping, the physical states and the normalization have to be derived together.

8.2 Matter

Status: part 8, matter

Proved, for every $\kappa > 1/2$: a posting gas held by its least-work support, and worldline matter that conserves postings, each carry a body at fixed small strength through second order (Section 4; [2, App. E]).

Identifications. The least-work law of the recognition cost read at the flat reference; the ratios of the reference occupation c^0 across the timelike classes, which rest on one premise about which ledger entries carry a posting; the split between body and carrier.

Open. A matter coupling and an equation of state that follow from the cost J without these identifications.

Matter enters the lattice through point-particle actions on the timelike edges [2, Sec. 3.5]. Its stress is fixed once a law says how much of each timelike class an occupation uses, and the recognition cost supplies one: the least-work carrier of a load [2, Thm. E.1]. That reading is an identification, and so are the reference ratios c^0, which fix how the carrier distributes over the eleven long classes. Worldline matter, whose postings move along worldlines [2, Thm. E.4], carries every load that does not lie on its blind fields [2, Thm. E.5]. What a derivation would add is the law itself: that the ledger's cost of an occupation, applied to the occupation that holds a body up, is the least-work law, and that the class ratios are forced.

The class ratios can be traced to one premise. The foundation's defect is an equal-weight sum of the cost J over ledger entries, so if an edge's occupation is spread additively over its entries, a reading of how occupation sits in the ledger (IDENTIFICATION), an edge spread over k entries costs at least $kJ(c/k)$: the reference occupation of an edge is its number of entries. The symmetry that exchanges the axes then forces equal weight for the 24 histories of a cell. What is not forced is which entries carry a posting. One entry per record pair of a cell, one per edge, and one per history passing through the edge give three different profiles across the long classes with two, three and four postings, $4:2:1$, $1:1:1$ and $1:1:2$; they differ, so the choice among them is a premise, and records, record order, symmetry and least work do not appear to fix it (INFERENCE). The last reading, that a posting is an entry of every history record through its record pair, the same entries as the action's sum over simplices, gives the ratios used here. It is the one identification left in c^0, and deriving it needs a statement of the ledger about which recognition events carry a posting.

8.3 Predictions

Status: part 9, predictions

Proved. At tree level every non-gauge long-wave mode propagates on the light cone up to a relative correction of order $(k_{\mathrm{ph}}b)^2$, and the static Newton kernel is correct to second order [2, Thms. 5.1 and 5.4]: no observable signature for lattice spacings up to 10^6 Planck lengths.

Computed, as a conditional diagnostic. If the lattice spacing and the map from the remainder to a mass term are read as below (two identifications), the computed remainders of the unrepaired path sum's model determinant correspond to a one-loop mass scale near 10^{28} eV, about fifty-one orders of magnitude above the measured bound (NUMERICAL). This is a diagnostic of the

unrepaired sum in the one-loop model of Section 3 under named maps, not a prediction of the theory.

Open. Every quantum prediction of a repaired path sum: the one-loop correction to the Newtonian potential for the lattice path sum itself, the graviton's one-loop mass and width (zero if part 5 closes), and lattice-scale corrections, which need the spacing b derived. Under two identifications, that light crosses one cell per record rank and the phase premise of [2, Sec. 3.5], the spacing at $\kappa = 1$ is $b = (2/\sqrt{3})\,\lambda_{\mathrm{rec}} \approx 1.92\,\ell_P$.

What would settle it. Parts 2 and 5 first; then the static two-mass amplitude of the repaired path sum, Born-subtracted, with its logarithmic coefficient.

The tree-level statements are predictions in the plain sense, and they are null at any reachable scale. A quantum prediction has to wait for the action. Once a repaired path sum exists, the step from its amplitude to a potential is standard: a static kernel of the form $A/q^2 + B\log q^2 + C$ plus a remainder that is smooth enough at $q = 0$ gives $V(r) = A/(4\pi r) - B/(2\pi r^3) + o(r^{-3})$, with the regulator removed before $r \to \infty$, so the relative correction is fixed by the ratio B/A of the lattice amplitude. The earlier paper shows that the one-loop amplitude assembled from the $\kappa = 1$ long-wave Feynman rules has the coefficient $41/(10\pi)$ of the continuum effective theory [17, 18], in an infrared-subtracted reading [2, Sec. 7]. That is a statement about the long-wave rules, not about the lattice path sum, whose doublers enter the two-mass channel; whether the path sum reproduces the coefficient is open, and the question can only be asked of an action whose graviton is massless at one loop. A massless graviton (part 5), shown with a ghost and gauge measure derived from the ledger rather than the model's, would itself be the first quantum prediction: no one-loop mass, to be compared with the bound on the graviton mass [25]; a shift of the cone or a width needs the momentum dependence of the self-energy as well, to be compared with GW170817 [26].

Three statements about any repaired path sum can be made now. A one-loop correction to the Newtonian potential of relative size $\xi(\ell_P/r)^2$ stays below the sensitivity of short-range tests of gravity at every distance they reach, for every $|\xi| \le 10^{40}$, so a test will not come from the Newton correction. The one-loop dispersion of the graviton fixes its speed through the difference between the spatial and temporal self-energy coefficients. Read with the identifications that the dressed zero is the observed wave and its phase speed the arrival speed, at a spacing of at most two Planck lengths any difference above 10^{-12}, for temporal coefficients up to 10 in size, contradicts the near-simultaneous arrival of gravitational and electromagnetic waves from GW170817 [26]. A repaired path sum at Planckian spacing therefore needs a one-loop light cone that is Lorentz invariant to about twelve digits, which the lattice's preferred frame does not guarantee (INFERENCE). And a remainder whose ordered limit vanishes on every transverse-traceless probe gives no one-loop mass term on any of them.

The unrepaired path sum also gives a conditional diagnostic. Read through two identifications, the lattice spacing[9] $b = (2/\sqrt{3})\,\lambda_{\mathrm{rec}}$ and the map that turns the zero-momentum remainder into a mass term, which is linear in the remainder and proportional to $(\ell_P/b)^2$, the computed remainders of Section 6.3 correspond to a one-loop mass scale of about 3×10^{28} eV for the native path sum and about 5×10^{28} eV with the derived weight, most of it from the imaginary part (NUMERICAL). Whether that scale is a mass or a width depends on the evolution of

[9] What fixes the value of Newton's constant in these units is the subject of [19].

Section 8.1; either way it exceeds the bound of 2.42×10^{-23} eV$/c^2$ at 90% credibility [25] by about fifty-one orders of magnitude. The statement is conditional four times over: on the two maps, on the computed values, on the action being the unrepaired one, and on the one-loop model of Section 3, with no ghost line and a unit-weight gauge term. It is not a prediction of the theory; it says that the path sum without a repaired action conflicts with measurement under these readings, and so measures what the action of part 2 has to remove.

9 From the foundation to each result

Status: part 10, one derivation

Done. Every result of this paper is stated with its premises, and Table 2 sorts them: what holds for every $\kappa > 1/2$, what needs the identification $\kappa = 1$, what rests on a model ingredient, and what rests on an open premise. For the earlier paper's results, and for Theorem 7.4(i) and (ii), a single theorem already states every result from the foundation's premises, with the identification $\kappa = 1$ and the observational bounds as stated premises and each result's own conditions kept.

Open. The same single derivation for every claim of this paper, with each identification named where it enters.

The foundation reaches this paper in three places. The lattice is its picture of events as records on four counters, which gives the Kuhn triangulation and the family g_κ; the value $\kappa = 1$ is the identification of Section 2, not a consequence. The cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ enters twice: as the cost of a hinge cycle in the measure of Section 5, under (I1) and (I2), and as the least-work law of the matter of Section 8.2, under the identifications listed there. Everything else, the Regge action, the Bloch analysis, the Ward identity and the finiteness theorem, is mathematics about the lattice and holds whatever the foundation says. That division is deliberate: it shows exactly which statements a derivation of κ, of the measure identifications and of a repaired action would turn into consequences of the ledger.

Table 2. Premises of the results. "All κ" means every $\kappa > 1/2$. The regulator is the Feynman damping $+i\varepsilon$, a model wherever it enters; "numerical" marks computed values; "pending" marks a result whose proof is complete and whose independent verification is not yet finished.

result	κ	other premises
corner theorem, four readings, two worlds (Theorems 2.1 to 2.3)	all κ	none
divergence of the anisotropy (Theorem 3.1)	$\kappa = 1$	gauge term with unit weight, flat measure, no ghost line, regulator (models)
record time keeps the doublers (Theorem 3.2)	$\kappa = 1$	regulator
the native model graviton is not massless (Theorem 3.3)	$\kappa = 1$	regulator
linear field, $G = 1/(8\pi s)$ [2, Thm. 4.6]	all κ	conserved stress family on a non-null cone of waves
exact solutions, Newton's potential [2, Thms. 4.8–4.10]	$\kappa = 1$	model medium; odd sides not divisible by 105

Table 2, continued

result	κ	other premises
second order with worldline matter (Section 4)	all κ	least-work law at the flat reference, ratios of c^0, body and carrier split (identifications); point postings, non-gravitating reference (models); in the continuum form, the zero wave of the second order balanced by a free constant
every long-wave vertex is Einstein's (Theorem 4.2)	all κ	a star-good background; sampled smooth jets (a statement about the field equations, not about solutions)
the vertex symmetry fails at second order (Theorem 4.3)	$\kappa = 1$	side-8 torus and the infinite lattice, local at one record; also the point mass's Taylor fields; its far field (numerical)
second differences are consistent (Theorem 4.1)	any	a jet-norm bound independent of the mesh
measure at Gaussian order, precision criterion (Theorems 5.1 and 5.2)	$\kappa = 1$; none	the two classes of Theorem 5.1, separately
the derived weight (Theorem 5.4)	$\kappa = 1$	(I1), (I2); channel count n_h (definition); the scaling of the phase
properties of the weight (Theorem 5.5)	$\kappa = 1$	dual-cycle counts; (iii) as marked
the weight on the uncut domain (Proposition 5.7)	$\kappa = 1$	fixed channel counts; the whole admissible cone with its Haar measure
kernel of the derived form (Theorem 5.6)	$\kappa = 1$	dual-cycle counts
Ward criterion, defect identity, real parts (Theorems 6.2 to 6.5)	$\kappa \neq \frac{1}{4}$; all	differentiability; for the lattice path sum, the regulator
no weight, no cubic correction (Theorems 6.6 and 6.7)	$\kappa = 1$	sides divisible by 4 for the failure of the symmetry
higher deficits leave the response unchanged (Theorem 6.10)	$\kappa = 1$	same slope, weight and regulator; regular determinants for the defects
what refinement does (Theorem 6.9)	$\kappa = 1$	(i) a tree-level coarse-graining exists (proved on the side-4 torus, open on larger tori); (iii) refinement invariance (identification)
one refinement step, the exact tension (Section 6.5), pending	$\kappa = 3/2$; $\kappa = 1$	side 4 to side 2
exact blocking multiplies the remainder by 16; no blocked fixed point, nor for block averages (Sections 6.5 and 6.7)	all κ	records on edges (identification); damping fixed on the fine lattice; regular fine determinants; with the damping fixed on the coarse lattice (pending): $\kappa = 1$, zone-averaged
computed remainders (Section 6.3)	$\kappa = 1$	numerical; for the weight, (I1), (I2), dual cycle, phase scale $a = \ln 2/(2\pi)$ (model)
the corner condition (Section 6.4), pending	$\kappa = 1$	even torus side; even curvature of class C^3 vanishing at the flat areas

Table 2, continued

result	κ	other premises
the holonomy route (Section 6.7), pending	$\kappa = 1$	a loop's phase read from its postings, one cycle per face (identifications); regulator
an action whose model graviton is massless (Target 6.8)	$\kappa = 1$	open
finiteness theorems (Theorems 7.1 to 7.3)	any family	(H), (D), (O), or the blow-up
the weighted responses are finite (Theorem 7.4)	$\kappa = 1$	coupling $a > 0$; real parts; the weight is the ledger's under (I1), (I2)
the complex observable (Proposition 7.5), pending	$\kappa = 1$	coupling $a > 0$; the weight is the ledger's under (I1), (I2)
record-time norm obstruction (Theorem 8.1)	$\kappa = 1$	spatial period $N \geq 1$
continuum-limit stability of the record-time roots with the derived weight (Section 8.1)	$\kappa = 1$	(I1), (I2), dual-cycle counts; Gaussian order; fixed weight strength; lattice-scale modes off the unit circle decouple (inference)
the obstruction with postings null (Proposition 8.2)	$\kappa = 1$	spatial period $N \geq 1$; linearized bare equation, open record time
tree-level propagation [2, Thms. 5.1 and 5.4]	$\kappa = 1$	none beyond the lattice

10 What remains

A quantum theory of gravity on the ledger needs four properties. One, Einstein gravity at long wavelengths, holds at linear order and at second order with matter, and every long-wave vertex of the action is Einstein's as field equations on sampled jets, while nonlinear lattice solutions are open (item 7); one, finiteness, holds for the real parts of the derived weight's one-loop responses in the diagram model of Section 3; two, a derived lattice, action and measure, and a massless graviton, are not yet earned. Here are the open statements in the order in which they decide the outcome. Each is stated so that it cannot be met trivially: the zero action, the quadratic truncation of the Regge action and the Regge action itself are each checked against it.

1. *An action with a vanishing Ward defect (part 2).* A phase built from the ledger's hinge records, with the classical limit of the Regge action (the same long-wave quadratic form, hence the linearized Einstein field and Newton's law), whose one-loop determinants are regular and whose summed Ward defect $\frac{i}{2}(\mathrm{WD}_2 - \mathrm{WD}_1)$ has ordered limit zero on every transverse-traceless probe, with every coefficient derived from the ledger (Target 6.8). The Regge action meets every classical requirement and fails this one, numerically; the zero phase has no classical limit. Two routes have been proposed: a perfect action fixed by invariance under refinement, the setting in which exact vertex symmetry is known to exist [9], and a local hinge charge with a term of second order in the deficit whose second and third variations at flat space are invariant under vertex moves on tori of side

divisible by 4. Five results narrow the search: no phase $cS + \sum_h \delta_h^3 \Gamma_h$ helps, since it has no exact vertex symmetry and, at the same slope and weight, exactly the remainder of cS (Theorems 6.7 and 6.10); refinement does not remove the doublers and cannot produce the repair from the identified lattice (Theorem 6.9); within the area-additive local charges the remainder depends on one number and, as computed, no value of it vanishes (Section 6.6); the remainder of any local charge depends only on its slope and even curvature, and the readings the ledger supplies fix the even curvature to 0 or βA, neither massless (Section 6.6; proof complete, verification pending); and a weight is a damping, so it cannot be part of a stable record-time evolution at the lattice scale (Section 8.1). A repair must change the quadratic action at the lattice scale while keeping its long-wave limit, with every coefficient derived. The local routes are largely closed (Section 6.7). Of the two non-local routes examined, neither supplies the repair at $\kappa = 1$: the record-history holonomy phase is the Regge phase, and the blocked action of the ledger's refinement map has no Einstein–Hilbert fixed point in either order of limits, the decisive one with the damping held fixed on the coarse lattice included (proofs complete, verification pending). What remains is a phase with an even part outside the ledger's present readings, or a phase whose slope map or even kernel is not that of any local charge. If no phase the ledger supplies can do this, proving that is a result about the programme, and we would state it as one.

2. *The massless graviton (part 5).* Given item 1 and the measure of Section 5, the graviton of the model determinant is massless at one loop if the doublers stay gapped and the remainder vanishes for the combined path sum. The second condition is item 1 applied to the full determinant, weight included. A ghost or gauge measure derived from the ledger, in place of the model's unit-weight gauge term with no ghost line, would have to be checked against the same condition. Reading it as a massless pole of a propagating graviton also needs the evolution of item 6.

3. *The value of κ (part 1).* A premise of the ledger, stated without reference to κ, that forces κ. By Section 2 any such premise that forces $\kappa = 1$ forces the four corner doublers, so a derivation of $\kappa = 1$ must come with the measure that gaps them; one that forces another value moves every classical result to that value, where they would have to be proved again.

4. *The measure identifications (part 3).* That the cost of a hinge record reads its area as the action does, and that the weight of a geometry is the Gibbs weight of its cost; and the number of channels at each hinge. With them the measure of Section 5 has no free coefficient. Beyond Gaussian order the weight also needs a domain on which it can be normalized: on the whole uncut cone of geometries with its Haar measure its mass is infinite (Proposition 5.7), so a derived restriction, a regulated definition, or the read of the ledger's own history law into geometry is required. That is separate from the one-loop results, which use the weight only at Gaussian order.

5. *Finiteness of the repaired theory (part 6).* The ordered one-loop limit for whatever closes item 1, with the weight of item 4, for the imaginary parts as well as the real parts, and in the native form on the eleven non-posting channels. The finiteness theorem of Section 7 reduces the real parts to hyperbolicity of the repaired operator's degenerate zeros, a corner condition and an origin condition, each checkable. Finiteness of the full path sum and of higher loops needs a further argument.

6. *Unitarity (part 7).* A record-time evolution of the path sum with a positive norm on its physical solutions that the tick preserves, or at least growth per unit physical time that vanishes in the continuum limit for every species, the doublers included. With the normalization of a unitary kernel it would make the imaginary part of the remainder vanish (INFERENCE), so this item and item 1 constrain each other; and since a weight is a damping, the measure of item 4 can be compatible with it only at long wavelengths, which is what the derived weight achieves for the record-time roots in the continuum limit at fixed strength, given its identifications and an inference about off-circle modes (Section 8.1).

7. *The nonlinear Einstein equations (part 4).* Lattice solutions at fixed finite strength, with matter that has equations of its own, converging to solutions of the full Einstein equations. Every long-wave vertex of the action is already the Einstein–Hilbert one (Theorem 4.2), so what is missing is the solutions: a continuum solution to compare with, the consistency of its lattice sample at fixed strength with a remainder bound in the jet norm, and, for vacuum exteriors, an escape from the second-order obstruction of Theorem 4.3.

8. *Matter from the cost (part 8).* A matter coupling and an equation of state that follow from the cost J without an identification; for the reference occupation, a statement of the ledger about which recognition events carry a posting.

9. *A prediction (part 9).* A measurable number from the repaired path sum: the one-loop correction to the Newtonian potential for the lattice path sum itself, not only for its long-wave rules, or a lattice-scale correction once the spacing is derived.

10. *One derivation (part 10).* A single chain from the foundation's premises to every claim above, with each identification named where it enters.

A Conventions and the structure of the proofs

A.1 Conventions and analytic ingredients

Notation. Several letters do double duty in the literature this paper draws on; Table 3 fixes their meaning here.

Definitions recalled from [2]. The flat Hessian $H_0 = D^2S(\ell^0)$ is invariant under lattice translations and acts on Bloch waves class by class: at a wave $k \in [-\pi, \pi]^4$ it is the 15×15 Hermitian Bloch matrix $B_\kappa(k)$, whose entries are trigonometric polynomials [2, Sec. 4.2]. A vertex displacement $\eta \in \mathbb{C}^4$ with Bloch phase $e^{ik\cdot x}$ changes the squared lengths by $(\Gamma(k)\eta)_d = 2\big(e^{ik\cdot d} - 1\big)d^\top\eta$, so $\Gamma(k)$ is the 15×4 matrix of vertex-move columns, and $B_\kappa\Gamma = 0$, $\Gamma^\dagger B_\kappa = 0$ (the Bianchi identity). The gauge-fixed operator in the Feynman-type gauge is $A = B + \Gamma\Gamma^\dagger$, the regulated propagator is $(A + i\varepsilon)^{-1}$ on all fifteen channels, and the native form fixes the four posting classes and works on the eleven others. A constant strain H acts on the edges

Table 3. Symbols that are easily confused.

symbol	meaning
$B(k)$, $B_\kappa(k)$	Bloch matrix of the flat Hessian at the wave k (Sections 2 and 3)
$\Gamma(k)$	the 15×4 matrix of vertex-move columns
$A(k)$, $A_\kappa(k)$	gauge-fixed operator $B + \Gamma\Gamma^\dagger$; $A_b(t)$ its strained form on the probe b (Section 7)
A_h, A_o	flat area of the hinge h; the four flat hinge areas (Sections 5 and 6.4)
n_h; n_H	channel count of the hinge h; normalization $\mathrm{tr}(g^{-1}Hg^{-1}H)$ of the probe H
$\mathscr{D}$, $\mathscr{D}(k)$	the derived weight form and its Bloch symbol (Section 5)
D, det	a function of constant strain, typically a one-loop determinant (Sections 3 and 6); the determinant
ε; a	Feynman damping, the regulator (MODEL); the phase coefficient that couples the weight to the action, never a spacing
b; s	lattice spacing (Section 8.3); action scale, with $G = 1/(8\pi s)$
$\mathscr{R}_H$, WD_1, WD_2	zero-momentum remainder on the probe H; first- and second-order Ward defects (Section 6)
H_E, H_{T_2}; f_E, f_{T_2}	spatial strain probes (Section 3); posting-null transverse-traceless probes (Section 6)

through $h_d = d^\top H d$. The anisotropy of Section 3 uses the spatial probes

$$H_E = \begin{pmatrix} 0 & 2 & -1 & -1 \\ 2 & 0 & -1 & -1 \\ -1 & -1 & 0 & 2 \\ -1 & -1 & 2 & 0 \end{pmatrix}, \qquad H_{T_2} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

with $H\mathbf{1} = 0$, $\mathrm{tr}(g_1^{-1}H) = 0$ and $n_H = \mathrm{tr}(g_1^{-1}Hg_1^{-1}H)$ equal to 24 and 8, one in each irreducible representation E and T_2 of the axis permutations on traceless strains, and the diagram model of [2, eq. (7)]: the graviton sunset and seagull and the scalar sunset and seagull at zero external momentum, divided by n_H, with the gauge term of unit weight, no ghost line, the Feynman sign of $i\varepsilon$ and one massless scalar for matter (MODEL). The remainder of Section 6 uses the posting-null transverse-traceless probes f_E and f_{T_2}, the strains f_E and f_3 of [2, Sec. 6.8], with $\tau = \mathrm{tr}(g^{-1}Hg^{-1}H)$ equal to $\frac{9}{2}$ and $\frac{3}{2}$, and $H_{\mathrm{tt}} = f_E + f_{T_2}$ with $\tau = 6$; on f_E the sums of s_e^2 over the classes with two, three and four bits are 9, 0, 0, and on f_{T_2} they are 9, 18, 0. The doubler vector at the axis corner πe_c is $v_c(d) = 2^{|d|-3}$ for $d_c = 0$ and 0 otherwise [2, Prop. 6.1]; $4v_c$ has integer entries, which is the normalization of the census in Theorem 5.3.

Exact computations. The corner theorem (Theorem 2.1) is proved by exact arithmetic: the corner operators are integer pencils in κ, their determinants factor, and at each corner a matrix of polynomials in κ turns the corner matrix into a diagonal one whose entries are nonzero constants times products of those factors, which certifies where it is regular; an exact $LDL^\top$ factorization gives the sign of the axis-corner determinant. The census of Theorem 5.3 is an identity of integer vectors on one cell, extended to the torus by periodicity, followed by exact real arithmetic. The no-go results, Theorems 6.6 and 6.7, reduce to the exact cubic value of [2, Thm. 6.23], computed over $\mathbb{Q}(\sqrt{3}, i)$ on one block of side 4.

Analytic proofs. The precision criterion, the Ward theorems and the defect identity are proved in the text. The finiteness theorem (Theorem 7.1) partitions the zone smoothly. Away from the zero set of $\det M$ the regulated integrand converges with a uniform bound; at a nondegenerate characteristic point and at a hyperbolic degenerate point the contour is deformed along the hyperbolic direction, which gives a local limit uniform in the volume; at the corners (D) makes the integrand regular; at the origin (O) is the local limit. Periodicity reduces the zone to finitely many pieces. At each fixed $\varepsilon > 0$ the integrand is smooth and periodic, so the volume average converges to the zone integral, and the local bounds carry the limit $\varepsilon \to 0^+$. The dissipative version (Theorem 7.2) uses that a damping $G \geq 0$ keeps every hyperbolic direction hyperbolic and removes every null vector it sees. The blow-up of Theorem 7.3 rescales the strain directions by $1/\lambda$ and reduces the origin to the leading symbol on an annulus, where hyperbolicity gives the local limit as before.

Computations. Every value marked numerical is a midpoint-rule evaluation of a zone integral: N^4 waves at the centres of the cells of side $2\pi/N$. The strain derivatives of the one-loop log-determinant come from jets of the Regge Hessian and of the derived weight along the strain, built from the exact Lorentzian angle and area formulas in multiple-precision arithmetic and checked against independent jets of the same Hessian; the trace formula $\Sigma_H = \frac{i}{2}\big[\mathrm{tr}(M^{-1}M'') - \mathrm{tr}\big((M^{-1}M')^2\big)\big]$ then gives the remainder wave by wave, and the identity $\mathscr{R} = \frac{i}{2}(\mathrm{WD}_2 - \mathrm{WD}_1)$, which holds on every grid to 10^{-15}, checks the jets. For the native path sum at the fixed regulators $\varepsilon = 0.1$, 0.05 and 0.025 the values are converged in the torus side; toward $\varepsilon \to 0^+$ the integral over the record-time frequency at fixed spatial wave is done exactly, by Jensen's formula on the 22 roots of the frequency pencil, on spatial grids up to 48^3, under the assumption that the strain derivatives commute with the limit [2, Sec. 6.8]. The finite part 0.27 ± 0.05 of the real part is a grid evaluation at $\varepsilon = 0^+$ on grids up to side 40, whose spacing cuts the corner logarithm off; at side 40 that logarithm contributes about -0.013, inside the error. With the derived weight the operator is invertible at every real wave other than the origin (Theorem 5.6), so the remainder is computed at $\varepsilon = 0$ on grids of side 8 to 24 that avoid the four planes $q_a \in 2\pi\mathbb{Z}$, where the eleven-channel operator is singular. Grids offset to within 0.005 of a plane change the value by less than 10^{-3}. The stated errors are the spread over the grids of side 16 and 24 and three randomly shifted grids of side 16; the grids of side 8 and 12 give $0.1143 + 24.687\,i$ and $0.1129 + 24.661\,i$. The scans over the parameter β, over constant even amplitudes and over curvature-squared strengths use the same evaluator, at the sides and regulators stated in Sections 6.3 and 6.6. None of these values is enclosed by interval arithmetic. The margins are large compared with the observed grid errors, $|\mathscr{R}| \geq 4.98$ on f_E over the scanned band of β and $\mathrm{Im}\,\mathscr{R} = 24.66$ with the derived weight, and that is why they are reported as evidence, not as theorems.

Archived computations. The exact data and the programs behind Theorem 2.1 (the corner matrices rebuilt from the cell numerators, their factored determinants, the polynomial certificates and the exact $LDL^\top$ factorizations), Theorem 5.3 (the integer vectors and triples of the census, on all four axes), Theorem 5.6 (the integer row table, the probe row and the exact ranks at the corners), Section 5 (the exact hinge weights and doubler damping), Section 7 (the corner data, the blow-up frame and the census of degenerate zeros of [2]), Sections 6.3 and 6.6 (the grid programs and their outputs) and Appendix A.3 (the null vector of the grow-

ing mode) are archived as the file `QGRecognitionLedger_computations_v1.1.zip` in the record doi: 10.5281/zenodo.23813297. Each item has a program that checks it independently and its expected output, and the SHA-256 checksum of every file is listed in `SHA256SUMS.txt` inside the archive. The archive has 5 922 824 bytes and the SHA-256 checksum

`d53c1ed89275540c2c762a06d26398d52fcd8a49d828357d5bfdc0bdd52638be`

A.2 The weighted diagrams

Fix $a > 0$, the unit-weight gauge term and the regulator. For the probe $b \in \{E, T_2\}$ put

$$L_b(t,q) = A_b(t,q) + \frac{i}{a}\,\mathscr{D}_b(t,q) + i\varepsilon I, \qquad R(q) = L_b(0,q)^{-1}.$$

The base operator is the same for both probes. Its first and second strain vertices are

$$C_b = \partial_t A_b(0) + \frac{i}{a}\,\partial_t \mathscr{D}_b(0), \qquad Q_b = \partial_t^2 A_b(0) + \frac{i}{a}\,\partial_t^2 \mathscr{D}_b(0).$$

Differentiating the log-determinant twice gives the normalized coefficient

$$F_b(q,\varepsilon) = \frac{i}{2d_b}\Big\{\mathrm{tr}(Q_b R) - \mathrm{tr}(C_b R C_b R)\Big\},$$

and the anisotropy of Theorem 7.4(ii) is $F_E - F_{T_2}$. In the kernel of Section 7, the E term has $X = -iC_E/(2d_E)$, $Y = C_E$ and $Z = iQ_E/(2d_E)$; the T_2 term has the signs of X and Z reversed. These are the full strain vertices, so both strain derivatives of the weight stay inside the diagrams. The local bounds at the origin apply to each vertex separately: the rescaled sunset vertices have three continuous derivatives and the rescaled seagull has two. The smooth periodic data therefore remain admissible when one probe, or one diagram, is selected, and the finiteness theorem applies to each selection: at every fixed $\varepsilon > 0$ its torus average converges to its zone integral, and as $\varepsilon \to 0^+$ the real part of the integral has a finite limit. Reversing the sign of the selected T_2 coefficient gives $\mathrm{Re}\, F_{T_2}$. This proves Theorem 7.4(iii). The argument gives no limit for the imaginary parts and says nothing about the value of the remainder.

A.3 The record-time growing mode

Index the fifteen nonempty edge classes by the binary digits of $d+1$, for $0 \le d < 15$, so that the classes 0, 3 and 4 are e_0, e_2 and $e_0 + e_2$. Put $r = \sqrt{2}$. A null polarization of the $\kappa = 1$ Bloch matrix at the phases $z = (r^{-1}, r^{-1}, r^{-1}, r^{-1})$ is

$$u_3 = 2 + 2\sqrt{2}, \qquad u_6 = -1, \qquad u_7 = -(2 + 2\sqrt{2}), \qquad u_{10} = 1, \qquad u_d = 0 \text{ otherwise},$$

checked by exact arithmetic in $\mathbb{Q}(\sqrt{2})$. With the phase convention of [2, Prop. 6.16], under which the field equation acts on $z^x D(z)u$ through the Bloch matrix at z^{-1}, the field $h_d(x) = r^{t(x)+|d|}u_d$ solves the linearized Regge equation.

It is not a vertex move of any shape. Suppose

$$h_d(x) = 2\sum_a d_a\big(\zeta_a(x+d) - \zeta_a(x)\big)$$

for some field ζ of covectors. The vanishing e_0 component makes ζ_0 invariant under e_0. The vanishing $e_0 + e_2$ component then gives, for $P = \zeta_2$,

$$P(x + 2e_0 + e_2) - P(x + e_0) = P(x + e_0 + e_2) - P(x).$$

The e_2 component gives $P(x + e_2) - P(x) = \frac{c}{2} r^{t(x)}$ with $c = r(2 + 2\sqrt{2}) \neq 0$. Subtract the displayed identity at $x = 0$ from the same identity at $x = e_2$ and insert these e_2 differences at 0, e_0, $e_0 + e_2$ and $2e_0 + e_2$. The result is

$$\frac{c}{2}(r - 1)(r^2 - 1) = 0,$$

which is false for $r = \sqrt{2}$. So the field survives the quotient by all vertex moves, at every spatial period $N \geq 1$, and Theorem 8.1 follows.

Versions. This is version 1.2 of the paper, doi: 10.5281/zenodo.23839819; every version is at doi: 10.5281/zenodo.23806703. Version 1.1 is doi: 10.5281/zenodo.23813297, with its preprint record at doi: 10.67347/pith.preprint.2026.000022; version 1.0 is doi: 10.5281/zenodo.23806704, with its preprint record at doi: 10.67347/pith.preprint.2026.000018.

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