Linearized Einstein gravity from Lorentzian Regge calculus on the Kuhn lattice, and the divergence of its one-loop graviton limit

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Abstract

We study a real-time path sum over squared edge lengths on the Kuhn triangulation of the periodic hypercubic lattice, with the Lorentzian Regge action and the flat metric $I - \kappa \mathbf{1}\mathbf{1}^{\top}$, at $\kappa = 1$, where the edges that advance one coordinate are null. The value $\kappa = 1$ is an identification, motivated by an order on those edges. Classically, linearized Einstein gravity is recovered as a long-wavelength limit of lattice solutions. On every torus of odd side not divisible by 105 the flat Hessian has no zero modes beyond constant strains and vertex moves. For band-limited conserved sources off the light cone, the long-wave limit of the linear lattice field is the de Donder solution, with an explicit stability constant. With a resting body and a model medium, the unconstrained Regge equations on all edges have exact solutions for small coupling; their weak-field long-wave limit is that field and gives Newton's potential with $G = 1/(8\pi s)$, where s is the action scale. These solutions are related to the path sum on its null stratum at linear order. At fixed source strength the long-wave equations are controlled uniformly in the torus size, while a lattice resonance rules out uniform stability at all wavelengths. At tree level the lattice has no observable signature for spacings up to 10^6 Planck lengths. At one loop, in a stated diagram model, the quantum limit fails. Four exact massless doublers at the corners of the Brillouin zone, whose squared mass is linear in the background strain, make the zero-momentum anisotropy of the graviton self-energy diverge like $\frac{5}{44\sqrt{335}\,\pi^2}\log(1/\varepsilon)$ as the regulator ε is removed after the volume, because every other degenerate zero of the operator is non-pinched; the proof's computations are exact. The long-wave self-energy also has a remainder that no gauge choice, ghost or covariant factor removes; what it means physically depends on whether the real-time path sum has a unitary evolution, which is not established. The one-loop two-mass amplitude assembled from the long-wave Feynman rules, which are the Einstein–Hilbert rules in lattice coordinates, has the coefficient $-\frac{41}{10\pi}G^2\hbar m_1 m_2/r^3$ in an infrared-subtracted Born reading. The lattice path sum at $\kappa = 1$ is not shown to inherit it: the doublers couple to the static field of a mass without derivatives, and in an exact local model their pair loop

carries a logarithm that would add a $\log q^2/q^4$ term to the static amplitude, more singular than the one that carries $41/(10\pi)$, unless the rest of the Brillouin zone cancels it. The path-sum measure is chosen, not derived.

1 Gravity at a fixed lattice spacing

Can Einstein gravity be obtained as the long-wavelength limit of a real-time sum over geometries on a lattice whose spacing is physical? We study one concrete lattice: the Kuhn triangulation of the periodic hypercubic lattice [2, 3], the triangulation with which Roček and Williams began linearized Regge calculus [5, 18], with the Lorentzian Regge action [1] and the flat metric $g_\kappa = I - \kappa \mathbf{1}\mathbf{1}^\top$ in lattice coordinates. At $\kappa = 1$ the edges that advance one coordinate are null, every other edge is timelike, and time runs along the main diagonal. The weight is e^{iaS}, with no Wick rotation. A complete answer needs eight things: a configuration space with its causal conditions, an action with its Lorentzian sign, a measure, a well-defined sum, a classical limit in which solutions (not only the action) converge to Einstein solutions, a source law that reproduces Newton's potential with G fixed by the action scale, at least one computed quantum quantity, and a prediction or an explicit statement that pure gravity predicts nothing observable. Table 1 says which of these are theorems, which are theorems under a stated model or identification, and which are open. Each result is stated in its section with the hypotheses it uses.

Which limit. The lattice spacing is physical. The path sum is finite only at a stated cutoff and diverges without one (Section 3.2), so it is an effective theory at fixed spacing and claims no ultraviolet completion. Shrinking the spacing at fixed combinatorics does not help: rescaling every squared length rescales the Regge action and leaves its stationary points unchanged. The limit in which Einstein gravity should appear is therefore a long-wavelength limit, with waves and sources spread over many spacings of a lattice whose spacing stays put. Where two limits are taken their order is stated; for the quantum statements it is fixed throughout: first the torus grows at a fixed positive regulator, then the regulator is removed.

The value of κ. The motivation for the lattice is a programme in which events are records kept on four counters advanced one step at a time [4]; the one-step edges, the *postings*, carry an order. That order is causal for the lattice interval exactly when $\kappa \geq 1$, and the interval reads it on an interval of positive duration only at $\kappa = 1$ (Section 2.2). That $\kappa = 1$ is the physical value is an identification, not a consequence of the definitions; we write *identified* $\kappa = 1$ and keep it visible. The motivation enters nowhere else, except as the source of one candidate construction of the measure (Section 3.6).

What is proved. Classically, linearized Einstein gravity is recovered as the long-wavelength limit of lattice solutions (Section 4). On every torus of odd side not divisible by 105 the flat Hessian has no zero modes beyond constant strains and vertex moves; for band-limited conserved sources off the light cone the long-wave limit of the linear lattice field is the de Donder solution, with an explicit stability constant; and with a resting body and a model

Table 1. The eight requirements and their status. "Theorem" means proved as stated in the section named; qualifiers name what the statement rests on.

requirement	status	where
1 configurations	fixed Kuhn graph with varying squared lengths and explicit causal conditions; the null stratum at identified $\kappa = 1$	Sections 2 and 3.1
2 action and sign	Regge action; Lorentzian sign from the causal range of the posting order, given the length family	Section 3.1
3 measure	open: chosen, not derived; at Gaussian order its gauge behaviour and density independence are theorems	Section 3.6
4 well-definedness	theorem at a stated cutoff; the sum diverges without it	Section 3.2
5 classical limit	theorem at linear order: exact solutions of the unconstrained Regge equations with a model medium, on the odd sides not divisible by 105, in the weak-field order; at fixed strength, theorems for the long-wave equations, uniformly in the torus size	Section 4
6 source	theorem: the linearized Einstein field of the source and Newton's law for compact bodies, $G = 1/(8\pi s)$, with the model medium	Sections 3.5 and 4
7 quantum quantity	theorem for the long-wave Feynman rules (the Einstein–Hilbert rules in record coordinates): the coefficient $41/(10\pi)$ in an infrared-subtracted Born reading. Not established for the lattice path sum at $\kappa = 1$: its corner doublers couple to the static field (theorem) and their pair loop carries $\log q^2$ (theorem in an exact local model), which would add a more singular term unless the rest of the zone cancels it	Section 7
8 prediction	tree level: no signature, with explicit bounds (theorem). Quantum part not met: the ordered one-loop graviton limit diverges at identified $\kappa = 1$ (theorem within a stated diagram model)	Sections 5 and 6

medium the unconstrained Regge equations have exact solutions for small coupling, whose weak-field long-wave limit is that field and gives Newton's potential with an explicit finite-size remainder. These solutions satisfy the Regge equations on all fifteen edge classes and are related to the path sum on its null stratum at linear order. At fixed small strength the long-wave equations are controlled uniformly in the torus size, while a lattice resonance rules out uniform stability at all wavelengths. At tree level every long-wave mode in every direction propagates on the light cone up to a relative correction of order $(k_{\mathrm{ph}} b)^2$, so there is no observable signature for spacings up to 10^6 Planck lengths (Section 5).

At one loop, within a stated diagram model, the quantum limit fails at identified $\kappa = 1$ (Section 6). The gauge-fixed operator has four exact massless doublers at the axis corners of the Brillouin zone, whose squared mass is linear in the background strain. Their pair loop makes the zero-momentum anisotropy of the graviton self-energy diverge like $\frac{5}{44\sqrt{335}\,\pi^2}\log(1/\varepsilon)$ in the ordered limit, because every other degenerate zero of the operator is not pinched; the proof's computations are exact. The long-wave self-energy also has a remainder that no gauge

choice, ghost or covariant factor removes; it is large and imaginary numerically, and what it means physically depends on whether the real-time path sum has a unitary evolution, which is not established. Finally, the one-loop two-mass amplitude assembled from the long-wave Feynman rules has the coefficient $-\frac{41}{10\pi}G^2\hbar m_1 m_2/r^3$ in an infrared-subtracted Born reading (Section 7). This is a statement about the long-wave rules, which are the Einstein–Hilbert rules in lattice coordinates. The lattice path sum at $\kappa = 1$ is not shown to inherit it: the doublers couple to the static field of a mass without derivatives, and in an exact local model their pair loop carries a logarithm at nonzero momentum, which would add a more singular term to the static amplitude unless the rest of the zone cancels it.

Relation to other work. Regge calculus has a long history as a discretization of general relativity [1, 28, 29, 24]. Its linearization about flat space on this lattice, its vertex-translation zero modes and its continuum limit were studied by Roček and Williams [5, 18], Barrett and Williams [19, 20], Christiansen [21], and in the finite-element literature [74, 73]; its gauge modes and their breaking away from flat space by Hamber and Williams [22], Dittrich, Freidel and Speziale [26], Morse [27], Bahr and Dittrich [9, 25] and Dittrich [72]; and the Newtonian potential of quantum Regge gravity, with a Euclidean measure, by Hamber and Williams [23]. Lorentzian Regge calculus goes back to Sorkin [30, 31], and Lorentzian simplicial path integrals are an active subject [32, 33, 34, 35]; causal dynamical triangulations sum over triangulations and use a Wick rotation [36, 37, 38, 39]. The present path sum does neither: it is real time on a fixed graph. What is new here is the Lorentzian lattice with null postings and its consequences: the exact flat kernel on an infinite family of tori; bounds uniform on every non-null cone; exact solutions with matter on every edge and their limits; bounds uniform in the torus size at fixed strength; the tree-level light cone in every direction with explicit constants; and, at one loop, exact bosonic doublers that make the ordered graviton limit diverge, a feature of real-time discretizations whose light cone reaches the boundary of the Brillouin zone. The one-loop correction to the Newtonian potential is the result of Donoghue [10] and of Bjerrum-Bohr, Donoghue and Holstein [11]; what we add is its derivation from the long-wave rules of this lattice in a precise infrared-subtracted reading, and the proof that natural alternative readings are ill-posed.

The paper is organized so that its parts can be read separately: the lattice and κ (Section 2), the path sum (Section 3), classical recovery (Section 4) and tree-level predictions (Section 5); the one-loop graviton (Section 6 and Appendix G); the one-loop coefficient (Section 7); and what remains (Section 8). Some proofs rely on exact computations whose data are too long to print; each such dependence is stated where it occurs, and the data, with independent programs that check them in exact arithmetic, are described at the end of the paper.

2 The lattice and the value of κ

2.1 The Kuhn lattice

Vertices are points $x \in (\mathbb{Z}/N)^4$, the side-N torus, or points of $\mathbb{Z}^4$ when no finite action is needed. We call the coordinates x_a *record coordinates* and $t_{\mathrm{r}}(x) = \sum_a x_a$ the *record time* of a vertex, for the reason given in Section 2.2. Write e_a for the unit vectors and $\mathbf{1} = (1, 1, 1, 1)$. Each unit cube is cut into 24 four-simplices by Kuhn's rule [2, 3]: for each ordering $(a_1, \dots, a_4)$

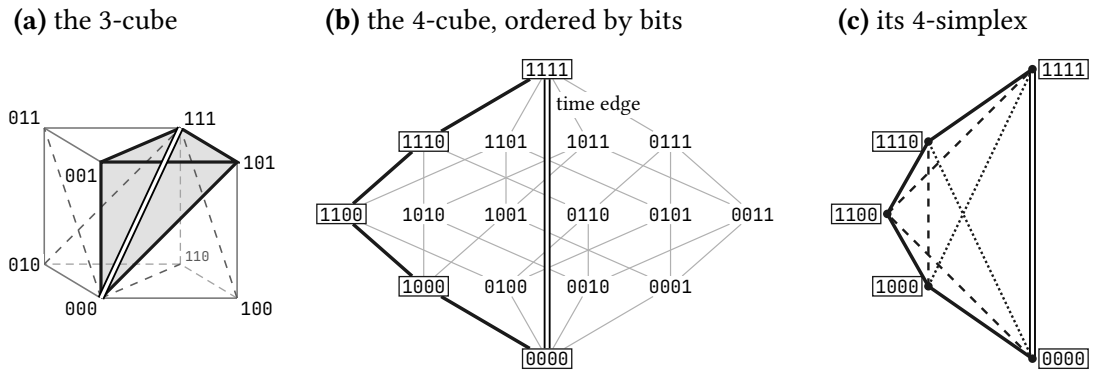

Figure 1. The Kuhn triangulation. (a) Kuhn's rule in three dimensions: each ordering of the axes gives a tetrahedron from `000` to `111`, and all six share the main diagonal (double line). (b) The four-dimensional unit cube ordered by inclusion of bits; a Kuhn simplex is a chain from `0000` to `1111`, and the time edge $(1,1,1,1)$ (double line) joins the ends of all 24 chains. (c) The ten edges of the highlighted simplex. At $\kappa = 1$ the solid one-bit edges are null and every other edge is timelike.

of the four axes, the simplex has the vertices x, $x+e_{a_1}$, $x+e_{a_1}+e_{a_2}$, $x+e_{a_1}+e_{a_2}+e_{a_3}$ and $x+\mathbf{1}$ (Fig. 1). Every edge joins x to $x+d$ for one of the fifteen *edge classes* $d \in \{0,1\}^4 \setminus \{0\}$; we write $e=(x,d)$ for that edge and $|d|$ for the number of ones in d. There are four classes with $|d|=1$, six with $|d|=2$, four with $|d|=3$ and the main diagonal $\mathbf{1}$. An edge with $|d|=1$ is a *posting*. This is the triangulation of the hypercubic lattice with which Roček and Williams began the study of linearized Regge calculus [5, 18]; here it carries a Lorentzian metric.

A *configuration* is a real squared length ℓ_e on every edge. The flat lattice of parameter κ has the metric

$$g_\kappa = I - \kappa\,\mathbf{1}\mathbf{1}^\top, \qquad g_\kappa(d,d) = |d|\,(1-\kappa|d|), \qquad \det g_\kappa = 1-4\kappa, \tag{1}$$

in record coordinates, and its squared lengths are $\ell^0_{(x,d)} = |d|(1-\kappa|d|)$. At $\kappa=1$ they are $0,-2,-6,-12$ for $|d|=1,2,3,4$: postings are null, every longer edge is timelike, and $\det g_1 = -3$. Time runs along the main diagonal; the unit future time vector is $\hat t = \mathbf{1}/(2\sqrt{3})$, with $g_1(\hat t,\hat t)=-1$. For $\kappa \neq \frac14$, $g_\kappa^{-1} = I + \frac{\kappa}{1-4\kappa}\mathbf{1}\mathbf{1}^\top$, and the *wave symbol* of g_κ is

$$q_\kappa(u) = g_\kappa^{-1}(u,u) = \sum_a u_a^2 + \frac{\kappa}{1-4\kappa}\Big(\sum_a u_a\Big)^2.$$

At $\kappa=1$ we write

$$q(u) = q_1(u) = \sum_a u_a^2 - \tfrac13\Big(\sum_a u_a\Big)^2, \tag{2}$$

which vanishes on the light cone. A covector with $\sum_a u_a = 0$ is *static*, constant along the time direction, and on nonzero static covectors $q_\kappa(u) = \sum_a u_a^2 > 0$.

For every $\kappa > \frac14$ the metric has exactly one negative direction, along $\mathbf{1}$. For every $\kappa > \frac12$ every triangle of the flat lattice is timelike, and each dihedral angle of a flat simplex at a triangle is a rotation angle in $(0,\pi)$; the null postings at $\kappa=1$ do not make any triangle null. Results proved for every $\kappa > \frac12$ therefore use a nondegenerate Lorentzian Regge action.

What is given. Three features of the lattice are inputs here: the Kuhn triangulation with one common diagonal orientation, the family of squared lengths $n(1-\kappa n)$, and the placement

of time on the main diagonal. The configuration space is a fixed graph with varying squared lengths; we never sum over triangulations. The value of κ is the one number of the lattice, and most theorems below use it, so we treat it separately.

2.2 The posting order and the value of κ

The motivation for this lattice comes from a programme in which physical events are records kept on four counters, each advanced by one step, a posting, at a time [4]. Four counters advanced one at a time from $(0,0,0,0)$ to $(1,1,1,1)$ have $4! = 24$ arrival orders, and these are exactly the 24 Kuhn simplices of a cell: a Kuhn simplex is a maximal chain in $\{0,1\}^4$ ordered by inclusion. This is the reason for the triangulation and for the names above. In the results the motivation enters through the choice of κ, by way of the following order, and otherwise only as the source of one candidate construction of the measure (Section 3.6).

An *event* is a pair $e = (t, x)$ of a time $t \in \mathbb{R}$ and a position x in the hyperplane $\{x \in \mathbb{R}^4 : \sum_a x_a = 0\}$. A posting on axis a advances the time by a fixed $\tau > 0$ and the position by $\sigma_a = e_a - \frac{1}{4}\mathbf{1}$. The *posting order* is the closure of the relation generated by chains of postings, at every dyadic refinement of the step (Appendix A). For an ordered pair of events (e, f) write its *content*

$$c_a(e,f) = (x_f - x_e)_a + \frac{t_f - t_e}{4\tau}, \qquad a = 1, \dots, 4,$$

the number of postings on each axis that the pair contains. The posting order is the orthant of contents: (e, f) is in it exactly when every $c_a(e, f) \geq 0$ (Appendix A). In contents the lattice interval is

$$Q_\kappa(w) = \sum_a w_a^2 - \kappa\Big(\sum_a w_a\Big)^2,$$

which is g_κ in record coordinates. A content w is *future causal* for Q_κ when $Q_\kappa(w) \leq 0$ and $\sum_a w_a \geq 0$.

Theorem 2.1 (The causal range)**.** *Let $\tau > 0$. Every pair in the posting order has future-causal content for Q_κ if and only if $\kappa \geq 1$. Hence $\kappa = 1$ is the least such value.*

Proof. A single posting has content e_a, and $Q_\kappa(e_a) = 1 - \kappa$, which is ≤ 0 only when $\kappa \geq 1$. Conversely, let $\kappa \geq 1$ and let w lie in the orthant. Then $\sum_a w_a \geq 0$, and $\sum_a w_a^2 \leq (\sum_a w_a)^2$ because all cross terms are nonnegative, so $Q_\kappa(w) \leq (1-\kappa)(\sum_a w_a)^2 \leq 0$. □

There is a plain way to see the number. The position moves by $|\sigma_a| = \sqrt{3}/2$ in each posting, and an interval of the form $|\Delta x|^2 - v_c^2(\Delta t/\tau)^2$, written in contents, is Q_κ with $\kappa = \frac{1}{4} + v_c^2$, because $|\Delta x|^2 = \sum_a w_a^2 - \frac{1}{4}(\sum_a w_a)^2$ and $\Delta t/\tau = \sum_a w_a$. So Theorem 2.1 says that the cone speed v_c, in units of position per tick, is at least the posting speed $\sqrt{3}/2$, and $\kappa = 1$ is the statement that the two are equal. The theorem bounds κ from below; it does not select $\kappa = 1$. A reading that does select it is the following. Say that Q_κ *reads the order* on a pair (e, f) if for every event g, the contents of (e, g) and (g, f) are both future causal for Q_κ exactly when (e, g) and (g, f) are both in the posting order: the interval's notion of "g lies causally between e and f" agrees with the order's.

Theorem 2.2 (Reading the posting order)**.** *Let $\tau > 0$ and $\kappa \in \mathbb{R}$.*

(i) *There is a pair (e, f) in the posting order with $t_e < t_f$ on which Q_κ reads the order if and only if $\kappa = 1$.*

(ii) *At $\kappa = 1$, a pair in the posting order is read exactly when its interval is null, $Q_1(c(e, f)) = 0$, that is, when its content is supported on a single axis.*

(iii) *No κ reads the order on every pair of the posting order.*

Proof sketch. For a single-axis content $v = me_a$ the order interval between 0 and v is the segment $[0, v]$, and Q_1 vanishes on that segment, so the Q_1-causal interval between 0 and v is the same segment. For $\kappa > 1$ the causal interval of a posting contains points off the segment, and for $\kappa < 1$ a posting is not causal at all (Theorem 2.1). For content spread over two or more axes the order interval is a box $\{0 \le z_a \le v_a\}$ with at least two sides, while the Q_κ-causal interval is a quadric region, never a box. □

Part (iii) has a simple geometric cause. The posting order is the closed orthant of contents, a polyhedral cone, while the future cone of Q_κ is round. At $\kappa = 1$, $Q_1(w) = -2\sum_{a<b} w_a w_b$, so on the orthant the interval vanishes exactly on the four posting rays, and the orthant never fills the future cone: the content $(2, 2, 2, -1)$ has $Q_1 = -12$ and positive sum, so it is future timelike, but no chain of postings reaches it. Part (i) is a weak criterion in one sense: at $\kappa = 1$ the pairs that are read are the null single-axis pairs, so what it selects is that postings are null.

The identification. Theorem 2.2 is mathematics about a reading and an order. It becomes the physical value $\kappa = 1$ through one identification: *the physical interval of the lattice reads the posting order on at least one interval of positive duration.* Under it every posting is null and runs along the null cone of the interval, the cone on which the long-wave graviton propagates (Theorem 5.1). Nothing in the definitions forces this. The posting order is the same orthant for every κ, and every $\kappa \ge 1$ makes it causal (Theorem 2.1), so no condition on the order alone can single out $\kappa = 1$; the identification is an additional physical premise: in the terms above, that the cone speed equals the posting speed. We argue for $\kappa = 1$ from the posting order and the graviton's cone only. Whether light shares that cone is a question about the photon sector, which this paper does not address.

We therefore write *identified* $\kappa = 1$ throughout. Every theorem stated at $\kappa = 1$ inherits the identification; theorems stated for $\kappa > 1/2$ do not depend on it. Of those, only $\kappa \ge 1$ is causal in the sense of Theorem 2.1.

2.3 Notation and units

The lattice carries no length unit of its own: squared lengths, waves and Brillouin-zone integrals are in *lattice units*, in which the edge classes have the squared lengths above and waves run over $[-\pi, \pi]^4$. The path-sum weight is e^{iaS}, with the *phase scale* a; the Regge action enters the field equations with an *action scale* s; and Section 3.5 shows that the lattice Newton constant is $G = 1/(8\pi s)$. Reading $\hbar = 1/a$ and $c = 1$ are definitions. Physical lengths enter only through the lattice spacing b, compared with the Planck length ℓ_P in Section 5. Table 2 lists the symbols that recur across sections; symbols used within one proof are defined there.

Table 2. Notation used across sections.

symbol	meaning	where
N; $e=(x,d)$; $\lvert d\rvert$	torus side; edge from x to $x+d$, class $d\in\{0,1\}^4\setminus\{0\}$; number of ones in d	Section 2
$\mathbf{1}$; g_κ; ℓ^0	$(1,1,1,1)$; flat metric $I-\kappa\mathbf{1}\mathbf{1}^\top$; flat squared lengths	Eq. (1)
q_κ, q	wave symbol $g_\kappa^{-1}(u,u)$; $q=q_1$	Eq. (2)
t_{r}; $\hat{t}$	record time $\sum_a x_a$; unit time vector at $\kappa=1$	Section 2
ℓ_e; S; A_h, $\theta_{\sigma,h}$, δ_h	squared length; Regge action; area, dihedral angle, deficit	Section 3.1
a; s; G; b	phase scale ($\hbar=1/a$); action scale; $1/(8\pi s)$; lattice spacing	Section 3.5
$\mathscr{A}_N(\kappa)$; null stratum	admissible configurations; posting-null configurations at $\kappa=1$	Section 3.1
H_0; $\mathscr{R}_N$	flat Hessian $D^2S(\ell^0)$ on the torus; constant strains plus vertex moves	Section 4
H	a constant strain (symmetric 4×4 matrix), profile $d\mapsto d^\top Hd$	Sections 4 and 6
$B(k)$, $\Gamma(k)$, $A(k)$	Bloch matrix of H_0, vertex-move columns, gauge-fixed operator	Section 4.2
ε	damping regulator of the path sum; limits $N\to\infty$, then $\varepsilon\to0^+$	Section 3.3
σ; λ; α	matter coupling; medium amplitude; fixed source strength	Section 4
J	the cost $J(x)=\frac{1}{2}(x+x^{-1})-1$	Section 4.12
K; v_c; $\delta_c(f)$	doubler cone form; axis-corner zero mode; its coupling to a profile f	Section 6
$\Delta(\varepsilon)$; $D(\varepsilon)$	zero-momentum anisotropy; its corner term	Section 6
η; t	Feynman regulator of the two-mass amplitude; momentum-transfer scale	Section 7

3 The path sum

3.1 Configurations, action and weight

Fix $N\geq3$ and the side-N torus. A configuration is a real squared length ℓ_e on each of the $15N^4$ edges $e=(x,d)$; write $n_e=|d|$ for the number of postings the edge contains, and $\ell^0_e=n_e(1-\kappa n_e)$ for the flat field. Say ℓ is *admissible* if every Kuhn simplex is timelike in all its faces (the Gram determinants of its ten triangles, five tetrahedra and of the simplex itself are negative), the main-diagonal edge and every edge with $n_e\geq2$ are timelike, and every posting has squared length of the sign of $1-\kappa$. Write $\mathscr{A}_N(\kappa)$ for this set. For $\kappa>1/2$ it contains the flat field and is a cone; for $\kappa>1$ the postings are timelike as well. At $\kappa=1$ the posting clause says that every posting is null, so

$$\mathscr{A}_N(1)=\{\ell\text{ admissible}\ :\ \ell_e=0\text{ for every posting }e\},$$

the *null stratum*. It has empty interior and Lebesgue measure zero in $\mathbb{R}^{15N^4}$; its natural measure is Haar measure on the subspace of posting-null fields.

The action is Regge's [1],

$$S(\ell)=\sum_{\text{slots }(\sigma,h)}A_h(\ell)\Big(\frac{2\pi}{n_h}-\theta_{\sigma,h}(\ell)\Big),\tag{3}$$

summed over the $240N^4$ pairs of a Kuhn simplex σ and one of its ten triangles h, with $A_h = \frac{1}{4}\big((\ell_1+\ell_2-\ell_3)^2 - 4\ell_1\ell_2\big)^{1/2}$ the area of the timelike triangle with squared edge lengths ℓ_1, ℓ_2, ℓ_3, $\theta_{\sigma,h}$ the dihedral angle of σ at h and n_h the number of simplices sharing h. It is homogeneous of degree one, $S(c\ell) = cS(\ell)$ for $c \geq 0$, and smooth on the admissible set. The weight is e^{iaS}, with a the phase scale. Two facts single out this action among local hinge actions: additivity over areas and deficits together with continuity forces the bilinear form $\sum_h A_h\delta_h$ in the areas and the deficits δ_h up to its coupling; and an even function of the deficit cannot agree with the signed Regge action at both signs of the deficit when the coupling, the area and the deficit are nonzero. Replacing the deficit by $\sinh\delta_h$ would also break the Schläfli cancellation that gives the Regge equations (Theorem 4.1).

The Lorentzian sign is not inserted separately. Within the length family $n(1-\kappa n)$, the causal-range theorem (Theorem 2.1) forces $\kappa \geq 1$ from the posting order alone, and every $\kappa > 1/4$ gives $\det g_\kappa = 1 - 4\kappa < 0$: one time direction, along the main diagonal. What is an input is the family itself and the Kuhn graph (Section 2).

3.2 Finite at a stated cutoff

For $L > 0$ let $\mathrm{box}_N(L) = \{\ell : |\ell_e| \leq L \text{ for all } e\}$.

Theorem 3.1 (The cutoff path sum). *Let $\|\ell^0\|_\infty < L$ and let μ be Lebesgue measure on $\mathscr{A}_N(\kappa) \cap \mathrm{box}_N(L)$ when $\kappa > 1/2$, $\kappa \neq 1$, or Haar measure on the posting-null subspace restricted to $\mathscr{A}_N(1) \cap \mathrm{box}_N(L)$ when $\kappa = 1$. Then μ is finite and nonzero; for every a and every bounded measurable observable O the functions e^{iaS} and Oe^{iaS} are μ-integrable; $Z(a) = \int e^{iaS}\,\mathrm{d}\mu$ is continuous in a with $|Z(a)| \leq \mu(\mathit{all})$; and*

$$|a|\,C_N L < \frac{\pi}{2}, \qquad C_N = 120\sqrt{13}\,\pi N^4,$$

implies $Z(a) \neq 0$, so normalized expectations exist and $\langle 1\rangle = 1$.

The window condition is sufficient for $Z(a) \neq 0$, not necessary, and the measure is the restricted Lebesgue (or Haar) measure named in the statement, a choice; what is known about the measure is collected in Section 3.6.

Proof. On the box every triangle has area at most $\sqrt{13}\,L/4$ and every slot share $2\pi/n_h - \theta$ has absolute value at most 2π, so $|S| \leq 240N^4 \cdot \frac{\sqrt{13}}{4}L \cdot 2\pi = C_N L$. Under the window condition the phase aS stays in $(-\pi/2, \pi/2)$, so $\mathrm{Re}\,Z(a) \geq \cos(|a|C_N L)\,\mu(\mathrm{all}) > 0$. The measure is nonzero because a neighbourhood of the flat field inside the box meets the admissible set in positive measure (for $\kappa = 1$, positive Haar measure in the posting-null subspace). Integrability and continuity follow from boundedness of the integrand on a finite measure. □

Without the cutoff the sum diverges: the admissible set has infinite measure, e^{iaS} is not absolutely integrable on it for any a, and along every ray $c \mapsto c\ell$ the truncated integrals $\int_0^R e^{iacS(\ell)}\,\mathrm{d}c$ have no limit as $R \to \infty$, because the phase is linear in c. So the sum is defined only at a stated cutoff; it is an effective theory at fixed spacing and claims no ultraviolet completion.

The window is small. At the phase scale $a = \ln 2/(2\pi) \approx 0.11$ in lattice units, the value of the phase premise of Section 3.5 at unit action scale, and a cutoff above the flat field ($L > 12$ at

$\kappa = 1$), the product $aC_N L$ exceeds $10^3 N^4$. Whether $Z \neq 0$ there is not decided. For every fixed a there is a ball around the flat field on which every nonzero finite measure has $Z \neq 0$, but its radius shrinks as $|a|$ grows, and no single ball is known to work for all phase scales. Whether the cutoff is physical is also open.

3.3 Gaussian order: the regulated construction

The quantum statements of this paper are made at Gaussian order around the flat field, with three regulators kept explicit: a fluctuation scale $t \to 0^+$, a damping $\varepsilon > 0$, and the torus side N. Fix any set R of retained edges, coordinates $x \in \mathbb{R}^R$ on the fields that vanish off R, and the weight

$$w_{t,\varepsilon}(x) = \mathbf{1}_{\mathrm{adm}}\big(\ell^0 + t\imath x\big)\, \exp\Big(\frac{ia}{t^2}\big(S(\ell^0 + t\imath x) - S(\ell^0)\big)\Big)\, e^{-\varepsilon|x|^2/2},$$

with $\imath$ the zero extension and $\mathbf{1}_{\mathrm{adm}}$ the indicator of the *admissible cut*: the open conditions of Section 3.1, that every simplex is timelike in all its faces and every edge with $n_e \geq 2$ is timelike, with the postings left free. The cut is open and contains the flat field. If R contains no posting, every $\ell^0 + t\imath x$ is posting-null and the cut is the null stratum $\mathscr{A}_N(1)$ on the retained directions; if R contains a posting, as the twisted slice of Theorem 3.3(iii) does, the retained posting coordinates take both signs and the integral runs over configurations outside the null stratum, and Section 3.4 says how such sums relate to it. Let A_R be the restriction of the flat Hessian $H_0 = D^2 S(\ell^0)$ to R.

Theorem 3.2 (Gaussian and regulator limits)**.** *Let $\kappa = 1$, $N \geq 3$, $a \in \mathbb{R}$, $\varepsilon > 0$, and let R be any set of edges.*

(i) As $t \to 0^+$, the normalized moments $\int x_i x_j w_{t,\varepsilon} / \int w_{t,\varepsilon}$ converge to $\big((\varepsilon - iaA_R)^{-1}\big)_{ij}$. No invertibility of A_R is assumed. The same holds after multiplying the weight by any bounded measurable density ρ on edge space that is continuous at ℓ^0 with $\rho(\ell^0) \neq 0$; $\rho(\ell^0)$ cancels in the ratio.

(ii) Let A be real symmetric, $a \neq 0$, and $j = Au$. Then $j^\top(\varepsilon - iaA)^{-1} j \to (i/a)\, u^\top j$ as $\varepsilon \to 0^+$. Conversely, if the limit exists then $j \in \operatorname{range} A$.

Proof. (i) Since S is stationary at ℓ^0 on retained directions, $(S(\ell^0 + t\imath x) - S(\ell^0))/t^2 \to \frac{1}{2} x^\top A_R x$ pointwise; the admissible cut is open and contains the flat field, so the indicator tends to one. The integrand is dominated by $(1 + |x|^2) e^{-\varepsilon|x|^2/2}$, and dominated convergence applies to numerator and denominator. The limiting denominator is a damped Fresnel integral, $\int e^{\frac{ia}{2} x^\top A_R x - \frac{\varepsilon}{2}|x|^2} \mathrm{d}x = (2\pi)^{|R|/2} \det(\varepsilon - iaA_R)^{-1/2} \neq 0$, and the ratio of the two limits is the inverse of the precision $\varepsilon - iaA_R$. A density continuous at ℓ^0 contributes the factor $\rho(\ell^0)$ to both. (ii) Write $A = \sum_k \lambda_k v_k v_k^\top$. Then $j^\top(\varepsilon - iaA)^{-1} j = \sum_k (j \cdot v_k)^2 / (\varepsilon - ia\lambda_k)$. For $\lambda_k = 0$, $j \cdot v_k = u^\top A v_k = 0$; for $\lambda_k \neq 0$ the term tends to $(i/a)(j \cdot v_k)^2/\lambda_k = (i/a)(u \cdot v_k)(j \cdot v_k)$. Summing gives $(i/a) u^\top j$. If $j \cdot v_k \neq 0$ for some $\lambda_k = 0$, the term $(j \cdot v_k)^2/\varepsilon$ diverges. □

So the source-exchange observable $\langle (j \cdot x)^2 \rangle$ has the ordered limit $(i/a)\, j \cdot u$ whenever the source is $j = A_R u$ for an edge field u that solves the linearized Regge equations on the retained directions. In units of Section 3.5, $i/(as) = 8\pi i G\hbar$: this is the gauge-invariant tree-level exchange of linearized gravity. Which retained set R is physical is the question of the next subsection.

3.4 The null stratum and its gauge slice

At $\kappa = 1$ the configurations of the path sum are posting-null, so the natural choice is $R =$ all non-posting edges. We call the Gaussian construction on this choice the *native* one. Vertex moves act on edge fields by $\xi \mapsto (\delta\ell_\xi)_e = 2\sum_a d_a\big(\xi_a(x+d) - \xi_a(x)\big)$, for a covector field ξ on the vertices: this is the first-order change of the squared lengths of the flat lattice when its vertex x is displaced by $g_\kappa^{-1}\xi(x)$. A source j is *conserved* if it annihilates every vertex move. For a field h and an axis a, the *cycle total* along a posting fibre is the sum of h over the N postings $(x + ke_a, e_a)$, $k = 0, \dots, N-1$.

Theorem 3.3 (What the null stratum sees)**.** *Let* $\kappa = 1$.

(i) For invertible metric data, a field h can be moved into the posting-null subspace, the tangent space of the null stratum, by a vertex move exactly when every cycle total of h vanishes. No vertex move takes every field into it. Both statements concern the linear action of vertex moves on edge fields, and they hold for every κ.

(ii) On the side-3 torus, for every $a \neq 0$ there is a conserved source $j = H_0 u$, with u a solution of the linearized equations on all edges (an ambient *solution), such that the exchange computed on the posting-null subspace does not tend to $(i/a)j \cdot u$ as $\varepsilon \to 0^+$.*

(iii) Let the twisted *subspace consist of the fields that vanish on every posting except one closing posting per fibre. Every edge field is a twisted field plus a vertex move, so the twisted subspace is a complete linear gauge slice; its seam value on each fibre equals the fibre's cycle total. The decomposition is unique up to the* residual *vertex moves, those whose component ξ_a is constant along every a-fibre: they change no posting, so they lie in the slice. For every conserved source with an ambient solution, the twisted exchange equals the ambient gauge-invariant exchange, and at every $\varepsilon > 0$ the normalized ambient Gaussian with the regulator $\varepsilon(|x|^2 + |g|^2)/2$ in slice coordinates (x, g), with g the coordinates of a vertex move in a fixed complement of the residual moves, equals the twisted regulated exchange. The change of variables is linear, so its Jacobian is constant, and conserved sources annihilate the residual moves, so the exchange does not depend on the complement chosen.*

(iv) There are C and, for each $\varepsilon > 0$, $\mu > 0$, both independent of N, such that for every source supported at layer distance at least m_0 from the seam the twisted and posting-null regulated exchanges differ by at most $(4|a|C/\varepsilon^2)\, e^{-\mu m_0} \sum_e j_e^2$.

Proof sketch. (i) A vertex move changes a fibre's postings by a telescoping sum, so it does not change the cycle total; conversely a field with zero cycle totals is a gradient along each fibre and can be removed fibre by fibre. A class selector with nonzero total cannot be removed. (ii) uses (i): a source whose ambient response has nonzero cycle total has a component the posting-null precision cannot reach, and Theorem 3.2(ii) then fails on that subspace. (iii) Subtract the vertex move that zeroes every posting but the closing one; gauge invariance of H_0 and of conserved j lets the slice variables g factor out of the Gaussian. (iv) is a Combes–Thomas estimate: the Hessian couples only edges of a common simplex, with row sums bounded by C; conjugating $\varepsilon - iaA$ by $e^{\mu\rho}$, with ρ the layer distance, perturbs it by at most $|a|C(e^{\mu} - 1) \le \varepsilon/2$, so the weighted resolvent is bounded by $2/\varepsilon$ and the restriction costs the stated amount. □

Item (iii) is gauge fixing, and it is legitimate. Its physical content lies in one IDENTIFICATION: that the configuration domain of the path sum is this slice. The twisted configurations are not in the null stratum, because for nonzero holonomy the closing posting has nonzero squared length, and the identification of Section 2.2, which makes postings null, supports the posting-null stratum rather than the twisted one. Item (ii) is a fixed-N effect with the regulator removed at that N; item (iv) says that in the prescribed order, $N \to \infty$ first, the two strata agree for sources far from the seam. We return to the large-volume limit in Section 3.6.

Stationary points of the constrained sum. On the null stratum the stationarity conditions are not the unconstrained Regge equations. If $\ell - \ell^0$ is posting-null and $DS(\ell)y = j(y)$ for every posting-null y, then for every y

$$DS(\ell)\, y = j(y) + \sum_{x,a} y_{(x,e_a)}\, m_{x,a}, \qquad m_{x,a} = DS(\ell)(\delta_{x,a}) - j(\delta_{x,a}),$$

with one multiplier per fixed posting. At linear order the multiplier sees only the fibre cycle totals; a critical field solves the ambient linearized equations exactly when every posting coefficient vanishes; and a constant diagonal stress has no ambient solution, so every critical point then carries tension. On the twisted slice the multiplier is the action's gauge defect, and critical points near flat solve the Regge equations to first order: for every $\delta > 0$ there is $r > 0$ such that $|\ell - \ell^0| < r$ implies $|DS(\ell)y - j(y)| \le \delta|\ell - \ell^0||y|$. Exact nonlinear critical points of the constrained sum are not constructed here. Section 4 constructs exact solutions of the *unconstrained* Regge equations instead, on all fifteen edge classes, with posting components that are not required to vanish: they lie in the Lorentzian chamber, not in the null stratum. Their relation to the path sum is the one just described. At linear order they are the ambient solutions whose exchange the twisted slice reproduces exactly (iii), and in the large-volume limit at fixed regulator the posting-null stratum gives the same exchange for sources far from the seam (iv). Beyond linear order no relation is proved.

3.5 Units and sources

With the action scale s, the action is sS and the linearized field equation for an edge source τ reads $s\,H_0 h = -\tau$. The lattice Newton clause, that every static mode of every solution for a point mass of mass M has $h_{tt}\, q(u) = 8\pi GM$, holds for exactly one coupling,

$$G = \frac{1}{8\pi s} \qquad (\kappa > 1/2,\ s > 0).$$

The phase scale is read as $\hbar = 1/a$, with $c = 1$; these two are definitions. If in addition $as = \ln 2/(2\pi\lambda_{\mathrm{rec}}^2)$ for a fixed length λ_{rec} of the motivating programme [4] (MODEL), the Planck area follows, $G\hbar = 1/(8\pi as) = \lambda_{\mathrm{rec}}^2/(4\ln 2)$. That premise fixes $G\hbar$, not G; it is a consistency of definitions, not a derivation of Newton's constant, and no result below uses it.

Matter enters through point-particle actions fixed in advance. A resting clock of density ρ on the time edges has action $\sum_x \rho_x\sqrt{-\ell_{(x,\mathbf{1})}}$, and an occupation $c \ge 0$ of the timelike edges, a *posting gas*, has the action $\sum_e c_e\sqrt{-\ell_e}$ of free point particles moving along the edges. Both are homogeneous of degree $1/2$; their sources are minus their exact derivatives, with edge stress $c_e/(2\sqrt{-\ell_e})$; and pairing a gas source with a vertex move gives $2\sum_x\sum_a \xi_{x,a}(\text{inflow} - \text{outflow})$,

so conservation is momentum balance at every vertex of the flat lattice. The ten constant-strain conditions on a source are equivalent to the vanishing of its integrated stress tensor $T = \sum_e j_e\, dd^\top$, and a nonnegative gas of timelike edges has $\operatorname{tr} T > 0$: a closed-universe Gauss obstruction that returns in Section 4. Resting dust has stress proportional to $\mathbf{1}\mathbf{1}^\top$, pure time direction, with no rest-frame stress.

3.6 The measure

Open section: the path-sum measure

What is required is a measure on configurations derived rather than chosen, on the domain the calculations use, with its behaviour under refinement and under gauge modes established. Settled at Gaussian order: the gauge behaviour and the independence from the density (below). Not settled: a derived measure; the regulator limit $\varepsilon \to 0^+$ after $N \to \infty$, which is the graviton question of Section 6; refinement of the spacing; and anything beyond Gaussian order.

The measure statements proved here hold for small fluctuations around flat space at identified $\kappa = 1$, on a periodic lattice of fixed side $N \geq 3$ unless a larger side or the limit $N \to \infty$ is named.

- On the twisted slice, a complete gauge slice for all edge fields, the exchange of every conserved source with an ambient solution tends to $(i/a)j \cdot u$ (Theorems 3.2 and 3.3); the native posting-null exchange does not on the side-3 torus (Theorem 3.3(ii)). The slice has an order-theoretic description. No strict order of the lattice points increases along every posting, since the postings of a fibre form a cycle. Ranking the points by $\sum_a x_a$ with $0 \leq x_a < N$ gives a strict order that increases along every posting except the closing posting of each fibre; keeping exactly those postings null selects the twisted slice, and the set is maximal, since adding a closing posting completes a fibre cycle. Other maximal sets are not classified.
- That value is unchanged by any bounded measurable density on edge space that is continuous and nonzero at the flat field, including Gibbs factors on edge configurations and indicators of open cutoffs around flat (Theorem 3.2(i)), and gauge fixing costs only a constant factor (Theorem 3.3(iii)). A weight acts on the propagator only if its exponent scales like the action; then a weight bounded below gives every mode a mass (Theorem 6.11).
- At every fixed $\varepsilon > 0$, for sources far from the seam, the twisted and posting-null exchanges agree up to an error that decays exponentially in the distance, uniformly in N (Theorem 3.3(iv)), so in the order $N \to \infty$ first the two strata give the same exchange. The bound degenerates as $\varepsilon \to 0^+$, so nothing here controls the second limit; and the volume grows at fixed spacing, which is not a refinement.

A ledger of records. One candidate source of a derived measure is a ledger of the records themselves, which we describe only as far as results are proved. A *ledger state* with a set X of further accounts is a pair of real arrays: the move of each of the four posting axes in each of the $4 + |X|$ accounts, and a shift of each account at each of the N^4 lattice points; it is a vector in $\mathbb{R}^{(4+N^4)(4+|X|)}$. The move v_k of an edge in account k is the sum of the moves of its axes plus the

change of the shift between its ends, and the *chart* reads the squared length $\sum_k v_k^2 - (\sum_k v_k)^2$ of the edge from its moves. The *balance* is the state in which each posting moves its own axis account by one and nothing is shifted; an edge of class d then reads $|d| - |d|^2$, the flat value. A *regular flat state* reads the flat field and has an onto chart derivative.

- The balance propagates nothing. Every first-order ledger motion there reads as a constant strain plus a vertex move, so the regulated exchange of a source has the form $\sum_i c_i^2/\varepsilon$ with real c_i fixed by the source, which diverges as $\varepsilon \to 0^+$ unless every c_i vanishes, and is then zero. On the side-3 torus an explicitly constructed folded ledger state, with two further accounts, reads the flat field and propagates a nonzero conserved source; its path sum does not impose the null condition on postings.

- A regular flat state needs at least eleven further accounts, by a dimension count: with at most ten, $(4 + N^4)(4 + |X|) < 15N^4$, the number of edges. For $N \geq 4$ regular flat states with at least eleven accounts exist in every ball around the balance; they dress the balance with small shifts of further accounts, compensated by a slight bending of the posting axes. They are not unique: raising every shift by a small constant changes no move of an edge, so a second one lies in the same ball. Around any one of them, the ledger's small-fluctuation path sum and the regulated Gaussian path sum on the twisted slice both give, as the regulator is removed at fixed N, the exchange $(i/a)j \cdot u$. What agrees is this value, not the two measures.

- For $N \geq 4$ there is a finite measure on ledger states whose reading at $\kappa = 1$ is the uniform measure on a ball around the flat field in the posting-null subspace. It is obtained by lifting a uniform measure through a local section of the chart, not computed from a ledger weight. On every torus of side at least 3 the small-fluctuation limit of the path sum of such a measure is the posting-null exchange, and as $N \to \infty$ at fixed ε the posting-null and twisted exchanges have one common limit. The section, the family of accounts and the regulator are still chosen.

These settle the gauge-mode behaviour, the density independence and the large-volume half of the ordered limit at Gaussian order. The missing links are a measure computed from a weight rather than lifted through a chosen section, and the regulator limit at infinite volume, which is where the results of Section 6 enter.

4 Classical recovery

The classical test asks for more than the right equations: the *solutions* of the lattice equations must converge to solutions of Einstein's equations, without an unproved stability constant [6, 7, 8]. This section proves that at linear order in the field.

Which equations. The equations solved here are the *unconstrained* Regge equations: stationarity of the action, with matter, under variations of all fifteen edge classes, at configurations in the *Lorentzian chamber*, where every simplex is nondegenerate and Lorentzian. Their posting components are not required to vanish, and in general do not, so they are not configurations of the path sum at $\kappa = 1$, whose stationary points on the null stratum carry a

multiplier on every posting (Section 3.4). The two are related at linear order: the linear lattice field is the ambient solution whose exchange the twisted slice of Theorem 3.3 reproduces exactly, and with which the posting-null exchange agrees in the large-volume limit. That is the only sense in which the solutions below concern the path sum.

What is proved. On every torus of odd side not divisible by 105 the flat Hessian has no zero modes beyond constant strains and vertex moves (Theorem 4.2). For band-limited conserved sources whose waves avoid the light cone, the long-wavelength limit of the linear lattice equations is the de Donder solution of the linearized Einstein equations, with an explicit stability constant (Theorem 4.6); this is the core of the section. On each torus of the admissible sides, a resting body with a uniform compensation and a model medium has exact solutions of the full coupled equations for small coupling, and their weak-field long-wave limit is the linear field, which for a compact static body gives Newton's potential with an explicit finite-size remainder (Theorems 4.8 to 4.10). The coupling goes to zero on each torus first, so the physical content of these limits is linear: the nonlinear solutions add existence on each torus, not nonlinear content in the limit. At $\kappa = 1$ no stability bound uniform in the torus size holds at every wavelength, because of a lattice resonance (Proposition 4.11); what holds uniformly at a fixed small source strength are the long-wave (Galerkin) equations, with control of the gauge directions, stability and the second-order field (Section 4.10).

What is not covered. The torus is periodic in time and the sources avoid the light cone, so the source problem is elliptic in character: radiation and retarded propagation are not part of these theorems, and free waves are treated only through their tree-level dispersion (Section 5). The matter that holds the gauge and strain rows is a model, and the solution theorems hold on the odd sides not divisible by 105, not on every side (Section 4.12).

Relation to earlier work. Roček and Williams introduced this triangulation for linearized Regge calculus and found its vertex-translation zero modes [5, 18]; Barrett and Williams studied the convergence of linearized Regge solutions [19]; Christiansen showed that linearized Regge calculus is a finite-element discretization of linearized gravity, with convergence results [21]; Hamber and Williams, Dittrich, Freidel and Speziale, Morse, and Bahr and Dittrich analysed its gauge modes and their breaking away from flat space [22, 26, 27, 9]. What is new here is the Lorentzian lattice with null postings: the exact flat kernel on an infinite family of tori, proved arithmetically; uniform bounds on every non-null cone with the sharp quartic shell near the cone; exact solutions with matter on every edge; and, at fixed strength, curvature-order bounds on the gauge directions uniform in the torus size, the lattice form of the broken diffeomorphism symmetry of [9, 27], together with the second-order field and its limit.

4.1 The Regge equations and the flat family

Theorem 4.1 (Stationarity gives Regge's equations)**.** *For every admissible* ℓ, $\mathrm{d}S(\ell)[\delta\ell] = \sum_h \delta_h(\ell)\,\mathrm{d}A_h(\ell)[\delta\ell]$ *for every variation, where* $\delta_h = \sum_{\sigma\ni h}(2\pi/n_h - \theta_{\sigma,h})$ *is the deficit at* h*. Hence* ℓ *is stationary under all edge variations exactly when* $\sum_h \delta_h\,\mathrm{d}A_h = 0$.

Proof. Differentiate Eq. (3). The terms with derivatives of the dihedral angles are

$$-\sum_{\sigma}\sum_{h\subset\sigma} A_h \,\mathrm{d}\theta_{\sigma,h},$$

and each inner sum vanishes by Schläfli's identity for the flat simplex σ. □

Write ℓ^0 for the flat field and $H_0 = D^2S(\ell^0)$ for its Hessian. Because S is homogeneous of degree one, $DS(c\ell) = DS(\ell)$ and $DS(\ell)[\ell] = S(\ell)$; differentiating gives $H_0\ell^0 = 0$. More is true. Moving the vertices of the flat lattice, or replacing g_κ by $g_\kappa + B$ for a small symmetric B, gives another flat lattice with all deficits zero. This *flat family* $\Psi(B,\xi)$ is stationary, so differentiating $DS(\Psi(q)) \equiv 0$ shows that its tangent space

$$\mathscr{R}_N = \{s_B + \delta\ell_\xi\}, \qquad s_B(x,d) = d^\top B d, \qquad \dim \mathscr{R}_N = 4N^4 + 6 \quad (N \geq 3,\ \kappa > 1/2),$$

the *radical* of constant strains and vertex moves, lies in $\ker H_0$. (On a curved Regge lattice vertex moves are not exact symmetries [9]; at the flat lattice they are.) Differentiating twice gives the identity $D^3S(\ell^0)(D\Psi a, D\Psi b, v) + H_0(D^2\Psi(a,b), v) = 0$ for all a, b, v.

These facts already constrain sources. If $DS(\ell^0 + u) = j$ on every edge, then for every $t \in \mathscr{R}_N$,

$$|j(t)| \leq L\,\|u\|^2\|t\|,$$

because $DS(\ell^0) = 0$ and $H_0 t = 0$. So a source must annihilate the radical at first order: conservation (vertex moves) and a discrete Gauss law (the constant strains, including $t = \ell^0$). Its radical component at second order is then fixed by the solution. Both consequences are proved obstructions below.

4.2 The flat Hessian in Bloch form

The flat Hessian is invariant under lattice translations, so it acts on Bloch waves class by class. For a wave $k \in [-\pi,\pi]^4$ and $v \in \mathbb{C}^{15}$, the edge field $h(x,d) = v_d\, e^{ik\cdot x}$ is mapped to $(H_0h)(x,d) = (B_\kappa(k)v)_d\, e^{ik\cdot x}$. This defines the 15×15 Hermitian *Bloch matrix* $B_\kappa(k)$, whose entries are trigonometric polynomials in k, since H_0 couples only edges of a common simplex; we write $B(k) = B_1(k)$, and also $B_\kappa(z)$ with $z_a = e^{ik_a}$. On the side-N torus the waves are $k \in (2\pi/N)\mathbb{Z}^4$ and H_0 is the direct sum of the $B_\kappa(k)$. The vertex move generated by $\xi(x) = \eta\, e^{ik\cdot x}$ is the Bloch field $(\Gamma(k)\eta)_d\, e^{ik\cdot x}$ with

$$(\Gamma(k)\eta)_d = 2\big(e^{ik\cdot d} - 1\big)\, d^\top \eta,$$

so $\Gamma(k)$ is the 15×4 matrix of vertex-move columns. Vertex moves are zero modes, so $B_\kappa(k)\Gamma(k) = 0$, and since $B_\kappa(k)$ is Hermitian, $\Gamma(k)^\dagger B_\kappa(k) = 0$; we call this the Bianchi identity. The gauge-fixed operator in the Feynman-type gauge is $A(k) = B(k) + \Gamma(k)\Gamma(k)^\dagger$. With $\Phi(k) = \mathrm{diag}(e^{ik\cdot d/2})$, the phases at the edge midpoints, $M(k) = \Phi(k)^\dagger A(k)\Phi(k)$ is real symmetric, by the point reflection of the lattice; we call this the *midpoint frame*. At $\kappa = 1$ the cell Hessian from which $B(k)$ is assembled is printed exactly in Appendix B.

4.3 The flat kernel

Every solution theorem below needs the radical to be the whole kernel of H_0. This is a property of the lattice, not a copy of linearized general relativity on the same torus, and it leaves the lattice with fewer zero modes than general relativity has there. Time is periodic along $\mathbf{1}$, so every *null torus wave*, an integer covector m with $q(m) = 0$ such as $m = (1, 1, 1, 0)$, with $q(m) = 3 - 9/3 = 0$, its permutations and multiples, carries two transverse polarizations that solve the linearized Einstein equations. On the sides of the next theorem the lattice has no zero mode at those waves beyond the radical, and along the null rays its determinant vanishes to higher order than off the cone (Proposition 4.5). For the solution theorems the missing modes help; they are a lattice effect, not agreement with general relativity. They also indicate what kind of problem the torus poses: a time-periodic hyperbolic problem whose solution operator has small divisors near the cone, as for time-periodic wave equations [68, 69]. At a null torus wave m the continuum operator vanishes, and only the lattice correction, of order $(2\pi|m|/N)^4$ rather than $(2\pi|m|/N)^2$, keeps the operator invertible; these small divisors grow with N. We take them to be why the constants of the solution theorems depend on N (INFERENCE, from the order count of Proposition 4.5; the constants are not computed in this form).

Theorem 4.2 (The flat kernel on odd tori)**.** *Let $\kappa = 1$ and let $N \geq 3$ be odd and not divisible by* 105*. Every edge field h on the side-N torus with $H_0(h, h') = 0$ for all h' is a constant strain plus a vertex move. In particular this holds on every prime side $p >$* 31104*.*

Through the Bloch decomposition the theorem says that one integer polynomial $F_1 \in \mathbb{Z}[z_1, \dots, z_4]$, the gauge-fixed Bloch determinant, has no zero at a torsion point $z \neq 1$ whose order divides N. Concretely, by the formula of Appendix B the entries of $16\sqrt{3}\,B(z)$ are integer Laurent polynomials in z. Let $B_E(z)$ be the 11×11 block of $B(z)$ on the classes with $|d| \geq 2$; the posting rows of $\Gamma(z)$ form the diagonal matrix $\Gamma_S(z) = \operatorname{diag}\big(2(z_a - 1)\big)$. In the basis formed by the four columns of Γ and the unit vectors of those eleven classes, the Bianchi identity makes A block triangular up to a Schur complement, which gives

$$|\det \Gamma_S(z)|^2 \det A(z) = \det\big(\Gamma(z)^\dagger \Gamma(z)\big)^2 \det B_E(z),$$

and $\det\big(16\sqrt{3}\,B_E(z)\big)$ is a nonzero rational constant times a monomial times $\prod_a (z_a - 1)^2 F_1(z)$, an identity between explicit Laurent polynomials that defines F_1. Since $\Gamma^\dagger\Gamma$ is invertible at every $z \neq 1$ on the unit torus, and both sides are continuous, $\det A(z)$ is a nonzero constant times $(z^{\mathbf{1}})^{-3} F_1(z) \det(\Gamma^\dagger\Gamma)^2$ at every z, with $z^{\mathbf{1}} = z_0 z_1 z_2 z_3$: away from $z = 1$ the gauge-fixed operator is singular exactly where F_1 vanishes. F_1 has 741 monomials, all exponents at most 6, and no zero of order 3. For prime sides above 31104 there is a short proof from a general fact about sparse integer polynomials, proved in Appendix C:

Lemma 4.3 (Torsion zeros of sparse polynomials)**.** *Let $f = \sum_{i=1}^{n} c_i z^{m_i} \in \mathbb{Z}[z_1, \dots, z_4]$ have every exponent entry in $\{0, \dots, 6\}$ and no zero at a point of order* 3 *other than* 1*. If p is prime with $p > n$ and $p >$* 31104 $= 4! \cdot 6^4$*, then $f(z) \neq 0$ for every $z \neq 1$ with $z^p = 1$.*

The lemma belongs to the study of vanishing sums of roots of unity and of torsion points on hypersurfaces [55, 56, 57, 58, 59, 60, 61]; its proof in Appendix C is short and self-contained.

The odd sides 3 to 13 are also covered by exact determinant certificates, one per Galois orbit of torsion points. For every odd side the proof is 2-adic. F_1 is invariant under the

permutations of the four coordinates, so it is a polynomial in their elementary symmetric functions $e = (e_1, \dots, e_4)$. On points of odd order the Galois automorphism $z \mapsto z^2$ acts on e through the Graeffe map, which lifts Frobenius modulo 2, and expanding $F_1(z) = 0$ along this structure gives, at a nontrivial zero of odd order, explicit polynomial conditions modulo 2 on $e(z)$, up to the fifth power of 2. If $3 \nmid N$, then z^3 is a zero of the same order, which gives a second set of conditions through the symmetric functions of the cubes. Reduced modulo a prime above 2, the two sets form one polynomial system in (e_2, e_3, e_4) over a field of characteristic 2. An elimination over $\mathbb{F}_2$, a finite tree of 353 identities between explicit polynomials, each checked by exact arithmetic, shows that in every field of characteristic 2 the system has solutions only on four one-parameter families and at one point. Reduction modulo 2 is injective on roots of unity of odd order, so the coordinates of z lie on the same families in characteristic 0; there the restriction of F_1 is an explicit polynomial in the family parameter, which has no zero at a nontrivial root of unity of odd order prime to 3, and the remaining point has order 5, where $F_1 = 9600$. If instead $5 \nmid N$, then z^5 is a zero of the same order, and the same scheme with the symmetric functions of the fifth powers gives a second system; its elimination, a tree of 372 identities, leaves the same four families and the point $e = (1, 1, 1)$, and these carry no nontrivial zero of odd order (for one family at points of order 3 this uses the certificate for the side 3). If $7 \nmid N$, then z^7 is a zero of the same order. The conditions at z up to the fifth power of 2 and those at z^7 at the first and fourth powers, reduced in the same way, form a third system; its elimination, a tree of 1598 identities, again leaves only the four families and the point $e = (1, 1, 1)$, on which there is no nontrivial zero of odd order, while four further orbits of points satisfy all but the last of its equations and violate that one. Every odd side not divisible by 105 is prime to 3, to 5 or to 7, so one of the three routes applies. The odd multiples of 105 are open. Even sides fail, for a reason that returns in Section 6:

Theorem 4.4 (Even tori)**.** *Let $\kappa = 1$ and $N \geq 4$ even. The field $h(x, d) = (-1)^{x_3} v(d)$, with $v(d) = 2^{|d|-1}$ on the seven classes with $d_3 = 0$ and $v(d) = 0$ otherwise, is a zero mode of H_0 that is neither a constant strain nor a vertex move.*

Along the null torus waves the polynomial F_1 can be written down in closed form.

Proposition 4.5 (Null torus waves)**.** *Let $\kappa = 1$ and $w = e^{is}$. On the null line $z = (w, w, w, 1)$, the phases of the waves $s(1, 1, 1, 0)$, and on its images under permutations of the coordinates,*

$$F_1 = (w - 1)^8 \, (5w^2 - 4w + 5)^2 \, (3w^6 - 6w^5 - 3w^4 + 4w^3 - 3w^2 - 6w + 3),$$

while on the axis line $z = (w, 1, 1, 1)$, which is not null, $F_1 = -8(w - 1)^4 (w + 1)^2$. For $0 < |s| \leq \arccos(3/5)$, $F_1 \neq 0$ on the null lines, A is nonsingular there, and every null vector of B there is a vertex move.

Proof. The two identities are exact expansions of the 741 terms of F_1. On the unit circle the roots of $5w^2 - 4w + 5$ have real part $2/5 < 3/5$. The sextic is palindromic; divided by w^3 it is $f(x) = 3x^3 - 6x^2 - 12x + 16$ in $x = w + w^{-1} = 2 \cos s$. Since $f'(x) = 9x^2 - 12x - 12 < 0$ for $-\frac{2}{3} < x < 2$ and $f(\frac{6}{5}) < 0$, f has no root with $\frac{6}{5} \leq x \leq 2$, that is with $\cos s \geq \frac{3}{5}$. So $F_1 \neq 0$ there, A is nonsingular by the factorization of $\det A$, and if $Bv = 0$, write $v = \Gamma y + r$ with $\Gamma^{\dagger} r = 0$; then $Ar = Br = 0$, so $r = 0$ and v is a vertex move. □

The statement concerns single waves, so it holds on every torus, the even sides and the multiples of 105 included. Along the axis line F_1 vanishes to order 4 at the origin, along the null lines to order 8. We read this as follows: the two transverse eigenvalues, of order $|\mathbf{k}|^2$ off the cone, are of order $|\mathbf{k}|^4$ on the null rays and not zero, so the lattice dispersion lies off the light cone at relative order $|\mathbf{k}|^2$ there, as along the paired direction (Theorem 5.5) (INFERENCE).

4.4 The linear limit on non-null cones

The linear theory is complete off the light cone, for every $\kappa > 1/2$. Let $B_\kappa(e^{iu})$ be the Bloch matrix of H_0 at the wave u and $M_B(u) = \Phi(u)^\dagger B_\kappa(e^{iu})\Phi(u)$ its real form in the midpoint frame (Section 4.2). Let $V : \mathbb{R}^{10} \to \mathbb{R}^{15}$ be the strain embedding $(VX)_d = d^\top X d$ of symmetric matrices X, V^+ a fixed left inverse of V, and $w_\kappa = -\frac{1}{4}\sqrt{4\kappa - 1}$. The *de Donder gauge-fixed form* is

$$A_{\mathrm{dD}}(u) = M_B(u) + \ell(u)\, Q_{\mathrm{dD}}\, \ell(u)^\top, \qquad Q_{\mathrm{dD}} = 2w_\kappa\, g_\kappa^{-1},$$
$$\ell(u)^\top h = X g_\kappa^{-1} u - \tfrac{1}{2}\operatorname{tr}(g_\kappa^{-1} X)\, u \qquad (X = V^+ h),$$

the Hessian plus the square of the de Donder divergence of the strain read from h. Let $\mathcal{G}(X,Y) = \operatorname{tr}(g_\kappa^{-1} X g_\kappa^{-1} Y) - \frac{1}{2}\operatorname{tr}(g_\kappa^{-1} X)\operatorname{tr}(g_\kappa^{-1} Y)$ be the DeWitt form of g_κ on symmetric matrices, $C' = w_\kappa \mathcal{G}$ in the basis of symmetric unit matrices, and $G_\infty = V C'^{-1} V^\top$. With $M_2(u)$ the quadratic part of $M_B(u)$ at $u = 0$, these satisfy the exact identity $V^\top\big(M_2(u) + \ell(u) Q_{\mathrm{dD}} \ell(u)^\top\big) V = q_\kappa(u)\, C'$. A *non-null cone* is a closed set P of covectors, invariant under positive scaling, on which $q_\kappa(u) \neq 0$ away from 0. A *stress family* on P is a map $u \mapsto T(u)$ from P to symmetric 4×4 matrices; it is *conserved on P* if $T(u)\, u = 0$ for every $u \in P$, the Fourier form of $\partial_\mu T^{\mu\nu} = 0$. The lattice source $\tau(u)$ of $T(u)$ is the gauge-orthogonal projection of the edge source $\frac{1}{2}\sqrt{|\det g_\kappa|}\, h_{\mu\nu} T^{\mu\nu}$ (Appendix D); let $s \neq 0$ be the action scale and $G = 1/(8\pi s)$.

Theorem 4.6 (The linear field on non-null cones)**.** *Let $\kappa > 1/2$, P a non-null cone and T a stress family continuous at 0 and conserved on P. There is $\delta > 0$ such that for $u \in P$, $0 < |u| < \delta$, the de Donder solution $h(u)$ solves $s\, B_\kappa(e^{iu}) h = -\tau(u)$, every other solution differs from it by a vertex move, and*

$$q_\kappa(u)\, h_{\mu\nu}(u) \longrightarrow 16\pi G\Big(T_{\mu\nu}(0) - \tfrac{1}{2} g_{\mu\nu} T(0)\Big) \qquad (u \to 0 \text{ in } P).$$

Quantitatively, $q_\kappa(u) A_{\mathrm{dD}}(u)^{-1} \to G_\infty$, and for every $\eta > 0$ there is $\delta > 0$ with $|q_\kappa(u)||h_i(u)| \leq |s|^{-1} \sum_j (|(G_\infty)_{ij}| + \eta)|b_j|$ for $h = -s^{-1} A_{\mathrm{dD}}(u)^{-1} b$ and $0 < |u| < \delta$ in P.

The stability bound is the closed-form matrix G_∞ plus any margin; it is not an assumed constant. The same statement holds in position space. For a finite set F of integer waves with $q_\kappa(m) \neq 0$, conserved stresses T_m and amplitudes c_m, the real-wave lattice field h_N built from the Bloch solutions at $u_m = -2\pi m/N$ solves the side-N equation against every test field, and for every $\eta > 0$ there is N_0 such that for $N \geq N_0$, at every edge,

$$\Big|\Big(\frac{2\pi}{N}\Big)^2 h_N(x,d) - h_\infty(X,d)\Big| < \eta, \qquad X = \frac{2\pi x}{N},$$
$$h_\infty(X) = \sum_{m \in F} \operatorname{Re}\big(c_m e^{im\cdot X}\big)\, q_\kappa(m)^{-1}\big(-s^{-1} G_\infty b_m\big), \tag{4}$$

with b_m the edge source of T_m. In strain coordinates each mode of h_∞ is $q_\kappa(m)^{-1}\, 16\pi G(T_m - \frac{1}{2} g\, \operatorname{tr}_g T_m)$: h_∞ is the de Donder solution of the linearized Einstein equations with the same

source. Here $N \to \infty$ is a long box at fixed spacing, so the waves grow longer in lattice units: a long-wavelength limit, not a refinement. The construction and source conventions are in Appendix D. Compared with the convergence results for linearized Regge calculus under refinement [19, 21], the statement is uniform on every non-null cone of a Lorentzian lattice, and the next paragraph shows how it degenerates at the cone.

Off the non-null cones the gauge-fixed solution operator is not uniformly bounded. For every $\kappa > 1/2$ no bound $|q_\kappa(u) A_{\mathrm{dD}}(u)^{-1}_{ij}| \le C$ holds on a punctured neighbourhood of 0: there are u with $c\|u\|^4 \le |q_\kappa(u)| \le 5\|u\|^4$ and $\det A_{\mathrm{dD}}(u) = 0$; a bound holds outside the quartic shell $|q_\kappa(u)| \ge c_1\|u\|^4$, and the exponent 4 is sharp among integer powers. These are zeros of the de Donder gauge-fixed operator, and the modes found there are lattice gauge modes, which a ghost operator removes; the statement is about the gauge-fixed solution operator, not about physical resonances near the cone. At $\kappa = 1$ every non-null ray tp meets, at some finite t, a wave where the null space of B_1 exceeds the vertex moves. Static waves show it most simply: along $u = (-t, t, 0, 0)$ the static Bloch operator has a non-gauge zero mode at

$$\cos t_* = \frac{9}{11} \qquad (t_* \approx 0.61,\ \text{wavelength} \approx 10.3 \text{ spacings}),$$

and its largest component is on the time edge, so a resting source is not solvable there; Appendix B gives the zero mode exactly. We call this the *static resonance*. It is a lattice artefact at about ten spacings, a spurious finite-wavelength mode of the kind familiar from finite elements and lattice field theory, and it is why the theorems below are long-wave statements.

4.5 Why a fixed source cannot simply be solved

Two proved obstructions show what any solution theorem has to supply.

Theorem 4.7 (Closed-torus Gauss law)**.** *Let $N \ge 3$, $\kappa > 1/2$, $\rho \ge 0$ a site density with $\rho_x > 0$ somewhere, and $B > 0$. There is $\delta > 0$ such that for every $\sigma \in (0, \delta]$, every u with $\|u\| \le B\sigma$ and every covector j' with $j'(\mathfrak{l}^0) \le 0$, the equation $DS(\mathfrak{l}^0 + u) = \sigma\,\rho^\flat + j'$ fails on some edge, where $\rho^\flat$ is the time-edge source of ρ. Moreover every solution in a fixed ball has $\|u\|^2 \ge c\,\sigma$.*

The mechanism is the bound of the previous subsection with $t = \mathfrak{l}^0$: the flat field pairs with a positive mass at order σ with a fixed sign, because the time edges have negative squared length, while the left side is $O(\|u\|^2)$. On a closed universe a static positive mass must be compensated (INFERENCE: this is the discrete form of $\int \rho = 0$ on a three-torus). The second obstruction is a discrete von Laue condition. Let a fixed source ray σj be compatible at first order and let w_1 be its linear response with self-energy $Q = H_0(w_1, w_1) \ne 0$. Then for small $\sigma > 0$ there is no solution of $DS(\mathfrak{l}^0 + u) = \sigma j$ with $\|u\| \le B\sigma$; and any compatible source $\sigma j + k$ that does have a slice solution within $O(\sigma)$, with a correction k along the radical directions, must satisfy $k(\mathfrak{l}^0) = -\frac{\sigma^2}{2} Q + O(|\sigma|^3)$. A self-gravitating static body needs compensating stresses at second order. Fixed sources with $Q \ne 0$ are dense among compatible ones; no physically selected source has been shown to have $Q \ne 0$.

4.6 Exact solutions with matter

Matter is given by actions fixed before any solution is sought, and the coupled equations are stationarity of $S - \sigma\mathscr{M}$ on every edge:

$$DS(\ell^0 + u)[v] = \sigma D\mathscr{M}(\ell^0 + u)[v] \qquad \text{for every } v \in \mathbb{R}^{15N^4}. \tag{5}$$

Nothing is projected away: the gauge rows, the strain rows and every other zero mode are imposed. Three pieces of matter enter.

- A *resting body*: clocks of static density ρ (invariant under $x \mapsto x + \mathbf{1}$) on the time edges, with action $\sum_x \rho_x \sqrt{-\ell_{(x,\mathbf{1})}}$, the point-particle action $-M\int \mathrm{d}\tau$ of static particles of proper mass $\sigma\rho_x$.
- Its *compensation*: the fixed linear functional equal to the flat clock source of the mean density $\bar{\rho}$. Theorem 4.7 says some compensation is needed; this uniform, non-responding one is a modelling choice.
- A *medium* on the timelike edges, $\mathscr{A}_+(\ell) = \sum_{e \text{ timelike}} \big(\gamma_d \tau_e^0 \tau_e + \frac{\gamma_d}{2}\ell_e\big)$, with $\tau_e = \sqrt{-\ell_e}$ the proper length, $\tau_e^0 = \sqrt{-\ell_e^0}$ its flat value ($\sqrt{2}, \sqrt{6}, \sqrt{12}$ at $\kappa = 1$) and class weights $\gamma_d > 0$. Theorems 4.8 to 4.10 hold for every choice of positive weights: the weights move the medium's thresholds and the solution at finite coupling, not the linear field or the limits. Exactly,
$$\mathscr{A}_+(\ell) = \mathscr{A}_+(\ell^0) - \frac{1}{2} \sum_{e \text{ timelike}} \gamma_d \big(\tau_e - \tau_e^0\big)^2,$$
so in $S - \sigma\lambda\mathscr{A}_+$ the medium is a Hooke spring on the timelike proper lengths with the flat lattice as its rest lengths. Its first variation at ℓ^0 vanishes, so it does not change the linear field.

Theorem 4.8 (Full coupled solutions on odd tori)**.** *Let $\kappa = 1$, let $N \geq 3$ be odd and not divisible by 105, and let ρ be a static density on the side-N torus. There is λ_0 such that for every $\lambda \geq \lambda_0$ there are a linear field h with $H_0 h$ equal to the body's flat source, a radical vector $r \in \mathscr{R}_N$, and constants $\sigma_0, B, C, r_0 > 0$, such that for every $\sigma \neq 0$ with $|\sigma| \leq \sigma_0$ there is a field u with*

(i) $\|u - \sigma(h + r)\| \leq C\sigma^2$ *and* $\|u\| \leq B|\sigma|$*;*

(ii) $\ell^0 + u$ *in the Lorentzian chamber (every simplex nondegenerate Lorentzian);*

(iii) Eq. (5) *on every edge, with* $\mathscr{M} = \lambda\mathscr{A}_+ + (\textit{body}) + (\textit{compensation})$*;*

(iv) u the only such field within $r_0|\sigma|$ of $\sigma(h + r)$.

Proof outline. Split $u = \tau + w$ with $\tau \in \mathscr{R}_N$ and w in a complement W. By Theorem 4.2, H_0 is nondegenerate on W, so the W-rows are solved by the implicit function theorem. The $\mathscr{R}_N$-rows vanish identically at orders 0 and 1: H_0 kills $\mathscr{R}_N$, and $D^3S(\ell^0)$ vanishes on $\mathscr{R}_N^3$ by the flat kernel and the second-derivative identity of the flat family. Their leading part is therefore of order σ^2 and reads $Q(\tau_0, \cdot) = \frac{1}{2}D^3S(w_1, w_1, \cdot) - D^2\mathscr{M}(w_1, \cdot)$ on $\mathscr{R}_N$, with reduced operator $Q(r, r') = D^2\mathscr{M}(r, r') - D^3S(w_1, r, r')$. If Q is nondegenerate on $\mathscr{R}_N$, a blow-up

$u = \sigma(w_1 + \tau_0) + \sigma^2(\cdots)$ and a scaled implicit function theorem give existence, the $O(\sigma^2)$ remainder and local uniqueness (a Lyapunov–Schmidt argument). On the flat family the reduced operator is the Hessian of the matter, $Q(D\Psi a, D\Psi b) = D^2(\mathscr{M} \circ \Psi)(0)(a, b)$. The medium's Hessian $-\sum \gamma_d v_e^2/(4(\tau_e^0)^2)$ is negative definite on timelike components, and a flat-family tangent that vanishes on every timelike edge vanishes; so for λ large the medium term dominates the bounded body terms and Q is nondegenerate. Smoothness of S near ℓ^0 and admissibility of $\ell^0 + u$ for small σ complete the argument. The side enters only through the flat kernel, so the same argument gives the theorem on every side where Theorem 4.2 holds. □

The medium holds the radical rows, and it holds them at second order in σ; the first-order field h is the body's own linear field and does not depend on λ. The solution u is a configuration of all fifteen edge classes. Nothing constrains its posting components to vanish, so it lies in the Lorentzian chamber and not in the null stratum, and Eq. (5) are the unconstrained Regge equations with matter, not the stationarity conditions of the path sum on the null stratum (Section 3.4).

4.7 The weak-field long-wave limit

Theorem 4.9 (Weak-field long-wave limit)**.** *Let $\kappa = 1$, $s \neq 0$, F a finite set of integer waves with $q(m) \neq 0$, T_m conserved at m, amplitudes c_m, and let h_∞ be the linearized Einstein field of Eq.* (4)*. For every $\eta > 0$ there is N_0 such that for every odd side $N \geq N_0$ not divisible by* 105*: the sampled lattice source is within η of the continuum source at every edge, and there are $r \in \mathscr{R}_N$ and $\sigma_0 > 0$ such that for every $0 < |\sigma| \leq \sigma_0$ the full coupled equations on the side-N torus, with the medium (at an amplitude chosen for that side and source) and the fixed source σj_N, have a solution u in the chamber with*

$$\left|\left(\frac{2\pi}{N}\right)^2\left(\frac{u_e}{\sigma} - r_e\right) - h_\infty(X_e, d_e)\right| < \eta \qquad \text{at every edge } e.$$

Proof. The argument of Theorem 4.8, with the fixed source in place of the body, uses the side only through the flat kernel. On each such side it gives $u = \sigma(h_N + r) + O_N(\sigma^2)$, where h_N is the linear lattice field. Then $(2\pi/N)^2(u/\sigma - r) - h_\infty = [(2\pi/N)^2 h_N - h_\infty] + (2\pi/N)^2 O_N(\sigma)$. Choose N_0 by Eq. (4) for $\eta/2$; on each side $N \geq N_0$ choose $\sigma_0(N)$ so that the second term is below $\eta/2$. □

The order matters and is stated: $\sigma \to 0$ on a fixed torus first, then $N \to \infty$ along the odd sides not divisible by 105, an unbounded family that contains every odd prime. Nothing is uniform in N; the medium's effects disappear because $\sigma\lambda \to 0$.

The resting body can itself be the source, with its target fixed before any lattice field is computed. For a static band-limited density $\rho(x) = \bar{\rho} + \sum_{m \in F} \mathrm{Re}(c_m e^{im \cdot X})$ with static nonzero modes, take as target the linearized Einstein field of resting dust of proper density 2ρ, each mode being $q^{-1} 16\pi G(T - \frac{1}{2} g \,\mathrm{tr}\, T)$. The factor 2 is a change of volume element: the static slice has proper volume element $\mathrm{d}Z/2$ in its coordinates Z, so $\int 2\rho \,\mathrm{d}V = \int \rho \,\mathrm{d}Z$. Reading a clock density as dust is the point-particle reading of the resting body above (MODEL). The body's linear lattice fields converge to this target after the same rescaling. With Theorem 4.8 the full coupled solutions with the body as the only source then converge to it along the odd sides not divisible by 105 in the sense of Theorem 4.9: weak-field order, modulo a radical vector,

and with the medium's amplitude chosen for the side and the body. The proof uses that a static body sources only the time edges, whose Bloch vector is exactly conserved on static rays, so the linear construction of Eq. (4) applies with a constant source.

4.8 Newton's law for a compact body

The last step of the source requirement is a body of finite extent and its potential $h_{00} = -2\Phi$ with $\Phi = -GM/r$.

Theorem 4.10 (Compact-body Newton field)**.** *Let $\kappa = 1$ and let g be a smooth static density of compact support in the proper ball of radius $R_0 > 0$ on the static slice, with mass $M = \int g$, centred at Z_0; let $R_1 > 0$ and $\eta > 0$. There is $v_0 > 0$ such that for every body size $0 < v \le v_0$ there is n_0 such that for every band $n \ge n_0$ the following holds for the lattice body formed by the body's own Fourier modes up to n, with its compensation. On every odd side $N \ge N_0$ not divisible by 105, for some N_0, with the medium at an amplitude chosen for that side and body, there are a radical vector $r \in \mathscr{R}_N$ and $\sigma_0 > 0$ such that for every $0 < |\sigma| \le \sigma_0$ the full coupled equations have a solution u as in Theorem 4.8, and at every site x and period shift p with $W = (Z_x + 2\pi p - Z_0)/v$, $\|W\| \le R_1$ and proper distance $R > R_0$,*

$$\Big| v\big(-\tfrac{1}{2} h_{tt}(x)\big) + \frac{GM}{R} \Big| \le \eta + \frac{G R_0 \int |g|}{R(R - R_0)}, \qquad G = \frac{1}{8\pi},$$

where $h_{tt}(x) = (2\pi/N)^2 (u - \sigma r)_{(x,\mathbf{1})}/(12\sigma)$ is the time-time readout of u with σr removed. If the body is centred, $\int g(y)\, y \,\mathrm{d}y = 0$, the remainder improves to $4GR_0^2 \int |g|/R^3$ at $R \ge 2R_0$.

So $h_{00} = -2\Phi$ and $\Phi = -GM/R$, with an explicit finite-size remainder. The chain of the proof is: the time-time component of the linearized Einstein field of a static body is -2φ with φ the box Poisson potential, $\sum_a \partial_a^2 \varphi = 4\pi G \cdot 2(\rho - \bar{\rho})$; the box potential of the band-limited body is an exact Riemann sum of the body's spectrum; as the body shrinks against the box the sums converge uniformly to the spectral integral; and the spectral integral is the Coulomb potential, $1/(4\pi |\det e|\, |e^{-\top} Z|)$ in slice coordinates with $|\det e| = 2$, by a heat-kernel computation. The coefficient $1/(8\pi)$ is computed along this chain, not inserted, and it is the lattice Newton coupling at unit action scale, the only one by Section 3.5.

The theorem says exactly this, and no more. The coupling goes to zero first on each torus, so its content is the first-order part of exact solutions. The vector r is a vertex move plus a constant strain, a lattice gauge transformation plus a flat deformation of the torus; its existence is proved and its size is not controlled, so $h_{00} = -2\Phi$ holds in a suitable lattice gauge. A gauge-free form holds after averaging along the time direction: a radical vector has the same sum along every orbit $x + k\mathbf{1}$, and the time-averaged readout of u/σ itself, minus one constant per torus fixed before σ, satisfies the same bounds, for the theorem's solution u. The medium is the spring of Theorem 4.8 at an amplitude chosen after the body and the side. The lattice body is a band truncation of a smooth compact body, and η comes with no explicit rate; a body supported on finitely many lattice sites is not covered, since at $\kappa = 1$ every compact lattice body of nonzero mass couples to the static resonance. Reading G as Newton's constant uses the action-scale premise of Section 3.5, and reading M as the body's mass uses the point-particle reading of a clock (MODEL).

4.9 What cannot be uniform at $\kappa = 1$

Every theorem so far lets the coupling go to zero on each torus before the torus grows. A source of fixed strength on a growing torus needs a stability bound uniform in N, and at $\kappa = 1$ the static resonance rules that out at the lattice scale.

Proposition 4.11 (No uniform stability at $\kappa = 1$)**.** *Let $\kappa = 1$.*

(i) No inf-sup bound uniform in N holds over all wavelengths: there is C such that on every large torus a static plane wave h within $2\pi/N$ of the static resonance satisfies $D^2S(\ell^0)(h,h')^2 \le (C/N)^2\|\nabla h\|^2\|\nabla h'\|^2$ for every h'.

(ii) For every $c \in (0, 1/4]$, on every large torus, a flat lattice within $16c$ of the $\kappa = 1$ lattice at every edge has an exact non-gauge zero mode.

A medium of fixed amplitude contributes to the rescaled Hessian only at order $\lambda\sigma(2\pi/N)^2$ and so cannot remove these modes, while a medium strong enough to remove them would act as a mass term for the long-wave field. We infer that the static rows off the band cannot be imposed at fixed strength by a perturbative argument, and that a fixed-strength theorem at $\kappa = 1$ has to be a statement about the long-wave equations together with control of the gauge and strain directions (INFERENCE). That is what the next subsection proves; imposing the remaining rows is open.

4.10 Fixed source strength

What does hold uniformly in N at a fixed small strength is a statement about the long-wave (Galerkin) equations, together with control of the gauge and strain directions at every order in the field.

Setting. Let $\kappa > 1/2$, $s \neq 0$, and let the source be static and band-limited: a finite set F of static integer waves off the light cone, conserved stresses T_m and amplitudes c_m, with lattice source j_N and limit field h_∞ as in Eq. (4). On the side-N torus the source is $\alpha(2\pi/N)^2 j_N$, where $\alpha > 0$ is the fixed strength. An edge field is *band-limited at radius δ* if its Fourier support lies on static waves $u = -2\pi m/N$, $\sum_a u_a = 0$, with $\max_a|u_a| < \delta$; the *band rows* are the equations tested against band-limited fields orthogonal to $\mathscr{R}_N$.

We measure fields by a *jet norm*. Write $\|f\|_A = \sum_m|\hat{f}(m)|$ for the sum of the absolute Fourier coefficients of a function on the vertices. The *metric read-out* of an edge field w is the symmetric matrix field $B_w(x)$ whose strain profile $d \mapsto d^\top B_w(x)d$ is the strain part of the class averages $\frac{1}{2}\big(w(x,d) + w(x-d,d)\big)$, for a fixed splitting of $\mathbb{R}^{15}$ into the ten strains and a five-dimensional complement; $P_w(x)_\rho = B_w(x+e_\rho) - B_w(x)$ are its first differences. The *stencil* of x is the set of edges $(x - c + c', d)$ with $c, c' \in \{0,1\}^4$, and the *stencil rest* $Z_w(x)$ is w on the stencil minus the prediction of the affine metric $B_w(x) + \sum_\rho(c' - c + \frac{1}{2}d)_\rho P_w(x)_\rho$ there. Then

$$\|w\|_{\mathrm{J}} = \sum_{a,b}\|(B_w)_{ab}\|_A + N\sum_{\rho,a,b}\|(P_w)_{\rho,ab}\|_A + N^2\sum_s\|Z_w(s)\|_A,$$

the last sum over the 3840 stencil slots, and $|w_e| \le 64\|w\|_{\mathrm{J}}$. A field is *smooth to third order at amplitude k*, with read-out bound k_0, if at every site $|B_w| \le k_0$, $|P_w| \le k/N$, the stencil rest and

the second-order Taylor remainders of B_w over offsets in $\{-3,\dots,3\}^4$ are at most k/N^2 (with a second-difference witness of that size), and the corresponding remainders of P_w are at most k/N^3: the scaling of a fixed smooth field sampled on a torus of side N.

Theorem 4.12 (Band solutions at fixed strength)**.** *In this setting there is $\delta_* > 0$, and for every $\delta \le \delta_*$ there are $\alpha_0, C, C_r > 0$, such that for every $0 < \alpha \le \alpha_0$ and $\eta > 0$ there is N_0 with the following property. For every $N \ge N_0$ there is exactly one band-limited field $w \perp \mathscr{R}_N$ with $\|w\|_{\mathrm{J}} \le C_r\alpha$ such that*

$$DS(\ell^0 + w)[v] = \alpha\Big(\frac{2\pi}{N}\Big)^2 j_N(v) \qquad \text{for every band-limited } v \perp \mathscr{R}_N,$$

and for this field there is $r \in \mathscr{R}_N$ with $|(w_e - r_e)/\alpha - h_\infty(X_e, d_e)| \le C\alpha + \eta$ at every edge. The constants $\delta_*, \alpha_0, C, C_r$ do not depend on N.

Theorem 4.13 (Gauge directions at fixed strength)**.** *Let $\kappa > 1/2$.*

(i) There are $C, \rho > 0$ such that for every $N \ge 3$, every background $\ell^0 + w$ that is smooth to third order at amplitude k with read-out bound $k_0 \le \rho$, and every $t \in \mathscr{R}_N$,

$$|D^2 S(\ell^0 + w)(t,t)| \le C\,\frac{k}{N^2}\,\|t\|^2.$$

(ii) In the setting of Theorem 4.12 there is C_k, independent of N and α, such that the band solution w is smooth to third order at amplitude $C_k\alpha$, with read-out bound $C_r\alpha$. So (i) applies to it with $k = C_k\alpha$ once $C_r\alpha \le \rho$.

Part (i) says that vertex moves and constant strains remain almost zero modes of the Regge Hessian at a curved background of fixed amplitude, at every wavelength and uniformly in the torus size: their energy is of curvature order, k/N^2, relative to their norm. It is the lattice form of the statement that the action is invariant under diffeomorphisms up to terms that need curvature. The proof makes this literal. For a vertex move generated by a covector field ζ, the adapted lattice Lie derivative of the background along $\eta = g^{-1}\zeta$ is an exact symmetry direction of the action of each cell; the difference between the vertex move and this Lie derivative is, up to a second-order rest, a metric perturbation proportional to the Christoffel symbols of the background contracted with η. That metric-like term is integrated by parts once on the lattice, using that the strained flat functional annihilates affine metrics, and the resulting sums are controlled by a Poincaré inequality on the four-torus, Korn's inequality and the coercivity of the strains. Part (ii) needs only first differences of the solution: second differences of the metric read-out are bounded by the stencil rest, which the jet norm already controls, and the first differences of w are small because the band problem is translation covariant, its fixed point depends Lipschitz-continuously on the source, and unit translates of a source made of torus-scale waves differ by phases within $2\pi|m|/N$ of 1.

With matter, the same control gives stability of the actual coupled solution. Let the medium $\mathscr{A}_+$ of Section 4.6 have weights $\gamma_d > 0$ and amplitude λ, write $\sigma = \alpha(2\pi/N)^2$, and call the constant strains together with the static vertex moves whose integer wave numbers satisfy $|m_a| < M$ the *radical band*.

Theorem 4.14 (Coupled solutions at fixed strength, and their stability)**.** *In the setting of Theorem 4.12, with medium weights $\gamma_d > 0$ and $M \geq 1$, there is $\delta_* > 0$, and for every $\delta \leq \delta_*$ there are λ_0 and C_0, C_w, C_t, C_k, independent of λ, α and N, such that for every $\lambda \geq \lambda_0$ there is $\alpha_0 > 0$ with the following property. For $0 < \alpha \leq \alpha_0$, $\eta > 0$ and all large N there is exactly one pair of a band-limited field $w \perp \mathscr{R}_N$ and a field τ in the radical band, with $\|w\|_{\mathrm{J}} \leq C_w\alpha$ and $\|\tau\|_{\mathrm{J}} \leq C_t\alpha$, that satisfies (i), and it satisfies (ii)–(iv):*

(i) the coupled equations $DS(\ell^0 + w + \tau)[v] = \sigma\big(\lambda\, D\mathscr{A}_+(\ell^0 + w + \tau)[v] + j_N(v)\big)$ hold for every band-limited $v \perp \mathscr{R}_N$ and every v in the radical band;

(ii) there is $r \in \mathscr{R}_N$ with $|(w + \tau - r)_e/\alpha - h_\infty(X_e, d_e)| \leq C_0(1 + \lambda)\alpha + \eta$ at every edge;

(iii) $w + \tau$ is smooth to third order at amplitude $C_k\alpha$;

(iv) for every $t \in \mathscr{R}_N$, with $p_{\min} > 0$ depending only on κ and the weights,

$$\frac{\sigma\lambda\, p_{\min}}{4} \sum_{e \text{ timelike}} t_e^2 \;\leq\; D^2S(\ell^0 + w + \tau)(t,t) - \sigma\lambda\, D^2\mathscr{A}_+(\ell^0 + w + \tau)(t,t).$$

Proof outline. The pair comes from a nested contraction: the band solver of Theorem 4.12 for w, and for τ a solver of the radical band rows, where the medium's Hessian is negative definite on timelike components. Its contraction constants do not depend on λ, which is why λ_0 can be fixed before α and N; the contraction also gives uniqueness in the stated balls. Translation covariance of the radical band, of the medium rows and of the loads, together with Lipschitz dependence of the pair on the source, gives (iii) as in Theorem 4.13(ii). Then Theorem 4.13(i) bounds the Regge part of the Hessian on $\mathscr{R}_N$ by $CC_k\alpha/N^2$ times $\|t\|^2$; a flat-family tangent is controlled by its timelike components, $\|t\|^2 \leq 701 \sum_{\text{timelike}} t_e^2$; and near the flat lattice the medium contributes a fixed positive multiple of $\sigma\lambda\, p_{\min}$ times the timelike sum. With $\sigma = \alpha(2\pi/N)^2$ the two terms scale alike in N and α, so the medium wins once $\lambda\, p_{\min}$ exceeds a fixed multiple of CC_k, which gives (iv). □

Read physically, (iv) says that the second variation of the coupled action at the coupled solution is coercive in every direction of the flat family, vertex moves at every wavelength and constant strains, on every large torus and with one medium amplitude: the solution is linearly stable in those directions. The positive term is the medium's. Regge's own second variation in these directions is only of curvature order (Theorem 4.13(i)), so the stability is supplied by the spring of Section 4.6, which is a MODEL. The equations tested against fields with zero average along the static orbits $x + k\mathbf{1}$ hold as well, but by symmetry alone: for every static configuration, the flat field included, both sides of those equations vanish separately, so they carry no information about the solution beyond its being static. Four things separate these statements from a full fixed-strength theorem. Not every edge is imposed: the static rows off the band, which include the static resonance, the rows of static vertex moves with some $|m_a| \geq M$, and the rows of any static zero mode of H_0 outside $\mathscr{R}_N$ (none is known on the admissible sides) are left out, and imposing the remaining radical rows needs a solver that does not lose derivatives, for which (iv) is the linear ingredient. The bounds cover the flat family only, which at $\kappa = 1$ on the odd sides not divisible by 105 is the whole kernel of H_0. The strength is fixed but small, and α_0, ρ are not computed, so the limit object is still the linearized Einstein field, now with an error $C\alpha$. And the medium is the spring of Section 4.6, with an amplitude independent of N and α but dependent on the profile of the source.

The next order. The band solution of Theorem 4.12 can be followed one order further, still uniformly in N. Write $\mathcal{S}_N$ for the band solver: for a linear load f on band-limited fields, $\mathcal{S}_N f$ is the band-limited field $u \perp \mathscr{R}_N$ with $D^2S(\ell^0)(u, v) = f(v)$ for every band-limited $v \perp \mathscr{R}_N$.

Theorem 4.15 (Second order at fixed strength)**.** *In the setting of Theorem 4.12, put*

$$w_1 = \mathcal{S}_N\Big[\alpha\Big(\frac{2\pi}{N}\Big)^2 j_N\Big], \qquad w_2 = -\mathcal{S}_N\Big[\tfrac{1}{2}D^3S(\ell^0)(w_1, w_1, \cdot)\Big].$$

So w_2 is the band-limited field orthogonal to $\mathscr{R}_N$ with $D^2S(\ell^0)(w_2, v) = -\frac{1}{2}D^3S(\ell^0)(w_1, w_1, v)$ for every band-limited $v \perp \mathscr{R}_N$. There are $C_2, C_3 > 0$, chosen before α, η and N, such that for every $\alpha \le \alpha_0$, every $\eta > 0$ and every $N \ge N_0$ as in Theorem 4.12, the band solution w satisfies $\|w_2\|_{\mathrm{J}} \le C_2\alpha^2$ and $\|w - w_1 - w_2\|_{\mathrm{J}} \le C_3\alpha^3$.

Proof outline. The band equations are the fixed-point problem $w = \mathcal{S}_N[f - R(w)]$, with f the load and $R(w) = DS(\ell^0 + w) - D^2S(\ell^0)(w, \cdot)$ the nonlinear part of the Regge residual. Its quadratic part is $R_2(w) = \frac{1}{2}D^3S(\ell^0)(w, w, \cdot)$, and a cubic Taylor estimate in the norm of absolute Fourier coefficients, uniform in N, bounds $R(w) - R_2(w)$ by $C\|w\|_{\mathrm{J}}^3/N^2$. The solver gains the factor N^2 back, as in Theorem 4.12. So $w - w_1 - w_2 = -\mathcal{S}_N\big[(R(w) - R_2(w)) + (R_2(w) - R_2(w_1))\big]$ is at most a constant times $(\|w\|_{\mathrm{J}} + \|w_1\|_{\mathrm{J}})^3$, because $R_2(w) - R_2(w_1) = \frac{1}{2}D^3S(\ell^0)(w - w_1, w + w_1, \cdot)$ and $\|w - w_1\|_{\mathrm{J}}$ is of order α^2. □

So through second order, in the jet norm and with constants independent of N and α, the band solution of the Regge equations is the band response to the source plus the band response to minus half the Regge cubic vertex on the first-order field: the lattice form of the second-order Einstein equation with the source held fixed. The scope is definite. The theorem concerns pure gravity, with no medium, and the band equations with a fixed external static source. The equation for w_2 is projected off the flat radical: the gauge (vertex-move) rows and the constant-strain rows are not imposed, nor are the static rows off the band. At second order the projection matters, because $R_2(w_1)$ has a part at zero wave number, and its constant-strain component carries the virial constraint. With a fixed external source we do not expect the full second-order equations to be solvable: by the standard argument from the Bianchi identity, the source would have to be conserved in the corrected metric (INFERENCE; not proved here). A physical second order needs matter that responds with its own equations.

As the torus grows, both orders have explicit limits. Assume in addition that no wave of the source occurs together with its negative, $F \cap (-F) = \emptyset$. Each pair $\pm m$ then carries one real conserved stress T_m with a complex amplitude c_m, so the cosine and sine parts of the source at $\pm m$ are proportional; static sources whose cosine and sine parts at one wave carry different stresses are not covered. Let L be the two-derivative part of the lattice cubic vertex: the coefficient of t^2 in the Bloch form of $D^3S(\ell^0)$ at waves of size t on three nearly metric legs, written with the phases taken at the edge midpoints. In that form the vertex has no term of first order in t, because the lattice is invariant under point reflection, and its zero-momentum term vanishes whenever two legs are strains, by the second-derivative identity of the flat family.

Theorem 4.16 (The second-order field as the torus grows)**.** *In the setting of Theorem 4.12, let $F \cap (-F) = \emptyset$. Put*

$$H_1(X) = \sum_{m \in F \cup -F} \mathrm{Re}\big(e^{im\cdot X} A_1(m)\big), \qquad H_2(X) = \sum_{p \neq 0} \mathrm{Re}\big(e^{ip\cdot X} A_2(p)\big),$$

the second sum over the waves $p = m + m'$ with $m, m' \in F \cup -F$, where

$$A_1(m) = \frac{P_m\, VC'^{-1}\, \Pi_m\, S_1(m)}{q_\kappa(m)}, \qquad A_2(p) = \frac{P_p\, VC'^{-1}\, \Pi_p\, S_2(p)}{q_\kappa(p)},$$

$$S_2(p)_r = \frac{1}{2} \sum_{m \in F \cup -F} L\big(A_1(m), A_1(p-m), E_r\big).$$

Here $S_1(m)$ is the strain source of the mode m, E_r runs over the ten strain directions, and Π_p, P_p remove the limiting gauge directions at the wave p, in strain and in edge coordinates. There is $\delta_* > 0$, and for every $\delta \le \delta_*$ there are $\alpha_0, C, C_r > 0$ independent of N, such that for every $0 < \alpha \le \alpha_0$ the band solutions w_N of Theorem 4.12, one on each large torus and fixed before η (the only band-limited field orthogonal to $\mathscr{R}_N$ with $\|w_N\|_{\mathrm{J}} \le C_r \alpha$ that solves the band rows), satisfy: for every $\eta > 0$ there is N_0 with

$$\left| \frac{w_N(e)}{\alpha} - H_1(X_e, d_e) + \alpha H_2(X_e, d_e) \right| \le C\alpha^2 + \eta \qquad \text{at every edge } e, \text{ for } N \ge N_0.$$

Moreover H_1 is the linearized Einstein field h_∞ of Eq. (4) *minus a linearized diffeomorphism.*

Proof outline. By Theorem 4.15 and the homogeneity of the band solver, $w_N = \alpha W_1 - \alpha^2 W_2 + O(\alpha^3)$ in the jet norm, with W_1 the band response to the unit source and W_2 the band response to $\frac{1}{2} D^3 S(\ell^0)(W_1, W_1, \cdot)$, both independent of α; the jet norm bounds every edge. It remains to show $W_1 \to H_1$ and $W_2 \to H_2$ uniformly in the edge. The cubic vertex on the torus is an exact convolution of three plane waves, so the second-order source has exact mode amplitudes, one half of the Bloch vertex summed over the pairs of first-order modes. On nearly metric legs the Bloch vertex divided by t^2 tends to L; the non-metric parts of the first-order modes are of order t^2, and only the metric rows of a source of order t^2 survive in the limit of the gauge-orthogonal lattice response, whose scaled inverse tends to the Green matrix of Theorem 4.6. At $p = 0$ the band solver sets the constant-strain part to zero, and the remaining zero-mode amplitude tends to 0. For H_1, conservation of the source makes Π_m act as the identity on it, so H_1 differs from h_∞ only by the edge reading of a first-order vertex move. □

So the second-order band field has a continuum limit, and the limit is explicit. Every step is a statement about lattice objects, and the constant C is fixed before α, η and N; the threshold N_0 depends on α, so the statement is not uniform in α. The scope is that of Theorem 4.15: band rows only, so H_2 is the second-order field projected off the gauge directions at every nonzero wave, without a zero mode, in the band solver's gauge. Nothing here proves $H_2 \neq 0$ for a given source. The theorem is stated through L. Separately, on metric strains (the squared-length changes $d^\top h d$ read at the edge midpoints) L is one half of the Einstein–Hilbert cubic vertex on the background g_κ, for every $\kappa > 1/2$, the same factor as at quadratic order. The two halves cancel against each other in A_2, because the Green matrix comes from the same lattice action, so H_2 is the second-order field of general relativity for the first-order field H_1, with

these projections; the algebra is exact (it is part of the supplementary computations), and the normalization of the continuum action is the reading we give it (INFERENCE). That the lattice graviton's whole long-wavelength cubic coupling is the Einstein–Hilbert one needs more. The couplings of two strains to the five non-metric edge modes per vertex have no part with zero or one derivative, at every $\kappa > 1/2$. But that these modes do not change the two-derivative part when they are integrated out needs a bounded propagator for them, which is not proved. Theorem 4.16 does not need it, since it concerns the band solution itself.

4.11 Matter with equations of its own

The medium of Section 4.6 holds the gauge and strain rows by a stiffness about the flat lattice, and it has no equations of its own. Appendix E shows what matter with an action of its own does at fixed strength. A posting gas, free point particles on the long edges ($|d| \geq 2$), held in place by a support stress that is the carrier of least cost J for its load, solves the long-wave equations at fixed strength, uniformly in the torus size, with no medium; it stays positive, gives the same first-order limit, and has its second order controlled on the lattice (Theorems E.1 to E.3). The support is a stress, like the internal stress of a solid, and not a flow of postings: matter whose postings move along lattice worldlines does not feel a family of transverse vertex moves, carries exactly the loads that vanish on them, and fails to carry the second-order load of an explicit two-mode body (Theorems E.4 and E.5). The least-cost principle and its background weights are identifications, so these results do not enter the linear limit above, and we state them in the appendix.

4.12 Scope of the classical results

This subsection is the one place where the qualifications of the classical results are collected. The theorems hold under a stated MODEL for the matter, and the requirement they address is met in that qualified sense.

- *Which equations.* The coupled solutions solve the unconstrained Regge equations with matter on all fifteen classes, in the Lorentzian chamber; they are related to the path sum on the null stratum at linear order only (Section 3.4).
- *The medium is a model.* With its fixed tension it is exactly a spring on the proper lengths of the timelike edges whose rest lengths are the flat lattice. It is a penalty that holds the gauge and strain rows, of the same kind as an elastic background about the flat lattice. It has no degrees of freedom of its own. The posting gas of Appendix E is matter with an action of its own, and at fixed strength it replaces the medium on the long-wave rows; its equation of state rests on the least-work identification, and its support is a stress, not a flow of postings (Theorem E.4).
- *What is fixed about it, and what is not.* Part of its content is fixed by the equations. A flat vacuum with a posting gas and any further term L requires $DL(\ell^0)$ to equal the gas's flat stress on all fifteen classes; this is the vacuum condition itself. The Gauss law alone forces it on the ten constant strains, with positive trace, so some tension is needed. The compensation is fixed on the radical: it vanishes on vertex moves and, on the constant strains, equals the flat source of the mean density, the stress of uniform dust at rest and

never a cosmological term Λg^{-1}; among uniform clock backgrounds only the mean density works. Off the radical any further conserved source may be added; its homogeneous part leaves the limit unchanged, and the inhomogeneous part is not addressed.

- *The same medium written with a cost.* Let $J(x) = \frac{1}{2}(x + x^{-1}) - 1$, the cost function of the motivating programme [4], which we call the *recognition cost*; it is convex, with $J(1) = J'(1) = 0$ and $J''(1) = 1$. The medium $-\sum_{e \text{ timelike}} \gamma_d(\tau_e^0)^2 J(\tau_e/\tau_e^0)$ has no tension term and the stiffness of the spring, and exactly $-J(\tau_e/\tau_e^0) = -(\tau_e - \tau_e^0)^2/(2\tau_e^0\tau_e)$, so it is again a rest-length spring about the flat lattice. Flat space is a vacuum for a J-cost medium only if every reference time is the flat proper time, which selects the reference but does not derive the medium. With it, one amplitude fixed before the body and the side, $\lambda > 835828938$, serves every odd side not divisible by 105 for every band-limited body with $|\bar{\rho}| + \sum|c_m| \le 1$; at fixed strength it can replace the spring outright, with every statement of Theorem 4.14 and the non-static rows, keeping at least half its stiffness for $|w_e| \le (2\kappa - 1)/4$. That a medium fills the timelike edges and pays J of each edge's proper-time ratio, with its weights and amplitude, is an identification (IDENTIFICATION); it is still a potential about a fixed reference, a model on the matter side and not a rewriting of the gravitational action.

- *Weak-field order.* In Theorems 4.8 to 4.10 the coupling goes to zero on each torus before the torus grows, and the constants σ_0, B, C, r_0 are proved on each torus but are not uniform in its size. At fixed strength, Section 4.10 gives N-uniform statements for the band rows, the radical band, the stability of the coupled solution in the flat-family directions and the second-order field with its limit, not a solution on every edge; Appendix E gives statements of the same kind with the gas in place of the medium, without the stability.

- *Which sides.* The solution theorems use the side only through the flat kernel. The flat kernel holds on every odd side not divisible by 105, and so do the coupled solutions with the resting body (Theorem 4.8), the weak-field long-wave limit for fixed test sources (Theorem 4.9) and for the body, and the compact-body Newton field (Theorem 4.10). These sides form an unbounded family that contains every odd prime. The odd multiples of 105 are open even for the flat kernel, and even sides fail. The fixed-strength statements of Section 4.10 and Appendix E impose only the long-wave rows and hold on every large torus.

- *Linear limit, in a lattice gauge.* The limit object is the linearized Einstein field, read after a lattice gauge transformation and a flat deformation are removed (or, for the time-time component, after time averaging up to one constant). Nonlinear Einstein solutions are not reached. At fixed strength the second-order band field is identified on the lattice uniformly in N and has an explicit limit (Theorems 4.15 and 4.16), for pure gravity with a fixed external source and the band rows only; nothing proves it nonzero. With the medium at a fixed amplitude the coupled pair of Theorem 4.14 is also determined through second order, and numerically the medium's contribution then has a nonzero, anisotropic long-wave limit of the form of a mass term (NUMERICAL). We infer that any medium anchored to the flat lattice gravitates at second order like a Lorentz-violating graviton mass, so that Einstein's second order needs matter with its own equations (INFERENCE), such as the posting gas of Appendix E, whose second order is controlled on the lattice (Theorem E.3(ii)).

5 What the lattice predicts at tree level: no signature

A lattice with a physical spacing breaks Lorentz invariance at the spacing. The question for experiment is whether any consequence survives at accessible wavelengths. For linearized pure gravity on the fixed $\kappa = 1$ Kuhn lattice the answer at tree level is no, in every direction, with explicit constants. Everything in this section concerns long waves, near the origin of the Brillouin zone; the massless modes at the corners of the zone are the subject of Section 6.

5.1 Frame

For a real covector u (a lattice phase) write the frequency $\omega(u) = u \cdot \hat{t} = \sum_a u_a/(2\sqrt{3})$, the spatial wave number $|\mathbf{k}| = |\mathbf{k}(u)|$ with $|\mathbf{k}|^2 = \sum_a u_a^2 - \frac{1}{4}(\sum_a u_a)^2$, and $|u|_1 = \sum_a |u_a|$. Then

$$q(u) = |\mathbf{k}|^2 - \omega^2, \qquad \sum_a u_a^2 = |\mathbf{k}|^2 + 3\omega^2, \qquad |u|_1^2 \le 4 \sum_a u_a^2.$$

In physical units $|\mathbf{k}| = k_{\mathrm{ph}} b$ with k_{ph} the physical wave number and b the lattice spacing, and the phase speed is $|\omega|/|\mathbf{k}|$. A *non-gauge zero mode* at u is a real $h \in \mathbb{R}^{15}$ with $B_1(u)h = 0$ that is not a vertex move; complex Bloch modes reduce to real ones because the midpoint-frame Bloch matrix is real.

5.2 Every direction

Theorem 5.1 (The tree-level cone in every direction)**.** *Let $\kappa = 1$. For every $u \in \mathbb{R}^4$ with $|u|_1 \le 10^{-3}$ and every non-gauge zero mode h at u,*

$$|q(u)| \le 3.4 \times 10^{13}\, |u|_1^4.$$

Proof outline. Subtract a vertex move so that h satisfies the de Donder condition exactly; the gauge block is diagonally dominant at small $|u|_1$. Split h into its strain part, in the range of V, and a complement. The complement equation is invertible on the nondegenerate block of the zeroth-order operator, so the complement is $O(|u|_1^2)$ times the strain part. Projecting the zero-mode equation onto strains and using the exact de Donder identity $V^\top(M_2 + \ell Q_{\mathrm{dD}} \ell^\top)V = q(u)C'$ of Section 4, the ghost readout and the Bianchi identity, $q(u)$ times a unit strain is bounded by $O(|u|_1^4)$ terms. Every constant is a row or entry bound of an exact rational table, and their product is $33\,827\,196\,049\,920 < 3.4 \times 10^{13}$. □

Corollary 5.2 (No slow, fast or extra branch)**.** *If $0 < |u|_1 \le 10^{-12}$ and h is a non-gauge zero mode at u, then $|\mathbf{k}| > 0$ and*

$$\left| \frac{|\omega|}{|\mathbf{k}|} - 1 \right| \le 5 \times 10^{15} |\mathbf{k}|^2.$$

Proof. By Theorem 5.1 and the frame identities, $||\mathbf{k}|^2 - \omega^2| \le 3.4 \times 10^{13} |u|_1^2 \cdot |u|_1^2 \le 4 \cdot 10^{-10}(|\mathbf{k}|^2 + 3\omega^2)$. If $|\mathbf{k}| = 0$ this forces $\omega = 0$ and $u = 0$; hence $|\mathbf{k}| > 0$, and $|\omega|$ lies within 1% of $|\mathbf{k}|$. Then $|u|_1^2 \le 4(|\mathbf{k}|^2 + 3\omega^2) \le 16.25\, |\mathbf{k}|^2$ and $|\omega| + |\mathbf{k}| \ge 1.99\, |\mathbf{k}|$, so $\big||\omega| - |\mathbf{k}|\big| = |q|/(|\omega| + |\mathbf{k}|) \le 3.4 \times 10^{13} \cdot 16.25^2\, |\mathbf{k}|^3/1.99 < 5 \times 10^{15} |\mathbf{k}|^3$. □

So at long wavelength every non-gauge mode in every direction propagates on the light cone up to a relative correction of order $(k_{\mathrm{ph}}b)^2$. There is no slow or fast branch, and near the origin of the zone ($|u|_1 \le 10^{-12}$) no additional low-frequency mode. Away from the origin the operator does have further massless modes: the doublers at the four axis corners (Proposition 6.1), each with its own cone; they are lattice-scale modes, the statements of this section do not cover them, and their effect is a one-loop question (Section 6). The rate is sharp in form: along the paired direction (Theorem 5.5) the cross mode's speed differs from 1 by exactly $\frac{5}{24}|\mathbf{k}|^2$ at leading order, so no bound $o(|\mathbf{k}|^2)$ holds in every direction.

Corollary 5.3 (Gravitational waves). *Let $0 < b \le 10^6 \ell_P$, with $\ell_P = 1.616255 \times 10^{-35}$ m. For every gravitational wave of frequency at most 5000 Hz, in every direction, every non-gauge mode of wavelength at least $2\pi \times 10^{13}$ spacings has phase speed within 2×10^{-50} of light, and over the Hubble distance 1.4×10^{26} m its phase drifts by at most 3×10^{-28} rad.*

These numbers follow from Corollary 5.2 with $|\mathbf{k}| \le \frac{100}{99}\, 2\pi f b/c \le 1.72 \times 10^{-33}$. The constants of Theorem 5.1 and Corollary 5.2 are products of row and entry bounds of exact rational tables; they are rigorous and crude. Along the paired direction, where the coefficients are exact (Theorem 5.5), the plus mode's speed differs from 1 by $\frac{59}{24}(k_{\mathrm{ph}}b)^2 \approx 7 \times 10^{-66}$ and its phase drifts by about 10^{-43} rad over the same distance; whether a bound of that size holds in every direction is open. All these numbers lie far below the GW170817 bound on the speed of gravity, $-3 \times 10^{-15} \le v/c - 1 \le 7 \times 10^{-16}$ [13]. One identification is needed to compare with photons: "light" above is the null cone $q = 0$ of g_1, and reading GW170817 as a constraint on it assumes photons propagate on the same cone. That is an identification about the photon sector, which this paper does not derive; it is not part of the identification $\kappa = 1$, which concerns the posting order and the graviton's cone (Section 2.2).

The static field. Let $\chi_{tt}(k)$ be the time-time response of the gauge-fixed $\kappa = 1$ operator to a time-edge source at static wave $k \in \mathbb{R}^3$ (slice coordinates), $q(k)$ the static quadratic form and $c_{\mathrm{N}} = 8\pi G_s \cdot 2M_p$ the Newton coefficient, $G_s = 1/(8\pi s)$, for a source of one-site proper mass $M_p \ne 0$.

Theorem 5.4 (Newton's law to second order). *Let $\kappa = 1$, $s \ne 0$, $M_p \ne 0$. For $0 < \|k\| \le 10^{-10}$,*

$$\left| \frac{\chi_{tt}(k)\, q(k)}{c_{\mathrm{N}}} - 1 \right| \le 2.2 \times 10^{26} \|k\|^2.$$

The correction is second order: the same algebra as Theorem 5.1 on static covectors, with $q \ge |u|_1^2/4$ and the exact Newton readout supplying the lower bound on the leading term. For spacing $b \le 10^6 \ell_P$ and probe lengths $\ell \ge 52\,\mu$m, the shortest separation at which the inverse-square law has been tested [15], the lattice waves satisfy $\|k\| \le b/\ell < 3.11 \times 10^{-25}$ and the relative correction per wave is below 3×10^{-23}. This is a bound on the response kernel wave by wave; a pointwise bound on the potential of an extended body needs a further step, which Theorem 4.10 supplies for band-limited smooth bodies, not with these explicit constants.

5.3 The paired direction

One direction can be solved exactly. Take $u = (z+t, z+t, z-t, z-t)$: then $q(u) = 4t^2 - \frac{4}{3}z^2$, $|\mathbf{k}| = 2t$, $\omega = 2z/\sqrt{3}$, and the light cone is $z = \sqrt{3}t$.

Theorem 5.5 (Paired-direction dispersion)**.** *Let $\kappa = 1$. Near the cone ($0 < t \le 2\times10^{-4}$, $|z - \sqrt{3}t| \le \sqrt{3}t/20$) and at low frequency ($0 < z \le 1/5$, $z \ne t$), every zero mode of the Bloch operator is a vertex move plus a plus-polarized mode plus a cross-polarized mode, and each polarized mode exists. Their phase and group speeds satisfy*

$$|v_{\mathrm{ph}}^{+} - 1 + \tfrac{59}{24}|\mathbf{k}|^2| < \tfrac{25}{16}|\mathbf{k}|^3, \qquad |v_{\mathrm{g}}^{+} - 1 + \tfrac{59}{8}|\mathbf{k}|^2| \le 4000|\mathbf{k}|^4,$$
$$|v_{\mathrm{ph}}^{\times} - 1 - \tfrac{5}{24}|\mathbf{k}|^2| < \tfrac{25}{16}|\mathbf{k}|^3, \qquad |v_{\mathrm{g}}^{\times} - 1 - \tfrac{5}{8}|\mathbf{k}|^2| \le 400|\mathbf{k}|^4,$$

and $v^{+} < v^{\times}$ for both speeds.

The plus mode is subluminal and the cross mode superluminal at order $(k_{\mathrm{ph}}b)^2$: the lattice graviton is birefringent along this direction. With $U = \sin^2(t/2)$, $V = \sin^2(z/2)$ the cross mode is the exact curve $-6UV + 3U + 4V^2 - V = 0$ and the plus mode a degree-six curve; the windows are root enclosures for $\eta = V/(3U) - 1$ (Appendix F). At $b \le 10^6\ell_P$ and $f \le 5000\,\mathrm{Hz}$ the two polarizations accumulate a phase difference of at most 1.2×10^{-43} rad over 1.4×10^{26} m. Whether every direction has a $(k_{\mathrm{ph}}b)^2$ law with definite coefficients, beyond the bound of Corollary 5.2, is open.

5.4 The quantum channel

At one loop the graviton dispersion could acquire a mass, a width or a shift of the cone, suppressed only by the loop factor and not by $(k_{\mathrm{ph}}b)^2$ (INFERENCE). Write the one-loop dressed transverse-traceless frequency in a given direction and polarization, on its real branch, as

$$|\omega(|\mathbf{k}|)^2 - m^2 - (1+\delta)^2|\mathbf{k}|^2| \le C|\mathbf{k}|^4 \qquad (0 < |\mathbf{k}| \le k_0). \tag{6}$$

Reading ω as that lattice frequency is an IDENTIFICATION; no theorem links Eq. (6) to a lattice computation, and δ and m^2 are not computed. What can be proved is what Eq. (6) would imply.

- With $e_{\mathbf{k}} = (|m^2|/|\mathbf{k}|^2 + C|\mathbf{k}|^2)/(1+\delta)$, the phase speed obeys $|\omega/|\mathbf{k}| - 1 - \delta| \le e_{\mathbf{k}}$. If $\delta - e_{\mathbf{k}} > 7\times10^{-16}$ or $\delta + e_{\mathbf{k}} < -3\times10^{-15}$, the channel contradicts GW170817.
- If the cone shift is a loop effect, $\delta = 8\pi(\ell_P/b)^2 d$ with d the dimensionless one-loop coefficient and b the spacing (IDENTIFICATION, from $8\pi G\hbar = 1/(as)$ per spacing squared). At $b \le 2\ell_P$, and with $e_{\mathbf{k}} \le 10^{-16}$, every $|d| \ge 10^{-14}$ is excluded.
- Reading m through $E^2 = p^2c^2 + m^2c^4$ in lattice units, $m^2 = (rb/\ell_P)^2$ with $r = m_g c^2/E_P$, the bound $m_g c^2 \le 1.27\times10^{-23}$ eV [14] excludes every $m^2 \ge 10^{-100}$ in lattice units at $b \le 2\ell_P$.

Since ω is real in Eq. (6), it forces $m^2 \ge 0$ and carries no width. The one-loop results of Section 6 lie outside this form: at identified $\kappa = 1$ the ordered zero-momentum anisotropy diverges

(Theorem 6.8), and the long-wave transverse-traceless self-energy has a gauge-independent part that is numerically large and imaginary (Section 6.8). Whether an imaginary part can be read as a width at all depends on the status of the real-time path sum as a quantum evolution in record time, which Section 6.8 discusses before drawing any observational conclusion. So the tree-level requirement is met, with an explicit statement that pure gravity on the lattice has no observable signature there, and its quantum part is not.

6 The quantum graviton at identified $\kappa = 1$

A massless graviton needs more than the right tree-level propagator. At one loop the self-energy must not generate a mass, a width or an anisotropy that survives the removal of the regulator. On the lattice this is a precise question, because the order of limits is fixed: first the torus grows, $N \to \infty$ at a fixed regulator $\varepsilon > 0$, and then $\varepsilon \to 0^+$. At identified $\kappa = 1$ the answer is negative: the zero-momentum anisotropy of the one-loop self-energy grows like $\log(1/\varepsilon)$, with an explicit coefficient, and the ordered limit does not exist (Theorem 6.8).

The mechanism. The gauge-fixed tree operator has four exact non-gauge zero modes at the axis corners πe_c of the Brillouin zone: massless *doublers*, each with its own Lorentzian cone (Section 6.2). A constant strain moves their zero at first order, so the doublers' squared mass is linear in the background strain (Section 6.3). Their one-loop vacuum energy then has a non-analytic term of the form $\delta^2 \log \delta$ in the strain δ at the flat lattice, as in the Coleman–Weinberg potential [54], and its second derivative, which is the zero-momentum self-energy, diverges logarithmically as the regulator is removed. In the language of Feynman integrals each corner is a pinch: the doubler eigenvalue vanishes quadratically there, with no direction along which it increases, so no deformation of the integration contour avoids the $i\varepsilon$ prescription. Every other degenerate zero of the operator has a direction along which all its vanishing eigenvalues increase (Theorem 6.7), so the contour can be moved off it, as in the analysis of Landau singularities [64, 65, 66]; those zeros give finite contributions, and nothing cancels the corners' logarithm. This paragraph is a reading of the proof, not part of it.

Bosonic doublers. Doublers are familiar for lattice fermions, where Nielsen and Ninomiya showed they are hard to avoid [12]. Here they arise for a boson, for a different reason: the zero set of a Lorentzian symbol, the light cone, is unbounded, and on the lattice it reaches the boundary of the Brillouin zone. A Euclidean lattice operator that is positive away from the origin has no such zeros. The doublers are therefore a feature of real-time discretizations, and at $\kappa = 1$ they sit exactly at the axis corners because the postings are null.

The rest of the section. The one-loop statements use a diagram model, which we state as one (Section 6.1). We then ask whether anything in the construction gaps the doublers (Sections 6.5 and 6.6: nothing native does), in which time they propagate (Section 6.7), what the long-wave graviton's own zero-momentum self-energy contains and what an imaginary value of it can mean for a real-time path sum (Section 6.8), and what happens at nonzero momentum (Section 6.9). Section 6.10 states what would have to change. Throughout, theorems, exact

computations, numerical results and inferences are kept apart; none of this section touches the classical results of Sections 4 and 5.

6.1 The quantity

Let $B(k)$, $\Gamma(k)$ and $A(k) = B(k) + \Gamma(k)\Gamma(k)^\dagger$ be the Bloch matrix, the vertex-move columns and the gauge-fixed operator in the Feynman-type gauge (Section 4.2), and $P(k) = (A(k) + i\varepsilon)^{-1}$ the regulated propagator, with the damping on all fifteen channels. The interaction vertices are the Bloch forms of the third and fourth derivatives of the cell action: with $d_{\sigma f}$ and $b_{\sigma f}$ as in Appendix B,

$$\hat{T}_{dd'd''}(k_1,k_2,k_3) = \sum_{\sigma} \sum_{f,f',f''} [d_{\sigma f} = d][d_{\sigma f'} = d'][d_{\sigma f''} = d''] \times \frac{\partial^3 S_\sigma}{\partial \ell_f \partial \ell_{f'} \partial \ell_{f''}}\Big|_{\ell^0} e^{i(k_1\cdot b_{\sigma f} + k_2 \cdot b_{\sigma f'} + k_3 \cdot b_{\sigma f''})},$$

and $\hat{U}$ is defined in the same way from the fourth derivatives. For a constant strain H with profile $h_d = d^\top H d$ we put $V_h(k)_{d'd''} = \sum_d h_d \hat{T}_{dd'd''}(0,k,-k)$ and $W_h(k)_{d''d'''} = \sum_{d,d'} h_d h_{d'} \hat{U}_{dd'd''d'''}(0,0,k,-k)$. The matter is one massless scalar φ on the vertices, with the cell action $-\frac{1}{48}\sqrt{|\det G|}\,(\mathrm{d}\varphi)^\top G^{-1} \mathrm{d}\varphi$, where G is the Gram matrix of the cell's edge vectors computed from its squared lengths. On the flat $\kappa = 1$ lattice its symbol is $\hat{\mathscr{D}}(k) = \sqrt{3}\big[\sum_a (2 - 2\cos k_a) - \frac{1}{3}\big(2 - 2\cos \sum_a k_a\big)\big]$ and its propagator $P_{\mathrm{s}}(k) = (-\hat{\mathscr{D}}(k) + i\varepsilon)^{-1}$; $\hat{v}_h(k)$ and $\hat{w}_h(k)$ are its vertices with one and two strain legs at zero wave, the first and second variations of the scalar's action through G. The *diagram model* is

$$\sigma_N(\varepsilon; H) = \frac{1}{n_H}\Big\langle -\tfrac{i}{2}\,\mathrm{tr}\big(V_h(k) P(-k) V_h(k)^\dagger P(k)^\top\big) + \tfrac{i}{2}\,\mathrm{tr}\big(W_h(k) P(k)^\top\big) - 2i\,P_{\mathrm{s}}(k)^2 |\hat{v}_h(k)|^2 + i\,P_{\mathrm{s}}(k)\,\hat{w}_h(k)\Big\rangle_N, \quad (7)$$

with $\langle\cdot\rangle_N$ the mean over the N^4 torus waves and $n_H = \mathrm{tr}(g_1^{-1} H g_1^{-1} H)$. Its four terms are the graviton sunset and seagull and the scalar sunset and seagull at zero external momentum; the coefficients are those of the second strain derivative of the one-loop effective action $\frac{i}{2}$ Tr log of the regulated Hessians, and a tadpole, in which a line absorbs the model's one-loop response to its own fluctuations, vanishes on the probes below. We take Eq. (7) as the definition of the model, and it is a choice (MODEL): the gauge term with unit weight, no ghost line, the Feynman sign of $i\varepsilon$, and one massless scalar for matter.

Two probes matter, both spatial ($H\mathbf{1} = 0$) and traceless ($\mathrm{tr}(g_1^{-1}H) = 0$):

$$H_E = \begin{pmatrix} 0 & 2 & -1 & -1 \\ 2 & 0 & -1 & -1 \\ -1 & -1 & 0 & 2 \\ -1 & -1 & 2 & 0 \end{pmatrix}, \qquad H_{T_2} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \qquad n_{H_E} = 24, \quad n_{H_{T_2}} = 8,$$

one in each of the irreducible representations E and T_2 of the axis permutations on traceless strains. On such probes every covariant quadratic form $\alpha'\,\mathrm{tr}(g^{-1}Hg^{-1}H) + \beta'(\mathrm{tr}\, g^{-1}H)^2$ equals $\alpha' n_H$, so after division by n_H a covariant self-energy takes the same value on both, and their difference

$$\Delta_N(\varepsilon) = \mathrm{Re}\big(\sigma_N(\varepsilon; H_E) - \sigma_N(\varepsilon; H_{T_2})\big)$$

measures a zero-momentum anisotropy; for a massless graviton with a Lorentz-invariant long-wave limit it must have a finite ordered limit. At each fixed $\varepsilon > 0$ the torus limit exists and equals a Brillouin-zone integral, $\lim_{N\to\infty} \Delta_N(\varepsilon) = \Delta(\varepsilon)$, and the matter-loop part has a finite ordered limit by itself. The question is whether $\Delta(\varepsilon)$ converges as $\varepsilon \to 0^+$.

6.2 Four exact zero modes

The Brillouin zone has sixteen half-period corners $\pi\delta$, $\delta \in \{0,1\}^4$. Call πe_c, with one odd coordinate, an *axis corner*.

Proposition 6.1 (Axis-corner zero modes). *Let $\kappa = 1$ and $c \in \{0,1,2,3\}$. Define $v_c \in \mathbb{R}^{15}$ by*

$$v_c(d) = \begin{cases} 2^{|d|-3}, & d_c = 0, \\ 0, & d_c = 1. \end{cases}$$

Then $B(\pi e_c)v_c = 0$ and $\Gamma(\pi e_c)^\dagger v_c = 0$, so $A(\pi e_c)v_c = 0$; the kernel of $A(\pi e_c)$ is spanned by v_c; $v_c^\dagger v_c = 31/16$; and v_c is not a vertex move.

Because v_c lies in the kernels of both B and $\Gamma^\dagger$, it is a zero mode of $B + \xi\Gamma\Gamma^\dagger$ for every gauge weight ξ: it is not an artefact of the gauge term. At the origin the kernel of A is exactly the ten constant strains, and A is invertible at every half-period corner with two, three or four odd coordinates, by exact factorizations of A at those corners; so the kernel has dimension 10 at the origin, 1 at each axis corner and 0 at the other eleven corners. We do not use the last fact below.

In position space the mode is $h(x,d) = (-1)^{x_c} v_c(d)$: a staggered field along axis c that lives on the edges of the orthogonal three-cube and doubles its amplitude with each extra bit. On an even torus πe_c is a lattice wave and this field is an honest zero mode of the flat Hessian that is neither a strain nor a vertex move; it is the zero mode that spoils the flat-kernel criterion on every even torus (Theorem 4.4). On an odd torus the corner itself is not a lattice wave, but the zero set around it is.

Proposition 6.2 (The corner cone). *Near $k = \pi e_0 + h$, the Schur complement of A onto the doubler direction is a smooth function $S(h)$ with*

$$S(h) = \frac{\sqrt{3}}{640}\, h^\top K h + o(|h|^2), \qquad K = \begin{pmatrix} -2 & 9 & 9 & 9 \\ 9 & 52 & -3 & -3 \\ 9 & -3 & 52 & -3 \\ 9 & -3 & -3 & 52 \end{pmatrix}, \qquad \det K = -1013375 = -55^2 \cdot 335,$$

and K has one negative and three positive eigenvalues; after completing squares, $h^\top K h = -2y_0^2 + \frac{55}{2}y_1^2 + \frac{55}{6}y_2^2 + \frac{335}{6}y_3^2$. Moreover, for every $\rho > 0$ there is a wave k with $|k - \pi e_0| < \rho$, all four phases $e^{ik_b} \neq 1$, and $\det A(k) = 0$.

Permutations of the four axes preserve the Kuhn lattice, g_1 and the vertex moves, and they permute the axis corners, so each axis corner carries the same cone in permuted coordinates. Each therefore carries a massless excitation with its own Lorentzian cone, whose time axis lies mostly along e_c: these are the doublers. They belong to the null postings: moving κ above 1 gives all four a mass at first order (Theorem 6.10).

6.3 How the doublers couple to constant backgrounds

A constant edge profile $f \in \mathbb{R}^{15}$ changes the squared length of class d by f_d; a constant strain H is the profile $f_d = d^\top H d$, and the strains form a ten-dimensional subspace. The first-order effect of f on the doubler at corner c is the Rayleigh derivative

$$\delta_c(f) = v_c^\dagger \, (\partial_f B)(\pi e_c)\, v_c,$$

the derivative along f of the Bloch matrix at the corner, paired with the zero mode. For a strain it is the first-order shift of the corner zero along the flat family $g_1 + tH$. Since the corner form K has one negative and three positive directions, a positive shift opens a gap, as a mass does, and a negative shift makes the doubler tachyonic.

Proposition 6.3 (Corner couplings and the sum rule)**.** *At $\kappa = 1$:*

(i) $\delta_c(H) = 0$ *at every corner for every strain of the irrep E and for the conformal strain $H = g_1$;*

(ii) for the T_2 probe the pairing is $\mp\sqrt{3}/32 \neq 0$, with sign $-$ at $c = 0, 1$ and $+$ at $c = 2, 3$;

(iii) for the strain $H = \mathbf{1}\mathbf{1}^\top$, which changes every squared length by $|d|^2$ and so moves κ, the shift is $-396/(1152\sqrt{3})$ at every corner;

(iv) for every real constant strain,

$$\sum_{c=0}^{3} \delta_c(H) = -\frac{396}{1152\sqrt{3}}\ \mathrm{tr}\, H;$$

(v) in particular the off-diagonal strain with $H_{01} = 1$, $H_{23} = -1$ (all $H_{aa} = 0$) shifts the four corners by $(+1, +1, -1, -1)/(16\sqrt{3})$.

The trace in (iv) has a direct meaning: H_{aa} is the change of the squared length of the posting along axis a, so $\mathrm{tr}\, H$ is the total change of the four posting lengths. The doubler couples to T_2 and to the strain $\mathbf{1}\mathbf{1}^\top$ without derivatives. A massless particle that couples without derivatives to a constant background produces, in a pair loop at zero external momentum in four dimensions, a logarithm of the infrared cutoff. The regulator ε is the only infrared cutoff here.

The sum rule is a statement about strains, that is, about flat backgrounds. A constant profile outside the strains describes a lattice that is not flat, and it can move all four corner zeros together.

Proposition 6.4 (Profiles that are not strains)**.** *Let $\kappa = 1$ and let f depend only on $|d|$, with values f_1, f_2, f_3, f_4. Then at every axis corner*

$$\delta_c(f) = \frac{-288 f_1 - 2592 f_2 - 2592 f_3 + 1728 f_4}{16 \cdot 1152\sqrt{3}}.$$

In particular the body-diagonal profile ($f_4 = 1$, all other values 0) leaves every posting null, is not a strain, and shifts all four corners by $+\sqrt{3}/32$.

The formula reproduces Proposition 6.3 on the strains among these profiles: $f_n = n^2$ is the strain $\mathbf{1}\mathbf{1}^\top$, and $f_n = n - n^2$ is g_1, which gives zero. Such a background is not a tree-level vacuum, since it is not flat. It matters because the theory produces one: see Section 6.6.

6.4 The logarithm

The local mechanism is a single integral, which can be done exactly.

Proposition 6.5 (The model double pole). *Let* $Q(y) = -y_0^2 + y_1^2 + y_2^2 + y_3^2$ *and, for* $\varepsilon > 0$,

$$I(\varepsilon) = \int_{\mathbb{R}^4} \frac{e^{-|y|^2}}{(Q(y) + i\varepsilon)^2}\, \mathrm{d}y.$$

Then

$$I(\varepsilon) = -\pi^2 \int_0^\infty e^{-\varepsilon\tau}\, \frac{\tau(1 + i\tau)}{(1 + \tau^2)^{3/2}}\, \mathrm{d}\tau,$$

so $\operatorname{Re} I(\varepsilon) \to -\pi^2$, *while* $\operatorname{Im} I(\varepsilon) - \pi^2 \log \varepsilon$ *converges as* $\varepsilon \to 0^+$. *In particular* $\operatorname{Im} I(\varepsilon) \to -\infty$, *with* $-\operatorname{Im} I(\varepsilon) \geq \pi^2 \log(1/\varepsilon)/(2\sqrt{2}\, e)$ *for* $0 < \varepsilon < 1$.

Proof. For $\operatorname{Im} z > 0$, $\int_0^\infty \tau e^{i\tau z}\, \mathrm{d}\tau = -1/z^2$; apply it with $z = Q(y) + i\varepsilon$. The integrand $\tau e^{-\varepsilon\tau} e^{-|y|^2}$ is absolutely integrable on $(0, \infty) \times \mathbb{R}^4$, so the order of integration may be exchanged, and the Gaussian integral factorizes:

$$\int_{\mathbb{R}^4} e^{-|y|^2 + i\tau Q(y)}\, \mathrm{d}y = \sqrt{\frac{\pi}{1 + i\tau}} \Big(\frac{\pi}{1 - i\tau} \Big)^{3/2} = \frac{\pi^2}{(1 - i\tau)\sqrt{1 + \tau^2}} = \frac{\pi^2 (1 + i\tau)}{(1 + \tau^2)^{3/2}},$$

with principal square roots. This gives the formula. Its real part has integrand $\tau e^{-\varepsilon\tau}(1+\tau^2)^{-3/2}$, dominated by an integrable function, with $\int_0^\infty \tau(1 + \tau^2)^{-3/2}\, \mathrm{d}\tau = 1$. Its imaginary part has integrand $e^{-\varepsilon\tau}\tau^2(1+\tau^2)^{-3/2} = e^{-\varepsilon\tau}\big((1+\tau)^{-1} + r(\tau)\big)$ with r integrable, and $\int_0^\infty e^{-\varepsilon\tau}(1+\tau)^{-1}\, \mathrm{d}\tau - \log(1/\varepsilon)$ converges. The lower bound follows from $\tau^2(1 + \tau^2)^{-3/2} \geq (2\sqrt{2}\,\tau)^{-1}$ for $\tau \geq 1$ and $e^{-\varepsilon\tau} \geq e^{-1}$ for $\tau \leq 1/\varepsilon$. □

Near πe_0 the structure of the actual integrand is exact. The determinant factors as $\det(A(\pi e_0 + h) + z) = (r(h) + z)\, \tilde{q}(h, z)$ with r the smooth doubler eigenvalue, $r(0) = 0$, and $\tilde{q}$ smooth and nonzero, so near the corner the regulated propagator has the single small denominator $r + i\varepsilon$, at k and at $-k$. For a constant strain H write $\mathscr{K}_{\varepsilon,H}$ for the graviton part (sunset and seagull, all fifteen channels) of the zone integrand of $\lim_N \sigma_N(\varepsilon; H)$; it includes the DeWitt denominator n_H of H. Near the corner

$$\mathscr{K}_{\varepsilon,H} = \frac{1}{n_H} \Big(-\frac{i}{2}\, \frac{N_2}{(r + i\varepsilon)^2 \tilde{q}^2} + \frac{i}{2}\, \frac{N_1}{(r + i\varepsilon)\, \tilde{q}} \Big),$$

with numerators N_1, N_2 that are smooth in the wave and in ε, and $N_2/\tilde{q}^2$ equals at the corner the *corner residue* $\rho_c(H) = \delta_c(H)^2/|v_c|^4$, the squared coupling of Proposition 6.3: 0 for the E probe and $3/3844$ for T_2, at every corner. At zero external momentum both lines of the sunset carry the same wave, so each corner contributes a double pole. Summing the four corners with the two probes' DeWitt normalizations defines the *doubler term*

$$D(\varepsilon) = \frac{\rho\, \operatorname{Im} I(\varepsilon)}{2|\det L|(2\pi)^4}, \qquad |\det L| = \frac{3\sqrt{1013375}}{1240^2},$$

$$\rho = \sum_{c=0}^{3} \Big(\frac{\rho_c(H_E)}{24} - \frac{\rho_c(H_{T_2})}{8} \Big) = -\frac{3}{7688},$$

where $|\det L|$ is the Jacobian of the chart of Proposition 6.6(i) below at the corner. By Proposition 6.5,

$$D(\varepsilon) = \frac{5}{44\sqrt{335}\,\pi^2}\,\log\frac{1}{\varepsilon} + O(1) \approx 6.3\times 10^{-4}\log\frac{1}{\varepsilon}. \tag{8}$$

This is a statement about the explicit function D. The next proposition makes it a statement about Δ near the corners and says exactly what the rest of the zone would have to do. Write $\mathscr{K}_\varepsilon = \mathrm{Re}\big(\mathscr{K}_{\varepsilon,H_E} - \mathscr{K}_{\varepsilon,H_{T_2}}\big)$ for the graviton part of the integrand of $\Delta(\varepsilon)$.

Proposition 6.6 (The four corners)**.** *Let $\kappa = 1$.*

(i) There are smooth coordinates $y = \phi(h)$ near $h = 0$, with $\phi(0) = 0$, in which the doubler eigenvalue is exactly the Lorentzian form: $r(\pi e_0 + h) = Q(\phi(h))$. Their Jacobian at the corner is $|\det D\phi(0)| = |\det L|$.

(ii) Each axis corner πe_c has a neighbourhood U_c such that, for every constant strain H and every cutoff χ of class C^8 with compact support in U_c,

$$\int \chi(k)\,\mathscr{K}_{\varepsilon,H}(k)\,\mathrm{d}k \;+\; \chi(\pi e_c)\,\frac{i\,\rho_c(H)}{2\,n_H|\det L|}\,I(\varepsilon)$$

has a finite limit as $\varepsilon \to 0^+$.

(iii) The transposition of axis 0 with axis c is an exact symmetry of the loop and carries πe_0 to πe_c. For the two probes the four corner model terms of (ii) are therefore equal, and their real parts, combined as in Δ and divided by $(2\pi)^4$, sum to $D(\varepsilon)$.

(iv) Take cutoffs χ_c as in (ii), equal to 1 near πe_c, with disjoint supports, and integrate over a period cell that has the four axis corners in its interior. For every $\varepsilon > 0$

$$\Delta(\varepsilon) = D(\varepsilon) + E(\varepsilon) + \Delta_{\mathrm{m}}(\varepsilon) + C(\varepsilon), \qquad C(\varepsilon) = \frac{1}{(2\pi)^4}\int\Big(1 - \sum_c \chi_c\Big)\mathscr{K}_\varepsilon\,\mathrm{d}k,$$

where $E(\varepsilon)$, the sum of the four corner remainders of (ii), and $\Delta_{\mathrm{m}}(\varepsilon)$, the matter loop, have finite limits as $\varepsilon \to 0^+$. Hence $\Delta(\varepsilon)$ has a finite limit as $\varepsilon \to 0^+$ if and only if $C(\varepsilon) + D(\varepsilon)$ has one, and $\Delta(\varepsilon) - D(\varepsilon)$ has one if and only if $C(\varepsilon)$ has one.

Proof outline. (i) By the Schur form of Proposition 6.2 and the corner value of the adjugate (the cofactor along the doubler direction is $\frac{16}{31}\tilde{q}(0,0)$), r vanishes to second order at the corner with quadratic part $\frac{\sqrt{3}}{1240}h^{\top}Kh$. Hadamard's lemma, applied twice to a cut-off copy of r, gives $r(\pi e_0 + h) = \sum_{i,j} h_i h_j S_{ij}(h)$ with S smooth and symmetric and $S(0) = \frac{\sqrt{3}}{1240}K$. In the coordinates in which K is diagonal (the completed squares of Proposition 6.2, a change of determinant 6), $S(0)$ has the diagonal $\lambda = \frac{\sqrt{3}}{1240}\big(-2, \frac{55}{2}, \frac{55}{6}, \frac{335}{6}\big)$. Completing the squares of $S(h)$ in the same order gives pivots that are smooth and have the signs $(-,+,+,+)$ near 0; their square roots times the completed linear forms are the coordinates y, and $|\det D\phi(0)| = 6\prod_a|\lambda_a|^{1/2} = 3\sqrt{1013375}/1240^2$. (ii) By the inverse function theorem ϕ is a diffeomorphism near 0. Changing variables turns the double-pole part of $\int \chi\mathscr{K}_{\varepsilon,H}$ into $\int w(y)\,(Q(y) + i\varepsilon)^{-2}\mathrm{d}y$ with a compactly supported weight w of class C^6, whose value at 0 is $\chi(\pi e_c)$ times $-\frac{i}{2}\rho_c(H)/(n_H|\det L|)$. The numerators depend on ε through a polynomial of degree two plus ε^3 times a bounded function;

the terms of order ε and ε^2 have vanishing limits because $\varepsilon I(\varepsilon) \to 0$, and the rest is controlled by $|r + i\varepsilon| \geq \varepsilon$. For the model step write $(Q + i\varepsilon)^{-2} = -\int_0^\infty \tau e^{i\tau(Q+i\varepsilon)}\,\mathrm{d}\tau$. The Gaussian part $w(0)e^{-|y|^2}$ gives $w(0)I(\varepsilon)$, and its cut-off tail decays like $\tau^{-5/2}$. The rest of w vanishes at 0; Hadamard's lemma and one integration by parts in each coordinate make its oscillatory integral $\int(\cdot)e^{i\tau Q}\mathrm{d}y$ decay like $\tau^{-5/2}$, so τ times it is integrable at infinity, uniformly in ε, and dominated convergence gives the limit. The single-pole part converges in the same way with one power of τ less. On U_c the factor $\tilde{q}$ does not vanish, so no other eigenvalue reaches zero there and nothing is dropped. (iii) The transposition permutes the edge classes, the vertex-move columns and the channels; it preserves B, Γ, the gauge term, the regulated propagator, the cubic and quartic vertices, the strain kernel and the DeWitt norm, which reduces to three finite table identities. It also preserves Lebesgue measure, so the integral of (ii) at πe_c with cutoff χ and strain H equals the one at πe_0 with cutoff $\chi \circ S$ and the transposed strain. The transported residues are those of Proposition 6.3: 0 and $3/3844$ at every corner. (iv) At fixed ε the torus limit $\Delta(\varepsilon)$ is the zone integral of the graviton sunset and seagull and the matter sunset and seagull; the tadpole vanishes on both probes. The integrand is periodic, so the period cell may be shifted to $[-\pi/2, 3\pi/2]^4$, which has the four axis corners in its interior. The corners are a distance π apart, so the cutoffs can be chosen with disjoint supports, and $1 = \sum_c \chi_c + \big(1 - \sum_c \chi_c\big)$ splits the integral with nothing dropped. The corner pieces are (ii) and (iii), and the matter loop converges. The two equivalences follow because E and Δ_{m} converge. □

So near the corners the logarithm is a theorem about Δ itself, not only about the model: the ordered anisotropy could have a finite limit only if the corner complement $C(\varepsilon)$ diverged like $-D(\varepsilon)$, cancelling the four corner logarithms up to a convergent remainder. It does not. The rest of the zone has a finite limit, and the reason is one local property of the operator's degenerate zeros. To state it, use the real symmetric form $M(k) = \Phi(k)^\dagger A(k)\Phi(k)$ of the midpoint frame (Section 4.2). Call a wave a *critical zero* if $\det M$ and its gradient both vanish there, and put $k_* = \frac{\pi}{3}(1,1,1,1)$. Call a critical zero k *hyperbolic* if there is a direction $v \in \mathbb{R}^4$ such that the derivative of M along v, compressed to the kernel of $M(k)$, is positive definite:

$$w^\top\big(\partial_v M(k)\big)w > 0 \qquad \text{for every nonzero } w \in \ker M(k).$$

Then every eigenvalue of M that vanishes at k increases along v, so v acts as a time direction for all of them at once. At the axis corners no direction does this, because the gradient of the doubler eigenvalue vanishes there; that is where the logarithm comes from. Say that an open set U of waves has a *finite local limit* if $\int \chi \mathscr{K}_\varepsilon$ converges as $\varepsilon \to 0^+$ for every compactly supported cutoff χ of class C^8 with support in U.

Theorem 6.7 (Hyperbolic critical zeros)**.** *Let $\kappa = 1$. Every critical zero other than the lattice copies of 0 and of the axis corners is hyperbolic.*

The proof is a finite case analysis of the tree operator, given in full in Appendix G; its structure is described after Proposition 6.9. With it the divergence follows.

Theorem 6.8 (The ordered anisotropy diverges)**.** *Let $\kappa = 1$. There is a real constant r_0 with*

$$\Delta(\varepsilon) = \frac{5}{44\sqrt{335}\,\pi^2}\log\frac{1}{\varepsilon} + r_0 + o(1) \qquad (\varepsilon \to 0^+). \tag{9}$$

Consequently $\Delta(\varepsilon) \to +\infty$*, and the ordered limit* $\lim_{\varepsilon\to 0^+}\lim_{N\to\infty}\Delta_N(\varepsilon)$ *does not exist as a real number. The doubler-subtracted ordered limit* $\lim_{\varepsilon\to 0^+}\lim_{N\to\infty}\big(\Delta_N(\varepsilon) - D(\varepsilon)\big)$ *exists.*

Proof. By Proposition 6.6(iv) it suffices to show that $C(\varepsilon)$ has a finite limit. Split the support of $1 - \sum_c \chi_c$ with a smooth partition of unity. Off the zero set of $\det A$ the kernel converges with a uniform bound. At a zero where the gradient does not vanish, the zero set is a smooth hypersurface carrying one simple zero eigenvalue, and the regulated kernel has a finite local limit. At the origin the cubic vertex vanishes on constant strains, because flat configurations have zero action, so the double pole there has homogeneity -2 and the local limit exists. Every remaining point is a critical zero other than the origin and the corners; by Theorem 6.7 it is hyperbolic, and by Proposition 6.9(i) below it has an open neighbourhood with a finite local limit. The kernel is periodic, so finitely many neighbourhoods suffice. Hence $C(\varepsilon)$, and with it $\Delta(\varepsilon) - D(\varepsilon)$, has a finite limit, and Proposition 6.5 gives Eq. (9). □

The local step is a theorem at every hyperbolic point. Near a wave at which M has exactly two zero eigenvalues, the two eigenvalues closest to zero are $a \pm (b_1^2 + b_2^2)^{1/2}$, with a, b_1, b_2 smooth: a is their mean and (b_1, b_2) the trace-free part of the 2×2 block of M on their spectral subspace, in a smooth frame. Call such a wave a *nondegenerate crossing* if the differentials of a, b_1, b_2 there are linearly independent.

Proposition 6.9 (Local limits at hyperbolic points)**.** *Let* $\kappa = 1$.

(i) Every hyperbolic critical zero has an open neighbourhood with a finite local limit.

(ii) Every nondegenerate crossing is hyperbolic.

(iii) Let k be a critical zero at which $M(k)$ has exactly two zero eigenvalues, and write $\Lambda_k(v)$ for $\partial_v M(k)$ compressed to $\ker M(k)$*. If* $\det \Lambda_k(v) > 0$ *for some v, then k is hyperbolic. This holds in particular if* $\det M(k + \tau v)/\tau^2$ *tends to limits of opposite signs as* $\tau \to 0$ *along two directions v.*

Proof outline. (i) M is a trigonometric polynomial, so it continues analytically to complex waves $k + i\eta v$. Write $M' = M(k')$ and $B' = \partial_v M(k')$ for k' near k. If $B = \partial_v M(k)$ is positive definite on $\ker M(k)$, a compactness argument gives $c, \delta > 0$ with $\|(M' + i\eta B' + E + i\varepsilon)w\| \geq c\eta\|w\|$ for $|k' - k| < \delta$, $0 < \eta \leq \delta$, every $\varepsilon \geq 0$ and every perturbation E with entries at most $\delta\eta$: on the kernel the imaginary part $\eta B + \varepsilon$ is coercive, and off it the real part is. With E the second-order remainder of the continuation, this gives $\det\big(M(k' + i\eta v) + i\varepsilon\big) \neq 0$ and $\|(M(k' + i\eta v) + i\varepsilon)^{-1}\| \leq C/\eta$, uniformly in $\varepsilon \geq 0$. The deformation and the regulator push the small eigenvalues the same way, which is the point of the condition. Now move the integration off the real waves along iv. Extend the cutoff χ almost analytically to third order, $\tilde\chi(k', \eta) = \sum_{j\leq 2}(i\eta)^j \partial_v^j \chi(k')/j!$. For the regulated kernel $F_\varepsilon(k', \eta)$ at $k' + i\eta v$, which satisfies $\partial_\eta F_\varepsilon = i\,\partial_v F_\varepsilon$, differentiating $\int \tilde\chi(\cdot, \eta) F_\varepsilon(\cdot, \eta)$ in η and integrating by parts along v leaves only $\frac{i}{2}\eta^2 \int \partial_v^3 \chi F_\varepsilon$, so

$$\int \chi F_\varepsilon(\cdot, 0) = \int \tilde\chi(\cdot, \eta_0) F_\varepsilon(\cdot, \eta_0) - \int_0^{\eta_0} \frac{i}{2}\,\eta^2 \int \partial_v^3 \chi F_\varepsilon(\cdot, \eta)\,\mathrm{d}\eta.$$

The kernel is a sum of products of at most two resolvent entries with smooth numerators, so $|F_\varepsilon(\cdot,\eta)| \le C/\eta^2$ uniformly in $\varepsilon \ge 0$, and $F_\varepsilon(\cdot,\eta) \to F_0(\cdot,\eta)$ for each $\eta > 0$. Both terms on the right then converge by dominated convergence, which gives the local limit, with all fifteen channels and the ε-dependent numerators kept. (ii) In the chart (a, b_1, b_2, ℓ) the compressed derivative along v is the 2×2 matrix $\mathrm{d}a(v) + \mathrm{d}b_1(v)\sigma_3 + \mathrm{d}b_2(v)\sigma_1$; when the three differentials are independent, v can be chosen with $\mathrm{d}a(v) = 1$ and $\mathrm{d}b_1(v) = \mathrm{d}b_2(v) = 0$. (iii) A symmetric 2×2 matrix with positive determinant is definite, so $\Lambda_k(v)$ or $\Lambda_k(-v) = -\Lambda_k(v)$ is positive definite. For the second statement, $\det M(k + \tau v)/\tau^2 \to \det\big(M(k) + UU^\top\big) \det \Lambda_k(v)$, with U an orthonormal basis of $\ker M(k)$, and the first factor, the product of the nonzero eigenvalues of $M(k)$, is not zero. □

At nondegenerate crossings the local limit also has a direct proof. In the chart above the kernel is the real part of $w(\varepsilon, y)/\big((y_0 + i\varepsilon)^2 - y_1^2 - y_2^2\big)^2$ with w jointly smooth, and since both small eigenvalues carry $+i\varepsilon$ the 2×2 block behaves like a retarded propagator: the representation

$$\frac{1}{\big((x + i\varepsilon)^2 - \rho^2\big)^2} = \int_0^\infty e^{i(x+i\varepsilon)\sigma} S(\rho, \sigma)\, \mathrm{d}\sigma, \qquad S(\rho,\sigma) = \frac{1}{4}\int_0^\sigma (\sigma^2 - v^2)\cos(\rho v)\, \mathrm{d}v,$$

followed by integration by parts in y_0 along lines on which the phase is not stationary, bounds the σ-integrand by a multiple of $(1 + \sigma^2)^{-1}$ uniformly in ε.

Which critical zeros occur, and how Theorem 6.7 is proved. Not every critical zero is a nondegenerate crossing. On the plane $k = (t, t, t, s)$ the determinant factors exactly into a symmetric-sector factor and the square of a doublet factor, both affine in $(\cos s, \sin s)$; eliminating s leaves $(8c^2 - 9)\, p_5(c) = 0$ in $c = \cos t$, with $p_5(c) = 72c^5 - 480c^4 + 129c^3 + 1241c^2 - 1356c + 396$, which changes sign on $[0.6, 0.61]$. The common zero is a critical zero at which the second-order form of $\det M$ has rank at most two, so it is not a nondegenerate crossing; numerically three eigenvalues of M vanish there, and computation finds sixteen such *triple points* per cell (COMPUTATION). Hyperbolicity is the condition that covers them. The proof of Theorem 6.7 (Appendix G) is a finite case analysis of the tree operator at its degenerate zeros; it involves no regulator and no loop integral. Lattice translations, axis permutations and $k \mapsto -k$ reduce it to two cases. With four distinct phases, a chart by the binary quartic with roots $-\tan(k_a/2)$ turns $\det M$ into an integer sextic, and every critical zero is a nondegenerate crossing. With a repeated phase, the exchange symmetry splits M into blocks of sizes 11 and 4, and every critical zero is a transversal crossing of the blocks' zero sets or one of finitely many exactly located points and small boxes, each with an explicit hyperbolic direction. The computations are exact and of four kinds: identities between integer polynomials, including ideal memberships; a sum-of-squares identity [67]; arithmetic in quadratic and biquadratic number fields; and rational enclosures. Some rational data, such as the centres of the boxes, were found numerically, which does not matter, since every inequality they enter is proved exactly.

Two further facts are exact. The scalar's operator does not vanish at the axis corners (its symbol there is $8\sqrt{3}/3$), and a ghost loop, were one added, could not cancel a corner logarithm either, since the vertex-move Gram determinant is nonzero away from $k = 0$.

The finite part r_0 is canonical: it involves no model subtraction. The computed anisotropy at $\varepsilon = 0.0125$ is 0.1541 ± 0.0008, of which the logarithmic term accounts for 0.0028 (NUMERICAL);

if the remainder there is already small, $r_0 \approx 0.151$ (INFERENCE). The logarithm is slow. It overtakes a finite part of that size only when $\log(1/\varepsilon)$ exceeds about 240, so at every regulator a computation can reach, the anisotropy is dominated by its finite part. The divergence is a property of the limit, and Theorem 6.8 proves it.

6.5 No gap from the construction

Could anything in the construction itself give the doublers a mass? At identified $\kappa = 1$ the native configuration space is the null stratum (Section 3.4), on which the four posting classes are frozen; its measure is flat Haar measure, its regulator a damping ε on the eleven remaining channels, its gauge the vertex moves, its action the Regge action. The damping is a regulator model (MODEL); nothing in the construction supplies a different one. Write $B_{\mathrm{nat}}(k)$ for the compression of $B(k)$ to the eleven non-posting classes. The native Gaussian precision on each wave is $\varepsilon \cdot 1 - ia\, B_{\mathrm{nat}}(k)$.

Theorem 6.10 (No native gap)**.** *Let* $\kappa = 1$.

(i) At every wave with all four phases $e^{ik_b} \neq 1$*, if* $\det A(k) = 0$ *then* $\det B_{\mathrm{nat}}(k) = 0$*: the null stratum is a transversal slice of the vertex-move orbits there and keeps every zero of the gauge-fixed operator.*

(ii) For every $\rho > 0$ *there is* k *with* $|k - \pi e_0| < \rho$*, all phases* $\neq 1$*, and* $\det B_{\mathrm{nat}}(k) = 0$*; at such* k *no nonzero vertex move lies in the null stratum, so the native kernel is not gauge. Hence the native precision at* $\varepsilon = 0$ *is singular at non-gauge waves arbitrarily close to the corner.*

(iii) A constant strain that keeps every posting null has $H_{aa} = 0$ *for all* a*, so by Proposition 6.3(iv) the four first-order shifts sum to zero: no flat posting-null background moves all four corner zeros in the same direction, and a nonzero shift of this kind makes at least one doubler tachyonic.*

(iv) A constant strain that shifts all four corners with one sign has $\operatorname{tr} H \neq 0$*, so it changes the posting lengths. The* κ *direction* $H = -\mathbf{1}\mathbf{1}^{\top}$ *moves each by* $+396/(1152\sqrt{3})$*, a mass for all four; it leaves* $\kappa = 1$.

The converse of (i) also holds, so near πe_0, off the three hyperplanes through it on which a phase equals 1, the native zero set is exactly the doubler cone; on those hyperplanes the native block is singular at every wave (Proposition 6.21). Suppose y is supported off the postings and $B_{\mathrm{nat}} y = 0$. Then By vanishes off the postings, and $\Gamma^{\dagger} By = 0$ forces its posting rows to vanish as well, because the posting rows of Γ are invertible when every phase differs from 1; so $By = 0$. If $\det A(k) \neq 0$, the kernel of B is the range of Γ (write $y = \Gamma z + w$ with $\Gamma^{\dagger} w = 0$; then $Aw = Bw = 0$), and a vertex move that vanishes on every posting is zero, so $y = 0$.

Item (iii) is about flat backgrounds only, and Proposition 6.4 shows that it cannot be widened: posting-null profiles that are not strains can shift all four zeros together.

The native construction is not the only candidate. Section 3 also builds the exchange on the twisted slice (Theorem 3.3(iii)), and a bare weight on configurations could add damping. On an even torus the corner πe_3 is a lattice wave and the doubler is an exact zero mode h_*, the field of Theorem 4.4, so these constructions can be tested on it directly. Let $X_\varepsilon(j)$ denote the

regulated exchange $j^{\top}(\varepsilon - iaA_R)^{-1}j$ of Theorem 3.2 on the retained set of the construction in question.

Theorem 6.11 (The twisted slice and bare weights)**.** *Let $\kappa = 1$, $N \geq 4$ even, and $a \neq 0$.*

(i) The cycle totals of h_* along the axis-0 fibres are $\pm N$, so h_* is not a posting-null field plus a vertex move. Its twisted representative x_0 is a nonzero kernel vector of the twisted operator, and $(\varepsilon - iaA)x_0 = \varepsilon x_0$: the twisted precision acts on the doubler by the regulator alone.

(ii) The doubler source $j_*(h) = \sum_e h_*(e)\, h(e)$ is conserved, and $\varepsilon X_\varepsilon(j_*)$ tends to a positive limit as $\varepsilon \to 0^+$ in the twisted construction and in the gauge-fixed ambient Gaussian, with any density as in Theorem 3.2(i). In each, the exchange of j_* has a pole in $1/\varepsilon$.

(iii) Let the weight be multiplied by $e^{-W(tx)/t^2}$ with $W \geq 0$ continuous and $W(tx)/t^2 \to \frac{1}{2}x^{\top}\mathscr{D}x$ as $t \to 0^+$. Then the normalized moments tend to $\big((\varepsilon + \mathscr{D}) - iaA_R\big)^{-1}$. If $\mathscr{D}x_0 = 0$, the doubler is undamped as in (i). If $\mathscr{D} \geq c > 0$, every exchange obeys $|X_\varepsilon(j)| \leq |j|^2/(\varepsilon + c)$ and has positive real part for $j \neq 0$, at every wavelength, and has a finite limit as $\varepsilon \to 0^+$.

(iv) For the on-site recognition-cost weight, $\mathscr{D} = b \cdot 1$ with $b > 0$, the ordered exchange of every nonzero conserved source differs from $(i/a)j \cdot u$, and the exchange between sources at lattice distance m decays like $e^{-\mu m}$ with $\mu > 0$ independent of N and ε.

A fixed inverse temperature does nothing at Gaussian order (Section 3.6); a weight acts on the propagator only if its exponent scales like the action, which is a premise of (iii) and (iv), not a consequence of anything derived here. Read physically, (iii) and (iv) say that a weight bounded below gives every mode a mass, the long-wave graviton included, and destroys the tree-level exchange of Theorem 3.2. A repair through the weight would need a Hessian that vanishes faster than $|k|^2$ on long-wave gravitons and is positive on the corner modes, so it would have to tell the corner modes from the long waves; none is derived (INFERENCE). The theorem is stated on even tori, where the corner is a lattice wave. For the ordered limit the link to odd sides and infinite volume runs through the large-volume theorem of Section 3.6: the twisted and posting-null exchanges have the same limit at each fixed ε, so the native cone of Theorem 6.10 is the twisted one (INFERENCE).

6.6 What the one-loop dynamics does to the doublers

The routes closed in Section 6.5 are static: they ask whether the action, the measure or a fixed background moves the corner zeros. The theory also moves them itself. At one loop the doubler at corner c acquires the self-energy

$$\Sigma_c(\varepsilon) = \frac{v_c^{\dagger}\,\Pi(\pi e_c;\varepsilon)\,v_c}{|v_c|^2},$$

where $\Pi(p;\varepsilon)$ is the one-loop self-energy matrix of the diagram model above at wave p, after the torus limit at fixed ε; it is the first-order shift of the doubler eigenvalue r, in the units of r. It has five pieces: the graviton sunset and seagull, the scalar sunset and seagull, and the tadpole, in which a propagator line absorbs the model's one-loop response to its own fluctuations. Two facts about the cubic vertex organize them. It is the third derivative of the action of a

cell, so it is symmetric in its three legs. And a leg at zero wave carrying a constant profile acts on the other two as a derivative of the Bloch matrix.

Proposition 6.12 (Zero-wave legs at the corners). *Let $\kappa = 1$, $c \in \{0, 1, 2, 3\}$, and let $t \in \mathbb{C}^{15}$ be a constant profile.*

(i) Inserting t into a propagator line at wave p gives the matrix $\sum_d t_d\,(\partial_{e_d} B)(p)$, the derivative of the Bloch matrix along t. At the corner it moves the zero, to first order, by $\delta_c(t)$, the complex-linear extension of the Rayleigh derivative of Section 6.3; for a profile that depends only on $|d|$ this is the formula of Proposition 6.4, the same at all four corners.

(ii) In the sunset with its graviton line at zero wave and both other legs on the doubler v_c, the vertex is δ_c: a zero-wave graviton with profile f couples to the doubler pair with strength $\delta_c(f)$, without derivatives. It is nonzero on the T_2 strains and on the strain $\mathbf{1}\mathbf{1}^\top$ (Proposition 6.3).

Proof. Both statements contract the cubic vertex with a constant profile on one leg, at wave 0, and with vectors at waves p and $-p$ on the other two. The derivative of the Bloch matrix along t is the same vertex with t placed on a possibly different leg. By the symmetry of the vertex in its legs the two contractions agree, and the pairing with v_c is the Rayleigh derivative by its definition. The last sentence of (i) is Proposition 6.4, extended complex-linearly. □

The tadpole and the corner logarithm. The model's one-loop response $t(\varepsilon)$ to its own fluctuations is a constant profile in the range of $B(0)$; it has one propagator and is infrared-finite. If it depends only on $|d|$, as the axis permutations suggest and the numerics confirm to rounding, it moves all four corner zeros by the same amount $\delta_c(t(\varepsilon))$ (Proposition 6.12(i)). Numerically that shift is small and mostly imaginary, about $-0.0015 - 0.0077i$ at $\varepsilon = 0.0125$ in the units of r, and not yet asymptotic (NUMERICAL). The graviton sunset at the corner is about fifty times larger, and power counting indicates a logarithm of its own: with the graviton line at small wave k the vertex tends to δ_c, a coupling without derivatives, the graviton propagator on strains is of order $|k|^{-2}$, and the doubler line near its corner is of order $|k|^{-2}$ by Proposition 6.2, so the integrand is homogeneous of degree -4 and

$$\Sigma_c(\varepsilon) = C \log \frac{1}{\varepsilon} + O(1), \qquad C \approx (1.8 + 2.2i) \times 10^{-3}, \tag{10}$$

the same at all four corners, with C the angular integral of a local model (NUMERICAL). That Eq. (10) holds is open: the direct sums over the zone at accessible regulators are pre-asymptotic, since the doubler form's smallest eigenvalue is near -0.009, and they neither confirm nor refute it (NUMERICAL). If it holds with $C \neq 0$, the doublers' first-order radiative shift has no finite ordered limit at one loop. What one loop does show is how a dressed doubler line would act.

Proposition 6.13 (Dressed doubler lines). *For $z \in \mathbb{C}$ let*

$$I(\varepsilon; z) = \int_{\mathbb{R}^4} \frac{e^{-|y|^2}}{(Q(y) + z + i\varepsilon)^2}\, \mathrm{d}y,$$

for every $\varepsilon > 0$ if $\operatorname{Im} z \geq 0$, and for $0 < \varepsilon < |\operatorname{Im} z|/2$ if $\operatorname{Im} z < 0$; thus $I(\varepsilon; 0) = I(\varepsilon)$. If $z \neq 0$, then $I(\varepsilon; z)$ has a finite limit as $\varepsilon \to 0^+$. Consequently the dressed doubler term

$$D_z(\varepsilon) = \frac{\rho \operatorname{Im} I(\varepsilon; z)}{2|\det L|(2\pi)^4},$$

which is D with each corner line $(r + i\varepsilon)^{-1}$ replaced by $(r + z + i\varepsilon)^{-1}$, has a finite limit as $\varepsilon \to 0^+$ for every $z \neq 0$, massive, tachyonic or complex, while $D_0 = D$ diverges. For $\operatorname{Im} z > 0$ the limit is $-\pi^2 \int_0^\infty e^{iz\tau}\, \tau(1 + i\tau)(1 + \tau^2)^{-3/2}\, \mathrm{d}\tau$, the formula of Proposition 6.5 with $e^{iz\tau}$ in place of $e^{-\varepsilon\tau}$, and as $z \to 0$ in the upper half-plane its imaginary part is $\pi^2 \log|z| + O(1)$.

Proof. If $\operatorname{Im} z \neq 0$ and $0 < \varepsilon < |\operatorname{Im} z|/2$, then $|Q + z + i\varepsilon| \geq |\operatorname{Im} z + \varepsilon| \geq |\operatorname{Im} z|/2$, so the integrand is bounded by $4e^{-|y|^2}/|\operatorname{Im} z|^2$ and dominated convergence gives the limit $\int e^{-|y|^2}(Q + z)^{-2}\, \mathrm{d}y$. Now let $z = x$ be real and nonzero. As in the proof of Proposition 6.5,

$$I(\varepsilon; x) = -\pi^2 \int_0^\infty e^{(ix-\varepsilon)\tau} g(\tau)\, \mathrm{d}\tau, \qquad g(\tau) = \frac{\tau(1 + i\tau)}{(1 + \tau^2)^{3/2}}.$$

Write $g(\tau) = i(1 + \tau)^{-1} + s(\tau)$. Then s is bounded and $s(\tau) = O(\tau^{-2})$ as $\tau \to \infty$, so its part converges by dominated convergence. For the rest put $w = ix - \varepsilon$, so that $\operatorname{Re} w < 0$, and integrate by parts:

$$\int_0^\infty \frac{e^{w\tau}}{1 + \tau}\, \mathrm{d}\tau = -\frac{1}{w} + \frac{1}{w} \int_0^\infty \frac{e^{w\tau}}{(1 + \tau)^2}\, \mathrm{d}\tau.$$

This converges as $\varepsilon \to 0^+$, because $w \to ix \neq 0$ and $|e^{w\tau}| \leq 1$. For $\operatorname{Im} z > 0$ the Laplace formula holds at $\varepsilon = 0$ with $e^{iz\tau}$ in place of $e^{-\varepsilon\tau}$. Put $u = -iz$, so $\operatorname{Re} u > 0$; then $\int_0^\infty e^{-u\tau}(1 + \tau)^{-1}\, \mathrm{d}\tau = e^u E_1(u) = -\log u - \gamma + O(|u| \log|u|)$ as $u \to 0$, and the imaginary part of $-\pi^2\big(\int_0^\infty e^{-u\tau} s(\tau)\, \mathrm{d}\tau + i\, e^u E_1(u)\big)$ is $\pi^2 \log|z| + O(1)$. The statements about D_z follow because D_z is a fixed multiple of $\operatorname{Im} I(\varepsilon; z)$. □

A dressing z in the upper half-plane thus replaces $\log(1/\varepsilon)$ in Eq. (8) by a term of size $\log(1/|z|)$. The proposition is about the model term of Eq. (8), with a constant dressing, the same at all four corners; the model term's relation to the actual integral is proved near the corners, and the rest of the zone has a finite limit (Theorem 6.8). A one-loop dressing depends on momentum and, if Eq. (10) holds, grows like $C \log(1/\varepsilon)$; the fixed-z limits do not apply to it, although a dressing whose imaginary part grows drives the model term to zero, since for $\operatorname{Im} z > 0$ the integrand of $I(\varepsilon; z)$ is bounded by $e^{-|y|^2}/(\operatorname{Im} z)^2$. Before asking what dressing the theory supplies, one more static candidate needs closing. A background with period two along an axis is not flat, it is not translation invariant, and it couples the corner mode to the zero-wave sector, where the strains are also exact zero modes.

Proposition 6.14 (Period-two backgrounds)**.** *Let $\kappa = 1$ and $c \in \{0, 1, 2, 3\}$, and let $u(x, d) = (-1)^{x_c} u(d)$ be a background at wave πe_c. To first order in u, on the space of exact zero modes spanned by v_c at wave πe_c and the constant strains at wave 0:*

(i) the corner mode alone gets no first-order shift;

(ii) if u couples v_c to a strain H with a nonzero coefficient m, the first-order form on the span of v_c and H is off-diagonal, so it takes both signs: one direction stiffens and one softens, and there is no uniform gap;

(iii) for the corner mode's own condensate, $u = v_c$, the coefficient is $m = \delta_c(H)$. For the strain $\mathbf{11}^\top$ it is $-396/(1152\sqrt{3}) \neq 0$, and for the posting-null off-diagonal strain of Proposition 6.3(v) it is $(+1,+1,-1,-1)/(16\sqrt{3})$ at the four corners, nonzero at each.

Proof. A translation by one step along axis c multiplies the background by -1, so to first order it couples waves that differ by πe_c and no others. It therefore connects πe_c only with $2\pi e_c \equiv 0$, and its first-order matrix has no block within either wave. This gives (i), and on the span of v_c and H the form is $\left(\begin{smallmatrix} 0 & m \\ \bar{m} & 0 \end{smallmatrix}\right)$, which gives (ii). For $u = v_c$ the coefficient contracts the cubic vertex with v_c on two legs, at waves $\pi e_c \equiv -\pi e_c$, and with H on the third, at wave 0. By the symmetry of the vertex this is $\delta_c(H)$, and Proposition 6.3(iii) and (v) give its values. □

A period-two background therefore pushes levels apart instead of lifting the four doublers together; the strain $\mathbf{11}^\top$ is the κ direction and changes the posting lengths, but the posting-null strain of (iii) shows that the conclusion holds inside the null stratum too. The proposition treats the exact zero modes at first order, and its premise that the first-order matrix has no block within one wave is the translation argument of the proof. Second order is not treated: it produces a constant profile, which Proposition 6.12(i) covers but which need not depend on $|d|$ alone, since a background at πe_c singles out axis c, together with a quartic contact term and second-order mixing. No order of perturbation theory in the translation-invariant model produces a period-two background, since its one-loop response is a constant profile; one could appear only through condensation of the corner mode, which is a question beyond perturbation theory (INFERENCE). Near the origin the same mixing would shift the long-wave strain moduli at first order in the background amplitude, which is a mass term for the long-wave graviton, so a condensate would not be a repair either (INFERENCE).

The size of the dressing. At accessible regulators the graviton-sunset part of the corner self-energy is about 0.2 in modulus and mostly imaginary (NUMERICAL; the seagull and scalar loops at the corner are not computed), while the tree-level doubler eigenvalue at distance q from the corner is at most $0.077|q|^2$. So the one-loop term exceeds the tree-level one within $|q| \lesssim 1.5$ of the corner, which contains the region producing the logarithm. We read this as saying that perturbation theory about the flat $\kappa = 1$ lattice is not controlled at the lattice scale (INFERENCE). A resummed doubler line would carry a dressing, and by Proposition 6.13 a constant nonzero dressing makes the doubler term finite; but the dressing would be as large as the operator it corrects, and its sign is not settled, so a finite anisotropy reached this way would not be a massless graviton with small corrections (INFERENCE).

6.7 The doublers have no common time with the graviton

Apart from what the doublers' loops do, one can ask in which time they propagate. The long-wave graviton propagates on the cone of the wave symbol q of Eq. (2). A covector θ with $q(\theta) < 0$ is a time direction for it; record time, the sum of the record coordinates, has the covector $\theta_r = (1,1,1,1)$ with $q(\theta_r) = -\frac{4}{3}$. The doubler at πe_0 propagates on the cone of $Q(h) = h^\top K h$ (Proposition 6.2), whose one negative direction lies near e_0.

Theorem 6.15 (No common time). *Let $\kappa = 1$, let q be the wave symbol of Eq. (2) and $Q(h) = h^\top K h$ with K as in Proposition 6.2.*

(i) For every covector k, with $S = k_1 + k_2 + k_3$,

$$30\,q(k) + Q(k) = 18\Big(k_0 - \frac{S}{18}\Big)^2 + \frac{85}{3}\sum_{1\le i<j\le 3}(k_i - k_j)^2 + \frac{275}{18}S^2.$$

Hence no nonzero covector θ has both $q(\theta) \le 0$ and $Q(\theta) \le 0$: the two cones share no timelike or null direction.

(ii) $q(\theta_r) = -\frac{4}{3}$ and $Q(\theta_r) = 190$: record time is timelike for the graviton and spacelike for the doubler.

(iii) Let $q(\theta) < 0$. Then $Q(\theta) > 0$, and there is $\xi \neq 0$ with $Q(\xi + s\theta) = Q(\xi) + s^2 Q(\theta) > 0$ for every real s. For $\theta = \theta_r$ one may take $\xi = (0, 1, -1, 0)$, with $Q(\xi + s\theta_r) = 110 + 190s^2$.

(iv) For θ and ξ as in (iii) and every $S_0 \ge 0$ there is $t_0 > 0$ such that $\det A(\pi e_0 + t(\xi + s\theta)) \neq 0$ for $0 < t < t_0$ and $|s| \le S_0$. The same holds at every axis corner with the permuted forms, and for the native operator B_{nat} at every axis corner, except at the at most three values of s at which the line meets a hyperplane on which a phase equals 1; for θ_r and $\xi = (0, 1, -1, 0)$ at πe_0 these are $s \in \{-1, 0, 1\}$.

(v) For every real μ and every θ with $q(\theta) < 0$ there is ξ with $Q(\xi + s\theta) + \mu > 0$ for every real s. If $\mu > 0$, already $\xi = 0$ gives the non-real roots $s = \pm i(\mu/Q(\theta))^{1/2}$ of $\mu + s^2 Q(\theta) = 0$.

Proof. (i) With $q(k) = \sum_a k_a^2 - \frac{1}{3}(k_0 + S)^2$ and $Q(k) = -2k_0^2 + 18k_0 S + 55\sum_{i\ge 1} k_i^2 - 3S^2$, both sides equal $18k_0^2 - 2k_0 S + 85\sum_{i\ge 1} k_i^2 - 13S^2$, using $\sum_{i<j}(k_i - k_j)^2 = 3\sum_i k_i^2 - S^2$. The right side vanishes only if $S = 0$, all k_i are equal and $k_0 = S/18$, that is, only at $k = 0$; if $q(\theta) \le 0$ and $Q(\theta) \le 0$ the left side is not positive, so $\theta = 0$. (ii) is direct. (iii) By (i), $Q(\theta)$ exceeds $-30q(\theta) > 0$. Since K has one negative and three positive eigenvalues and $Q(\theta) > 0$, the Q-orthogonal complement of θ is three-dimensional with one negative and two positive directions, so it contains ξ with $Q(\xi) > 0$, and the cross term $2s\,\xi^\top K\theta$ vanishes. For θ_r, $K\theta_r = (25, 55, 55, 55)$ is orthogonal to $(0, 1, -1, 0)$, and $Q(0, 1, -1, 0) = 110$. (iv) The kernel of $A(\pi e_0)$ is one-dimensional (Proposition 6.1), so near the corner the block of A complementary to ν_0 is invertible and $\det A$ vanishes exactly where the Schur complement of Proposition 6.2 does. On $h = t(\xi + s\theta)$ with $|s| \le S_0$, $Q(h) = t^2(Q(\xi) + s^2 Q(\theta)) \ge t^2 Q(\xi)$ while $|h| \le t(|\xi| + S_0|\theta|)$, so for small t the $o(|h|^2)$ term of Proposition 6.2 is smaller than $\frac{\sqrt{3}}{640}t^2 Q(\xi)$ and the Schur complement does not vanish. The axis permutations preserve q and carry the statement to the other corners. Off the hyperplanes on which a phase equals 1, $B_{\mathrm{nat}}(k)$ is singular exactly when $A(k)$ is (Theorem 6.10(i) and its converse). Every coordinate of a graviton time covector is nonzero: $\theta_a = 0$ would give $\sum_{b\neq a}\theta_b^2 < \frac{1}{3}\big(\sum_{b\neq a}\theta_b\big)^2 \le \sum_{b\neq a}\theta_b^2$. So for small t the line $\pi e_c + t(\xi + s\theta)$ meets the hyperplane of axis $b \neq c$ only where $\xi_b + s\theta_b = 0$, at one value of s, and never meets that of axis c; for θ_r and $\xi = (0, 1, -1, 0)$ the three values are where $t(1 + s)$, $t(s - 1)$ or ts vanishes. (v) For $\mu \ge 0$ take ξ from (iii); for $\mu < 0$ replace it by $c\xi$ with $c^2 Q(\xi) > -\mu$. □

Part (iv) is the statement that matters. The doubler cone reaches every neighbourhood of the corner (Proposition 6.2), but it is not a graph over the spatial momenta of any time the graviton uses: along lines through the corner in a graviton time direction the operator has no real zero. Numerically the doubler's own time axis points along $(1, -0.171, -0.171, -0.171)$, 16.5° from the posting covector e_0 and 76.5° from record time (NUMERICAL, Euclidean angles in

record coordinates). In the language of partial differential equations, the principal symbol Q of a corner mode's slowly varying envelope is not hyperbolic with respect to any time direction of the graviton, so by the theorem of Lax and Mizohata [16, 17] (for constant coefficients, the Hadamard–Gårding criterion [62, 63]) its record-time initial-value problem is ill-posed in the continuum limit (INFERENCE). On the lattice at fixed spacing there is no ill-posedness in that sense; near the corner, at spatial offset t, one expects one mode to grow and one to decay at a rate of about $t(Q(\xi)/Q(\theta))^{1/2}$ per record tick (INFERENCE), a rate that vanishes at the corner. Part (v) concerns the leading-order model $Q+\mu$ and bears on the operator only for a gap $|\mu|$ small compared with the lattice scale: such a gap of either sign leaves non-real record-time roots near the corner, so for κ slightly above 1, where the doublers acquire a positive shift (Theorem 6.10(iv)), each corner mode becomes a record-time tachyon with rate of order $(\kappa-1)^{1/2}$ (INFERENCE, at leading order).

Record time is not a hyperbolic evolution parameter at the lattice scale either, at $\kappa=1$ or at $\kappa=3/2$, the other value we examine, and this puts Theorem 6.15 in proportion. Continue the Bloch matrix B_κ of the flat Regge Hessian at κ, and the vertex-move columns Γ, to complex phases $z=(z_0,\dots,z_3)$, $z_a=e^{ik_a}$. A vector $v\neq 0$ with $B_\kappa(z)v=0$ is read, through the Bloch symbol, as the solution $h_x=vz^x$ of the linearized lattice equations on $\mathbb{Z}^4$, and as a physical one when v is not a vertex move $\Gamma(z)c$. For the graviton we use this standard reading without carrying out the step from the symbol to lattice solutions (INFERENCE); for the scalar below the solution is checked directly. On the record line $z=(w,w,w,w)$ the solution is $vw^{t_{\mathrm{r}}(x)}$, where $t_{\mathrm{r}}(x)=x_0+x_1+x_2+x_3$ is the record time: it has no spatial dependence and grows in record time when $|w|>1$. The value $\kappa=3/2$ enters the next statement as a hypothesis under examination, not as a proposal.

Proposition 6.16 (Record time at the lattice scale, and $\kappa=3/2$)**.** *(i) For every $\kappa>1/2$ the massless lattice scalar of the diagram model, written on $\mathbb{Z}^4$ up to a positive factor,*

$$(\mathscr{D}_\kappa f)(x)=\sum_a\big(2f(x)-f(x+e_a)-f(x-e_a)\big)-\beta\big(2f(x)-f(x+\mathbf{1})-f(x-\mathbf{1})\big),$$

with $\beta=\kappa/(4\kappa-1)$, has an exact solution $f(x)=w^{t_{\mathrm{r}}(x)}$ on $\mathbb{Z}^4$ with $|w|>1$.

(ii) At $\kappa=1$ with $x_0=\sqrt{2}$, and at $\kappa=3/2$ with $x_0=(1+\sqrt{10})/3$, the positive root of $3x^2-2x-3$, each of $B_\kappa(x_0,x_0,x_0,x_0)$ and $B_\kappa(x_0^{-1},x_0^{-1},x_0^{-1},x_0^{-1})$ has a nonzero kernel vector that is not a vertex move. Read as above, these are physical solutions $vx_0^{\pm t_{\mathrm{r}}(x)}$ of the linearized equations.

(iii) At $\kappa=3/2$ the gauge-fixed operator $A_{3/2}(k)=B_{3/2}(k)+\Gamma(k)\Gamma(k)^\dagger$ is invertible at all fifteen half-period corners $k\in\{0,\pi\}^4\setminus\{0\}$; its determinant is negative at the corners with three coordinates equal to π and positive at the others. Hence each corner has a neighbourhood on which $\det A_{3/2}$ *does not vanish.*

(iv) At $\kappa=3/2$ and every root ρ of

$$p_8(x)=x^8+\tfrac{25}{9}x^7+\tfrac{100}{27}x^6-\tfrac{475}{81}x^5-\tfrac{2554}{243}x^4-\tfrac{475}{81}x^3+\tfrac{100}{27}x^2+\tfrac{25}{9}x+1,$$

the matrix $B_{3/2}(\rho(-1,1,1,1))$ has a nonzero kernel vector that is not a vertex move. p_8 has the real roots $\rho_1\in(1.4337,1.4339)$ and $\rho_2=\rho_1^{-1}\in(0.6974,0.6975)$.

Proof. (i) On the record line $f(x\pm e_a) = w^{\pm1} f(x)$ and $f(x\pm\mathbf{1}) = w^{\pm4} f(x)$, so $\mathscr{D}_\kappa f = s(w) f$ with $s(w) = 4(2 - w - w^{-1}) - \beta(2 - w^4 - w^{-4})$. With $y = w + w^{-1}$ one has $w^4 + w^{-4} = y^4 - 4y^2 + 2$, hence $2 - w^4 - w^{-4} = y^2(2 - y)(2 + y)$ and

$$s(w) = (2 - y)\big(4 - \beta y^2(y + 2)\big).$$

For $\kappa > 1/2$, $\frac{1}{4} < \beta < \frac{1}{2}$. The cubic $c(y) = \beta y^2(y + 2) - 4$ has $c(2/3) = \frac{32}{27}\beta - 4 < 0$ and $c(2) = 16\beta - 4 > 0$, so it has a real root $y_0 \in (2/3, 2)$. Its other two roots have sum $-(y_0 + 2)$ and product $4/(\beta y_0)$, which equals $y_0(y_0 + 2)$ because $\beta y_0^2(y_0 + 2) = 4$; their discriminant is $(y_0 + 2)^2 - 4y_0(y_0 + 2) = (y_0 + 2)(2 - 3y_0) < 0$. So they are not real. If $|w| = 1$, then $w + w^{-1} = 2\,\mathrm{Re}\, w$ is real; hence a solution w of $w + w^{-1} = y$ with y non-real has $|w| \neq 1$, and since w and w^{-1} both solve it, one of them has $|w| > 1$. For that w, $s(w) = 0$ and $\mathscr{D}_\kappa w^{t_\mathrm{r}(x)} = 0$.

(ii)–(iv) are exact finite computations. In (ii) the kernel vectors have entries in $\mathbb{Q}(x_0)$ and are checked by multiplication in that field. Each vanishes on the posting class e_0 and on the class $e_0 + e_2$ but not on e_2. A vertex move has, on the class d, the entry $2(z^d - 1)\, d^\top \eta$ for a covector η; vanishing on e_0 and on $e_0 + e_2$ forces $\eta_0 = \eta_2 = 0$ when $x_0 \neq \pm1$, and then it vanishes on e_2 too, so these kernel vectors are not vertex moves. In (iii) the sign of each determinant comes from an exact $LDL^\dagger$ factorization of $\sqrt{5}\, B_{3/2} + \Gamma\Gamma^\dagger$, whose entries at the corners are rational. Neither invertibility nor the sign depends on the positive weights of the two terms: since $\Gamma^\dagger B = 0$ and B is Hermitian, B acts on the orthogonal complement of the range of Γ and $\Gamma\Gamma^\dagger$ on that range, so the determinant is the product of the two restricted determinants. In (iv) one vector with entries in $\mathbb{Q}[x]/(p_8)$ is checked to be a kernel vector of $B_{3/2}(x(-1, 1, 1, 1))$ there, so it is one at every root, and a linear test that vanishes on every vertex move takes the value 1 on it. The intervals for ρ_1, ρ_2 follow from sign changes of p_8, which is palindromic, so its roots come in pairs ρ, ρ^{-1}. □

Read as a solution, ρ_1 gives at the corner πe_0 a physical mode staggered along axis 0 that grows by a factor of about 1.434 per record tick; that this real pair continues the $\kappa = 1$ corner zero, and that other branches at the same spatial momentum grow by about 2.320 per tick, are numerical (NUMERICAL). Across the spatial zone, at every sampled momentum, at least 14 of the 24 record-time roots lie off the unit circle at both values of κ (NUMERICAL). We read the cause in the multi-tick edges (INFERENCE): the body diagonal spans four record ticks and the light cone of g_κ is wider than the cone the lattice edges generate, so an explicit record-time evolution has parasitic multi-tick modes, the lattice analogue of violating a Courant condition [40].

So the graviton's difficulty in record time splits in two (INFERENCE). The infrared part sits on $\kappa = 1$: the corner doublers are massless, share no time direction with the graviton, and feed the logarithm of Section 6.4; at $\kappa = 3/2$, and at $\kappa = 5/4$ by the same kind of exact computation, the corners are regular. The lattice-scale part does not sit on $\kappa = 1$: growth in record time is present for the graviton at both values and for the scalar at every $\kappa > 1/2$. Moving to $\kappa = 3/2$ would remove the corner doublers; it would not make record time a well-posed evolution parameter for the lattice operator.

6.8 The long-wave graviton at zero momentum

The doublers live at the edge of the zone. The long-wave graviton has a zero-momentum question of its own. In the continuum, a constant strain $g \mapsto g + B$ is a linear change of coordinates, so the one-loop effective action of constant strains can depend on B only through the volume, that is, through $\det(g + B)$: it can produce a cosmological term and nothing else. Anything that is not of this form is a mass term for the transverse-traceless graviton at zero momentum. The lattice question is whether such a remainder appears.

For a function D of a constant strain, nonzero at $B = 0$, and a strain H with $\operatorname{tr}(g^{-1}H) = 0$, put

$$\Sigma_H[D] = \frac{i}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log D(tH)\Big|_{t=0}, \qquad \sigma_{\mathrm{dil}}[D] = \frac{i}{2}\frac{\mathrm{d}}{\mathrm{d}t}\log D(tg)\Big|_{t=0},$$

$$R_H[D] = \Sigma_H[D] + \tfrac{1}{4}\operatorname{tr}(g^{-1}Hg^{-1}H)\,\sigma_{\mathrm{dil}}[D].$$

When D is the one-loop Gaussian determinant of the strained flat lattice (per site, a product over torus waves), $(i/2)\log D$ is the one-loop effective action of constant strains, Σ_H is the zero-momentum self-energy along H and σ_{dil} the response to a uniform dilatation. Write $g = g_1$, so that $\det g = -3$.

Theorem 6.17 (The zero-momentum remainder)**.** *Let $\kappa = 1$ and let H be a strain with $\operatorname{tr}(g^{-1}H) = 0$.*

(i) $R_H[cD] = R_H[D]$ for every constant $c \neq 0$; $R_H[DW] = R_H[D] + R_H[W]$; and $R_H[\varphi(\det(g + \cdot))] = 0$ for every φ with $\varphi(-3) \neq 0$ that is differentiable near -3 with φ' differentiable at -3. So a cosmological term, a power of $|\det(g + B)|$ in the measure and any factor that depends on B only through $\det(g + B)$ contribute nothing to R_H.

(ii) Let $N \geq 3$ and let k be a torus wave with every phase $e^{ik_a} \neq 1$. For every symmetric B near 0 the strained Bloch Hessian at k annihilates the strained vertex-move columns from both sides. The native determinant (the strained Hessian compressed to the eleven non-posting classes; at such waves holding the postings fixed is a hard gauge), the Gaussian-gauge determinant with the gauge term $\xi\Gamma\Gamma^{\dagger}$ for every $\xi > 0$, and the same with the Faddeev–Popov determinant of any strain-independent gauge function, all have the same R_H at $\varepsilon = 0$, wherever they do not vanish at $B = 0$.

(iii) A linear map that preserves g and maps each posting direction e_a to a multiple of itself is $+1$ or -1.

Proof. (i) The first two statements follow from $\log(cD) = \log c + \log D$ and $\log(DW) = \log D + \log W$. For the third put $f = \log\varphi$ and $\tau_H = \operatorname{tr}(g^{-1}Hg^{-1}H)$. Since $\operatorname{tr}(g^{-1}H) = 0$, $\det(g+tH) = -3\big(1-\frac{t^2}{2}\tau_H+O(t^3)\big)$, so the second derivative of $f(\det(g+tH))$ at 0 is $3\tau_H f'(-3)$; and $\det(g+tg) = -3(1+t)^4$, so the first derivative of $f(\det(g+tg))$ at 0 is $-12f'(-3)$. Hence $R_H = \frac{i}{2}\big(3\tau_H f'(-3) - \frac{1}{4}\tau_H\cdot 12f'(-3)\big) = 0$. (ii), sketch. The kernel relation is the derivative, at the strained flat lattice, of the fact that moving the vertices of a flat lattice leaves it flat. We read the strained vertex moves as displacements of the vertices of the strained flat lattice, with squared lengths measured by $g + B$. A displacement ζ then changes the squared length of the edge from x to $x + d$ by $2d^{\top}(g + B)(\zeta_{x+d} - \zeta_x)$, so at each wave the strained columns are

$\Gamma_B(k) = \Gamma^\circ(k)(g + B)$, where $(\Gamma^\circ(k)\eta)_d = 2(e^{ik\cdot d} - 1)\, d^\top \eta$ does not depend on B, and the posting rows of Γ° form the diagonal matrix $2\,\mathrm{diag}(e^{ik_a} - 1)$, invertible under the phase condition. Now let M be a Hermitian matrix with $M\Gamma = 0$ for a 15×4 matrix Γ whose posting block Γ_P is invertible, and let C be a gauge function with $C\Gamma$ invertible. Then

$$\det M_{\mathrm{np}}\, |\det C\Gamma|^2 = \det(M + C^\dagger C)\, |\det \Gamma_P|^2,$$

where M_{np} is the block on the non-posting classes. To see it, pass to the basis formed by the columns of Γ and the eleven non-posting unit vectors, a change of determinant $\pm \det \Gamma_P$; there $M + C^\dagger C$ has the blocks $(C\Gamma)^\dagger C\Gamma$, $(C\Gamma)^\dagger CE$, $(CE)^\dagger C\Gamma$ and $M_{\mathrm{np}} + (CE)^\dagger CE$, with E the non-posting unit vectors, and the Schur complement of the first block is M_{np} because $C\Gamma$ is square and invertible. Scaling C by $\xi^{1/2}$ multiplies the right side by a power of ξ. With $\Gamma = \Gamma_B$, every factor relating the three determinants is a constant times a power of $\det(g + B)$, and (i) removes it. (iii) Such a map is $\mathrm{diag}(\lambda_a)$, and preserving g requires $\lambda_a \lambda_b g_{ab} = g_{ab}$. At $\kappa = 1$, $g_{ab} = -1$ for $a \neq b$, so $\lambda_a \lambda_b = 1$ for every pair $a \neq b$, which forces all λ_a to be equal to 1 or all to -1. □

Parts (i) and (ii) say that R_H is a property of the path sum, not of how it is gauge-fixed: no gauge choice, ghost, cosmological counterterm or covariant measure factor changes it. Part (ii) holds wave by wave at $\varepsilon = 0$ and needs every phase to differ from 1. The waves on which a phase equals 1, at most $4N^3$ of the N^4, contribute only through the regulator, each by an amount bounded uniformly in the side at fixed $\varepsilon \neq 0$, so they drop out of the per-site remainder as $N \to \infty$. At $\varepsilon > 0$ the native and Gaussian-gauge constructions carry different regulators, both models (MODEL): at every wave with all phases different from 1, the regulated Gaussian-gauge determinant is a strain-independent constant times the native determinant damped by $i\varepsilon\,\Omega_k^{-1}$, with $\Omega_k = 1 + X_k X_k^\dagger$ positive definite and X_k the non-posting block of the vertex-move columns times the inverse of their posting block. That the two ordered limits agree is not proved; numerically the gap between them shrinks roughly linearly in ε (NUMERICAL), and agreement would not by itself show that either limit exists. Part (iii) explains why nothing forces R_H to vanish: the continuum argument needs every linear change of coordinates to be a symmetry, while on the lattice a linear map must also keep each null posting along its own axis, and only ± 1 do. We find no other symmetry that would force $R_H = 0$ (INFERENCE). We evaluate R_H only on posting-null probes: for a strain that changes the posting lengths the remainder depends on how the strain is followed beyond first order, through the plain trace of its second-order term, which vanishes inside the native construction (INFERENCE, by direct algebra on the permutation-invariant form of the one-loop strain source).

The dilatation half. One of the two terms of R_H can be computed exactly. Write $M_k(B)$ for the native 11×11 block at wave k and strain B, and $D_k(B) = \det\big(M_k(B) + i\varepsilon\big)$ for its regulated determinant; the per-site remainder is the mean of $R_H[D_k]$ over the torus waves.

Proposition 6.18 (The dilatation half)**.** *Let $\kappa = 1$ and $N \geq 1$.*

(i) $M_k(s\,g) = (1 + s)^{-1} M_k(0)$ *for every $s > -1$ and every wave k.*

(ii) *For $\varepsilon \neq 0$, $\sigma_{\mathrm{dil}}[D_k] = -\frac{i}{2} \sum_{j=1}^{11} \lambda_j/(\lambda_j + i\varepsilon)$, where the λ_j are the eigenvalues of the Hermitian matrix $M_k(0)$; hence $-\frac{11}{2} \leq \mathrm{Im}\, \sigma_{\mathrm{dil}}[D_k] \leq 0$.*

(iii) *At $\varepsilon = 0$, $R_H[D_k]$ is purely imaginary at every wave at which D_k does not vanish near $B = 0$.*

(iv) *For a posting-null probe with $\operatorname{tr}(g^{-1}Hg^{-1}H) = 6$ the per-site remainder obeys $\operatorname{Im} R_H \geq \operatorname{Im}\Sigma_H - \frac{33}{4}$. Hence, if there is $c > 0$ with $\operatorname{Im}\Sigma_H \geq \frac{33}{4} + c$ for every small $\varepsilon > 0$ on all sufficiently large tori, then R_H does not tend to 0 in the ordered limit.*

(v) *For the probe of (iv) and every $\varepsilon \neq 0$, the per-site Σ_H, σ_{dil} and R_H converge, as the side grows, to the integrals over the Brillouin zone of $\Sigma_H[D_k]$, $\sigma_{\mathrm{dil}}[D_k]$ and $R_H[D_k]$, which are continuous in k. So R_H tends to 0 in the ordered limit if and only if its zone integral tends to 0 as $\varepsilon \to 0^+$, and the hypothesis of (iv) holds if and only if there is $c > 0$ with $\operatorname{Im}\int \Sigma_H[D_k]\,\mathrm{d}k/(2\pi)^4 \geq \frac{33}{4} + c$ for every small $\varepsilon > 0$.*

(vi) *As $\varepsilon \to 0^+$ the zone integral of $\sigma_{\mathrm{dil}}[D_k]$ tends to $-\frac{11}{2}i$, so the covariant subtraction $\frac{6}{4}\sigma_{\mathrm{dil}}$ tends to $-\frac{33}{4}i$ in the ordered limit. Hence R_H tends to 0 in the ordered limit if and only if the zone integral of $\Sigma_H[D_k]$ tends to $\frac{33}{4}i$.*

Proof. (i) The Regge action of a cell is the sum over its triangles of area times deficit angle. As a function of the squared edge lengths it is homogeneous of degree one: areas scale like squared lengths, and angles do not change under a common scaling. The dilated background $g + sg$ has squared lengths $(1 + s)\ell^0$, so the Hessian in the squared lengths, homogeneous of degree -1, is multiplied by $(1 + s)^{-1}$; compressing to the eleven classes and passing to waves commute with this factor. (ii) By (i), the derivative of $\log\det\big((1 + s)^{-1}M_k(0) + i\varepsilon\big)$ at $s = 0$ is $-\operatorname{tr}\big((M_k(0) + i\varepsilon)^{-1}M_k(0)\big) = -\sum_j \lambda_j/(\lambda_j + i\varepsilon)$, and for real λ, $\operatorname{Im}\big(-\frac{i}{2}\lambda/(\lambda + i\varepsilon)\big) = -\frac{1}{2}\lambda^2/(\lambda^2 + \varepsilon^2) \in [-\frac{1}{2}, 0]$. (iii) For real t the blocks $M_k(tH)$ and $M_k(tg)$ are Hermitian, so at $\varepsilon = 0$ their determinants are real; where these do not vanish, the derivatives of their logarithms are real, so Σ_H and σ_{dil} are $i/2$ times real numbers. (iv) Per wave, and hence per site, $R_H = \Sigma_H + \frac{6}{4}\sigma_{\mathrm{dil}}$, and by (ii) the mean of $\operatorname{Im}\sigma_{\mathrm{dil}}$ lies in $[-\frac{11}{2}, 0]$. If $\operatorname{Im}\Sigma_H \geq \frac{33}{4} + c$, then $\operatorname{Im} R_H \geq c$. (v) For $\varepsilon \neq 0$ the regulated determinant $D_k(B)$ does not vanish at $B = 0$, since the eigenvalues of $M_k(0)$ are real, and it is jointly twice continuously differentiable in the strain and the wave: the entries of $M_k(B)$ are trigonometric polynomials in k whose coefficients are smooth in B near 0. So the per-wave quantities are continuous on the zone, the per-site means over the torus waves are Riemann sums of them, and these converge to the zone integrals. The two equivalences follow by comparing a limit with the terms of its sequence. (vi) By (ii) the zone integrand of σ_{dil} is $-\frac{i}{2}\sum_j \lambda_j/(\lambda_j + i\varepsilon)$, each term of modulus at most 1 and tending to 1 wherever $\lambda_j \neq 0$. The eigenvalues of $M_k(0)$ are all nonzero wherever $\det M_k(0) \neq 0$. That determinant is a real-analytic function of k, and it does not vanish identically: at $k = (\pi, \pi, \pi, \pi)$ every phase is -1, the block is $3^{-1/2}$ times an explicit rational matrix, and that matrix has an explicit rational left inverse. So it vanishes only on a null set, and dominated convergence gives the limit $-\frac{i}{2} \cdot 11$. The equivalence follows from (v) and $R_H = \Sigma_H + \frac{6}{4}\sigma_{\mathrm{dil}}$. □

So, with no torus side left, a massless long-wave graviton at zero momentum is one explicit statement: the zone integral of the transverse-traceless self-energy tends to its covariant value $\frac{33}{4}i$, its real part to 0 and its imaginary part to $\frac{33}{4}$. One inequality on that integral would refute it: that its imaginary part stays above $\frac{33}{4}$ by a fixed margin for every small ε. The inequality is sufficient, not necessary, since the real part could fail as well. We have not proved it: the Lorentzian block is indefinite, so the integrand has no sign, and the limit involves finite parts across the moving zero sets of the eleven bands.

The value. Take H to be a transverse-traceless strain for a wave along $(1,1,1,0)$, in its posting-null representative, with $\mathrm{tr}(g^{-1}H) = 0$ and $\mathrm{tr}(g^{-1}Hg^{-1}H) = 6$, and add the damping $i\varepsilon$ to the strained eleven-class block (MODEL). The graviton part of R_H is $0.75 + 6.71i$, $0.62 + 6.93i$ and $0.45 + 7.03i$ at $\varepsilon = 0.1$, 0.05 and 0.025, converged in the torus side (NUMERICAL). To reach smaller ε, the integral over k_0 at fixed spatial wave is done exactly, by Jensen's formula on the 22 roots of the frequency pencil; on grids up to 48^3 this gives $\mathrm{Im}\,\Sigma_H = 15.01$, 15.13 and 15.29 at $\varepsilon = 0.003$, 0.001 and 0, the last assuming that the strain derivatives commute with $\varepsilon \to 0^+$. With the exact subtraction of Proposition 6.18(vi) this gives $\mathrm{Im}\,R_H \approx +7.0$, about 7 above the threshold of Proposition 6.18(iv) (NUMERICAL). The real part falls toward 0 in these data, but whether it vanishes in the ordered limit is open. The model's scalar adds $1.31i$, so R_H is about $+7i$ in the graviton sector and about $+8.3i$ with the scalar (NUMERICAL). The axis permutations split the posting-null traceless strains into a two-dimensional and a three-dimensional representation, the probe is the sum $f_E + f_3$ of one strain from each, with $\mathrm{tr}(g^{-1}Hg^{-1}H)$ equal to $\frac{9}{2}$ and $\frac{3}{2}$, and a massless graviton needs the remainder to vanish on both; the computed parts are about $+5.2i$ to $+5.6i$ on f_E and $+1.3i$ to $+1.8i$ on f_3, both positive (NUMERICAL). That the ordered limit of the native remainder vanishes, as a massless graviton at one loop would need, is open, and these values are evidence against it.

What an imaginary remainder can mean. In a theory with a unitary evolution, a positive imaginary part of the self-energy on the graviton's shell is a decay width, and a width of order one in lattice units would damp every gravitational wave. That reading presupposes that the real-time path sum defines a unitary evolution in record time, with a transfer matrix. Two facts proved above speak against the presupposition. On a finite torus at $\varepsilon = 0$ the remainder is purely imaginary (Proposition 6.18(iii)): there is no Wick rotation and the frequencies are compact, so the one-loop effective action $\frac{i}{2}\log D$ is i times a real number, and an imaginary self-energy is the generic outcome of a real Lorentzian determinant, not by itself an absorptive part. And the Bloch symbol has non-gauge solutions that grow in record time at the lattice scale, for the graviton at $\kappa = 1$ and $\kappa = 3/2$ and for the scalar at every $\kappa > 1/2$ (Proposition 6.16), as an explicit lattice evolution does when it violates a Courant condition [40]. Together these suggest that the real-time lattice path sum has no unitary transfer-matrix interpretation in record time (INFERENCE). A Euclidean lattice action with reflection positivity has one [70, 71], and causal dynamical triangulations obtain theirs through a Wick rotation [36, 37, 38, 39]; Lorentzian simplicial path integrals without one face the same question [32, 33, 34, 35]. Two readings remain, and we do not decide between them. If the path sum has a unitary record-time evolution that we have not found, the remainder is a width, and observed gravitational waves, which cross cosmological distances, exclude it through the identification of Section 5.4. If it has none, the path sum at identified $\kappa = 1$ is not a consistent quantum theory in record time, a different and arguably stronger negative conclusion. Either way the remainder is a Lorentz-violating term generated at one loop with no symmetry to forbid it, the situation of radiatively generated Lorentz violation in theories with a preferred frame, where violations suppressed at tree level reappear unsuppressed at loop order unless a parameter is tuned [41, 42, 43].

The real part and the corners. Write $\bar{\Sigma}(\varepsilon)$ for the zone integral of $\Sigma_H[D_k]$ on the probe above, so that, by Proposition 6.18(vi), a massless long-wave graviton at zero momentum is

the statement $\bar{\Sigma}(\varepsilon) \to \frac{33}{4}i$. Its real part has a mechanism of its own at the axis corners.

Proposition 6.19 (The corners of the transverse-traceless self-energy)**.** *Let $\kappa = 1$ and let H be the probe above.*

(i) If $\operatorname{Re}\bar{\Sigma}(\varepsilon)$ does not tend to 0 as $\varepsilon \to 0^+$, then R_H does not tend to 0 in the ordered limit. This holds in particular if $\operatorname{Re}\bar{\Sigma}(\varepsilon) + c\log(1/\varepsilon)$ converges for some $c \neq 0$.

(ii) At an axis corner πe_c the columns $\Gamma^\circ e_a$ of the three axes $a \neq c$, with Γ° as in the proof of Theorem 6.17, have no posting entries, and for every small symmetric strain B the strained native block annihilates their non-posting parts. At πe_1 the native block is $3^{-1/2}$ times an explicit rational matrix whose kernel is exactly the span of these three vectors, and the posting Schur complement of the full block there is $\frac{2}{\sqrt{3}}\big(I - \frac{1}{3}\mathbf{1}\mathbf{1}^\top\big)$ on $\mathbb{R}^3$, which has a single null direction; at πe_2 that direction is again the only one.

(iii) The first-order coupling $\tilde{p}_c$ of the probe to that null direction, the derivative along tH of the Schur complement taken on it, is 0, $\frac{1}{2\sqrt{3}}$, $-\frac{1}{2\sqrt{3}}$, 0 at $c = 0, 1, 2, 3$.

Proof outline. (i) By Proposition 6.18(vi), $R_H \to 0$ in the ordered limit exactly when $\bar{\Sigma}(\varepsilon) \to \frac{33}{4}i$, which forces $\operatorname{Re}\bar{\Sigma}(\varepsilon) \to 0$. (ii) The posting part of $\Gamma^\circ e_a$ is $2(z_a - 1)$ at the posting of a and 0 elsewhere, and at πe_c it vanishes for $a \neq c$. The strained Bloch Hessian annihilates the strained vertex-move columns $\Gamma^\circ(g + B)$, hence the columns of Γ°, so the strained native block annihilates the non-posting parts of these three. The block at πe_1, its kernel and the Schur complement are exact rational computations from the cell Hessian. (iii) The kernel does not move with the strain, and by the gauge identity the posting columns stay orthogonal to it, so the Schur complement and its null direction extend smoothly in t. By first-order perturbation of a simple eigenvalue, the derivative of the Schur complement along tH, taken on its null direction w_0, is $\xi_c^\top(\partial_H M)\,\xi_c/|w_0|^2$, with ξ_c the null vector of the full zero-strain block that completes w_0 and $|w_0|^2 = 3$; this pairing is evaluated exactly from the cubic vertex. □

So the probe reaches the corners πe_1 and πe_2 and no others. Near each of them the determinant of the native block is $\prod_{a\neq c}|z_a - 1|^2$ times a function that begins $h^\top A_c h + t\,\tilde{p}_c$, where $A_c = \frac{\sqrt{3}}{120}K_c$ is the cone form of Proposition 6.2 with axis c in the role of axis 0, and the damping reaches the cone mode from the side of $+i\varepsilon$ (COMPUTATION). The double pole $-\tilde{p}_c^{\,2}/(\mu + i0)^2$ in the cone eigenvalue μ then gives, as in Proposition 6.5, a logarithm that the factor $\frac{i}{2}$ makes real: each coupled corner contributes $-\tilde{p}_c^{\,2}\log(1/\varepsilon)/(64\pi^2\sqrt{|\det A_c|}) = -\frac{5}{44\sqrt{335}\,\pi^2}\log(1/\varepsilon)$, so that

$$\operatorname{Re}\bar{\Sigma}(\varepsilon) = -\frac{5}{22\sqrt{335}\,\pi^2}\log\frac{1}{\varepsilon} + O(1) \approx -1.26\times 10^{-3}\log\frac{1}{\varepsilon}. \tag{11}$$

This is a derivation, not a theorem, and we keep it as a premise. It has two open parts: the local asymptotics near the two coupled corners, derived here on paper, and the convergence of everything else in the zone. If Eq. (11) holds, part (i) shows that the long-wave graviton is not massless at one loop, through the real part. The same local model gives no logarithm in the imaginary part (INFERENCE). The sign comes from the side from which the damping reaches the cone mode, a property of the regulator model. The logarithm is tiny: between $\varepsilon = 0.1$ and $\varepsilon = 0.001$ it moves $\operatorname{Re}\bar{\Sigma}$ by about 0.006, while the computed real parts move by 0.13 to 0.17 per halving of ε at the regulators reached, so no computation here tests it.

Measure weights. By Theorem 6.17(i), a bare weight on configurations multiplies the determinant and adds its own remainder, and only a weight that is not a function of $\det(g+B)$ contributes. A weight W changes the one-loop effective action $\frac{i}{2}\log D$ by $-i\log W$, so it enters as DW^{-2}. Write $h_r = r(1-r)$ for the flat squared length of a class with r bits; the non-posting classes have $r \in \{2,3,4\}$, six, four and one of them.

Proposition 6.20 (Local measure weights)**.** *Let $\kappa = 1$ and $W = \exp\bigl(\sum_e \varphi_{r(e)}(\ell_e)\bigr)$, the sum running over the non-posting edges, with $r(e)$ the number of bits of the class of e and each φ_r twice differentiable at h_r. Put $u_r = \varphi_r'(h_r)$, $v_r = \varphi_r''(h_r)$ and $U = u_2 + 2u_3 + u_4$.*

(i) Per site, W adds $-i\bigl(9v_2 - \frac{27}{2}U\bigr)$ to the remainder on f_E and $-i\bigl(9v_2 + 18v_3 - \frac{9}{2}U\bigr)$ on f_3.

(ii) If W is stationary and log-concave at the flat lattice ($u_r = 0 \geq v_r$), both additions have nonnegative imaginary part. The recognition-cost weight $\exp\bigl(-\beta\sum_e J(\ell_e/h_{r(e)})\bigr)$ with $\beta \geq 0$ adds $\frac{9}{4}i\beta$ and $\frac{11}{4}i\beta$.

(iii) The power weight $\prod_e |\ell_e|^{-\alpha}$ adds $\frac{81}{8}i\alpha$ and $\frac{11}{8}i\alpha$, hence $\frac{23}{2}i\alpha$ on $f_E + f_3$. A power weight with one exponent cancels the remainders of both polarizations only if they stand in the ratio $81 : 11$.

(iv) Every measure on the null stratum that is invariant under its translations is a constant multiple of the flat measure, and a weight with this invariance is constant on the stratum. A weight that depends on each cell only through the volumes of its simplices is, on strained flat configurations, a function of $\det(g+B)$. Neither changes R_H.

Proof. Strains act linearly on squared lengths: along tH the edge e has $\ell_e = h_{r(e)} + ts_e$ with $s_e = d^{\top}Hd$, and along tg it has $(1+t)h_{r(e)}$. Hence, per site, $\Sigma_H[W^{-2}] = -i\sum_e v_{r(e)}s_e^2$ and $\sigma_{\mathrm{dil}}[W^{-2}] = -i\sum_e u_{r(e)}h_{r(e)} = 12iU$, so the remainder of W^{-2} is $-i\bigl(\sum_e v_{r(e)}s_e^2 - 3\tau_H U\bigr)$ with $\tau_H = \mathrm{tr}(g^{-1}Hg^{-1}H)$. For a strain with zero diagonal and $\sum_{a<b} H_{ab} = 0$, which is what posting-null and traceless mean at $\kappa = 1$, $s_e = 2H_{ab}$ on the class $e_a + e_b$, $s_e = -2\sum_b H_{cb}$ on the class that omits axis c, and $s_e = 0$ on the body diagonal; also $\tau_H = 2\sum_{a<b} H_{ab}^2 - \frac{2}{3}\sum_c\bigl(\sum_b H_{cb}\bigr)^2$. The strains of E have zero row sums. With the normalization of the probes, the sums of s_e^2 over the classes with two, three and four bits are 9, 0, 0 on f_E, with $\tau_H = \frac{9}{2}$, and 9, 18, 0 on f_3, with $\tau_H = \frac{3}{2}$. This gives (i). For (ii), $U = 0$ and $v_r \leq 0$ leave $-i\sum v_r s_e^2$, whose imaginary part is nonnegative; for the recognition cost $u_r = 0$ and $v_r = -\beta/h_r^2$. For (iii), $u_r = -\alpha/h_r$ and $v_r = \alpha/h_r^2$, so $U = \frac{11}{12}\alpha$. (iv) A translation-invariant locally finite measure on a real affine space is a multiple of Lebesgue measure. Simplex volumes of a strained flat cell are those of the unstrained cell times $|\det(g+B)/\det g|^{1/2}$, and Theorem 6.17(i) applies. □

Of the weights examined, the native flat measure adds nothing, a recognition cost of edge intervals adds a positive imaginary part on both polarizations, the sign that R_H has numerically, and simplex-volume factors are covariant; none of them cancels R_H. A cancelling weight would have to break the translation invariance of the null stratum, could not, given the computed sign, be the Gibbs factor of a cost minimized at the flat lattice, and would have to meet one condition for each polarization (INFERENCE). A single power exponent cancels the sum $f_E + f_3$ but not each polarization, since the computed ratio is far from $81/11$ (NUMERICAL). Such a weight would be an identification tuned to a computed number, not a derivation.

Other values of κ. The arguments of this subsection use $\kappa = 1$ only through the form of g and hold for every $\kappa > 1/2$, with a strengthened (iii): an isometry of g_κ that sends each posting to a lattice edge vector is $\pm$ an axis permutation, and the axis permutations do not force $R_H = 0$. At $\kappa = 3/2$, for a transverse-traceless probe that keeps the postings fixed, the same exact frequency integration gives an imaginary remainder of 7.6 in the normalization of the $\kappa = 1$ probe, or 5.1 in a test confined to posting-fixed strains (the two tests differ there by an exact multiple of the source of the strain that changes κ), against 7.0 at $\kappa = 1$ (NUMERICAL). Across $\kappa \in [0.6, 3]$ no sampled value makes the remainders of both polarizations vanish (NUMERICAL). So, on this evidence, the long-wave remainder does not single out $\kappa = 1$, and moving to $\kappa = 3/2$ removes the corner doublers but not R.

In sum: the native path sum predicts an order-one, Lorentz-violating one-loop term for long-wave gravitons at zero momentum, unless the bare measure weight cancels it. That is established numerically, not as a theorem. What is proved is that gauge fixing and ghosts cannot remove it, that its real part vanishes at $\varepsilon = 0$ on every finite torus, that one inequality on one explicit zone integral, or a logarithm in its real part, would prove it nonzero, and that none of the weights examined cancels it. The native block at a strained background is the compression of the strained Hessian, with the damping $i\varepsilon$ added to it (MODEL).

6.9 Nonzero momentum: soft modes on the phase-one hyperplanes

At zero momentum the waves on which a phase equals 1 drop out of the ordered limit (Section 6.8). At nonzero momentum they do not. On the hyperplane $k_a = 0$ holding the postings fixed is not a gauge, and the native block keeps an exact zero mode there.

Proposition 6.21 (Soft modes on the hyperplanes)**.** *Let $\kappa = 1$ and let k be a real wave with phases $z_b = e^{ik_b}$. For an axis a let $u_a(k)$ be the non-posting part of the vertex move $\Gamma^\circ(k)e_a$ of the proof of Theorem 6.17, $b_a(k)$ the non-posting part of the column of $B(k)$ at the posting of a, and $B_{aa}(k)$ the diagonal entry of $B(k)$ there. Then*

$$B_{\mathrm{nat}}(k)\,u_a = -2(z_a - 1)\,b_a, \qquad b_a^\dagger u_a = -2(z_a - 1)\,B_{aa},$$
$$\langle u_a, B_{\mathrm{nat}}(k)\,u_a\rangle = 4|z_a - 1|^2 B_{aa} = 16\sin^2\tfrac{k_a}{2}\,B_{aa}.$$

In particular, at every wave with $k_a = 0$, $u_a(k)$ is a zero mode of the native block, nonzero unless $k = 0$.

Proof. The Bianchi identity gives $B(k)\Gamma_a = 0$ for $\Gamma_a = \Gamma^\circ(k)e_a$, whose posting part is $2(z_a - 1)$ at the posting of a and 0 at the other postings. The non-posting rows of $B\Gamma_a = 0$ give the first identity, and the row of the posting of a gives the second, since B is Hermitian. Then $\langle u_a, B_{\mathrm{nat}}u_a\rangle = -2(z_a - 1)\,\overline{b_a^\dagger u_a} = 4|z_a - 1|^2 B_{aa}$. On the class $d = e_a + e_b$ the entry of Γ_a is $2(z_a z_b - 1)$, which is nonzero at $k_a = 0$ when $z_b \neq 1$. □

On the hyperplane $k_a = 0$ the vertex move $\Gamma^\circ(k)e_a$ has no posting component, since its posting entry is $2(z_a - 1) = 0$: these soft modes are the residual vertex moves of the posting-null condition, the same moves, with ξ_a constant along a-fibres, that make the twisted slice of Theorem 3.3(iii) non-unique. Off the hyperplane the mode is no longer null, and its Rayleigh quotient is exactly $16\sin^2(k_a/2)\,B_{aa}$, with no remainder; that the eigenvalue rises in the same

way is numerical (NUMERICAL). These are vertex-move modes of the quadratic action, soft in one direction and flat in the three directions along the hyperplane, and the damping $i\varepsilon$ is their only regulator (MODEL). A one-dimensional integral shows what that can do: for $c > 0$ the imaginary part of $\int_0^1 \mathrm{d}q/(cq^2 + i\varepsilon)$ is at least $1/(2\sqrt{c\varepsilon})$ in absolute value for small ε, so a slab $|k_a| \lesssim \sqrt{\varepsilon/c}$ about each hyperplane contributes a term of order $\varepsilon^{-1/2}$ to a one-loop integral whose residue on the hyperplane does not vanish. At zero momentum on strains that residue vanishes; at nonzero momentum it does not. Numerically, on the graviton's cone at momentum t the native two-point function has a real part of about $-1.0 \times 10^{-3}\, t^2 \varepsilon^{-1/2}$ after the tadpole insertion, with the t^2 scaling holding to a few per cent (NUMERICAL); the term reaches the size of the order-one imaginary part only for ε of a few times 10^{-7} at $t = 0.1$.

For the tadpole the step from these numbers to a theorem reduces to two premises. Write $j(\varepsilon)$ for the native one-loop source of a constant background, the torus limit at fixed ε of one damped native propagator contracted with the cubic vertex, and $\psi(\varepsilon) = -A_0^+ j(\varepsilon)$ for the shift of the background it induces on the eleven classes, with A_0 the native block at zero wave and A_0^+ its pseudo-inverse. On each hyperplane the exact residue of the source on the soft mode and an explicit curvature C_a, a ratio of determinants built from B, define a slab model $m(\varepsilon)$; the theorem below uses C_a only through its definition. The model satisfies: $\sqrt{\varepsilon}\, m_d(\varepsilon)$ tends to an explicit coefficient c_d, an integral over the hyperplanes with weight $1/\sqrt{|C_a|}$, whenever $C_a \neq 0$ almost everywhere and that integral converges absolutely.

Theorem 6.22 (The native tadpole under two premises)**.** *Let $\kappa = 1$, with the damping $i\varepsilon$ on the eleven non-posting classes (MODEL). Suppose*

(S) for every non-posting class d, $\sqrt{\varepsilon}\,\big(\mathrm{Re}\, j_d(\varepsilon) - m_d(\varepsilon)\big) \to 0$ as $\varepsilon \to 0^+$; and

(C) $C_a \neq 0$ almost everywhere on the hyperplanes, the weighted residue is integrable, and $\sum_d \psi_d c_d \neq 0$ for some real ψ in the range of A_0.

Then $|\psi(\varepsilon)| \to \infty$ as $\varepsilon \to 0^+$; in particular the native one-loop shift has no regulator limit. Premise (S) follows in turn from a bound on a spectral moment of the native block away from the slabs together with an estimate on the slabs.

Proof outline. The slab model converges after multiplication by $\sqrt{\varepsilon}$ by the one-dimensional slab lemma, with a uniform bound and dominated convergence over the hyperplanes; with (S) this gives $\sqrt{\varepsilon}\,\mathrm{Re}\, j(\varepsilon) \to c$. Pairing through A_0^+, (C) makes $\sqrt{\varepsilon}\,\psi(\varepsilon)$ tend to a nonzero limit. For the last statement, the tadpole vertex is Hermitian, so the absorptive part of the source away from the slabs is an eigenvalue sum $\frac{1}{2}\sum_j m_j \varepsilon/(\lambda_j^2 + \varepsilon^2)$, which a moment of $\|A^{-1}\|$ of order $p > \frac{1}{2}$ makes $o(\varepsilon^{-1/2})$. □

The premises are not proved. Numerically the coefficient of (C) is about 0.40 to 0.44, and (C) is a statement about the value of one integral, not about a sign (NUMERICAL). For the two-point function on the cone the hyperplane residue meets a threshold and is only principal-value integrable, so the argument above does not apply there.

What makes this more than a regulator artefact is that the soft modes are not gauge. They are vertex moves, and the vertex moves are exact zero modes of the flat Hessian, but they are not symmetries of the interacting action.

Theorem 6.23 (Vertex moves are not symmetries). *(i) Let S be of class C^3 near a critical point x_0, and let X be a vector field of class C^2 with $DS(x)[X(x)] = 0$ for all x near x_0. Then $D^3S(x_0)[h, h, X(x_0)] = 0$ for every h with $D^2S(x_0)h = 0$.*

(ii) Let $\kappa = 1$, let N be divisible by 4, and let S be the Regge action on the side-N torus. At the flat field there are a zero mode $x_1 = \mathrm{Re}\big(v_* e^{ip_*\cdot y}\big)$ of the Hessian, with $p_* = (\frac{\pi}{2}, \pi, \pi, \pi)$ and v_* an explicit null vector of the Bloch matrix at p_* with entries in $\mathbb{Z}[i]$, and the vertex move g generated by the displacement $(-1)^{y_0}e^0$, also a zero mode, with

$$D^3S(\ell^0)[x_1, x_1, g] = -\frac{409600}{\sqrt{3}}\Big(\frac{N}{4}\Big)^4 \neq 0.$$

(iii) In the setting of (ii) let g_b be the vertex move generated by $(-1)^{y_0}e^b$, so $g_0 = g$. Then $D^3S(\ell^0)[x_1, x_1, g_b] = (307200/\sqrt{3})(N/4)^4$ for $b = 1, 2, 3$, so

$$D^3S(\ell^0)\Big[x_1, x_1, \sum_{b=0}^{3} a_b g_b\Big] = \frac{102400}{\sqrt{3}}\Big(\frac{N}{4}\Big)^4\big(-4a_0 + 3(a_1 + a_2 + a_3)\big),$$

whose kernel on the span of the four moves is three-dimensional and contains $3g_0 + 4g_1$.

By (i) and (ii), no infinitesimal symmetry of the action near the flat field takes the value g there; in particular no symmetry of the interacting $\kappa = 1$ action has all vertex moves among its generators.

Proof. (i) Differentiate $F(x) = DS(x)[X(x)] \equiv 0$ twice at x_0 along h: $D^3S(x_0)[X(x_0), h, h] + 2D^2S(x_0)[DX(x_0)h, h] + DS(x_0)[D^2X(x_0)(h, h)] = 0$, and the last two terms vanish because $D^2S(x_0)h = 0$ and $DS(x_0) = 0$. (ii) and (iii) The cubic form is assembled cell by cell from the exact third derivatives of the cell action; the values on one block of side 4 are exact computations over $\mathbb{Q}(\sqrt{3}, i)$, and since the fields have period 4 the sum over the torus is $(N/4)^4$ times the block sum. □

Scope. The wave p_* exists exactly on the sides divisible by 4, and per vertex the defect $-1600/\sqrt{3}$ does not dilute as such a torus grows. On the odd sides of Theorem 4.2 every zero mode of the Hessian is a constant strain plus a vertex move, and $D^3S(\ell^0)$ vanishes on that radical, so the test of (i) is empty there; whether the vertex moves generate a symmetry of the action on odd tori is not decided by it. Sides of the form $4k + 2$ are not treated. The reading of the soft modes below therefore rests on the sides divisible by 4, and, for other sides and the infinite lattice, on the inference that the symmetries of the action do not depend on the side (INFERENCE).

The cubic test does not exclude every symmetry whose generator leaves the posting-null stratum. By (iii), on the four moves $g_0, \dots, g_3$ the functional $w \mapsto D^3S(\ell^0)[x_1, x_1, w]$ is proportional to $(-4, 3, 3, 3)$, and its kernel contains combinations such as $3g_0 + 4g_1$. The moves g_1, g_2, g_3 vanish on every posting, so the posting part of $3g_0 + 4g_1$ is that of $3g$, and it leaves the stratum in the same direction as g; a symmetry whose generator at the flat field is such a combination is not excluded by the cubic test. Within these limits the theorem decides how the soft modes are to be read. If the posting-null stratum is the configuration space, as the native construction of Section 6.5 takes it, they are physical directions, and a term of order $\varepsilon^{-1/2}$ is a property of the native measure: the native path sum would then have no one-loop

graviton two-point function at nonzero momentum (INFERENCE). Reading the constraint instead as a gauge slice of the unconstrained edge space would need a symmetry to slice. By Theorem 6.23 the vertex-move orbits are not that symmetry; a symmetry of another kind, with a generator such as $3g_0 + 4g_1$ at the flat field, is not excluded.

6.10 What is established, and what would have to change

The premises of this section are identified $\kappa = 1$, the Kuhn triangulation, the bilinear Regge action, the diagram model Eq. (7), one loop, and Gaussian order wherever a measure enters. Within them the main result is a theorem: the ordered zero-momentum anisotropy diverges like $\frac{5}{44\sqrt{335}\,\pi^2}\log(1/\varepsilon)$, with a finite part (Theorem 6.8), because four massless corner doublers that nothing in the construction gaps couple to strains without derivatives, while every other critical zero of the operator is hyperbolic. The further results point the same way. The doublers have no real record-time frequency near the corners (Theorem 6.15). The long-wave zero-momentum remainder R does not depend on the gauge fixing or on any covariant or examined measure weight, and numerically it is large and imaginary (Section 6.8). At nonzero momentum the vertex moves are not symmetries of the interacting action (Theorem 6.23, with its scope), and the soft modes they leave make the native one-loop shift diverge under two stated premises (Theorem 6.22).

At nonzero external momentum the same corner structure gives a logarithm of the momentum. In the exact local model of a doubler pair, the loop has a regulator limit at every transfer off the doubler cone, with $i\pi^2 \log t^2$ at small transfer t, no regulator limit on the cone, and no logarithm once the pair is gapped (Proposition 7.7). We read the zero-momentum divergence of Theorem 6.8 as the shadow of this logarithm, a non-analytic term whose cone is the doubler's and not the graviton's (INFERENCE); transferring the model statement to the lattice integrand is the open step named in Section 7.5.

Still open within the same premises:

- the corner self-energy Eq. (10) and its sign, and the seagull and scalar loops at the corner;
- for R: the threshold inequality of Proposition 6.18(iv), a bound on one explicit zone integral; its real part; the existence of the ordered limit and its independence of the regulator; and the corner logarithm Eq. (11) in its real part;
- at nonzero momentum: the premises (S) and (C) of Theorem 6.22, the two-point function on the cone, and the transfer of the pair-loop model to the lattice integrand;
- the value of the finite part r_0, whose existence is proved (numerically about 0.151);
- matter beyond one scalar: matter with no zero modes at the corners adds infrared-finite pieces and cannot cancel a logarithm whose source is the graviton's own massless pair (INFERENCE), though it does add to R; a fermion sector is not treated;
- actions that are not bilinear in area and deficit, condensation of the corner mode, and second-order shifts along E strains.

A massless graviton with controlled one-loop corrections would need one of the following changes.

- *The identification $\kappa = 1$.* The doublers belong to it: moving κ above 1 gaps all four at first order (Theorem 6.10(iv)), and at $\kappa = 3/2$ every half-period corner is regular (Proposition 6.16(iii)). The rest stays: lattice-scale growth in record time occurs at both values, and the imaginary part of the long-wave remainder survives at $\kappa = 3/2$ with about the same size (NUMERICAL). The classical results proved with $\kappa = 1$ lattice objects would have to be derived again. We examine $\kappa = 3/2$ as a hypothesis, not as a proposal, and on the present evidence a change of κ alone would not give a massless graviton.
- *A measure weight that is not covariant*, tuned to cancel R on both polarizations and, for the doublers, positive on the corner modes while vanishing faster than $|k|^2$ on long waves (Theorem 6.11(iii), Proposition 6.20). None of the weights examined does this, and such a weight would be chosen to cancel a computed number, not derived.
- *The identification of the lattice spacing.* Every loop effect enters physical units through $(\ell_P/b)^2$ (Section 5.4); a different spacing changes the physical size of R and of the doubler terms, not their presence.
- *Physics beyond one loop.* The doublers' self-energy is as large as their tree-level operator at accessible regulators, so a resummation or a condensation of the corner modes could change the picture. It would have to remove the massless corner modes from the long-wave dynamics, which a small gap does not do (Theorem 6.15(v)), and leave $R = 0$.

The conclusion concerns identified $\kappa = 1$ within the stated model. The definitions of Section 2 do not fix κ, so it is not a statement about them; it is why the paper's positive results are classical and its one-loop graviton result is an obstruction.

7 The one-loop correction to Newton's potential

General relativity treated as an effective field theory predicts a definite quantum correction to the potential between two masses [10, 11],

$$V(r) = -\frac{Gm_1m_2}{r}\Big(1 + \frac{3G(m_1+m_2)}{rc^2}\Big) - \frac{41}{10\pi}\,\frac{G^2\hbar\, m_1m_2}{r^3c^3}.$$

The one-loop ultraviolet divergences of gravity [53] are local and do not touch it. The coefficient $41/(10\pi)$ comes from the non-analytic $\log q^2$ term of the one-loop amplitude, and it does not depend on the ultraviolet completion provided the light spectrum is that of general relativity: the graviton and the two masses. This section shows that the one-loop amplitude assembled from the $\kappa = 1$ long-wave Feynman rules has this coefficient, in a precise infrared-subtracted reading at small momentum transfer, and that the natural first readings of the question are false, which is why the reading matters. The light spectrum of the lattice at $\kappa = 1$ is not that of general relativity: it contains the four massless doublers of Section 6, which couple to strains without derivatives. So the theorem below is a statement about the long-wave rules, not about the lattice path sum. Section 7.5 shows that the doublers do enter the two-mass channel. They couple to the static field of a mass without derivatives, and in an exact local model their pair loop adds a term to the static amplitude that is more singular at small momentum transfer than the logarithm read here; whether the lattice integrand follows the model is open. The potential itself depends on how it is defined from the amplitude [44, 45, 46, 47, 48, 49, 50, 51, 52]; we use the Born reading of [11] and state it as a reading.

7.1 The amplitude

Momentum transfers are static, $Q = tP$ with $P \in \mathcal{S} = \{P \in \mathbb{R}^4 : \sum_a P_a = 0\}$, $P \neq 0$, and $t > 0$ the long-wave scaling parameter; on $\mathcal{S}$ the wave symbol is $q(P) = \sum_a P_a^2$. Two scalars of masses μ_1, μ_2 in lattice units carry Breit energies $M_i(t) = \sqrt{\mu_i^2 + t^2 q(P)/4}$. The one-loop amplitude has the classes of Table 3: box and crossed box, two triangles, the double seagull, four vertex corrections, and the graviton vacuum polarization with its ghost loop; the iterated tree exchange, the Born term, is subtracted. Every line carries the same Feynman regulator $\eta > 0$. For a diagram integrand $\mathcal{J}$ and a region S of loop covectors L we write

$$V_{\mathcal{J},S}(\eta, t) = -\frac{1}{(2\pi)^4} \frac{1}{4ia\, M_1(t) M_2(t)} \int_S \mathcal{J}(\eta, t, L)\, \mathrm{d}L.$$

Loop diagrams are integrated over the cube $|L|_\infty \leq \Lambda$. The Born term is integrated over the cylinder

$$\mathrm{Cyl}(R, X) = \big\{L : \ |L - \tfrac{1}{4}(\textstyle\sum_a L_a)(1,1,1,1)|_\infty \leq R, \ \ |\textstyle\sum_a L_a| \leq 2X\big\},$$

a spatial cube of half-side R times a time window. The *restricted range* is $0 < X < 2\sqrt{3}\min(\mu_1, \mu_2)$: on the time axis the antiparticle pole of matter line i sits at $|\sum L/2| = 2\sqrt{3}M_i(t) \geq 2\sqrt{3}\mu_i$, so a restricted cylinder holds no antiparticle pole and covers only the static region the iterated Born term describes.

Reading a logarithm. The coefficient is read at small t from a family that also contains the inverse powers of renormalized tree terms and lattice cutoffs. Define

$$(\mathcal{T} f)(t) = 512 f(8t) - 224 f(4t) + 28 f(2t) - f(t), \qquad (\Delta f)(t) = \big(2f(2t) - f(t)\big) - \big(2f(4t) - f(2t)\big).$$

$\mathcal{T}$ annihilates t^{-2}, t^{-3}, t^{-4} and multiplies t^{-1}, constants and $\log t^2$ by 21, 315 and 315 (up to a constant); Δ annihilates t^{-1} and constants and sends $\log t^2$ to $-\log 4$. Say f has *long-wave log coefficient* c if $(\Delta\mathcal{T} f)(t) \to -315\, c \log 4$ as $t \to 0^+$. A function of the form $a_4 t^{-4} + a_3 t^{-3} + a_2 t^{-2} + \alpha t^{-1} + c \log t^2 + A + o(1)$ has coefficient c; the coefficient is unique, linear, and depends only on the germ of f at 0^+. Because $|Q|^2 = t^2 q(P)$, it is the coefficient of $\log |Q|^2$.

Subtracting the regulator logarithm. At fixed t the infrared region of the box produces a $\log \eta$. For a family $C(\eta, t)$ and a number c_0, write $C \in \mathrm{IR}(c_0)$ if there are functions d, F_0 with

$$C(\eta, t) + d(t) \log \eta \ \longrightarrow \ F_0(t) \quad (\eta \to 0^+) \text{ for every } t > 0,$$

F_0 of long-wave coefficient c_0, and d of long-wave coefficient 0. The condition on d makes the reading invariant under $\eta \mapsto k\eta$; it is not an arbitrary subtraction. Write $C \in \mathrm{IR}_{\mathrm{germ}}(c_0)$ if the same holds for $t \in (0, t_0)$, for some $t_0 > 0$ chosen after all other parameters. The germ reading is implied by the all-t reading with the same coefficient, is unique and additive, and keeps everything the long-wave coefficient sees.

7.2 The long-wave Feynman rules

Every line and vertex of the amplitude is the long-wavelength limit of an object of the lattice theory, with two exceptions named at the end. Scale the loop covectors and the masses by ζ and the regulator by ζ^2, and let $\zeta \to 0^+$. The objects V, ℓ, Q_{dD}, w_κ, $\mathcal{G}$ and $C' = w_\kappa \mathcal{G}$ are those of the de Donder gauge-fixed form in Section 4.4.

Graviton line. The regulated two-point function of the Regge operator with the de Donder gauge term $\ell\, Q_{\mathrm{dD}}\, \ell^\top$ tends, at every loop covector L off the light cone, to the de Donder line $V\big(\eta\, V^\top V - ias\, q(L)\, C'\big)^{-1} V^\top$ in strain form. On strains the lattice quadratic form is w_κ times the Fierz–Pauli form, and the Fierz–Pauli form plus the gauge term is q times the DeWitt form. This is the Einstein–Hilbert quadratic action in de Donder gauge with unit weight, and its source term gives $G = 1/(8\pi s)$.

Scalar line. The simplicial scalar field has the cell action $-\frac{1}{48}\sqrt{|\det G_\sigma|}\; d\phi^\top G_\sigma^{-1} d\phi$ on each four-simplex σ, as in Section 6.1, with G_σ the Gram matrix of the four edge vectors of σ from one vertex and $d\phi$ the differences of ϕ along them, together with its mass term. Its Bloch symbol is $\sqrt{4\kappa - 1}\big(\sum_a (2 - 2\cos u_a) - \frac{\kappa}{4\kappa - 1}(2 - 2\cos \sum_a u_a) + \mu^2\big)$, and its regulated line tends to $1/\big(\eta + ia\sqrt{4\kappa - 1}(q(u) + \mu^2)\big)$: the Klein–Gordon line with the cell volume $\sqrt{|\det g|}$.

Matter vertices. The one-graviton vertex is the first variation of the matter action in the edge lengths. On a strain H it tends to $\frac{1}{2}\sqrt{|\det g|}\, H_{ab} T^{ab}$ with $T^{ab} = w^a w^b - \frac{1}{2} g^{ab}(w^2 + \mu^2)$, the stress-tensor coupling. The seagull is the mixed second variation; its long-wave limit is computed exactly from the cell formula.

Three-graviton vertex. The cubic vertex is the plane-wave third variation of the Regge action, with exact momentum conservation. On three strain legs its zeroth- and first-order terms vanish, and its second-order term is half the cubic vertex of $\int \sqrt{|g|} R$. The quadratic and cubic long-wave terms of $s\,S$ are therefore those of $\frac{s}{2}\int \sqrt{|g|} R = \frac{1}{16\pi G}\int \sqrt{|g|} R$, so this is the Einstein–Hilbert vertex with the same G as the propagator. The cubic vertex is a property of the nonlinear action, not of the tree-level operator.

Four-graviton vertex. The quartic cell form is the fourth variation of the action. It enters only the tadpole, which carries no $\log q^2$ and is omitted.

Ghosts. The ghost line is the regulated inverse of $\ell^\top \Gamma$ and tends to the de Donder ghost line. The ghost vertex is not supplied by the path-sum measure: its Faddeev–Popov factor is independent of the graviton field, so it has no ghost coupling. A lattice generator built from the vertex displacement of the flat family (exact at constant strain) and parallel transport has the de Donder ghost vertex as its long-wave limit. It is not shown to be a symmetry of the interacting action, so the ghost vertex used here is a choice of the gauge-fixing procedure at curved configurations.

Inputs. Two ingredients are imposed rather than derived. Every line carries the Feynman $i\eta$: the lattice regulator's own pole prescription puts strain poles on both sides of the real axis, and the reading with it fails (Theorem 7.1). The ghost vertex is the second. The scalar action and the gauge readout are model choices (MODEL).

So the long-wave rules are the Einstein–Hilbert rules in harmonic gauge, written in record coordinates with the metric $g_1 = I - \mathbf{1}\mathbf{1}^\top$, with $G = 1/(8\pi s)$, $\hbar = 1/a$ and $m = \hbar\mu$. What

this section takes from the lattice is the long-wavelength limit of each rule, the cubic vertex included. The loop integrals are continuum integrals.

7.3 The readings that fail

Theorem 7.1 (The original prescription fails)**.** *In the original prescription (a Born* cube *in place of the cylinder, no subtraction, and the regulator applied separately to the matter lines and to the strain-regularized graviton lines), the statement "for all positive parameters and every nonzero static P the regulator limit exists for every $t > 0$ and has long-wave coefficient $-\frac{41}{20\pi}G\hbar$ times the tree coefficient" is false. For every loop cutoff, in the equal-mass family $\mu(x) = \sqrt{4/3 - |x|^2/4}$ with $s = a = t = 1$ near $P_\star = (\frac{1}{2}, -\frac{1}{2}, 0, 0)$, the potentials with Born cubes of half-sides $1/4$ and $3/4$ cannot both have regulator limits at every nearby transfer.*

The two potentials differ only by the Born integral over the shell between the cubes. Averaged against a bump in the transfer, that shell grows at least like $\log(1/\eta)$, with leading coefficient proportional to an exact positive rational, $3 \cdot \frac{1189}{1152} \cdot \frac{1609}{1152}$; a Baire-category argument converts divergence of the averages into failure at some transfer. What fails is the existence of the regulator limit, not the value of a coefficient.

Replacing the cube by the cylinder and subtracting $d(t)\log\eta$ is the natural repair. It does not survive at all t.

Theorem 7.2 (A $\log^2$ at a face centre)**.** *Let $s = a = 1$, $\mu_1 = \mu_2 = \sqrt{55/48}$ and $P = (-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, \frac{3}{4})$, and let $V_{\mathrm{sh}}(\eta, t)$ be the Born potential over* $\mathrm{Cyl}(\frac{3}{4}, 1)$ *minus that over* $\mathrm{Cyl}(\frac{17}{20}, 1)$*; both cylinders are in the restricted range. There are $c > 0$, C and $\eta_0 > 0$ with*

$$\|V_{\mathrm{sh}}(\eta, 1)\| \geq c\log^2\eta - C \qquad (0 < \eta < \eta_0).$$

Consequently no all-t reading can hold for every restricted cylinder, whatever coefficient it assigns: two such statements would put their difference, this shell, in $\mathrm{IR}(0)$*. This refutes the all-t restricted statement, its box-and-crossed form, the cylinder-shell statement and the unrestricted corrected statement.*

At $t = 1$ the static point $Q = tP$ lies at the centre of a face of the inner cylinder. Integrating over the time component pinches the two matter poles from opposite sides of the real axis, and together with the soft graviton line the half-space on one side of the face produces $\log^2\eta$. No $d(t)\log\eta$ removes a $\log^2\eta$. At this point the Breit mass is $2/\sqrt{3}$ and the tree and forward couplings at $L = Q$ are the exact rationals $-\sqrt{3} \cdot 5005/4608$ and $-\sqrt{3} \cdot 7633/4608$, so the coefficient is not accidentally zero. The theorem is proved at this point; that every face-centre configuration behaves the same way is an INFERENCE.

The face-centre time $t = R/|P|_\infty$ is large, while the coefficient only reads small t. Two small-t failures are also known, and the reading below must not dodge them: at $X = 2\sqrt{3}\mu_i$ the antiparticle pole reaches the cylinder's time boundary and is expected to produce a $(\log t)/t$ term (outside the restricted range), and the box and crossed box read separately each keep a $t^{-2}\log t^2$ from the soft line, which cancels only in their sum (numerical evidence; the cancellation in the sum is proved).

Table 3. The one-loop classes and their long-wave log coefficients X, in units of $u = \sqrt{3}\mu_1\mu_2/(960\pi^2 as^2)$, with $x = \mu_2/\mu_1$, and the corresponding coefficients c of $V \supset -cG^2\hbar m_1 m_2/(\pi r^3)$: ours, $c = X/30$, and those of [11] (BDH). Box and crossed box, the triangles and the double seagull agree class by class. The vertex corrections and the vacuum polarization differ individually and agree in their sum; the two calculations use different off-shell three-graviton vertices ([11] the background-field vertex and the 't Hooft–Veltman vacuum polarization, here the Einstein–Hilbert vertex in de Donder gauge). The box and crossed box are stated only together: their separate values are exact computations, and their separate infrared-subtracted readings are expected to fail, since each keeps a $t^{-2}\log t^2$ from the soft line that cancels only in the sum; the $x + 1/x$ parts cancel between them. The graviton loop gives $133u$ and the ghost loop $-30u$ of the vacuum polarization in our computation; only the sum $103u$ is proved and used.

class	X	$c = X/30$	BDH c
box − cylinder Born	$235 - 70(x + 1/x)$		
crossed box	$235 + 70(x + 1/x)$		
box + crossed box (far part 360, soft points 110)	470	$47/3$	$47/3$
two triangles	-840	-28	-28
double seagull	660	22	22
four vertex corrections	-270	-9	-7
matter-loop type; seagull type	150; -420	5; -14	$5/3$; $-26/3$
vacuum polarization	103	$103/30$	$43/30$
vertex corrections + vacuum polarization	-167	$-167/30$	$-167/30$
total	123	$41/10$	$41/10$

7.4 The germ theorem

Let $u = \sqrt{3}\,\mu_1\mu_2/(960\pi^2 as^2)$ and $x = \mu_2/\mu_1$.

Theorem 7.3 (The Born shell is harmless at small t)**.** *For all $s, a, \mu_1, \mu_2 > 0$, every nonzero static P, and all $R, X, R', X' > 0$ with $X, X' < 2\sqrt{3}\min(\mu_1, \mu_2)$, there is $t_0 > 0$ such that for every $t \in (0, t_0)$ the shell potential between $\mathrm{Cyl}(R, X)$ and $\mathrm{Cyl}(R', X')$ has a regulator limit $F_0(t) = A/t + B + o(1)$. Its long-wave coefficient is 0, with no subtraction.*

Theorem 7.4 (Box, crossed box and Born term)**.** *For all $s, a, \mu_1, \mu_2 > 0$ (equal masses included), every nonzero static P, every $\Lambda, R > 0$ and every $0 < X < 2\sqrt{3}\min(\mu_1, \mu_2)$, the Feynman-regulated box minus its cylinder Born term plus the crossed box lies in $\mathrm{IR}_{\mathrm{germ}}(470\,u)$.*

Theorem 7.5 (The one-loop coefficient)**.** *Under the hypotheses of Theorem 7.4, the whole restricted potential, all classes of Table 3 with the cylinder Born term subtracted, lies in $\mathrm{IR}_{\mathrm{germ}}(123\,u)$, and*

$$123\,u = \Big(-\frac{41}{20\pi}G\hbar\Big)\cdot\Big(-\frac{\sqrt{3}}{2}\frac{\mu_1\mu_2}{s}\Big), \qquad G = \frac{1}{8\pi s},\ \hbar = \frac{1}{a}\ :$$

the coefficient is $-\frac{41}{20\pi}G\hbar$ times the tree coefficient.

Proof outline of Theorems 7.3 to 7.5. Shells. A shell between two cylinders splits into shells in R at fixed X and one thin shell in X. Choose $t_0 \le R^2/(8R'(|P'|_1 + 1))$, with P' the spatial part of

P; then on an R-shell the static points and the soft lines stay out of the region, the antiparticle poles stay beyond the time window for every t, the particle poles lie on opposite sides with no pinch, and only integrable endpoint logarithms remain, with an η- and t-independent majorant. On a thin X-shell both matter lines stay a fixed gap off shell and the two soft graviton lines give a pure bubble; expanding the smooth weight in t and in the loop momentum, the would-be $\log t$ from its linear term cancels by the central symmetry of the cube, leaving $A/t + B + o(1)$. This is why the germ window ($t_0 = 9/272 \approx 0.033$ at the refuting point of Theorem 7.2) never reaches the face-centre time $R/|P|_\infty$.

One reference cylinder. The combined box–Born–crossed family at (R, X) equals the family at the reference cylinder $(\min \mu/4, \sqrt{3} \min \mu)$ plus a shell, so by Theorem 7.3 it suffices to treat the reference cylinder.

Decomposition. For every $\eta > 0$ the regulated potential equals, exactly, a far part plus integrals over a small ball around the two soft points $L = 0$ and $L = Q$. Far parts have continuous regulator limits on $[0, t_0]$, hence coefficient 0 beyond the explicit far contribution. On the soft ball the integrand is written exactly as a regular part, the two divided differences of the matter lines and a pinch term; near each soft point it differs from an explicit model, homogeneous of degree -3, by $O(1/G(w))$ uniformly in η. The model's $\log \eta$ is extracted exactly by scaling: if $H(\eta, y) = \eta^{-3/2} H(1, y/\sqrt{\eta})$ with $H(1, \cdot)$ locally integrable and asymptotic to a degree-(-3) function H_0 with an integrable difference at infinity, then $\int_{B_r} H(\eta) + \frac{A}{2} \log \eta \to A \log r + \text{const}$ with $A = \int_{S^2} H_0$. The remainder converges by dominated convergence with an η-independent majorant. The soft points' t^{-2} and t^{-1} terms cancel between box and crossed box exactly.

Values. The far parts give $(180 \mp 120(x + 1/x))u$ for box and crossed box, the soft constants $(\frac{55}{2} \pm 25(x + 1/x))u$ twice, and the regular, Born, divided-difference and pinch remainders give 0 or the far value; the total is $470u$ for every pair of masses, the crossed far-node partial fraction being replaced by a contour integral at equal masses. The remaining classes and the vacuum polarization are proved in the stronger all-t reading, with the coefficients of Table 3; the all-t reading implies the germ reading, and the sum is $123u$. The per-class values rest on exact computations that are not reproduced here; they are part of the material described in the data statement at the end of the paper. □

7.5 The lattice at $\kappa = 1$: the doublers in the two-mass channel

The light spectrum of the lattice at $\kappa = 1$ contains the four doublers of Proposition 6.1. A long-wave graviton with profile f couples to a doubler pair at the corner πe_c with the strength $\delta_c(f)$ of Proposition 6.3, without derivatives (Proposition 6.12). The question is whether the field exchanged between two masses is one they see.

Proposition 7.6 (The doublers see the static field)**.** *Let $\kappa = 1$. The long-wave field of a mass M at rest, in the de Donder gauge, is $h(u) = q(u)^{-1}\, 16\pi G \big(gTg - \frac{1}{2}\operatorname{tr}(gT)\, g\big) + o\big(q(u)^{-1}\big)$ on static waves u, with $g = g_1$ and $T = M\hat{t}\hat{t}^{\top} = \frac{M}{12}\mathbf{1}\mathbf{1}^{\top}$; that is,*

$$h(u) = \frac{2}{s\, q(u)} \Big(\frac{3M}{4}\, \mathbf{1}\mathbf{1}^{\top} + \frac{M}{2}\, g_1 \Big) + o\big(q(u)^{-1}\big).$$

Its coupling to the doubler pair at every axis corner is

$$\delta_c(h) = -\frac{33\,M}{64\sqrt{3}\,s\,q(u)} + o\big(q(u)^{-1}\big), \qquad c = 0, 1, 2, 3,$$

the same at all four corners and nonzero for $M \neq 0$.

Proof. The static waves $\sum_a u_a = 0$ form a non-null cone, on which $q(u) = \sum_a u_a^2 > 0$, and T is constant and conserved there ($Tu = 0$), so the first statement is Theorem 4.6. With $g_1\mathbf{1} = -3 \cdot \mathbf{1}$ one has $g_1 T g_1 = \frac{3M}{4}\mathbf{1}\mathbf{1}^\top$ and $\operatorname{tr}(g_1 T) = \frac{M}{12}\mathbf{1}^\top g_1 \mathbf{1} = -M$, and $16\pi G = 2/s$. The map δ_c is linear, vanishes on g_1 and equals $-396/(1152\sqrt{3})$ on $\mathbf{1}\mathbf{1}^\top$ (Proposition 6.3); so $\delta_c\big(\frac{3M}{4}\mathbf{1}\mathbf{1}^\top + \frac{M}{2}g_1\big) = -\frac{33M}{128\sqrt{3}}$, and the factor $2/(s\,q(u))$ gives the value. □

The reason is visible in the trace: $\operatorname{tr} h \neq 0$, so the field of a mass changes the squared lengths of the null postings, and by the sum rule of Proposition 6.3 the four shifts add to a nonzero total. In a posting-null representation of the same field ($H_{aa} = 0$) the four couplings sum to zero but need not vanish one by one.

The pair loop itself is the second ingredient. Near each axis corner the doubler eigenvalue is a nondegenerate quadratic form of signature $(1, 3)$ (Proposition 6.2). In coordinates in which it is $\Omega(y) = -y_0^2 + y_1^2 + y_2^2 + y_3^2$ (the form written Q in Proposition 6.5; here Q is the transfer), the local model of the pair loop at external momentum p, with a gap parameter ν and a Gaussian vertex weight, is

$$\beta_\nu(\eta, p) = \int_{\mathbb{R}^4} \frac{e^{-|y|^2}\,\mathrm{d}y}{\big(\Omega(y) + \nu + i\eta\big)\big(\Omega(y+p) + \nu + i\eta\big)},$$

with $\nu = 0$ for the massless doubler.

Proposition 7.7 (The pair loop at nonzero momentum)**.** *Let $p = tY$ with $Y \in \mathbb{R}^4$, $Y \neq 0$, and $t > 0$.*

(i) If $\Omega(Y) \neq 0$, the regulator limit $\beta_0(t) = \lim_{\eta\to 0^+} \beta_0(\eta, tY)$ exists for every $t > 0$, and $\beta_0(t) = i\pi^2 \log t^2 + C_Y + o(1)$ as $t \to 0^+$: the coefficient of $\log t^2$ is $i\pi^2$ in every direction off the cone.

(ii) If $\Omega(Y) = 0$, then $\operatorname{Im} \beta_0(\eta, tY) \to -\infty$ as $\eta \to 0^+$ for every t: there is no regulator limit.

(iii) If $\nu \neq 0$, the regulator limit exists for every t with $t^2|\Omega(Y)| \leq 2|\nu|$, zero included, and is continuous at $t = 0$: a gapped pair carries no logarithm.

The proof writes the loop with a Feynman parameter x and a Schwinger parameter T; the shifted complex Gaussian integrals are exact. The transform $\int_0^\infty e^{-zT} k(T)\,\mathrm{d}T$ of the kernel $k(T) = T(1 + iT)/(1 + T^2)^{3/2}$ equals $-i \log z$ plus a function continuous on the closed right half-plane, origin included. The logarithm of $z = \eta - i\,t^2 x(1-x)\Omega(Y)$ then gives (i) and (ii), and a gap term keeps z away from 0, which gives (iii).

Theorem 7.8 (The doublers and the long-wave reading)**.** *A function of the form $\gamma t^{-k} \log t^2 + a_4 t^{-4} + a_3 t^{-3} + a_2 t^{-2} + \alpha t^{-1} + A + R(t)$ with $\gamma \neq 0$, $k \in \{1, 2, 3, 4\}$ and $t^k R(t)$ convergent as $t \to 0^+$ has no long-wave log coefficient. In particular, for $\gamma \neq 0$, $t \mapsto \gamma\,\beta_0(t)/t^k$ has none.*

Proof. Both $\mathscr{T}$ and Δ are finite combinations of dilations $f \mapsto f(\lambda t)$, so $\Delta\mathscr{T}$ is linear and maps functions that are $o(t^{-k})$ to functions that are $o(t^{-k})$. It annihilates $t^{-4}, \dots, t^{-1}$ and constants, and a direct evaluation gives $\Delta\mathscr{T}(t^{-k}\log t^2) = \varkappa_k \log 2 \cdot t^{-k}$ with $\varkappa_k = 21, -\frac{9}{4}, \frac{21}{32}, -\frac{315}{512}$ for $k = 1, 2, 3, 4$. Write $R(t) = \rho t^{-k} + o(t^{-k})$ with $\rho = \lim t^k R(t)$. Then $(\Delta\mathscr{T} f)(t) = \gamma\varkappa_k \log 2 \cdot t^{-k} + o(t^{-k})$, which is unbounded as $t \to 0^+$ because $\gamma\varkappa_k \neq 0$. For the second statement, Proposition 7.7(i) gives $\gamma\beta_0(t)/t^k = i\pi^2\gamma t^{-k}\log t^2 + \gamma C_Y t^{-k} + R(t)$ with $t^k R(t) \to 0$. □

In the static two-mass amplitude the doubler pair loop enters the vacuum polarization between two long-wave propagators, each of order $q(u)^{-1}$, and couples to both without derivatives (Proposition 7.6). The four corners contribute with one sign: the couplings are equal and the corner cones are permutations of one another. If near each corner the lattice integrand agrees with the model of Proposition 7.7 up to terms without $t^{-4}\log t^2$, and the rest of the zone adds no compensating $t^{-4}\log t^2$, the static potential has a term $\gamma t^{-4}\log t^2$ with $\gamma \neq 0$. By Theorem 7.8 it then has no long-wave log coefficient at all, and the lattice form of requirement 7 of Table 1 is false at $\kappa = 1$. Both hypotheses are open. The evidence is the exact corner form (Proposition 6.2) and the fact that every other critical zero of the operator is hyperbolic (Theorem 6.7). If the direction of the transfer lies on a doubler cone, the model has no regulator limit at all.

Read physically, a $\log q^2/q^4$ term of the static amplitude is more singular at small momentum transfer than the $\log q^2$ that gives the r^{-3} term, so at long distance it dominates it (INFERENCE). The same power counting gives $\log q^2/q^2$ from the vertex corrections and $\log q^2$ at the order of Table 3 from the corner regions of the box, crossed box, triangles and double seagull (INFERENCE). In the local model a gap removes the logarithm (Proposition 7.7(iii)). At $\kappa = 1$ nothing native supplies one (Theorem 6.10), and moving κ above 1 gives all four doublers a mass at first order but abandons the identification.

7.6 From $123u$ to $41/(10\pi)$

The lattice amplitude is converted to physical units by the relations of Section 3.5: $G = 1/(8\pi s)$, forced by the lattice Newton clause; $\hbar = 1/a$ and $c = 1$, by definition; masses $m_i = \hbar\mu_i$ (MODEL); and the normalization N of the amplitude, fixed by matching the tree, $N \cdot (-\frac{\sqrt{3}}{2}\mu_1\mu_2/s) = -4\pi G m_1 m_2$, which gives $N = \hbar^2/\sqrt{3}$. Then

$$\frac{Nu}{2G^2\hbar\, m_1 m_2} = \frac{\mu_1\mu_2/(960\pi^2 a^3 s^2)}{\mu_1\mu_2/(32\pi^2 a^3 s^2)} = \frac{1}{30}.$$

With the Fourier pair $c\log|q|^2 \mapsto -c/(2\pi r^3)$ and the physical log coefficient $c = N \cdot 123u$,

$$V_q(r) = -\frac{123\,Nu}{2\pi r^3} = -\frac{123}{30}\cdot\frac{G^2\hbar\, m_1 m_2}{\pi r^3} = -\frac{41}{10\pi}\,\frac{G^2\hbar\, m_1 m_2}{r^3}.$$

Corollary 7.9 (The quantum coefficient)**.** *Read the lattice amplitude physically through the following identifications: $\kappa = 1$; one units map for tree and loop, under which the physical log coefficient is N times the lattice one with t read as the physical $|q|^2$; the Fourier pair above; and the Born reading, under which the amplitude normalized by $4aM_1M_2$ minus the cylinder Born term is the physical two-body potential. With $m_i = \hbar\mu_i$, the r^{-3} term of the physical potential is $-\frac{41}{10\pi}G^2\hbar m_1 m_2/(r^3c^3)$, and within the Born reading this coefficient is unique.*

What this establishes, and what it does not, should be kept apart. Theorem 7.5 is unconditional mathematics about the amplitude assembled from the long-wave rules, which are the Einstein–Hilbert rules in record coordinates with two chosen inputs (Section 7.2). It is not a statement about the lattice path sum at $\kappa = 1$. There the doublers couple to the static field (Proposition 7.6), and in the local model their pair loop carries a logarithm at nonzero momentum (Proposition 7.7), which would add a $\log q^2/q^4$ term to the static amplitude unless the rest of the zone cancels it (Section 7.5). The corollary adds identifications, each named; the classical part of the potential enters by definition, and only the r^{-3} term is fixed. The final $41/(10\pi)$ does not depend on N or on the mass normalization. Nothing is claimed at finite t. Three links between the long-wave amplitude and the path sum are open: the regulator limits of the lattice amplitude and its uniform remainder against the long-wave amplitude, which at $\kappa = 1$ contain the doubler corners; the identification of the amplitude with the moments of the path sum of Section 3, including its ghost factor; and the graviton self-energy at zero momentum, whose ordered anisotropy diverges at identified $\kappa = 1$ (Section 6).

8 What the path sum still needs

At identified $\kappa = 1$ the classical side succeeds at linear order and the quantum side fails, and both outcomes trace to the null postings. Classically they give the lattice its causal structure and leave its long-wave sector that of linearized gravity. Quantum mechanically they leave four corner doublers massless: by the sum rule of Proposition 6.3, a constant strain moves all four with one sign only if it changes the posting lengths, that is, only if it leaves $\kappa = 1$.

The results in one statement.

Theorem 8.1 (Summary)**.** *Assume the identification of Section 2.2: the lattice interval reads the posting order on an interval of positive duration. Then $\kappa = 1$ (Theorem 2.2), and on the Kuhn lattice with the Regge action the following hold.*

(1) Configurations and the path sum. *Every posting is null and every longer edge timelike; the path sum is finite at a stated cutoff (Theorem 3.1); at Gaussian order its moments and exchanges have the limits of Theorem 3.2; and the null stratum, its gauge behaviour and the twisted slice are as in Theorem 3.3. The measure is chosen, not derived.*

(2) Classical recovery. *On every torus of odd side not divisible by* 105 *the flat kernel is the constant strains and vertex moves (Theorem 4.2); off the light cone the linear lattice field converges to the de Donder solution of the linearized Einstein equations (Theorem 4.6). The unconstrained Regge equations on all fifteen classes, with a resting body and a model medium, have exact solutions in the Lorentzian chamber for small coupling, whose weak-field long-wave limit is the linearized Einstein field (Theorems 4.8 and 4.9) and, for a compact body, Newton's potential with $G = 1/(8\pi s)$ (Theorem 4.10); they are related to the path sum on the null stratum at linear order (Theorem 3.3). No stability bound uniform in the torus size holds at all wavelengths (Proposition 4.11); at fixed small strength the long-wave equations have solutions uniformly in the torus size, with controlled gauge directions, stability supplied by the medium and an explicit second-order limit (Theorems 4.12 to 4.16), and with a posting gas held by a least-cost support, under an identification, in place of the medium (Theorems E.1 to E.5).*

(3) Tree level. *Every non-gauge long-wave mode in every direction propagates on the light cone up to a relative correction of second order in the wave number, and the static Newton kernel is correct to second order (Theorems 5.1 and 5.4 and Corollaries 5.2 and 5.3).*

(4) The one-loop graviton. *There are four exact massless non-gauge doublers at the axis corners (Propositions 6.1 and 6.2); nothing in the construction gaps them (Theorems 6.10 and 6.11); every other critical zero of the operator is hyperbolic (Theorem 6.7); so in the diagram model the ordered zero-momentum anisotropy diverges like* $\frac{5}{44\sqrt{335}\,\pi^2}\log(1/\varepsilon)$*, with a finite part, and the ordered one-loop graviton limit does not exist (Theorem 6.8). The doublers' cones share no time direction with the graviton's (Theorem 6.15); the long-wave remainder is unchanged by gauge choice, ghosts and covariant weights (Theorem 6.17 and Propositions 6.18 and 6.20); and the vertex moves are not symmetries of the interacting action on tori of side divisible by* 4 *(Theorem 6.23).*

(5) The one-loop coefficient. *The amplitude assembled from the long-wave Feynman rules, each the long-wave limit of the lattice rule except the Feynman* $i\eta$ *and the ghost vertex, which are chosen (Section 7.2), has in the infrared germ reading the long-wave coefficient* $-\frac{41}{20\pi}G\hbar$ *times the tree coefficient (Theorem 7.5); under named identifications its potential has the term* $-\frac{41}{10\pi}G^2\hbar m_1 m_2/r^3$ *(Corollary 7.9); the original prescription and every reading required at all momentum transfers fail (Theorems 7.1 and 7.2). At* $\kappa = 1$ *the corner doublers couple to the static field without derivatives (Proposition 7.6) and, in an exact local model, their pair loop carries a logarithm at nonzero momentum (Proposition 7.7), so the coefficient is not established for the lattice path sum there.*

The theorem collects results proved above, and its proof is theirs. What it does not contain is as definite: $\kappa = 1$ as a consequence of the definitions, a nonzero long-wave remainder, any repair of the graviton sector, a derived measure, nonlinear Einstein solutions in the limit, matter whose equation of state needs no identification, and a quantum prediction at identified $\kappa = 1$; and its fifth item concerns the long-wave rules, not the lattice path sum.

Four questions decide what the programme can eventually claim.

1. *The value of* κ*.* The quantum results are tied to it. At $\kappa = 3/2$ the corners are regular, while lattice-scale growth in record time and the long-wave remainder remain (Section 6.10). A derived value without massless corner modes and with a vanishing remainder would reopen the quantum limit; a value chosen to make the graviton work would not, and the classical results would have to be derived again at it.
2. *The quantum graviton within the model.* The doublers' own self-energy and its sign; the threshold inequality that would prove the long-wave remainder nonzero (Proposition 6.18); whether the logarithm of the doubler pair loop at nonzero momentum (Proposition 7.7) carries over to the lattice integrand, in the self-energy and in the static two-mass channel (Section 7.5); and whether the real-time path sum has a unitary evolution in record time at all (Section 6.8).
3. *The measure.* A measure derived rather than chosen, with its regulator limit at infinite volume and its behaviour beyond Gaussian order (Section 3.6).

4. *Matter and every edge.* At fixed strength the remaining radical rows and the static rows off the band should be imposed, with a solver that does not lose derivatives, for which Theorem 4.14 supplies the linear input. Einstein's second order needs matter with equations of its own: the medium itself, at fixed amplitude, gravitates at second order like a Lorentz-violating graviton mass (INFERENCE, from a numerical computation; Section 4.12), and the posting gas of Appendix E is the candidate whose second order is controlled on the lattice.

Data availability

The exact computations on which the theorems rely, with independent checking programs, are deposited with this preprint at `https://doi.org/10.5281/zenodo.23764491` as a supplement. The supplement gives the $\kappa = 1$ cell Hessian of Appendix B as an exact table, with the normalizations of $B(k)$, $\Gamma(k)$ and the midpoint frame, and re-derives it and the cubic vertex from the Regge action. It contains the polynomial F_1, the small-side certificates and the three elimination trees of Theorem 4.2; the closed forms behind Theorem 4.6; the rational tables behind Theorems 5.1 and 5.4 and the polynomials of Appendix F; the exact Bloch data of Section 6, including Theorem 6.23; the per-class values of Table 3; and the polynomials, ideal memberships, sum-of-squares Gram matrices and rational enclosures of Appendix G. Each item comes with a short program that checks it in exact arithmetic. Analytic steps of the proofs are not part of the supplement, and neither are results labelled numerical.

A The posting order

Let $\mathbf{1} = (1,1,1,1)$, $\sigma_a = e_a - \mathbf{1}/4$ and $\Sigma = \{x \in \mathbb{R}^4 : \sum_a x_a = 0\}$. The dyadic sites are $\mathscr{D} = \{2^{-k}(m - \frac{1}{4}(\sum_a m_a)\mathbf{1}) : k \in \mathbb{N},\ m \in \mathbb{N}^4\}$. At depth n a posting takes time $\tau 2^{-n}$ and moves the position by $2^{-n}\sigma_a$ for one a; $\mathscr{R}_n$ is the reflexive transitive closure of these steps between events whose positions lie in $\mathscr{D}$, with no absolute clock grid. The posting order is the closure of $\bigcup_n \mathscr{R}_n$. For events $e = (t, x)$ and $f = (t', x')$ put $w_a(e, f) = x'_a - x_a + (t' - t)/(4\tau)$.

Proposition A.1 (Completion)**.** *For every $\tau > 0$, (e, f) lies in the posting order if and only if $x, x' \in \Sigma$ and $w_a(e, f) \geq 0$ for every a.*

Proof. A posting on axis a adds 2^{-n} to w_a and nothing to the other contents, so every finite chain has nonnegative content on each axis, and the condition is closed. Conversely, from a dyadic starting site, a chain that posts $\lfloor 2^n w_a \rfloor$ times on axis a at depth n approximates any pair with nonnegative contents, including the boundary. □

The order does not involve κ, and by Theorem 2.1 every $\kappa \geq 1$ makes all of it causal. In the terms of Section 2.2, the interval with cone speed v_c in position per tick is Q_κ with $\kappa = \frac{1}{4} + v_c^2$, and the posting speed is $|\sigma_a| = \sqrt{3}/2$; the choices $v_c = \sqrt{3}/2$ and $v_c = 1$ give $\kappa = 1$ and $\kappa = 5/4$, with a single posting null in the first and of interval $-1/4$ in the second. So no condition on the posting order alone forces $\kappa = 1$; the identification of Section 2.2 is a condition on the interval's reading of the order.

B The flat Hessian at $\kappa = 1$

This appendix gives the flat Hessian exactly, so that the Bloch data used in Sections 4 to 6 can be recomputed. Label the vertices of a Kuhn simplex $0, \dots, 4$ along its chain, $v_0 = x$ and $v_j = x + e_{a_1} + \cdots + e_{a_j}$, and its ten edges by the pairs ij with $0 \le i < j \le 4$, in the order $01, 02, 03, 04, 12, 13, 14, 23, 24, 34$. The edge ij has class $v_j - v_i$, with $j - i$ ones. The action of a simplex is $S_\sigma = \sum_{h \subset \sigma} A_h(2\pi/n_h - \theta_{\sigma,h})$, a function of its ten squared lengths, and all 24 simplices of a cell have the same Hessian in these labels. At the flat $\kappa = 1$ lattice it is

$$\left.\frac{\partial^2 S_\sigma}{\partial \ell_{ij}\,\partial \ell_{kl}}\right|_{\ell^0} = \frac{1}{16\sqrt{3}}\,C, \qquad C = \begin{pmatrix} 2 & -2 & 1 & 0 & 0 & 3 & -2 & -6 & 2 & 2 \\ -2 & -4 & 4 & -1 & 6 & -6 & 2 & 3 & -2 & 2 \\ 1 & 4 & 0 & -2 & -6 & 0 & 2 & -3 & 2 & -2 \\ 0 & -1 & -2 & 2 & 2 & 2 & -2 & 2 & -1 & 0 \\ 0 & 6 & -6 & 2 & -6 & 6 & -3 & 0 & 3 & -6 \\ 3 & -6 & 0 & 2 & 6 & 0 & 0 & 6 & -6 & 3 \\ -2 & 2 & 2 & -2 & -3 & 0 & 0 & -6 & 4 & 1 \\ -6 & 3 & -3 & 2 & 0 & 6 & -6 & -6 & 6 & 0 \\ 2 & -2 & 2 & -1 & 3 & -6 & 4 & 6 & -4 & -2 \\ 2 & 2 & -2 & 0 & -6 & 3 & 1 & 0 & -2 & 2 \end{pmatrix},$$

rows and columns in the order above. The flat squared lengths of the simplex, $\ell^0_{ij} = (j-i)(1-j+i)$, lie in the kernel of C, as homogeneity of degree one requires. Write $d_{\sigma f}$ for the class and $b_{\sigma f}$ for the starting vertex, relative to x, of edge f of the simplex σ. The Bloch matrix of Section 4.2 is

$$B(k)_{dd'} = \sum_\sigma \sum_{f,f'} [d_{\sigma f} = d]\,[d_{\sigma f'} = d']\,\frac{C_{ff'}}{16\sqrt{3}}\,e^{ik\cdot(b_{\sigma f'} - b_{\sigma f})},$$

the sum running over the 24 simplices of one cell, and $\Gamma(k)$ and the midpoint frame are as in Section 4.2. From these formulas one recovers, by direct computation, the Bianchi identity, the ten constant strains as the kernel of $B(0)$, the axis-corner zero modes v_c of Proposition 6.1 and the cone form K of Proposition 6.2.

The static resonance. At the static wave $u = (-t_*, t_*, 0, 0)$ with $e^{it_*} = \zeta = (9 + 2\sqrt{10}\,i)/11$, so that $\cos t_* = 9/11$, the vector W with

$$\begin{aligned}
&440\,W_{1000} = -1131 + 429\zeta && 440\,W_{0100} = -429 - 429\zeta && 440\,W_{1100} = 1760 \\
&440\,W_{0010} = 440 && 440\,W_{1010} = -87 + 33\zeta && 440\,W_{0110} = -33 - 33\zeta \\
&440\,W_{1110} = 2640 && 440\,W_{0001} = 440 && 440\,W_{1001} = -87 + 33\zeta \\
&440\,W_{0101} = -33 - 33\zeta && 440\,W_{1101} = 2640 && 440\,W_{0011} = 880 \\
&440\,W_{1011} = 1305 - 495\zeta && 440\,W_{0111} = 495 + 495\zeta && 440\,W_{1111} = 3960,
\end{aligned}$$

indexed by the classes $d = d_0 d_1 d_2 d_3$, satisfies $B(u)W = 0$ and $\Gamma(u)^\dagger W = 0$: it is a non-gauge zero mode, and its largest entry is on the time edge **1**. In position space it is the edge field $W_d\,e^{iu\cdot x}$, static, with period $2\pi/t_* \approx 10.3$ in the coordinate difference $x_1 - x_0$.

C The flat kernel on prime tori

On the side-N torus the Hessian decomposes into 15×15 Bloch blocks at the torsion points $z \in \mu_N^4$. At $z = 1$ the kernel is the constant strains. At $z \neq 1$ the radical contributes exactly the vertex moves, and the flat-kernel property holds at z exactly when the gauge-fixed Bloch matrix is nonsingular there. After clearing denominators its determinant is the value at z of one integer polynomial $F_1 \in \mathbb{Z}[z_1, \dots, z_4]$ with 741 monomials and all exponents at most 6. An exact certificate, the values of F_1 modulo the prime 1073741827 at representatives of the 40 Galois orbits of points of order 3, all nonzero, shows that F_1 has no zero at a point of order 3 other than 1. On prime sides $p > 31104$, Theorem 4.2 therefore follows from Lemma 4.3.

Proof of Lemma 4.3. Let $\zeta = e^{2\pi i/p}$ and $z = (\zeta^{j_1}, \dots, \zeta^{j_4})$ with $j \in (\mathbb{Z}/p)^4$, $j \neq 0$, and suppose $f(z) = 0$.

Class sums vanish. Let $r_i \in \{0, \dots, p-1\}$ with $r_i \equiv \langle j, m_i \rangle \pmod p$, and group the monomials into classes by r_i. Then $f(z) = g(\zeta)$ with $g(T) = \sum_i S_r T^r$, where S_r is the sum of the coefficients c_i in class r. Since $\deg g \leq p-1$ and the minimal polynomial of ζ over $\mathbb{Q}$ is $1 + T + \cdots + T^{p-1}$, $g(\zeta) = 0$ forces all p coefficients of g to be equal. There are at most $n < p$ monomials, so some class is empty and that common value is 0: every class sum vanishes.

Differences. Let $V \subseteq \mathbb{Q}^4$ be spanned by the differences $m_i - m_k$ of exponents in the same class; their entries lie in $[-6, 6]$ and each satisfies $\langle j, m_i - m_k \rangle \equiv 0 \pmod p$.

If $V = \mathbb{Q}^4$, pick four independent differences as the rows of an integer matrix W. Each of the 24 terms of its determinant has absolute value at most 6^4, so $0 < |\det W| \leq 4! \cdot 6^4 = 31104 < p$. From $Wj \equiv 0$ and the adjugate, $(\det W)\, j \equiv 0 \pmod p$, and $p \nmid \det W$, so $j \equiv 0$: a contradiction.

If $V \neq \mathbb{Q}^4$, pick a primitive integer vector $y \perp V$; not all its entries are divisible by 3. Put $w = (\omega^{y_1}, \dots, \omega^{y_4})$ with $\omega = e^{2\pi i/3}$, so $w^3 = 1$ and $w \neq 1$. Exponents in one class have the same $\langle y, m_i \rangle$, so $f(w) = \sum_{\text{classes}} \omega^{\langle y, m_{\text{class}} \rangle} S_r = 0$, contradicting the hypothesis on points of order 3. □

The argument needs every nontrivial torsion point to have order exactly p, so it says nothing about composite sides; the odd ones not divisible by 105 are covered by the 2-adic argument outlined after Theorem 4.2, and even sides fail for the reason given in Theorem 4.4.

D Sources and the long-wave construction

For a symmetric stress T the edge stress has entries $\mathrm{stress}(T)_{e_a} = T^{aa} - \sum_{b \neq a} T^{ab}$ and $\mathrm{stress}(T)_{e_a + e_b} = T^{ab}$ $(a < b)$, and zero on classes with three or four ones; it satisfies $\sum_d \mathrm{stress}(T)_d\, d^\top H d = \mathrm{tr}(TH)$ for symmetric H. The edge source is $b(T) = \frac{1}{2}\sqrt{|\det g_\kappa|}\, \mathrm{stress}(T)$, the coupling $\frac{1}{2}\sqrt{|\det g|}\, h_{\mu\nu} T^{\mu\nu}$. The lattice source at wave u converts $b(T)$ to the midpoint frame, with phases $e^{id \cdot u/2}$, and projects off the gauge columns Γ_M: $r \mapsto r - \Gamma_M (\Gamma_M^\top \Gamma_M)^{-1} \Gamma_M^\top r$. For a stress conserved along the wave ray the projected source tends to $b(T)$ as $u \to 0$.

The real-wave lattice field of Eq. (4) is

$$h_N(x,d) = \sum_{m\in F} \mathrm{Re}\Big(e^{im\cdot X} c_m \,\frac{h_m(u_m)_d}{z^d}\Big), \qquad u_m = -\frac{2\pi m}{N}, \quad z^d = \prod_a e^{iu_{m,a}d_a},$$

with $h_m(u_m)$ the Bloch solution for the conserved lattice source. The division by the edge phase is part of the reconstruction. The rescaled amplitudes $(2\pi/N)^2 h_m(u_m)_d/z^d$ converge to the real limit $q_\kappa(m)^{-1}(-s^{-1}G_\infty b_m)$, uniformly in the edge class, and one N_0 serves every amplitude vector with $\sum|c_m| \le 1$.

E Matter held by a support stress

This appendix collects the results on matter with an action of its own (Section 4.11). Throughout it the *long classes* are the eleven edge classes with $|d| \ge 2$, and the *long edges* are the edges of those classes. For $\frac{1}{2} < \kappa \le 1$ they are exactly the timelike classes of the flat lattice; for $\kappa > 1$ the postings are timelike as well, and they are not included.

The medium of Section 4.6 holds the gauge and strain rows by a stiffness about the flat lattice, and it has no equations of its own. At fixed strength matter with an action of its own can do that job. Let a *posting gas* (Section 3.5) be an occupation c of the long edges with the action $\mathscr{G}_c(\ell) = \sum_e c_e\sqrt{-\ell_e}$, which is the action of free point particles when c counts postings moving along the edges (MODEL: the point-particle reading of a posting). The resting body of Section 4.6 is the gas on the time edges. The gas has stress $c_e/(2\sqrt{-\ell_e})$ on the edge e, and the coupled equations are $DS(\ell^0+w) = \sigma\big(D\mathscr{G}_c(\ell^0+w)+b\big)$ with a fixed compensation b.

Free postings cannot hold a body up. An edge that advances n postings has proper time $\sqrt{\kappa - 1/n}$ per unit of coordinate time, largest on the time edge, so the gas action alone puts its postings at rest (INFERENCE, from this inequality); and the static stress of postings at rest annihilates every vertex move, so it balances no load that a vertex move feels. A body is held up by a *support*: an occupation of the long edges whose stress keeps it in momentum balance on the radical band of Section 4.10. In the coupled equations the support enters the radical band rows only through its stress, and that stress enters every band row as well, which is what makes it matter and not a dropped equation. Which support is realized is the body's equation of state, and we take it from the recognition cost J of Section 4.12. Give each long class d a background occupation $c^0_d > 0$ and charge a carrier c the cost

$$\mathscr{C}(c) = \sum_{e \text{ long}} c^0_{d(e)}\, J\big(c_e/c^0_{d(e)}\big). \tag{12}$$

The *least-work principle* says that the realized carrier is the one of least cost among the positive carriers that put the same load on the radical band. This principle, with the cost weighted by the background as in Eq. (12), is not derived here (IDENTIFICATION), and neither are the ratios of c^0 between the classes.

Theorem E.1 (The least-work carrier). *Let $\kappa > 1/2$, let ℓ be negative on the eleven long classes, and let μ be a field in the radical band with $\mu_e < \sqrt{-\ell_e}$ on every long edge. Put*

$$x_e = \frac{c^0_{d(e)}}{\sqrt{1-\mu_e/\sqrt{-\ell_e}}} \qquad \text{on the long edges.}$$

(i) x is positive, and $J'(x_e/c^0_{d(e)}) = \mu_e/(2\sqrt{-\ell_e})$: the strain conjugate to the carrier is a constant strain plus a vertex move.

(ii) Let c be positive on the long edges, zero elsewhere, and let it put the same load on the radical band as x: $\sum_e (c_e - x_e)\, t_e/(2\sqrt{-\ell_e}) = 0$ for every t in the radical band. Then

$$\mathscr{C}(c) = \mathscr{C}(x) + \sum_{e \text{ long}} \frac{(c^0_{d(e)})^2\,(x_e - c_e)^2}{2x_e^2\, c_e},$$

so x is the unique carrier of least cost for its load.

(iii) At the flat lattice there are $f_1, K > 0$, independent of N, such that every load f on the radical band of size at most f_1 (in the norm of absolute Fourier coefficients) is carried by such an x, with $\|\mu\| \le K\|f\|$. Its stress beyond the background is, up to a correction of order $\|f\|^2$, the stress S of least weighted norm $\sum_e S_e^2/W_{d(e)}$ among the stresses on the long edges that balance f, with

$$W_d = \frac{c^0_d}{4(\tau^0_d)^2}.$$

Scaling c^0 leaves this leading stress unchanged, so only the ratios of c^0 enter at this order; for $c^0 \propto (\tau^0)^2$ it is the balancing stress of least plain norm.

Proof. (i) $J'(y) = \frac{1}{2}(1 - y^{-2})$, so the stated identity reads $1 - (c^0/x)^2 = \mu/\sqrt{-\ell}$, whose positive solution is x; it exists because $\mu_e < \sqrt{-\ell_e}$. (ii) For $a, x, c > 0$ direct algebra gives

$$aJ(c/a) - aJ(x/a) - J'(x/a)\,(c - x) = \frac{a^2(x - c)^2}{2x^2 c}.$$

Sum over the long edges with $a = c^0_{d(e)}$. By (i) the linear terms add up to $\sum_e \mu_e(c_e - x_e)/(2\sqrt{-\ell_e})$, which vanishes because μ lies in the radical band and c puts the same load on it as x. Equality forces $c = x$. (iii), outline. At ℓ^0 the stress of x beyond the background is $\frac{c^0}{2\tau^0}\big((1 - \mu/\tau^0)^{-1/2} - 1\big)$, whose linear part is $W\mu$ because $J''(1) = 1$. A stress $W\mu$ with μ in the radical band is the balancing stress of least weighted norm, by the Lagrange condition for that norm. So the load condition is a fixed-point equation $\mu = \mu_1 - \mathscr{Q}(\mu)$ on the radical band, with $W\mu_1$ the least-norm stress and $\mathscr{Q}$ the weighted solution for the part of the stress beyond linear order. The weighted solver on the radical band is bounded uniformly in N, so a contraction gives μ with $\|\mu - \mu_1\| \le K\|\mu_1\|^2$. Replacing c^0 by θc^0 multiplies W by θ and leaves the weighted least-norm stress unchanged. □

With the least-work support the long-wave equations can be solved at fixed strength, with every constant independent of the torus size and of the scale of the background.

Theorem E.2 (A body held by its least-work support at fixed strength)**.** *Let $\kappa > 1/2$, $M \ge 1$, c^0 as above and $m, m' > 0$. There are $\delta_*, \alpha_0, K > 0$, independent of N and of θ, with the following property. Let $\delta \le \delta_*$ and $N \ge 3$ with $2\pi M/N \le 1/16$; let c_1 be a body occupation on the long classes of size at most m (the sum over the classes of its absolute Fourier coefficients), b a compensation of size at most m' compatible with c_1, which means that the flat stress of c_1 minus b, the field $c_1/(2\sqrt{-\ell^0}) - b$, annihilates every radical band row; let $\theta > 0$, $0 < \alpha \le \alpha_0 \min(1, \theta)$*

and $\sigma = \alpha(2\pi/N)^2$. There are a band-limited field $w \perp \mathscr{R}_N$ with $\|w\|_{\mathrm{J}} \le K\alpha$ and a field μ in the radical band with $\theta\|\mu\|_{\mathrm{J}} \le K\alpha$ and $\mu_e < \tau_e^0$, such that:

(i) let x be the carrier of Theorem E.1 for the background θc^0 at ℓ^0, and $s(w)$ the support that carries at $\ell^0 + w$ the stress that $x - \theta c^0$ carries at ℓ^0; then the coupled equations with the gas $c_1 + s(w)$ hold on every band row and every radical band row;

(ii) x is positive and is the unique carrier of least cost for its load;

(iii) $\theta c^0 + s(w) \ge \frac{3}{4}\,\theta c^0$ on every long edge.

In particular, if the background is part of the body, $c_1 = \rho + \theta c^0$ with $\rho \ge 0$, then $c_1 + s(w) \ge \rho + \frac{3}{4}\theta c^0$, and the whole gas is nonnegative on every edge.

Proof outline. The field solves the band rows through the band solver of Theorem 4.12, with the support's stress in the load, and the multiplier μ solves the radical band rows through Theorem E.1(iii) with the background θc^0; together they form one contraction. The background scale drops out of every constant: the removal of the weighted least-norm stress from a load does not change when the weights are scaled, and in the correction of the support the factor $\sigma\theta$ cancels. Compatibility makes the load on the radical band of order α, so $\theta\|\mu\|$ is of order α, and $\alpha \le \alpha_0\theta$ keeps μ and w small enough that the carrier at the actual geometry stays within a quarter of the background. □

Theorem E.3 (The supported body: first and second order)**.** *Let $\kappa > 1/2$, $M \ge 1$ and c^0 as above.*

(i) Let ρ be the static band-limited body described after Theorem 4.9, with the background θc^0 carried in the body and the compensation equal to the flat source of the mean density plus the flat stress of the background. There is $\delta_* > 0$, and for every $\delta \le \delta_*$ there are $\alpha_0, C, K > 0$ independent of θ, such that for every $\theta \in (0, 1]$, $0 < \alpha \le \alpha_0\theta$ and $\eta > 0$, and all large N, there are a band-limited field $w \perp \mathscr{R}_N$ with $\|w\|_{\mathrm{J}} \le K\alpha$ that solves the coupled equations with the least-work support on every band row and every radical band row, and a vector r in the kernel of H_0, of uncontrolled size, with

$$|(w_e - r_e)/\alpha - h^{\rho}(X_e, d_e)| \le C\alpha + \eta \qquad \text{at every edge,}$$

where h^{ρ} is the linearized Einstein field of that body.

(ii) In the setting of Theorem E.2 there are $\alpha_1, K_2 > 0$, independent of N and θ, such that for $\alpha \le \alpha_1 \min(1, \theta)$ the solution satisfies $\|w - w_1 - w_2\|_{\mathrm{J}} \le K_2(\alpha^3 + \alpha^3/\theta)$. Here w_1 is the band response to the flat stress of the body minus the compensation, and

$$w_2 = -\mathscr{S}_N\Big[P_W\Big(\tfrac{1}{2}D^3S(\ell^0)(w_1, w_1, \cdot) + \sigma G(w_1)\Big)\Big],$$

with $G(w)$ the linear response of the body's stress to the field and P_W the removal, from a load, of the W-weighted least-norm stress that balances it on the radical band. A background carried in the body changes w_2 by $O(\theta\alpha^2)$.

So at fixed strength, uniformly in the torus size, Regge gravity coupled to a posting gas held by its least-work support has solutions on the long-wave rows, and their first-order field is the linearized Einstein field of the body's rest mass, with no medium anywhere. The first order does not test the equation of state: at the flat field the body's stress minus the compensation already annihilates the radical band, so the support vanishes there and enters only at the next order. The least-work law shows itself in the second order, in the positivity of the gas and in its minimality. The scope is the following.

- *Identifications.* The least-work principle and the background weighting of the cost (IDENTIFICATION); the split of the gas into a resting body and a carrier of stress (IDENTIFICATION); the reading of the flat-reference carrier at the actual geometry with its stress held fixed (IDENTIFICATION), which differs from the least-cost carrier of the actual geometry by $O(\alpha^2)$ in occupation and $O(\alpha^3)$ in the field, beyond the orders used (INFERENCE); the carrier of the compensation, whose value the Gauss law forces (IDENTIFICATION); and $\kappa = 1$ (IDENTIFICATION; the theorems hold for every $\kappa > 1/2$).
- *Open.* The ratios of c^0 between the classes; the least-work carrier for loads that are not small against the background ($\alpha \gtrsim \theta$), and whether every carrier of least cost has the form of Theorem E.1; positivity and least cost for the solution of Theorem E.3(i), which that statement does not assert; the limit of w_2/α^2 as the torus grows; and the coupled solutions in which matter made only of postings on worldlines holds a static body, which Theorem E.5 prepares but does not give.
- *Rows.* As in Theorem 4.14, the static rows off the band and the radical rows of static vertex moves with some $|m_a| \geq M$ are not imposed. Nor are the rows of fields that vary along the time diagonal: for a static body they hold by symmetry, as there, but Theorem E.2 does not assume the body static, and for a body that is not static they do not hold in general.
- *Stability.* A gas at rest has no stiffness on static vertex moves, so the coercivity of Theorem 4.14(iv), which the medium supplied, is not established for the gas.

With the least plain-norm stress in place of least work and no compensation term, the same construction gives solutions too, but the support then carries the body's whole integrated stress, as the Gauss and von Laue rows require, and it is a signed occupation; the least-work carrier with its background is what keeps the gas positive.

The support is a stress, not a flow of postings. Read as postings moving along lattice worldlines, an occupation has to conserve them: as many postings reach each vertex as leave it. For an occupation c of the long edges write

$$\phi_c(y) = \sum_d c_{(y-d,d)} - \sum_d c_{(y,d)}$$

for its net posting flow into the vertex y, the sums running over the eleven long classes, and call c a *worldline occupation* if $\phi_c = 0$ at every vertex. Every closed lattice walk along long edges gives one, and so do a static resting body and every occupation that is constant on each class. The supports above are not of this kind, and for a load with a component along a certain family of vertex moves no occupation of this kind can take their place, because worldline

occupations do not feel those moves. Write v_η for the vertex move generated by a covector field η, the edge field $(v_\eta)_{(y,d)} = 2\,d^\top(\eta_{y+d} - \eta_y)$ at the flat lattice (the operator Γ° of the proof of Theorem 6.17), and $\tau_n = \sqrt{n(n\kappa - 1)}$ for the flat proper time of a class with n bits. For axes $a \neq b$ let χ^{ab} be the covector with entries $2\tau_2 - \tau_3$ at a and b and $\tau_3 - \tau_2$ at the other two axes. A long class d *separates* a and b if $d_a \neq d_b$; six of the eleven do.

Theorem E.4 (Postings that move along worldlines)**.** *Let $\kappa > 1/2$, let $a \neq b$ be axes, and let c be an occupation of the long edges.*

(i) $d \cdot \chi^{ab} = \tau_{|d|}$*, the flat proper time of d, for every long class d that separates a and b.*

(ii) Let ψ be a function on the vertices that does not change along the long edges whose class does not separate a and b, for example any function of $y_a - y_b$. Then

$$\sum_e \frac{c_e}{2\sqrt{-\ell_e^0}} \left(v_{\psi\,\chi^{ab}}\right)_e = \sum_y \psi(y)\,\phi_c(y).$$

So every worldline occupation pairs to zero with every such blind move *$v_{\psi\,\chi^{ab}}$, and no worldline occupation, of either sign, balances a load that does not vanish on some blind move. For $M \geq 2$ the radical band contains blind moves, for instance the one with $\psi(y) = \cos\big(2\pi(y_a - y_b)/N\big)$, so at the flat field there are loads on the radical band that no worldline occupation balances.*

(iii) Let ℓ be negative on the long classes, and let $\mu = \beta\, v_{\varphi\mathbf{1}}$ lie in the radical band, with $\beta > 0$, $\mu_e < \sqrt{-\ell_e}$ on every long edge and φ a function on the vertices that changes along some long edge (such φ exist for $M \geq 2$). Then the carrier x of Theorem E.1 has $\sum_y \varphi(y)\,\phi_x(y) > 0$, so it is not a worldline occupation.

(iv) Let ℓ and μ be as in Theorem E.1, and let ψ be a function on the vertices with $\mu_e/\sqrt{-\ell_e} + 2(\psi_{y+d} - \psi_y) < 1$ on every long edge $e = (y, d)$. Put

$$x_e^\psi = c_d^0 \left(1 - \frac{\mu_e}{\sqrt{-\ell_e}} - 2(\psi_{y+d} - \psi_y)\right)^{-1/2}.$$

Then $J'(x_e^\psi/c_d^0) = \mu_e/(2\sqrt{-\ell_e}) + \psi_{y+d} - \psi_y$, and among the positive occupations that put the same load on the radical band as x^ψ and have the same posting flow at every vertex, x^ψ is the unique one of least cost.

Proof. (i) A class d with n bits that separates a and b contains exactly one of them and $n-1$ of the other two axes, so $d\cdot\chi^{ab} = (2\tau_2 - \tau_3) + (n-1)(\tau_3 - \tau_2)$, which is τ_2 for $n = 2$ and τ_3 for $n = 3$; $n = 4$ does not occur. (ii) The left side is $\sum_{(y,d)} c_{(y,d)}\,(d\cdot\chi^{ab}/\tau_{|d|})\,(\psi_{y+d} - \psi_y)$. On separating classes the middle factor is 1 by (i); on the others the last factor vanishes. Summing $\sum_{(y,d)} c_{(y,d)}(\psi_{y+d} - \psi_y)$ by parts gives $\sum_y \psi(y)\phi_c(y)$. The function $\cos\big(2\pi(y_a - y_b)/N\big)$ is static, has integer wave numbers $\pm(e_a - e_b)$, and changes along every long edge that separates a and b, so the move is a nonzero element of the radical band when $M \geq 2$, and the load $t \mapsto \sum_e t_e\,(v_{\psi\,\chi^{ab}})_e$ on the radical band does not vanish on it. (iii) Here $\mu_e = 2\beta|d|\,(\varphi_{y+d} - \varphi_y)$, and $x_e - c_d^0$ has the sign of μ_e, so $\sum_e (x_e - c_{d(e)}^0)(\varphi_{y+d} - \varphi_y) > 0$. On the torus $\sum_e c_{d(e)}^0(\varphi_{y+d} - \varphi_y) = 0$, class by class. Summing by parts as in (ii), $\sum_y \varphi(y)\phi_x(y) = \sum_e x_e(\varphi_{y+d} - \varphi_y) > 0$. A static φ with integer wave

numbers $\pm(e_a - e_b)$ is an example when $M \geq 2$, with β small. (iv) The derivative formula is (i) of Theorem E.1 with $\mu_e/\sqrt{-\ell_e}$ replaced by $\mu_e/\sqrt{-\ell_e} + 2(\psi_{y+d} - \psi_y)$. For a competitor c, the identity in the proof of Theorem E.1(ii) leaves the linear terms $\sum_e \big(\mu_e/(2\sqrt{-\ell_e}) + \psi_{y+d} - \psi_y\big)(c_e - x^{\psi}_e)$. The first part vanishes because the loads agree, and the second equals $\sum_y \psi(y)\big(\phi_c(y) - \phi_{x^\psi}(y)\big) = 0$ because the flows agree. So $\mathscr{C}(c) - \mathscr{C}(x^{\psi})$ is the sum of the nonnegative quadratic terms of that identity, which vanishes only at $c = x^{\psi}$. □

So the theorems of this appendix stand as stated, and the support they use is a stress, not a flow of postings. The least-work carrier changes the number of postings at the vertices (iii), so the point-particle reading applies to the resting body, a worldline occupation, and not to the support. The support is a stress on the long edges whose distribution the least-work identification fixes, like the internal stress of a solid, and the bodies of Theorems E.2 and E.3 are prescribed profiles that it holds in place. Matter made only of postings on worldlines can do less. Along the blind moves its momentum balance is posting conservation (ii), an invariant of the eleven lattice velocities that a continuum medium does not have, like the spurious conserved quantities of the simplest lattice gases, and its least-work law is (iv), with the vertex potential ψ as the multiplier of posting conservation. The next theorem says exactly which loads it carries, and that it carries them at fixed strength with bounds uniform in the torus size. Call a field r in the radical band *worldline-blind* if some function φ on the vertices has $r_e/(2\sqrt{-\ell^0_e}) + \varphi_{y+d} - \varphi_y = 0$ on every long edge $e = (y, d)$: a field that a change of the vertex potential hides from the long edges. A wave m is *static* if $m_0 + m_1 + m_2 + m_3 = 0$, and the *blind lines* are the static waves proportional to $e_a - e_b$.

Theorem E.5 (What worldline matter carries)**.** *Let $\kappa > 1/2$, let $c^0 > 0$ on the long classes, let $M \geq 1$, and let $N \geq 2$ with a radical band of radius δ, $0 < \delta \leq 1/16$ and $N\delta \leq 2\pi M$, at the flat lattice.*

(i) A load f on the radical band is balanced by the flat stress of a worldline occupation, of either sign, if and only if f vanishes on every worldline-blind field. The worldline-blind fields have Fourier amplitudes only on the blind lines, and the blind moves of Theorem E.4 are worldline-blind.

(ii) There are $f_1, K > 0$, independent of N, such that every load f of size at most f_1 (in the norm of absolute Fourier coefficients) that vanishes on every worldline-blind field is balanced, with the stress measured from c^0, by the carrier x^{ψ} of Theorem E.4(iv) for some μ in the radical band and some ψ, with $\|\mu\|_{\mathrm{J}} \leq K\|f\|$ and $|\psi_{y+d} - \psi_y| \leq K\|f\|$ on every edge. This carrier is a worldline occupation, so by Theorem E.4(iv) it is the positive worldline occupation of least cost that carries f.

(iii) In the setting of Theorem E.3(ii) with a static body, let the second-order load be the load of $\frac{1}{2}D^3S(\ell^0)(w_1, w_1, \cdot) + \sigma G(w_1)$ on the radical band. If no wave of the body and no sum of two of them is a nonzero point of a blind line, the second-order load is balanced by a worldline occupation on every large torus. Let instead $\kappa = 1$, $M \geq 2$, and let the body have the waves $\pm q, \pm q'$ with unit coefficients, $q = (2, 0, -1, -1)$ and $q' = (-1, -1, 1, 1)$. Then on every large torus no worldline occupation, of either sign, balances the second-order load, alone or plus any multiple of the stress response of a uniform class-weighted background to w_1; at the blind wave $q + q' = (1, -1, 0, 0)$ the blind component of the load, normalized, tends to a

nonzero multiple of $3\tau_2 - 2\tau_3$. *For a body with one wave along* $e_0 - e_1$, *and for the body with the waves* $(1, 0, -1, 0)$ *and* $(0, -1, 1, 0)$, *which is symmetric under the exchange of the first two axes, the blind component tends to* 0.

Proof outline. (i) Linearize the carrier as $c^0\,\Theta(r,\varphi)$ with $\Theta(r,\varphi) = r/(2\sqrt{-\ell^0}) + \Delta\varphi$ on the long edges. Balance on the radical band together with posting conservation says that Θ is the representer, among the fields $\Theta(r,\varphi)$, of the functional $\Theta(r,\varphi) \mapsto f(r)$ in the inner product $\sum_e c_e^0 a_e b_e$; it is well defined exactly when f vanishes on the fields r with $\Theta(r,\varphi) = 0$, and finite-dimensional duality gives the equivalence. Necessity holds for every worldline occupation by Theorem E.4(ii)'s summation by parts. At a static wave m the long-wave form $\sum_d c_d^0 (d\cdot m)^2\big((d\cdot\xi)/\tau_{|d|} + \phi\big)^2$ over the long classes vanishes only at 0 unless m lies on a blind line, where its zeros are the multiples of $(\chi^{ab}, -1)$ by Theorem E.4(i); the other patterns of vanishing $d\cdot m$ would force $3\tau_2 = 2\tau_3$, which fails for every $\kappa > 1/2$. (ii) A Korn inequality for worldline matter: for μ in the radical band orthogonal to the worldline-blind fields, $\sum_e \mu_e^2 \le C\sum_e c_e^0 \Theta(\mu,\psi)_e^2$, mode by mode with one constant for the finitely many band waves, from the coercivity of the long-wave form off the blind lines, the orthogonality on them, and $(4/\pi^2)\theta^2 \le |e^{i\theta} - 1|^2 \le \theta^2$. It gives the linear law with bounds uniform in N; the projection onto the fields $\Theta(r,\varphi)$ is bounded on each Fourier mode, so the quadratic part of J is a contraction in the norm of absolute Fourier coefficients, and a fixed point gives x^ψ. (iii) The blind-free case is (i) applied to the second-order load, whose amplitudes lie on the body's waves and their pairwise sums. For the explicit bodies the first-order amplitudes are rational and exact; the blind component of the second-order load is $\frac{\sqrt{3}}{4}$ times a sum of cubic cores of the Regge vertex, each $\pm\frac{1}{6}$ for the two-mode body and 0 for the other two, and the lattice blind row, normalized, converges to it; a uniform background's first-order response has no blind row on large tori. □

Matter made of postings on worldlines can therefore hold a static body only if the body's load has no blind component. The blind moves see only transverse forces: a gradient force density pairs to zero with a blind move at leading order in the wave number (INFERENCE), so a static body held by such matter alone needs an equilibrium profile, as a barotropic fluid body has. At second order the blind component of the load is, by the continuum Ward identity, a component of the Newtonian force density $\rho\nabla\Phi$ along $e_a - e_b$, which vanishes for bodies of one Fourier mode and for symmetric bodies and not for a generic body (INFERENCE); the exact cases of Theorem E.5(iii) agree, including the ratio -1 between the parts of the blind load along $e_0 + e_1$ and $e_2 + e_3$ for the two-mode body. We infer that static bodies held by postings on worldlines alone generically do not exist on the Kuhn lattice at second order (INFERENCE); a coupled solution with worldline matter in place of the least-work support is not proved here.

F The paired-direction polynomials

On the paired covector $u = (z+t, z+t, z-t, z-t)$ set $U = \sin^2(t/2)$, $V = \sin^2(z/2)$, $\eta = V/(3U) - 1$ and $|\mathbf{k}| = 2t$. In their symmetry sectors the near-cone mode equations of the $\kappa = 1$ Bloch operator are exactly $G_\times(U,V) = 0$ and $G_+(U,V) = 0$ with

$$G_\times = -6UV + 3U + 4V^2 - V,$$

$$\begin{aligned} G_+ = {} & 128U^3V^3 - 192U^3V^2 + 80U^3V - 8U^3 \\ & - 512U^2V^4 + 832U^2V^3 - 432U^2V^2 + 88U^2V + 20U^2 \\ & - 512UV^5 + 1792UV^4 - 1584UV^3 + 280UV^2 + 20UV - 6U \\ & + 1024V^6 - 2816V^5 + 2400V^4 - 648V^3 + 44V^2 + 2V. \end{aligned}$$

The plus-sector determinant carries the further factor $-36\sin^2\frac{z+t}{2}\sin^2\frac{z-t}{2}$, nonzero in the near-cone region. On a root in that region

$$-\tfrac{238}{3}U - 10^5U^2 < \eta_+ < -\tfrac{238}{3}U + 10^5U^2, \qquad 6U - 10^3U^2 < \eta_\times < 6U + 10^3U^2.$$

The group speed is $v_g(G) = -G_U \sin t/(\sqrt{3}\, G_V \sin z)$, equal to $z'(t)/\sqrt{3}$ along a differentiable root branch by implicit differentiation. The enclosures and trigonometric remainders give the windows of Theorem 5.5; the coefficients $-59/24$, $5/24$ (phase) and $-59/8$, $5/8$ (group) are their limits.

G The critical zeros of the tree operator

This appendix proves Theorem 6.7. Throughout $\kappa = 1$, $M(k)$ is the real symmetric 15×15 operator of Section 4.2, $\partial_v M$ its derivative along $v \in \mathbb{R}^4$, $z_a = e^{ik_a}$, $z^{\mathbf{1}} = z_0z_1z_2z_3$, and F_1 is the integer polynomial of Appendix C.

Exact computations. The proof uses explicit integer polynomials too long to print. Their identities are proved by one elementary lemma.

Lemma G.1 (Kronecker substitution)**.** *Let $P \in \mathbb{Z}[x_1, \dots, x_n]$ have degree less than D in each variable and coefficients of absolute value less than $B \geq 2$. If $P(B, B^D, \dots, B^{D^{n-1}}) = 0$, then $P = 0$.*

Proof. The monomial x^a goes to $B^{k(a)}$ with $k(a) = a_1 + a_2D + \cdots + a_nD^{n-1}$, and distinct exponent vectors with entries below D give distinct $k(a)$. If $P \neq 0$, let cB^k be the term with the largest k. The other terms add up to at most $(B-1)(1 + B + \cdots + B^{k-1}) = B^k - 1$ in absolute value, which is less than $|c|\, B^k$. □

Every polynomial identity below is proved with Lemma G.1: the degree and coefficient bounds of the difference of the two sides are proved from their shapes, and its value at the integer point is computed exactly. An *ideal membership* is such an identity $de = \sum_i c_i g_i$ with explicit integer polynomials c_i; it shows that e vanishes wherever every g_i vanishes and d does not. Positivity is shown by an explicit sum of squares or by rational enclosures, and numbers in $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{105}, \sqrt{-3})$ are computed exactly through their rational coordinates. We state each identity with its size and its consequence and do not reproduce the coefficients.

Symmetries. For $n \in \mathbb{Z}^4$, $M(k + 2\pi n) = S\,M(k)\,S$ with S diagonal, with entry $(-1)^{d\cdot n}$ on the class d; for a permutation π of the axes, $M(\pi k) = P\,M(k)\,P^{\top}$ with P the induced permutation of the edge classes; and $M(-k) = M(k)$. The derivatives follow: $\partial_\nu M(k + 2\pi n) = S\,\partial_\nu M(k)\,S$, $\partial_{\pi\nu} M(\pi k) = P\,\partial_\nu M(k)\,P^{\top}$ and $\partial_{-\nu} M(-k) = \partial_\nu M(k)$. So each map carries critical zeros to critical zeros, the lattice copies of the origin and of the corners to themselves, and a hyperbolic direction at k to one at the image of k. If two phases of a critical zero agree modulo 2π, a permutation and a translation move it to the hyperplane $\mathscr{P} = \{k_0 = k_1\}$. It therefore suffices to treat critical zeros with four distinct phases (part I) and critical zeros on $\mathscr{P}$(part II).

Part I: four distinct phases. Let

$$\Xi(k)(X, Y) = \prod_{a=0}^{3} \Big(X \cos\frac{k_a}{2} + Y \sin\frac{k_a}{2} \Big) = \sum_{j=0}^{4} S_j X^{4-j} Y^j,$$

the binary quartic with roots $X/Y = -\tan(k_a/2)$. Its value at $(1, i)$ is $e^{i(k_0+k_1+k_2+k_3)/2}$, so $h(S) = (S_0 - S_2 + S_4)^2 + (S_1 - S_3)^2$, the squared modulus of that value, equals 1: the quartic is never divisible by $X^2 + Y^2$.

(I.1) *The chart.* There is an integer form $\hat{p}$ of degree six in S, with 103 terms, such that $(z^{\mathbf{1}})^{-3} F_1(e^{ik}) = 512\,\hat{p}(\Xi(k))$ for every k. This is a polynomial identity in the half-angle variables $e^{ik_a/2}$, proved by Lemma G.1 with $B = 2^{82}$ and $D = 23$. With the factorization of the determinant through F_1 and a Gram determinant of the vertex moves, it gives $\det M(k) = \varrho(k)\,\hat{p}(\Xi(k))$, with ϱ continuous and positive off the lattice copies of the origin. When the phases are distinct the differential of Ξ is injective, and its image together with the radial direction S spans $\mathbb{R}^5$. By Euler's identity $\sum_j S_j \partial_j \hat{p} = 6\hat{p}$, k is then a critical zero exactly when all five partial derivatives of $\hat{p}$ vanish at $\Xi(k)$. At such a point $\det M(k + \tau v)/\tau^2 \to \frac{1}{2}\varrho(k)\,\mathrm{Hess}\,\hat{p}\,(D\Xi\, v, D\Xi\, v)$ as $\tau \to 0$, and the radial direction lies in the kernel of $\mathrm{Hess}\,\hat{p}$, again by Euler.

(I.2) *Where the singular points lie.* Let E be the ideal generated by 13 explicit integer forms, seven cubics and six quartics, whose zero set is a curve C_{10} of degree 10, and let R be generated by 10 explicit forms whose zero set is eight points. Every real S with $h(S) \neq 0$ at which all partials of $\hat{p}$ vanish has $E(S) = 0$ or $R(S) = 0$. The proof is three families of identities: 29 ideal memberships $\lambda_i h^2 g_i \in (\partial_0 \hat{p}, \dots, \partial_4 \hat{p})$ for quintics g_i, with integers $\lambda_i \neq 0$; one identity $\sum_l \mu_l \varphi_l^2 = \sum_i a_i g_i$ for 24 quartics φ_l, with every $\mu_l > 0$; and 130 memberships $\lambda E_i R_j \in (\varphi_1, \dots, \varphi_{24})$, one for each pair of generators. At such an S the first family makes every g_i vanish; the second, a sum of positive multiples of real squares, then makes every φ_l vanish; and the third makes every product $E_i(S)R_j(S)$ vanish. No coefficient exceeds 68 bits.

(I.3) *The eight points.* Explicit polynomial identities show that at a real point where every generator of R vanishes, $S_4 \neq 0$ and the quartic is a perfect square. A quartic with four distinct roots is not a square. So at a critical zero with distinct phases, $\Xi(k)$ lies on C_{10}.

(I.4) *The Hessian.* Let e_3 be the sum of the principal 3×3 minors of $\mathrm{Hess}\,\hat{p}$ and D the discriminant of the quartic. There is an identity of forms of degree 14 with integer coefficients,

$$\Lambda\big(a\,e_3 - c\,D^2|S|^2 - \sigma\big) = \sum_{i=1}^{13} h_i E_i,$$

with constants $\Lambda, c > 0$, an explicit quadratic form a, explicit cofactors h_i, and σ a sum of squares of 53 forms of degree seven whose integer Gram matrices are positive definite. At a real point of C_{10} the right side vanishes, so $a\,e_3 = c\,D^2|S|^2 + \sigma \geq c\,D^2|S|^2$, which is positive when $D \neq 0$. So $e_3 \neq 0$, and Hess $\hat{p}$ has rank at least three at every real point of C_{10} with distinct roots. (The real points of the curve at which e_3 vanishes all have a repeated root.)

(I.5) *Two zero eigenvalues on the curve.* Let B_E be the 11×11 block of the Bloch matrix, in the phase frame, on the eleven edge classes that are not single axes; it is Hermitian. On C_{10} with $D \neq 0$ the coefficient $S_4 = \prod_a \sin(k_a/2)$ does not vanish, so no phase is $\equiv 0$ and the single-axis block of the vertex-move columns, diag $2(z_a - 1)$, is invertible. A kernel vector of B_E, extended by zero and projected off the vertex-move columns, is then a kernel vector of A, and distinct kernel vectors stay distinct. The sum of the principal minors of order ten of B_E, cleared of a monomial factor, equals $48\,(z^{\mathbf{1}})^6$ times a form F_8 of degree eight evaluated at the coefficients of the quartic, up to a factor that does not vanish, and $F_8 \in E$ by explicit cofactors. The identity is proved at 21 integer values of z_3 by Lemma G.1 in (z_0, z_1, z_2), with $B = 2^{121}$ and $D = 21$, the minors at the integer point being computed by triangular factorizations that are themselves verified as exact matrix identities, and it extends to every z_3 because both sides have degree at most 20 in z_3. So on $\Xi^{-1}(C_{10})$ the trace of the adjugate of B_E vanishes, and so does its determinant, by the factorization of the determinant. A Hermitian matrix with a one-dimensional kernel spanned by u has adjugate $c\,uu^\dagger$, with $c \neq 0$ the product of its nonzero eigenvalues, and that adjugate has nonzero trace. So B_E, and with it A and M, has at least two zero eigenvalues there.

(I.6) *Conclusion of part I.* Let k be a critical zero with distinct phases. By (I.3) and (I.5), $M(k)$ has at least two zero eigenvalues. If it had three, $\det M$ would vanish to third order at k; by (I.1) the form Hess $\hat{p}$ would then vanish on the image of $D\Xi$ and on the radial direction, hence identically, against (I.4). So $M(k)$ has exactly two zero eigenvalues. Let $\Lambda_k(v)$ be $\partial_v M(k)$ compressed to its kernel. As in the proof of Proposition 6.9(iii), $\det M(k+\tau v)/\tau^2 \to c_k \det \Lambda_k(v)$, with $c_k \neq 0$ the product of the nonzero eigenvalues of $M(k)$; by (I.1) and (I.4) this is a quadratic form of rank at least three in v. Writing $\Lambda_k(v) = \left(\begin{smallmatrix} \alpha & \beta \\ \beta & \gamma \end{smallmatrix}\right)$ with α, β, γ linear in v, $\det \Lambda_k(v) = \left(\frac{\alpha+\gamma}{2}\right)^2 - \left(\frac{\alpha-\gamma}{2}\right)^2 - \beta^2$ has at most one positive square, so a form of this kind with rank three takes positive values. By Proposition 6.9(iii), k is hyperbolic.

Part II: the hyperplane $k_0 = k_1$. (II.1) *The two blocks.* On $\mathscr{P}$ the exchange of the first two bits of the edge classes commutes with $M(k)$, and with $\partial_v M(k)$ for $v \in \mathscr{P}$. Seven classes are fixed and eight are exchanged in pairs. Let V_e be the 15×11 integer matrix whose columns are the seven fixed unit vectors and the four sums of exchanged pairs, and V_o the 15×4 matrix of the four differences. Then $[V_e\,V_o]^\top [V_e\,V_o] = \mathrm{diag}(1^7, 2^8)$, so $\det[V_e\,V_o]^2 = 256$, and since $V_e^\top M V_o = 0$,

$$256\,\det M = \det X_e\,\det X_o, \qquad X_e = V_e^\top M V_e, \quad X_o = V_o^\top M V_o.$$

Write $x = e^{ik_0} = e^{ik_1}$, $y = e^{ik_2}$, $w = e^{ik_3}$. There are integer polynomials P_8 and P_{16} with $F_1(x, x, y, w) = P_8 P_{16}$, an identity proved by Lemma G.1 ($B = 2^{82}$, $D = 23$). Off the origin $\det X_o$ is a nonzero multiple of $P_8 R_o^2$, with R_o a polynomial that vanishes on the torus only at the origin; this is a 4×4 determinant identity, proved by Lemma G.1 ($B = 2^{50}$, $D = 22$). With the factorization of $\det M$ and continuity, $\det X_e$ is a nonzero multiple of P_{16} off the origin.

(II.2) *Three cases.* Since $\det M$ is invariant under the exchange, its derivative across $\mathscr{P}$ vanishes, so $k \in \mathscr{P}$ is a critical zero when $\det M(k) = 0$ and the derivative of $\det M$ along every $v \in \mathscr{P}$ vanishes. With $L_e(v)$ and $L_o(v)$ the derivatives of $\det X_e$ and $\det X_o$ along $v \in \mathscr{P}$, the second condition reads $L_e(v) \det X_o + \det X_e \, L_o(v) = 0$ for $v \in \mathscr{P}$. So one of the following holds: (a) $\det X_e = \det X_o = 0$, and the linear forms L_e, L_o on $\mathscr{P}$ are independent; (b) $\det X_e = \det X_o = 0$, and L_e, L_o are dependent; (c) exactly one of the two determinants vanishes, and its derivative vanishes along all of $\mathscr{P}$.

(II.3) *Transversal crossings.* In case (a) $L_e \neq 0$. By Jacobi's formula $L_e(v) = \operatorname{tr}\big(\operatorname{adj} X_e \, \partial_v X_e\big)$, so $\operatorname{adj} X_e \neq 0$, the kernel of X_e is a line spanned by some u_e, and $\operatorname{adj} X_e = c_e u_e u_e^{\top}$ with $c_e \neq 0$; hence $L_e(v) = c_e \, u_e^{\top} (\partial_v X_e) u_e$. The same holds for X_o, with u_o and c_o. The kernel of $M(k)$ is spanned by $n_e = V_e u_e$ and $n_o = V_o u_o$, and $n_e^{\top} (\partial_v M) n_o = 0$ for $v \in \mathscr{P}$, because $\partial_v M$ commutes with the exchange, which fixes n_e and reverses n_o. So in this basis $\Lambda_k(v)$ is diagonal, with entries $L_e(v)/c_e$ and $L_o(v)/c_o$, and since L_e and L_o are independent some $v \in \mathscr{P}$ makes both equal to 1.

(II.4) *The special points.* In case (c) with $\det X_o = 0$, either $R_o = 0$, which on the torus happens only at the origin, or P_8 vanishes together with its three derivatives along the torus of $\mathscr{P}$. By exact elimination the singular points of $P_8 = 0$ on that torus are the origin and four points with coordinates in $\mathbb{Q}(\sqrt{105}, \sqrt{-3})$. At each of the four the kernel of M is exactly the span of two explicit vectors over that field, as an exact inverse of M plus the projection onto them shows, and along an explicit direction the compressed derivative is the symmetric 2×2 matrix with first entry $\frac{31\sqrt{3}}{338} + \frac{305\sqrt{35}}{1014} > 0$ and determinant $\frac{865\sqrt{105} - 8573}{72} > 0$, which is positive definite.

In case (c) with $\det X_e = 0$, P_{16} vanishes together with its three derivatives along the torus of $\mathscr{P}$. On the plane $y = w$ exact elimination leaves only the origin. Off it, the x-coordinate of such a point is a root of $(x-1)(x^2+1) f(x) g(x)$, with f and g explicit integer polynomials of degrees 10 and 14. The root $x = 1$ gives the two axis corners πe_2 and πe_3, and $x = \pm i$ gives one point each. The roots of f and g on the unit circle, four and two, are isolated in rational arcs with Newton bounds; on these components the product yw is determined exactly by x (for f, $88128\, yw = \psi(x)$ with an explicit integer polynomial ψ of degree nine), after which y and w follow from $y + w$ and yw. Every such point lies in one of sixteen boxes of radius 2^{-40} in (x, y, w); the roots of f give the triple points.

In case (b), $R_o = 0$ gives the origin, and a singular point of $P_8 = 0$ is one of the points already treated. Otherwise $P_8 = P_{16} = 0$ with parallel derivatives along $\mathscr{P}$. On $y = w$ the elimination polynomial in x is $(x-1)^4 (x^2-2)(2x^2-1)(x^2-x+1)$. Its roots on the unit circle are $x = 1$, which forces $y = 1$, the origin, and $x = e^{\pm i\pi/3}$, which forces $y = x$: the lattice copies of $\pm k_*$. At k_* the kernel of M is exactly the span of two integer vectors, and on it the derivative of M along $-(1,1,1,1)$ is one half of the identity, by exact arithmetic in $\mathbb{Q}(\sqrt{3})$ with every entry computed from the cell Hessian; at $-k_*$ the same holds along $+(1,1,1,1)$. Off $y = w$ the tangency points lie in the eight of those boxes that contain triple points.

(II.5) *The boxes.* One perturbation bound handles every wave in a box.

Lemma G.2 (A robust hyperbolic direction)**.** *Let M and D be complex $n \times n$ matrices, N an $n \times k$ matrix, and Y and A matrices of sizes $n \times n$ and $k \times n$, with $\|1 - YM - NA\| \leq \vartheta < 1$, all norms Euclidean or the operator norms they induce. Put $\delta = \vartheta \|N\| / (1 - \vartheta)$, and let λ be a lower bound for the Hermitian part of $N^{\dagger} D N$. If $\lambda - \big(\|N^{\dagger} D\| + \|DN\|\big)\delta - \|D\| \delta^2 > 0$, then $\operatorname{Re} \eta^{\dagger} D \eta > 0$ for*

every nonzero η with $M\eta = 0$.

Proof. If $M\eta = 0$ then $\eta = Na + r$ with $a = A\eta$ and $r = (1 - YM - NA)\eta$, so $\|r\| \le \vartheta\|\eta\| \le \vartheta(\|N\|\|a\| + \|r\|)$ and $\|r\| \le \delta\|a\|$. If $a = 0$ then $\|\eta\| = \|r\| \le \vartheta\|\eta\|$ and $\eta = 0$. Otherwise $\operatorname{Re}\eta^{\dagger}D\eta \ge \lambda\|a\|^2 - (\|N^{\dagger}D\| + \|DN\|)\,\delta\|a\|^2 - \|D\|\delta^2\|a\|^2 > 0$. □

The sixteen boxes are the images of four base boxes under $k \mapsto -k$, which conjugates the phases, and under the exchange of the last two coordinates. For each base box, rational enclosures of the entries of $M(k)$ and of $\partial_\nu M(k)$, which are explicit trigonometric polynomials, hold uniformly for every $k \in \mathscr{P}$ whose phases x, y, w lie within 2^{-40} of the centre's, and Lemma G.2, applied with an explicit integer direction ν, rational N, Y, A and entrywise bounds that dominate the norms it uses, gives $w^{\top}(\partial_\nu M(k))w > 0$ for every nonzero null vector at every such k. The margins:

base point	$\arg x/\pi$	columns of N	largest row sum of $\lvert 1 - YM - NA\rvert$	λ
first triple point	0.185263	3	2.0×10^{-8}	0.3215
second triple point	0.293388	3	2.3×10^{-8}	0.3823
first even point	$1/2$	2	2.3×10^{-8}	0.3924
second even point	0.324086	2	5.6×10^{-8}	0.1164

The other terms of the condition of Lemma G.2 are below 10^{-4}. The triple points of Section 6.4 lie on $\mathscr{P}$ in the boxes about the first two base points, where the approximate kernel has three columns.

(II.6) *Conclusion.* On $\mathscr{P}$, every critical zero other than the lattice copies of the origin and of the corners is a transversal crossing, one of the four points of $\mathbb{Q}(\sqrt{105}, \sqrt{-3})$, a lattice copy of $\pm k_*$, or a wave in one of the sixteen boxes, and each is hyperbolic. With part I and the symmetries this proves Theorem 6.7.

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