Invariant Couplings and Recorded-Exchange Control

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Abstract

A symmetry specifies which interactions are compatible with a system; it does not specify which interactions its controls can produce. We resolve this distinction for a finite recognition apparatus with a main signal, a retained record and an independent clock. An explicit unitary change of coordinates turns the two signal ports into four equivalent copies of a four-dimensional symplectic representation. The invariant source Hamiltonians are therefore exactly the Hermitian 4×4 coefficient matrices acting between those copies. The original controls preserve one decoded record component, excluding even a simple exchange allowed by the symmetry. Adding forward waits under one fixed positive-strength recorded-exchange Hamiltonian removes this obstruction: every invariant source unitary, with identity on the clock, lies in the operator-norm closure of the resulting finite programs. The proof derives the required operations from the stated inventory: a clock echo removes unwanted clock motion, and a record-restoring subroutine supplies control of the main register. We also carry the original energy readout through the same programs, retaining its reference and positivity conditions.

1 From symmetry to interactions

Consider a signal that can exchange amplitude with a record while a separate clock runs. We call the main signal and its record together the *source*. The record may already contain information. An operation on this apparatus must therefore be specified on all three registers, including their occupied states. A formula that works only when the record starts at zero does not tell us what the operation does to an existing record.

There are two questions. First, which interactions preserve the apparatus's internal symmetry? Second, which of those interactions can be assembled from its available operations? A *program* means a finite sequence of those operations and waits. Some interactions allowed by the symmetry are inaccessible to the original control inventory. After one precisely specified exchange mode is added, every invariant source unitary can be approximated. The intermediate operations need not themselves preserve the symmetry.

The obstruction can be pictured before any group theory is needed. In suitable coordinates, the source consists of four vector components x_0, x_1, x_2, x_3. The first two describe the main signal; the last two describe its record. The original dynamics couple x_1 to $sx_2 + cx_3$, but preserve the orthogonal combination

$$d = cx_2 - sx_3.$$

Here c and s are fixed positive decoder coefficients with $c^2 + s^2 = 1$. Exchanging x_1 and x_3 changes d on some inputs. No program that always preserves d can perform that exchange, even approximately on every input. An exchange acting directly on the recorded amplitudes can change this previously preserved component. Sections 4–5 show why adding it yields the entire permitted source-unitary family, rather than only this one exchange.

The apparatus comes from Recognition Science (RS), which studies the structure required for distinctions, comparisons and retained records. No prior RS definitions are needed here: the carrier, symmetry, matrices, controls and readout used in the proofs are specified below. The source symmetry was obtained by requiring a change of frame to remain compatible with a source carrying arbitrary records [1]. We use that representation as our starting point and derive its interaction consequences.

The distinction between symmetry and controllability is standard in quantum control [2, 3]. Likewise, operations on repeated representation spaces and exchange-based encoded control have an established history [4, 5]. Our result concerns this particular source, its actual decoder and its complete retained registers. Its content is the explicit chain from a symmetry-compatible coupling to an admitted program and then to the original readout. The control conclusion establishes arbitrary accuracy: for any target and error tolerance, one finite program approximates that target on every unit-norm input. This is what we mean by *operator-norm closure*. A bound on program length or execution time remains a separate problem.

2 The apparatus in coordinates

2.1 Three registers, with no discarded sector

The state is an amplitude vector

$$w = (f, r, q) \in \mathscr{H} = \mathbb{C}^8_{\text{main}} \oplus \mathbb{C}^8_{\text{record}} \oplus \mathbb{C}^8_{\text{clock}}, \qquad \|w\|^2 = \|f\|^2 + \|r\|^2 + \|q\|^2. \tag{1}$$

The three registers label blocks of one amplitude vector: eight entries for the main signal, eight for the record and eight for the clock. They are combined by a direct sum, so $\mathscr{H}$ has complex dimension 24. In particular, the record is a coherent signal port with its own complex amplitudes; it is not a measured classical label. Both its amplitude and phase are retained in the calculation, whether or not it starts empty. All operators are complex-linear, and $A^\dagger$ denotes the adjoint for the displayed inner product. We write $\mathrm{U}(n)$ for the unitary matrices ($A^\dagger A = I_n$) and $\mathrm{SU}(n)$ for those with determinant one. A Hermitian, or self-adjoint, matrix satisfies $A^\dagger = A$; a skew-Hermitian matrix satisfies $A^\dagger = -A$.

Number the coordinates of each eight-component signal from 0 to 7. For $v \in \mathbb{C}^8$, define its odd and even parts under a shift by four slots:

$$a(v)_j = \frac{v_j - v_{j+4}}{2}, \qquad b(v)_j = \frac{v_j + v_{j+4}}{2}, \qquad 0 \le j < 4. \tag{2}$$

Thus $v = (a + b, -a + b)$ and $\|v\|^2 = 2\|a\|^2 + 2\|b\|^2$. The words "odd" and "even" refer to this half-cycle, not to odd- and even-numbered slots.

The following maps put the odd and even components into the same internal coordinate convention:

$$L(v) = (v_0 + iv_1,\ v_2 - iv_3,\ v_0 - iv_1,\ v_2 + iv_3), \tag{3}$$

$$S = \mathrm{diag}(1, 1, -1, -1). \tag{4}$$

The interaction coordinates are

$$x_0 = SLa(f), \quad x_1 = Lb(f), \quad x_2 = La(r), \quad x_3 = Lb(r). \tag{5}$$

Write $T(f, r) = (x_0, x_1, x_2, x_3)$. Because $L^\dagger L = 2I_4$, equations (2)–(5) make T unitary. The map is onto: recover $a(f) = L^{-1}Sx_0$, $b(f) = L^{-1}x_1$, and similarly recover the record, then use $v = (a + b, -a + b)$. This is a change of coordinates on the entire source. Unless original slots are explicitly specified, all source matrices below use these interaction coordinates.

2.2 The symmetry acts within each copy

The symplectic factor consists of unitary matrices that preserve a fixed antisymmetric bilinear form. Its matrix description is

$$\Omega = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \mathrm{Sp}(2) = \{G \in \mathrm{U}(4) \,:\, G^T \Omega G = \Omega\}. \tag{6}$$

For $G \in \mathrm{Sp}(2)$ and $|z| = 1$, the source symmetry is

$$x_j \longmapsto zGx_j \quad (j = 0, 1, 2, 3), \qquad q \longmapsto q. \tag{7}$$

The pairs (G, z) and $(-G, -z)$ represent the same transformation. There are no further identifications: $zG = I_4$ implies $G = z^{-1}I_4$, and the symplectic condition gives $z^2 = 1$. The represented source group is therefore

$$\mathcal{G} = (\mathrm{Sp}(2) \times \mathrm{U}(1))/\{(I_4, 1), (-I_4, -1)\}.$$

Each x_j has four internal coordinates transformed by zG. The label j tells us which copy carries those coordinates. An interaction may mix the copy labels while acting identically on their internal coordinates. Keeping those two roles distinct is the key to the classification.

For a 4×4 matrix M, define its source action by

$$\big(\mathscr{M}(M)x\big)_a = \sum_{b=0}^{3} M_{ab}x_b. \tag{8}$$

In tensor notation this is $M \otimes I_4$ on $\mathbb{C}^4_{\text{copies}} \otimes \mathbb{C}^4_{\text{internal}}$. This tensor notation describes the sixteen source coordinates after T; the clock is still a separate direct summand.

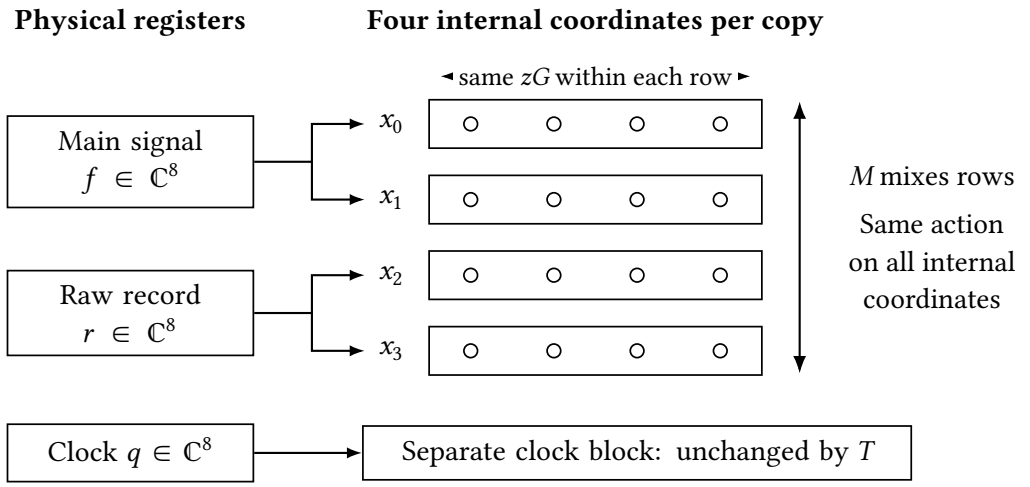

Figure 1. Two distinct actions on the same source. Each row is one complete vector x_j from equation (5); its four circles stand for the four complex internal coordinates. The symmetry applies the same zG within every row. An invariant coupling M mixes the rows and acts identically on each internal coordinate. The chart retains all sixteen source coordinates and leaves the separate clock block unchanged.

3 All fixed invariant Hamiltonians

A fixed operator is *invariant* when it commutes with every transformation (7): applying the operator before or after the symmetry gives the same result. "Fixed" means that the operator itself is held constant while the state changes frame. This question differs from transforming an external field or the operator's coefficients along with the state.

Theorem 3.1 (Invariant coupling classification)**.** *A source operator commutes with every element of $\mathcal{G}$ if and only if, in coordinates T, it is $\mathscr{M}(M)$ for a unique $M \in M_4(\mathbb{C})$. It is self-adjoint if and only if $M^\dagger = M$, and unitary if and only if $M^\dagger M = I_4$. On the full carrier* (1)*, every invariant self-adjoint operator has the unique form*

$$(M \otimes I_4) \oplus C_0, \qquad M^\dagger = M, \quad C_0^\dagger = C_0, \tag{9}$$

where C_0 acts on the clock.

Proof. We first show directly that a 4×4 matrix B commuting with every $G \in \mathrm{Sp}(2)$ is scalar. The group contains

$$P_1 = \mathrm{diag}(i, -i, i, -i), \qquad P_2 = \mathrm{diag}(i, -i, -i, i),$$

as well as Ω and the permutation that exchanges $0 \leftrightarrow 2$ and $1 \leftrightarrow 3$. Their membership follows by substitution in (6). Commutation gives $B_{ab}((P_\ell)_{bb} - (P_\ell)_{aa}) = 0$ for $\ell = 1, 2$. The joint diagonal labels are all distinct, so every off-diagonal entry of B vanishes. Commutation with Ω then equates entries $0, 1$ and entries $2, 3$. The pair exchange equates the two pairs. Thus $B = \lambda I_4$.

Write a source operator as a 4×4 array of internal blocks B_{ab}. Commutation with (7) says that each block commutes with every G. Hence $B_{ab} = M_{ab} I_4$, which is precisely (8). Conversely,

scalar internal blocks commute with zG. Uniqueness follows by applying the operator to an input supported in one copy. Since T is unitary, adjoints and products give the stated self-adjointness and unitarity criteria in the original physical inner product.

Finally, the group contains the transformation $-I_{16}$ on the source and I_8 on the clock. Commuting with this transformation forces both source–clock off-diagonal blocks to vanish. The clock representation is trivial, so its diagonal block is otherwise unrestricted. □

The source Hamiltonian family contains sixteen real coefficients: four real diagonal entries and six complex off-diagonal entries subject to conjugate symmetry. This is the space of Hermitian coefficient matrices, whose associated skew-Hermitian generators form $\mathfrak{u}(4)$. It is a different object from the source symmetry algebra $\mathfrak{sp}(2) \oplus \mathfrak{u}(1)$, which has dimension eleven. The former mixes equivalent copies; the latter acts within each copy. Adding a control mode can enlarge the reachable operations without changing this given symmetry representation.

Symmetry alone permits every clock block C_0 in (9). The control theorem below concerns the source family with the clock retained. It does not infer control of an arbitrary C_0 from this classification.

4 The specified controls and the missing exchange

4.1 The fixed decoder and the two waiting modes

Put $\phi = (1+\sqrt{5})/2$ and $\gamma = \sqrt{1-\phi^{-4}}$. The decoder constants are fixed by the source:

$$c = \frac{1}{\phi\sqrt{\phi}\,\gamma}, \qquad s = \frac{\phi}{\phi\sqrt{\phi}\,\gamma}, \qquad c^2 + s^2 = 1, \quad s = \phi c > 0. \tag{10}$$

Their particular arithmetic matters in Appendix A: repeated uses of this one fixed decoder approximate a whole circle of rotations. On an eight-component signal, the decoder is

$$(Dv)_j = cv_j + sv_{j+4}, \quad (Dv)_{j+4} = -sv_j + cv_{j+4} \qquad (0 \le j < 4). \tag{11}$$

In particular, the coordinates in (5) use the *raw* record r. The odd component of the inverse-decoded record $D^{-1}r$, after applying L, is $cx_2 - sx_3$.

Let $\theta = \arccos(-\phi^{-2}) > 0$. In interaction coordinates, the dimensionless source Hamiltonian is $H_s = i\theta J$, where

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -s & -c \\ 0 & s & 0 & 0 \\ 0 & c & 0 & 0 \end{pmatrix}. \tag{12}$$

Thus the source flow at dimensionless time u is $e^{\theta u J} \otimes I_4$. The independent clock has a fixed orthonormal Fourier basis in which its dimensionless generator and flow are

$$C = \frac{\pi}{4}\,\mathrm{diag}(0,1,0,3,0,-3,0,-1), \qquad Q(u) = e^{-iuC}. \tag{13}$$

In particular $Q(u+8) = Q(u)$. A source wait advances both systems:

$$W(u) = (e^{\theta u J} \otimes I_4) \oplus Q(u), \qquad u \ge 0. \tag{14}$$

If $\tau_0 > 0$ is the clock's time unit, $u = t/\tau_0$ and the physical source Hamiltonian is $(\hbar/\tau_0)H_s$.

In the original eight-slot coordinates, the inventory also contains three instantaneous main-register operations:

$$(Pv)_j = v_{j+1 \pmod 8}, \qquad V = \mathrm{diag}(1,1,1,1,i,i,1,1), \qquad D. \tag{15}$$

The cyclic shift P will be called the *tick*. Each operation acts on f and is the identity on r and q. These are the specified ideal operations; their finite pulse durations and hardware implementation are not part of the inventory. Let $\mathscr{C}_{\mathrm{old}}$ be the operator-norm closure of all finite products of these three operations and the forward waits (14).

We add one waiting mode at one fixed strength $e > 0$:

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \tag{16}$$

$$X_e(t) = (e^{-ietK/\hbar} \otimes I_4) \oplus Q(t/\tau_0), \qquad t \geq 0. \tag{17}$$

This is the raw recorded-exchange Hamiltonian eK with its concurrent clock advance. It couples the main even component x_1 to the raw record even component x_3. The extension assumes that this mode can be selected separately from the source wait. The larger closure generated by the original inventory and (17) is $\mathscr{C}_e$. There is no continuously adjustable coupling strength in this definition: e is fixed and waiting time is selectable.

4.2 A record component proves the obstruction

Proposition 4.1 (An exchange excluded by the original inventory)**.** *Every operator in $\mathscr{C}_{\mathrm{old}}$ preserves $d = cx_2 - sx_3$ on every input. Let S_{13} be the 4×4 permutation matrix swapping labels $1, 3$. Its full action $(S_{13} \otimes I_4) \oplus I_8$ is invariant under $\mathscr{G}$ but is not in $\mathscr{C}_{\mathrm{old}}$.*

Proof. The main-register gates leave x_2, x_3 unchanged. Under the source flow, equation (12) gives

$$\dot{x}_2 = \theta s x_1, \qquad \dot{x}_3 = \theta c x_1, \qquad \dot{d} = \theta(cs - sc)x_1 = 0.$$

The clock does not enter d. Preserving d is a closed linear condition on an operator, so it survives products and operator-norm limits. The exchange is $\mathscr{M}(M)$ for the coefficient permutation M swapping $1, 3$, so it is invariant by Theorem 3.1. But on $x_1 = v \neq 0$, $x_0 = x_2 = x_3 = 0$, the exchange changes d from 0 to $-sv \neq 0$. □

The added mode performs precisely this source exchange at $t = \pi\hbar/(2e)$, because $K^2 = 2K$ and

$$e^{-i\alpha K} = I_4 + \frac{e^{-2i\alpha} - 1}{2} K, \qquad e^{-i\pi K/2} = I_4 - K = S_{13}. \tag{18}$$

Its simultaneous clock advance must still be compensated. The next section derives that compensation from the original controls.

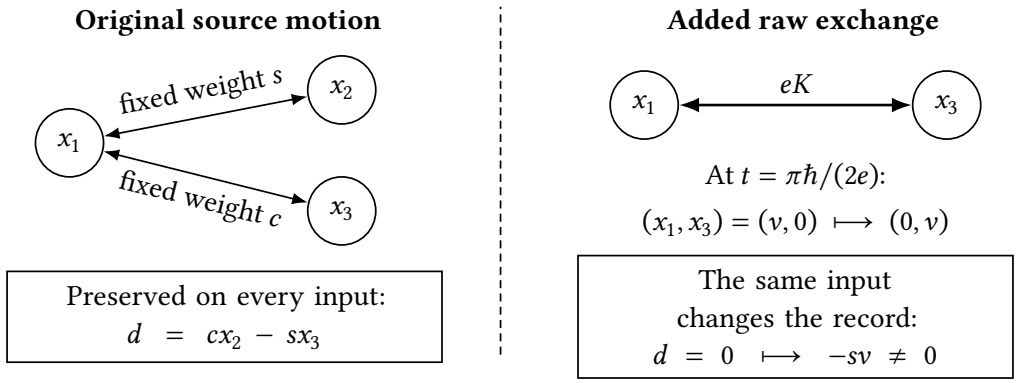

Figure 2. The obstruction and the operation that crosses it. The two source edges have locked weights: their flow preserves $cx_2 - sx_3$. The added exchange acts on the raw 1, 3 pair and can change that combination. The right-hand source swap also advances the clock; the echo in Figure 3 supplies the necessary clock compensation. The idle copy x_0 is omitted here.

5 From the inventory to every invariant source unitary

Membership in $\mathscr{C}_e$ means that for each $\varepsilon > 0$ a finite program has full-operator error below ε. The same program then approximates the target on every bounded input, including occupied records and independently prepared references. The invariant target may intentionally change the record, as the exchange does. Identity on the record is required only for the main-control subroutine in Lemma 5.2; identity on the clock is required for every target in the theorem. These are identities of the limiting operators. An approximating finite program need only approach them to the chosen accuracy.

Theorem 5.1 (Complete invariant source control)**.** *For every fixed $e > 0$ and every $M \in \mathrm{U}(4)$,*

$$\widehat{M} = (M \otimes I_4) \oplus I_8 \in \mathscr{C}_e. \tag{19}$$

Consequently, for every Hermitian $A \in M_4(\mathbb{C})$ and every $t \in \mathbb{R}$, the source flow $(e^{-itA} \otimes I_4) \oplus I_8$ lies in $\mathscr{C}_e$. Moreover $\mathscr{C}_{\mathrm{old}} \subsetneq \mathscr{C}_e$.

The proof follows three steps. First cancel the clock motion while retaining the source operation. Then obtain independent main-register rotations while restoring every incoming record. Finally combine those rotations with the two waiting modes to generate all invariant source directions. Each matrix used in that last calculation is thereby tied to an operation from the inventory.

5.1 Separate the source motion from the clock

Let F multiply the main register by -1 and leave the record and clock unchanged. The fixed decoder has dense powers in its rotation circle, and therefore $F \in \mathscr{C}_{\mathrm{old}}$; Appendix A proves this without an adjustable decoder angle. In interaction coordinates F changes the signs of x_0, x_1. It satisfies $FJF = -J$ on the source and commutes with the clock. Hence, for $v \geq 0$,

$$W(v)FW(v)F = I_{16} \oplus Q(2v). \tag{20}$$

All factors belong to the original closure. Products are written in operator order, so the rightmost operation acts first.

Every real clock time has a nonnegative representative modulo 8. Thus $I_{16} \oplus Q(u)$ belongs to $\mathscr{C}_{\mathrm{old}}$ for every real u. Multiplying a forward $W(u)$ by the compensating clock operation gives $e^{\theta u J} \otimes I_4$ with the clock retained. Conjugation by F reverses the sign of u. The complete signed source flow is therefore available in the original closure.

The same compensation removes the clock advance from $X_e(t)$. Since $e^{-ietK/\hbar}$ has period $\pi\hbar/e$, its nonnegative times cover all signed times modulo that period. The complete signed exchange flow, with identity on the clock, belongs to $\mathscr{C}_e$. In particular, the clock-retained exchange in Proposition 4.1 lies in $\mathscr{C}_e$, proving strict containment.

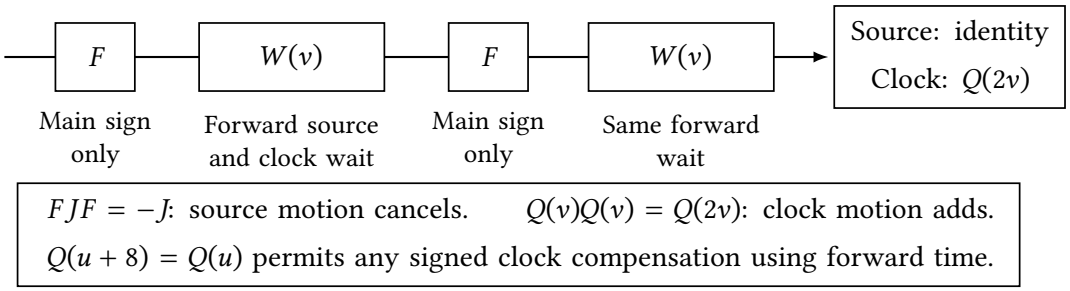

Figure 3. A clock echo derived from the original inventory. Each F is itself admitted as a limit of powers of the fixed decoder. The diagram depicts the exact limiting identity $W(v)FW(v)F = I_{16} \oplus Q(2v)$, not a claim that a finite decoder word always equals F. Applying the compensating clock operation retains the clock while the desired source or exchange motion acts.

5.2 The original controls supply the two main directions

We need two independent operations on the copy pair (x_0, x_1). They must come from the given main-register gates and source waits, with their effects on existing records accounted for.

Lemma 5.2 (Main control with record and clock retained)**.** *For every $U \in \mathrm{SU}(8)$ acting on the original main register,*

$$U \oplus I_{8,\mathrm{record}} \oplus I_{8,\mathrm{clock}} \in \mathscr{C}_{\mathrm{old}}.$$

Proof. Appendix A derives local real rotations and relative phases from P, V, D. A source quarter turn, followed by P and the inverse source turn, supplies an additional cycle on the main odd sector. Together these operations generate $\mathfrak{su}(8)$ on the main register. The quarter-turn construction also acts on a populated record: its record companion is a fixed cycle C_r of order four. Every word in these operations acts as $U_w \oplus C_r^{n(w)}$ on main and record. Thus repeating the whole word four times returns *every* record exactly. Given $U \in \mathrm{SU}(8)$, choose a fourth root in SU(8), approximate that root by main words, and repeat each full word four times. Their limits are $U \oplus I_8$. The source motions used here have already been admitted with clock identity by (20), so the resulting full target also retains the clock. The appendix supplies the coordinate identities and generation argument. □

An $A \in \mathrm{SU}(2)$ acting on (x_0, x_1) and identically on each internal coordinate is a unitary operation on the main register with determinant $(\det A)^4 = 1$. The main part of T is unitary,

so Lemma 5.2 admits it. In particular the two coefficient directions

$$R_{01} = E_{01} - E_{10}, \qquad D_{01} = i(E_{00} - E_{11}) \tag{21}$$

are available with record and clock retained. Here E_{ab} has one entry 1 in row a, column b and zeros elsewhere.

5.3 Four admitted directions generate the complete algebra

For distinct a, b, write

$$R_{ab} = E_{ab} - E_{ba}, \quad I_{ab} = i(E_{ab} + E_{ba}), \quad D_{ab} = i(E_{aa} - E_{bb}), \qquad [A, B] = AB - BA.$$

Call a skew-Hermitian coefficient matrix A *admitted* when $(e^{tA} \otimes I_4) \oplus I_8 \in \mathscr{C}_e$ for every real t. The product limits given below show that admitted directions are closed under real linear combinations and commutators. Physically, the commutator limit alternates short operations and their reverses to isolate a further control direction. The admitted directions form a real Lie algebra, and it contains

$$R_{01}, \quad D_{01}, \quad J, \quad iK. \tag{22}$$

The first two came from Lemma 5.2; the last two came from the compensated waiting modes. Their nonzero physical scales θ and $e/\hbar$ do not change their real Lie span.

Two identities expose the missing record direction:

$$[D_{01}, iK] = -R_{13}, \qquad J + cR_{13} = -sR_{12}. \tag{23}$$

Since $s \neq 0$, both R_{13} and R_{12} are generated. Together with R_{01} they connect all four copy labels. The identities $[R_{ab}, R_{bc}] = R_{ac}$ for distinct labels propagate real rotations along this connected graph. All R_{ab} follow. Also $[R_{13}, D_{01}] = I_{13}$. Commuting this imaginary symmetric direction with the real rotations supplies all I_{ab}; for example $[R_{ab}, I_{bc}] = I_{ac}$ for distinct labels. Finally $[R_{ab}, I_{ab}] = 2D_{ab}$. These matrices span all traceless skew-Hermitian 4×4 matrices.

There is also a central direction, which the traceless calculation alone would miss. Since $\operatorname{tr}(iK) = 2i$, the matrix $iK - \frac{1}{2}iI_4$ is traceless and skew-Hermitian. It is already in the generated $\mathfrak{su}(4)$, so subtraction supplies iI_4. The generated algebra is exactly $\mathfrak{u}(4)$.

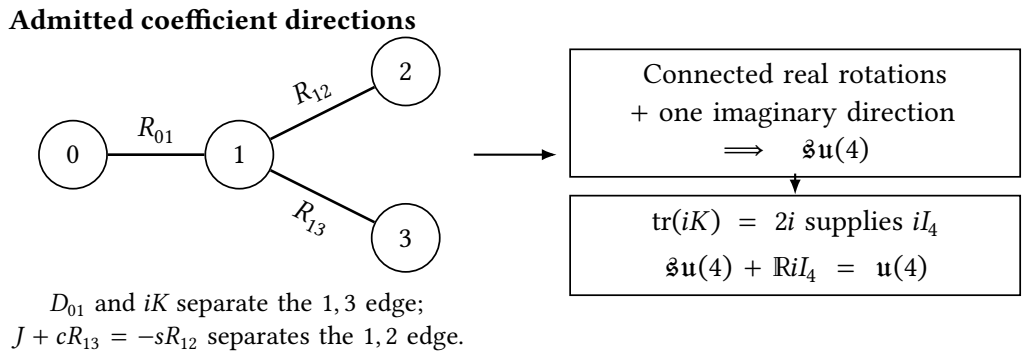

Figure 4. Why one added mode is sufficient. The vertices are copy labels, each carrying four internal coordinates. Once the two record edges are independently available, commutators connect every pair and generate all traceless skew-Hermitian coefficients. The exchange's nonzero trace supplies the remaining scalar direction. This is the interaction algebra; it is distinct from the eleven-dimensional source symmetry algebra.

Completion of Theorem 5.1. For completeness, the passage from these directions to finite-program closure uses only finite-dimensional unitary limits. If the full signed one-parameter flows of A and B are in a closed control set, then

$$(e^{tA/n}e^{tB/n})^n \longrightarrow e^{t(A+B)}.$$

Likewise,

$$e^{hA}e^{hB}e^{-hA}e^{-hB} = I + h^2[A, B] + O(h^3).$$

For $t \geq 0$, take $h = \sqrt{t/n}$ and repeat this product n times; the limit is $e^{t[A,B]}$. Exchanging A and B supplies negative times. These facts follow by multiplying the matrix exponential series and taking the indicated product limits. Thus admitted signed directions form a real Lie algebra. All approximating factors can in turn be approximated by words from the inventory, since multiplication is continuous and $\mathscr{C}_e$ is closed.

The preceding calculation gives every skew-Hermitian coefficient direction. Every $M \in \mathrm{U}(4)$ has a skew-Hermitian logarithm: diagonalize M unitarily, choose real arguments of its eigenvalues, and conjugate back the diagonal matrix of their imaginary arguments. Its exponential is M, proving (19). Taking the logarithmic direction $-iA$ proves the Hamiltonian statement. Strict containment was proved using S_{13}. □

The coefficient target is invariant throughout its one-parameter evolution. Individual gates in an approximating program need not be invariant. This distinction is essential: the control inventory may temporarily leave the symmetry-compatible family to assemble a final operation within it.

6 Carry the energy readout through the same program

To connect the control theorem to a measurement, apply the same program to the signal and its calibration references. We retain the source and clock generator already specified, including the scalar baseline used in the original apparatus. In dimensionless energy units, put

$$H_\beta = (H_s \otimes I_4) \oplus C + \beta I_{24}, \qquad \mathscr{E}_\beta(w) = \langle w, H_\beta w \rangle. \tag{24}$$

The inner product is conjugate-linear in its first argument. The real number β is a fixed baseline; its original calibration is retained. Multiplication by $\hbar/\tau_0$ converts this generator to physical energy units and cancels in the ratios below.

For the admitted target $\widehat{M}$ in (19), direct substitution gives

$$\mathscr{E}_\beta(\widehat{M}w) = \langle x, ((M^\dagger H_s M) \otimes I_4)x \rangle + \langle q, Cq \rangle + \beta\big(\|x\|^2 + \|q\|^2\big). \tag{25}$$

Thus the interaction changes the source energy read by conjugating H_s. The clock term and scalar baseline are unchanged. This is a transformation law for the measured energy, not a claim that the interaction conserves H_s. A zero-source clock reference $(0, 0, q)$ is fixed by $\widehat{M}$, so its measured baseline is retained as well.

The reference protocol compares a recorded signal preparation w_{sig} with a fixed finite set of recorded reference preparations $w(\zeta_j)$. Here $|\zeta_j| = 1$ labels the phase used to prepare reference j. Let $a_j \in \mathbb{R}$ be their fixed calibration weights, and let $b_{\mathrm{sig}}, b_{\mathrm{ref}} \in \mathbb{R}$ be the original baseline corrections. For an applied operator A, the corrected readings and their ratio are

$$\begin{aligned} N(A) &= \mathscr{E}_\beta(Aw_{\mathrm{sig}}) - b_{\mathrm{sig}}, \\ R(A) &= \sum_{j=1}^{m} a_j \mathscr{E}_\beta(Aw(\zeta_j)) - b_{\mathrm{ref}}, \\ \rho(A) &= \frac{N(A)}{R(A)}. \end{aligned} \tag{26}$$

Any finite expression formed by adding and rescaling reference readings has this form. The same operator A acts on the signal and every reference, including their records and clocks. Thus the ratio compares different preparations subjected to one operation; it does not compare separately tailored controls. The preparations, weights and baseline corrections remain fixed as A changes. The original nonnegative meter interpretation requires $N(A) \geq 0$ and $R(A) > 0$. These hypotheses concern the corrected readings; positivity of an uncorrected energy alone does not imply them.

Proposition 6.1 (A common program retains the read law)**.** *Fix the finite recorded preparations, reference expression and baseline corrections just described. If A_n are finite programs from the extended inventory and $A_n \to \widehat{M}$ in full operator norm, then $N(A_n) \to N(\widehat{M})$ and $R(A_n) \to R(\widehat{M})$. If $R(\widehat{M}) > 0$, their ratios converge to $\rho(\widehat{M})$. If also $N(\widehat{M}) > 0$, the sufficiently late approximants satisfy both original meter conditions. At $N(\widehat{M}) = 0$, ratio convergence remains valid, but it does not guarantee nonnegative finite-approximation numerators.*

Proof. For each fixed preparation w, operator-norm convergence implies $A_n w \to \widehat{M}w$. The finite-dimensional quadratic form (24) is continuous. A finite weighted sum and subtraction

of a fixed baseline preserve continuity. A strictly positive limiting denominator stays positive eventually, so division is continuous there. The same argument gives eventual numerator positivity when its limit is strictly positive. Convergence to zero alone gives no such sign conclusion. Every preparation uses the same A_n, because the approximation is of the full operator, independently of the prepared input. □

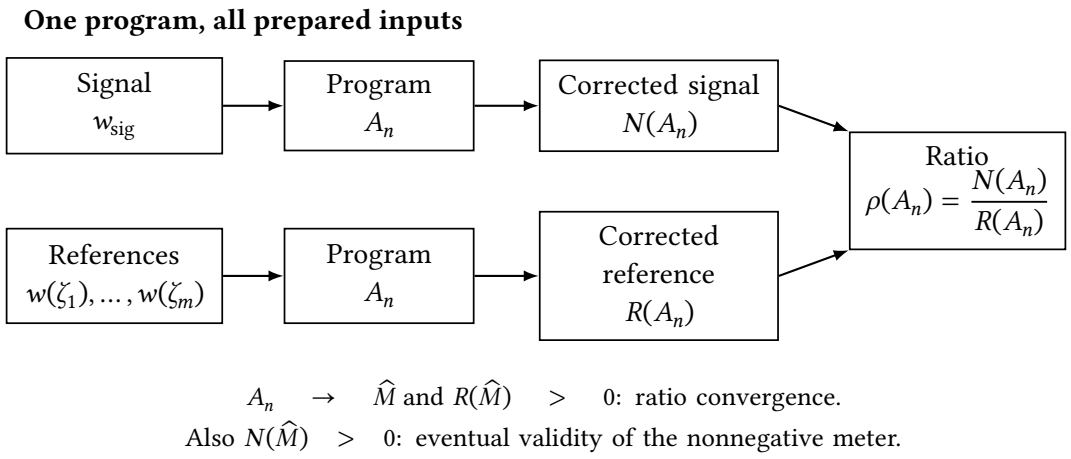

Figure 5. The original comparison protocol is retained through the control limit. A single program A_n acts on the signal and every reference preparation, each with its full record and clock. Finite addition and real scaling preserve continuity. At a zero target numerator the quotient still converges, but a finite approximant need not yet satisfy the nonnegativity condition.

The theorem therefore transports both the operation and its original comparison readout. It does not replace phase-dependent reference shots by a new fitted reference or give different programs to the signal and the reference. The required sign conditions remain visible at the point where the quotient is interpreted as a meter reading.

7 What the interaction result establishes

The finite apparatus now has a complete answer to a specific interaction question. Its fixed symmetry-compatible source Hamiltonians are exactly the Hermitian four-copy couplings. The original control inventory cannot realize that family: it preserves a decoded record component that an allowed exchange changes. One added positive-strength exchange waiting mode, together with the original operations, admits every invariant source unitary in full-carrier operator closure. The proof accounts for occupied records, concurrent clock motion and a common readout program.

The result supplies exact constraints for subsequent physical derivations. A proposed source interaction can be checked against (8); an implementation claim must identify its control modes and their action on all retained registers. A proposed readout must satisfy (25) and retain its reference conditions. These are separate requirements, and satisfying one does not settle the others.

Connecting this result to matter and forces requires a further physical construction: for example, an identified spacetime carrier with a derived locality law and a demonstrated relation between its field dynamics and these finite couplings. An electromagnetic identification or a

particle mass spectrum would have to follow from that construction. A device claim would additionally need switching dynamics and quantitative finite-program bounds. The present theorem supplies the exact finite interaction structure those next derivations must respect.

A How the original inventory controls the main register

This appendix proves the control facts used in Lemma 5.2. The construction first turns the fixed decoder into local rotations, then uses a source excursion to connect their separate coordinate pairs, and finally restores the record. This last step is necessary because the excursion changes a populated record. All closures below are operator-norm closures in finite dimension.

A.1 A fixed decoder supplies a rotation circle

Let $\lambda = c + is$ with c, s from (10). The identities $s = \phi c$, $c^2 + s^2 = 1$ and $\phi^2 = \phi + 1$ imply

$$5c^2s^2 = 1, \qquad \lambda^4 + \overline{\lambda}^4 = -\frac{6}{5}.$$

If λ had finite order, both terms on the left would be algebraic integers. Their sum would be an algebraic integer too, but a rational algebraic integer is an integer, whereas $-6/5$ is not. Thus λ has infinite order. An infinite cyclic subgroup of the circle is dense: a proper closed subgroup has a smallest positive angular spacing and is finite. The closure of positive powers is the same subgroup, since compactness supplies powers tending to 1 and hence approximations to the inverse.

Let $D(z)$ denote (11) with $c = \Re z$ and $s = \Im z$, for $|z| = 1$. Then $D^n = D(\lambda^n)$, so every $D(z)$ is in the original main-control closure. In particular $D(-1) = -I_8$, proving the required admission of F.

The tick localizes this simultaneous rotation. Direct multiplication on the four opposite-slot pairs gives

$$D(z)PD(z^{-1})P^{-1} = D_{3,7}(z^2), \tag{27}$$

where $D_{3,7}(z^2)$ rotates only slots $3, 7$ and fixes the other six. The tick reverses the rotation orientation only at the wraparound pair; the other three pairs cancel. Every circle element has a square root and $P^{-1} = P^7$. Conjugation by further powers of P therefore supplies an arbitrary real rotation on each opposite-slot pair.

Conjugate the rotation on slots $0, 4$ by V and $V^{-1} = V^3$. Its two off-diagonal entries become $-i\Im z$, giving an independent imaginary rotation axis on that pair. Tick conjugation transports this construction to each opposite pair. In the odd/even coordinates (2), the real rotations mix a_j, b_j, while this imaginary axis becomes a pair of opposite phases. No new tunable gate has been added: both circles are limits of words in P, V, D.

A.2 A source excursion joins the four pairs

Let $u_* = \pi/(2\theta)$ and let $\mathscr{R}(u)$ be the source flow with the clock retained, already admitted by (20). Define the full source operation

$$B = \mathscr{R}(-u_*)\,(P \oplus I_{8,\mathrm{record}})\,\mathscr{R}(u_*).$$

Use decoded source coordinates $(a(f), b(f), a(D^{-1}r), b(D^{-1}r))$ in this paragraph. Substitution in the source rotation gives

$$B : (a, b, \alpha, \delta) \longmapsto (P_- a,\ b,\ \alpha,\ P_+ \delta), \tag{28}$$

where

$$P_-(v_0, v_1, v_2, v_3) = (v_1, v_2, v_3, -v_0), \qquad P_+(v_0, v_1, v_2, v_3) = (v_1, v_2, v_3, v_0).$$

In particular the main action is the signed odd-sector cycle $P_- \oplus I_4$. The record action is $I_4 \oplus P_+$ in decoded coordinates, hence a fixed raw-record operator C_r with $C_r^4 = I_8$. A populated record generally changes under one use of B.

Number the main odd/even coordinates by $0, 1, 2, 3, 4, 5, 6, 7$, with j representing a_j and $j + 4$ representing b_j. The local controls supply the real rotation edges $(j, j + 4)$ and opposite phases on each such pair. Conjugating the edge $(0, 4)$ by the signed odd cycle supplies $(1, 4), (2, 4), (3, 4)$, with signs absorbed by reversing rotation time. This graph is connected. The bracket identities used in Section 5, now on eight labels, give every real rotation, every imaginary symmetric off-diagonal direction and every traceless imaginary diagonal direction. They span $\mathfrak{su}(8)$. A special unitary matrix has a traceless skew-Hermitian logarithm: choose eigenvalue arguments whose sum is zero, changing one argument by an integral multiple of 2π if necessary. The product-limit argument therefore admits every main SU(8) action in the closure of words in P, V, D and the main part of B.

A.3 Return an arbitrary record before taking the limit

All three original main gates have identity record action, while every use of B contributes the same C_r. A full source word therefore has the form

$$U_w \oplus C_r^{n(w)},$$

where $n(w)$ counts its uses of B. Repeating this *whole word* four times yields $U_w^4 \oplus I_8$. This return is independent of the incoming record.

For a target $U \in \mathrm{SU}(8)$, choose a traceless skew-Hermitian logarithm Y and put $U_* = e^{Y/4}$. Then $U_* \in \mathrm{SU}(8)$ and $U_*^4 = U$. Choose main words $U_{w_n} \to U_*$ and repeat each corresponding full word four times. Continuity of the fourth power gives

$$U_{w_n}^4 \oplus I_8 \longrightarrow U \oplus I_8.$$

Every B used here belongs to the original timed closure with clock identity. Approximating its occurrences by the already admitted timed programs, and then taking limits, proves Lemma 5.2 on the complete 24-dimensional carrier. The ideal four-repeat words return records exactly; finite timed approximations to their source excursions are only required to converge to that full target.

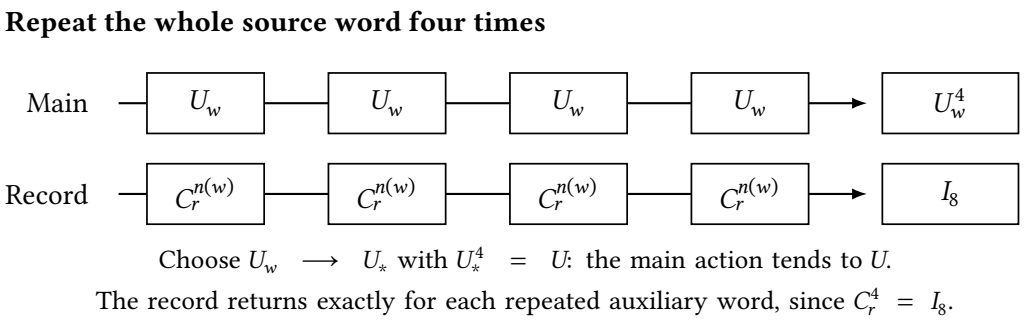

Figure 6. Why an occupied record can be returned. Each column is one complete source word, with main action U_w and record companion $C_r^{n(w)}$. Repeating that word four times returns every record, independently of its initial contents. Approximating a fourth root of the desired main unitary gives the desired limit. These are the auxiliary words built with clock-compensated source operations; their actual finite timed implementations converge to the same full target.

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