One Cost, One Composition Law: States, Lattice Fields, Holonomies and Counted Ledgers

Jonathan Washburn

Recognition Physics Research Institute, Austin, Texas, USA

jon@recognitionphysics.org

Abstract

We take as premise that a comparison of two positive quantities with ratio x reads $J(x) = \frac{1}{2}(x + x^{-1}) - 1$, the unique solution of the composition law $J(xy) + J(x/y) = 2J(x)J(y) + 2J(x) + 2J(y)$ with $\lim_{t\to 0} 2J(e^t)/t^2 = 1$, and derive what this cost and this law determine under explicit premises. Pairwise readings fix a configuration up to a common unit and a global inversion, and determine it exactly on the domain containing a fixed unit item, with all entries at least one. On a periodic lattice the total cost S of a logarithmic field is the quadratic form of $A = -\sum_\mu J(T_\mu)$, with T_μ the unit shift, plus a remainder beginning at fourth order; only constant fields are stationary. The Hessian of S at every field dominates $2A$, so by the Brascamp–Lieb inequality the variance of every linear observable in the mean-zero finite-volume ensemble $e^{-\beta S}$ is bounded by its Gaussian value. No nonconstant real entire function of the reading, in particular no Boltzmann factor $e^{-\beta J}$, gives a positive-definite time kernel on the line, so the standard one-step transfer operator of $e^{-\beta S}$ on the unrestricted slice space is not e^{-H} for a self-adjoint H (its absolute value is), whereas Wilson's circle kernel is positive definite; the quadratic part has dispersion $\sinh^2(E/2) = \sum_j \sin^2(k_j/2)$, with $E \le |k|$ and, where the dispersion is differentiable, group velocity below one in lattice units. Every nonconstant step cost satisfying the formal law is $J(T^n)$ for a single shift T^n, and calibration selects the nearest neighbour. On $U(1)$ holonomies the nonconstant finite character sums G obeying the cosine equation, the shifted form of the law, are $\cos n\theta$, $n \ge 1$; unit calibration selects $n = 1$, and with the sign premise, opposite to that of the lattice action, the plaquette action is Wilson's $1 - \cos\theta$. With the $SU(N)$ action taken as the negative average of the eigenvalue costs it gives $1 - \frac{1}{N}\operatorname{Re}\operatorname{tr} U$, while on the whole group the half-trace of $SU(2)$ obeys the law and for $N \ge 3$ the only continuous solutions on $SU(N)$ are $G = 0$ and $G = 1$. On ledgers of integer counters whose counts are counters plus one, the readable ledgers are those posted from the empty ledger. The composition law enters the classification of J, of step costs and of holonomy costs; the other results use the level sets, convexity or analytic form of J, or follow from the stipulated counting and posting rules. No real-time evolution is derived.

1 Introduction

A physical description needs a rule for when two descriptions are of the same state, a field equation, and a starting point. We study state identification, a lattice action and a posting order under separate explicit premises, using one reciprocal cost for their comparison readings and its composition law where costs are composed. Recognition Science takes every physical event to be a comparison; no result below uses that wider statement. We assume, as premise (P0) below, that a comparison of two positive quantities with ratio x reads

$$J(x) = \frac{x + x^{-1}}{2} - 1 = \frac{(x-1)^2}{2x}, \qquad x > 0. \tag{1}$$

This is the unique function on the positive reals that obeys the composition law

$$J(xy) + J(x/y) = 2J(x)J(y) + 2J(x) + 2J(y) \tag{2}$$

with unit curvature in logarithmic coordinates, $\lim_{t\to 0} 2J(e^t)/t^2 = 1$; no continuity assumption is needed beyond this curvature condition (Proposition 1; the result is classical [2, 4], and [1] is a recent treatment). With $C = J + 1$ and $t = \log x$, (2) is the classical cosine equation $C(s+t) + C(s-t) = 2C(s)C(t)$ [2, 3]. The cost vanishes only at balance, $x = 1$; it is positive elsewhere; and it cannot tell a ratio from its reciprocal, $J(x) = J(1/x)$. The results below do not all use the same property of J: some use only that it increases with $|\log x|$, some only its curvature, and the composition law fixes the step operator, the holonomy cost and J itself. Section 8 records which result uses which.

The results rest on J together with the following premises, each used where stated. For a real combination G of shifts or characters we write the cost as $G - 1$ and the coefficients of G as g_k.

(P0) *Comparison.* A comparison of two positive quantities with ratio $x > 0$ reads $J(x)$.

(P1) *Identity.* Every available examination of a configuration of positive quantities on a fixed labelled set has an outcome that is a function of its profile, the readings of its pairwise comparisons. The pairwise comparisons are themselves available examinations, so configurations with different profiles are different states. Its identification of states has two parts: (P1a) configurations that differ by a common unit are the same state, and (P1b) configurations that differ by a global inversion are the same state; with Theorem 6 they give the converse, so two configurations are the same physical state exactly when their profiles agree (Principle 5). Labels are part of the state. Sections 3 and 6 use both parts; on lattice fields Section 4 uses only (P1a).

(P2) *Action.* On the periodic lattice $(\mathbb{Z}/L\mathbb{Z})^D$, the total cost S of a logarithmic field, the unweighted sum of the costs of its nearest-neighbour comparisons, is its Euclidean action: fields modulo the common shift carry the weight $e^{-\beta S}$, with respect to the induced Lebesgue measure on the mean-zero hyperplane $\sum_x t_x = 0$ (which represents, up to a constant, the quotient of the invariant measure $\prod_x dq_x/q_x$ by common scaling), and a coupling $\beta > 0$ that the cost does not fix. Where energies are discussed, $D = 4$ with one axis read as Euclidean time, and the energies of the quadratic part of S are those of its transfer matrix.

(P3) *Step operators.* A step cost $G(T) - 1$ obeys (2) as an identity in two commuting, independent shifts, with unit calibration $\sum_k k^2 g_k = 1$. It is not derived from (P0)–(P2); it characterizes the step operator independently of (P2). On each residue class $r+n\mathbb{Z}$, $J(T^n)$ is a nearest-neighbour operator, so on the full lattice it decomposes into n independent chains; unit calibration relative to the given one-site spacing selects $n = 1$.

(P4) *Counts.* The configuration of a counted system contains a unit item of value one; each of an account's two counts is its integer counter plus one; a post adds one item, and posts are the only updates.

(P5) *Holonomies.* A $U(1)$ holonomy $e^{i\theta}$ is a comparison of phases transported around a plaquette, and its cost is $G(\theta) - 1$, with G a finite real combination of characters; the cost $F = G - 1$ obeys (2) for independent holonomies in its additive-angle form, equivalently $G(\theta+\varphi) + G(\theta-\varphi) = 2G(\theta)G(\varphi)$, with unit calibration $G''(0) = -1$; the plaquette action is $-F$. This sign is opposite to that of (P2) and is part of the premise. For $U \in U(N)$ with eigenvalues $e^{i\alpha_j}$ the plaquette action is the negative average of the eigenvalue costs, $W(U) = -\frac{1}{N}\sum_j [G(\alpha_j) - 1]$; this additivity and averaging are part of the prescription.

We prove the following.

- **The state.** Pairwise readings classify configurations up to a common unit and a global inversion (Theorem 6); under (P1) these classes are physical states. On the counted domain, a unit entry equal to one and every entry at least one, they determine it exactly (Corollary 10).
- **The lattice field.** Under (P2), the total cost is the quadratic form of $A = -\sum_\mu J(T_\mu)$ plus a remainder beginning at fourth order; only constant fields are stationary, and the Euclidean lattice equation of the quadratic part is $\sum_\mu J(T_\mu)u = 0$ (Theorem 11). Variances of linear observables under $e^{-\beta S}$ are bounded by their Gaussian values (Proposition 12), and the standard one-step transfer operator on the unrestricted slice space is not positive (Proposition 13). Under (P3), every nonconstant step cost satisfying the formal law is $J(T^n)$, the cost of a shift by n sites, and calibration gives $n = 1$ (Theorem 14). The quadratic part has $E \le |k|$ and, where the dispersion is differentiable, group velocity below one (Theorem 16). No real-time evolution is derived.
- **The holonomy.** Under (P5), the law restricts the $U(1)$ plaquette function to $\cos n\theta$; calibration gives $n = 1$, and the sign premise gives Wilson's action (Theorem 17). For $U(N)$ the eigenvalue prescription is Wilson's action by the identity $\operatorname{Re}\operatorname{tr} U = \sum_j \cos\alpha_j$ (Theorem 18). On the whole group the half-trace of $SU(2)$ obeys the law, and for $N \ge 3$ the only continuous solutions on $SU(N)$ are $G = 0$ and $G = 1$ (Proposition 19, Corollary 20).
- **The least ledger.** Under (P4), the readable ledgers are exactly those with nonnegative counters. They have a unique least element, fixed by the offset in (P4), every readable ledger is posted from it, and readings against the unit detect every nonzero common shift between readable ledgers (Theorem 21).

Most of the underlying mathematics is classical: congruence of point sets on a line, the cosine equation and its solutions on groups, the lattice Laplacian with its dispersion and Green kernel, gradient models with convex potentials, and Wilson's action. The paper is a dictionary under explicit premises; Section 8 pairs each result with its classical antecedent.

2 The cost

We extend J to nonzero complex numbers by the same formula, $J(z) = \frac{1}{2}(z + z^{-1}) - 1$. On the positive axis and on the unit circle it is real:

$$J(e^t) = \cosh t - 1 \ \geq 0, \qquad J(e^{i\theta}) = \cos\theta - 1 \ \leq 0. \tag{3}$$

The composition law (2) is an identity of Laurent polynomials in x and y, so it holds for all nonzero complex x, y and for commuting invertible operators.

Proposition 1 (Uniqueness)**.** *The cost J is the only function $F : (0,\infty) \to \mathbb{R}$ that obeys* (2) *with* $\lim_{t\to 0} 2F(e^t)/t^2 = 1$.

Proof. Put $C(t) = F(e^t) + 1$. On an abelian group every nonzero solution of the cosine equation is $\frac{1}{2}(\chi + \chi^{-1})$ for a homomorphism χ into the nonzero complex numbers [4]. If C is real, the sets where χ is real and where $|\chi| = 1$ are subgroups covering $\mathbb{R}$, so one of them is $\mathbb{R}$. In the first case $\chi(t) = \chi(t/2)^2 > 0$, so $\chi = e^A$ and $C = \cosh\circ A$ with A additive; in the second, $C = \operatorname{Re}\chi \leq 1$. The curvature condition excludes $C = 0$, for which $2F(e^t)/t^2 = -2/t^2$, and every $C \leq 1$, for which $2F(e^t)/t^2 \leq 0$. For $C = \cosh\circ A$ it forces $\cosh A(t) \to 1$, so A is continuous at 0 and therefore linear, $A(t) = ct$, and then $c^2 = 1$. So $F(e^t) = \cosh t - 1$, that is $F = J$. □

Lemma 2 (Level sets)**.** *For $x, y > 0$, $J(x) = J(y)$ if and only if $y = x$ or $y = x^{-1}$. On $[1,\infty)$ the cost is injective.*

Proof. $J(e^s) = \cosh s - 1$, and cosh is even and strictly increasing on $[0,\infty)$. If $x, y \geq 1$ and $y = x^{-1}$ then $x = y = 1$. □

The cost has two symmetries. Multiplying both quantities of a comparison by the same positive number leaves the reading unchanged; this is the freedom of the *unit*. Exchanging the two quantities replaces x by $1/x$; this is the freedom of *orientation*, which side is written first.

Proposition 3 (An orientation average)**.** 1. *For every real t, $J(e^t) + 1 = \frac{1}{2}(e^t + e^{-t})$, the average of $e^{\sigma t}$ over the two orientations $\sigma = \pm 1$, and for all real s, t,*

$$\tfrac{1}{2}\Big[\big(J(e^{s+t}) + 1\big) + \big(J(e^{s-t}) + 1\big)\Big] = \big(J(e^s) + 1\big)\big(J(e^t) + 1\big). \tag{4}$$

2. *For any function F on the positive reals and any $x, y > 0$, $\frac{1}{2}\big[(F(xy) + 1) + (F(x/y) + 1)\big] = (F(x) + 1)(F(y) + 1)$ holds exactly when F obeys* (2) *at (x, y).*

Proof. (1) Averaging $e^{\sigma s + \tau t}$ uniformly over the four pairs (σ, τ) gives $\frac{1}{4}(e^{s+t} + e^{s-t} + e^{-s+t} + e^{-s-t})$, which equals both sides of (4). (2) Expanding the product turns one identity into the other. □

By (P0) a comparison reads J of its ratio, and $J(x) = J(1/x)$, so a single comparison does not read its own orientation; $J + 1$ is its exponential averaged over both. Equation (4) writes the cosh addition formula as an average over two sign choices; it is an algebraic identity and carries no premise of statistical independence. This representation is not specific to J: every nonzero solution of the cosine equation on an abelian group has the form $\frac{1}{2}(\chi + \chi^{-1})$ [4], and each is such an average. It explains the form of the law rather than selecting J, and it says nothing about the joint statistics of physical comparisons (Section 8).

3 Same readings, same state

Definition 4 (Configuration and profile)**.** A *configuration* is a family $q = (q_i)_{i \in I}$ of positive quantities on a fixed labelled set I. Its *profile* is the function $(i, j) \mapsto J(q_i/q_j)$, the readings of all pairwise comparisons.

Principle 5 (Identity of states)**.** Every available examination of a configuration has an outcome that is a function of its profile. Two configurations on the same labelled set are the same physical state exactly when their profiles agree.

We adopt the principle as a premise. On the unrestricted configuration domain, its access hypothesis rules out examinations that distinguish configurations with the same profile, such as the sum-versus-part reading below; it does not rule out all composite readings, since invariant monomial readings factor through the profile. On the counted domain of Corollary 10 the profile is injective, so every function of the configuration factors through it. Operationally, it says that every examination map factors through the profile (for a probabilistic examination, its outcome distribution does) and that the pairwise comparisons themselves are available; the identification of physical states is then a stipulation on top of this access assumption. It is motivated by the idea that two occurrences are distinct exactly when some examination can tell them apart. By (P0), an available pairwise comparison has the stated reading; the assertion that these readings determine every available examination is (P1), not a consequence of (P0). Its first sentence is an access hypothesis, and it carries the content. A comparison of a sum with a part, $J\big((q_1 + q_2)/q_3\big)$, reads 0 on $(1, 2, 3)$ and $\frac{49}{36}$ on $(6, 3, 2)$, and an unbalanced product such as $J(q_1 q_2/q_3)$ sees the unit. J of a monomial of degree zero, $J\big(\prod_i q_i^{a_i}\big)$ with finitely many nonzero a_i and $\sum_i a_i = 0$, such as $J(q_1 q_2/q_3^2)$, is unchanged by $q \mapsto cq$ and by $q \mapsto c/q$, so it is a function of the profile (Corollary 9); the sum example shows that degree zero alone does not suffice. Reciprocal invariance makes globally inverted configurations indistinguishable by pairwise readings, and (P1b) additionally identifies them as physical states by excluding distinguishing examinations; a reading that recorded the ratio itself, or its signed logarithm, would distinguish them. Whether pairwise comparisons exhaust the physically available examinations is open (Section 8).

Theorem 6 (The profile is a complete invariant)**.** *Let q and q' be configurations on the same labelled set. Their profiles agree if and only if there is a constant $c > 0$ such that either $q'_i = c\,q_i$ for all i, or $q'_i = c/q_i$ for all i.*

Proof. Write $t_i = \log q_i$ and $t'_i = \log q'_i$. By Lemma 2, the profiles agree exactly when $|t_i - t_j| = |t'_i - t'_j|$ for all i, j. The two stated forms satisfy this. Conversely, suppose all pairwise distances

agree; if I is empty there is nothing to prove. Fix i_0 and put $u_i = t_i - t_{i_0}$, $u'_i = t'_i - t'_{i_0}$, so that $|u_i| = |u'_i|$ and $|u_i - u_j| = |u'_i - u'_j|$. If every u_i vanishes, $q' = c\,q$ with $c = q'_{i_0}/q_{i_0}$. Otherwise choose k with $u_k \neq 0$ and let $\varepsilon \in \{\pm 1\}$ satisfy $u'_k = \varepsilon u_k$. For any i, $u'_i = \pm u_i$. If $u_i \neq 0$ and $u'_i = -\varepsilon u_i$, then $|u_i + u_k| = |u'_i - u'_k| = |u_i - u_k|$, which forces $u_i u_k = 0$, a contradiction. Hence $u' = \varepsilon u$, so $t'_i = \varepsilon t_i + a$ and $q'_i = e^a q_i^{\varepsilon}$. □

Remark 7. The configurations $(1, 2, 3)$, $(2, 4, 6)$ and $(6, 3, 2)$ have the same profile, with readings $J(2) = \frac{1}{4}$, $J(3) = \frac{2}{3}$ and $J(\frac{3}{2}) = \frac{1}{12}$ in matching places; the third is $6/q$ and is not a multiple of the first. The configuration $(1, 2, 4)$ has a different profile. The theorem uses only that $J(x)$ is a strictly increasing function of $|\log x|$: it is the congruence theorem for labelled points on a line [5], read through the cost.

Proposition 8 (What a profile reads)**.** *Let $x, y > 0$ and let q be a configuration.*

1. Relative orientation. *$J(xy) = J(x/y)$ if and only if $x = 1$ or $y = 1$. Moreover, for $a = t_i - t_j$ and $b = t_j - t_k$ with $ab \neq 0$, where $t = \log q$, the profile gives $|a|$, $|b|$ and $|a + b|$, and hence the sign of ab.*

2. Offsets. *If $c \neq 0$ and every $q_i + c$ is positive, adding c to every quantity leaves the profile unchanged if and only if all the q_i are equal.*

3. Balance. *Every comparison in q reads zero if and only if all its quantities are equal.*

Proof. (1) By Lemma 2, equality means $xy = x/y$ or $xy = y/x$, that is $y^2 = 1$ or $x^2 = 1$; and $|a + b|$ equals $|a| + |b|$ or $\big||a| - |b|\big|$ according as $ab > 0$ or $ab < 0$, two values that differ when $ab \neq 0$. (2) For $a \neq b$ positive with $a + c, b + c > 0$, equality $J\big((a + c)/(b + c)\big) = J(a/b)$ would need $(a + c)/(b + c) = a/b$, so $c(b - a) = 0$, or $(a + c)/(b + c) = b/a$, so $(a - b)(a + b + c) = 0$ and $a + c = -b < 0$; if all q_i are equal both profiles vanish. (3) $J(q_i/q_j) = 0$ exactly when $q_i = q_j$. □

Adding a constant to every quantity is an operation on configurations in a fixed unit, not on states: $(1, 2, 3)$, $(6, 3, 2)$ and $(2, 4, 6)$ are one state, but adding one to each gives three different states. Part (2) therefore concerns systems with a fixed unit, such as counted systems.

Corollary 9 (Every symmetric reading is a function of the profile)**.** *Let ρ be a reading of configurations on I with $\rho(c/q) = \rho(q)$ and $\rho(cq) = \rho(q)$ for all $c > 0$. Then ρ takes equal values on configurations with equal profiles, so it is a function of the profile on the set of realized profiles.*

Proof. By Theorem 6, two configurations with the same profile are related by $q' = cq$ or $q' = c/q$. □

The identification of Theorem 6 must be compatible with how states change. Consider two counts and a change that adds one item to the first. The configurations $(1, 1)$ and $(2, 2)$ have the same profile, every reading zero, but after the change they become $(2, 1)$ and $(3, 2)$, which read $J(2) = \frac{1}{4}$ and $J(\frac{3}{2}) = \frac{1}{12}$. So the identification cannot survive when the size of a change is fixed in a unit outside the configuration. Under (P4) the unit is inside: a count is a count of items against one item, and the configuration of a counted system includes that unit item.

Corollary 10 (The unit item fixes the configuration)**.** *Let the labelled set contain a unit item $\star$ with $q_\star = 1$, and let every quantity be at least one. Two such configurations are the same state if and only if they are equal. So each state has at most one representative in this domain, and every map of the domain into itself, in particular every post, induces a map on those states.*

Proof. By Theorem 6, $q' = cq$ or $q' = c/q$. Evaluating at $\star$ gives $c = 1$. In the first case $q' = q$. In the second, $q'_i = 1/q_i$ with $q_i \geq 1$ and $q'_i \geq 1$, so $q_i = q'_i = 1$ for every i. □

The normalization $q_\star = 1$ removes the unit freedom and the bound $q \geq 1$ removes the inversion; if the state relation quotients only by common scaling, as the lattice model of Section 4 does, $q_\star = 1$ fixes the representative without the lower bound. A unit entry alone does not suffice: $(1; 2, 3)$ and $(1; \frac{1}{2}, \frac{1}{3})$ share a profile. Comparing a count with the unit item determines the count uniquely: for $n \geq 1$, $n = 1 + J(n) + \sqrt{J(n)(J(n)+2)}$ by Lemma 2. So two descriptions of the same state by positive configurations on one labelled set differ by a common unit, possibly composed with a global inversion, and by nothing else; on the counted domain they coincide.

4 The lattice field

Let D and L be positive integers and e_μ the unit step along axis μ of the lattice $\mathbb{Z}^D$ or of its periodic version $(\mathbb{Z}/L\mathbb{Z})^D$. The shift T_μ moves a field by one site, $(T_\mu f)(x) = f(x + e_\mu)$. Evaluating the Laurent polynomial (1) at the shift gives the operator

$$J(T_\mu) = \tfrac{1}{2}\big(T_\mu + T_\mu^{-1}\big) - 1, \qquad \big(2J(T_\mu)f\big)(x) = f(x + e_\mu) + f(x - e_\mu) - 2f(x), \tag{5}$$

so $2\sum_\mu J(T_\mu)$ is the nearest-neighbour lattice Laplacian. We write $A = -\sum_\mu J(T_\mu)$, a non-negative operator for the real inner product $\langle u, v\rangle = \sum_x u_x v_x$ on the periodic lattice (and on $\ell^2(\mathbb{Z}^D)$). The composition law holds as an identity of translations, $J(T^{a+b}) + J(T^{a-b}) = 2J(T^a)J(T^b) + 2J(T^a) + 2J(T^b)$ for all integers a, b, because the shifts along one axis commute.

4.1 The total cost and its quadratic part

A lattice configuration of positive quantities $q_x = e^{t_x}$ is described by its logarithmic field t. Each site compares itself with its neighbour along each axis, at ratio $e^{t_{x+e_\mu} - t_x}$, and by (P2) the total cost is the unweighted sum of these costs (for $L = 2$ each neighbouring pair is counted twice).

Theorem 11 (The total cost)**.** *On the periodic lattice $(\mathbb{Z}/L\mathbb{Z})^D$, let*

$$S(t) = \sum_x \sum_\mu J\big(e^{t_{x+e_\mu} - t_x}\big) = \sum_x \sum_\mu \big(\cosh(t_{x+e_\mu} - t_x) - 1\big). \tag{6}$$

1. *$S \geq 0$, with equality exactly when every comparison is balanced, that is when t is constant.*
2. *With $(\Delta_\mu t)_x = t_{x+e_\mu} - t_x$,*

$$S(t) = \langle t, At\rangle + R(t), \qquad 0 \leq R(t) \leq \frac{1}{24}\sum_x \sum_\mu (\Delta_\mu t)_x^4 \cosh\big((\Delta_\mu t)_x\big), \tag{7}$$

where $\langle t, At\rangle = \frac{1}{2}\sum_x \sum_\mu (\Delta_\mu t)_x^2$ *and* $R(t) = \sum_x \sum_\mu \big[\cosh((\Delta_\mu t)_x) - 1 - \frac{1}{2}(\Delta_\mu t)_x^2\big]$. *The Hessian of* S *at balance is* $2A = -2\sum_\mu J(T_\mu)$.

3. *The stationarity condition* $\partial S/\partial t_y = 0$ *is*

$$\sum_\mu \Big[\sinh\big(t_y - t_{y-e_\mu}\big) - \sinh\big(t_{y+e_\mu} - t_y\big)\Big] = 0, \tag{8}$$

and its linearization at balance, $t = su$ *with* $s \to 0$, *is the lattice equation* $\sum_\mu J(T_\mu)u = 0$.

4. *The stationary fields on the periodic lattice are exactly the constant fields.*

5. *The solutions of the lattice equation* $\sum_\mu J(T_\mu)u = 0$ *on the periodic lattice are exactly the constant fields.*

Proof. (1) Each term is $\cosh - 1 \geq 0$, zero exactly at zero argument, and a field with all neighbouring differences zero is constant. (2) Summation by parts, using that the sum over x is invariant under $x \mapsto x + e_\mu$, gives $\sum_x u_x(u_{x+e_\mu} + u_{x-e_\mu} - 2u_x) = -\sum_x (u_{x+e_\mu} - u_x)^2$. The remainder is a sum of $\cosh a - 1 - a^2/2 = \sum_{m\geq 2} a^{2m}/(2m)!$, which lies between 0 and $a^4 \cosh a/24$ by comparing coefficients. The second derivative of $S(su)$ at $s = 0$ is $2\langle u, Au\rangle$. (3) Differentiate in t_y and reindex. The derivative of (8) along su at $s = 0$ is $\sum_\mu \big[(u_y - u_{y-e_\mu}) - (u_{y+e_\mu} - u_y)\big] = -2\sum_\mu (J(T_\mu)u)_y$. (4) Pairing (8) with t and reindexing gives $\sum_y t_y \partial S/\partial t_y = \sum_x \sum_\mu (\Delta_\mu t)_x \sinh(\Delta_\mu t)_x$, a sum of nonnegative terms that vanishes only when every difference does. (5) Pairing with u gives $\langle u, Au\rangle = \frac{1}{2}\sum_x \sum_\mu (\Delta_\mu u)_x^2 = 0$. □

At quadratic order this uses only the unit curvature of J: every even C^2 cost F of the log ratio with $F(0) = 0$ and $F''(0) = 1$ gives the same Hessian at balance. The full form of J enters through the remainder R, whose series is fixed. Parts (4) and (5) show that on a finite periodic lattice the content of S is not classical stationarity. It is the Euclidean weight $e^{-\beta S}$ of (P2), a classical statistical weight; the local equation (8) and its linearization describe fields on the infinite lattice or with fixed boundary data.

The total cost is a sum of profile entries, so it is a function of the profile, and it is unchanged by the common shift $t \mapsto t + a$, which (P1) identifies; this is why (P2) places the weight on fields modulo the common shift, which also removes the constant mode of A. (P1) would further identify the global inversion $t \mapsto -t + a$, a symmetry of S. We do not impose it on lattice fields. In the Gaussian Fock representation of the nonzero modes of the quadratic part, field inversion gives parity $(-1)^n$ to an n-particle excitation; if inversion is implemented by projecting onto invariant vectors, an additional choice, only even total particle number survives. Whether it is an identification of physical states is part of the open question of observational completeness (Section 8).

Proposition 12 (Fluctuations of the full weight)**.** 1. *For every field* t *and direction* u, *the second derivative of* $s \mapsto S(t + su)$ *at* $s = 0$ *is* $\sum_x \sum_\mu \cosh\big((\Delta_\mu t)_x\big)(\Delta_\mu u)_x^2$, *which is at least* $2\langle u, Au\rangle$.

2. *Let* μ_β *be the probability measure proportional to* $e^{-\beta S}$ *on the hyperplane* $\sum_x t_x = 0$, *with* $L \geq 2$. *For every* v *with* $\sum_x v_x = 0$, *the variance of* $\langle v, t\rangle$ *under* μ_β *is at most* $\langle v, (2\beta A)^{-1} v\rangle$,

its variance under the Gaussian weight $e^{-\beta\langle t,At\rangle}$ on the same hyperplane. In particular, with $r = x - y$ and k ranging over $(2\pi/L)\mathbb{Z}^D$ modulo 2π,

$$\mathrm{Var}_{\mu_\beta}(t_x - t_y) \le \frac{1}{\beta L^D} \sum_{k \neq 0} \frac{1 - \cos(k \cdot r)}{\sum_j (1 - \cos k_j)}.$$

Proof. (1) Differentiate twice; $\cosh \ge 1$, and $\sum_x \sum_\mu (\Delta_\mu u)_x^2 = 2\langle u, Au\rangle$. (2) By Theorem 11(5), A is positive definite on the hyperplane, and $S \ge \langle t, At\rangle$ because $\cosh a - 1 \ge a^2/2$, so μ_β is normalizable. By (1), βS restricted to the hyperplane is smooth with Hessian at least $2\beta A$ there. The Brascamp–Lieb inequality [14], applied as for anharmonic crystals [15], gives $\mathrm{Var}(\langle v, t\rangle) \le \mathbb{E}\langle v, (\mathrm{Hess}\, \beta S)^{-1} v\rangle \le \langle v, (2\beta A)^{-1} v\rangle$, the second step because inversion reverses the order of positive-definite operators, and $(2\beta A)^{-1}$ is the covariance of $e^{-\beta\langle t,At\rangle} = e^{-\frac{1}{2}\langle t, 2\beta A t\rangle}$. The formula follows from $Ae^{ik\cdot x} = \sum_j (1 - \cos k_j) e^{ik\cdot x}$ and $|e^{ik\cdot x} - e^{ik\cdot y}|^2 = 2 - 2\cos(k \cdot r)$. □

For linear observables the nonlinearity of the cost can only reduce variances. As $L \to \infty$, for fixed r, the bound tends to $\beta^{-1} \int_{[-\pi,\pi]^D} \frac{1-\cos(k\cdot r)}{\sum_j (1-\cos k_j)} \frac{d^D k}{(2\pi)^D}$, which is finite in every dimension and, for $D \ge 3$, at most $2\beta^{-1} \int \frac{d^D k}{(2\pi)^D} \big(\sum_j (1-\cos k_j)\big)^{-1} < \infty$ for every r. Since $1 - \cos(k\cdot r) \le |k|^2 |r|^2/2$ and $\sum_j (1 - \cos k_j) \ge 2|k|^2/\pi^2$ on $[-\pi, \pi]^D$, the finite-volume bound is at most $\pi^2 |r|^2/(4\beta)$, uniformly in L in every dimension. For $D \ge 3$ the same lower bound gives $\mathrm{Var}(t_x - t_y) \le (\pi^2/\beta) L^{-D} \sum_{k \neq 0} |k|^{-2}$ with k in $[-\pi, \pi]^D$; the lattice momenta with $2\pi s/L \le |k| < 2\pi(s+1)/L$ number $O(s^{D-1})$, so the sum is $O\big(L^{2-D} \sum_{s \le \sqrt{D}L} s^{D-3}\big) = O(1)$, and $\mathrm{Var}(t_x - t_y) \le C_D/\beta$ uniformly in L, x and y. It applies to any limiting difference distribution that is constructed; it is not a construction of an infinite-volume field. For gradient models whose potential has second derivative bounded above and below, the large-scale limit is a Gaussian free field with a renormalized stiffness [18, 19]; here the second derivative cosh is bounded below but not above, so those theorems do not apply directly, and the corresponding statement is not proved here.

Proposition 13 (The time kernel of the full weight)**.** *Call a real symmetric kernel K positive definite when every finite form $\sum_{i,j} v_i v_j K(t_i, t_j)$ with real weights is nonnegative (for such kernels this is equivalent to the complex-weight condition). Along the time axis, $e^{-\beta S}$ couples successive slices through the one-site kernel $K_\beta(s,t) = \exp\big(-\beta(\cosh(s-t) - 1)\big)$.*

1. *For every nonconstant entire function Φ that is real on the real axis, the kernel $\Phi\big(J(e^{s-t})\big)$ is not positive definite. In particular, for no $\beta > 0$ is K_β positive definite; for $\beta = \frac{1}{4}$, the points $0, \log 2, 2\log 2$ and the weights $(1, -2, 1)$ give the form $6 - 8e^{-1/16} + 2e^{-9/32} < 0$.*
2. *Let V be the spatial action of a finite slice. For no $\beta > 0$ is the one-step transfer kernel of the slice positive definite:*

$$\mathscr{T}(t,s) = e^{-\beta V(t)/2} \prod_x K_\beta(t_x, s_x)\, e^{-\beta V(s)/2}.$$

Proof. (1) Since $\cosh(\log 2) = \frac{5}{4}$ and $\cosh(2\log 2) = \frac{17}{8}$, the form is as displayed. The Taylor bounds $e^{-1/16} \ge 1 - \frac{1}{16} + \frac{1}{512} - \frac{1}{24576}$ and $e^{-9/32} \le \sum_{m=0}^{4} (-\frac{9}{32})^m/m!$ bound it above by $-\frac{70367}{12582912}$.

For general Φ, let $f(u) = \Phi(\cosh u - 1)$, a real, even, entire function with $f(u + 2\pi i) = f(u)$. The form of f at the points $0, h, \dots, nh$ with weights $(-1)^i\binom{n}{i}$ equals $\sum_{m=-n}^{n}(-1)^m\binom{2n}{n+m}f(mh) = (-1)^n\delta_h^{2n}f(0)$, the $2n$-th central difference, which is $(-1)^n h^{2n} f^{(2n)}(0) + O(h^{2n+2})$. So a positive-definite f has $(-1)^n f^{(2n)}(0) \geq 0$ for every n. Its Taylor series converges on the imaginary axis to $f(iy) = \sum_n (-1)^n f^{(2n)}(0)\, y^{2n}/(2n)!$, a sum of nonnegative terms; at $y = 2\pi$ it equals $f(0)$, so every term with $n \geq 1$ vanishes. Then all derivatives of positive order of the even function f at 0 vanish, f is constant, and so is Φ, since an entire function constant on $[0, \infty)$, the range of $\cosh u - 1$, is constant. For K_β, $\Phi(z) = e^{-\beta z}$. (2) For the given β, let points a_i and weights v_i give K_β a negative form, as (1) supplies. Take slice configurations $t^{(i)}$ that agree off one site and equal a_i there, and put $d_i = e^{-\beta V(t^{(i)})/2} > 0$. Their Gram matrix under $\mathscr{T}$ is $d_i K_\beta(a_i, a_j) d_j$, and the weights v_i/d_i give the same negative form. □

A one-step transfer operator e^{-H} with H self-adjoint is positive. On the unrestricted slice space $L^2(\mathbb{R}^M, dt)$ of a slice with M sites, the standard one-step operator is $T = M_D C_F M_D$, with kernel $\mathscr{T}$, where M_D is multiplication by $e^{-\beta V/2}$ and C_F is convolution by $\prod_x f_\beta(u_x)$, $f_\beta(u) = e^{-\beta(\cosh u - 1)}$. It is bounded and self-adjoint; smooth functions concentrated near the points of a negative Gram form give it a negative quadratic form, so by part (2) it is not positive and is not e^{-H}. It is injective, because the Fourier transform of f_β is a nonzero entire function whose real zeros are discrete, so $H = -\log|T|$ is defined by spectral calculus, with $T^{2m} = e^{-2mH}$; it describes transfer over even numbers of steps and loses the sign of the one-step spectral values. Relating H to the centered ensemble of (P2) requires a convention for the spatially uniform mode and is open.

The Gaussian kernel $e^{-\beta(s-t)^2/2}$ of the quadratic part is positive definite. It is $\Phi(J(e^{s-t}))$ with $\Phi(r) = e^{-\frac{\beta}{2}[\operatorname{arcosh}(1+r)]^2}$, which is analytic near $[0, \infty)$ but not entire, since it branches at $r = -2$; so the entire hypothesis in part (1) cannot be dropped. Wilson's circle kernel is positive definite too: $e^{-\beta(1-\cos(\theta-\varphi))} = e^{-\beta}\sum_n I_n(\beta) e^{in(\theta-\varphi)}$ with $I_n(\beta) = \sum_{m\geq 0}(\beta/2)^{2m+|n|}/\big(m!\,(m+|n|)!\big) > 0$; a strictly positive transfer matrix for Wilson's action is classical [16]. On the circle the sign premise of (P5) gives a positive one-step kernel; on the line no nonconstant real entire function of the reading does. (P2) assigns the energies of Section 4.3 to the quadratic part.

For the Boltzmann case, with $\hat{f}(p) = \int e^{-ipu} f(u)\, du$, Bochner's theorem identifies the failure of positive definiteness with negative values of $\hat{f}_\beta(p) = 2e^{\beta} K_{ip}(\beta)$, known from the theory of Bessel functions of imaginary order [21]; part (1) is broader, and its proof is elementary.

4.2 The step operators

By (P3), a translation-invariant step cost is given by a real coefficient sequence $g : \mathbb{Z} \to \mathbb{R}$, read as the formal series $G(T) = \sum_k g_k T^k$, with cost $G(T) - 1$. The law $G(T_1T_2) + G(T_1T_2^{-1}) = 2G(T_1)G(T_2)$ is imposed in the space of formal double series in two commuting independent shifts T_1, T_2, for instance two axes of $\mathbb{Z}^D$ with $D \geq 2$ acting on the indicator of the origin. Both sides are defined there for every sequence: $G(T_1T_2)$ and $G(T_1T_2^{-1})$ are supported on the two diagonals, and $G(T_1)G(T_2)$ is an outer product. Comparing coefficients of $T_1^a T_2^b$ gives

$$[a = b]\, g_a + [a = -b]\, g_a = 2\, g_a\, g_b \qquad (a, b \in \mathbb{Z}), \tag{9}$$

where $[\,\cdot\,]$ is 1 when the condition holds and 0 otherwise. On a single periodic lattice the powers T^k are not independent ($T^L = 1$), and the coefficient comparison does not apply there.

Theorem 14 (The composition-respecting step costs)**.** *The solutions of* (9) *are exactly $g = 0$ (cost -1), the constant $G = 1$ (cost 0), and $G(T) = \frac{1}{2}(T^n + T^{-n})$ for a positive integer n. So every nonconstant solution is supported on a single opposite pair; finite support and symmetry are consequences, not assumptions. The nonconstant solutions give the cost $G(T) - 1 = J(T^n)$ of a shift by n sites, with curvature $\sum_k k^2 g_k = n^2$; the unit calibration holds exactly for the nearest-neighbour cost $J(T)$. The same classification holds when the law is imposed on ratios: if g has finite support and $G(x) = \sum_k g_k x^k$ satisfies $G(xy) + G(x/y) = 2G(x)G(y)$ for all $x, y > 0$, then g satisfies* (9).

Proof. Taking $a = b = k \neq 0$ gives $g_k = 2g_k^2$, so $g_k \in \{0, \frac{1}{2}\}$; $a = b = 0$ gives $g_0 \in \{0, 1\}$. For $a \neq 0$, taking $b = -a$ gives $g_a = 2g_a g_{-a}$, so $g_{-a} = \frac{1}{2}$ whenever $g_a \neq 0$. If $a \neq \pm b$ then $g_a g_b = 0$, so the support is at most one pair $\{n, -n\}$, and $a = 0$, $b \neq 0$ shows $g_0 g_b = 0$. Each listed g satisfies (9), and the paired solution has curvature $\frac{1}{2}n^2 + \frac{1}{2}n^2 = n^2$. For the functional form, fix $y > 0$ and read $G(xy) + G(x/y) - 2G(x)G(y) = \sum_a g_a(y^a + y^{-a} - 2G(y))x^a$ as a Laurent polynomial in x vanishing on $(0, \infty)$. Its coefficients vanish, so $g_a(y^a + y^{-a}) = 2g_a G(y)$ for all a and $y > 0$; comparing coefficients of y^b gives (9). □

Within this class, the law restricts the support to one opposite pair, and the unit calibration selects the nearest-neighbour pair. Calibration alone would not fix the form of the step cost: the halved fourth-order improved stencil, with cost $G - 1$ given by the coefficients $(-1, 16, -30, 16, -1)/24$ on the shifts $T^{-2}, \dots, T^2$, has unit curvature but violates (9), since $g_1 = \frac{2}{3} \neq 2g_1^2$. Theorem 11 uses the nearest-neighbour comparisons of (P2) and not (P3); Theorem 14 characterizes the same operator independently.

4.3 Waves and speed

Proposition 15 (Plane waves)**.** *Let $z \in (\mathbb{C} \setminus \{0\})^D$ and $f(x) = \prod_\mu z_\mu^{x_\mu}$ on $\mathbb{Z}^D$. Then $J(T_\mu)f = J(z_\mu)f$, and $\sum_\mu J(T_\mu)f = 0$ if and only if $\sum_\mu J(z_\mu) = 0$.*

Proof. $T_\mu^{\pm 1} f = z_\mu^{\pm 1} f$, and $f(0) = 1$. □

These are pointwise solutions on $\mathbb{Z}^D$, not finite-action configurations. Take $D = 4$ with axis 0 for Euclidean time, $z_0 = e^E$, and spatial phases $z_j = e^{ik_j}$ with real k_j. By (3) the condition becomes

$$\cosh E - 1 = \sum_{j=1}^{3} \big(1 - \cos k_j\big), \qquad \text{equivalently} \qquad \sinh^2 \tfrac{E}{2} = \sum_{j=1}^{3} \sin^2 \tfrac{k_j}{2}, \tag{10}$$

so the nonnegative branch is $E(k) = 2\,\mathrm{arsinh}\big(\sum_j \sin^2(k_j/2)\big)^{1/2}$, and $\cosh E = 3$ at $k = (\pi, 0, 0)$. This exponent is the characteristic exponent of the linearized lattice equation. In the standard construction of the free lattice field, which (P2) adopts for the quadratic part of S, $E(k)$ is the excitation energy of the oscillator mode with nonzero spatial momentum k [6, 7]. The common-shift identification of (P2) removes only the spacetime constant. Writing $t_{n,\mathbf{x}} = a_n + \eta_{n,\mathbf{x}}$ with

$\sum_{\mathbf{x}} \eta_{n,\mathbf{x}} = 0$ on a slice of M sites, the quadratic action keeps $\frac{M}{2}\sum_n (a_{n+1} - a_n)^2$ for the spatially uniform mode, a free particle in Euclidean time rather than an oscillator, which needs separate treatment.

The action of (P2) carries one unweighted term per axis. The composition law constrains a weight only when the weight is placed inside the comparison cost. The scaled cost wJ obeys (2) only for $w \in \{0, 1\}$, since its left side gives $2wJ(x)J(y)$ where the right side gives $2w^2 J(x)J(y)$; and a weighted step operator $wJ(T_0) = G_w(T_0) - 1$, with $G_w = (1-w) + \frac{w}{2}(T_0 + T_0^{-1})$, obeys the law of (P3) only for $w \in \{0, 1\}$, by Theorem 14. The reparametrized costs $J(x^c)$ and $J(T^n)$ obey the law too; the first gives the time Hessian $-2c^2 J(T_0)$, the second has curvature n^2, and the unit calibration leaves $|c| = 1$, where both signs give the same cost, and $n = 1$. Couplings a_μ placed outside the comparison cost, in an action $\sum_{x,\mu} a_\mu J(e^{(\Delta_\mu t)_x})$, are not constrained by the law: equal weighting of the axes is the content of (P2). For a time weight $w > 0$ the dispersion condition becomes $w(\cosh E - 1) = \sum_j (1 - \cos k_j)$, equivalently $w \sinh^2(E/2) = \sum_j \sin^2(k_j/2)$, with nonnegative branch $E_w(k) = 2\,\mathrm{arsinh}\big(\sum_j \sin^2(k_j/2)/w\big)^{1/2}$.

Theorem 16 (Dispersion bounds)**.** *For a time weight $w > 0$, every real solution of $w(\cosh E - 1) = \sum_j (1 - \cos k_j)$, of either sign, satisfies $|k|^2\big(1 - \frac{|k|^2}{12}\big)e^{-|k|^2/(2w)} \le wE^2 \le |k|^2$, so the long-wave phase speed of the branch E_w is $1/\sqrt{w}$; and $\cosh E = 1 + 2/w$ at $k = (\pi, 0, 0)$. For the unweighted law $w = 1$:*

1. *every real solution satisfies $|k|^2\big(1 - \frac{|k|^2}{12}\big)e^{-|k|^2/2} \le E^2 \le |k|^2$, so for $k \neq 0$ the phase speed $E/|k|$ of the branch $E \ge 0$ is at most one lattice site per lattice time step and tends to one as $k \to 0$;*

2. *wherever E is differentiable in k, which excludes the points $k \in (2\pi\mathbb{Z})^3$ where E has a cone, $\sinh E\, \partial E/\partial k_j = \sin k_j$ and the group velocity satisfies $\sum_j (\partial E/\partial k_j)^2 < 1$.*

Proof. The value at $(\pi, 0, 0)$ is arithmetic, and $wE^2 \le |k|^2$ follows from $\cosh E - 1 \ge E^2/2$ and $1 - \cos k \le k^2/2$. For the lower bound, $1 - \cos k \ge k^2/2 - k^4/24$ and $\sum_j k_j^4 \le |k|^4$ give $\sum_j (1 - \cos k_j) \ge \frac{1}{2}|k|^2(1 - |k|^2/12)$, while $\cosh E - 1 \le \frac{1}{2}E^2 e^{E^2/2} \le \frac{1}{2}E^2 e^{|k|^2/(2w)}$; multiplying the dispersion condition through gives the lower bound for every w, and $w = 1$ gives (1). Near $k = 0$ the bounds give $E = |k| + O(|k|^3)$ on the nonnegative branch. For (2), differentiate (10). With $u_j = 1 - \cos k_j \in [0, 2]$ and $U = \sum_j u_j$, we have $\sinh^2 E = U^2 + 2U$ and $\sum_j \sin^2 k_j = \sum_j u_j(2 - u_j) = 2U - \sum_j u_j^2 < \sinh^2 E$ when $U > 0$. □

The bounds $E \le |k|$ and $|\nabla_k E| < 1$ are the standard subluminal dispersion of the massless lattice free field and depend on the cost only through its unit curvature, with the unweighted nearest-neighbour placement of (P2) fixed. The lower bound in (1) is a long-wave statement; it is empty for $|k|^2 \ge 12$. The bound in (2) is a group-velocity bound for the dispersion branch of the quadratic part; it is not a front-propagation theorem, which would need a specified real-time evolution. One site per time step becomes a physical speed once the lattice spacings in space and time are given. The lattice equation has hypercubic symmetry, and its leading continuum term $\frac{1}{2}|k|^2$ is rotation invariant. For $w < 1$ the long-wave speed $1/\sqrt{w}$ exceeds one (Figure 1).

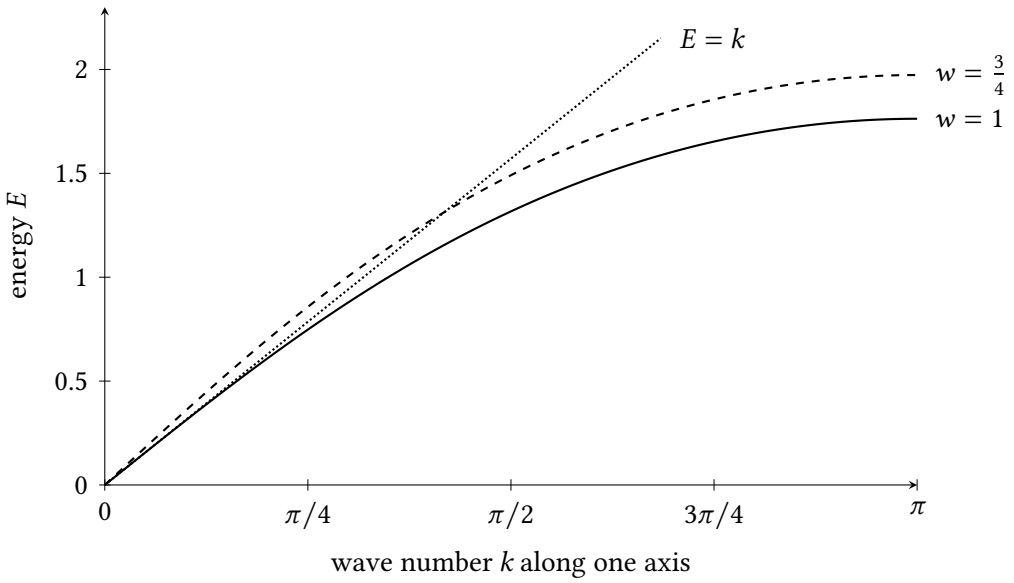

Figure 1. The nonnegative branch E_w for wave vector $(k, 0, 0)$, in lattice units (time step equal to the spacing). Solid: the unweighted law, $\cosh E - 1 = 1 - \cos k$, which meets $\cosh E = 3$ at the zone boundary and stays on or under the reference line $E = k$. Dashed: time comparisons weighted $w = \frac{3}{4}$, reaching $\cosh E = \frac{11}{3}$ and rising above $E = k$ at long wavelength with slope $2/\sqrt{3}$. Dotted: the reference line $E = k$.

The same spatial stencil has a second, distinct use. If $H = -\sum_j J(T_j)$ is chosen as a nonrelativistic hopping Hamiltonian in lattice units, an identification outside (P0)–(P5), its eigenvalue on $e^{ik\cdot x}$ is

$$\varepsilon(k) = -\sum_j J(e^{ik_j}) = \sum_j \bigl(1 - \cos k_j\bigr) = \tfrac{1}{2}|k|^2 + O(|k|^4), \tag{11}$$

quadratic at long wavelength, whereas the transfer-matrix energy $E = \operatorname{arcosh}(1 + \varepsilon)$ of the quadratic part is linear. The Green kernel of H on $\mathbb{Z}^3$ is one third of the Green function of simple random walk, since $H = 3(I - P)$ with P its transition operator, and that function is asymptotic to $3/(2\pi|x|)$ [8, Section 26, P26.1], [9, Theorem 4.3.1]. So the kernel falls off as $1/(2\pi|x|)$; if a static potential obeys the lattice Poisson equation $H\phi = \rho$, this is the Coulomb form.

5 The holonomy

On a lattice gauge field with $U(1)$ link variables, the holonomy around a plaquette p is $U_p = e^{i\theta_p}$; by (P5) it is a comparison of phases transported around the plaquette. A gauge transformation $\theta_\mu(x) \mapsto \theta_\mu(x) + \chi(x + e_\mu) - \chi(x)$ leaves every plaquette angle $\theta_p = \theta_\mu(x) + \theta_\nu(x + e_\mu) - \theta_\mu(x + e_\nu) - \theta_\nu(x)$ unchanged. Under (P5), a cost of the holonomy has the form $G(\theta) - 1$ with $G(\theta) = \sum_k g_k e^{ik\theta}$, and the composition law for two independent holonomies reads $G(\theta + \varphi) + G(\theta - \varphi) = 2G(\theta)G(\varphi)$.

Theorem 17 (Wilson's plaquette action)**.** *If g has finite support and $G(\theta + \varphi) + G(\theta - \varphi) = 2G(\theta)G(\varphi)$ for all real θ, φ, then g satisfies* (9)*, so $G = 0$, $G = 1$, or $G(\theta) = \cos n\theta$ with n a positive integer. Under the unit calibration $G''(0) = -1$, the cost of the holonomy is $G - 1 = J(e^{i\theta}) =$*

$\cos\theta - 1$, *and its negative is Wilson's plaquette action [11],*

$$W(\theta) = 1 - \cos\theta = -J(e^{i\theta}). \tag{12}$$

For every real θ, $0 \le \frac{1}{2}\theta^2 - W(\theta) \le \frac{1}{24}\theta^4$*; so for a smooth field with* $\theta_p = a^2 F_{\mu\nu} + O(a^3)$*, where* a *is the lattice spacing,* $W(\theta_p) = \frac{1}{2}a^4 F_{\mu\nu}^2 + O(a^5)$.

Proof. The characters $e^{ik\theta}$ are linearly independent, so the argument of Theorem 14 applies with $e^{i\theta}$ in place of x: fix φ, compare coefficients of $e^{ia\theta}$, then of $e^{ib\varphi}$. The calibration $G''(0) = -\sum_k k^2 g_k = -1$ selects $n = 1$. For the bound let $r(\theta) = \cos\theta - 1 + \theta^2/2$; then $r(0) = r'(0) = 0$ and $r'' = 1 - \cos\theta \ge 0$, so $r \ge 0$, and $\theta^4/24 - r$ has second derivative $r \ge 0$ with the same initial data. □

The finite-support hypothesis can be replaced by continuity: the continuous real solutions of the cosine equation on $\mathbb{R}$ are 0, $\cos c\theta$ and $\cosh c\theta$ [2], so 2π-periodicity leaves 0 and $\cos n\theta$ with $n \ge 0$, and calibration selects $n = 1$. Villain's periodic-Gaussian action W_V [10], normalized by $W_V(0) = 0$, is not among the classified actions. Its weight is proportional to $\sum_{\ell\in\mathbb{Z}} e^{-\kappa(\theta+2\pi\ell)^2/2}$ for some $\kappa > 0$, with Fourier coefficients proportional to $e^{-m^2/(2\kappa)}$, all nonzero; this excludes the constant actions and $1 - \cos n\theta$ with $n \ge 2$, whose weights have Fourier coefficients only at multiples of n. For $n = 1$ the weight $e^{-(1-\cos\theta)}$ has coefficients proportional to $I_m(1)$, whose consecutive ratios decay like $1/(2m)$, while the periodic-Gaussian ratios $e^{-(2m+1)/(2\kappa)}$ decay exponentially. So $G_V = 1 - W_V$ does not obey the cosine equation. The sign is a premise: $J \le 0$ on the circle, and the weight $e^{-\beta\sum_p W(\theta_p)}$ with the nonnegative action $W = -J$ favours the identity holonomy, while the opposite sign would favour $\theta_p = \pi$.

For a nonabelian holonomy U there are two readings. Evaluating the cost on the holonomy matrix, as $J(T_\mu)$ is evaluated on the shift, gives $J(U) = \frac{1}{2}(U + U^{-1}) - 1$, whose eigenvalues are $J(\lambda_j) = \cos\alpha_j - 1$ for the eigenvalues $\lambda_j = e^{i\alpha_j}$ of U; (P5) prices each eigenvalue by the $U(1)$ cost and averages. In general the eigenvalues of UV cannot be obtained by pairing those of U and V, so this prescription is not a composition law on the group. The law can instead be imposed on the whole group, $G(UV) + G(UV^{-1}) = 2G(U)G(V)$ for all U, V.

Theorem 18 (The eigenvalue prescription is Wilson's action)**.** *Under the eigenvalue prescription of (P5), for* $U \in U(N)$ *with eigenvalues* $e^{i\alpha_j}$*,*

$$W(U) = -\frac{1}{N}\sum_{j=1}^{N}\big(G(\alpha_j) - 1\big) = -\frac{1}{N}\operatorname{tr} J(U) = 1 - \frac{1}{N}\operatorname{Re}\operatorname{tr} U,$$

Wilson's plaquette action for $U(N)$ *and* $SU(N)$*. The factor* $1/N$ *is a normalization that the coupling* β *can absorb.*

Proof. By Theorem 17, $G(\alpha) = \cos\alpha$, and $\sum_j(\cos\alpha_j - 1) = \operatorname{Re}\sum_j e^{i\alpha_j} - N = \operatorname{Re}\operatorname{tr} U - N$. □

Proposition 19 (The law on the whole group)**.** *1. For* 2×2 *complex matrices with* $\det V = 1$, $\operatorname{tr}(UV) + \operatorname{tr}(UV^{-1}) = \operatorname{tr} U \operatorname{tr} V$*. So* $G(U) = \frac{1}{2}\operatorname{tr} U$ *obeys* $G(UV) + G(UV^{-1}) = 2G(U)G(V)$ *on* $SL(2,\mathbb{C})$*, in particular on* $SU(2)$*, and its cost* $\frac{1}{2}\operatorname{tr} U - 1$ *is minus Wilson's action for* $SU(2)$.

2. *For $N \geq 3$ the normalized trace $f(U) = \frac{1}{N}\operatorname{Re}\operatorname{tr} U$ violates the law at $U = V = \operatorname{diag}(i, -i, 1, \dots, 1) \in SU(N)$: $f(U^2) + f(I) = (2N-4)/N$, while $2f(U)^2 = 2(N-2)^2/N^2$.*

Proof. (1) For $\det V = 1$, $V^{-1} = \operatorname{adj} V = (\operatorname{tr} V)I - V$ by the Cayley–Hamilton theorem, so $\operatorname{tr}(UV) + \operatorname{tr}(UV^{-1}) = \operatorname{tr}\big(U(\operatorname{tr} V)I\big) = \operatorname{tr} U \operatorname{tr} V$. (2) $U^2 = \operatorname{diag}(-1, -1, 1, \dots, 1)$, so $\operatorname{tr} U^2 = N - 4$, $\operatorname{tr} I = N$ and $\operatorname{tr} U = N-2$; the difference is $(2N-4)/N - 2(N-2)^2/N^2 = 4(N-2)/N^2 > 0$. □

The continuous solutions of the law on these groups are known. On a compact group every nonzero continuous solution is $\frac{1}{2}\operatorname{tr}\rho$ for a continuous homomorphism ρ into $SU(2)$, up to conjugation, and a reducible ρ gives $\frac{1}{2}(\chi + \bar{\chi})$ for a character χ [12]; on $SU(2)$ the nonconstant continuous solution is $\frac{1}{2}\operatorname{tr} U$ alone [13].

Corollary 20 (No nonconstant cost on $SU(N)$, $N \geq 3$)**.** *For $N \geq 3$ the only continuous functions $G : SU(N) \to \mathbb{C}$ with $G(UV) + G(UV^{-1}) = 2G(U)G(V)$ for all U, V are $G = 0$ and $G = 1$.*

Proof. By the representation theorem just quoted, a nonzero G is $\frac{1}{2}\operatorname{tr}\rho$ with $\rho : SU(N) \to SU(2)$ continuous. Its kernel is a closed normal subgroup, and the proper closed normal subgroups of $SU(N)$ are finite and central. A finite kernel would make ρ an injective immersion of the Lie algebra of $SU(N)$, of dimension $N^2 - 1 > 3$, into that of $SU(2)$, which is impossible; so the kernel is all of $SU(N)$ and $G = 1$. □

Imposed on the whole group with continuity, the law therefore selects among nonconstant costs $\cos\theta - 1$ on $U(1)$ with unit calibration and $\frac{1}{2}\operatorname{tr} U - 1$ on $SU(2)$, and admits no nonconstant cost on $SU(N)$, $N \geq 3$; there Wilson's action comes from the eigenvalue prescription of (P5).

6 Counted ledgers

A ledger is a formal model of a counted record: a system whose quantities are counts of identical items, compared against one item. It has d accounts, d a positive integer, each with a debit counter D_i and a credit counter C_i, both integers; the *empty* ledger has all counters zero. A post raises one counter by one. Differences of counters do not determine the common offset of the counters: adding the same integer to every counter changes no difference.

By (P4) each count is its counter plus one, $D_i + 1$ and $C_i + 1$, and the configuration of a ledger consists of the unit item and the $2d$ counts. A ledger is *readable* when every count is positive. An account is *balanced* when its two counters are equal, and a ledger is balanced when every account is. The *shift* of a ledger by an integer c adds c to every counter and leaves the unit item fixed. Among the readings are the account readings $J\big((D_i + 1)/(C_i + 1)\big)$ and the unit readings $J(D_i + 1)$, $J(C_i + 1)$.

Theorem 21 (Counted ledgers)**.** 1. *A ledger is readable if and only if all its counters are nonnegative, if and only if it is reached from the empty ledger by posts.*

2. *Among readable ledgers the empty ledger is the unique least one, counter by counter, and the only one with no readable predecessor (a readable ledger from which one post leads to it). More generally, if every count is the counter plus an integer n, the least readable ledger is the one in which every count equals one, every readable ledger is reached from it by posts, and it is the empty ledger exactly when $n = 1$.*

3. *Under (P1), two readable ledgers are the same state if and only if they are equal, and every post maps the readable ledgers into themselves.*

4. *If a readable ledger and its shift by a nonzero integer c are both readable, every unit reading changes: $J(D_i + 1 + c) \neq J(D_i + 1)$ and $J(C_i + 1 + c) \neq J(C_i + 1)$. The account reading of account i changes if and only if that account is unbalanced; every account reading is zero exactly when every account is balanced.*

5. *On a balanced account whose two counts equal $m > 0$, a debit post gives count ratio $(m + 1)/m$ and a credit post gives $m/(m + 1)$; the account reading after either post is $1/\big(2m(m + 1)\big)$, which equals $J(2) = \frac{1}{4}$ exactly when $m = 1$.*

Proof. (1) For integers, $D_i + 1 > 0$ exactly when $D_i \geq 0$. Each post raises one counter by one, so ledgers reached from the empty ledger have nonnegative counters; conversely, D_i debit posts and C_i credit posts on each account reach a ledger with nonnegative counters. (2) With offset n the readable ledgers are those with every counter at least $1 - n$; their least element has every count one, and every other readable ledger has a readable predecessor obtained by undoing one post. (3) Corollary 10, since every count is at least one. (4) The unit readings change by Lemma 2, since J is injective on $[1, \infty)$. For the account readings, apply the two-quantity argument in the proof of Proposition 8(2) to $a = D_i + 1$, $b = C_i + 1$ (the shift moves both counts and not the unit item); a balanced account reads zero before and after the shift. (5) $J\big((m + 1)/m\big) = J\big(m/(m + 1)\big) = 1/\big(2m(m + 1)\big)$. □

The zero of the cost does not name the start: every balanced ledger has zero account readings. Under (P4), positivity of the counts makes the empty ledger the least readable ledger of the posting order: every count, debit and credit, equals one, the value of the unit item, and every readable ledger is posted from it. This least element is fixed by the offset in (P4) and uses no property of J; it is the minimum of the stipulated posting order, and that a physical history began there is not derived. The unit readings make the count of every account visible, and with it every nonzero common shift between readable ledgers.

Proposition 22 (Two posts on one account)**.** *Let the ledger be readable, let Σ exchange debits and credits on every account, and let $P_{D,k}$ and $P_{C,k}$ be the debit and credit posts on account k. Then $P_{C,k} = \Sigma P_{D,k} \Sigma$. If account k is balanced, the two posts give the same account reading of k, while their labelled states are distinct. If every account is balanced, Σ fixes the ledger and carries one outcome to the other. If account k is unbalanced, the two posts give different account readings of k.*

Proof. The conjugation identity is the definition of Σ. For a balanced account the two ratios are $(a + 1)/a$ and $a/(a + 1)$ with $a = D_k + 1$, which have equal cost; the unit readings of the debit count differ, $J(a + 1) \neq J(a)$. If every account is balanced, Σ fixes the ledger. For an unbalanced account put $a = D_k + 1 \neq b = C_k + 1$; equal costs after the two posts would need $(a + 1)/b = a/(b + 1)$, so $a + b + 1 = 0$, or $(a + 1)/b = (b + 1)/a$, so $(a - b)(a + b + 1) = 0$, both impossible. □

The exchange Σ acts on the labelled states and preserves the identification of (P1), since it permutes profile entries. It is not imposed as an identification of states: labels are part of the state, so the two posts on a balanced account are distinct states.

7 Time-reflection symmetry

Let Θ reflect the time axis, $(\Theta f)(x_0, \mathbf{x}) = f(-x_0, \mathbf{x})$. Then $\Theta T_0 \Theta = T_0^{-1}$ and $\Theta T_j \Theta = T_j$ for the spatial shifts, so Θ commutes with $\sum_\mu J(T_\mu)$, and every solution of the lattice equation reflected in time is again a solution. On plane waves this is $J(z_0^{-1}) = J(z_0)$. The lattice equation does not select a time orientation. Posts only add, so ledgers are partially ordered by posting away from their least element; identifying that order with the lattice time axis is not derived here.

8 What the premises determine, and what remains

Theorem 23 (What the premises determine)**.**

1. *Under (P1), the state of a configuration is its profile; two configurations are the same state exactly when they differ by a common unit, possibly composed with a global inversion; on the counted domain the profile determines the configuration (Theorem 6, Corollaries 9 and 10).*

2. *Under (P2), the total cost is $\langle t, At\rangle$ plus a remainder beginning at fourth order, with $A = -\sum_\mu J(T_\mu)$ and Hessian $2A$ at balance; only constant fields are stationary and only constants solve $\sum_\mu J(T_\mu)u = 0$ on the periodic lattice (Theorem 11); the Hessian at every field dominates $2A$, and variances of linear observables under $e^{-\beta S}$ are bounded by their Gaussian values (Proposition 12); no nonconstant real entire function of the reading gives a positive-definite time kernel, and the standard one-step transfer operator on the unrestricted slice space is not positive (Proposition 13). Under (P3), every nonconstant composition-respecting step cost is $J(T^n)$, and calibration selects $J(T)$ (Theorem 14). The nonnegative dispersion branch of the quadratic part has $E \le |k|$ and group velocity below one wherever E is differentiable, that is for $k \notin (2\pi\mathbb{Z})^3$ (Theorem 16).*

3. *Under (P5), the $U(1)$ holonomy cost is $J(e^{i\theta})$ and the plaquette action is $1 - \cos\theta$ (Theorem 17); with the eigenvalue prescription the action of $U \in U(N)$ is $1 - \frac{1}{N}\operatorname{Re}\operatorname{tr} U$ (Theorem 18). On the whole group the half-trace of $SU(2)$ obeys the law, and for $N \ge 3$ the only continuous solutions $G : SU(N) \to \mathbb{C}$ are $G = 0$ and $G = 1$, that is costs -1 and 0 (Proposition 19, Corollary 20).*

4. *Under (P4), the readable ledgers are those reached from the empty ledger; it is their unique least element, in which every count equals one; under (P1) the state of a readable ledger is the ledger; every nonzero common shift between readable ledgers changes every unit reading, and the account reading of an account changes if and only if the account is unbalanced (Theorem 21); the two posts on an account are intertwined by the exchange of debit and credit (Proposition 22).*

5. *The lattice equation is reflection invariant; posting partially orders ledgers, and its identification with physical time is not derived (Section 7).*

Proof. Each item collects the cited results. □

Which property of the cost each result uses. Theorem 6 uses only that $J(x)$ increases strictly with $|\log x|$. Theorem 21(1)–(2) use only integer counts and the posting order, so

the least readable ledger is fixed by the offset in (P4); part (3) adds (P1) and uses Lemma 2 through Corollary 10. Theorem 21(4) and Proposition 22 use only the level sets of Lemma 2. The Hessian at balance, the linearized equation and Theorem 16 use only the unit curvature and nearest-neighbour placement; Proposition 12 uses the bound $\frac{d^2}{da^2} J(e^a) = \cosh a \geq 1$. The composition law enters in Proposition 1, Theorems 14 and 17, Proposition 19 and Corollary 20; Theorem 18 adds the eigenvalue prescription. Proposition 13(1) uses only that the kernel is an entire function of the reading $\cosh(s-t)-1$; part (2) adds that the spatial Boltzmann factors do not vanish. The specific form of J enters in the remainder R, in the explicit witness of Proposition 13, and in the value of Theorem 21(5).

What is classical and what is new. Theorem 6 is the congruence theorem on a line; Proposition 3 is the addition formula for cosh; Proposition 1 is the classical solution of the cosine equation with a calibration; Theorems 14 and 17 are coefficient forms of the classification of cosine functions on the line and the circle; Theorem 18 is the identity $\operatorname{Re}\operatorname{tr} U = \sum_j \cos \alpha_j$; Proposition 19(1) is the Cayley–Hamilton theorem, and the classification of continuous solutions on compact groups is due to Davison and Yang [12, 13]; the Laplacian, its dispersion, transfer-matrix energy and Green kernel, and Wilson's action are standard; Theorem 11(4) and (5) follow from convexity; Proposition 12 is the Brascamp–Lieb argument for anharmonic crystals [14, 15] applied to S, a gradient model with convex potential $\cosh -1$ [17, 18, 19]; for K_β, Proposition 13(1) is equivalent to the sign changes of $K_{ip}(\beta)$ [21], and a positive transfer matrix for Wilson's action is classical [16]; Theorem 21 is elementary. Since $J(q_i/q_j)+1 = \cosh(t_i - t_j)$ is minus the Minkowski product, in signature $(-,+)$, of the points $(\cosh t, \sinh t)$ of the hyperbola H^1, Theorem 6 says that the isometries of H^1 are translations and reflections, and S is the nearest-neighbour H^1 sigma model, whose bond term is the t-part of the bond term of the $H^{2|2}$ model [22, 23, 24]; the full measures differ. The contribution is the organization of these classical results under explicit premises, through one cost and one composition law, and an elementary statement and proof of Proposition 13(1): no nonconstant real entire function of the reading gives a positive-definite time kernel on the line. It also follows from Bochner's theorem and the theory of analytic characteristic functions [20, Section 7.1]: for a real, even, positive-definite entire f, its symmetric Bochner measure ν gives $f(iy) = \int \cosh(py)\, d\nu(p) > f(0)$ for $y \neq 0$ unless ν is a nonnegative measure supported at 0; the period $2\pi i$ of the reading gives $f(2\pi i) = f(0)$, so f is constant. Its Boltzmann case is equivalent to the known sign changes of $K_{ip}(\beta)$, and the Gaussian counterexample shows that analyticity on the whole plane is what it uses. The presentation brings the results together through these identifications: J at the shift is half the second difference and $-2\sum_\mu J(T_\mu)$ is the Hessian of the total cost at balance; $\frac{1}{N}\operatorname{tr} J(U) = -W(U)$ on unitary holonomies, while the law on the whole group separates $SU(2)$ from $SU(N)$, $N \geq 3$; J of counts against a unit reads counted states exactly; and the Hessian bound $\cosh \geq 1$ places the full weight under the Brascamp–Lieb comparison with its quadratic part.

What remains. The premises leave the following questions open.

1. *Observational completeness.* (P1) assumes that the available examinations are functions of the profile. Corollary 9 is conditional on that invariance; on the counted domain injectivity concerns the chosen variables. The precise question is whether additional

examinations distinguish a common scale or the global inversion; after restriction to scale-invariant examinations only the inversion remains; on lattice fields, if inversion is additionally implemented by projecting the Gaussian Fock space onto invariant vectors, only even total particle number survives; that implementation is not derived here.

2. *The nonlinear theory.* A useful characterization of the generator $H = -\log|T|$, its relation to the centered ensemble and to the quadratic part, and the large-scale limit of $e^{-\beta S}$; the cited homogenization results assume a second derivative bounded above.

3. *The joint law of interacting recognizers.* The cost fixes the readings of each comparison. It does not fix the correlations between recognizers, the systems that make comparisons, that have not compared with each other; Proposition 3 supplies no independence premise for them.

4. *The law of posting.* Which post occurs, and with what weight, is not fixed by the cost; Proposition 22 shows which posts are genuine alternatives.

5. *Placement and dimension.* The cost prices comparisons; it does not say which quantities are compared. The lattice, its spatial dimension three (spacetime dimension four in Section 4.3), the gauge group and the placement of recognizers are inputs here, and no result links the ledgers to the lattice field.

6. *Normalization.* The law constrains the cost of each comparison and any weight placed inside it. The coupling β, external bond weights a_μ and the couplings of different sectors are inputs.

9 Conclusion

By (P0) one function prices every comparison of positive quantities, and its composition law, imposed on step operators and on holonomies, selects their costs, given unit calibration and, for holonomies, the sign premise. Under explicit premises (that states are profiles of readings, that the total cost of a lattice field is its Euclidean action, that step and holonomy costs obey the composition law with the eigenvalue prescription for nonabelian holonomies, and that a counted system contains a unit item and integer counts) the following hold. The state is the profile of readings, exact on the counted domain. The Euclidean lattice equation of the quadratic part has the cost of the shift as its operator: nearest-neighbour by (P2), and independently by (P3) with calibration. Its dispersion is $\sinh^2(E/2) = \sum_j \sin^2(k_j/2)$, with $E \le |k|$ and, where the dispersion is differentiable, group velocity below one in lattice units. The Hessian of the full action satisfies $\operatorname{Hess} S(t) \ge 2A$, so on the mean-zero hyperplane the Brascamp–Lieb inequality bounds linear variances by those of the Gaussian action; and no nonconstant real entire function of the reading gives a positive-definite time kernel on the line, so the standard one-step transfer operator on the unrestricted slice space is not e^{-H} for a self-adjoint H, while on the circle Wilson's kernel is positive. Imposed on the $U(1)$ cost, the law gives Wilson's action; imposed on the whole group, among nonconstant continuous solutions on $SU(2)$ it gives the half-trace, while for $N \ge 3$ its only continuous solutions on $SU(N)$ are $G = 0$ and $G = 1$; the $U(N)$ action $1 - \frac{1}{N}\operatorname{Re}\operatorname{tr} U$ follows from the separate eigenvalue prescription. The holonomy cost enters the action with the opposite sign from (P2), and

that sign is a premise. The posting order has a least readable ledger, in which every count equals one, fixed by the offset in (P4). What remains is a list of open questions: whether pairwise comparisons exhaust the examinations of a state, what the transfer generator and large-scale limit of the nonlinear weight are, how recognizers interact, which posts occur, where recognizers sit, and how strongly each sector is weighted.

References

[1] J. Washburn and M. Zlatanović, Uniqueness of the canonical reciprocal cost, *Mathematics* **14**(6), 935 (2026). doi:10.3390/math14060935.

[2] J. Aczél, *Lectures on Functional Equations and Their Applications*, Academic Press, New York (1966).

[3] H. Stetkær, *Functional Equations on Groups*, World Scientific, Singapore (2013).

[4] Pl. Kannappan, The functional equation $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ for groups, *Proc. Amer. Math. Soc.* **19**, 69–74 (1968). doi:10.1090/S0002-9939-1968-0219936-1.

[5] L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Clarendon Press, Oxford (1953).

[6] I. Montvay and G. Münster, *Quantum Fields on a Lattice*, Cambridge University Press, Cambridge (1994).

[7] B. A. Berg and Z. McDargh, On the energy momentum dispersion in the lattice regularization, *J. High Energy Phys.* **12**, 042 (2012). arXiv:1207.0320.

[8] F. Spitzer, *Principles of Random Walk*, D. Van Nostrand, Princeton (1964).

[9] G. F. Lawler and V. Limic, *Random Walk: A Modern Introduction*, Cambridge Studies in Advanced Mathematics **123**, Cambridge University Press, Cambridge (2010).

[10] J. Villain, Theory of one- and two-dimensional magnets with an easy magnetization plane. II. The planar, classical, two-dimensional magnet, *J. Phys. (Paris)* **36**, 581–590 (1975). doi:10.1051/jphys:01975003606058100.

[11] K. G. Wilson, Confinement of quarks, *Phys. Rev. D* **10**, 2445–2459 (1974).

[12] T. M. K. Davison, D'Alembert's functional equation on topological groups, *Aequationes Math.* **76**, 33–53 (2008). doi:10.1007/s00010-007-2916-4.

[13] D. Yang, Factorization of cosine functions on compact connected groups, *Math. Z.* **254**, 655–674 (2006).

[14] H. J. Brascamp and E. H. Lieb, On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation, *J. Funct. Anal.* **22**, 366–389 (1976).

[15] H. J. Brascamp, E. H. Lieb and J. L. Lebowitz, The statistical mechanics of anharmonic lattices, *Bull. Int. Statist. Inst.* **46**, 393–404 (1975).

[16] M. Lüscher, Construction of a selfadjoint, strictly positive transfer matrix for Euclidean lattice gauge theories, *Commun. Math. Phys.* **54**, 283–292 (1977).

[17] T. Funaki and H. Spohn, Motion by mean curvature from the Ginzburg–Landau $\nabla\phi$ interface model, *Commun. Math. Phys.* **185**, 1–36 (1997).

[18] A. Naddaf and T. Spencer, On homogenization and scaling limit of some gradient perturbations of a massless free field, *Commun. Math. Phys.* **183**, 55–84 (1997).

[19] G. Giacomin, S. Olla and H. Spohn, Equilibrium fluctuations for $\nabla\phi$ interface model, *Ann. Probab.* **29**, 1138–1172 (2001).

[20] E. Lukacs, *Characteristic Functions*, 2nd ed., Griffin, London (1970).

[21] T. M. Dunster, Bessel functions of purely imaginary order, with an application to second-order linear differential equations having a large parameter, *SIAM J. Math. Anal.* **21**, 995–1018 (1990).

[22] T. Spencer and M. R. Zirnbauer, Spontaneous symmetry breaking of a hyperbolic sigma model in three dimensions, *Commun. Math. Phys.* **252**, 167–187 (2004).

[23] M. Disertori, T. Spencer and M. R. Zirnbauer, Quasi-diffusion in a 3D supersymmetric hyperbolic sigma model, *Commun. Math. Phys.* **300**, 435–486 (2010).

[24] C. Sabot and P. Tarrès, Edge-reinforced random walk, vertex-reinforced jump process and the supersymmetric hyperbolic sigma model, *J. Eur. Math. Soc.* **17**, 2353–2378 (2015).