

What a Point Source Does to the Lattice: Newton's Kernel, One Edge per Tick, and the Sign of Time

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Abstract

Recognition Science derives physical structure from what a consistent record of distinguishable events requires. In it, space is a cubic lattice of cells whose state advances in discrete steps called ticks, and the gravitational field is a deformation of the lattice's geometry, governed by a fixed lattice action: the Regge action of a canonical triangulation, with nothing adjusted but one overall scale. This paper asks what a single source does to that field. On the transverse-traceless deformations, the ones that carry gravitational waves, the field equation for a field that does not change from tick to tick is the lattice Poisson equation. Its exact response to a unit source at one cell, the lattice Green function, falls off as $1/(4\pi r)$ in every direction: Newton's kernel, with no constant adjusted. A posting, the pair of opposite unit sources left when one unit moves to a neighbouring cell, has no $1/r$ field, so only the net count is felt far away. The action treats its four lattice directions alike, three of space and one along which the ticks run. Read as a rule that steps forward tick by tick, the same operator has one response from rest, and that response spreads at most one edge per tick. But for every non-uniform spatial wave the rule also has a solution that grows without bound, while, wave by wave, the operator's own Green function decays toward both past and future at that same rate. A field that evolves in ticks therefore needs time to enter with the opposite sign from space, the sign of the wave equation. The kernel found here is that of the gravitational-wave amplitude. In general relativity the potential of a static mass lives in a different component of the metric, the time-time component; deriving its source and strength on the lattice is the next step.

1 The question

Every theory of gravity has to say what a mass does to the space around it. Newton's answer is a potential that falls off as $1/r$; Einstein's is a curvature whose weak-field limit reproduces that potential. A theory in which space is made of discrete cells has to give its own answer, and the answer has to come from the field equation the theory already has.

Recognition Science is such a theory. It derives physical structure from what it takes to keep a consistent record of distinguishable events, and this paper uses three of its results as given. First, space is a three-dimensional cubic lattice of cells, called *voxels*, and its state advances in discrete steps called *ticks*. Space has three dimensions because one complete pass of the theory's recording process traces a closed loop, and that loop must link the path of the process doing the recording, so that the pass leaves a record no continuous deformation can remove; closed loops can link only in three dimensions [1]. Second, the gravitational field is a deformation of the lattice's geometry, governed by a fixed discrete action. The four-dimensional cubic lattice, whose directions are the

three of space and the one along which the ticks run, is cut into simplices by the Kuhn triangulation: each hypercube is divided into the $4! = 24$ simplices that follow the orders in which its four axes can be stepped through (Figure 4) [10]. This triangulation is the one the cubic lattice determines by itself: its vertices are the lattice points, and each simplex is a path that steps once along each axis. The action is the Regge action of that triangulation, the discrete form of the Einstein–Hilbert action [11]. In Regge calculus, curvature sits at the hinges, the triangles along which simplices meet, as a deficit angle: the amount by which the angles of the simplices around a hinge fall short of a full turn. The action sums hinge area times deficit angle, and nothing in it is adjusted except one overall scale. It treats all four directions alike, so it is Euclidean, with no sign that singles out time; Section 7 is about what that means. Third, the theory keeps account of what happens at each voxel in whole-unit counts, in a record it calls its *ledger*. The ledger is double-entry: every transfer is entered as a debit at one voxel and an equal credit at a neighbouring voxel, so it lowers one count by one, raises the other by one and leaves the total unchanged [4]. What one unit is in physical terms, an amount of mass or energy, is fixed only when the ledger is connected to stress-energy, which is one of the open steps of Section 8. From these, the paper derives what a single source does to the field.

Companion papers worked out what this action does when nothing sources it [2, 3]. For small deformations of the flat lattice the field equation is linear, and its operator is the second variation of the action, the action’s quadratic part. On the transverse-traceless deformations (traceless and divergence-free; they carry the two polarizations of a gravitational wave), that operator acts as the four-dimensional nearest-neighbour Laplacian at every wavelength, with the coefficient the Einstein–Hilbert action of general relativity has. So gravitational waves are massless, carry two polarizations and travel at the speed of light at long wavelength. That statement about propagation needs one more step. A Euclidean operator describes waves only after it is continued to real frequencies: the fourth wave number k_t is replaced by $i\omega$, so that the Euclidean condition $k_t^2 + |\mathbf{k}|^2 = 0$, which no real nonzero wave satisfies, becomes $\omega^2 = |\mathbf{k}|^2$ at long wavelength, as in a Wick rotation. [2] adopts that continuation as a premise, and Section 7 shows what the Euclidean operator does when it is run directly as a rule in ticks. A third paper gives light a local update rule on the same grid, each tick computing every voxel from its neighbours [5]. A fourth reads the deficit angle at each hinge as the logarithm of a ratio of quantities the ledger records there, up to a correction of third order in the mesh size, and so connects the Regge action to the theory’s own record; it recovers the Einstein–Hilbert action on transverse-traceless modes and it records that this is not yet the sourced Einstein equation [4]. None of these papers says what happens when something sits on the lattice and sources the field. The simplest such source is a *unit count*, one unit at a single voxel, and its field can be found exactly.

The answer in brief. On the transverse-traceless deformations the field equation at rest, for a field that does not change from tick to tick, is the lattice Poisson equation. A unit source at one voxel has an exact solution, the lattice Green function W , which plays the part the potential of a point charge plays in the continuum. Its far field is $1/(4\pi|n|)$ in every direction. That is Newton’s kernel, and no constant is chosen to make it so: W is fixed by the lattice operator alone (Section 4). A *posting*, one unit at a voxel and minus one at the next, has zero net count, and its field falls off strictly faster than $1/|n|$; the $1/|n|$ field belongs to the net count alone. The transverse-traceless amplitude responds to a unit source with W/κ , where κ is the coefficient the action gives the lattice Laplacian, so its far-field amplitude is $1/(4\pi\kappa)$ (Section 5).

The unit count enters here as a test source, the way a point charge is used to learn what a field equation does before one knows what carries charge. The paper finds what the operator does with

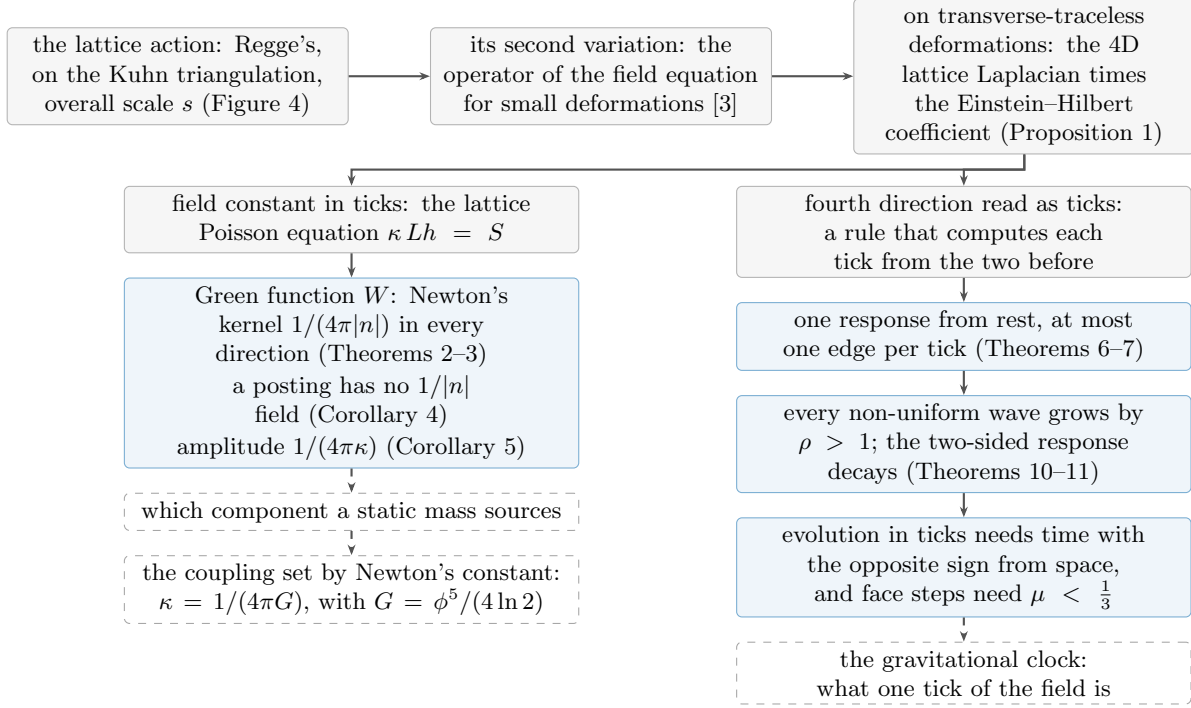


Figure 1: The argument at a glance. The top row is the operator, from the action to its transverse-traceless branch. The left column reads it at rest, the right column reads the fourth direction as ticks. Shaded boxes are the results proved here; dashed boxes are the steps that remain (Section 8). Symbols are defined in Table 1 and Section 3: κ is the coupling the action gives the Laplacian, h the field's amplitude, S its source and μ the weight of space relative to time; G is Newton's constant in the theory's units and ϕ the golden ratio (Section 8), and a face step is a tick that reaches only the six face neighbours of a voxel.

a source; how a ledger count couples to the field, and with what strength, is left open. The field it sources is also not yet the one a mass sources. In general relativity a static mass sources the time-time component of the metric, whose static kernel is the same $1/(4\pi r)$; carrying the result to that component is the first open step of Section 8.

Reading the fourth lattice direction as the sequence of ticks turns the four-dimensional operator into a recursion, a rule that computes each tick from the two before it (Section 6). Started from rest, with zero field at the first two ticks, it has exactly one solution, and its support grows by at most one lattice edge per tick. But the operator is Euclidean: time enters it with the same sign as space. For every non-uniform spatial wave the recursion has a source-free solution that grows by a factor $\rho > 1$ each tick, and it amplifies any initial data on that growing branch without bound. What the operator does define, its Green function in space and time, decays as $\rho^{-|t|}$ toward both past and future; Section 7 proves both statements wave by wave. So the Euclidean operator cannot itself be the rule that steps the field from tick to tick. A field that evolves tick by tick needs time to enter with the opposite sign from space: the sign of the wave equation, whose waves oscillate instead of growing. Section 8 names what is left: which component of the metric a mass sources, the identification of the far-field amplitude with Newton's constant, and the gravitational clock, the rule by which the field advances in time. Figure 1 lays out the argument.

term	meaning
voxel, tick	a cell of the cubic grid, labelled by $n = (n_0, n_1, n_2) \in \mathbb{Z}^3$, and one time step; side and step are 1
count, ledger	the whole-unit amount at each voxel, kept in the theory's double-entry ledger, where every transfer is a debit at one voxel and an equal credit at a neighbour; the net count is the sum of the counts
Δ , L	the face Laplacian $\Delta f(n) = \sum_j [f(n + e_j) - 2f(n) + f(n - e_j)]$, and the positive operator $L = -\Delta$
zone, symbol	the cube $B = [-\pi, \pi]^3$ of wave vectors (the Brillouin zone); $\sigma(k) = \sum_j 2(1 - \cos k_j) = \sum_j 4 \sin^2(k_j/2)$, the factor by which L multiplies the wave $\cos(k \cdot n)$
unit count, posting	the sources δ_0 (one at the origin) and $p = \delta_0 - \delta_{e_0}$ (one at the origin, minus one at the neighbour $e_0 = (1, 0, 0)$)
Green function	$W(n) = (2\pi)^{-3} \int_B \cos(k \cdot n) / \sigma(k) dk$, the field of a unit count
ℓ^1 distance	$\ n\ _1 = n_0 + n_1 + n_2 $, the number of edges on a shortest lattice path from the origin
TT branch	the transverse-traceless deformations: traceless and divergence-free, two polarizations at each wave vector; they carry gravitational waves
Euclidean, hyperbolic	time entering an operator with the same sign as space, as in a four-dimensional Laplacian; and with the opposite sign, as in the wave equation
spatial weight μ	the weight of space relative to time in the four-dimensional operator; the action treats the four directions alike, which is $\mu = 1$, the lattice's own weight
face step	a tick that reaches only the six face neighbours of a voxel
growth factor ρ	the larger root of $\rho^2 - (2 + \mu\sigma)\rho + 1 = 0$; its inverse is the smaller root

Table 1: Terms used throughout.

2 A source on a line, and at a voxel

For a field that does not change from tick to tick, the field equation of the lattice action on the transverse-traceless deformations is the lattice Poisson equation (Section 3 derives it). Solving that equation for a point source first puts the answer in view before the derivation. Start in one dimension. A unit source at the origin of a row of cells asks for a field h with

$$-[h(x+1) - 2h(x) + h(x-1)] = \delta_{x0}.$$

Away from the origin the second difference vanishes, so h is a straight line on each side, and at the origin its slope drops by one. The solutions are $h(x) = -|x|/2 + a + bx$. None of them vanishes far away: on a line the field of a source grows in size with distance instead of dying out. Gravity's $1/r$ is a three-dimensional fact.

On the cubic lattice the equation is $Lh = \delta_0$, that is,

$$6h(n) - \sum_j [h(n + e_j) + h(n - e_j)] = \delta_{n0}. \quad (1)$$

Figure 2 draws the stencil of (1) and the two sources of Section 3. The equation and the source are both unchanged by the 48 rotations and reflections of the cube about the origin, and the field that decays far away is unique, so the field of a unit count has the same symmetry. Its six values next to the origin are therefore equal, and (1) at $n = 0$ reads $6h(0) - 6h(e_0) = 1$:

$$h(e_0) = h(0) - \frac{1}{6}.$$

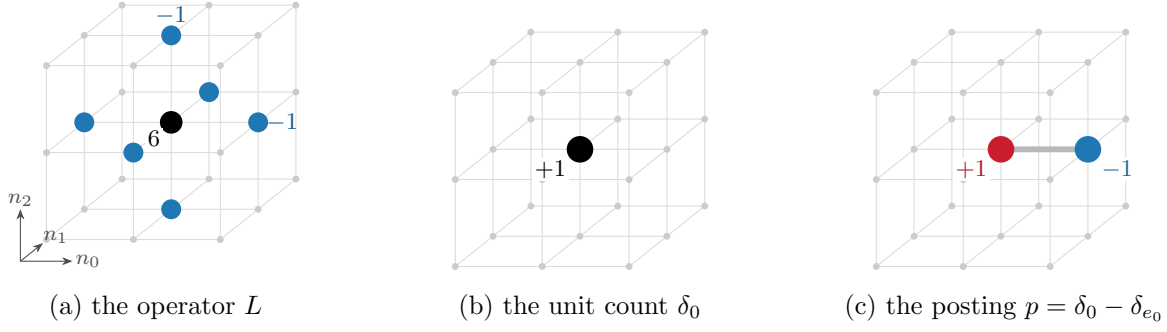


Figure 2: The operator and the two sources, on a $3 \times 3 \times 3$ block of voxels. (a) The face Laplacian L of (1) weighs a voxel by 6 and each of its six face neighbours (blue) by -1 ; the other twenty voxels of the block do not enter. (b) The unit count, one unit at the origin. (c) The posting, $+1$ at the origin and -1 at the face neighbour $e_0 = (1, 0, 0)$, with total zero.

That is one exact relation. The rest of the field needs the Green function. Its value at the source is Watson's integral, $W(0) = (2\pi)^{-3} \int_B dk / \sigma(k) = 0.2527310 \dots$ [14], finite where the continuum potential is infinite, and so $W(e_0) = 0.0860643 \dots$. One lattice spacing from the source, $4\pi W(e_0) = 1.0815$, only 8.2% above the value 1 that Newton's kernel $1/(4\pi|n|)$ gives at $|n| = 1$. Figure 3 shows the field on the plane through the source.

3 The field equation and its sources

This section derives, from the lattice action, the operator used in the rest of the paper, and defines its two sources.

Where the operator comes from. The gravitational field of the lattice is a set of small changes to the lengths of the edges of the Kuhn triangulation, which cuts each four-dimensional hypercube into 24 simplices [10] (Figure 4). The lattice action is s times the Regge action of those edge lengths [11], with s its overall scale. Expanded about the flat lattice, its second-order part is a finite table of couplings between nearby edges, the same at every vertex, and a table that is the same at every vertex is diagonalized by plane waves. A plane wave on the four-dimensional lattice is multiplied by a fixed phase at each step along an axis: e^{ik_j} along the spatial axis j , and $w = e^{ik_t}$ along the axis that will be read as time. We collect the four phases as $z = (e^{ik_0}, e^{ik_1}, e^{ik_2}, w)$. Fifteen edges start at each vertex, one for each nonempty set of axes an edge can span: four axes, six face diagonals, four body diagonals and the long diagonal of the hypercube. The companion paper [3] reduces the second variation to ten variables in two exact steps. The long diagonal drops out: every triangle that contains it has a right angle opposite it, and the area of a right triangle does not change, to first order, when only its hypotenuse does. The four body diagonals couple only to themselves, through a block that is the same at every wave vector, so they can be solved for and eliminated, like a variable with no dynamics of its own. The ten axis and face-diagonal edges that remain are traded, by an invertible change of variables, for the ten components of a symmetric 4×4 matrix $h_{\mu\nu}$: the metric perturbation in trace-reversed form, $h_{\mu\nu} - \frac{1}{2}\delta_{\mu\nu} \text{tr} h$, which on the traceless perturbations used below is the metric perturbation itself. Ten metric components can stand for ten edges because a constant metric perturbation h changes the squared length of the edge with vector D by $D^\top h D$, and the four axes and six face diagonals already determine h . In

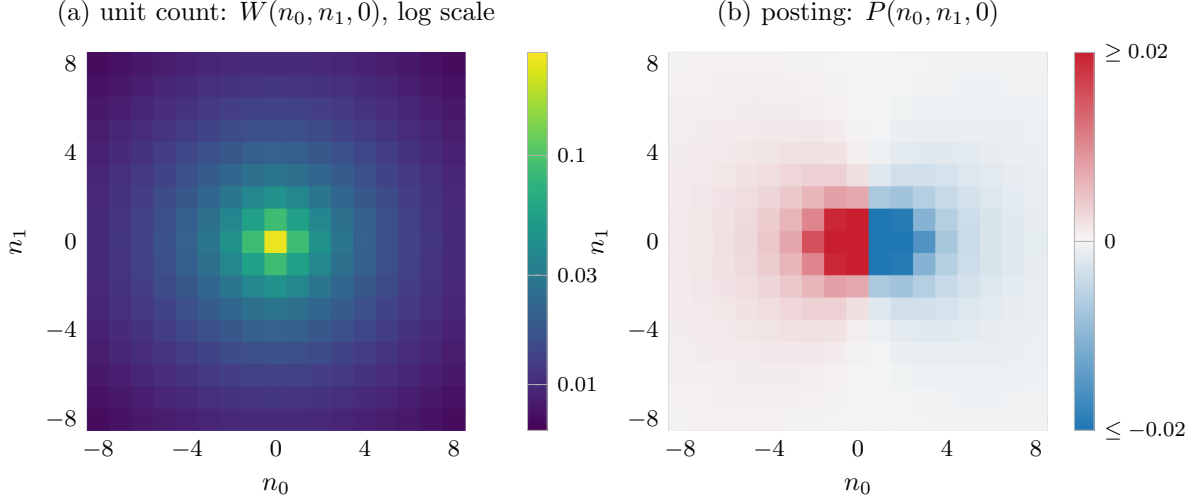


Figure 3: The field on the plane $n_2 = 0$ through the source, for $|n_0|, |n_1| \leq 8$. (a) The Green function W , the field of a unit count at the origin, on a logarithmic colour scale: largest at the source, $W(0) = 0.2527$, and decaying alike in every direction. (b) The field $P(n) = W(n) - W(n - e_0)$ of a posting, $+1$ at the origin and -1 at $e_0 = (1, 0, 0)$: positive on the side of the $+1$, negative on the side of the -1 . The colour scale saturates at ± 0.02 ; the two source cells carry $\pm \frac{1}{6}$. Values computed from the integral formula for W (Section 4).

these variables the second variation at the phases z is exactly $-\frac{s}{2} K(z)$, at every wave number, with

$$K(z) = \mathcal{L}(z) \mathbf{V} - \mathbf{C}^\sharp(z) \mathbf{C}(z), \quad \mathcal{L}(z) = (2 - w - w^{-1}) + \sigma(k) = 4 \sin^2 \frac{k_t}{2} + \sum_j 4 \sin^2 \frac{k_j}{2}.$$

This is the weak-field operator of general relativity in Fourier space, $(k_t^2 + |k|^2) \mathbf{V} - \mathbf{C}^\sharp \mathbf{C}$, with the squared four-dimensional wave number $k_t^2 + |k|^2$ replaced by $\mathcal{L}$ and each derivative by one lattice difference [3]. Its three pieces:

- $\mathcal{L}$ is the four-dimensional face symbol, the factor by which the four-dimensional nearest-neighbour Laplacian multiplies the plane wave.
- $\mathbf{V}$ is a constant matrix, the same at every wave vector: the form $\bar{h} \cdot \mathbf{V} h = \frac{1}{2} \sum_{\mu\nu} |h_{\mu\nu}|^2 - \frac{1}{4} |\text{tr } h|^2$ that the Laplacian multiplies in the linearized Einstein–Hilbert action written in harmonic gauge.
- $\mathbf{C}(z)$ is the lattice divergence of h , with rows $(\mathbf{C}h)_\mu = (z_\mu - 1)h_{\mu\mu} + \sum_{\nu \neq \mu} (1 - z_\nu^{-1})h_{\mu\nu}$: the continuum divergence $\partial^\nu h_{\mu\nu}$, with each derivative replaced by a forward or backward difference. Setting it to zero is the lattice form of the harmonic (de Donder) gauge condition.

The adjoint $\mathbf{C}^\sharp(z) = \mathbf{C}(\bar{z})^\top$ describes the deformations that change no geometry. Moving the vertices, a change of coordinates on the lattice, changes the edge lengths without changing any deficit angle to first order, and the perturbations it produces are the vectors $\mathbf{V}^{-1} \mathbf{C}^\sharp \xi$, one for each displacement ξ of the vertices [3]. Since $(\mathbf{V}^{-1} \mathbf{C}^\sharp \xi) \cdot \mathbf{V} h = \bar{\xi} \cdot \mathbf{C} h$, the condition $\mathbf{C} h = 0$ says that h is orthogonal, in the form $\mathbf{V}$, to every such change of coordinates. We write $\|h\|^2 = \sum_{\mu\nu} |h_{\mu\nu}|^2$, each off-diagonal entry counted twice. What follows needs one fact: on the deformations that carry gravitational waves the term $\mathbf{C}^\sharp \mathbf{C}$ drops out, and what remains is a constant times the face symbol $\mathcal{L}$. The next proposition states this exactly.

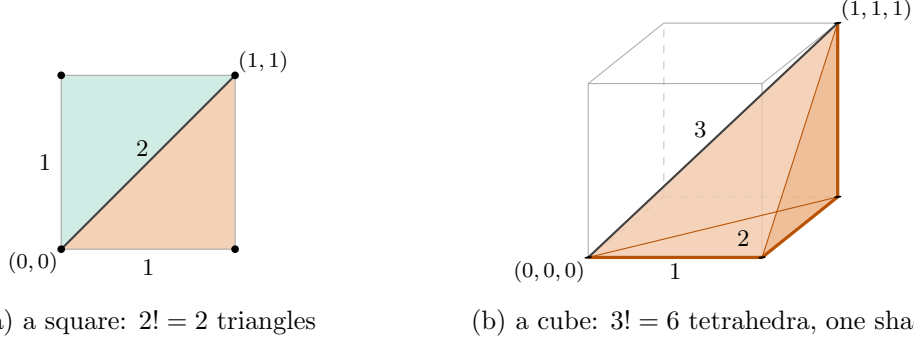


Figure 4: The Kuhn triangulation in two and three dimensions. Each simplex belongs to one order of stepping along the axes from the corner $(0, \dots, 0)$ to the opposite corner; the shaded tetrahedron in (b) steps along n_0 , then n_1 , then n_2 (thick path). Every edge joins two corners, and its squared length is the number of axes it spans: 1 for cube edges, 2 for face diagonals, 3 for the body diagonal. In four dimensions the same rule cuts each hypercube into $4! = 24$ simplices with squared edge lengths 1, 2, 3, 4, and the lattice action is a function of those lengths [3].

Proposition 1 (The transverse-traceless branch carries the face Laplacian). *Let h be a spatial metric perturbation, with no time components, that is traceless and satisfies $C(z)h = 0$ at one time phase. Then $C(z)h = 0$ at every time phase, and $K(z)h = \mathcal{L}(z)Vh$ for every w . At each spatial wave vector $k \neq 0$ such perturbations form a two-dimensional space, and on them the second variation of the action is*

$$-\frac{s}{4} \left[(2 - w - w^{-1}) + \sigma(k) \right] \|h\|^2,$$

the Einstein–Hilbert coefficient $-\frac{1}{4}$, in the normalization of [3], times the four-dimensional face symbol.

Proof. On a spatial perturbation every entry $h_{t\nu}$ vanishes, so the time row of C is zero and the three spatial rows involve only the spatial phases. The condition $Ch = 0$ therefore does not depend on w . For such h , $C^\sharp Ch = 0$ and $Kh = \mathcal{L}Vh$. On a traceless perturbation $\bar{h} \cdot Vh = \frac{1}{2}\|h\|^2$, so the second variation is $-\frac{s}{2} \cdot \mathcal{L} \cdot \frac{1}{2}\|h\|^2$. The three conditions $Ch = 0$ and the trace condition are independent for $k \neq 0$, which leaves two of the six spatial components free. $\square$

Two features of Proposition 1 matter here. The spatial part of the symbol is exactly $\sigma(k)$, the symbol of the face Laplacian, at every momentum and not only at long wavelength. And the time and space parts enter with the same sign, because the symbol is that of a Euclidean operator, positive in all four directions.

A transverse-traceless field is a superposition of modes, each with one of the two polarizations of Proposition 1 and an amplitude; the function h giving the amplitudes over space and time is its *amplitude profile*. From here on h denotes this profile, a scalar function of voxel and tick, and no longer the metric matrix. We write the operator on a field with amplitude profile h as

$$\alpha [D_t h + \mu L h], \quad D_t h(t) = 2h(t) - h(t+1) - h(t-1), \quad (2)$$

where D_t is the second difference in time with the sign that makes it positive, like L ; its symbol is the time part $2 - w - w^{-1}$ of $\mathcal{L}$. The normalization is $\alpha = -s/4$ and the spatial weight is μ , which is 1 in Proposition 1. Other values of μ weigh the space differences against the time difference differently; the action does not give them, but every statement below holds for each $\mu > 0$, and Section 7 asks what weight of space relative to time an evolution in ticks would need, which is

a question about them. So we keep μ general. On fields with an amplitude profile, (2) is exact, mode by mode, and it does not depend on which two polarizations are chosen at each wave vector: on the two-dimensional space of Proposition 1 the operator is a multiple of the identity, so in any basis each amplitude obeys the same scalar equation. The basis matters only for which tensor field a given profile describes. Because the polarizations depend on the direction of the wave vector, a source localized in the profile, such as δ_0 , is not a point source of the metric: as a tensor it is the transverse-traceless projection of a point source, and that projection is spread over space.

The sourced equation. A source S , a given function of voxel and tick, enters the field equation linearly: the operator (2) applied to h equals S . A field constant in time has $D_t h = 0$, so at rest the sourced equation is

$$\kappa Lh = S, \quad \kappa = \alpha\mu. \quad (3)$$

The coupling κ is fixed by the action, through the Einstein–Hilbert coefficient $-\frac{1}{4}$, the action scale s and the lattice’s own weight $\mu = 1$. The scale s is the one constant of the action still to be identified, against Newton’s constant, and that identification also says how a physical source is normalized against κ and fixes the sign of κ , which is the sign of $-s$ (Section 8). The shape of the response does not depend on either: it is W , divided by κ .

Two sources. In symbols, the unit count is δ_0 , one unit at the origin, and the posting is $p = \delta_0 - \delta_{e_0}$, one unit at a voxel and minus one at its neighbour. We use p as the test source for one ledger transfer between those two cells, the debit and the credit of Section 1 entered as opposite unit sources. Its total count is zero on every finite set of voxels that contains both sites.

4 The field at rest

Everything in this section is exact: the Green function of a unit count, its far field in every direction, and what a balanced pair leaves behind.

Theorem 2 (The field of a unit count). *The function $W(n) = (2\pi)^{-3} \int_B \cos(k \cdot n) / \sigma(k) dk$ is finite at every voxel and solves $LW = \delta_0$.*

Proof. On the zone, $1 - \cos t \geq 2t^2/\pi^2$ for $|t| \leq \pi$ (Jordan’s inequality), so $\sigma(k) \geq (4/\pi^2)|k|^2$ and $1/\sigma(k) \leq (\pi^2/4)/|k|^2$. In three dimensions $1/|k|^2$ is integrable near the origin, since the volume element contributes $|k|^2 d|k|$; so $\cos(k \cdot n)/\sigma(k)$ is integrable for every n and W is finite. The Laplacian is a finite combination of shifts, so it passes under the integral. Each shift multiplies $\cos(k \cdot n)$ by a phase, and the six shifts together give $L \cos(k \cdot n) = \sigma(k) \cos(k \cdot n)$. Away from $k = 0$, a set of measure zero, σ cancels, and

$$LW(n) = (2\pi)^{-3} \int_B \cos(k \cdot n) dk = \delta_{n0},$$

because the integral of $\cos(k \cdot n)$ over the zone is $(2\pi)^3$ at $n = 0$ and vanishes for every other integer n (by Fubini, one axis at a time). $\square$

Theorem 3 (Newton’s kernel, in every direction). *$|n| W(n) \rightarrow 1/(4\pi)$ as $|n| \rightarrow \infty$, in whatever direction n goes. Equivalently, $4\pi|n| W(n) \rightarrow 1$ along every sequence of distinct voxels.*

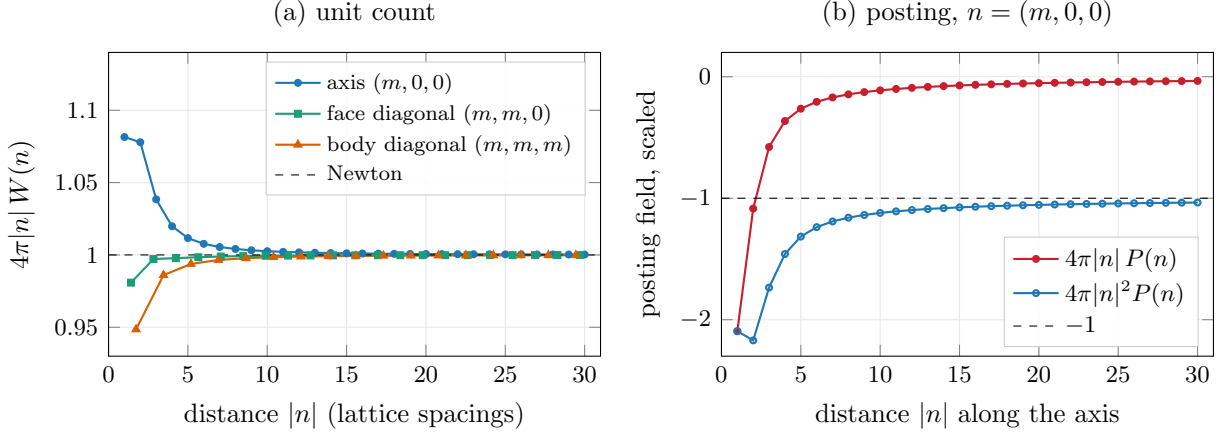


Figure 5: The far field. (a) $4\pi|n|W(n)$ along an axis, a face diagonal and a body diagonal, up to $|n| = 30$. All three tend to 1, the Newton value: at $|n| = 30$ they are 1.00028, 0.99993 and 0.99981. The shortfall falls off as $|n|^{-2}$, with opposite signs on the axes and the diagonals. (b) For the posting along the axis, $4\pi|n|P(n)$ tends to 0 (Corollary 4), and $4\pi|n|^2P(n)$ approaches -1 , the dipole field of the continuum. Values computed from the heat-kernel form $W(n) = \int_0^\infty \prod_j e^{-2t} I_{n_j}(2t) dt$, with I_m the modified Bessel function of the first kind; this form reproduces Watson’s $W(0)$ to twelve digits.

The proof is in Appendix A. Its idea is to compare $1/\sigma$ with the continuum kernel $e^{-|k|^2}/|k|^2$, whose transform is exactly $(2\pi^2/|n|)\text{erf}(|n|/2)$. The difference is a function with an integrable derivative, and integrating by parts along the axis on which $|n_j|$ is largest shows that it contributes nothing at order $1/|n|$.

Corollary 4 (A posting has no $1/|n|$ field). *The field $P(n) = W(n) - W(n - e_0)$ solves $LP = p$, and $|n|P(n) \rightarrow 0$ as $|n| \rightarrow \infty$.*

Proof. L commutes with shifts, so $LW(\cdot - e_0) = \delta_{e_0}$ and $LP = p$. For the decay, $|n|W(n - e_0) = |n - e_0|W(n - e_0) \cdot |n|/|n - e_0|$. The first factor tends to $1/(4\pi)$ by Theorem 3, since $|n - e_0| \rightarrow \infty$ too, and the second tends to 1, since $||n| - |n - e_0|| \leq 1$. So $|n|P(n) \rightarrow 1/(4\pi) - 1/(4\pi) = 0$. $\square$

Physical reading. At large distance the lattice Green function becomes the continuum Green function of the Laplacian, the kernel of Newton’s potential: $1/(4\pi r)$ per unit source, the same in every direction although the lattice has preferred axes. The anisotropy of the lattice survives only at short range; Figure 5(a) shows the approach to Newton’s kernel along the three kinds of lattice direction. A posting produces no $1/r$ field at all. Only the net count is seen from far away, and balanced pairs leave nothing at order $1/r$. The computed field of a posting approaches the continuum dipole field, which falls off as $1/|n|^2$ (Figure 5(b)).

How close, how soon. Does the lattice leave a trace in tests of Newton’s law? The computed values in Figure 5 put the field of a unit count within 0.3% of Newton’s kernel at ten lattice spacings and within 0.03% at thirty, and the shortfall falls off as $|n|^{-2}$, the rate of the classical expansion of the lattice Green function [18]. The theory puts the lattice spacing at $\sqrt{4 \ln 2} \approx 1.67$ Planck lengths, about 2.7×10^{-35} m [2, 7]. The number comes from entropy: the theory’s record holds one bit, an entropy of $\ln 2$, on each lattice face (each unit square of the cubic lattice), and setting the Bekenstein–Hawking entropy of a face, its area over four Planck areas, equal to $\ln 2$ gives a face area of $4 \ln 2$ Planck areas (Section 8). The shortest separations at which the inverse-square law

has been tested, near $52 \mu\text{m}$ [22], are about 2×10^{30} lattice spacings, where the lattice correction to the kernel is of order 10^{-61} . The estimate is about the kernel. The potential those experiments test lives in the time-time component of the metric, and it inherits this correction if that component obeys the same lattice Laplacian, as every component obeys the same operator in harmonic gauge in the continuum (Section 8).

5 The transverse-traceless response

On the transverse-traceless branch, the static equation (3) has the same Green function, divided by the coupling.

Corollary 5 (The transverse-traceless response to a unit source). *Let $\kappa \neq 0$. The profile $h = W/\kappa$ solves $\kappa Lh = \delta_0$, and $|n|h(n) \rightarrow 1/(4\pi\kappa)$ as $|n| \rightarrow \infty$. With $\kappa = \alpha\mu = -s\mu/4$, the far-field amplitude is $1/(4\pi\kappa) = -1/(\pi s\mu)$.*

Proof. $\kappa L(W/\kappa) = LW = \delta_0$ by Theorem 2, and $|n|h(n) = |n|W(n)/\kappa \rightarrow 1/(4\pi\kappa)$ by Theorem 3. $\square$

The amplitude is the inverse of the coupling: the stiffer the action, the weaker the response to a given source. In Einstein's equation the metric response to matter is likewise proportional to the inverse of the Einstein–Hilbert coefficient, which is $1/(16\pi G)$. Here the action fixes the coefficient: it is the $-\frac{1}{4}$ of Proposition 1 times the action scale. In general relativity a static mass has no anisotropic stress and sources no transverse-traceless field at all, so Corollary 5 is a statement about the static operator of the field and its kernel, not yet about the field of a mass; Section 8 takes up the component a mass does source.

6 Reading one direction as time

Section 4 treated the field as static. In the theory the fourth lattice direction is the one along which the ticks run, so the same operator should also say how the field changes from one tick to the next. The operator (2) treats the four lattice directions alike up to the weight μ . Reading the fourth direction as the sequence of ticks turns the sourced equation into a rule that computes each tick from the two before it:

$$2h(t+1) - h(t+2) - h(t) + \mu Lh(t+1) = S(t+1), \quad t = 0, 1, 2, \dots, \quad (4)$$

for a source $S(1), S(2), \dots$ given tick by tick. (We have divided by α , which rescales the source.) Solved for the latest tick,

$$h(t+2) = 2h(t+1) - h(t) - \mu \Delta h(t+1) - S(t+1). \quad (5)$$

Without the source this is the face leapfrog $u(t+2) = 2u(t+1) - u(t) + r^2 \Delta u(t+1)$ of [5], the standard second-order scheme in which one tick reaches the six face neighbours of a voxel. There r is the Courant number, the wave speed times the tick over the lattice spacing, and here $r^2 = -\mu$, a negative square that no real wave speed gives. The sign of r^2 is the sign with which time enters relative to space, and here it is the Euclidean one; the opposite sign, $r^2 = +\mu$, is the hyperbolic one of the wave equation. Figure 6(a) draws one tick of the rule with each sign.

Theorem 6 (One response from rest). *For every weight μ and every source, (4) has exactly one solution with $h(0) = h(1) = 0$: the one computed tick by tick by (5).*

Proof. The forward rule solves (4) by construction. If h and h' both solve it from rest, their difference d solves the source-free equation with $d(0) = d(1) = 0$, and (5) gives $d(t+2)$ from $d(t+1)$ and $d(t)$; by induction $d \equiv 0$. $\square$

Theorem 7 (One edge per tick). *If the source vanishes outside the ℓ^1 ball of radius R at every tick, the response from rest at tick $t+1$ vanishes outside the ℓ^1 ball of radius $R+t$.*

Proof. If f vanishes outside the ball of radius R' , so does $f(n \pm e_j)$ for every n with $\|n\|_1 > R' + 1$, so Lf vanishes outside the ball of radius $R' + 1$. At ticks 0 and 1 the response is zero. If $h(t+1)$ and $h(t+2)$ vanish outside the balls of radii $R+t$ and $R+t+1$, then by (5) every term of $h(t+3)$, including the source $S(t+2)$, vanishes outside the ball of radius $R+t+2$. $\square$

Proposition 8 (First response and zero mode). *The response from rest first appears at tick 2 and is $h(2) = -S(1)$. A source that is the same number u at every voxel and every tick produces $h(t) = -\frac{t(t-1)}{2}u$ at every voxel.*

Proof. At $t = 0$, (5) gives $h(2) = -S(1)$, since $h(0) = h(1) = 0$. For the uniform source the field is uniform, L annihilates it, and (5) becomes $h(t+2) = 2h(t+1) - h(t) - u$, which $-t(t-1)u/2$ satisfies from $h(0) = h(1) = 0$. $\square$

Proposition 9 (Static fields are stationary solutions). *If $\mu Lh_0 = S_0$, the field $h(t) = h_0$ at every tick solves (4) with the source S_0 at every tick. In particular W/μ , the static response to a unit count in the units of (4), held constant, solves (4) for a unit count at every tick.*

Proof. For a field constant in time the three time terms of (4) cancel, $2h_0 - h_0 - h_0 = 0$, leaving $\mu Lh_0 = S_0$. $\square$

Physical reading. Read as ticks, the field equation computes the present from the past and nothing else, and influence travels at most one lattice edge per tick (Figure 6). The static field of Section 4 is the time-independent solution of the same equation, not a separate law. It is not where the response from rest goes, though: started from rest under a constant unit count, every non-uniform wave of the response picks up a part on the growing branch of Section 7, and the response does not settle onto W/μ . The uniform source shows the price of a source with nonzero total: on a finite periodic lattice the static equation $Lh = S$ has a solution only when S sums to zero, because summing Lh over the lattice gives zero (each difference between neighbours appears once with each sign), and the tick rule turns a nonzero total into a field that grows quadratically. The posting, with total zero, never feeds that mode.

7 Why time must enter with the opposite sign

In a continuum theory with a Euclidean action, waves are obtained by a Wick rotation: the time coordinate is continued to imaginary values, a step taken by hand. On a lattice whose fourth direction is a sequence of ticks, one can ask instead whether the operator can serve directly as the law that steps the field forward. The tick rule is well defined and local, but it cannot serve as that law, because of its sign. A rule that advances a field in time should not amplify every wave without limit, and this one does.

Theorem 10 (Every non-uniform spatial wave grows). *Let $\mu \geq 0$ and let k be any wave vector. Let ρ be the larger root of*

$$\rho^2 - (2 + \mu\sigma(k))\rho + 1 = 0, \quad \rho = 1 + \frac{\mu\sigma}{2} + \sqrt{\left(1 + \frac{\mu\sigma}{2}\right)^2 - 1}. \quad (6)$$

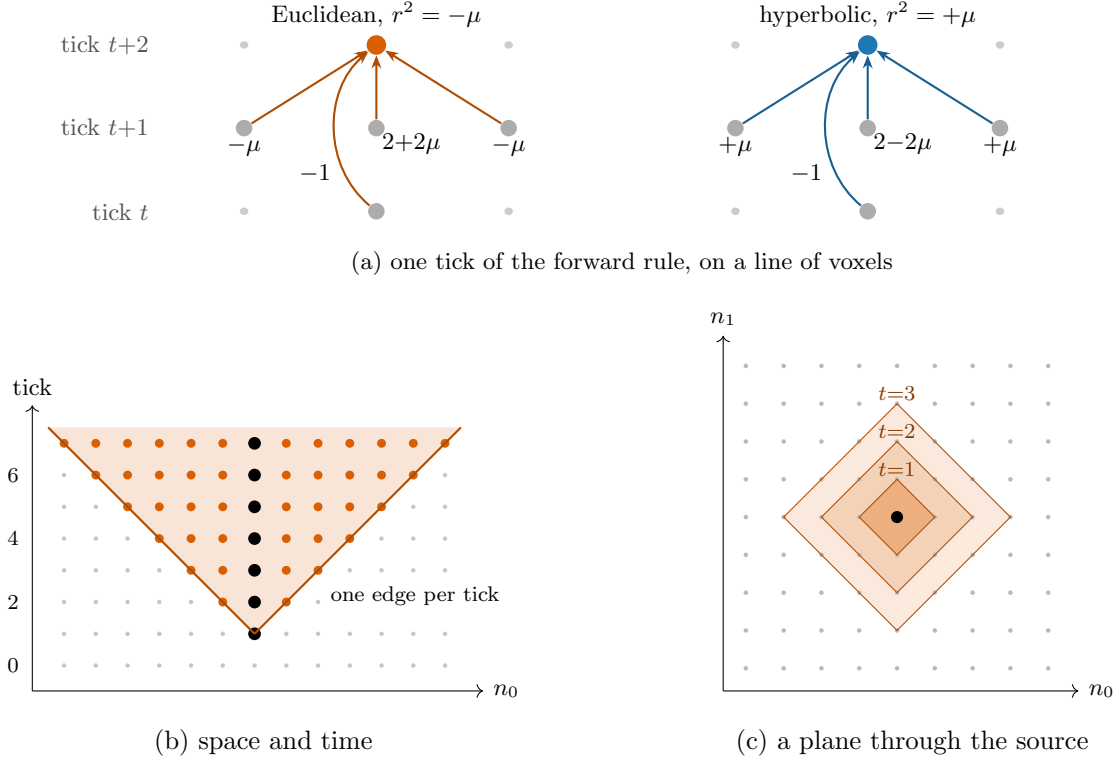


Figure 6: Reading one direction as time. (a) The forward rule (5) computes the field at tick $t+2$ (coloured) from the voxel and its neighbours at tick $t+1$ and the voxel at tick t , drawn here on one line of voxels; in three dimensions the centre carries $2 \pm 6\mu$ and each of the six face neighbours $\mp\mu$. The two readings differ only in the sign of the neighbour weight, and that sign decides whether waves oscillate or grow (Section 7). (b) A source that sits at the origin (black) from tick 1 on: the response at tick $t+1$ lies within t edges of the source (Theorem 7), inside the wedge, and its first value, at tick 2, sits on the source itself ($h(2) = -S(1)$, Proposition 8). (c) The same bound on the plane through the source at fixed tick: the ℓ^1 balls of radius $t = 1, 2, 3$, one lattice edge further each tick; in three dimensions they are octahedra.

Then $h(t, n) = \rho^t \cos(k \cdot n)$ solves (4) with no source, and $\rho > 1$ whenever $\mu\sigma(k) > 0$. Writing $\rho = e^\omega$, the rate ω satisfies $\cosh \omega = 1 + \mu\sigma(k)/2$, that is,

$$\sinh^2 \frac{\omega}{2} = \mu \sum_j \sin^2 \frac{k_j}{2}. \quad (7)$$

For the alternating wave $k = (\pi, 0, 0)$, whose sign flips from voxel to voxel along an axis, $\sigma = 4$ and $\rho = 1 + 2\mu + 2\sqrt{\mu(1+\mu)} = e^{2\operatorname{arsinh}\sqrt{\mu}} \geq 1 + 4\mu$. At the lattice's own weight $\mu = 1$ this is $3 + 2\sqrt{2} \approx 5.83$ per tick.

Proof. $L \cos(k \cdot n) = \sigma(k) \cos(k \cdot n)$, so substituting $\rho^t \cos(k \cdot n)$ into (4) with $S = 0$ gives $\rho^t \cos(k \cdot n) [2\rho - \rho^2 - 1 + \mu\sigma\rho]$, which vanishes by (6). If $\mu\sigma > 0$ the square root in (6) is positive and $\rho > 1 + \mu\sigma/2 > 1$. The two roots multiply to 1 and add to $2 + \mu\sigma$, so $\cosh \omega = (\rho + \rho^{-1})/2 = 1 + \mu\sigma/2$, and $\cosh \omega - 1 = 2\sinh^2(\omega/2)$ gives (7) because $\sigma = \sum_j 4\sin^2(k_j/2)$. At $\sigma = 4$ the root is $1 + 2\mu + 2\sqrt{\mu(1+\mu)}$; with $\sinh(\omega/2) = \sqrt{\mu}$ this is $e^{2\operatorname{arsinh}\sqrt{\mu}}$, and $\sqrt{\mu(1+\mu)} \geq \mu$ gives the bound $1 + 4\mu$. $\square$

Every wave vector of the zone except $k = 0$, the uniform wave, has $\sigma(k) > 0$. So at every positive weight, every non-uniform spatial wave has a source-free solution that grows exponentially

from tick to tick. Each wave's data split into a growing and a decaying branch (the two roots ρ and ρ^{-1} of (6)), and the rule (5) amplifies any initial data with a component on a growing branch without bound. The largest factor belongs to the shortest wave, $k = (\pi, \pi, \pi)$ with $\sigma = 12$: at $\mu = 1$ it is $7 + 4\sqrt{3} \approx 13.9$ per tick. On the lattice the rate is therefore bounded, and the rule is exponentially unstable rather than ill posed; it is the continuum limit, where wave numbers are unbounded, that turns this into Hadamard's ill-posed Cauchy problem for Laplace's equation [20]. There a wave of wave number k grows as $e^{|k|t}$, so data that are arbitrarily small but vary rapidly enough produce arbitrarily large solutions within any fixed time. Equation (4) is a four-dimensional Poisson equation, and a Poisson equation is not an evolution equation.

What a Poisson equation does define is a response that decays in all directions, time included. Wave by wave, it is the following.

Theorem 11 (The two-sided response, wave by wave). *Let $\mu > 0$, let $\sigma = \sigma(k) > 0$, and let ρ be as in (6). Then*

$$K(t) = \frac{\rho^{-|t|}}{\rho - \rho^{-1}}, \quad t \in \mathbb{Z},$$

solves $(2 + \mu\sigma)K(t) - K(t+1) - K(t-1) = \delta_{t0}$, and $K(t) \cos(k \cdot n)$ solves (4) on all ticks $t \in \mathbb{Z}$ with the source $\delta_{t0} \cos(k \cdot n)$. It decays as $\rho^{-|t|}$ in both directions of time, at the rate at which the forward rule grows.

Proof. Write $r = \rho^{-1}$, the smaller root, so $\rho + r = 2 + \mu\sigma$ and $\rho r = 1$. For $t \neq 0$ the three values $K(t), K(t \pm 1)$ are r^m, r^{m-1}, r^{m+1} over $\rho - r$ in some order, with $m = |t| \geq 1$, and $(\rho + r)r^m - r^{m+1} - r^{m-1} = r^{m-1}(\rho r - 1) = 0$. At $t = 0$ the left side is $[(\rho + r) - 2r]/(\rho - r) = 1$. The spatial factor contributes $\mu L \cos(k \cdot n) = \mu\sigma \cos(k \cdot n)$, which turns the scalar identity into (4) with the stated source. $\square$

Superposing these waves over the zone gives the four-dimensional Green function of (2), decaying in space and time together. The superposition decays more slowly than any single wave: long waves have ρ close to 1, so their parts decay slowly, and together they give a power law in place of the per-wave rate $\rho^{-|t|}$. For $\mu > 0$ the four-dimensional symbol vanishes only at the origin and its inverse is integrable, and the classical random-walk construction gives a kernel bounded by a constant times $1/(1 + t^2 + |n|^2)$ [19].

The opposite sign. Figure 7(a) compares the two readings wave by wave. With $r^2 = +\mu$ instead of $-\mu$, (5) is the face leapfrog of [5], and its waves oscillate instead of growing: a wave $e^{i(k \cdot n - \omega t)}$ solves it when $\sin^2(\omega/2) = \mu \sum_j \sin^2(k_j/2)$, the real-frequency counterpart of (7). Its frequencies are all real when the right side never exceeds one, that is, when $3\mu \leq 1$; at $3\mu = 1$ the shortest wave sits at $\omega = \pi$ and grows in proportion to the tick number, so bounded evolution needs $3\mu < 1$ [5]. Its long waves then travel at $\sqrt{\mu} < 1/\sqrt{3}$ voxel per tick. So an evolution of this field in ticks needs time to enter with the sign opposite to space, which the Euclidean action does not supply by itself. With face steps, ticks that reach only a voxel's six face neighbours as this rule does, it also needs a spatial weight below $1/3$.

At the lattice's own weight $\mu = 1$ neither sign gives a stable evolution in face steps. At the shortest wave, $k = (\pi, \pi, \pi)$, the spatial side of the relation $4 \sin^2(\omega/2) = \mu\sigma(k)$ is 12 and the temporal side is at most 4; [4] records the same failure. A clock that runs continuously, in a time not divided into ticks, has no such bound: $\partial_t^2 h = -\mu L h$ has the real frequencies $\omega^2 = \mu\sigma(k)$ at every weight, and a continuous clock is the reading adopted for the theory's real-time statements [2, 4]. In that reading the field's time is continuous and is not stepped tick by tick. Which reading

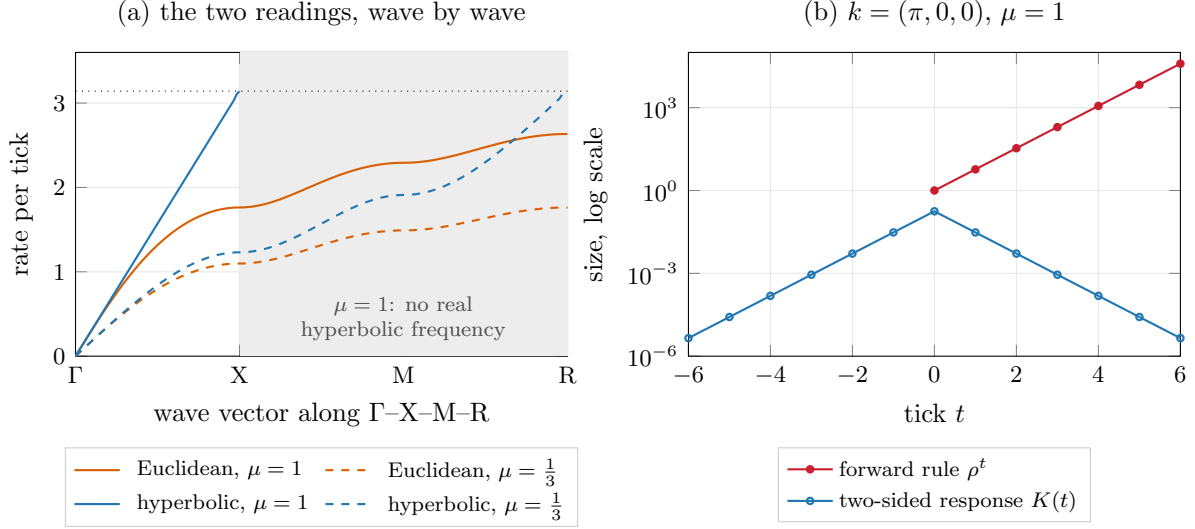


Figure 7: The sign of time. (a) Along the path $\Gamma = (0, 0, 0)$, $X = (\pi, 0, 0)$, $M = (\pi, \pi, 0)$, $R = (\pi, \pi, \pi)$ through the zone: the Euclidean rule's growth rate $\omega = \log \rho$ from (7) (orange), and the frequency of the hyperbolic rule, from $\sin^2(\omega/2) = \mu \sum_j \sin^2(k_j/2)$ (blue), at the lattice's weight $\mu = 1$ (solid) and at $\mu = \frac{1}{3}$ (dashed). The Euclidean rule grows on every wave but Γ , up to $\rho = 7 + 4\sqrt{3}$ at R when $\mu = 1$. At $\mu = 1$ the hyperbolic rule has no real frequency beyond X (shaded); at $\mu = \frac{1}{3}$ its frequencies are real all the way to R , where ω reaches the ceiling π (dotted). (b) For the alternating wave at $\mu = 1$, the forward rule grows by $3 + 2\sqrt{2}$ per tick, while the two-sided response of Theorem 11 decays by the same factor in both directions from its source at $t = 0$, where $K(0) = 1/(4\sqrt{2})$.

is the field's clock, a continuous time or a tick rule of some reach, is the third of the steps that remain.

8 What the source still has to be

Three steps stand between this kernel and the field of a mass: the component a mass sources, the strength of the coupling against Newton's constant, and how the field advances in time.

Mass and the time-time component. In general relativity a static mass sources the time-time component of the metric and the trace of its spatial part, not the transverse-traceless part, and the Newtonian potential lives in the time-time component. In harmonic gauge every component of the trace-reversed perturbation obeys the same scalar wave operator, so its static kernel is the $1/(4\pi r)$ of Theorem 3. On the lattice, this paper proves the static operator to be the face Laplacian on the transverse-traceless branch only. The next derivation is the same computation for the remaining components of the second variation about the flat lattice, with a mass identified as a source of the time-time component. The ledger enters there. [4] reads the curvature at each hinge from what the ledger records there, so a mass is to be a pattern of ledger entries; how its count converts to stress-energy in physical units is still to be derived.

The Newton normalization. Newton's law $\Phi = -GM/r$ is the statement that, for the field $-\Phi$ sourced by mass density, the operator coefficient is $1/(4\pi G)$ and the far-field amplitude of a unit source is G . The theory fixes G in its native units, where the lattice edge and the tick are 1, so that $c = 1$. Two of its results enter. The quantum of action is $\hbar = \phi^{-5}$: the theory's coherence quantum

$E_{\text{coh}} = \phi^{-5}$, the energy unit on which it builds particle masses as powers of ϕ , times one tick [8], where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. The golden ratio is the theory’s scaling constant: in a ladder of levels where one rule takes each level to the next and each new level joins two earlier ones, it is the only ratio between levels when comparing the two parts of a join has a positive cost [9]. And the records of a patch of lattice faces carry exactly one independent bit per face, so the entropy per unit area is $\eta = \ln 2$ [7]. Jacobson’s relation $G = c^3/(4\hbar\eta)$, which obtains Einstein’s equation from the entropy per unit area of local horizons [6], then gives $G = 1/(4\phi^{-5} \ln 2) = \phi^5/(4 \ln 2)$. Corollary 5 gives the amplitude $1/(4\pi\kappa)$ with κ fixed by the action. The identification asks that $\kappa = 1/(4\pi G)$, up to the factor and sign that relate the transverse-traceless amplitude to the potential; that is a statement about the action scale s , and it is open.

The gravitational clock. Section 7 shows that the Euclidean reading of the action cannot be run tick by tick, and that the hyperbolic reading with face steps needs $\mu < 1/3$. Light on the same grid travels one voxel per tick only with a tick that reaches the corner neighbours [5]: a cube step, which reaches all 26 voxels around a voxel, where a face step reaches 6. Whether one tick of the gravitational field is a face step, a cube step or a continuous flow is still to be derived, from how the ledger itself is updated from one tick to the next.

9 Relation to earlier work

The Green function of the cubic lattice is classical. Watson evaluated its value at the origin in closed form [14], McCrea and Whipple related it to random walks [15], Duffin developed discrete potential theory on the lattice and its asymptotics [16], Maradudin and coauthors tabulated it for the simple cubic lattice [17], and Martinsson and Rodin gave its asymptotic expansion [18]. Lawler and Limic give the sharp random-walk estimates in every dimension [19]. Theorem 3 is the leading term of that classical asymptotics; the proof in Appendix A is a self-contained Fourier argument. The ill-posedness of the Cauchy problem for elliptic equations is Hadamard’s [20], and the stability bound of the face leapfrog is the Courant–Friedrichs–Lewy condition [21].

Regge calculus replaces a curved manifold by a simplicial complex with fixed edge lengths [11]; Roček and Williams showed that its weak-field limit reproduces linearized general relativity [12], and Hamber and Williams computed the Newtonian potential between static sources in quantum Regge gravity by Monte Carlo methods [13]. The difference here is the origin of the lattice and of its operator. The lattice, its triangulation and the form of its action, with the Einstein–Hilbert coefficient of Proposition 1, are fixed by the theory (Section 1), not chosen as a regularization; only the overall scale s of the action waits on the identification with Newton’s constant (Section 8). So the static kernel, the posting’s neutrality, the reach of one tick and the sign condition are consequences of an operator that was not built to produce them.

10 Consequences

The gravitational-wave amplitude of this lattice has a sourced equation. At rest its solution is Newton’s kernel, with a far-field amplitude fixed by the action. Read in ticks, the same equation shows that evolution needs the opposite sign: a Euclidean lattice action gives a static field and a response that decays in time, but no waves that oscillate. Waves need time with the opposite sign and, with face steps, a spatial weight below the lattice’s own. What remains is to let a mass be the source: to carry the time-time component through the full second variation of the lattice action,

to fix the action scale against Newton's constant, and to derive the clock the field's propagation needs.

A Proof of Theorem 3

Write $v \in \mathbb{R}^3$ for the voxel n regarded as a vector, and let $g(k) = e^{-|k|^2}/|k|^2$ on $\mathbb{R}^3$. Split

$$(2\pi)^3 W(n) = \int_{\mathbb{R}^3} \cos(k \cdot v) g(k) dk + \int_{\mathbb{R}^3} \cos(k \cdot v) d(k) dk =: \Phi(v) + D(n), \quad d = \mathbf{1}_B/\sigma - g.$$

Both integrands are integrable: g is, because the volume element supplies $|k|^2$, and $\mathbf{1}_B/\sigma$ is, by the proof of Theorem 2.

The Gaussian part. In polar coordinates,

$$\Phi(v) = \int_0^\infty \int_{S^2} \cos(r \hat{k} \cdot v) e^{-r^2} d\hat{k} dr = \frac{4\pi}{|v|} \int_0^\infty \frac{\sin(r|v|)}{r} e^{-r^2} dr = \frac{2\pi^2}{|v|} \operatorname{erf}\left(\frac{|v|}{2}\right),$$

using $\int_{S^2} \cos(r \hat{k} \cdot v) d\hat{k} = 4\pi \sin(r|v|)/(r|v|)$ and $\int_0^\infty \sin(ar) e^{-r^2} dr/r = \frac{\pi}{2} \operatorname{erf}(a/2)$. So $|v|\Phi(v) \rightarrow 2\pi^2$.

The remainder. Near $k = 0$ both $1/\sigma$ and g equal $1/|k|^2$ up to bounded terms, since $\sigma(k) = |k|^2 - \frac{1}{12} \sum_j k_j^4 + O(|k|^6)$ and $1 - e^{-|k|^2} = O(|k|^2)$. So d is bounded near the origin and its first derivatives are $O(1/|k|)$ there, which is integrable in three dimensions. Away from the origin, d is smooth inside the zone, equals $-g$ outside it, and its derivatives are integrable.

Fix an axis j with $|n_j| = \max_i |n_i|$; then $n_j \neq 0$ for $n \neq 0$, and $|n| \leq \sqrt{3}|n_j|$. Integrate by parts in k_j : since $\partial_{k_j} \sin(k \cdot v) = n_j \cos(k \cdot v)$,

$$n_j D(n) = \left[\text{boundary terms} \right] - \int_{\mathbb{R}^3} \sin(k \cdot v) \partial_j d(k) dk.$$

There are two places where boundary terms could arise. At the faces $k_j = \pm\pi$ of the zone, the term is $\sin(k \cdot v)/\sigma(k)$ evaluated at $k_j = \pi$ minus its value at $k_j = -\pi$. Both factors take the same values there: σ is even and 2π -periodic in k_j , and $\sin(k \cdot v)$ changes by the phase $2\pi n_j$, an integer multiple of 2π . So the face terms cancel. At the origin, excise a ball of radius ε ; d is bounded, so the boundary term over its surface is $O(\varepsilon^2)$ and vanishes as $\varepsilon \rightarrow 0$. Hence

$$n_j D(n) = -S_j(n), \quad S_j(n) = \int_{\mathbb{R}^3} \sin(k \cdot v) \partial_j d(k) dk.$$

$\partial_j d$ is integrable, so $S_j(n) \rightarrow 0$ as $|n| \rightarrow \infty$ by the Riemann–Lebesgue lemma, and

$$|n| |D(n)| \leq \sqrt{3} |n_j| |D(n)| = \sqrt{3} |S_j(n)| \leq \sqrt{3} (|S_0(n)| + |S_1(n)| + |S_2(n)|) \rightarrow 0.$$

Conclusion. $(2\pi)^3 |n| W(n) = |v|\Phi(v) + |n|D(n) \rightarrow 2\pi^2$, so $|n|W(n) \rightarrow 2\pi^2/(2\pi)^3 = 1/(4\pi)$. Nothing in the argument depends on the direction in which $|n|$ grows.

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