

Gravity from the Recognition Ledger: A Finite Sum over Geometries with Einstein's Equations at Every Order

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Abstract

The records of a ledger, whose four counters advance one posting at a time, form the Kuhn lattice, with flat metric $I - \kappa \mathbf{1}\mathbf{1}^{\top}$. Quantum gravity on it is a sum over geometries whose action gives each triangle, a hinge, a charge set by its area and deficit angle; Regge's is their product times a scale s. This paper asks why that charge. For it, the author's earlier preprints prove: at $\kappa = 1$, over a finite lattice's geometries with every posting null, the Lorentzian sum with a model measure is finite on every finite window of squared lengths; and at every $\kappa > \frac{1}{2}$, at leading order in the lattice spacing, the metric part of the field equation is Einstein's at every order in the field amplitude (equations, not solutions), with Newton's law at $G = 1/(8\pi s)$. The field equation varies every edge, so the two results concern different sets of configurations, and no reduction between them is proved.

Two routes answer, both proved here. Gauge consistency: for a charge that is deficit times a function of area, the gauge tests on the Euclidean lattice, $\kappa = 0$, the cubic one on that lattice's own graviton shell, fix the function's value, slope and curvature at each flat area to Regge's, up to a common scale: the action through cubic order there. The record clause: if one posting on a hinge carries an action proportional to its area, at every real area, such a charge, or one depending only on area times deficit, is exactly Regge's, scale included, whatever κ; at $\kappa = 1$ its long-wave cubic vertex is Einstein's.

Beyond long wavelength the $\kappa = 1$ lattice's two polarizations travel at different, direction-dependent speeds (proved, unpublished). At the record clause's spacing, 1.665 Planck lengths up to an uncomputed factor, this correction passes the 2015 cosmic-ray bound of Kostelecký and Tasson by about a million on that bound's own case, an iron nucleus, and sits at or beyond it for a proton; the bound applies only if matter is not slowed as the waves are (not computed here). The paper claims no unitarity, black holes or massless graviton: at $\kappa = 1$ the computed one-loop term at zero momentum is nonzero (numerical), and whether that excludes a massless graviton rests on three open inputs and is argued on two of five uniform strains.

The value of κ is an identification with two readings, neither derived: if one tick is one posting, $\kappa = 1$, used here; if a tick is a debit with its credit on another account, $\kappa = \frac{1}{2}$, where none of the Lorentzian results above is established.

1 Gravity on the ledger and the question of its action

Events are records kept on a ledger. Four counters advance one step at a time, and each step is a *posting*. The readings of the four counters are the coordinates of a point, so the records the counters generate form a four-dimensional lattice, and that lattice carries an order before it carries any geometry [1, paper 4, Secs. 1–2]. One step of this identification is still to be proved: the correspondence of the four-account cell record with the three-account ledger of a single recognizer, which derives the lattice from the ledger [1, paper 4, Sec. 2], [5, Sec. 1]. Gravity is what the geometry of that lattice does. Give every edge a squared length and you have a *configuration*; an action weighs each configuration with a phase; and quantum gravity is the sum over configurations.

Two phrases in the title say what this sum has been shown to do, and both results are proved in earlier preprints by the author. *A finite sum over geometries*: on the lattice with $\kappa = 1$ (defined in Section 2), on a torus of fixed side N and over the geometries that keep every posting null, the regulated Lorentzian sum over Regge geometries exists and is finite on every finite window. Its normalization vanishes only on a discrete set of phase scales [2].

Einstein's equations at every order: the lattice's leading two-derivative field equation has fifteen components, one for each edge class, and ten of their combinations form a symmetric matrix, its metric part. For every $\kappa > \frac{1}{2}$ that metric part is Einstein's equation with its full nonlinear dependence on the metric [4, 5]. Here every order means every order in the field amplitude, at the leading order in the lattice spacing; it concerns the field equations, not lattice solutions. Its sourced linear field gives Newton's law with $G = 1/(8\pi s)$ [6, 5]. And its long-wave quadratic and cubic vertices are those of $\frac{1}{2}\int \sqrt{|g|}\, R$ [7]. Sections 3 and 4 state them precisely.

Both rest on one action, Regge's: each triangle of the lattice, or *hinge*, contributes its area times its deficit angle, the curvature the hinge carries. A reader is entitled to ask why this action. Any rule $C(A, \delta)$ that gives a hinge a share of the action depending on its area and deficit defines a lattice action of the same hinge-sum form. We call such a rule a *hinge charge*. Regge's is the charge $s\,A\delta$ with an action scale s, and most of the others are not Einstein's.

Section 5 answers with complete proofs, by two routes. The first is *gauge consistency*. Einstein's theory cannot tell apart two metrics that differ by a change of coordinates; on the lattice the matching move, a *vertex move*, displaces the lattice's points. Gauge consistency asks that two consequences of that blindness survive on the lattice. At quadratic order a vertex move costs no energy, alone or added to any other field. At cubic order, the cubic interaction of three waves vanishes when one of them is a vertex move and the other two obey the linearized field equation. We test both. Through cubic order, a charge built from one function of the area enters the action only through that function's jets. The jets are its value, slope and curvature at the four flat hinge areas. Among such charges the tests select Regge's charge, and do so through cubic order; they are run on the Euclidean member of the lattice family, with the cubic test posed on that member's own graviton shell (Section 5.4).

The ledger's own cost of comparing two records, the recognition cost, suggests a hinge charge of its own. That charge passes the quadratic test and fails the cubic test. Among ratio charges (defined in Section 5.1), the only correction to that charge that keeps its quadratic action and passes the cubic test brings its jets to Regge's. The correction is added by this paper and is

fixed by the gauge test, not derived from the recognition cost; the corrected charge is Regge's through cubic order and is no longer the cost's own (Section 5.8).

The second route is the ledger's *record clause*. It says that, seen from a hinge, one posting is a fixed fraction of a full turn around that hinge, and that the posting carries an action proportional to the hinge's area. By itself the clause fixes the charge at one deficit only. It selects Regge's charge exactly when the charge has either of two forms. The charge may depend on the hinge only through its content $A\delta$, the product of the hinge's area and deficit; then twice the area at half the deficit costs the same. Or it may charge each unit of deficit at a rate that depends on the area alone; then equal-area hinges pay the same per unit of deficit. The two routes meet: the first fixes the jets, and the second fixes the whole function and its scale.

Figure 1 draws the chain from the ledger to this paper's results, with its inputs. Sections 3 and 4 recall the results about Regge's action from the earlier preprints, and Sections 6 to 9 cover the graviton identity, the one-loop graviton, the inputs, and what is not claimed. A reader who wants the new argument first can go from Section 2 to Section 5, reading Section 4.3 before Section 5.6.

Every box of Fig. 1 is stated on one lattice: the fixed Kuhn graph, with real squared lengths and time running along **1**, where all four counters advance together. The boxes do not all range over the same configurations. The finite sum of Section 3 runs over the null stratum: at $\kappa = 1$ it holds the squared length of every posting edge at zero and integrates over the other eleven edge classes. The field equation of Section 4 varies all fifteen edge classes, and the setting's premise counts every small-amplitude field as a physical configuration. A uniform strain that changes the squared length of a posting is such a field. It lies outside the null stratum, and no periodic vertex move brings it back, because to first order a vertex move leaves the average squared length of every edge class unchanged. This paper does not prove, and cites no proof, that the sum over the null stratum is a gauge-fixed form of a sum over all fifteen edge classes. That reduction, with its Faddeev–Popov factor, is open, and until it is proved the finite sum and the field equation are results about two different sets of configurations.

The lattice has one parameter, κ, which fixes its flat metric (Section 2), and each result is stated at given values of it. The gauge tests behind the heavy box are run at $\kappa = 0$, the Euclidean member of the family. The record-clause selection concerns functions, and no value of κ enters it. The other results are stated on the Lorentzian lattice: the finite sum, the slope and the cubic vertex at $\kappa = 1$, and the field equation and Newton's law at every $\kappa > \frac{1}{2}$. What carries from $\kappa = 0$ to $\kappa = 1$ is the cubic vertex of Regge's action (Section 5.6), not the gauge test: the jets that the gauge tests fix are jets at the Euclidean flat areas, and two of the four flat areas of the $\kappa = 1$ lattice are not among them (Section 5.5).

Which value of κ describes gravity is an identification, and two readings of it are on record (Section 2). If one tick of the ledger is one posting, light's step is a posting, a posting is null, and $\kappa = 1$; this paper uses that reading. If one tick is a double entry, a debit with its credit on another account, light's step is a record of two postings, the edge with two postings is null, and $\kappa = \frac{1}{2}$; that is the working value of the law of motion on the same ledger [13]. Neither reading is derived, and this paper does not decide between them. Its theorems hold at the values they name whichever reading is right. At $\kappa = \frac{1}{2}$ the Lorentzian results listed above are

not established; Table 4 in Appendix D sets the paper's objects and results at the two values side by side.

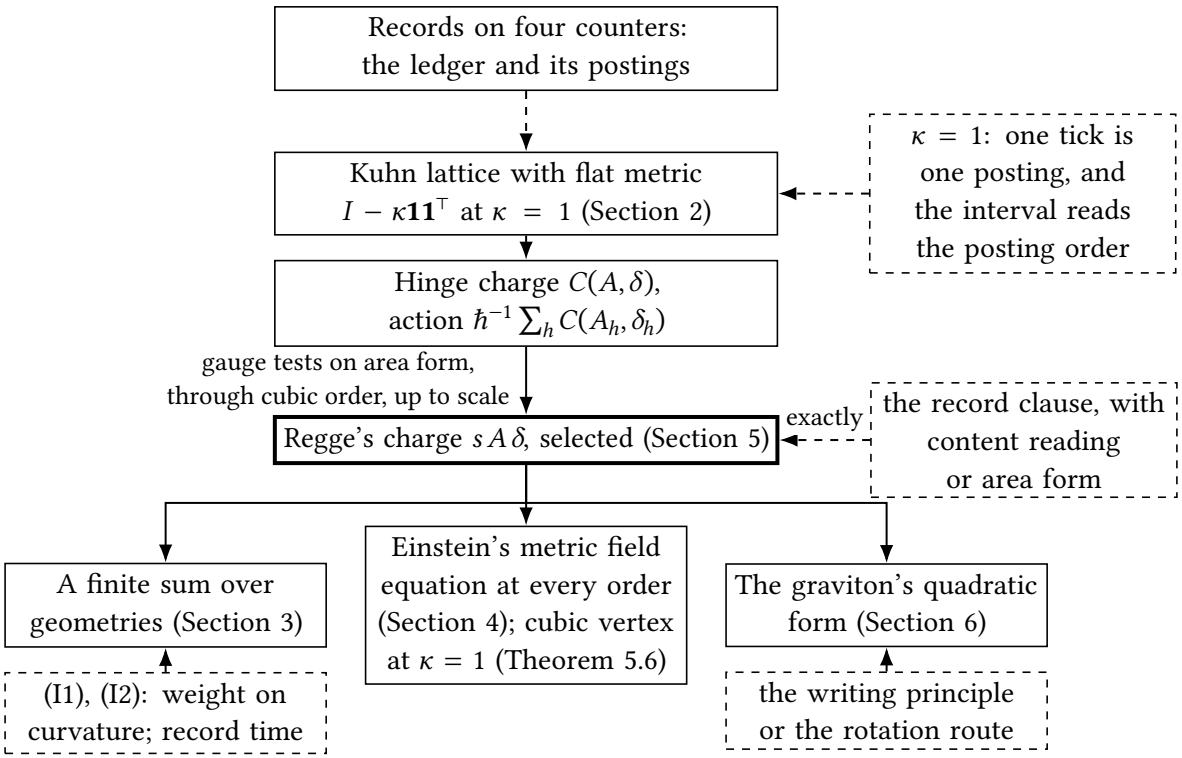

Figure 1. The chain of the paper. The top three boxes, the ledger, the lattice and the hinge charge, are the setting, with every small-amplitude field a physical configuration (a premise, Section 8). Below them, solid boxes and arrows are theorems, and the heavy box is proved in this paper (Section 5). Dashed means an identification or a premise whose derivation is open: dashed boxes are inputs, each named in Section 8, entering where their arrows point, and the dashed arrow from the ledger to the lattice is the identification of Section 1. The finite sum ranges over a smaller set of configurations than the setting's, the null stratum, and the reduction between the two is open (Section 1). The value $\kappa = 1$ is one of two readings on record; on the other, $\kappa = \frac{1}{2}$, the finite sum, the field equation and the cubic vertex are not established (Section 2).

How to read the status of a claim. A *theorem* is a proved statement with every hypothesis beside it. The theorems, lemmas, propositions and corollaries proved here have complete proofs in the paper. Where a proof is an exact computation on tables, the paper gives the tables' definitions, the witnesses and the equations that result; the tables themselves are not printed (see "Data availability" at the end). Results recalled from the cited papers carry the status those papers state. The author's earlier papers cited here, references [1] to [12], are preprints deposited on Zenodo (one of them also on Pith); none has appeared in a peer-reviewed journal. The law-of-motion paper [13] is a later preprint by the author, public on the institute's site and likewise not peer reviewed. Where this paper calls a result *published*, it means only that the result is public in one of those preprints. The quartic results of Section 5.10 are proved in a companion paper [12] or in work not yet published, and some one-loop results of Section 7 come from work not yet published; each is labelled where it occurs. Five more words mark what is not a theorem: NUMERICAL, a computation; INFERENCE, a reasoned step that is not one of the theorems; IDENTIFICATION, a physical reading that the definitions do not force

($\kappa = 1$ is one); MODEL, a choice of mathematical object; and OPEN, a precise statement that is neither proved nor refuted. Section 8 lists every claim of the paper with its status (Table 3) and every input with where it enters (Fig. 3). Two more words appear there: a *premise* is a stated assumption on which a result rests and which the paper does not derive, and a *definition* fixes a count or a name.

2 The lattice, the action and the record clause

Everything in this paper happens on one fixed lattice. The choice it examines most closely is the hinge charge: the rule that turns a hinge's area and deficit angle into an action. This section describes the lattice and its flat members, states the value $\kappa = 1$ at which this paper works and the second value on record, $\kappa = \frac{1}{2}$, defines the hinge charge, and states the record clause, the ledger's statement of what one posting on a hinge carries. A last paragraph lists four equivalent descriptions of $\kappa = 1$.

Records and the Kuhn lattice. Vertices are points x of the torus $(\mathbb{Z}/N)^4$, with $N \geq 3$ so that no edge joins a point to itself and no two edges join the same pair of points. The coordinates x_a, $a = 0, 1, 2, 3$, are the readings of the four counters, and e_a is one step of counter a: $e_0 = (1, 0, 0, 0)$, $e_1 = (0, 1, 0, 0)$, and so on. The torus is a periodic regulator. On it N steps of one counter return to the start, so it carries no order and no record time. The order that the postings give, and the record time that counts them, belong to the unwrapped lattice $\mathbb{Z}^4$, where x precedes y when postings lead from x to y. Advancing the four counters one at a time, each by one step, takes x to $x + \mathbf{1}$, $\mathbf{1} = (1, 1, 1, 1)$, and can be done in $4! = 24$ orders. Each order passes through three intermediate points between x and $x + \mathbf{1}$, so it visits five points in all, and these are the vertices of one of the 24 simplices of Kuhn's triangulation of the unit cube [21, 22]. Every edge joins x to $x + d$ for one of the fifteen edge classes $d \in \{0, 1\}^4 \setminus \{0\}$, and $|d|$, the number of ones, counts its postings. A single step of one counter is an edge with one posting. The edge $d = (1, 0, 1, 0)$ has two postings: it joins the start to the end of two steps, on the first and third counters, taken in either order. A configuration ℓ assigns a real squared length to every edge. The graph is fixed; we never sum over triangulations.

The flat lattices. The flat lattice of parameter κ gives the edge (x, d) the squared length

$$\ell_\kappa(x, d) = g_\kappa(d, d) = |d|\,(1 - \kappa|d|), \qquad g_\kappa = I - \kappa\,\mathbf{1}\mathbf{1}^\top, \qquad \det g_\kappa = 1 - 4\kappa.$$

For $\kappa < 1/4$ the metric is Euclidean; $\kappa = 0$ is the standard Kuhn lattice. For $\kappa > 1/4$ it is Lorentzian with time along $\mathbf{1}$, and for $\kappa > 1/2$ every triangle is timelike [5]. At $\kappa = 1$ every posting is null, that is, it has squared length zero. The four flat hinge areas are $\frac{1}{2}, 1, \frac{3}{2}, \sqrt{3}$ at $\kappa = 1$ [7] and $\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$ at $\kappa = 0$. In general a hinge whose two shorter edges have m and n postings has squared area $\frac{1}{4}mn\,|1 - \kappa(m + n)|$, with (m, n) one of $(1, 1)$, $(1, 2)$, $(1, 3)$ and $(2, 2)$: the two edges have squared lengths $m(1 - \kappa m)$ and $n(1 - \kappa n)$ and inner product $-\kappa mn$, so their Gram determinant is $mn\,(1 - \kappa(m + n))$. Section 5 evaluates a charge, with its slope and curvature, at the four flat areas.

The value $\kappa = 1$. This paper works at $\kappa = 1$. That value is an IDENTIFICATION, made on the transfer reading of light's lattice step (explained below): the definitions do not force it. The postings give the lattice an order before it has any geometry, and the lattice interval gives it light cones; the value $\kappa = 1$ is the one at which the interval reads the posting order. Equivalently, it is the least value at which every pair of events that the postings put in order is timelike or null and future-directed [5, Thm. 2.2]. A single posting shows what that asks. A step of one counter has squared length $1 - \kappa$, which is positive for $\kappa < 1$: the interval would then call two events spacelike that the postings put one after the other. At $\kappa = 1$ every posting is null, and every combination of postings is null or timelike. The ledger's foundation, as the cited paper states it, admits worlds at $\kappa = 1$ and at $\kappa = 3/2$ with the same premises [5, Thm. 2.3], so the value is an input.

What one tick is, and the value $\kappa = \frac{1}{2}$. The argument just given asks the interval to read the order of single postings: no single posting may outrun light. That is a premise about what one tick of the ledger times, and nothing derives it. Two readings are on record.

One tick is one posting. Then light's step is a posting, a posting is null, and $\kappa = 1$, as above. This is the reading this paper uses.

One tick is a double entry, a debit with its credit on another account: a record of two postings on two accounts. Then light's step is that record. The edge with two postings has squared length $2(1 - 2\kappa)$, so it is null at $\kappa = \frac{1}{2}$. A single posting then has squared length $\frac{1}{2}$: taken alone it is spacelike and outruns light, and the bound is asked of the two-posting record instead. This is the working value of the law of motion on the same ledger [13, Sec. 3.3]: every law completed there at nearest reach carries light at $\kappa = \frac{1}{2}$, and that paper concludes that $\kappa = 1$ would have to be derived to return.

Which of the two a tick is, is not settled, and neither reading is derived (OPEN). This paper does not choose. Its theorems are statements about the lattice at the values of κ they name, and they hold whichever reading is right; what the choice decides is which of them describe gravity. At $\kappa = \frac{1}{2}$ this paper's results on the Lorentzian lattice are not established, apart from the long-wave quadratic form, which Section 5.10 reports at every κ with $4\kappa - 1 \neq 0$. The field equation and Newton's law are proved for $\kappa > \frac{1}{2}$, and the finite sum at $\kappa = 1$. At $\kappa = \frac{1}{2}$ the hinge of two single postings has no area, and the table behind the cubic vertex has no value (Theorem 5.6(b)). Table 4 in Appendix D lists this paper's objects and results at the two values.

Which reading of light's step. Light's lattice step is the rule that carries the light field from one tick to the next. It weighs the field's change between ticks with a time-face weight and its change between neighbouring sites with a space-face weight; their ratio r sets the speed of a long wave, $\sqrt{r}$ sites per tick.

The step can be read in two ways. On the *real-time reading*, the step is an update computed tick by tick: a leapfrog rule that gives the field at the next tick from its values at nearby sites at the two ticks before, so that a disturbance moves at most a fixed distance per tick. Such a rule is stable only for small ratios, below $r = \frac{1}{3}$ (Appendix D). On the *transfer reading*, the

reading this paper uses, the step is the transfer kernel of the lattice field: the weight that the sum over the field's histories gives to the passage from one tick to the next. That kernel multiplies each wave by a fixed factor per tick, and the factor fixes the wave's frequency; it is a property of the sum, not an update computed site by site, and it gives every wave a real frequency at every ratio $r > 0$.

The choice matters because the ratio is tied to κ: the lattice is taken to fix r, with $r = 1$ at $\kappa = 1$, and the real-time bound $r < \frac{1}{3}$ corresponds to $\kappa < \frac{1}{2}$. So on the transfer reading the value $\kappa = 1$ is allowed, and on the real-time reading it is not, which would leave only $\kappa < \frac{1}{2}$, outside the range $\kappa > \frac{1}{2}$ that Sections 3 and 4 assume. The transfer reading's frequencies, moreover, belong to no two-step update of finite reach: for polynomial laws this is proved in work not yet published, and for laws with a real symbol it rests on an argument (Appendix D). Besides the reading of the tick, two assumptions carry the value $\kappa = 1$, and each is an IDENTIFICATION: reading r as a value of κ, the *step identification*, and taking the transfer reading, the *light identification*. Whether the ledger forces the transfer reading is open (OPEN).

Hinge charges. The hinges are the triangles. A hinge h of a configuration near a flat lattice has an area $A_h(\ell)$ and a deficit angle $\delta_h(\ell)$, which is 2π minus the sum of the dihedral angles of the simplices that share it. The deficit angle is the curvature the hinge carries, and every flat lattice has $\delta_h = 0$.

Two special forms of charge will matter. A charge in area form charges each unit of deficit at a rate that depends only on the hinge's area. A content-reading charge depends on area and deficit only through their product $A\delta$, the hinge's content. Regge's charge is both.

Definition 2.1 (Hinge charge)**.** A hinge charge, or *area-and-deficit charge*, is a function $C : \mathbb{R}^2 \to \mathbb{R}$, $(A, \delta) \mapsto C(A, \delta)$. Its action is $\Phi_C(\ell) = \hbar^{-1} \sum_h C(A_h(\ell), \delta_h(\ell))$. Regge's charge is $C_{\mathrm{R}}(A, \delta) = s\, A\, \delta$ with a positive action scale s; its action is $(s/\hbar)$ times the Regge action $S_{\mathrm{R}} = \sum_h A_h \delta_h$ [23]. A charge is in *area form* if $C(A, \delta) = \varphi(A)\, \delta$ for one function φ, and it is *content-reading* if $C(A, \delta) = F(A\delta)$ for one function F.

Only positive areas occur on the lattice; a charge is defined on all of $\mathbb{R}^2$ so that the statements below are about functions.

The record clause. A posting is one step of one counter. The record clause says that, seen from a hinge, a posting is an equal share of a turn around that hinge: with $k \geq 1$ postings per turn, one posting is one k-th of the turn, a deficit $2\pi/k$. The lattice's unit of length is the recognition length λ_{rec}. The clause makes one assignment: a full turn of k postings on a face of area λ_{rec}^2 carries one bit, the action $\hbar \ln 2$. That fixes the scale of the action. For instance, with $k = 12$, the value the lattice takes below, one posting on such a face carries a twelfth of a bit, $\hbar \ln 2/12$. On a hinge of area A a posting carries $A/\lambda_{\mathrm{rec}}^2$ times as much. In general one posting is a k-th of that turn, and the action is proportional to the area, so one posting on a hinge of area A carries the *posting action*:

$$a(A) = s_{\mathrm{p}}\, A\, \frac{2\pi}{k}, \qquad s_{\mathrm{p}} = \frac{\hbar \ln 2}{2\pi \lambda_{\mathrm{rec}}^2}.$$

This is [9, Thm. 7.2], stated there at $k = 12$; the argument is the same for every k. On the lattice we take $k = 12$: a posting is a twelfth of a full turn, a deficit $\pi/6$. That fits the $\kappa = 1$ lattice's flat dihedral angles, which are multiples of $\pi/6$ [9, Sec. 6.2]; the choice is part of the record clause's identification.

Definition 2.2 (The record clause)**.** A hinge charge meets the record clause at k if one posting on a hinge of area A carries the posting action at every area: $C(A, 2\pi/k) = s_{\mathrm{p}}\, A\,(2\pi/k)$ for every A.

The record clause ties Newton's strength to the recognition length. If the action $s_{\mathrm{p}} \sum_h A_h \delta_h$ is read as Einstein–Hilbert's, which is an INFERENCE, then $G\hbar = \lambda_{\mathrm{rec}}^2/(4 \ln 2)$ [9, Thm. 7.2], so the clause fixes the length λ_{rec} in terms of G and does not predict G [8]. Here G is the coupling of the lattice action. A laboratory measures the long-wave coupling ZG, where Z is a positive factor that the quantum path sum fixes and that is not yet computed. So a measured value fixes G and λ_{rec} only up to Z [8, Sec. 2]. The record clause is an IDENTIFICATION.

Negative areas. The clause is read physically only on positive areas. The cited form of the clause, [9, Prop. 7.3], is stated on positive areas and adds a statement for credits: a posting adds one to a counter, and a credit takes one away, so it carries the opposite deficit and the opposite action. Definition 2.2 asks the clause at every real area. At negative areas the clause does the work of the credit statement: for a charge that depends only on the content $A\delta$, reversing the area and reversing the deficit have the same effect. On content-reading charges the two forms therefore agree, and Theorem 5.5(a) uses negative areas to fix the function F at negative arguments.

Four descriptions of $\kappa = 1$. The value $\kappa = 1$ has four equivalent descriptions [5, Thm. 2.2]: one for the lattice's spectrum (i), two for its causal order (ii) and (iii), and one for the number itself (iv); the example above is description (iii). The spectral description (i) belongs to the setup because the corner doublers it names are where the one-loop problem of Section 7 starts. For $\kappa > 1/2$, with a fixed positive time advanced by each posting,

(i) the lattice has a massless corner doubler: an extra massless mode at a corner of the momentum cell, where the wave number is $\pm\pi$ along one axis, the shortest wavelength the lattice carries; in operator terms, it is a second zero of the gauge-fixed operator besides the graviton at the centre;

(ii) the lattice interval reads the posting order on some pair of positive duration. Here an event is any real time and position, not only a record of the lattice, and one event precedes another when the passage between them spends a nonnegative amount on each of the four counters; this is the closure of the posting order. The interval reads the order on a pair when the events it puts causally between the two are exactly the events the order puts between them;

(iii) κ is the least value at which every pair in the posting order has future-causal content, meaning that the displacement the postings make between the pair is timelike or null and future-directed for the interval;

(iv) $\kappa = 1$.

For $\kappa \geq 1$ these are also equivalent to (v): some history of postings is null.

Description (ii) needs the continuum of events. Among the lattice's records alone it does not single out $\kappa = 1$. Between a record and its successor by one posting the interval puts no third record at any $\kappa \geq 1$. Between a record and its successor by two postings on different counters it puts exactly the two records that the postings pass through, at every κ from 1 up to, but not including, 3. Nor does the interval read the order on every pair: at $\kappa = 1$ the record $(2, 2, 2, -1)$ lies causally between 0 and $(3, 3, 3, 0)$, and no posting reaches it from 0.

Graviton doubling on a lattice is not new: Menotti and Pelissetto found it in the flat-background expansion of Euclidean lattice gravity built from vierbeins and Lorentz links [18, 20]. What is particular to the Kuhn lattice with Regge's action is that its corner doublers occur at exactly one value of κ, the value at which the interval reads the posting order [5, Thms. 2.1 and 2.2].

3 A finite sum over geometries

The sum over geometries exists, at a stated regulator. On every finite window of squared edge lengths it is a finite-dimensional, absolutely convergent oscillatory integral. Its normalization is an entire function of the phase scale and vanishes only on a discrete set, so expectation values of bounded observables are defined at every other phase scale. The window is a regulator, stated and not derived. This section recalls that result and then what is known about the sum's evolution in record time, which stops short of a unitary evolution.

Status: the finite sum and its record time (published as preprints)

Proved [2]. The admissible geometries, the Regge action and its first variation, a model measure, and finiteness at a stated regulator, at $\kappa = 1$ on the null stratum.

Model and conditional in that paper. The fixed Kuhn graph with real squared lengths; the measure and its flat bare weight; the regulator; the Lorentzian sign, given a causal reading of the tick order.

Proved [3]. With the derived weight, the ledger's own weight on curvature ((I1) and (I2) below), the growth per unit physical time read from the roots of the Gaussian path sum vanishes as the spacing shrinks, at every fixed nonzero weight strength: continuum root stability in record time (the end of this section explains it) [3, Thm. 6.9]. That the derived weight has a nonzero strength is proved in [3, Thm. 6.2].

Open. The uncut sum; a derived regulator; the reduction of a sum over all fifteen edge classes to the null stratum, a gauge slice with its Faddeev–Popov factor; a positive physical Hilbert space and a norm-preserving record-time evolution (Section 9 recalls a later report that bears on it).

What is summed. The fixed graph is the Kuhn lattice on a torus of side N, with edge set E_N of $15N^4$ edges [2]. A geometry is a squared length on every edge, a point of $\mathbb{R}^{E_N}$, and not every assignment counts: the admissible geometries are those in which every simplex is a nondegenerate Lorentzian simplex with the causal character of the lattice. For $\kappa > \frac{1}{2}$, $\kappa \neq 1$, they form an open semialgebraic cone, with an explicit ball around the flat lattice, and every admissible simplex has signature $(3, 1)$ [2, Thms. 2.2–2.5]. At $\kappa = 1$ every posting edge is null,

which confines the geometries to the linear subspace V_0 on which every posting edge has squared length zero. The admissible set is then the null stratum Adm_1, a cone relatively open in V_0 [2, Thm. 5.13(iv)]. So at $\kappa = 1$ the sum integrates over eleven of the fifteen edge classes and holds the four posting classes fixed. The field equations of Section 4 vary all fifteen, and the paper does not derive the one setting from the other (Section 1).

The phase and the measure. Each geometry enters the sum with a phase, set by its action, and a measure on squared lengths says how geometries are counted. The ledger's derived weight, an extra positive factor, is a third ingredient; it enters under Record time below, through (I1) and (I2). The action is Regge's. It is smooth and homogeneous of degree one on the cone, and by Schläfli's identity its first variation is the sum over hinges of the deficit times the variation of the area, $dS_{\mathrm{R}} = \sum_h \delta_h \, dA_h$ [2, Thms. 3.1–3.3]. Its Lorentzian sign is conditional: the tick order fixes the time axis, and the sign enters through a causal reading of the cell [2, Thm. 3.4].

The measure is a model measure. One choice is Lebesgue measure on the cone (Haar measure of V_0 at $\kappa = 1$), which part (c) of the theorem below uses. The other is the push-forward of the ledger's states. A state says how the ledger's accounts move, one account for each counter and a finite family of further accounts. A chart reads a geometry from each state. Restrict the states to a box of given radius, and give each state in it a bare weight, flat or the ledger's own. The measure of a set of admissible geometries is then the total weight of the states that read as one of them. With the flat weight it is the volume of those states. This measure changes with the account family, the box radius and the bare weight [2, Sec. 4]. The bare weight is part of the model measure, not the derived weight.

The theorem has three parts. Part (a) is general: it needs only a finite measure on which the action is bounded. Part (b) is the bound on a window of squared lengths. Part (c) is the result for the lattice at $\kappa = 1$, where the bound of (b) applies.

Theorem 3.1 (The sum is finite on every window; [2])**.**

(a) For a nonzero finite measure μ on which $|S| \le M$, the normalization $Z_\mu(a) = \int e^{iaS} \, d\mu$ is an entire function of the phase scale a with $Z_\mu(0) = \mu(\mathbb{R}^{E_N}) > 0$; its real zeros form a closed discrete set, and off that set $\langle O \rangle_\mu(a)$ is defined for every bounded measurable observable [2, Thm. 5.4].

(b) On a window $|\ell_e| \le L$ the action is bounded, $|S| \le C_N L$ with $C_N = 120\sqrt{13}\,\pi N^4$ [2, Thm. 5.1].

(c) At $\kappa = 1$, Haar measure of V_0 restricted to $\mathrm{Adm}_1 \cap \{|\ell_e| \le L\}$, for every L above the size of the flat field, is finite and nonzero, the weight and bounded observables are integrable, and $Z \neq 0$, $\langle 1 \rangle = 1$ throughout $|a| \, C_N L < \pi/2$ [2, Thm. 5.13].

Without the window the sum is not absolutely integrable, on the open cone or on the null stratum [2, Thms. 5.6 and 5.13(iv)].

Two remarks keep the theorem at its size. First, part (a) is immediate once the measure is finite and the action bounded: the integrand has absolute value one, and the power series of e^{iaS} converges under the integral. What is specific to the lattice is in (b) and (c): the bound on

the action up to the boundary of the null stratum, and the fact that the window cuts out a set of finite, nonzero volume in V_0. Second, the theorem gives no control as the window or the lattice grows. The interval of phase scales on which it guarantees $Z \neq 0$, $|a| < \pi/(2C_N L)$, shrinks to zero as L or N increases, since C_N grows as N^4.

Record time. The record count of the four counters is a time that exists before geometry is assigned, so one can ask how the sum evolves in it. At Gaussian order, a norm preserved by a record tick requires every physical exponential solution to change, from one tick to the next, by a unimodular factor, a number of absolute value one. The bare Regge equation fails this: at $\kappa = 1$ a spatially uniform physical solution grows by $\sqrt{2}$ per tick [3].

The third ingredient, the derived weight, changes the picture. It is a weight on geometries built from the ledger's cost of recording curvature, and it rests on two identifications, set out in Section 8.

The first, (I1), says what that cost is. The curvature of a hinge is its deficit, and the ledger pays to record it. Take one face of area λ_{rec}^2 on the hinge. The record on it is a *cycle*, a closed loop of channels around the hinge, one channel for each four-simplex that contains the hinge; the number n_h of channels is four or six on the Kuhn lattice, a count fixed by definition. The hinge's deficit is shared among the n_h channels, so that for a small deficit each carries about its n_h-th part, and it goes onto the cycle with no extra factor, which is what *unit coupling* means. The ledger charges each channel its recognition cost for carrying its share, so for a small deficit one cycle costs $\delta_h^2/(2n_h)$ at leading order.

(I1) counts the cost as the action counts area, which means three things. It adds over the pieces of a hinge. It never decreases as the area grows. And each face of area λ_{rec}^2 carries one such cycle, so a hinge of area A carries $A/\lambda_{\mathrm{rec}}^2$ cycles. The costs of the hinges add to the cost of a geometry.

The second, (I2), turns that cost into a weight: the Born weight of a geometry is the Gibbs weight of its ledger cost, so a geometry whose records cost more weighs less.

With this weight, the continuum limit and the lattice at fixed spacing behave differently. In the continuum limit, at Gaussian order, the growth per unit physical time vanishes as the spacing shrinks, a statement about the roots of a Gaussian equation. A mode's record factor is its factor per tick. The statement holds at fixed weight strength for every non-gauge Gaussian mode family whose record factor approaches the unit circle [3].

At fixed lattice spacing, on the lattice itself, the weighted equation is not record-time stable. This failure of stability holds at every spatial period at which the weighted equation has a physical exponential solution away from the zero wave, the spatially uniform mode [3, Thm. 6.9].

The estimates behind the continuum statement are at fixed coupling and fixed physical momentum; they neither prove a nonempty physical graviton branch nor justify integrating over all momenta. They construct no positive physical Hilbert space and no interacting unitary evolution [3].

4 Einstein's equations at every order

Does the lattice's field equation reproduce Einstein's? At leading order in the lattice spacing its metric part does, with every nonlinear term in the field. Add a source and the linear field gives Newton's law; around flat space, the quadratic and cubic vertices are those of $\frac{1}{2}\int \sqrt{|g|}\, R$. All of this is published in the preprints cited below. Section 5 then asks why the action has this form.

Status: Einstein's equations (published as preprints)

Proved. The metric field equation at every order in the field [4, 5]; the sourced linear field and Newton's law with $G = 1/(8\pi s)$ [6, 5]; Regge's slope and the Einstein–Hilbert quadratic and cubic vertices from flat-space stationarity [7].

Meaning of "every order". Every order in the field amplitude, for every small-amplitude field, which the setting takes to be a physical configuration (a premise, Section 8), at the leading order of the expansion in the lattice spacing. It is a statement about the field equations, not about lattice solutions.

Open. Lattice solutions at finite strength converging to solutions of the full Einstein equations.

4.1 The field equation at every order

Sample a smooth metric onto the edges of the Kuhn lattice at lattice spacing b: around a point the field has a value g, first derivatives P and second derivatives Q, and each edge reads the field at its midpoint. The lattice field equations, the derivatives of the Regge action along the fifteen edge classes, have an expansion in b; let Φ_d be the coefficient of order b^2 of the equation of class d. The fifteen equations, one for each edge class, are more than the ten components of a symmetric 4×4 matrix. Weighting the equation of class d by $d\,d^{\top}$ and adding gives such a matrix, their *metric part* $\sum_d d\,d^{\top}\Phi_d$; five combinations of the equations do not enter it and are treated separately below. The theorem compares this matrix with Einstein's tensor of the same field.

Theorem 4.1 (Einstein's metric field equation at every order; [4], [5, Thm. 4.2])**.** *Let $\kappa > 1/2$ and let g be a symmetric metric such that every metric on the segment from g_κ to g is Lorentzian with nondegenerate flat Kuhn simplices. For all symmetric first and second derivatives P and Q, the metric part of the order-b^2 coefficient of the lattice field equations on the sampled field equals $-\frac{1}{2}$ times the Einstein density $\sqrt{|\det g|}\, G^{\mu\nu}$ of the 2-jet (g, P, Q), with every nonlinear term included.*

The identity follows from the linear identity at each constant background and the vanishing of both sides on jets of flat coordinate changes [4]. Its factor $-\frac{1}{2}$ is the one that $\frac{1}{2}\int \sqrt{|g|}\, R$ produces, since $\delta \int \sqrt{|g|}R/\delta g_{\mu\nu} = -\sqrt{|g|}\, G^{\mu\nu}$. A further extension to every regular background is proved in [4].

The theorem covers the metric part of the lattice equations, ten of their fifteen combinations. It is a statement about a smooth metric sampled onto all fifteen edge classes, each of them varied freely: the ten combinations are projections of fifteen unconstrained equations. It says nothing about variations confined to the null stratum of Section 3, on which the four

posting classes are held fixed. The other five combinations are not Einstein's. Near flat space a correction in the five non-metric combinations of edge classes can solve them at this order, the leading order in the field amplitude. This is proved in [4, Prop. 4.1]. Beyond that order the five equations are not eliminated here. On a sampled vacuum metric three of the five non-metric combinations are driven by the Weyl curvature, which is why a sampled metric needs that correction (NUMERICAL in [5]).

The theorem identifies the long-wave differential operator. It does not construct lattice solutions, and one attempt shows that this is a separate question. Two expansion parameters are in play, the field amplitude t and the lattice spacing b, and two corrections. The non-metric correction adjusts the edges at first order in t, so that a sampled field solves the linearized equations; the second-order correction d sought below would extend that solution to second order in t.

The attempt. Fix one record y and its thirty edges, the edges of the fifteen classes that start at y and those that end at y. Take, at $\kappa = 1$, the weak field h of a static cubic potential. Here h is a vacuum field given by a polynomial, so that every lattice sum below is exact. Sample it onto the edges together with a specified non-metric correction of the kind described above. The correction lies in the non-metric combinations of edge classes. It solves the linearized lattice equations on those thirty edges. Look for the second-order correction, an edge field d such that $\ell_0 + th + \frac{1}{2}t^2 d$ solves the full lattice equations on the same edges through order t^2.

The failure. No such d exists [5, Thm. 4.3]. The vertex moves are the reason. They cost nothing at quadratic order, alone or added to any other field, so the second-order equations can be solved only if their source pairs to zero with every vertex move. That source is the third variation of the action on h and h, and at y it does not pair to zero with every vertex move. At second order in t, then, the vertex moves stop being a symmetry of the lattice equations. The cubic Taylor fields of a point mass's potential fail in the same way [5, Sec. 4].

Where the defect sits in b. Einstein's equations carry two derivatives of the metric. The defect is a lattice artifact one order further, with one more derivative and so one more power of b; at the order of Einstein's equations the same pairing vanishes in vacuum (an inference in [5], where the expansion in the spacing is not written out).

Earlier work. Feinberg, Friedberg, Lee and Ren proved that lattice gravity approaches the continuum theory as the links shrink, on regular and irregular lattices, and gave the deviation as a power series in the squared link length [28]; Theorem 4.1 is the leading term of such an expansion on the Kuhn lattice, with every nonlinear term in the field. Brewin's continuous-time formulation develops the Bianchi identities of Regge calculus and uses them to show that the evolution equations conserve the time constraint [29]; Gentle reviews Regge's equations as they are used in numerical relativity, with their strengths and weaknesses [30]. The failure found above, that vertex moves stop being a symmetry once the lattice is curved, is known in general: Bahr and Dittrich showed that the vertex-displacement symmetries of flat Regge solutions are broken on solutions with curvature [31], and studied improved and perfect actions, which keep them [32].

4.2 Where solutions exist

Solutions do exist, in three settings, which ask more and more of the matter. A linear response to a source gives Newton's law. The full equations with a model medium have exact solutions for small coupling, with the medium supplying their stability. And worldline matter replaces the medium by matter with an equation of its own, at the price of solving the equations only through second order in the field amplitude, on a band of long waves.

The sourced linear field and Newton's law. The weak-field limit of Regge calculus on hypercubic lattices is linearized gravity [15, 16]. On the Kuhn lattice, take $\kappa > 1/2$ and a stress family continuous at the origin of wave-number space and conserved on a non-null cone of waves (a source that obeys energy and momentum conservation). The linear lattice equations $s\,B_\kappa h = -\tau$ have the de Donder solution for small waves, and $q_\kappa(u)\,h_{\mu\nu}(u) \to 16\pi G\big(T_{\mu\nu}(0) - \frac{1}{2}g_{\mu\nu}T(0)\big)$ as $u \to 0$ within the cone, with $G = 1/(8\pi s)$ [6, 5].

The full equations with a model medium. At $\kappa = 1$, on every torus of odd side not divisible by 105, the full Regge equations on all fifteen edge classes, with a resting body and a model medium, have exact solutions for small coupling. On such a torus the flat Hessian has no zero modes besides constant strains and vertex moves (on even sides further zero modes appear, the doublers among them; 105 is a limit of the proof, not a known failure) [6, Thm. 4.2]. The solutions lie in the Lorentzian chamber and are unique near their linear approximation. Their weak-field long-wave limit is the linearized Einstein field, and for a compact body, outside the body, $-\frac{1}{2}h_{tt} = -GM/R$ plus an explicit finite-size remainder [6, 5]. In these solutions the model medium, a model matter field, supplies their stability; it has no equation of its own.

Worldline matter. At second order in the field amplitude the model medium can be replaced by matter that has an equation of its own, *worldline matter*. Its body is a static set of resting *matter postings*, with finitely many Fourier modes; matter postings are point particles counted on edges, not the single-counter steps of Section 2. A *carrier* of further matter postings holds the body up (Appendix C describes it), and the construction solves the lattice equations of a *band*, the set of static waves of long wavelength that it keeps.

The result holds for every $\kappa > \frac{1}{2}$, at small strength, and for bodies whose modes avoid six lines of momentum space (the blind lines of Appendix C: static modes supported on only two coordinate axes, on which worldline matter cannot act on every lattice equation). Then, on every large enough torus, the coupled lattice problem has a solution of the equations of a fixed long-wave band. The first two coefficients of that solution converge to fields that solve Einstein's equation with the matter's own stress. They solve it through second order in the amplitude, up to an error of third order [4, Thm. 5.1].

The solution is not an exact solution of all the lattice equations. The obstruction above concerns exact solutions of all the lattice equations, those along vertex moves included. This result is not affected by it: it imposes the equations of a long-wave band, or lets matter carry

the load along the vertex moves [5, Sec. 4]. Appendix C gives the construction, the result in full, its five premises (three identifications and two models) and what is not proved.

4.3 Flat-space stationarity and Regge's slope

One more published result fixes the charge's slope with respect to the deficit, a different derivative from the area derivatives that make up the jets of Section 5.1. It asks only that flat space stay a vacuum when the lattice is stretched in time. Take a hinge charge C of class C^3 with $C(A,0) = 0$ and write $\sigma(A) = \partial_\delta C(A,0)$. Part (a) of the theorem says that the slope is Regge's. Part (b) says that, on long waves, the second derivative of the charge's action is a fixed multiple of the Fierz–Pauli form; part (c) says that the third derivative is a fixed multiple of Einstein's cubic vertex. Both multiples are proportional to the same constant c.

Theorem 4.2 (Regge's slope and the long-wave vertices; [7, Thm. 3.1])**.** *Suppose that for every $\kappa > 1$ the flat lattice ℓ_κ on the torus of side 3 is a stationary point of Φ_C, and put $c = \sigma(1)/\hbar$. Then:*

(a) $\sigma(A) = \sigma(1)A$ *for every* $A \geq 1/2$*;*

(b) on midpoint-sampled cosine waves around the flat $\kappa = 1$ *lattice, for every nonzero wave, the same on both legs,*

$$N^{-2} D^2 \Phi_C \to c\,(-\sqrt{3}\pi^2/2)\,\mathrm{FP}_{g_1};$$

(c) on the same waves and for every momentum-conserving triple of nonzero waves,

$$N^{-2} D^3 \Phi_C \to c\,(\pi^2/2)\,V_{g_1}.$$

Here FP *is the Fierz–Pauli form and V_g the cubic vertex of Einstein's first-order Lagrangian $\sqrt{|\det G|}\, G^{ab}(\Gamma^c_{ad}\Gamma^d_{bc} - \Gamma^c_{ab}\Gamma^d_{cd})$, volume factor included.*

Both constants are those of one action: through cubic order on plane waves around the flat $\kappa = 1$ lattice, the action of the charge is $(\sigma(1)/\hbar)\cdot\frac{1}{2}\int\sqrt{|g|}\,R\,d^4y$ in lattice units [7, Rem. 3.2]. Reading the physical action $\frac{1}{2}\sigma(1)\int\sqrt{|g|}R$ as $(16\pi G)^{-1}\int\sqrt{|g|}R$ is an INFERENCE. It gives $G = 1/(8\pi\sigma(1))$, and for Regge's charge $G = 1/(8\pi s)$, the value Newton's law gives on the same lattice. The stationarity premise fixes neither the size of $\sigma(1)$ nor its sign: $\sigma(1) > 0$, the sign of attraction, is a separate premise [7, Rem. 3.2]; for Regge's charge $s > 0$ by Definition 2.1.

Given $C(A,0) = 0$, stationarity at every stretched lattice is equivalent to Regge's slope on the hinge areas that the stretched lattices have [7, Thm. 5.1]. The theorem leaves free the part of the charge that is even in the deficit, such as a term proportional to δ^2, and every higher deficit derivative. How much is left depends on the charge's form (Definition 2.1). For a charge in area form, $C = \varphi(A)\,\delta$ with φ of class C^3, the slope is the function itself, $\sigma = \varphi$, and there is no even part and no higher deficit derivative. Under the theorem's premise, then, $\varphi(A) = \varphi(1)\,A$ at every area $A \geq \frac{1}{2}$: the charge is already Regge's on every area from $\frac{1}{2}$ up, to its scale and sign, which the premise does not fix. That range holds the four flat areas of the Euclidean lattice and the four of the $\kappa = 1$ lattice; below $\frac{1}{2}$ the premise says

nothing. For a content-reading charge, $C = F(A\delta)$ with F of class C^3 and $F(0) = 0$, the slope is $\sigma(A) = F'(0)\,A$ whatever F is: every such charge has Regge's slope and meets the premise, and the theorem constrains F no further.

Section 5 asks what fixes a charge without the stationarity premise, which it does not use. Its gauge route asks only the two gauge tests, on the one Euclidean lattice, and its record route asks the record clause; Section 5.5 sets the three premises side by side.

5 Regge's action is selected, not chosen

The lattice does not fix its own action. Any hinge charge gives a lattice action, and Sections 3 and 4 used Regge's. This section shows what singles Regge's out, in three steps (Fig. 2 summarizes them), and then tests the result against four neighbouring cases.

1. *Read a charge through its jets* (Section 5.1). Up to cubic order, a charge in area form enters the action only through a few numbers at the flat hinge areas, its *jets*.
2. *Test for gauge consistency* (Sections 5.2 to 5.4). Among charges built from one function of the area, only Regge's jets pass the quadratic and cubic tests. A *witness*, in the cubic test, is an explicit choice of waves at which the cubic vertex is evaluated; the box below gives the scope.
3. *Close the gap exactly, with the record clause* (Section 5.5). The tests see only jets, so they fix the charge through cubic order, up to its scale, and no further. The record clause fixes the charge exactly, scale included, when the charge is read by its content $A\delta$ or by one function of the area.

The main route ends at Section 5.5, and Section 5.6 carries its cubic vertex to the lattice $\kappa = 1$. The other subsections test its reach, each by one question. Section 5.7 asks what the tests miss when each class of hinge has its own function. Section 5.8 asks whether a natural rival, the charge of the recognition cost (the ledger's cost of comparing two records), passes them. Section 5.9 asks which direction they leave free: a change of the coupling passes them to first order and is not an area-and-deficit charge. Section 5.10 asks what happens one order further, for the lattice's quartic table and for a changed coupling.

Several proofs are exact computations on explicit tables; a first reading can follow the argument from the statements and the paragraphs between them.

Status: the selection of Regge's charge (proved here)

Proved here. Every theorem, lemma, proposition and corollary of this section, with the proof given; where a proof is an exact computation, it is a computation on tables that the paper defines and does not print (see "Data availability").

Scope. The gauge tests are proved on the Euclidean member $\kappa = 0$ of the lattice family, at the exact quadratic level and at three explicit cubic witnesses. Among charges built from one function of the area for every hinge they select Regge's jets, up to a common scale λ, and so Regge's charge *through cubic order*. The three witnesses sit on the Euclidean graviton's shell; for a charge whose quadratic form has another shell the cubic test is a requirement posed on the Euclidean shell, not that charge's own solvability condition (Section 5.4). The jets fixed are

jets at the Euclidean flat areas; the stationarity premise of Section 4.3, which this section does not use, reaches further within area form (Section 5.5). Broader classes of charges are taken up in Sections 5.7 to 5.9. The $\kappa = 1$ statement proved here is the cubic vertex (Theorem 5.6). The record-clause route (Theorem 5.5) does not depend on κ and is exact at all orders.

Open. The cubic gauge test on the $\kappa = 1$ lattice itself; the cubic test of a charge with $u_3 \neq 0$ on its own shell; charges whose factor depends on more than the hinge's own three edges; a selection by the gauge tests beyond cubic order.

5.1 Hinge charges are read through their jets

A charge in area form, $C = \varphi(A)\,\delta$, enters the action around a flat lattice through the values of φ and its first two derivatives at the flat hinge areas, and through nothing else up to cubic order. Write a perturbation of the squared lengths as $\ell_0 + t\ell$ and expand $A = a + tA' + \frac{1}{2}t^2A'' + \dots$ and $\delta = t\delta' + \frac{1}{2}t^2\delta'' + \frac{1}{6}t^3\delta''' + \dots$, using $\delta = 0$ on the flat lattice. Then

$$\frac{d^2}{dt^2}\big[\varphi(A)\delta\big]_{t=0} = \varphi(a)\,\delta'' + 2\varphi'(a)\,A'\delta',$$
$$\frac{d^3}{dt^3}\big[\varphi(A)\delta\big]_{t=0} = \varphi(a)\,\delta''' + 3\varphi'(a)\big(A'\delta'' + A''\delta'\big) + 3\varphi''(a)\,A'^2\delta'.$$

Every term carries a derivative of δ, because $\delta = 0$ on the flat lattice, and φ enters only through $\varphi(a)$, $\varphi'(a)$ and $\varphi''(a)$. We call $(p_0, p_1, p_2) = (\varphi(a)/a,\ \varphi'(a),\ a\varphi''(a))$ the *jets* of the charge at the flat area a: three numbers that measure its value, slope and curvature there. They are scaled so that Regge's charge $\lambda A\delta$, which is linear in the area, has jets $(\lambda, \lambda, 0)$ at every area; we call these *Regge's jets*.

Both derivatives can be computed one simplex at a time, and only the jets come from the charge; everything else is fixed geometry of one simplex. Every area and every dihedral angle of a hinge depends only on the edges of the simplices that contain it, and the deficit is 2π minus a sum of dihedral angles. So both derivatives split into shares of single simplices: a quadratic share Q, from the second derivative, and a cubic share T, from the third. Each share is a weighted sum. The weights are the jets, which are the charge's data. The terms they weight are fixed polynomials in the ten edge perturbations of the simplex, set by its geometry alone. We call such a polynomial a *table* and give it by the array of its coefficients. A *share* is one simplex's contribution for a given charge: the sum of tables weighted by that charge's jets, below. For Regge's charge, with jets $(\lambda, \lambda, 0)$, the cubic share is λ times the sum of the tables $S_{0,\alpha}$ and $S_{1,\alpha}$ over the flat areas. For one simplex with flat triangle areas a_t and dihedral angles θ_t, and with jets $p_{j,\alpha}$ at the flat area α, the shares are

$$Q = \textstyle\sum_\alpha\big(p_{0,\alpha}Q_{0,\alpha} + p_{1,\alpha}Q_{1,\alpha}\big), \qquad T = \sum_\alpha\big(p_{0,\alpha}S_{0,\alpha} + p_{1,\alpha}S_{1,\alpha} + p_{2,\alpha}S_{2,\alpha}\big).$$

The quadratic tables are $Q_{0,\alpha} = -\sum_{t\in\alpha} a_t\theta_t''$ and $Q_{1,\alpha} = -2\sum_{t\in\alpha} A_t'\theta_t'$. The cubic tables are $S_{0,\alpha} = -\sum_{t\in\alpha} a_t\theta_t'''$, $S_{1,\alpha} = -3\sum_{t\in\alpha}(A_t'\theta_t'' + A_t''\theta_t')$ and $S_{2,\alpha} = -3\sum_{t\in\alpha} A_t'^2\theta_t'/a_t$. In each, the sums run over the triangles of flat area α.

Section 5.2 turns the tables into gauge-test equations: summed over a cell and expanded in plane waves, they give a quadratic form and a cubic vertex that are linear in the jets. The derivatives of areas and angles are taken along a straight line in the simplex's squared edge

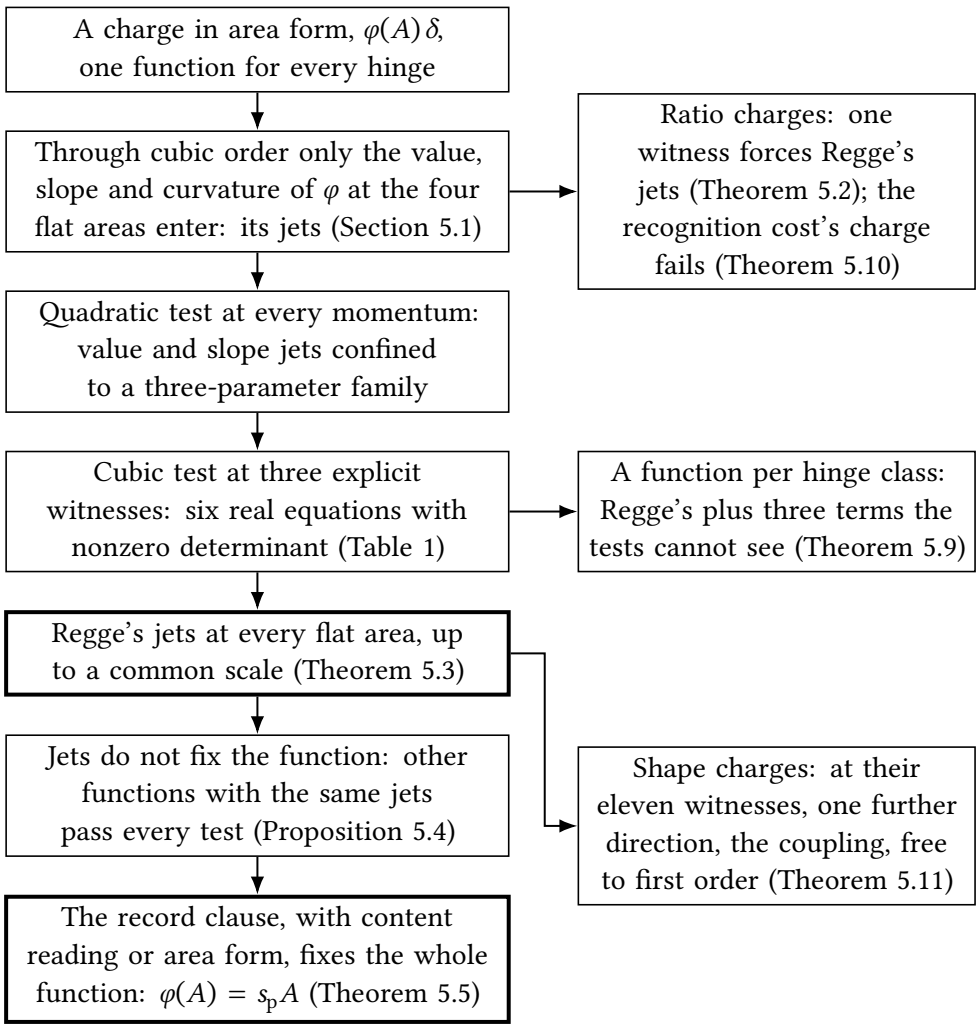

Figure 2. The selection argument at a glance: the three steps of Section 5 down the left, its two conclusions in heavy boxes; the neighbouring classes of charges on the right.

lengths and computed in closed form from its Gram matrix. At the Euclidean Kuhn simplex every table is an explicit polynomial with rational coefficients.

Three classes of charges will matter: the first is the main case, the second shows what the tests cannot see, and the third contains the recognition cost's charge. One comparison separates them. Take two hinges with the same area and the same deficit. One function for every hinge charges them alike. One function per hinge class may charge them differently if they are in different classes. A ratio charge may charge them differently if their flat areas differ. The three classes are these.

- *One function for every hinge*, $\varphi(A)\delta$: jets $p_{j,\alpha}$ at the four flat areas.
- *One function per hinge class*, $\varphi_c(A)\delta$, a weaker premise: each class of hinges may have its own function, so two hinges of equal area may be charged differently. For example, the Euclidean Kuhn lattice has six classes of hinges: one class has area $\frac{1}{2}$, one has area 1, and two classes, exchanged by central inversion, share each of the areas $\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{3}}{2}$. Two hinges of equal area from the two classes of such a pair are charged by the same function under the first premise and may be charged by different functions under the second.

- *Ratio charges*, $a_0\,\psi(A/a_0)\,\delta$, where a_0 is the hinge's own flat area: one function ψ, the same for every hinge, of the ratio of a hinge's area to its own flat area. Their jets are $(\psi(1), \psi'(1), \psi''(1))$ at every hinge; we write $(c_0, c_1, c_2) = (\psi(1), \psi'(1), \frac{1}{2}\psi''(1))$, and Regge's values, at unit scale, are $(1, 1, 0)$. The recognition cost's charge of Section 5.8 is a ratio charge through cubic order.

A fourth kind, the shape charges of Section 5.9, comes later: their cost also depends on the hinge's shape, so they can separate even two hinges of one class.

5.2 The two gauge tests

Einstein's theory cannot tell apart two metrics that differ by a change of coordinates. On the lattice the matching move, a vertex move, is a displacement of the lattice points, and gauge consistency asks that vertex moves remain a symmetry wherever Einstein's theory needs them to. We test it twice, at quadratic and at cubic order, and the next two paragraphs say why. Both tests are run at $\kappa = 0$, the Euclidean member, where every table has exact rational coefficients; Section 5.6 carries the cubic vertex to $\kappa = 1$, not the test.

Why the quadratic test. Vertex moves carry a flat lattice to a flat lattice, so a charge whose quadratic action does not annihilate them would let pure gauge change the energy of a field it is added to. Regge's charge does annihilate them.

Why the cubic test. The quadratic operator has no response along a vertex move, so the cubic source must have none either. The reason is that at second order in the field amplitude the field equation $D^2S(h_2, \cdot) + D^3S(h_1, h_1, \cdot) = 0$ is solvable only if the source pairs to zero with every zero mode of D^2S, the vertex moves included. This is the linearized Bianchi identity. On long waves it is the vanishing of the cubic vertex when one of its three waves is pure gauge and the other two are on shell, which means that they obey the linearized field equation. The three waves are the *legs* of the vertex.

Both tests use plane waves. A plane wave with symmetric polarization H and momentum K gives the edge f of a simplex the amplitude $(d_f^{\top} H d_f)\, e^{iK\cdot m_f}$, with d_f the edge's step and m_f its midpoint in the cell. Summing a table over the 24 simplices of a cell gives a form in the waves. For the quadratic table Q it is the plane-wave quadratic form. For the cubic table T, expanding the phases to second order in the momenta gives the *long-wave cubic vertex* $\mathscr{V}_T(H_1, H_2, H_3; K_1, K_2, K_3)$, $K_1 + K_2 + K_3 = 0$.

Convention. We take the vertex as the part of order two in the momenta of one sixth of the cell sum, which is the cubic term of a Taylor expansion. The parts of order zero and one vanish for every charge that is a function of the area times the deficit, with one function for every hinge, one for each class or a ratio charge's: a strain that is uniform, or linear in position, leaves the sampled lattice flat to first order in its gradient, so every deficit vanishes to that order whatever the function, and the part of order two is the leading one. We leave out the factor $i^2 = -1$ that the two derivatives contribute. We fix this normalization and sign now; the tests ask only for zeros, and the sign matters only in Section 5.6. So the long-wave cubic vertex of an action whose cubic share is T is $-\mathscr{V}_T$.

A vertex move displaces the lattice points. Under the displacement $\xi e^{iK\cdot x}$ of each point x, the squared length of the edge from x to $x+d$ changes at first order by $2\,(d\cdot\xi)\big(e^{iK\cdot(x+d)}-e^{iK\cdot x}\big)$, with the Euclidean dot product, since the tests are run at $\kappa=0$. In the midpoint form above, with $m=x+\frac{1}{2}d$, the edge therefore has the amplitude

$$4i\,(d\cdot\xi)\,\sin\!\big(\tfrac{1}{2}K\cdot d\big)\,e^{iK\cdot m}.$$

This is the vertex move at momentum K, and the quadratic test uses it as it stands, at every momentum. At long wavelength it is i times the plane wave of the pure-gauge polarization $K\otimes\xi+\xi\otimes K$, whose amplitude is $2\,(d\cdot K)(d\cdot\xi)\,e^{iK\cdot m}$; the two differ at third order in the momentum. The long-wave cubic vertex keeps the second order only, so in the cubic test the gauge leg is the polarization $K\otimes\xi+\xi\otimes K$. At finite momentum the two are different vectors, and Regge's quadratic form annihilates the vertex move, not the sampled polarization. The plane-wave form of a table is its *symbol*; a table of zero symbol has no plane-wave response.

Why witnesses. The two on-shell legs have null momenta, and on the Euclidean lattice the only real null momentum is zero. The long-wave vertex is a polynomial in the momenta, however, so complex null momenta can witness the polynomial condition. We use Gaussian-integer null momenta, complex momenta whose real and imaginary parts are integers, so that every value below is an exact Gaussian rational. A witness is one explicit choice of waves, with their polarizations and momenta, at which the cubic vertex is evaluated. Because the vertex is linear in the charge's table, each witness turns the test into linear equations in the jets, and Section 5.4 solves the system that the three witnesses form. The witnesses are necessary conditions for a charge whose quadratic form has the Euclidean graviton's shell, the momenta with $\sum_a K_a^2=0$ on which the witnesses' two on-shell legs sit: for such a charge, passing them does not by itself prove gauge consistency, but failing one excludes it. A charge whose quadratic form has another shell is tested off its own shell; Section 5.4 says what the test then shows. Regge's charge passes both tests. Its quadratic form is the Regge Hessian, and its long-wave cubic vertex is one half of Einstein's $\Gamma\Gamma$ vertex (Theorem 5.6), which vanishes with one pure-gauge leg when the other two are on shell.

Definition 5.1 (The gauge tests)**.** The *quadratic test*: the plane-wave form of Q annihilates the vertex move $4i\,(d\cdot\xi)\sin(\frac{1}{2}K\cdot d)\,e^{iK\cdot m}$ for every real momentum K and every real ξ. The *cubic test*: the long-wave cubic vertex of T vanishes with one pure-gauge leg at three witnesses,

- W_1: legs $(\eta',\eta',K_3\otimes e_1+e_1\otimes K_3)$, $\eta'=\mathrm{diag}(-1,1,1,1)$, $e_1=(0,1,0,0)$, momenta $K_1=(0,3,4,5i)$, $K_2=(0,5,12,13i)$, $K_3=-K_1-K_2$;
- W_2: the same with $e_2=(0,0,1,0)$ in place of e_1;
- W_3: transverse-traceless legs h_1,h_2 (Appendix A) at the null momenta $(3,4,0,5i)$ and $(5,0,12,13i)$, with the gauge leg $K_3\otimes e_1+e_1\otimes K_3$.

Coordinates are numbered from zero, as in Section 2: in W_1 and W_2 the entry -1 of η' and the zero entry of K_1 and K_2 sit at $a=0$, and e_1 is the step of the second counter.

5.3 Ratio charges

The simplest family to test is the ratio charges, whose jets are the same three numbers (c_0, c_1, c_2) at every hinge. These are the jets (p_0, p_1, p_2) of Section 5.1, with $c_2 = p_2/2$. For them the cubic table is a combination of three sums over a simplex, and the first witness already forces Regge's jets.

Let $X_1 = \sum_t A_t' \theta_t''$, $Y_1 = \sum_t A_t'' \theta_t'$ and $Z_1 = \sum_t A_t'^2 \theta_t' / A_t$ over the ten triangles of a simplex. Schläfli's identity $\sum_t A_t \theta_t' = 0$, differentiated twice, gives $-\sum_t a_t \theta_t''' = Y_1 + 2X_1$. So a ratio charge has the per-simplex cubic table

$$T_\psi = c_0 \, (Y_1 + 2X_1) - 3c_1 \, (X_1 + Y_1) - 6c_2 \, Z_1.$$

Theorem 5.2 (Ratio charges)**.** *At the witness W_1 the three tables take the values*

$$\mathscr{V}(X_1) = -\tfrac{112}{3} - \tfrac{160}{9} i, \qquad \mathscr{V}(Y_1) = \tfrac{56}{3} + \tfrac{80}{9} i, \qquad \mathscr{V}(Z_1) = -\tfrac{40}{3} - \tfrac{80}{9} i.$$

A ratio charge passes the cubic test at W_1 if and only if $c_0 = c_1$ and $c_2 = 0$: its jets are Regge's.

Proof. The vertex depends on a table only through its cubic polynomial and is linear in it, so the witness value of T_ψ is

$$(c_0 - c_1)\Big(-56 - \tfrac{80}{3} i\Big) + c_2\Big(80 + \tfrac{160}{3} i\Big).$$

Its imaginary part vanishes exactly when $c_0 - c_1 = 2c_2$, and then its real part is $-32 \, c_2$. Both vanish exactly when $c_2 = 0$ and $c_0 = c_1$. The three witness values are exact evaluations of the per-triangle closed forms of the first and second derivatives of the areas and angles of the Kuhn simplex, summed over the 24 simplices of a cell. □

5.4 One function of the area for every hinge

Now the main case, one function φ for every hinge. The quadratic test confines the value and slope jets at the four flat areas to a three-parameter family, with coefficients u_1, u_2, u_3. The cubic test at the three witnesses then gives six real linear equations in six combinations of the unknowns: $u_1 + u_2$, u_3 and the four curvature jets z_α (Table 1). They force all six to vanish, which leaves one free parameter, the scale λ, and sets the curvature jets to zero.

Theorem 5.3 (One function of the area)**.** *Let φ be twice differentiable at the four flat areas of the Euclidean Kuhn lattice. The charge $\varphi(A)\delta$ passes the quadratic test at every momentum and the cubic test at W_1, W_2 and W_3 if and only if its jets at all four flat areas are $(\lambda, \lambda, 0)$ for one real λ: Regge's jets at scale λ.*

Proof. Quadratic. The quadratic table is linear in the eight jets $(p_{0,\alpha}, p_{1,\alpha})$, through eight exact rational 10×10 tables. It is gauge null at every momentum exactly when the jets lie in the span of

$$v_1 = (2, 1 \,|\, 2, 1 \,|\, 2, 1 \,|\, 2, 1), \qquad v_2 = (1, 0 \,|\, 1, 0 \,|\, 1, 0 \,|\, 1, 0), \qquad v_3 = (2, 1 \,|\, \tfrac{5}{2}, \tfrac{1}{2} \,|\, 3, 0 \,|\, 3, 0),$$

listed area by area in the order $\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$. The quadratic table of v_1 vanishes identically, by Schläfli's identity; Regge's jets are $v_1 - v_2$, so v_2 is minus Regge's direction up to v_1. Gauge nullity at every momentum is a finite set of polynomial identities in the Bloch phases, and the span is exact linear algebra on them. Written as equations on the jets, with a, b, c, d for the areas $\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$, the quadratic test is the five independent conditions

$$p_{1,a} - 2p_{1,b} + p_{1,d} = 0, \qquad p_{1,c} = p_{1,d}, \qquad p_{0,c} = p_{0,d},$$
$$2(p_{0,b} - p_{0,a}) = p_{1,a} - p_{1,d}, \qquad p_{0,d} - p_{0,a} = p_{1,a} - p_{1,d},$$

whose solutions are exactly the span of v_1, v_2 and v_3.

Cubic. Write $(p_{0,\alpha}, p_{1,\alpha}) = u_1 v_1 + u_2 v_2 + u_3 v_3$ and $z_\alpha = p_{2,\alpha}$. Three identities among the per-area tables hold exactly: $\sum_\alpha (2S_{0,\alpha} + S_{1,\alpha}) = X_1 - Y_1$, $\sum_\alpha S_{0,\alpha} = Y_1 + 2X_1$, and the v_3-combination of the S_0 and S_1 tables is one explicit table V_3. So the cubic test at a witness is

$$(u_1 + 2u_2)\,\mathscr{V}(X_1) + (u_2 - u_1)\,\mathscr{V}(Y_1) + u_3\,\mathscr{V}(V_3) + \textstyle\sum_\alpha z_\alpha\,\mathscr{V}(S_{2,\alpha}).$$

At all three witnesses $\mathscr{V}(Y_1) = -\frac{1}{2}\mathscr{V}(X_1)$, so the first two terms are $\frac{3}{2}(u_1 + u_2)\,\mathscr{V}(X_1)$. Separating real and imaginary parts at the three witnesses gives six real linear equations in the six unknowns $(u_1 + u_2,\ u_3,\ z_{1/2},\ z_{\sqrt{2}/2},\ z_{\sqrt{3}/2},\ z_1)$, with the coefficients of Table 1. The determinant of that system is $-27\,866\,927\,400\,397\,976\,309\,514\,240 \neq 0$, so $u_1 + u_2 = 0$, $u_3 = 0$ and every $z_\alpha = 0$: the jets are $u_1(v_1 - v_2)$ with vanishing curvature jets, Regge's with $\lambda = u_1$. Conversely Regge's jets pass both tests. □

Which shell the cubic test uses. The quadratic test leaves the three-parameter family $u_1 v_1 + u_2 v_2 + u_3 v_3$, and its members do not share one graviton. The table of v_1 has zero symbol. The directions v_2 and v_3 carry, each up to a factor, Regge's quadratic form on the Euclidean lattice and on the $\kappa = 1$ lattice. At long wavelength the quadratic form of $u_2 v_2 + u_3 v_3$ is, up to a factor, the Fierz–Pauli form of the flat metric $g_{\kappa'}$ of Section 2 with $\kappa' = u_3/(u_3 - u_2)$; this holds for $u_2 \neq u_3$ and $u_2 \neq -3u_3$, where that metric is defined and invertible.

Table 1. The cubic test of Theorem 5.3 as six real equations. Each row is the real or imaginary part of the test at one witness; the columns multiply $u_1 + u_2$, u_3 and the four curvature jets. The static witnesses, W_1 and W_2, do not see the curvature jet at area 1; the transverse-traceless witness, W_3, does.

row	$u_1 + u_2$	u_3	$z_{1/2}$	$z_{\sqrt{2}/2}$	$z_{\sqrt{3}/2}$	z_1
W_1, Re	-56	-360	-8	0	48	0
W_1, Im	$-\frac{80}{3}$	-2356	-864	80	$\frac{2432}{3}$	0
W_2, Re	-280	228	992	-504	-240	0
W_2, Im	$-\frac{352}{3}$	-4139	-1344	352	$\frac{3328}{3}$	0
W_3, Re	$-\frac{946463965}{8}$	$-\frac{1504452249}{4}$	14665886	$\frac{971809893}{8}$	$-\frac{37294969}{2}$	$\frac{55612715}{2}$
W_3, Im	$-\frac{998974207}{8}$	$-\frac{41233057}{4}$	390527308	$-\frac{1786358293}{8}$	1780182	$-\frac{68444255}{2}$

For $u_3 = 0$ the charge's quadratic form is a multiple of Regge's on the Euclidean lattice. The on-shell legs of W_1, W_2 and W_3 solve its linearized field equation, the cubic test is the solvability condition of Section 5.2, and Theorem 5.3 says that among these charges only Regge's jets meet it.

For $u_3 \neq 0$ the charge's own waves are not null for the Euclidean form. Apply the charge's long-wave linearized field equation to a leg that solves the Euclidean one, at a momentum with $\sum_a K_a^2 = 0$: what remains is u_3 times a term that vanishes on none of the witnesses' four on-shell legs. So those legs do not solve that charge's linearized field equation, and the cubic test is not its solvability condition. Theorem 5.3 excludes such a charge because it fails a test posed on the Euclidean shell. Whether a charge with $u_3 \neq 0$ could be gauge consistent on its own shell is not tested here. The same reading applies to Theorem 5.9, which uses the same three witnesses.

What the cubic test adds. The quadratic test does not see the curvature jets at all, because the quadratic table uses only the value and slope jets (Section 5.1). For every real c the function $\varphi_c(A) = A + c\prod_\alpha (A-\alpha)^2$, the product running over the four Euclidean flat areas, has Regge's value and slope jets at each of them and passes the quadratic test. Its curvature jet at α is $2c\,\alpha\prod_{\beta\neq\alpha}(\alpha-\beta)^2$, which is not zero for $c \neq 0$, and by Theorem 5.3 it then fails the cubic test. The weak-field analysis of Roček and Williams is at the quadratic order [15, 16] and so cannot tell these functions apart; the cubic test does. In the continuum, compare Boulanger, Damour, Gualtieri and Henneaux, who showed from the gauge symmetry of the free theory that the only consistent self-interactions of massless spin-two fields with at most two derivatives are those of Einstein's theory [33].

Theorem 5.3 is the main result of the gauge route. One might ask whether rotation invariance, which Einstein's theory has, already selects Regge's charge. It does not. The recognition cost's hinge charge of Section 5.8 has a rotation-invariant long-wave cubic vertex and still fails the cubic test; gauge consistency, not isotropy, does the selecting.

Two questions remain on that route: how much of the charge the jets determine, and what fixes the rest. Section 5.5 answers both at once: the jets leave the function free away from the flat areas, and the record clause closes the gap exactly.

5.5 Through cubic order only, and the exact route

The tests read a charge only through its jets at the flat areas, and many functions share Regge's jets. On each hinge, add to the area factor A of Regge's charge a term that vanishes to third order at that hinge's own flat area a_0, such as $(A-a_0)^3/a_0^2$: the value, slope and curvature at a_0 do not change, so every test is passed, and the charge is not Regge's. Part (a) of the proposition below tunes the added term to each hinge's own flat area, in a ratio charge. Part (b) uses one function for every hinge, whose added term must vanish to third order at all four flat areas at once. Part (c) shows that a single function tuned to the flat area 1 alone does not have Regge's jets at area $\frac{1}{2}$.

Proposition 5.4 (Jets do not fix the charge)**.** *Let* $\psi_3(x) = x + (x-1)^3$.

(a) *For every flat area $a_0 > 0$, the ratio charge $a_0\psi_3(A/a_0)\,\delta = \left[A + (A-a_0)^3/a_0^2\right]\delta$ has Regge's jets $(1,1,0)$ at a_0. It is not content-reading, and it does not meet the record clause at any $k \geq 1$.*

(b) *For every finite set S of positive areas, the single function $\varphi_S(A) = A + \prod_{\alpha\in S}(A-\alpha)^3$ has Regge's jets at every $\alpha \in S$. It is not content-reading and does not meet the record clause.*

(c) *The single function $\varphi_1(A) = A + (A-1)^3$, used for every hinge, has Regge's jets at area 1 but not at area $\frac{1}{2}$ at any scale, so on the Euclidean lattice it is the ratio form of (a), not the single function, that passes.*

Proof. (a) With $u = A/a_0 - 1$ the factor is $a_0(1 + u + u^3)$, with value a_0, slope $1 + 3u^2 = 1$ and curvature $6u/a_0 = 0$ at $u = 0$. At $A = 0$ it equals $-a_0 \neq 0$, while a content-reading charge in area form is linear through the origin and a charge meeting the record clause in area form is $s_{\mathrm{p}}A$ (Theorem 5.5). (b) Every factor $(A-\alpha)^3$ vanishes with its first two derivatives at its own root, so $q(A)^3$, $q = \prod_{\alpha\in S}(A-\alpha)$, contributes nothing to the value, slope or curvature at any $\alpha \in S$. At $A = 0$, $\varphi_S(0) = q(0)^3 \neq 0$ since $0 \notin S$. (c) $\varphi_1(\frac{1}{2}) = \frac{3}{8}$ would need $\lambda = \frac{3}{4}$, while $\varphi_1'(\frac{1}{2}) = \frac{7}{4}$ would need $\lambda = \frac{7}{4}$. □

The charge of (a), with a_0 the flat area of each hinge, has Regge's jets, and so does the charge of (b), with S the four flat areas. The quadratic and cubic tables depend on a charge only through its jets (Section 5.1), so both examples have exactly Regge's tables. They pass every test of Theorems 5.2 and 5.3, and the class tests of Section 5.7 as well. Gauge consistency therefore gives Regge's charge only *through cubic order*: it fixes the jets that small amplitudes see, up to the scale λ, and leaves the rest of the function free. The record clause closes the gap exactly.

The jets that the tests fix are jets at the four flat areas of the Euclidean lattice, $\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$. The $\kappa = 1$ lattice has the flat areas $\frac{1}{2}, 1, \frac{3}{2}, \sqrt{3}$, and the tests say nothing about a charge at $\frac{3}{2}$ or at $\sqrt{3}$. The function φ_S of part (b), with S the four Euclidean flat areas, shows it. It passes every Euclidean test, and at $\frac{3}{2}$ and at $\sqrt{3}$ its value and slope jets differ and its curvature jet is not zero, so its jets there are Regge's at no scale. The gauge route alone therefore does not make the action Regge's through cubic order around the $\kappa = 1$ lattice. Theorem 5.6 is a statement about Regge's action, and on this section's two routes it applies to the charge selected here only once the record clause, with its premise on the charge's form, has fixed $\varphi(A) = s_{\mathrm{p}}A$ at every area (Theorem 5.5).

The stationarity premise of Section 4.3, which this section does not use, excludes those charges within area form: a charge $\varphi(A)\,\delta$ of class C^3 for which every stretched flat lattice is stationary has $\varphi(A) = \varphi(1)\,A$ at every area from $\frac{1}{2}$ up (Theorem 4.2(a), with $\sigma = \varphi$), and φ_S does not, so some stretched flat lattice is not a stationary point of its action. Within area form, then, three premises lead to Regge's charge, by arguments none of which uses another's premise, and with different reach: stationarity of every stretched flat lattice gives the function at every area from $\frac{1}{2}$ up, $\frac{3}{2}$ and $\sqrt{3}$ included, with its scale and sign free; the gauge tests give its jets at the Euclidean flat areas, through cubic order, with the scale free; the record clause gives the function at every area, scale included. Their reach is nested. A charge in area form that

meets the record clause is $s_{\mathrm{p}}A\delta$ and meets the other two, and one of class C^3 that meets the stationarity premise has Regge's jets at the Euclidean flat areas and so passes the gauge tests. Neither converse holds: φ_S passes the tests and is not stationary, and $\lambda A\delta$ with $\lambda \neq s_{\mathrm{p}}$ is stationary and does not meet the clause.

The clause fixes the charge at one deficit only, the deficit $2\pi/k$ of a single posting, for every area. A premise on the charge's form carries that value to every deficit: either the charge depends on area and deficit only through the content $A\delta$, or its cost per unit deficit is one function of the area. With the clause and either premise, the charge is Regge's.

Theorem 5.5 (The record clause selects Regge's charge)**.** *Let $k \geq 1$ and let C be a hinge charge.*

(a) C is content-reading and meets the record clause at k if and only if $C(A,\delta) = s_{\mathrm{p}}A\delta$.

(b) If $C = \varphi(A)\delta$ is in area form, C meets the record clause at k if and only if $\varphi(A) = s_{\mathrm{p}}A$ for every A.

(c) Given the record clause, the three statements "C is content-reading", "C is in area form" and "$C = s_{\mathrm{p}}A\delta$" are equivalent.

(d) For a ratio charge with references $a_0(c) > 0$ on the hinge classes c and one function ψ, the record clause on one class makes the charge $s_{\mathrm{p}}A\delta$ on every class.

(e) $\lambda A\delta$ meets the record clause at k if and only if $\lambda = s_{\mathrm{p}}$.

Proof. (a) Regge's charge at s_{p} is $F(A\delta)$ with $F(x) = s_{\mathrm{p}}x$, and it meets the clause. Conversely, given $C(A,\delta) = F(A\delta)$ and the clause, for every real x put $A = kx/(2\pi)$: then $F(x) = C(A, 2\pi/k) = s_{\mathrm{p}}A(2\pi/k) = s_{\mathrm{p}}x$, so $C(A,\delta) = s_{\mathrm{p}}A\delta$. (b) The clause reads $\varphi(A)(2\pi/k) = s_{\mathrm{p}}A(2\pi/k)$, and $2\pi/k \neq 0$. (c) Content reading and the clause give $s_{\mathrm{p}}A\delta$ by (a), which is in area form; area form and the clause give $s_{\mathrm{p}}A\delta$ by (b), which is content-reading. (d) By (b) applied to the class c_0, $a_0(c_0)\psi(r) = s_{\mathrm{p}}\, a_0(c_0)\, r$ for every r, so $\psi(r) = s_{\mathrm{p}}r$, and then $a_0(c)\psi(A/a_0(c)) = s_{\mathrm{p}}A$ on every class. (e) is (b) with $\varphi(A) = \lambda A$. □

Given the record clause, the two premises on the charge's form, the content reading and the area form, coincide (part (c)). Without the record clause, the gauge tests fix a charge in area form only through cubic order, up to its scale (Theorem 5.3). The record clause fixes the scale as well (Theorem 5.5(e)).

The theorem should be read for what it is. Its proof is a substitution: the clause asks linearity in the area at one deficit, and the premise on the form carries that to every deficit. Definition 2.2 asks the clause at every real area; only part (a) uses the negative ones, to fix F at negative arguments, where they stand in for the credit statement of Section 2, and on positive areas (b) gives $\varphi(A) = s_{\mathrm{p}}A$ with no such extension. And the record route does not use the gauge tests: with the clause and either premise the charge is Regge's, so on this route the Euclidean tests of Sections 5.2 to 5.4 are redundant. The two routes are independent arguments with different premises, not two halves of one argument. The gauge route asks nothing of the ledger beyond the area form and reaches the jets at the Euclidean flat areas; the record route asks the clause, an IDENTIFICATION, and reaches the whole function.

Neither the record clause nor the premise on the charge's form suffices alone. The record clause does not settle the cubic tests by itself: it fixes the charge only on the line $\delta = 2\pi/k$,

far from the linear regime, and it is the content reading or the area form that carries those values to every deficit. Regge's charge plus $A^2(1-\cos k\delta)$ meets the clause but is neither content-reading nor in area form, so the clause alone does not give Regge's charge. The zero charge is both content-reading and in area form but meets no clause, so the form premise alone does not give Regge's charge [9, Prop. 7.3].

5.6 The cubic vertex at $\kappa = 1$, and one normalization

The gauge-selection theorems above (Theorems 5.2 and 5.3) live on the Euclidean member of the lattice family; the record-clause theorem does not depend on κ. The paper's lattice is $\kappa = 1$. The theorem below carries the vertex calculation across to $\kappa = 1$, not the gauge test itself, which stays open. It concerns Regge's action. For a charge known only through its Euclidean jets it says nothing, since two of the $\kappa = 1$ flat areas, $\frac{3}{2}$ and $\sqrt{3}$, are not Euclidean flat areas (Section 5.5). Its part (a) is the result: at $\kappa = 1$, as at $\kappa = 0$, the long-wave cubic vertex of the Regge action is one half of Einstein's $\Gamma\Gamma$ vertex. What that half means comes first. Scaling the quadratic and cubic parts of an action together changes only its overall scale, which sets G; the self-coupling is the ratio of the cubic to the quadratic part. The quadratic part carries the same half, so the ratio is Einstein's: the half is a normalization, not a change of the self-coupling (Remark 5.7). Two independent lattice computations give (a); the conversions after the theorem compare them. The other parts support it: (b) continues the Euclidean table to $\kappa = 1$, with one common factor, (c) records two facts about the continued table, and (d) converts between two ways of writing Einstein's vertex.

Conventions. Every vertex in this subsection is a cubic plane-wave vertex, a function of the three legs' polarizations and momenta, written without the background volume factor. The two sides of the theorem treat the factor $i^2 = -1$ of the two derivatives differently, and a sign in the table makes up the difference.

Einstein's vertex. For a symmetric inverse metric γ, $\tilde{V}_\gamma(H;K)$ is the cubic plane-wave vertex of Einstein's first-order Lagrangian $\sqrt{|\det G|}\, G^{ab}(\Gamma^c_{ad}\Gamma^d_{bc} - \Gamma^c_{ab}\Gamma^d_{cd})$ on the background with inverse metric γ. It keeps the factor i^2: on a plane wave each derivative is i times a momentum. Written out, let $2\Gamma_{e,ab}(H;K) = K_a H_{eb} + K_b H_{ea} - K_e H_{ab}$ be the doubled lowered symbol of a leg, twice the Christoffel symbol with all indices down and without its factor i, and for three matrices M, M', M'' and two legs j, k let

$$X_{jk}(M,M',M'') = \sum_{a,b,c,d,e,f} M^{ab}\, M'^{ce}\, M''^{df}\Big(2\Gamma_{e,ad}(j)\,2\Gamma_{f,bc}(k) - 2\Gamma_{e,ab}(j)\,2\Gamma_{f,cd}(k)\Big),$$

with $2\Gamma(j)$ short for $2\Gamma(H_j;K_j)$. Then

$$\tilde{V}_\gamma = -\tfrac{1}{4}\sum_{(i;\,j,k)}\Big[\tfrac{1}{2}\operatorname{tr}(\gamma H_i)\,X_{jk}(\gamma,\gamma,\gamma) - X_{jk}(\gamma H_i\gamma,\gamma,\gamma) - X_{jk}(\gamma,\gamma H_i\gamma,\gamma) - X_{jk}(\gamma,\gamma,\gamma H_i\gamma)\Big],$$

summed over the six ways to name one leg i and order the other two as (j,k). The leg i supplies the first-order change of the volume factor, $\frac{1}{2}\operatorname{tr}(\gamma H_i)$, or of one inverse metric, $-\gamma H_i\gamma$; the legs j and k supply the two symbols. All three inverse metrics change, the one that contracts the two symbols and the two that raise their first indices. The factor $-\frac{1}{4}$ is $(i/2)^2$.

The lattice's vertex. $\mathscr{V}_{T(\kappa)}$ is the long-wave cubic vertex of the table $T(\kappa) = \sum_t(\theta_t'' A_t' + 2\theta_t' A_t'')$ at g_κ, with the common factor $\sqrt{1-4\kappa}$ removed; by part (b) its coefficients are rational in κ, so $T(1)$ is defined by continuation from the Euclidean side. As fixed in Section 5.2, $\mathscr{V}_T$ leaves the factor i^2 out, so an action whose cubic share is T has the vertex $-\mathscr{V}_T$. The table $T(\kappa)$ is minus Regge's cubic share. The two signs cancel: $\mathscr{V}_{T(\kappa)}$ is the Regge action's own long-wave vertex with the factor i^2 in place, and the theorem compares it with Einstein's vertex written the same way. If the factor is left out of Einstein's vertex too, the identity of part (a) reads $\mathscr{V}_{T(1)} = -\frac{1}{2}$ times that vertex; the minus sign is the one between T and Regge's share. The conversions to the Regge action and to the forms used elsewhere follow the theorem.

Theorem 5.6 (The cubic vertex at $\kappa = 1$)**.**

(a) *The long-wave cubic vertex of* $T(1)$ *is* $\frac{1}{2}\tilde{V}_{g_1^{-1}}$, $g_1^{-1} = I - \frac{1}{3}\mathbf{1}\mathbf{1}^\top$, *for all symmetric polarizations and all momentum-conserving triples. The two-derivative cubic symbol of the Lorentzian Regge action at* $\kappa = 1$, *computed directly on the Lorentzian simplex, is* $\frac{1}{2}V_{g_1} = \frac{\sqrt{3}}{2}\tilde{V}_{g_1^{-1}}$*: it is* $\sqrt{3} = \sqrt{|\det g_1|}$ *times the vertex of* $T(1)$*, the volume factor that* $T(1)$ *omits. At* $\kappa = 0$ *the long-wave cubic vertex of* $T(0)$ *is* $\frac{1}{2}\tilde{V}_I$.

(b) *For every* $\kappa < \frac{1}{4}$ *the sign-reversed per-simplex Regge cubic* $\sum_t(\theta_t'' A_t' + 2\theta_t' A_t'')$ *at the Kuhn simplex of* g_κ, *with true derivatives, is* $\sqrt{1-4\kappa}$ *times a table* $T(\kappa)$ *whose 220 coefficients are rational functions of* κ *with poles only at* $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$. $T(0)$ *is the Euclidean table;* $T(1)$ *is the* $\kappa = 1$ *table, the analytic continuation of the Euclidean-signature cubic with the common factor* $\sqrt{\det g_\kappa} = \sqrt{1-4\kappa}$ *removed.*

(c) $T(1)$ *vanishes on constant metric strains summed over a cell, and its long-wave vertex has no term of order zero or one in the momenta: it starts at two derivatives.*

(d) *For every symmetric matrix* γ, *the cubic vertex of the* $\Gamma\Gamma$ *density written with the Christoffel symbols of the full metric,* $\Gamma^a_{bc} = \frac{1}{2}G^{ae}(\partial_b G_{ec} + \partial_c G_{eb} - \partial_e G_{bc})$ *with* G^{ae} *the inverse of* G, *expanded around the background with inverse metric* γ *and divided by the background volume factor, equals* $\tilde{V}_\gamma$ *as written above with the doubled lowered symbols* $2\Gamma_{e,ab}$. *So the vertex* V_g *of Theorem 4.2, written there with Christoffel symbols, is* $\sqrt{|\det g|}\,\tilde{V}_{g^{-1}}$, *and* $V_{g_1} = \sqrt{3}\,\tilde{V}_{g_1^{-1}}$.

Reading the theorem: two conversions.

- *Sign.* Up to the removed factor, $\mathscr{V}_{T(\kappa)}$ is the Regge action's own long-wave vertex: Regge's cubic share is $-T(\kappa)$, by Schläfli's identity differentiated twice (Section 5.1), and under the normalization of Section 5.2 a share $-T$ has the vertex $+\mathscr{V}_T$. So (a) says that the Regge action's long-wave cubic vertex is one half of Einstein's, with the same sign and with the factor i^2 in both, at $\kappa = 1$ as at $\kappa = 0$.
- *Volume factor.* The convention omits $\sqrt{|\det g|}$; two forms used elsewhere keep it. By (d), Einstein's vertex V_g of Theorem 4.2, written with Christoffel symbols, is $\sqrt{|\det g|}\,\tilde{V}_{g^{-1}}$; at $\kappa = 1$, where $\sqrt{|\det g_1|} = \sqrt{3}$, $V_{g_1} = \sqrt{3}\,\tilde{V}_{g_1^{-1}}$. The Regge action's cubic symbol computed

directly on the Lorentzian simplex at $\kappa = 1$ keeps the same $\sqrt{3}$, which $T(1)$ omits; with it restored the two computations of (a) agree.

Part (a) is reduced by symmetry: the six permutations of the three legs reorder the waves, and the twenty-four relabellings of the axes rename the coordinates, which leaves twenty representative triples of polarizations.

Proof. We prove (b)–(d) first. (b) The dual normals of the Kuhn simplex at g_κ are $(1-4\kappa)^{-1}$ times a matrix polynomial in κ, and every ingredient of the closed forms is homogeneous in them; Jacobi's complementary minor is $(1-4\kappa)^{-1}Q_t/4$. So each triangle's contribution is $\sqrt{1-4\kappa}$ times a cubic whose coefficients are rational in κ, and the ten triangles add to $T(\kappa)$ by 220 univariate rational-function identities.

(c) The cell sum of $T(1)$ on a constant strain is an exact rational identity, and the orders zero and one of the vertex vanish term by term.

(d) In the cubic vertex one leg supplies a first-order change and the other two supply the derivatives in the two symbols. The first-order change of the volume factor is $\frac{1}{2}\operatorname{tr}(\gamma H)$ times that factor, and the first-order change of each inverse metric is $-\gamma H\gamma$. There are three inverse metrics: the one that contracts the two symbols and the one inside each symbol, since G^{ae} is the inverse of the full metric. At lowest order a symbol on a plane wave is $\Gamma^a_{bc} = \frac{i}{2}\gamma^{ae}\,2\Gamma_{e,bc}(H;K)$, so each $\Gamma\Gamma$ contraction is $(i/2)^2 = -\frac{1}{4}$ times the doubled one with the same three inverse metrics. These are the four terms in the bracket of $\tilde{V}_\gamma$ for one choice of legs. Summing the six choices gives equality for every γ; with $|\det g_1| = 3$ and $g_1^{-1} = I - \frac{1}{3}\mathbf{1}\mathbf{1}^\top$ the second claim follows.

(a) The defect $\mathscr{V}_{T(1)} - \frac{1}{2}\tilde{V}_{g_1^{-1}}$ is linear in each leg and invariant under the six permutations of the legs and the 24 relabellings of the axes, because g_1^{-1} is invariant under every permutation of the axes. So it suffices to check twenty representatives: every ordered triple of symmetric basis matrices is a leg permutation of a relabelling of one of them. On each representative the twenty-four relabelled shares of the identity simplex add to one half of the Einstein vertex, as quadratic forms in the momenta. The case $\kappa = 0$ is the same argument with $\gamma = I$. The direct computation on the Lorentzian simplex gives $\frac{1}{2}V_{g_\kappa}$, off shell, for every $\kappa > \frac{1}{2}$ [7, Sec. 4.3 and Thm. 3.1(iii)]; at $\kappa = 1$, (d) turns it into $\frac{\sqrt{3}}{2}\tilde{V}_{g_1^{-1}}$. □

Remark 5.7 (One normalization: the factor $\frac{1}{2}$)**.** The factor $\frac{1}{2}$ in (a) is a normalization, not a halving of the self-coupling, as the opening of this subsection explains. $\Gamma\Gamma$ is $\sqrt{|g|}\,R$ up to a total derivative, which drops out of a momentum-conserving vertex, so (a) says that the lattice action $\sum_h A_h\delta_h$ is $\frac{1}{2}\int\sqrt{|g|}\,R$ through cubic order: "half of Einstein's $\Gamma\Gamma$ vertex" in (a) and "$\frac{1}{2}\int\sqrt{|g|}\,R$" in Theorem 4.2 are the same half. The quadratic part carries the same $\frac{1}{2}$ [7, Thm. 3.1(ii) and Rem. 3.2], and so does the field equation, $-\frac{1}{2}\sqrt{|g|}G^{\mu\nu}$ (Theorem 4.1); this is the classical normalization of Regge calculus [24, 25]. With the action scale s, $s\sum_h A_h\delta_h = \frac{s}{2}\int\sqrt{|g|}R = (16\pi G)^{-1}\int\sqrt{|g|}R$ at $G = 1/(8\pi s)$, the coupling of the linear field and of Newton's law. The same $\frac{1}{2}$ appears at quartic order (Section 5.10).

5.7 One function per hinge class

What if each class of hinge may carry its own function? The tests then leave three free numbers besides the scale. All three sit in the part of the charge that is odd under central inversion, and neither test sees them.

Central inversion $x \mapsto \mathbf{1} - x$ maps the Kuhn cell to itself. It sends each simplex to a simplex. An edge slot is an edge position within a simplex; inversion sends each slot to the reversed slot, keeps every edge step, and reflects every midpoint $m \mapsto \mathbf{1} - m$. The lemma has a short mechanism: the long-wave vertex depends on the phases of its three legs only through their sum, and inversion changes the sign of that sum, which leaves its square unchanged.

Lemma 5.8 (Inversion is invisible to the cubic vertex)**.** *For every per-simplex cubic table, the slot-reversed table has the same long-wave cubic vertex at every momentum-conserving triple and all polarizations. The plane-wave quadratic form of a slot-reversed table is the form at the opposite momentum.*

Proof. For one monomial and one ordered triple of slots, the six products of Taylor orders that make up the long-wave vertex add to $E_0E_1E_2\,(z_0+z_1+z_2)^2/2$, the second-order part of a product of exponentials, so only the sum $z_0+z_1+z_2$ of the three phases matters. Under inversion the phase of leg i on the reversed slot is $y_i - x_i$, with $y_i = \sum_a K_{i,a}$ and x_i the original phase; momentum conservation gives $\sum_i y_i = 0$, so the sum becomes $-\sum_i x_i$, whose square is the same. Reindexing the sum over simplices by the inversion closes the first claim. For the quadratic form, reindex the simplices and both slots: the steps are unchanged and the midpoint difference changes sign. □

With the lemma in hand, split each pair of classes exchanged by inversion into its mean and its difference: the means are governed by Theorem 5.3, and the differences are odd under inversion.

Theorem 5.9 (One function per class)**.** *A charge $\varphi_c(A)\delta$ with one function per hinge class passes the quadratic test at every momentum and the cubic test at W_1, W_2, W_3 if and only if its jets (p_0, p_1, p_2) are, for real $\lambda, \mu, \nu_1, \nu_2$,*

$$(\lambda, \lambda, 0) \text{ at areas } \tfrac{1}{2} \text{ and } 1; \qquad (\lambda \mp \tfrac{\mu}{2},\ \lambda \mp \tfrac{\mu}{2},\ \pm\nu_1) \text{ at } \tfrac{\sqrt{2}}{2}; \qquad (\lambda \mp \mu,\ \lambda \mp \mu,\ \pm\nu_2) \text{ at } \tfrac{\sqrt{3}}{2},$$

the two signs on the two classes of a pair exchanged by inversion. The part even under inversion is Regge's; μ, ν_1 and ν_2 are odd under inversion and invisible to both tests.

Proof. Quadratic. Exact linear algebra on the class tables shows that gauge nullity at every momentum puts the inversion-even jets, the means over each inverted pair, into the span of Theorem 5.3, and the odd differences on the single line $p_0 - p_0' = p_1 - p_1'$ with the area-$\frac{\sqrt{3}}{2}$ difference twice the area-$\frac{\sqrt{2}}{2}$ one; that line, μ, has zero symbol. Conversely the stated jets give class sums equal to a fixed multiple of λ times the Euclidean Regge table, which is gauge null. *Cubic.* Each pair of inverted class tables sums to the area table, and slot reversal exchanges the two classes of a pair; by Lemma 5.8 the two have equal long-wave vertices at every witness, each one half of the area table's. So the class test is the test of Theorem 5.3 on the even jets, and that theorem forces them to be Regge's. □

So the uniqueness class has to be *one function of the area for every hinge.* Read class by class, a charge can carry three inversion-odd terms that the long-wave tests cannot see.

The contrast is between one shared function and separate ones. A function shared by all classes constrains them together, so a premise on it acts on every class at once; separate functions keep their differences unless a premise reaches each class. Section 5.5 adds two premises, the content reading $C = F(A\delta)$ and the record clause. Three cases show what they do to the three inversion-odd terms: the first two remove them, the third leaves them.

- *The content reading, with one F for every hinge.* The charge is $\varphi_c(A)\delta$, and the content reading says it is $F(A\delta)$. Setting $\delta = 1$ gives $F = \varphi_c$ for every class, so the classes share one function, and $F(A\delta) = F(A)\delta$ makes it linear.
- *The record clause on every class.* It makes each $\varphi_c = s_{\mathrm{p}}A$ (Theorem 5.5(b), class by class).
- *Not enough: the content reading class by class*, one F for each class. The weightings $\varphi_c = s_cA$ are content-reading class by class and invisible to both tests; their slopes are those of Theorem 5.9, λ on the classes of areas $\frac{1}{2}$ and 1 and $\lambda \mp \frac{\mu}{2}$, $\lambda \mp \mu$ on the inverted pairs.

5.8 The recognition cost's hinge charge

A natural rival to Regge's charge comes from the recognition cost, the ledger's cost of comparing two records whose ratio is x; it is zero when they agree, at $x = 1$. A hinge charge must combine a hinge's area and deficit, and the recognition cost does so by pairing the two as records. Does the charge it suggests pass the gauge tests? At quadratic order it does; at cubic order it fails, and a counterterm that leaves the quadratic action alone repairs it.

The recognition cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ has an odd part: the half-difference of its values on a product and on a quotient, $J_{\mathrm{odd}}(x, y) = \frac{1}{2}\big(J(xy) - J(x/y)\big) = \frac{1}{4}(x - x^{-1})(y - y^{-1})$. Pair, hinge by hinge, the area record $x = A/a_0$ (the hinge's area against its flat value) with the holonomy record $y = e^{\delta}$ (the deficit angle written as a ratio). The result is the hinge charge $C_J(A, \delta) = \frac{a_0}{2}\, J_{\mathrm{odd}}(A/a_0, e^{\delta})$, normalized so that its quadratic action is Regge's. Write $\psi_J(x) = \frac{1}{4}(x - x^{-1})$. One technical point: since $J_{\mathrm{odd}}(x, y)$ has a pole at $x = 0$, C_J is defined for $A > 0$, the areas that occur on the lattice; any extension to $A \le 0$ makes it a hinge charge in the sense of Definition 2.1, and nothing below depends on the extension.

By part (a) below, the charge splits into the ratio charge of ψ_J and a remainder R_J. The remainder is a product of a factor that vanishes at the flat area and a factor of third order in the deficit, so it enters neither the quadratic nor the cubic action, and through cubic order C_J is the ratio charge of ψ_J. To repair C_J we may add a *ratio correction*, a ratio charge that we name by its triple of jets $c^* = (c_0^*, c_1^*, c_2^*)$. Uniqueness in the theorem below is uniqueness of this triple, the correction's jets; the function displayed with it is one representative. The correction is this paper's addition. Nothing in the recognition cost supplies it: its jets are fixed by the cubic test, and they are Regge's jets minus the cost's own, $(1, 1, 0) - (0, \frac{1}{2}, -\frac{1}{4})$. So the corrected charge is not the recognition cost's charge. What the theorem shows is that no ratio correction brings the cost's charge through the cubic test without giving it Regge's jets.

Theorem 5.10 (The recognition cost's charge and its counterterm)**.**

(a) The charge splits as $C_J = a_0\psi_J(A/a_0)\,\delta + R_J$ *with* $R_J = a_0\psi_J(A/a_0)\,(\sinh\delta - \delta)$.

(b) $\psi_J(1) = 0$, $\psi_J'(1) = \frac{1}{2}$, $\psi_J''(1) = -\frac{1}{2}$: $(c_0, c_1, c_2) = (0, \frac{1}{2}, -\frac{1}{4})$. *These are not Regge's jets at any scale, and* C_J *fails the cubic test at* W_1.

(c) A ratio correction c^* *has a quadratic table of zero symbol exactly when* $c_0^* = 2c_1^*$. *The only such triple of jets with which* $C_J + c^*$ *passes the cubic test at* W_1 *is* $c^* = (1, \frac{1}{2}, \frac{1}{4})$; *the charge* $\big[a_0 + \frac{1}{2}(A - a_0) + \frac{1}{4}(A - a_0)^2/a_0\big]\delta$ *is one ratio correction with these jets. With it,* $\psi_J + \psi^*$ *has Regge's jets* $(1, 1, 0)$.

(d) For every ratio correction with the jets c^*, *whatever function* ψ^* *represents it, the corrected charge* $a_0\big[\psi_J(A/a_0)\sinh\delta + \psi^*(A/a_0)\,\delta\big]$ *is not content-reading, and it agrees with Regge's charge through cubic order only. Whether* $\psi_J + \psi^*$ *is Regge's function depends on the representative: for the quadratic one displayed in (c) it is not, and for* $\psi^*(x) = \frac{3}{4}x + \frac{1}{4}x^{-1}$, *which has the same jets,* $\psi_J + \psi^* = x$ *on* $x > 0$.

Proof. (a) $J_{\text{odd}}(x, e^\delta) = \frac{1}{2}(x - x^{-1})\sinh\delta$, so $C_J = a_0\psi_J(A/a_0)\sinh\delta$. (b) $\psi_J'(x) = \frac{1}{4}(1 + x^{-2})$ and $\psi_J''(x) = -\frac{1}{2}x^{-3}$. Regge's jets at scale λ would need $\psi_J(1) = \lambda$ and $\psi_J'(1) = \lambda$, which are 0 and $\frac{1}{2}$. By Theorem 5.2 the charge fails, since $c_0 \neq c_1$; its witness value is 8, from $c_0 - c_1 = -\frac{1}{2}$ and $c_2 = -\frac{1}{4}$: real part $28 - 20$, imaginary part zero. (c) The uniform quadratic table of a ratio charge has zero symbol exactly when $c_0 = 2c_1$ (the direction ν_1 of Theorem 5.3). By Theorem 5.2, $C_J + c^*$ passes exactly when $c_0^* = c_1^* + \frac{1}{2}$ and $c_2^* = \frac{1}{4}$; with $c_0^* = 2c_1^*$ this gives $c^* = (1, \frac{1}{2}, \frac{1}{4})$, and $(0, \frac{1}{2}, -\frac{1}{4}) + (1, \frac{1}{2}, \frac{1}{4}) = (1, 1, 0)$. (d) Let ψ^* be any function, and suppose that the corrected charge were $F(A\delta)$ for every $A > 0$ and every δ. At $A = a_0$, where $\psi_J(1) = 0$, this reads $F(a_0\delta) = a_0\psi^*(1)\,\delta$, so F is linear. At $A = 2a_0$, where $\psi_J(2) = \frac{3}{8}$, it then reads $\frac{3}{8}\sinh\delta = \big(2\psi^*(1) - \psi^*(2)\big)\delta$ for every δ, which is false, because $\sinh 2 \neq 2\sinh 1$. So the remainder R_J of (a) keeps the charge from being content-reading whatever ψ^* is. With the jets c^* the corrected charge has Regge's jets, so it agrees with Regge's charge through cubic order; at fourth order R_J contributes $(A - a_0)\,\delta^3/12$, which no term of the form $\psi^*(A/a_0)\,\delta$ cancels. For the representative displayed in (c), $\psi_J + \psi^*$ takes the values 1 and $\frac{17}{8}$ at 1 and 2, so it is not linear; with that representative the corrected charge takes the values $a_0(\frac{3}{8}\sinh 1 + \frac{7}{4})$ and $2a_0$ at $(A, \delta) = (2a_0, 1)$ and $(a_0, 2)$, which have the same content; the two would be equal only if $\sinh 1 = \frac{2}{3}$, while $\sinh 1 > 1$. For $\psi^*(x) = \frac{3}{4}x + \frac{1}{4}x^{-1}$ the jets are $\psi^*(1) = 1$, $\psi^{*\prime}(1) = \frac{1}{2}$ and $\frac{1}{2}\psi^{*\prime\prime}(1) = \frac{1}{4}$, and $\frac{1}{4}(x - x^{-1}) + \frac{3}{4}x + \frac{1}{4}x^{-1} = x$; the corrected charge is then $A\delta + R_J$. □

The recognition cost's charge reproduces Regge's quadratic action exactly: its difference from Regge, $(-1, -\frac{1}{2}, -\frac{1}{4})$, has $c_0 = 2c_1$. It fails at cubic order. Two kinds of half should be kept apart here. Regge's own half is common to the quadratic and cubic parts, a normalization (Remark 5.7). The recognition cost's charge keeps Regge's quadratic part and halves the cubic coupling, so it changes the self-coupling: a genuine halving. Its long-wave cubic vertex is one quarter of Einstein's $\Gamma\Gamma$ vertex, half of Regge's coupling, plus a discrepancy Δ, in the split given below.

Could a redefinition of the metric absorb Δ? Two kinds of change are harmless at cubic order. A redefinition of the metric that adds terms quadratic in the strain changes the cubic vertex only by *field-equation vertices*: vertices that contain the linearized field equation of one leg, and so vanish when all three legs are on shell. Write $G^1(h)$ and $S^1(h)$ for the linearized Einstein tensor and scalar of a leg. There are four field-equation vertices, each summed over the six assignments of the legs: $(h_i h_j) : G^1(h_k)$, $\operatorname{tr} h_i\,(h_j : G^1(h_k))$, $(h_i : h_j)\,S^1(h_k)$ and $\operatorname{tr} h_i \operatorname{tr} h_j\,S^1(h_k)$. A change of the coupling changes the cubic vertex by a multiple of Einstein's vertex. Δ is not of either kind. It splits as

$$\Delta = -\tfrac{1}{8}\sum \operatorname{tr} h_i\,(h_j : G^1(h_k)) - \tfrac{1}{32}\sum \operatorname{tr} h_i \operatorname{tr} h_j\,S^1(h_k) + R.$$

The first two terms are field-equation vertices, which a redefinition removes, and R, a sum of five explicit invariant monomials in the legs and their momenta, is what remains. One configuration shows that R cannot be removed. Take the strains E_{01}, E_{02}, E_{23}, where E_{ab} has the entry 1 at (a,b) and at (b,a) and 0 elsewhere, and the momenta

$$(0,-1,0,0), \qquad (0,1,0,-1), \qquad (0,0,0,1).$$

There Einstein's vertex and all four field-equation vertices vanish while $\Delta = -\frac{1}{4}$, so no combination of them equals Δ.

Within ratio charges, then, the recognition cost's halved cubic coupling cannot be kept: the only counterterm that passes the cubic test without touching the quadratic action restores Regge's jets, and the charge it gives is Regge's through cubic order, not the recognition cost's. The uniqueness in (c) is a statement about ratio corrections; the next subsection shows a correction of another kind.

5.9 The coupling direction

Two triangles show what this subsection adds. With squared edge lengths $(1,1,2)$ and $(4, \frac{1}{4}, \frac{17}{4})$ they have the same area $\frac{1}{2}$ but different shapes. A charge built from one function of area and deficit charges them alike at equal deficit; a cost that reads the shape need not. For instance, the factor $A + L_0 \cdot \nabla A$, with A the area and $L_0 = (1,1,2)$ the squared lengths of the lattice's own hinge of area $\frac{1}{2}$, is 1 on the first triangle and $\frac{25}{16}$ on the second.

So far every hinge was charged by one function of its area and deficit (Definition 2.1), or, in Section 5.7, by one such function for each class of hinge. Here a hinge's cost may also depend on its shape. A *shape charge* gives each hinge h of class c the cost $f_c(L_h)\,\delta_h$, with L_h the hinge's three squared edge lengths and f_c one function for each class. The charges in area form are among them, since a hinge's area is a function of its squared edge lengths; but, as the two triangles show, two hinges of one class with the same area and deficit may now cost different amounts, and a shape charge that uses this freedom is not an area-and-deficit charge, a hinge charge in the sense of Definition 2.1.

The question is whether the tests still single out Regge's charge. They leave one more direction, and it is the gravitational coupling itself. Shrink every departure from the reference lattice by the factor $1-\varepsilon$ and rescale the action by $1+2\varepsilon$: to first order this keeps the quadratic action and changes only the cubic coupling, by the factor $1-\varepsilon$ (Corollary 5.12), whereas the common

scale λ multiplies the whole action. The first-order effect of the operation is a change of the action along one direction Y, the *coupling direction*. Y charges each hinge through the factor $A + L_0 \cdot \nabla A$ above, with L_0 the reference squared lengths of its class (written out below), so it is a shape charge but not an area-and-deficit charge, nor one function of area and deficit for each class: no such function reproduces its cost.

What survives is this. Among shape charges the quadratic test and the cubic test at eleven shape witnesses leave only Y besides Regge's, up to inversion-odd terms that they cannot see (Theorem 5.11(c)), and to first order Y is exactly as consistent as Regge's. Section 5.10 takes the question one order further.

To write Y down, take one four-simplex with reference squared lengths L_0 and current squared lengths L, and write $\Psi(L) = \sum_t A_t(L)\big(\theta_t(L) - \theta_t(L_0)\big)$. It is minus the simplex's share of the Regge action, with angles measured from L_0. Write also $\Psi_Y(L) = \sum_t \big(A_t(L) + L_0 \cdot \nabla A_t(L)\big)\big(\theta_t(L) - \theta_t(L_0)\big)$. The hinge sum $Y = \sum_h \big(A_h + L_0 \cdot \nabla A_h\big)\delta_h$ is the shape charge whose factor for class c is $A + L_0 \cdot \nabla A$, with L_0 the reference squared lengths of that class. It has the quadratic jets $(2, 1)$ of Section 5.1 at every flat area. At the flat lattice its factor has the value $2a$, since $L_0 \cdot \nabla A = a$ there by Euler's identity, and the same gradient ∇A as the area itself. Unlike an area function with these jets, the factor has zero second derivative at the flat lattice along every line. This follows from Euler's identity for the gradient, which is homogeneous of degree zero.

For part (c) of the next theorem each f_c is twice differentiable at the flat Euclidean hinge, and the jets of a shape charge are the value, gradient and Hessian of each f_c there: six classes, each with one value, three gradient components and six Hessian components, sixty numbers in all. Its cubic test uses eleven *shape witnesses*: each has two on-shell transverse-traceless legs, built from integer spinors, and one gauge leg along a coordinate axis. Each witness is fixed by two null momenta, two polarization vectors and the axis of the gauge leg; their exact values enter the proof of part (c) and are printed in Appendix E. Their two on-shell legs are null for the Euclidean form, like those of W_1, W_2 and W_3, so the reading of Section 5.4 applies here as well: a shape charge whose quadratic form has another shell is tested off its own shell.

Theorem 5.11 (The coupling direction)**.**

(a) The per-simplex quadratic table of Y is identically zero, and its per-simplex cubic table is exactly minus Regge's; so its long-wave cubic vertex is minus Regge's for every symmetric polarization and every momentum-conserving triple.

(b) For every reference simplex L_0 and every nondegenerate simplex L, with $\ell = L - L_0$,

$$\frac{d}{dg}\Big[g^{-2}\,\Psi(L_0 + g\ell)\Big]_{g=1} = -\Psi_Y(L).$$

(c) A shape charge $\sum_h f_c(L_h)\delta_h$ passes the quadratic test at every momentum and the cubic test at the eleven shape witnesses if and only if two conditions hold:

(i) the part of its jets even under central inversion is λ times Regge's plus μ times Y's, for real λ and μ;

(ii) the odd part has a quadratic table of zero symbol.

Corollary 5.12 (The coupling, to first order)**.** *$S_{\mathrm{R}} + \varepsilon Y = (1 + 2\varepsilon)\, S_{\mathrm{R}}\big(L_0 + (1 - \varepsilon)\ell\big) + o(\varepsilon)$: to first order in ε, adding Y keeps the quadratic action and multiplies the cubic coupling by $1 - \varepsilon$.*

Theorem 5.11 and Corollary 5.12 are proved together. Part (a) follows from (b) in two lines, (b) is a calculation for one simplex, the corollary sums (b) over the simplices, and (c) is exact linear algebra in the sixty jets.

Proof. (a) follows from (b), which is proved next. Expand $\Psi(L_0 + \ell) = \sum_{n\geq 2} \Psi_n(\ell)$ in parts homogeneous of degree n in ℓ; the sum starts at $n = 2$ because Ψ vanishes at L_0 and so does its first variation, by Schläfli's identity. Then $g^{-2}\Psi(L_0 + g\ell) = \sum_n g^{n-2}\Psi_n(\ell)$, and (b) gives $\Psi_Y(L_0 + \ell) = \sum_n (2 - n)\,\Psi_n(\ell)$. So the quadratic part of Ψ_Y is zero and its cubic part is $-\Psi_3$: the per-simplex quadratic table of Y vanishes and its cubic table is minus Regge's, at every reference simplex. The tables say the same: the jets $(2, 1)$ at every area give a quadratic table that vanishes entry by entry, and the sixteen cubic tables that make up Y and Regge's add to zero coefficient by coefficient. (b) Let $F(g) = \Psi(L_0 + g\ell)$; along the line, with $'$ the derivative in the direction ℓ at L, $F'(1) = \sum_t \big[A_t'(\theta_t - \theta_t(L_0)) + A_t\theta_t'\big]$. Schläfli's identity along every direction gives $\sum_t A_t\theta_t' = 0$. Heron's area is homogeneous of degree one in the squared lengths, so Euler's identity gives $L \cdot \nabla A_t = A_t$ and $A_t' = \ell \cdot \nabla A_t = A_t - L_0 \cdot \nabla A_t$. Then $\frac{d}{dg}[F/g^2]_{g=1} = F'(1) - 2F(1) = \sum_t \big(-A_t - L_0 \cdot \nabla A_t\big)\big(\theta_t - \theta_t(L_0)\big) = -\Psi_Y(L)$. Corollary 5.12 follows from (b) by a first-order expansion. The deficit is minus the sum of the angle changes, so $S_{\mathrm{R}} = -\sum_\sigma \Psi_\sigma$ and $Y = -\sum_\sigma \Psi_{Y,\sigma}$ over the simplices. By (b), $G(g) = g^{-2}S_{\mathrm{R}}(L_0 + g\ell)$ has $G(1) = S_{\mathrm{R}}$ and $G'(1) = -Y$, so $G(1 - \varepsilon) = S_{\mathrm{R}} + \varepsilon Y + o(\varepsilon)$, while $G(1 - \varepsilon) = (1 + 2\varepsilon)\,S_{\mathrm{R}}(L_0 + (1 - \varepsilon)\ell) + O(\varepsilon^2)$. The quadratic part of the right side is $(1 + 2\varepsilon)(1 - \varepsilon)^2 S_2 = S_2 + O(\varepsilon^2)$ and the cubic part $(1 + 2\varepsilon)(1 - \varepsilon)^3 S_3 = (1 - \varepsilon)S_3 + O(\varepsilon^2)$.

(c) This is exact linear algebra in the sixty jets, as for Theorem 5.9; central inversion exchanges the classes of a pair and reverses the order of a hinge's edges. Gauge nullity at every momentum is a finite set of linear equations in the jets. At each witness the cubic value is linear in the jets with exact Gaussian-rational coefficients, and gives two real equations. Exact linear algebra reduces the system to two conclusions. First, the even part of the jets is λ times Regge's plus μ times Y's. Second, the class sums of the odd part vanish, which is zero symbol. The coefficients λ and μ are read off from three of the jets. Conversely, Regge's quadratic table is gauge null and Y's vanishes by (a); every cubic test row is invariant under inversion, so the odd part drops out, and Regge's charge and Y pass every witness, Y because its vertex is minus Regge's. □

So Y is the gravitational coupling, read as a deformation of the action: to first order it is exactly as consistent as Regge, and the gauge tests leave it free. The statement is first order in ε; we prove nothing about a finite change of the coupling within hinge sums, and a computation reported below bears on it.

Two steps settle the coupling within area-and-deficit charges (Definition 2.1). First, the free direction is excluded: Y is not an area-and-deficit charge, as the two triangles above show, so restricting to area-and-deficit charges leaves it out. Second, the strength is fixed. Among area-and-deficit charges in area form, the gauge tests leave Regge's jets up to a common scale λ (Theorem 5.3). The record clause, with the area form or the content reading, makes the charge $s_{\mathrm{p}} A\delta$ (Theorem 5.5), and in particular $\lambda A\delta$ meets the record clause at $k \geq 1$ exactly when $\lambda = s_{\mathrm{p}}$ (Theorem 5.5(e)).

The same direction shows the limit of Theorem 5.10(c). Recall that the recognition cost's charge has one half of Regge's cubic vertex plus the discrepancy Δ, and that the ratio correction c^* of that theorem restores Regge's jets. Adding half the coupling direction to c^* cancels Δ.

Corollary 5.13 (The corrected charge without the discrepancy)**.** *The correction $c^* + \frac{1}{2}Y$ has a quadratic table of zero symbol and long-wave cubic vertex $-\Delta$. The recognition cost's charge plus this correction has one half of Regge's long-wave cubic vertex at every momentum-conserving triple, and it passes the cubic test at W_1, W_2 and W_3.*

Proof. The long-wave cubic vertex is linear in the table, so follow it in three stages. *The charge alone*: C_J has one half of Regge's vertex plus Δ (Section 5.8). *With the ratio correction*: by Theorem 5.10(c), $C_J + c^*$ has Regge's jets and so Regge's tables and Regge's vertex; so c^* alone has Regge's vertex minus that of C_J, which is one half of Regge's minus Δ. *With the coupling correction*: by Theorem 5.11(a), $\frac{1}{2}Y$ has minus one half of Regge's vertex, so $c^* + \frac{1}{2}Y$ has vertex $-\Delta$, and $C_J + c^* + \frac{1}{2}Y$ has one half of Regge's vertex, which vanishes with one pure-gauge leg wherever Regge's does, at the three witnesses in particular. The quadratic table of c^* has zero symbol since $c_0^* = 2c_1^*$, and that of Y vanishes identically. □

So with a correction that is not an area-and-deficit charge at all, since Y is not one, the halved coupling can be kept at cubic order. That is where the proved statement stops.

5.10 Quartic order

This subsection takes the normalization of Remark 5.7 one order further and then asks whether the halving of Section 5.8 can be continued. It has three parts: one normalization at quartic order, tree-level scattering, and a changed coupling, which meets an obstruction on listed configurations.

Three objects. Each statement below names the object it concerns.

- The *quartic table*: a per-simplex table of fourth order in the strain.
- The *table action*: the metric-sector action assembled from the lattice's tables of orders two, three and four.
- The *Lorentzian lattice action*: the $\kappa = 1$ lattice action itself, a sum over cells of one cell function.

Results proved in the companion paper [12] carry that citation, results proved in work not yet published are marked *(proved, unpublished)* below, and what is open is marked OPEN where it occurs. The two actions are made differently. The table action takes, at each order from two to four, that order's table, a polynomial in one simplex's ten edge perturbations (Section 5.1), sums it over the simplices and adds the three orders. The Lorentzian lattice action is the $\kappa = 1$ action itself: one cell function summed over the cells, with its own quadratic, cubic and quartic cell forms. The established conversion is that each of these cell forms is $\sqrt{3} = \sqrt{|\det g_1|}$ times the Regge-action form of the corresponding table (*the table action*, below), so through quartic order the two actions differ only by that volume factor.

Two signs to keep apart. The cubic and quartic tables are Taylor tables of Ψ, which is minus the simplex's share of the Regge action (Section 5.9); the Regge-action form of a table is therefore minus the table. And a long-wave vertex in the convention of Section 5.2 leaves out the factor $i^2 = -1$ of its two derivatives. The cell function of the Lorentzian lattice action is that action's own share, not Ψ. Where a statement below about that action carries a minus sign, the sign comes from one of these two conventions, and each such place says which.

One normalization at quartic order. In short:

- the quartic table, against Einstein's quartic vertex: its contact vertex is one half of it;
- the table action, against Einstein's action and against $\int \sqrt{|g|}R$: it is one half of each through quartic order, at $\kappa = 0$ and at $\kappa = 1$;
- the Lorentzian lattice action itself: it is one half of Einstein's action through quartic order, with the volume factor $\sqrt{|\det g_1|} = \sqrt{3}$ that the tables omit.

The statements below give the factors and the status of each. Together they carry the half of cubic order to quartic order, which supports reading it as a normalization rather than a change of the self-coupling.

The quartic table. At $\kappa = 0$ it is the quartic Taylor table of Ψ, Regge's own up to the sign just named. At $\kappa = 1$ it is the quartic term of the Lorentzian lattice action up to the factor $-\sqrt{3}$: along every direction, the fourth-order term of that action's cell function is $-\sqrt{3}$ times the table, which is $+\sqrt{3}$ times the table's Regge-action form [12]. The table's long-wave four-graviton contact vertex, in the convention of Section 5.2, is exactly one half of the quartic vertex of Einstein's action in its first-order form, the $\Gamma\Gamma$ Lagrangian of Section 5.6, for every κ with $(2\kappa-1)(3\kappa-1)(4\kappa-1) \neq 0$, all polarizations and all momenta summing to zero, real or complex [12]. As at cubic order (Section 5.6) the two signs cancel, and this says that Regge's own quartic vertex is one half of Einstein's. At $\kappa = 1$ it therefore holds for the quartic term of the Lorentzian lattice action too, up to the factor $-\sqrt{3}$. In the same convention, which leaves out the factor i^2, the long-wave quartic vertex of that action's cell function is $-\frac{\sqrt{3}}{48}$ times Einstein's quartic vertex at $\kappa = 1$ in the cell function's own normalization, which carries a factor $\frac{1}{24}$ that the table's convention omits, so $-\frac{\sqrt{3}}{2}$ times it in the table's convention [12]. With the factor i^2 in place this is the statement used in the rest of this subsection: the Lorentzian lattice action's own four-graviton vertex is $+\frac{\sqrt{3}}{2}$ times Einstein's quartic vertex. The two numbers $-\frac{\sqrt{3}}{2}$ and $+\frac{\sqrt{3}}{2}$ describe one action in two conventions.

The table action. With the quadratic and cubic identities of Section 5.6, the table action is one half of Einstein's action in its first-order form through quartic order at long wavelength, at $\kappa = 0$ and at $\kappa = 1$ [12]: one normalization of the whole action. At $\kappa = 1$ the same holds for the Lorentzian lattice action itself, with the volume factor that the tables omit: its quadratic, cubic and quartic cell forms are each $\sqrt{3} = \sqrt{|\det g_1|}$ times the Regge-action forms of the corresponding tables, so the half at every order is a single $\frac{1}{2}$ times $\sqrt{|\det g_1|}$ [12].

From the first-order form to $\int\sqrt{|g|}R$. At second, third and fourth order the vertices of $\sqrt{|g|}R$ itself are those of the first-order form, on every background, with the volume factor $\sqrt{\det g}$ in Euclidean and $\sqrt{-\det g}$ in Lorentzian signature, for all polarizations and all momenta summing to zero; the quadratic one is $-\frac{1}{2}$ times the Fierz–Pauli form [12]. The two densities differ by a total derivative, which has no contact term when the momenta sum to zero. So the table action is one half of $\int\sqrt{|g|}R$ itself through quartic order at long wavelength, at $\kappa=0$ and at $\kappa=1$. The quartic table is Regge's own quartic Taylor table, up to the common factor $\sqrt{1-4\kappa}$, at every Euclidean $\kappa<\frac{1}{4}$ [12], and at $\kappa=1$ it is the Lorentzian lattice action's quartic term up to the factor $-\sqrt{3}$ (above).

Tree-level scattering. Two amplitudes appear: the table action's, from its vertices and the lattice's own propagator, and at $\kappa=1$ the Lorentzian lattice action's own. From the table action's vertices and propagator a tree four-graviton amplitude is defined at long wavelength, in the metric sector; on shell it does not depend on the gauge-fixing parameter, on the Euclidean or the Lorentzian background (proved, unpublished). In the respects stated here the lattice's long-wave gravitons scatter as general relativity's do.

The lattice's own amplitude. At $\kappa=1$ the tree four-graviton amplitude built from the Lorentzian lattice action's own three- and four-graviton vertices and its own gauge-fixed propagator is, at long wavelength, $\sqrt{3}=\sqrt{|\det g_1|}$ times the tree amplitude of $\frac{1}{2}\int\sqrt{|g|}R$, Einstein's at $G=1/(8\pi s)$ (Remark 5.7), and pure-gauge gravitons decouple from it (proved, unpublished). This is the lattice's own amplitude, not a table identified with one. Its four-graviton vertex is, to leading order at long wavelength, $\frac{\sqrt{3}}{2}$, one half of $\sqrt{|\det g_1|}$, times the quartic vertex of Einstein's action in its first-order form, as for the quartic table (proved, unpublished).

Gauge legs decouple. The amplitude vanishes whenever one leg is a pure-gauge mode, at every physical configuration whose three channel invariants are nonzero [12]; this follows from the quartic Noether identity of Einstein's first-order action, which holds for all symmetric complex polarizations and all conserved complex momenta. The same holds at $\kappa=1$: Einstein's vertices satisfy the cubic and quartic Noether identities on the $\kappa=1$ background, and the long-wave amplitude at $\kappa=1$ decouples pure-gauge gravitons at every physical configuration (proved, unpublished).

Helicity. On the Euclidean lattice the amplitude has general relativity's helicity structure at every generic configuration: it vanishes unless exactly two gravitons have positive helicity, and then it equals general relativity's two-minus, two-plus formula, $-\frac{1}{256}\langle m\,m'\rangle^4[p\,p']^4/(stu)$, with $\langle m\,m'\rangle$ and $[p\,p']$ the spinor brackets of the two negative- and the two positive-helicity legs, s,t,u the channel invariants, and the helicity polarizations normalized so that a leg's two polarizations pair to $-\frac{1}{2}$ [12]. On the $\kappa=1$ lattice the amplitude has the same structure: it vanishes unless exactly two gravitons have positive helicity, and otherwise equals general relativity's two-minus, two-plus formula (proved, unpublished).

Every κ. Einstein's vertices satisfy the cubic and quartic Noether identities on every background g_κ, and the lattice's own long-wave quadratic form is one half of Einstein's at every κ with $4\kappa-1\neq 0$ (both proved, unpublished). So at every κ with $(2\kappa-1)(3\kappa-1)(4\kappa-1)\neq 0$ the table action's tree four-graviton amplitude, from its vertices and its own propagator, is general

relativity's: gauge invariant, with the helicity selection and the two-minus, two-plus formula (proved, unpublished). For the action built at each $\kappa > \frac{1}{2}$ from its own cell function, as the Lorentzian lattice action is at $\kappa = 1$, the quadratic, cubic and quartic cell forms are each $\sqrt{4\kappa - 1} = \sqrt{|\det g_\kappa|}$ times the Regge-action forms of the corresponding tables [12], and its full tree four-graviton amplitude, with the contact term, is $\sqrt{4\kappa - 1}$ times the table action's and gauge invariant (proved, unpublished). At every κ outside $\{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\}$ the five non-metric edge modes are massive (their block has determinant $3\kappa(2\kappa-1)^3/(16(4\kappa-1)^4)$), and with all fifteen edge classes on the internal line and in the legs the long-wave four-graviton amplitude is the metric-sector amplitude, Einstein's, and pure-gauge gravitons decouple (proved, unpublished). At $\kappa = 0$ the decoupling fails: the body diagonal is a zero mode.

Earlier work. Roček and Williams showed that Regge calculus on the hypercubic lattice, the Kuhn lattice at $\kappa = 0$, reproduces linearized general relativity in the weak-field limit [15, 16]. In that expansion the hyperbody diagonal, the edge along $\mathbf{1}$, does not enter the action at second order [17, Sec. II], so its fluctuations are a zero mode. Cheeger, Müller and Schrader proved that the Regge action converges to the Einstein–Hilbert action, in the sense of measures, as suitable piecewise flat spaces approach a smooth one [24]. On the Euclidean lattice the identities of this subsection also follow from that convergence theorem; here they come with explicit tables and a finite, exact proof, and on the Lorentzian lattice, at $\kappa = 1$ and at every $\kappa > \frac{1}{2}$, they lie beyond its reach [12].

The title's every-order claim rests on Theorem 4.1, not on these results.

A changed coupling. Can the halved coupling be kept at quartic order? The question is asked within the shape charges of Theorem 5.11(c), and it has three parts. The quadratic and cubic actions must stay as they are. The hinge costs may change only through their sixty third-derivative jets (six classes, each with the ten third derivatives of its function of three squared lengths). And the change must supply the long-wave quartic gauge variation that continuing the halved coupling $g = \frac{1}{2}$ to quartic order needs.

The target. Rescaling the edges shows what that variation should be. The action $g^{-2}S_{\mathrm{R}}(L_0 + g\ell)$ has the quadratic part of S_{R}, g times its cubic part and g^2 times its quartic part; Corollary 5.12 is its first-order form. At $g = \frac{1}{2}$ its quartic part is one quarter of Regge's, and so is its long-wave quartic gauge variation. One step is an INFERENCE: that a continuation of the halved coupling needs exactly this variation, and that the same holds for other finite changes of the coupling.

The computation. The quartic gauge variation of the lattice's quartic table lies outside the space of long-wave quartic gauge variations that the sixty third-derivative jets supply. This holds on finitely many listed four-leg configurations, each with three transverse-traceless legs and one gauge leg and all momenta null. That space is linear, so every nonzero multiple of the variation lies outside it too, one quarter in particular; at $\kappa = 0$ and $\kappa = \frac{1}{10}$ it is 32-dimensional. This is proved at $\kappa = 0$, at $\kappa = \frac{1}{10}$ and on the Lorentzian lattice at $\kappa = 1$, by exact computation configuration by configuration [12]. So, if the target above is the variation needed, no hinge cost of this family keeps the halved coupling at quartic order on those configurations. On the Lorentzian lattice, where the five non-metric edge modes are massive, the obstruction also

survives them, whatever admixture of them is chosen, and at $\kappa = \frac{1}{10}$, where they are massive too, it survives them as well (all proved, unpublished).

The three settings. They differ in the quartic table the computation uses. At $\kappa = 0$ it is Regge's own quartic Taylor table, and at $\kappa = \frac{1}{10}$, as at every Euclidean $\kappa < \frac{1}{4}$, it is that table up to the common factor $\sqrt{1-4\kappa}$, here $\sqrt{3/5}$ [12]. At $\kappa = 1$ the computation uses the lattice's quartic table, which is the quartic term of the Lorentzian lattice action up to the factor $-\sqrt{3}$ [12], so there the obstruction is a statement about the lattice action itself. In all three settings the obstruction concerns a changed coupling; it does not touch Regge's own action.

6 The graviton's quadratic form is the recognition cost's long-wave form

Section 5.8 used the recognition cost to build a hinge charge, and that charge fails at cubic order. Here the cost enters differently: as the cost of a strain, at quadratic order in the long-wave limit. Two quantities are compared: the graviton's quadratic form, the Fierz–Pauli form of linearized gravity, and the recognition cost's long-wave form, its cost for a strain, a departure of the metric from flat space. The published result, stated below with its scope, is an identity between the two quantities on the two physical polarizations of a wave in the lattice's rest frame. In the Fierz–Pauli normalization the graviton's quadratic form is twice the recognition cost's long-wave form, times the squared wave covector.

Write η for the squared wave covector of the identity just stated. At a null wave $\eta = 0$ and both sides vanish, so on shell the identity carries no information: it concerns the off-shell kinetic operator on the polarizations [9, Sec. 6]. The cited paper identifies the graviton with the cost the ledger charges for comparing a geometry with flat space, read on its two physical polarizations (an IDENTIFICATION): the graviton is not a field added to the ledger.

Status: the graviton's quadratic form (published as a preprint)

Proved [9]. On transverse-traceless strains the Fierz–Pauli form is $\eta \cdot 2J_{\mathrm{lw}}$, $J_{\mathrm{lw}}(X) = \frac{1}{2}\operatorname{tr} X^2$, for every cost with $J''(1) = 1$. With Regge's action at $\kappa = 1$, the long-wave flat Hessian on such strains is $-\frac{\sqrt{3}\pi^2}{2}\eta(m) \cdot 2Q(H)$ for every charge Q of spatial strains that has the lattice's symmetries and charges the plus and cross strains of one face wave as J does. The precise premise is stated below.

Identification. The cited paper identifies the graviton with the cost the ledger charges for comparing a geometry with flat space; the record charge, at which the writing rules are read, is a second identification.

Premise routes (not derived). Alternatives for supplying the premise: two rules for how the ledger writes a strained step (least-cost writing; one-comparison writing with local shares; together, the writing principle), or the rotation route, which needs no writing rule.

The recognition cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ is the one continuous cost, zero at $x = 1$, that obeys its composition law and the calibration $J''(1) = 1$ [14, 9]. Read on a metric compared with flat space, its long-wave form on a strain X is $J_{\mathrm{lw}}(X) = \frac{1}{2}\operatorname{tr} X^2$. The identity in the box

above uses J only through $J''(1) = 1$, so it holds for every cost that charges each principal stretch through an f with $f(1) = f'(1) = 0$ and $f''(1) = 1$. By itself it therefore does not test that identification [9].

The ledger writes each step of the strained lattice as a record, a list of postings, credits and ticks that add up to the step: a posting adds one to a counter, a credit takes one away, and a tick advances all four counters at once and reads no strain. The charge of a record is an IDENTIFICATION [9, Sec. 3.4]: each posting or credit pays a fixed comparison charge, and one that carries a share u of the strain pays in addition $\frac{1}{2}u^2$, the quadratic part of the recognition cost J.

A small example shows how a record is chosen. Take the double step $d = e_a + e_b$, which advances two counters, and let it read the strain $s = d^\top H d$; the shares of a record's postings add up to s. Every record of d has an even number of entries, postings and credits together, at least two. A record with two postings, for instance one on each of the two counters, and shares $u_1 + u_2 = s$ pays $2\rho + \frac{1}{2}(u_1^2 + u_2^2)$, with $\rho = 6(\cosh 1 - 1)$ the comparison charge of one posting; this is least, $2\rho + s^2/4$, when the shares are equal, $s/2$ each. A record with four entries pays at least $4\rho + s^2/8$, which is more while $s^2 < 16\rho$. So for a small strain the cheapest record of this step has two postings that share the strain equally [9, Sec. 3.4]. Whether the ledger chooses the cheapest record is a premise, the first of the routes below; applied to the plus and cross strains of one face wave, at a record charge that is itself an identification, it gives the two numbers on which the lattice identity rests.

The identity involves three things that are fixed separately. Regge's action fixes the shape: the whole transverse-traceless form as a function of the strain. J's calibration sets only the unit. The strength is the record clause's: neither Regge's action, nor J's calibration, nor the premises on the ledger's charge, introduced below, fix it.

In precise form, the premise concerns one wave and one pair of strains, and two numbers are enough. Given the lattice's symmetries, the ledger's charge on that pair fixes it on every transverse-traceless strain of every wave. Take the face wave $m = (1, 1, -1, -1)$ and the face axes $p_2 = (1, -1, 1, -1)$ and $p_3 = (1, -1, -1, 1)$. Its two polarizations are the plus strain $H'_+ = p_2 p_2^\top - p_3 p_3^\top$, diagonal in the frame of the face axes, and the cross strain $H'_\times = p_2 p_3^\top + p_3 p_2^\top$, off-diagonal; the recognition cost charges them alike, $J_{\mathrm{lw}}(H'_+) = J_{\mathrm{lw}}(H'_\times) = 16$.

Suppose that the ledger's charge Q of spatial strains has the lattice's symmetries, charges the plus strain as J does, $Q(H'_+) = 16$, and charges the cross strain alike, $Q(H'_\times) = Q(H'_+)$ (no birefringence at long wavelength).

Then $Q = J_{\mathrm{lw}}$ on the transverse-traceless strains of every wave, and with Regge's action at $\kappa = 1$ the scaled flat Hessian there tends to $-\frac{\sqrt{3}\pi^2}{2}\eta(m) \cdot 2Q(H)$ [9, Lemma 3.5, Prop. 6.2 and Thm. 6.1]. The wave and the pair matter: two such numbers at a wave along a threefold axis, or for a rotated pair at the face wave, do not suffice [9, Sec. 3.4].

Three premise routes give the two numbers. They are alternatives, and none is derived [9, Sec. 3.4]. The first two follow from the way the ledger chooses the record that writes a strained step; together we call them the *writing principle*. The third, the *rotation route*, needs no writing rule and no record charge.

Least-cost writing. The ledger chooses the record of least charge, which for a small strain has the fewest postings, each carrying an equal share of the strain.

One-comparison writing with local shares. Each step is one comparison event of at most two postings on distinct counters, and locality of the shares makes them equal [9, Props. 3.6 and 3.7]. Under either rule the two numbers follow, at a record charge that is itself an identification.

The rotation route. The charge is unchanged by every proper rotation of space, which gives no birefringence at long wavelength. It charges a stretch along each face axis as J does, with independent face axes, which gives the face calibration.

The routes agree on every traceless spatial strain and are alternatives, not parts of one cost.

Beyond long wavelength: a birefringent correction. At long wavelength the lattice graviton propagates as Einstein's does; beyond it the $\kappa = 1$ lattice predicts a correction that Einstein's theory does not have. Take the rest frame of g_1, with time along $\mathbf{1}$, a wave of wave number k moving along a spatial unit vector n orthogonal to $\mathbf{1}$, and $I_3 = \sum_\mu n_\mu^3$, $I_4 = \sum_\mu n_\mu^4$; write b for the lattice spacing. The two polarizations travel at the phase speeds

$$v_\pm = 1 - c_\pm(n)\,k^2b^2 + O(k^4b^4), \qquad c_\pm = \frac{79}{96} + \frac{\sqrt{3}}{4}\,I_3 + \frac{29}{24}\,I_4 \pm \frac{4}{3}\sqrt{\left(\frac{5}{4} - I_4\right)^2 - \frac{4}{3}I_3^2},$$

in units of the long-wave speed, that of the light cone of g_1 (proved in work not yet published). Here n is the direction in which the wave moves, and that is opposite to the spatial part K_s of its wave covector K when $\mathbf{1}\cdot K > 0$: at the point $x = t\,\mathbf{1} + y$, with y orthogonal to $\mathbf{1}$, the phase $K\cdot x$ is $(\mathbf{1}\cdot K)\,t + K_s\cdot y$, so the crests move along $-K_s$ as t grows, and $n = -K_s/|K_s|$. Precisely: along every real null direction of g_1, a wave through fourth order in the wave vector that is not pure gauge, with all fifteen edge classes, exists for exactly two values of the fourth-order term of $g_1^{-1}(K,K)$, namely $2|K_s|^4c_\pm$ with K_s the spatial part of the wave covector in lattice units, and the odd orders vanish. For example, for the covector $K = (1,4,1,3)$, a wave moving along $n \propto (5,-7,5,-3)$, $|K_s|^2 = \frac{27}{4}$, $c_+ = \frac{1744}{729}$ and $c_- = -\frac{16}{243}$, so the two values are 218 and -6: the two polarizations disperse differently and neither keeps to the light cone. For the covector $(1,1,1,0)$, on a threefold axis, both coefficients are $\frac{16}{9}$, and the polarizations share one value. The two speeds agree only along the four threefold axes, in both directions of each, and differ most, by $\frac{8}{3}k^2b^2$, along the twofold axes. The slower polarization is always slower than the long-wave speed, c_+ running from $\frac{23}{18}$ to about 2.49; the faster one is faster along about one direction in seven, c_- running from $-\frac{5}{24}$ to $\frac{16}{9}$; and n and $-n$ differ.

The correction is not special to $\kappa = 1$. For every $\kappa \notin \{0, \frac{1}{4}, \frac{1}{2}\}$ and every real light-cone direction of g_κ, the splitting of the two polarizations is proportional to $\kappa(3\kappa-1)(5\kappa-1)/(2\kappa-1)$. So every $\kappa > \frac{1}{2}$ is birefringent at this order (proved in work not yet published; for example, at $\kappa = \frac{3}{4}$ along $(2,0,1,1)$ the two values are 13 and $\frac{43}{16}$). The factor vanishes at three values, and none of them is a lattice of the kind this paper treats. At $\kappa = 0$ and $\frac{1}{5}$ the metric g_κ is Euclidean and has no real light-cone direction. At $\kappa = \frac{1}{3}$ it is Lorentzian, but the value lies outside the range $\kappa > \frac{1}{2}$ in which every triangle is timelike, and the flat lattice is degenerate there: every hinge whose edges have one, two and three postings has flat area $\sqrt{(1-3\kappa)/2} = 0$,

the derivatives of the dihedral angles at such a hinge are singular, and the cubic table has a pole (Theorem 5.6(b)). The single branch found at $\kappa = \frac{1}{3}$, for example along $(-1, 5, -1, 3)$, is therefore a property of the wave operator's plane-wave form continued in κ as a rational function, not of a nondegenerate Regge lattice. This is the fourth-order term in the wave vector, beyond the two-derivative vertices: the long-wave results of Section 5.10, the quadratic, cubic and quartic vertices one half of Einstein's and the tree amplitude Einstein's at every $\kappa > \frac{1}{2}$, are unchanged. So the lattice predicts Einstein's theory at leading order, with an anisotropic, polarization-dependent correction suppressed by $(kb)^2$.

Its size. The paper fixes the lattice spacing in physical units only through the record clause: the lattice's unit of length is λ_{rec}, and on reading the action as Einstein–Hilbert's, an INFERENCE, $G\hbar = \lambda_{\mathrm{rec}}^2/(4\ln 2)$, with G the lattice coupling and ZG the measured one (Section 2). So $b = \lambda_{\mathrm{rec}} = (4\ln 2/Z)^{1/2}\,\ell_{\mathrm{P}} \approx 1.665\,Z^{-1/2}\ell_{\mathrm{P}}$, with $\ell_{\mathrm{P}} = 1.616 \times 10^{-35}$ m the Planck length [42]; at $Z = 1$ that is 2.69×10^{-35} m. Z is not yet computed, so b is fixed only up to it. For a quantum of energy E, $(kb)^2 = 4\ln 2\,(E/E_{\mathrm{P}})^2/Z$, with $E_{\mathrm{P}} = 1.221 \times 10^{19}$ GeV the Planck energy [42]. At 100 Hz, in the band of the ground-based detectors, $(kb)^2 \approx 3\times10^{-81}/Z$, so the two polarizations differ in phase speed by at most about $8.5 \times 10^{-81}/Z$ of the long-wave speed; for a graviton of 10^{20} eV, the order of the energies of the most energetic cosmic rays, $(kb)^2 \approx 2 \times 10^{-16}/Z$ and the difference is at most $5 \times 10^{-16}/Z$.

Arrival times follow the group speed, the speed of a wave packet, not the phase speed. Along a fixed direction the frequency is proportional to $k\,v_{\pm}$, so the group speed is $d(kv_{\pm})/dk = 1 - 3\,c_{\pm}(n)\,k^2b^2$: its correction is three times that of the phase speed. The two packets therefore differ in speed by $3(c_+ - c_-)k^2b^2 = 8\,\sigma(n)\,k^2b^2$, with $\sigma(n) = \sqrt{(\frac{5}{4} - I_4)^2 - \frac{4}{3}I_3^2}$, which runs from 0 on the threefold axes to 1 on the twofold axes.

Against measurement. Three published bounds bear on the correction. Two come from the arrival times of gravitational waves and lie far above a correction at the spacing $b = \lambda_{\mathrm{rec}}$. The third comes from cosmic rays and does not.

Arrival with light. The arrival of GW170817 together with GRB 170817A, in 2017, puts the difference between the speeds of gravity and light between -3×10^{-15} and $+7 \times 10^{-16}$ of the speed of light [26]. At 100 Hz, if light keeps to the long-wave cone to that accuracy (its own correction on the lattice is not computed here) and the signal carried the slower polarization, whose packet is slower than the cone by $3c_+k^2b^2$ with c_+ between $\frac{23}{18}$ and about 2.49, this requires only $b \lesssim 1$ cm: $3c_+(kb)^2 \le 3 \times 10^{-15}$ gives $b \le 1.34$ cm at the least c_+ and $b \le 0.96$ cm at the greatest, with $k = 2\pi \times 100\,\mathrm{Hz}/c = 2.1 \times 10^{-6}\,\mathrm{m}^{-1}$.

Splitting of the arrival. Kostelecký and Mewes read a bound on a birefringence growing as the square of the frequency from GW150914: its chirp peak, about 3 ms wide at about 100 Hz, shows no splitting, from a source at redshift 0.09 [27], about 410 Mpc away by the discovery analysis of 2016 [39]. That reading assumes that the source excited both modes; it is not a measurement of two polarizations. The catalogue of 2019 gives the distance as 440^{+150}_{-170} Mpc [40]. Applied to the lattice under the same assumption, a splitting of the arrival times below 3 ms over a travel time of 4.2×10^{16} s asks that $8\,\sigma(n)\,(kb)^2 \le 7 \times 10^{-20}$, that is,

$$b \lesssim \frac{4.5 \times 10^{-5}\ \mathrm{m}}{\sqrt{\sigma(n)}}$$

for a source in the direction n as seen from the lattice; at the catalogue's distance the number is 4.3×10^{-5} m. In the most sensitive orientation, a twofold axis, this is $b \lesssim 4.5\times10^{-5}$ m. Along a threefold axis the splitting vanishes at this order and the event gives no bound. The orientation of the lattice relative to the source is not known, so one event gives no unconditional bound on b; it gives this one for each assumed orientation.

At $b \approx 1.665\, Z^{-1/2}\ell_{\mathrm{P}}$ these two conditions hold with room to spare. At $Z = 1$ the spacing lies below 1 cm by a factor 3.7×10^{32} and below 4.5×10^{-5} m by a factor 1.7×10^{30}. Since b grows as $Z^{-1/2}$, the two would fail only for Z below about 7×10^{-66} and 4×10^{-61}.

Cosmic rays that do not radiate. The sharpest published bound is of another kind. If gravitational waves are slower than a body's own limiting speed, a fast enough body radiates them, as a charge radiates light in a medium, and loses its energy; that cosmic rays of the highest energies arrive therefore bounds the slowing. Kostelecký and Tasson worked out the rate of that radiation and the bounds that follow [38]. For a slowing that grows as the square of the wave number they write the waves' phase speed as $1-\frac{1}{2}s(\hat{n})\,k^2$ and bound the part of s that does not depend on the direction: $\bar{s}^{(6)}_{00} < 2\times10^{-31}\,\mathrm{GeV}^{-2}$, where a slowing that is the same in every direction has $s=\bar{s}^{(6)}_{00}/\sqrt{4\pi}$ [38, Table 3 and Eqs. (20), (45)]. The bound is one-sided, it is given without a confidence level, and it is stated for a deliberately cautious case: the most energetic event of their sample, of 320 EeV, read as an iron nucleus in which one parton with a tenth of a nucleon's energy radiates, so that the radiating energy is $E_f = E/560$, over a distance of 10 Mpc. It scales as $E_f^{-7/2}$ and as the inverse square root of the distance [38, Eq. (44)]. Two of its assumptions matter here: the radiating body keeps its own limiting speed, and the slowing is the same for the two polarizations.

On the $\kappa = 1$ lattice the slower polarization is slower than the long-wave speed in every direction, so the test applies if matter keeps to the long-wave cone. The phase speeds $1-c_{\pm}(n)\,k^2b^2$ give $s_{\pm}(n) = 2c_{\pm}(n)\,b^2$. The mean of the two coefficients, averaged over directions, is exactly $\frac{41}{30}$. To see it, write n in orthonormal coordinates (x,y,z) of the hyperplane orthogonal to $\mathbf{1}$: then $I_3 = 3xyz$, which averages to zero, and $I_4 = \frac{1}{4}(x^4+y^4+z^4)+\frac{3}{2}(x^2y^2+y^2z^2+z^2x^2)$, which averages to $\frac{1}{4}\cdot\frac{3}{5}+\frac{3}{2}\cdot\frac{3}{15}=\frac{9}{20}$ by the sphere's moments $\langle x^4\rangle=\frac{1}{5}$ and $\langle x^2y^2\rangle=\frac{1}{15}$; so the mean is $\frac{79}{96}+\frac{29}{24}\cdot\frac{9}{20}$. As the letter's coefficient that is $\bar{s}^{(6)}_{00} = 2\sqrt{4\pi}\,\frac{41}{30}\,b^2 \approx 9.69\,b^2$, which at $Z=1$ is $1.8\times10^{-37}\,\mathrm{GeV}^{-2}$. The lattice splits the two polarizations at the same order, and its slowing depends on the direction; the letter's rate is worked out for a slowing without birefringence, and its bound on $\bar{s}^{(6)}_{00}$ supposes that this coefficient alone is not zero. The comparison is therefore an estimate. Over directions the slower polarization's coefficient c_+ runs from $\frac{23}{18}$ to about 2.49, that is, from about 0.93 to 1.82 times the mean $\frac{41}{30}$, and on that spread we take the comparison to be good to a factor of about two. What holds then depends on what the 320 EeV event was.

- *An iron nucleus, the letter's case.* The lattice's coefficient is below the bound by a factor 1.1×10^{6}: six orders of magnitude in the coefficient, three in the length. The bound asks $b \lesssim 2.8\times10^{-32}$ m, about a thousand times the spacing at $Z=1$, and it fails only for Z below about 9×10^{-7}.

- *A proton, with a tenth of its energy on the radiating parton.* The bound is tighter by $56^{7/2} \approx 1.3 \times 10^6$, and at $Z = 1$ the lattice's coefficient sits at it: 1.2 times the bound for the mean of the two polarizations, and 1.1 to 2.2 times it for the slower polarization alone, depending on the direction. No margin is left.
- *A proton radiating at its full energy*, the case the letter itself names when it calls its bounds conservative. The bound is tighter by $560^{7/2} \approx 4.2 \times 10^9$, and at $Z = 1$ the lattice's coefficient exceeds it by a factor of about 3.7×10^3.

This paper takes no position on which the event was. What it can say is what is missing. The bound applies only if matter is *not* slowed on the lattice as the waves are: a body whose own limiting speed falls with its energy as fast as the waves' does not outrun them and does not radiate. The second-order term of a body's own motion on this lattice is not computed in this paper, nor is light's, and that one calculation decides whether the cosmic-ray bound applies at all. The published bounds on a term of that kind for matter and for light themselves are not assessed in this paper either. Until it is done, the correction is consistent with the bound on the letter's own cautious case and is not shown to be consistent beyond it.

7 The one-loop graviton: what is established and what is open

Section 6 gave the graviton's quadratic form, the ledger's cost of a strain. The question here is whether the sum over geometries keeps the graviton massless at one loop, that is, whether its propagator keeps a long-wave pole. A computed number at zero momentum cannot settle that alone. For the path sum without the weight, if the imaginary part of the one-loop term at zero momentum is bounded below by a positive constant m, and near zero momentum the self-energy changes by less than m, there is no long-wave pole on the strains tested, those of type E (step 3 of the chain below); both bounds are open premises, and the computed numbers point only to the first. The chain is a statement about the strains of type E. It does not control the strains of type T_2, on which a pole is not excluded (*What the chain covers*, below). Under the paper's premises on the action, with a third open input, a proof that the remainder's real part is not zero, the near-flat one-loop graviton is not massless (below). Every result of this section marked as from work not yet published comes from one body of work, which we call the unpublished one-loop work. The paper claims no massless graviton, and its title claims none.

The test. Zero momentum means a strain that is uniform over the lattice, a constant change of the cell's shape that keeps its volume. The number tested is the one-loop correction to the quadratic form on such a strain, the self-energy of the transverse-traceless strains at zero momentum, which we call the *remainder*. The one-loop diagram model has a unit-weight gauge term, no ghost line and the Feynman regulator, and the remainder is taken per lattice site and in lattice units. The *zero-momentum masslessness test* asks whether the remainder vanishes in the *ordered limit*: first the torus grows at a fixed regulator $\varepsilon > 0$, then $\varepsilon \to 0$ [5].

What the number measures. The real part of the remainder is the lattice's resistance to a uniform transverse-traceless strain, a one-loop *shape stiffness*. In the effective action of constant strains it is a mass term [5]. It is also a test of covariance: a one-loop determinant that obeys the Ward identity of volume-preserving linear vertex maps has zero remainder [5, Thms. 6.2–6.4], so a nonzero remainder is a dependence on the cell's shape at fixed volume that no covariant effective action has. It is not by itself the mass of a propagating graviton, which is a property of the pole, the subject of the criterion in the chain below. Here the graviton is the propagating transverse-traceless mode, and a massless graviton would be a long-wave pole of its propagator.

The two types of strain. The test uses probe strains that keep every posting edge null and change the six edges with two postings, one for each pair of counters; the longer edges follow from these. The six pair off into three complementary couples, such as $d = (1, 1, 0, 0)$ with $(0, 0, 1, 1)$. A strain of type E changes the two edges of each couple alike, and a strain of type T_2 changes them oppositely; these are the two types into which the lattice's symmetries, the permutations of the counters, split the transverse-traceless strains. The combined probe is the sum of one probe of each type, and the E probe is its type-E part. Write $\Sigma_E(0)$ for the zero-momentum one-loop term on the E probe; it depends on the strength of the weight.

The settings of the calculations. Three choices distinguish the calculations.

- *Weight*: none, the *native model*, the path sum with the phase alone [10, 5]; or the *derived weight*, the ledger's weight on curvature under (I1) and (I2) [5], whose strength can be scaled from zero to the derived value.
- *Gauge*: the *posting gauge* fixes the four posting classes and works on the eleven others; it leaves a family of zero modes on whole planes of momentum space.
- *Prescription*: at nonzero momentum the treatment of those modes is a choice; of the four tried, only one has a finite ordered limit, a complete covariant gauge with a gauge parameter, the *complete gauge*.

Four calculations appear below, named by source and weight: the *published weighted*, the *published unweighted*, the *new weighted* and the *new unweighted* calculation. The step-1 table (Table 2) lists them with these choices and is the map for the rest of the section.

How a weight acts. A real factor μ in the measure can act on the zero-momentum term in two ways, and the two must be kept apart.

Its *direct term*: by its own value on the strained flat lattice the factor adds $-i \log \mu$ to the one-loop effective action, and so adds to the term $-i$ times the second variation of $\log \mu$ along the strain. For a real factor this is imaginary: it moves the imaginary part of the term and leaves the real part alone. Only a direct term could cancel the term outright.

Its *determinant effect*: if the factor is kept inside the Gaussian operator, the second variation of $\log \mu$ in the fluctuating variables is added to that operator and changes the one-loop determinant, and so the computed term. This can move both parts. The derived weight is

kept inside the operator, since it stands at the order of the action. It is real, and it takes the computed real part of $\Sigma_E(0)$ from 0.33 to 0.047 (step-1 table).

The powers of the hinge areas considered below enter by their own value only: the second variations of the logarithm of a hinge's area are not put inside the Gaussian operator. For the posting gauge's Faddeev–Popov factor and the functions of the volume, what is stated below is likewise their direct term. So the sentence "a real factor moves only the imaginary part" is true of direct terms and false of a real factor inside the operator.

One convention completes the definition. The second variation along a strain is taken along the strain that keeps the cell's volume, as the test requires. A straight line of metrics along a trace-free strain changes the volume at second order, so the volume-keeping strain differs from the line, at that order, by a multiple of the dilatation. With that term included a function of the volume has no direct term.

The chain of the argument. Four steps lead from computed numbers to the conclusion, all for the path sum without the weight; a separate branch after the chain compares the sum with the derived weight. The paragraphs after that give the material behind each step.

1. *Computed terms, without the weight.* The zero-momentum one-loop term on the transverse-traceless probe of type E is not zero in either unweighted calculation, the new one and the published one (NUMERICAL; the values are in the step-1 table).
2. *Two premises.* A computed number is not the path sum's term itself, so the conclusion rests on two named premises, both open. The *imaginary-part bound*: without the weight, the zero-momentum one-loop term on the E probe has an imaginary part bounded below by a positive constant m; numerically that imaginary part is at least 5.65. The *loop modulus*: as the momentum moves away from zero, the one-loop self-energy on the E probe, which shifts the pole, stays close to its value at zero momentum, $|\Sigma_E(k) - \Sigma_E(0)| \le \omega(|k|)$ with $\omega(s) \to 0$ as $s \to 0$. A bounded change is not enough. With the tree term bounded by $c_T|k|^2$, the change must stay below the lower bound of the first premise, $c_T|k|^2 + \omega(|k|) < m$, and this inequality fixes the radius within which there is no pole. The premise is stated in the complete gauge, the one prescription tried that is finite there.
3. *A criterion.* The criterion concerns one number at a time: the two-point function on a probe, which is the tree term plus the self-energy on that probe. A long-wave pole on the probe requires the zero-momentum one-loop term on it to vanish: if the imaginary part of that term is at least $m > 0$, the two-point function has no zero wherever $c_T|k|^2 + \omega(|k|) < m$, an explicit radius around zero momentum. With the tree term computed exactly, the two premises therefore exclude a long-wave pole on the E probe. The criterion and this implication are proved in work not yet published; the premises are open.
4. *A conditional conclusion.* Given both premises, the path sum without the weight has no long-wave pole on the strains of type E. A massless graviton needs the zero-momentum term to vanish on all five transverse-traceless constant strains, so in that sense its graviton is not massless at one loop; a pole carried by the strains of type T_2 alone is not excluded. No factor the ledger fixes can cancel the zero-momentum term, since each one's direct term along the constant strains that carry it is zero (proved in work not yet published),

so the obstruction lies in the Regge action, not in the weight. The two premises remain open.

What the chain covers. The five transverse-traceless constant strains are the two of type E and the three of type T_2, and by the lattice's symmetry the zero-momentum term is one number on each type. The chain controls the first only. A nonzero term on E does not exclude a pole carried by the T_2 strains. That would need the same two premises on T_2; and at nonzero spatial momentum, where a wave's two polarizations combine strains of both types, it would need a lower bound on the smallest singular value of the whole five-by-five block at zero momentum that exceeds the block's change nearby. Neither is supplied here. Without the weight, moreover, the T_2 term has no finite ordered limit (step 4 in detail), so for that sum the question on T_2 is not posed by a finite number.

The weighted branch. With the derived weight the new weighted calculation's term is not zero either (NUMERICAL; step-1 table), and as the weight is scaled from zero to its derived strength the term never vanishes (step 4 in detail). This branch bears on step 4's statement about the weight; the chain's conclusion concerns the path sum without it.

Two conclusions. The chain's conclusion is about one path sum, the one without the weight: given the imaginary-part bound and the loop modulus, it has no long-wave pole on the strains of type E. The paper's premises on the action support a broader conclusion, about the near-flat one-loop graviton of the action they fix. The action is a sum of hinge charges, with no charge at zero deficit, and the measure is real; with the record clause and the form of the charge, the action is Regge's (Theorem 5.5). A massless graviton needs the zero-momentum term on the five transverse-traceless constant strains to vanish. By its direct term a real factor in the measure moves only the imaginary part of that term, and only a phase term on those strains could move its real part directly (step 4 in detail); the premises write no such term. This is a statement about direct terms. A real factor kept inside the Gaussian operator moves the real part too, as the derived weight does (*How a weight acts*), so it does not follow that no real measure could bring the real part to zero. That is the role of the third input, a proof that the remainder's real part is not zero for the measure at hand. Under these premises the near-flat one-loop graviton is not massless, in the sense that the term does not vanish on all five strains, given three open inputs: the imaginary-part bound, the loop modulus in the complete gauge, and that proof. The implication is proved in the unpublished one-loop work; the inputs are open, and the computed real parts are NUMERICAL. Among the factors examined, those the ledger fixes and the powers of the hinge areas, none makes the term vanish. Through a direct term a massless one-loop graviton would need both a premise the paper does not make and a departure from the hinge-charge model, a phase term on those five strains; a real factor inside the Gaussian operator that brings the real part to zero is not excluded in general, and is examined only for the derived weight and its rescalings.

Step 1 in detail: four calculations. The table compares the four calculations behind the chain and its weighted branch; the paragraphs after it give, for each in the table's order, its probe, its treatment of the zero modes, its result and what remains unresolved. Every entry is a computed number, not an ordered limit.

Table 2. The step-1 table: the four calculations behind the chain and its weighted branch.

quantity	how it is computed	value	status
published weighted: remainder on the combined probe	derived weight; $\varepsilon = 0$ on grids that avoid the singular planes [5, Sec. 6.3]	$(0.1130 \pm 0.0002) + (24.656 \pm 0.002)\,i$	NUMERICAL, published
published weighted: its type-E part	the same	$0.0467 + 17.608\,i$	NUMERICAL, published
published unweighted: type E	no weight [5, Sec. 6.3]	about $0.31 + 5.67\,i$	NUMERICAL, published
new weighted: $\Sigma_E(0)$	derived weight; any of the prescriptions tried for the posting gauge's zero modes	$0.047 + 23.8\,i$	NUMERICAL, unpublished
new unweighted: $\Sigma_E(0)$	zero weight: phase, regulator and the posting Faddeev–Popov factor; no doubler logarithm on type E	$0.33 + 11.84\,i$	NUMERICAL, unpublished

The published weighted calculation. It works on the combined probe, with the derived weight, in the posting gauge; there the weighted operator stays singular on four planes of momentum space. It does not treat the zero modes that live on those planes: it computes at $\varepsilon = 0$ on grids that avoid them. The remainder is $(0.1130 \pm 0.0002) + (24.656 \pm 0.002)\,i$, with type-E part $0.0467 + 17.608\,i$, and a real normalization of the amplitude at each background changes only its imaginary part [5, Sec. 6.3 and Thm. 6.5]. What remains unresolved: the published finiteness theorem concerns the gauge-fixed responses on all fifteen edge classes, not this eleven-class remainder, whose ordered limit still needs one statement that is open, a finite local limit of the weighted operator at the origin [5, Sec. 6.3]. That statement would follow from a uniform bound on shells around zero momentum, or from a dissipation condition at each of the four points where the light cone touches the singular planes, and both are open [5, Sec. 7].

The published unweighted calculation. It is the published native model's term on the E probe, without the weight (NUMERICAL, [5, Sec. 6.3]). What remains unresolved: it too is a computed number, while the imaginary-part bound of step 2 asks that the imaginary part of the path sum's term itself stay above a positive constant.

The new weighted calculation. It works on the E probe, with the derived weight and a prescription for the posting gauge's zero modes (step 3 in detail). Its value is the same in every prescription tried (NUMERICAL, from the unpublished one-loop work). Its real part agrees with the published weighted type-E value to the digits quoted; the imaginary parts differ, for a reason not yet identified. What remains unresolved: nothing published says whether a value made in a prescription is an ordered limit, and that it is the same in every prescription tried does not settle the question.

The new unweighted calculation. It works on the E probe at zero weight, with the phase, the regulator and the posting gauge's Faddeev–Popov factor alone (NUMERICAL, from the unpublished one-loop work). It is finite although the native sum has no finite ordered limit

(step 2): that divergence is carried by the T_2 part of the combined probe, at the corner doublers, and the type-E part carries none (INFERENCE, [5, Sec. 6.3]; step 4 in detail). What remains unresolved: it too is a computed number, and it is not the ordered limit of the native remainder, whose probe has a T_2 part as well.

Step 2 in detail: the limit and the premises. Two published statements frame the limit. Without any weight, the native Regge path sum at $\kappa = 1$ fails the test: its per-site transverse-traceless remainder does not have ordered limit zero [10, 5]. That excludes zero as the limit; it does not say that a finite limit exists, and for the native sum no finite limit exists. "Not massless" there means the failure of that specified ordered limit; reading a complex zero-momentum response as a pole mass or a width needs a propagation law, the self-energy's dependence on momentum and a normalization [10]. The derived weight, built from the ledger's cost of curvature under (I1) and (I2), gaps the four corner doublers, the extra massless modes at the corners of the momentum cell (Section 2), and gives the real parts of the model's gauge-fixed one-loop responses on all fifteen edge classes finite ordered limits [5, Sec. 7]. The first premise of step 2 concerns the path sum's term itself, not a computed value, and none of the four calculations supplies it (step 1 in detail); the second, the loop modulus, is a separate matter (below).

Step 3 in detail: how a self-energy bears on a pole. In the path sum's convention the real part of the remainder moves the real pole and the imaginary part is a width, exactly when the normalization of the inverse propagator is real, as for a self-energy independent of momentum. This is proved in work not yet published; reading the pole as a physical mass and width still needs the propagation law above. The pole lies where the inverse propagator, the quadratic form corrected by the self-energy, vanishes, so its position depends on the self-energy at small nonzero momentum, starting with its coefficient of order k^2, and not on the zero-momentum value alone. The tree side is computed exactly: on the rest line, at zero spatial momentum and frequency ω, the tree transverse-traceless term is $3\sqrt{3}\,a(1-\cos\omega)(2\cos\omega-1)$, with a the phase scale, with an explicit bound and the light-cone coefficient $-\frac{9}{8}\sqrt{3}\,a$, the tree input of step 3's implication (proved in the unpublished one-loop work).

The posting gauge's zero modes. The pole's position depends on the self-energy's coefficient of order k^2 (above), and in the posting gauge that coefficient diverges like $\varepsilon^{-1/2}$ as the regulator is removed (NUMERICAL, from work not yet published). The cause is a family of modes that the gauge leaves unfixed: the quadratic operator and the weight share null vectors on whole planes of momentum space, those on which one component of the momentum vanishes [5], not only at isolated points. These modes have to be treated before a pole can be located, which is why the new weighted calculation is made with a prescription for them. They are pure gauge: they are vertex moves that keep the posting edges fixed, and they carry no cohomology of the flat lattice (proved in work not yet published).

What the modes are at zero momentum. At nonzero momentum each of these modes is a vertex move: it changes the edges only as a change of coordinates would, so it is pure gauge. As the momentum goes to zero along its plane, the change it makes to the edges, divided by

the size of the momentum, tends to a constant strain, a uniform change of the cell's shape that keeps every posting null. A uniform translation of the points changes no edge, so these limits are not vertex moves; they change the flat lattice itself, and so they are physical moduli. More precisely, a constant strain is what a vertex move growing linearly across the lattice does to the edges, and on no periodic lattice is it a periodic vertex move. The six constant strains that keep every posting null are the uniform rescaling of all four postings, the dilatation, and five transverse-traceless ones: the three of type T_2 rescale the four postings unequally, and the two of type E move the postings' directions (both proved in the unpublished one-loop work). Only at zero momentum, then, do these modes become those six constant strains, and the posting slice has an explicit Faddeev–Popov factor (both proved in the unpublished one-loop work). The quotient by the modes is forced and exact at zero momentum only.

Prescriptions at nonzero momentum. At nonzero momentum the unpublished one-loop work reduces their treatment to a named premise, a choice of prescription, and tries four. Three have no ordered limit. The fourth, a complete covariant gauge with a gauge parameter, has a finite ordered limit, and the gauge dependence of its coefficient of order k^2 is 3×10^{-4} (NUMERICAL). The argument uses only this distinction: the bound the criterion needs lives at nonzero momentum, where the prescription acts, and only a prescription with a finite ordered limit can supply it. So the second premise, the loop modulus, is stated in the complete gauge, the one prescription tried that is finite at nonzero momentum; whether it holds is open.

Step 4 in detail: where the obstruction lies. *The result without the weight.* The new unweighted calculation's term (step-1 table) is not zero. Wave by wave its real part vanishes at $\varepsilon = 0$ (proved in the unpublished one-loop work), so the real part of the total arises only as the regulator is removed. Scaling the weight from zero to its derived strength, $\Sigma_E(0)$ never vanishes; its imaginary part rises from 11.84 to 23.8 (NUMERICAL).

Its relation to the published remainder. It is the derived weight that gaps the four corner doublers [5]; without it, in the published native model, their contribution makes the real part of the remainder fall like $-r\log(1/\varepsilon)$ with $r > 0$ [5, Sec. 3]. In the unpublished one-loop work the type-E pair carries no doubler logarithm, while T_2 carries a small $\log(1/\varepsilon)$ (NUMERICAL); so the corner divergence of the published native remainder can sit in its T_2 part, and the finite type-E value is not in tension with it. The value is still not the ordered limit of the published remainder, whose probe has a T_2 part as well.

No factor the ledger fixes cancels the term. The weight's determinant effect changes the computed term, in both parts: between zero weight and the derived strength its real part goes from 0.33 to 0.047 and its imaginary part rises from 11.84 to 23.8 (above), and the term stays nonzero. Only a direct term along the five transverse-traceless constant strains could cancel it. The candidate factors are the derived weight, the posting gauge's Faddeev–Popov factor and the functions of the volume. For each of them the direct term along those strains is identically zero (proved in work not yet published), so none can cancel the one-loop term unless it is already zero, and the computed values say it is not (NUMERICAL). Given the premises of the chain, then, the obstruction to a massless one-loop graviton lies in the Regge action on this lattice, not in the derived weight. The published analysis points the same way: without the weight the native remainder does not have ordered limit zero [5, Thm. 3.3], and that paper

separates the corner doublers, a problem of the measure and of κ, from the nonzero long-wave remainder, a problem of the action [5, Sec. 3].

Why the action, as reasoning. The one-loop functional loses the vertex-move invariance because Regge's vertex moves are symmetries only of flat configurations (INFERENCE; the witness, the nonzero vertex with one gauge leg in Appendix B, is proved in the unpublished one-loop work).

What a massless one-loop graviton needs. Through a direct term, only a phase on the five transverse-traceless constant strains, beside the hinge charge, can move the real part of the remainder, and the paper's premises on the action exclude one: a hinge charge with no charge at zero deficit and a real measure write no phase there (proved in the unpublished one-loop work, with the counts behind it). A real factor that enters by its own value cannot move that part. The class examined is ultralocal: a power of each hinge's area, with one exponent for each of the six classes of hinges, six coefficients in all. These factors enter by their own value. The second variations of the logarithm of a hinge's area are taken along the volume-keeping strain and are not put inside the Gaussian operator (*How a weight acts*). In that class real exponents move only the imaginary part (closed forms proved in the unpublished one-loop work). The two derived members, the Born weight's per-hinge normalizations with exponents $-\frac{1}{2}$ and $-\frac{1}{4}$ (its normalization is open under (I2)), give the type-E term $0.047 - 4.89\,i$ and $0.047 + 6.36\,i$ (NUMERICAL), and only a tuned family of complex exponents cancels it. A real factor kept inside the Gaussian operator is outside this statement; the derived weight is one, and it moves the real part.

Record preservation. The vertex-move symmetry can be asked of the ledger as a premise on records: vertex moves that preserve the records are indistinguishable, up to a function of $\det g$. What this covers depends on what a record holds. The constant strains at issue keep every posting edge null but change other edges, so they preserve the records if a record holds the postings only, and not if it holds the edge data as well; the ledger's texts fix neither reading (OPEN). The question does not decide the graviton. For the one-loop effective weight, record preservation is exactly the vanishing of the remainder on the strains it covers, so, with the real part not zero (NUMERICAL), it is false under the paper's premises on the action. For the bare weight it holds on the flat family, which says nothing about the remainder, and fails outright as the principle that configurations with the same postings cannot be told apart, since Regge's action tells them apart at second order: the tree term of step 3 at frequency π is $-18\sqrt{3}\,a \neq 0$ (all proved in the unpublished one-loop work). Adopted as a premise, record preservation would need a phase term on those strains, a departure from the hinge-charge model, and would fix it linearly, $\zeta_E = -R_E/9$ on the type-E strains and $\zeta_T = -R_T/3$ on the type-T_2 strains, with ζ_E and ζ_T the coefficients of that phase term and R_E and R_T the remainder's values on the two types in the normalization of the unpublished one-loop work (an algebraic identity proved there).

Appendix B collects the results of the unpublished one-loop work that the chain does not use: two numbers at nonzero momentum, neither of which is an ordered-limit statement, so neither supplies either premise; the Ward identities of the vertex moves; and the stationarity of the flat background at one loop.

What stays open. The following remain open.

- The first premise, the imaginary-part bound: a positive lower bound on the imaginary part of the zero-momentum one-loop term on the E probe without the weight. The computed values point to it (numerically at least 5.65), but none is the path sum's term itself (steps 1 and 2).
- The second premise, the loop modulus: in the complete gauge, the self-energy's change near zero momentum stays below the lower bound of the first premise (step 2).
- The third input, a proof that the remainder's real part is not zero; the computed real parts are NUMERICAL.
- The strains of type T_2: the premises and the criterion are stated on the E probe. Excluding a pole on T_2, or on all five strains at nonzero spatial momentum, needs the corresponding bounds (*What the chain covers*).
- A real factor in the measure that is kept inside the Gaussian operator, other than the derived weight and its rescalings: whether one could bring the remainder's real part to zero is not examined.
- Which reading of records the ledger's texts fix, the postings only or the edge data as well; under the paper's premises it does not decide the graviton (step 4 in detail).
- The source of the difference between the imaginary parts of the published weighted and the new weighted type-E values.
- Publication of the unpublished one-loop work, and a value of $\Sigma_E(0)$ with rigorous error bounds.

Earlier lattice work. The same zero-momentum question was asked of gauge-theoretic lattice gravity, with vierbeins and Lorentz links in place of Regge's edge lengths: there Menotti and Pelissetto found that the one-loop Slavnov–Taylor identity can be satisfied for a particular choice of the local measure and of the cosmological constant [19], [20, Sec. 2.6]. Here the action is Regge's on the $\kappa = 1$ Kuhn lattice, and the derived weight is fixed by the named identifications (I1) and (I2) rather than chosen.

On the Lorentzian side, Sorkin posed the time-evolution problem of Regge calculus [34], and later defined Lorentzian angles in a way that includes lightlike vectors, the case of the postings at $\kappa = 1$ [35]. Borissova and Dittrich evaluated the Lorentzian Regge path integral at one loop around a flat background by Pachner moves; they built a local measure that makes it independent of the triangulation in three dimensions and found obstacles to one in four [37]. Their effective actions for the continuum limit of spin foams, built from area metrics, give the length metric an action of Einstein–Weyl type [36]. The present section asks a narrower question on one fixed lattice, the zero-momentum term on its uniform strains, and does not use Pachner moves.

8 Every input, named

Every result of this paper rests on named inputs. Figure 3 lists them, with what each does and where it enters, and Table 3, after it, lists every result, with its status and where it is proved. One published result has premises of its own: the worldline-matter result of Section 4 rests on three identifications and two models, stated with it in Appendix C and not repeated in the figure. The models are chosen mathematical objects; the identifications are physical readings that the definitions do not force.

Two remarks qualify the list. First, both selections take a premise on the form of the charge. The gauge route takes the area form, with no record clause (Theorem 5.3). The record route takes the record clause together with either the content reading or the area form; under the clause the two coincide (Theorem 5.5). Second, the record clause gives Newton's strength only for an action already in Regge's form: it fixes the length λ_{rec}, while G itself stays an input measured in the laboratory, up to the factor Z of Section 2 [8, Sec. 2].

input: what it says	status	finite sum	record time	field equation	Newton's law	slope, vertices	gauge selection	$\kappa = 1$ vertex	record selection	graviton's form	one loop
$\kappa = 1$: one tick is one posting, and the interval reads the posting order [5, Thm. 2.2]; on the transfer reading of light's step, with the step identification. On the other reading on record, a tick of two postings, $\kappa = \frac{1}{2}$ (Section 2)	IDENTIFICATION; which reading: OPEN	●	●	○	○	●		●		●	●
(I1): a hinge record's cost reads its area as the action does; channel count n_h a definition [5, Sec. 5]	IDENTIFICATION		●								●
(I2): Born weight = Gibbs weight e^{-C}; normalization open [5, Sec. 5]	IDENTIFICATION		●								●
record clause: one posting carries $s_{\mathrm{p}} A\,(2\pi/k)$ (Definition 2.2)	IDENTIFICATION				○				●	○	●
form of the charge: area form or content reading (Definition 2.1)	premise						●		●		●
setting: every small-amplitude field is a physical configuration, which the ledger's count fields alone do not give	premise		●	●	●	●	●	●		●	●
writing principle or rotation route (Section 6)	premise									●	
record charge at which the writing rules are read (Section 6)	IDENTIFICATION									●	
graviton read as the ledger's cost of comparing a geometry with flat space (Section 6)	IDENTIFICATION									○	
lattice: fixed Kuhn graph, real squared lengths, time along **1**	MODEL	●	●	●	●	●	●	●		●	●

input: what it says	status	finite sum	record time	field equation	Newton's law	slope, vertices	gauge selection	$\kappa = 1$ vertex	record selection	graviton's form	one loop
measure and regulator: model measure, a stated window	MODEL	•									
one-loop diagram model: unit gauge weight, no ghosts, Feynman regulator	MODEL										•
zero-mode prescription at nonzero momentum: the complete gauge, the one finite there (Section 7)	premise										•
imaginary-part bound: without the weight, the zero-momentum term on the E probe has its imaginary part bounded below by a positive constant (numerically at least 5.65)	premise, OPEN										•
loop modulus: in the complete gauge, the self-energy's change near zero momentum is below that bound	premise, OPEN										•
nonzero real part: the remainder's real part is not zero	premise, OPEN										•
section		3	3	4	4	4	5.2–4 5.7–9	5.6	5.5	6	7

Figure 3. Every input and where it enters. A filled circle • marks an input used in a result's statement or premises; an open circle ○ marks one used only to read the result physically: $\kappa = 1$ for the field equation and Newton's law, which hold at every $\kappa > \frac{1}{2}$, the record clause for reading Newton's strength as a value of λ_{rec} and for the strength of the graviton's form, and the graviton's reading for the graviton's form. The gauge selection is proved on the Euclidean member, so $\kappa = 1$ does not enter it; the record-clause selection is about functions and uses no lattice.

Table 3. The claims of this paper and their status. "Published" results are proved in the cited paper, a preprint by the author that has not appeared in a peer-reviewed journal (Section 1), and keep the status that paper states; "proved here" means a complete proof in this paper.

claim	status	where
the value of κ: 1 if one tick is one posting, the reading used here; $\frac{1}{2}$ if one tick is a double entry of two postings [13]	IDENTIFICATION on either reading; which reading holds: OPEN. At $\kappa = \frac{1}{2}$ the Lorentzian results below are not established, the long-wave quadratic form apart (Table 4)	Sec. 2, App. D
finite regulated Lorentzian sum over Regge geometries at $\kappa = 1$; normalization entire in the phase scale	theorem, published [2]; regulator, measure, Lorentzian sign: model or conditional	Sec. 3
record-time roots of the Gaussian path sum	theorem, published [3], at its stated scope (roots, not a norm)	Sec. 3

claim	status	where
Einstein's metric field equation at every order in the field, at every $\kappa > \frac{1}{2}$	theorem, published [4, 5]	Sec. 4
sourced linear field, Newton's law, $G = 1/(8\pi s)$, at every $\kappa > \frac{1}{2}$	theorem, published [6, 5]	Sec. 4
worldline matter, at every $\kappa > \frac{1}{2}$: Einstein's equation with the matter's stress through second order, on a long-wave band	theorem, published [4, Thm. 5.1]; three identifications and two models (Appendix C)	Sec. 4, App. C
Regge's slope and the Einstein–Hilbert vertices around the $\kappa = 1$ lattice, from flat-space stationarity at every $\kappa > 1$	theorem, published [7]	Sec. 4
among one-area-function charges, exactly Regge's jets at the Euclidean flat areas, up to scale, pass the quadratic test and the cubic test at three witnesses on the Euclidean graviton's shell (Euclidean member)	proved here	Thms. 5.2, 5.3
jets do not fix the charge; the record clause, with content reading or area form, fixes it exactly	proved here	Prop. 5.4, Thm. 5.5
the $\kappa = 1$ cubic vertex is half of Einstein's $\Gamma\Gamma$ vertex; one normalization	proved here	Thm. 5.6
a separate function for each hinge class leaves three inversion-odd terms	proved here	Thm. 5.9
the recognition cost's charge fails the cubic test; one ratio correction restores Regge's jets, and its jets are unique	proved here	Thm. 5.10
the coupling direction: free under the tests to first order; with Regge's, the only even direction among shape charges with one function per class at the eleven shape witnesses	proved here	Thm. 5.11
the recognition cost's charge with the ratio correction and half the coupling direction added: one half of Regge's long-wave cubic vertex, and it passes the cubic test at the three witnesses	proved here	Cor. 5.13
graviton's quadratic form is, on its two polarizations, η times twice the recognition cost's long-wave form	identity proved; the graviton's identification with that cost an IDENTIFICATION; published [9]	Sec. 6
graviton beyond long wavelength at $\kappa = 1$: Einstein's at leading order, then an anisotropic, birefringent correction, phase speeds $1 - c_{\pm}(n)(kb)^2$; every $\kappa > \frac{1}{2}$ birefringent at this order	at $\kappa = 1$ proved in work not yet published, in every direction; across the family proved in work not yet published; its physical size rests on the record clause (IDENTIFICATION), an INFERENCE and Z. Against the cosmic-ray bound of [38] at $Z = 1$: within it by six orders in the coefficient if the most energetic event was an iron nucleus, at the bound if a proton; whether the bound applies rests on matter's own second-order term, not computed	Sec. 6

claim	status	where
one loop: the native sum fails the zero-momentum test; the derived weight gaps the doublers	published [10, 5]; the remainder's value NUMERICAL	Sec. 7
one loop, without the weight: given two open premises (a positive lower bound on the imaginary part of the zero-momentum term; the loop modulus in the complete gauge), no long-wave pole on the strains of type E for the path sum; the strains of type T_2 are not controlled	the reduction proved in work not yet published; both premises open; the computed values NUMERICAL	Sec. 7
one loop: no factor the ledger fixes cancels the zero-momentum term by a direct term, so the obstruction lies in the Regge action	proved in work not yet published; that the term is not zero NUMERICAL; why the action: INFERENCE	Sec. 7
one loop, under the premises on the action: the near-flat one-loop graviton is not massless, in the sense that the zero-momentum term does not vanish on all five uniform strains, given three open inputs (the two premises and a proof that the real part is not zero)	proved in work not yet published; the three inputs open; type T_2 not controlled	Sec. 7
quartic order: the quartic table's long-wave contact vertex is one half of the quartic vertex of $\sqrt{\lvert g\rvert}R$ itself, at $\kappa = 0$ and $\kappa = 1$; the table action is one half of $\int \sqrt{\lvert g\rvert}R$ through quartic order	proved in [12] (long wavelength, metric sector)	Sec. 5.10
quartic order: the $\kappa = 1$ lattice action is one half of Einstein's action through quartic order, with the volume factor $\sqrt{3}$	proved in [12] (long wavelength, metric sector)	Sec. 5.10
tree scattering: the table action's amplitude is gauge invariant with general relativity's helicity structure at every κ off three poles; the cell-function action's own amplitude, contact term included, is $\sqrt{4\kappa - 1}$ times the table action's and gauge invariant at every $\kappa > \frac{1}{2}$ ($\sqrt{3}$ times that of $\frac{1}{2}\int \sqrt{\lvert g\rvert}R$ at $\kappa = 1$); at every κ outside $\{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\}$, with all fifteen edge classes, the amplitude is Einstein's (at $\kappa = 0$ the heavy modes do not decouple)	proved in work not yet published; the table action's amplitude on the Euclidean lattice in [12] (tree level, long wavelength)	Sec. 5.10
quartic order: within shape charges and on listed configurations, no completion of a changed coupling	proved in [12] at $\kappa = 0$, $\frac{1}{10}$ and 1 on listed configurations, with the massive non-metric modes at $\kappa = \frac{1}{10}$ and 1 in work not yet published; that this is the variation needed: INFERENCE	Sec. 5.10

claim	status	where
$\kappa = 1$ (one tick read as one posting; on the transfer reading of light's step); (I1); (I2); record clause; the charge's form; the setting (every small-amplitude field a physical configuration); writing principle or rotation route; zero-mode prescription; the one-loop inputs (imaginary-part bound, loop modulus, nonzero real part); channel count n_h	IDENTIFICATION (form, setting, writing principle, rotation route, zero-mode prescription: premises, not derived; the one-loop inputs: OPEN; n_h: a definition)	Sec. 8

9 Where this leaves the question, and what the paper does not claim

The finite sum of Section 3 and the field equation and vertices of Section 4 are results about Regge's action; Section 5 says why that action. Two routes lead to it, with different reach. On the Euclidean lattice, in local tests posed on that lattice's own graviton shell, gauge consistency fixes the jets of a charge built from one function of the area, through cubic order and up to a common scale. The record clause, with a charge read by its content or by one function of the area, fixes the whole function and its scale exactly. Of the two, only the second makes the action Regge's on the $\kappa = 1$ lattice; its proof is a substitution, and its content is the record clause, an IDENTIFICATION, with the premise on the charge's form. So for that charge the finite sum and the metric field equation at every order hold, each as stated: the finite sum at $\kappa = 1$, over the null stratum, and the field equation at every $\kappa > \frac{1}{2}$, over all fifteen edge classes, both at the scale s_{p} that ties G to the recognition length (Section 2). The reduction between the two sets of configurations is open (Section 1). Beyond long wavelength the $\kappa = 1$ lattice predicts a departure from Einstein's theory: an anisotropic, birefringent correction to the graviton's speed of relative size $(kb)^2$. If the lattice spacing is the recognition length, the correction lies far below the bounds from the arrival times of gravitational waves. Against the cosmic-ray bound of [38] it passes by six orders of magnitude in the coefficient if the most energetic event was an iron nucleus and has no margin if it was a proton; whether that bound applies rests on matter's own second-order term on the lattice, which is not computed (Section 6).

Eight points fix the scope of these results.

- *The value of κ.* Not claimed: that $\kappa = 1$ is the lattice of gravity. It is this paper's identification, on the reading that one tick is one posting. On the other reading on record, a tick of two postings, the value is $\kappa = \frac{1}{2}$ [13], and there this paper's Lorentzian results are not established, the long-wave quadratic form apart (Section 2, Table 4). Which reading holds is open.
- *One set of configurations.* Not claimed: that the finite sum and the field equation concern the same configurations. The sum runs over the null stratum, where every posting keeps squared length zero; the field equation varies all fifteen edge classes. The reduction of the second setting to the first, a gauge slice with its Faddeev–Popov factor, is open (Section 1).

- *The lattice from the ledger.* Not claimed: the derivation of the four-counter lattice from the ledger (the four-account to three-account correspondence, Section 1).
- *Unitarity.* The paper constructs no positive physical Hilbert space and no unitary or reflection-positive evolution in record time. The bare Regge equation at $\kappa = 1$ has a growing record-time solution, and the derived weight gives continuum-limit root stability only [3]. Unitary record time is Target 6.1 of [11], open. Work not yet published, reported in [13, Sec. 8.2], finds that Regge's action on a Kuhn cell has no positive transfer operator at any $\kappa > \frac{1}{2}$. This paper does not use that statement and has not checked it; if it stands, a positive evolution cannot come from Regge's action on this cell as it is.
- *Black holes.* No lattice black hole and no strong static exterior is constructed; Targets 3.10 and 3.11 of [11] are open.
- *Every input named.* $\kappa = 1$ (one tick read as one posting; on the transfer reading of light's lattice step, with the step and light identifications of Section 2), (I1), (I2), the channel count, the record clause, the premise on the charge's form, the setting's premise that every small-amplitude field is a physical configuration, and the writing principle or the rotation route are inputs (Section 8), and Appendix C states the premises of the worldline result. Three further inputs carry numbers or choices that the paper does not fix: Newton's constant G, a laboratory input, measured as ZG with Z not yet computed (Section 8); the worldline result's reference occupations, whose ratios between classes are chosen, not derived (Appendix C); and the zero-mode prescription at nonzero momentum, a named premise (Section 7). The one-loop conclusion rests in addition on three open inputs, the imaginary-part bound, the loop modulus and a proof that the remainder's real part is not zero (Section 7). Deriving the inputs is later work.
- *A massless graviton.* Not claimed; under the paper's premises it is excluded, conditionally. In the one-loop diagram model, a reduction proved in work not yet published gives no long-wave pole on the strains of type E for the path sum without the weight from two premises, both open: a positive lower bound on the imaginary part of its zero-momentum term on the E probe (numerically at least 5.65), and, in the complete gauge, a bound that keeps the self-energy's change near zero momentum below that lower bound. The strains of type T_2 are not controlled by this reduction, so a pole carried by them alone is not excluded. Under the premises on the action, a hinge-charge action with a real measure made Regge's by the record clause and the form of the charge, the near-flat one-loop graviton is not massless, in the sense that the term does not vanish on all five strains, given these two and a third open input, a proof that the remainder's real part is not zero (Section 7). Through a direct term, a massless graviton would need both a new premise and a departure from the hinge-charge model, a phase term on the five transverse-traceless constant strains; a real factor kept inside the Gaussian operator is examined only for the derived weight. No factor the ledger fixes cancels the zero-momentum term. The computed values are NUMERICAL; without the weight the published native model fails the zero-momentum test [5, Thm. 3.3].
- *Nonlinear lattice solutions.* "Every order" is a statement about the field equations (Theorem 4.1), not about lattice solutions at finite strength, which remain open.

The question this leaves is how far gauge consistency alone reaches: the cubic test on the $\kappa = 1$ lattice itself, charges whose factor depends on more than a hinge's own edges, and a selection of the charge by the gauge tests beyond cubic order. The quartic results reported here, the lattice's contact vertex (Section 5.10) and the obstruction for a changed coupling (Section 5.10), are not such a selection.

Where the question went. Later work on the same ledger, not yet published, does not stay on this paper's lattice. It works at $\kappa = \frac{1}{2}$, the working value of [13]; there the hinge of two single postings has no area, and Regge's action on this cell cannot be used as it stands. That work takes another lattice of records and a cost on its bond lengths that weighs each hinge by its area, an adopted reading. The cost carries no correction of the kind that Theorem 5.10 adds, and its cubic term is Einstein's by exact computation. The question that Section 7 leaves open is open there too, in a sharper form. Measurement asks that the graviton's mass lie below $2.42 \times 10^{-23}\ \mathrm{eV}/c^2$, at 90% credibility [41], which is a squared mass below 1.1×10^{-101} in units of $\hbar^2/(c^2 b^2)$ at $Z = 1$, and that gravitational waves keep light's speed [26]. The laws that work now has for light and matter do not keep the graviton massless after quantum corrections: their one-loop sums, computed numerically, meet neither condition. What would keep it massless is stated exactly, a weight of each history that depends on the geometry alone, and it is open. The present paper uses none of this.

A The transverse-traceless witness

This appendix gives the data of the transverse-traceless witness W_3, the one that sees the curvature jet at area 1; the last two rows of Table 1 evaluate the cubic test there. The witness W_3 has the null momenta $K_1 = (3, 4, 0, 5i)$, $K_2 = (5, 0, 12, 13i)$, $K_3 = -K_1 - K_2$, the gauge leg $K_3 \otimes e_1 + e_1 \otimes K_3$ with $e_1 = (0, 1, 0, 0)$, and the symmetric polarizations

$$h_1 = \begin{pmatrix} -321 + 240i & -228 + 70i & -100 - 125i & -200 - 375i \\ -228 + 70i & 171 - 240i & 75 & 150 \\ -100 - 125i & 75 & -75 & 75 \\ -200 - 375i & 150 & 75 & 225 \end{pmatrix},$$

$$h_2 = \begin{pmatrix} 482 + 312i & -120 + 65i & -60 - 65i & -60 + 130i \\ -120 + 65i & -457 - 312i & 50 & -25 \\ -60 - 65i & 50 & 25 & 25 \\ -60 + 130i & -25 & 25 & -50 \end{pmatrix}.$$

Each is traceless and transverse to its momentum, $\sum_a K_{1,a} h_{1,ab} = 0$ and $\sum_a K_{2,a} h_{2,ab} = 0$ for every b, and both momenta are null for the Euclidean form, $\sum_a K_a^2 = 0$. At this witness Regge's long-wave cubic vertex is 0.

B Further one-loop results

This appendix collects the results of the unpublished one-loop work of Section 7 that the chain of that section does not use.

Two numbers at nonzero momentum. The unpublished one-loop work reports two numbers at nonzero momentum (NUMERICAL); neither enters the chain.

No zero of the corrected inverse propagator. It has no zero for momenta of ℓ^1-norm below 0.92, nor anywhere it was sampled. This is not an ordered-limit statement.

The position of the pole. A leading-order estimate from the published weighted calculation's type-E term $0.0467 + 17.608\,i$ shifts the squared wave covector by $\delta\eta = 0.217 + 81.9\,i$, whose imaginary part depends on the integration variables (here the squared intervals). Its root, at $|\omega| \approx 9.05$ in the rest frame, lies outside the Brillouin zone, which ends at $2\pi/\sqrt{3} \approx 3.63$ along the rest line, so the estimate places no pole. It leaves out any complex part of the coefficient of order k^2, which in the posting gauge diverges (Section 7, the posting gauge's zero modes).

The vertex moves in the interactions. The unpublished one-loop work proves two further statements.

Two gauge legs. The second-order Ward identity with two gauge legs holds at every pair of momenta, as an exact identity proportional to the external wave's force, so the contraction vanishes on shell.

One gauge leg. With one gauge leg the identity fails: the vertex between two zero modes and one gauge leg is $(1 + i)\,\frac{11\sqrt{3}}{6} \neq 0$. So, by a two-line argument (INFERENCE), the vertex moves are not a symmetry of curved solutions, as Section 4 found for the classical equations at second order.

The background. Flat $\kappa = 1$ is still a stationary point of the phase at one loop, along every uniform strain of the metric: the transverse-traceless tadpoles, the one-loop terms that are linear in the strain, vanish exactly for all 24 axis permutations (proved in the unpublished one-loop work).

C The worldline-matter result in detail

This appendix gives what Section 4 summarizes about worldline matter: the construction, the result in full, its premises and what is not proved.

Worldline matter. At second order in the field amplitude the model medium can be replaced by matter that has an equation of its own, *worldline matter*. It is made of *matter postings*: a matter posting is a unit of matter, a point particle counted on an edge [5, Sec. 4], and not the single-counter step of Section 2. The body is a static set of resting matter postings with finitely many Fourier modes. A carrier holds the body up; it consists of matter postings

that run along worldlines on the eleven timelike edge classes, and what runs into a vertex runs out. Its occupation of an edge is the number of its matter postings on that edge. The carrier's occupation of the timelike edges follows the recognition cost's law of least work. The construction solves the lattice equations of a *band*, the set of static waves of long wavelength that it keeps.

What is proved. For every $\kappa > \frac{1}{2}$, at small strength, and for bodies whose modes avoid six lines of momentum space, the coupled lattice problem has on every large enough torus a solution of the equations of a fixed long-wave band. Its first two coefficients converge to fields that solve Einstein's equation with the matter's own stress, the carrier's included, through second order in the amplitude, up to an error of third order. At the zero wave the carrier takes the whole constant-strain load, so no free constant enters [4, Thm. 5.1]. The six lines are the *blind lines*: static modes supported on only two coordinate axes, the multiples of $e_a - e_b$. The solution is not an exact solution of all the lattice equations; the technical note below says what it leaves out.

The premises. Five premises are used [5, Sec. 4], three identifications and two models.

- *Identification.* The carrier's least-work problem is solved on the flat lattice, and the stress it gives each edge is held fixed as the field deforms the edges: the law of least work, read at the flat reference and carried at fixed stress.
- *Identification.* The carrier's reference occupation on each of the eleven timelike classes; the ratios between classes are chosen, not derived.
- *Identification.* The split into which matter postings count as the body and which as the carrier that holds it up.
- *Model.* Matter postings are point particles.
- *Model.* The carrier's reference occupation is not a source of the field, so that only the carrier's departure from that reference gravitates.

Technical note: what is not proved. The solution is not an exact nonlinear solution, and it is not shown to satisfy every lattice equation. Three terms say which equations it leaves out. A *row* is one lattice equation: the derivative of the action along one direction of the squared edge lengths, set equal to the matter's load along that direction. A *radical row* is a row along a direction on which the flat quadratic action vanishes: a vertex move or, at zero wave number, a constant strain. A *blind row* is a radical row along which worldline matter exerts no load. The equations left out form two groups [4], [5, Sec. 4]:

- the equations outside the band that the construction solves: the rows off the band, the radical rows of higher wave number and the non-static rows;
- for compact bodies, the blind rows of the band.

D The two readings of light's step

This appendix gives the detail behind the paragraph on light's lattice step in Section 2; a first reading can skip it.

Read as a real-time step, $1-\cos\omega\tau = r\sum_j(1-\cos k_j)$, the lawful face-weight ratio is exactly $0 < r < \frac{1}{3}$. This is the three-dimensional Courant bound $6r \le 2$, with $r = \frac{1}{3}$ excluded because the two-step map of the corner mode then has no eigenbasis (proved in work not yet published). Under the step identification that corresponds to $\kappa < \frac{1}{2}$, outside the range $\kappa > \frac{1}{2}$ that the results of Sections 3 and 4 assume. The transfer reading, for its part, is not the step of a local update. Along an axis, for every $r > 0$, no polynomial P gives $\cos E = P(\cos k)$ on its shell $\cosh E - 1 = r(1-\cos k)$ (proved in work not yet published). For laws with a real symbol a two-step law of finite range would give such a polynomial (an argument). So the transfer reading's frequencies belong to no finite-range two-step law. The identification $\kappa = 1$ therefore rests on a reading of light's step that a tick of finite reach cannot give.

One posting or two: the value $\kappa = \frac{1}{2}$. The two readings above concern how light's step is computed. A second pair of readings concerns what the step is, and it carries the second value of κ on record (Section 2). If one tick is one posting, light's step is the edge with one posting, of squared length $1-\kappa$, and it is null at $\kappa = 1$. If one tick is a double entry, a debit with its credit on another account, light's step is the record of two postings, the edge with two postings, of squared length $2(1-2\kappa)$, and it is null at $\kappa = \frac{1}{2}$ [13]. Under the step identification $\kappa = \frac{1}{2}$ is the ratio $r = \frac{1}{3}$, the end of the real-time range that the bound excludes. Neither reading of the tick is derived.

Table 4 sets this paper's objects at the two values side by side. Its upper rows follow from the two formulas of Section 2: an edge with n postings has squared length $n(1-\kappa n)$, and a hinge whose two shorter edges have m and n postings has squared area $\frac{1}{4}mn\,|1-\kappa(m+n)|$. At $\kappa = \frac{1}{2}$ the hinge of two single postings has no area: each posting has squared length $\frac{1}{2}$, their sum is null and orthogonal to both, and the plane they span is null. Regge's action is a sum over hinges of area times deficit, so on this cell it cannot be used at that value as it stands. At $\kappa = \frac{1}{2}$ the Lorentzian results in the lower rows of the table are not established, the long-wave quadratic form apart; the last column says what stops each.

Table 4. This paper's objects at $\kappa = 1$ and at $\kappa = \frac{1}{2}$. Upper rows: the flat lattice. Lower rows: where each result stands at the two values.

object or result	at $\kappa = 1$	at $\kappa = \frac{1}{2}$
squared length $n(1-\kappa n)$ of the edges with $n = 1, 2, 3, 4$ postings	$0,\ -2,\ -6,\ -12$	$\frac{1}{2},\ 0,\ -\frac{3}{2},\ -4$
the null edge	one posting	two postings
$\det g_\kappa = 1-4\kappa$; volume factor $\sqrt{\lvert\det g_\kappa\rvert}$	-3; $\sqrt{3}$	-1; 1
inverse metric	$I - \frac{1}{3}\mathbf{1}\mathbf{1}^\top$	$I - \frac{1}{2}\mathbf{1}\mathbf{1}^\top$, the metric itself

object or result	at $\kappa = 1$	at $\kappa = \frac{1}{2}$
flat hinge areas for $(m, n) = (1, 1), (1, 2), (1, 3), (2, 2)$	$\frac{1}{2}$, 1, $\frac{3}{2}$, $\sqrt{3}$	0, $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, 1
light's ratio r under the step identification	1	$\frac{1}{3}$, the excluded end of the real-time range
finite sum (Theorem 3.1)	proved, on the stratum where one-posting edges are null	the theorem is stated at $\kappa = 1$ only
field equation, Newton's law, worldline matter (Section 4)	proved, as at every $\kappa > \frac{1}{2}$	the hypothesis $\kappa > \frac{1}{2}$ excludes it
slope and long-wave vertices (Theorem 4.2); cubic vertex (Theorem 5.6)	proved	stated around the $\kappa = 1$ lattice; the table $T(\kappa)$ has no value: the poles of its coefficients lie at $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$
long-wave quadratic form (Section 5.10)	one half of Einstein's	reported there at every κ with $4\kappa - 1 \neq 0$, so also here (from work not yet published)
quartic and tree-level statements (Section 5.10); determinant of the block of the five non-metric modes, $3\kappa(2\kappa - 1)^3/(16(4\kappa - 1)^4)$	at the status given there; $\frac{1}{432}$	excluded there by name; 0, so the block is singular
birefringent correction (Section 6)	proved in work not yet published	its formula for the splitting has $2\kappa - 1$ in the denominator and is not stated there
one-loop numbers (Section 7)	computed	not computed
gauge selection (Theorem 5.3)	run at $\kappa = 0$; the cubic test on either Lorentzian lattice is open	
record-clause selection (Theorem 5.5)	holds: no value of κ enters	
the record clause's twelfth of a turn, $k = 12$	fits the flat dihedral angles, multiples of $\pi/6$	not examined

E The eleven shape witnesses

This appendix gives the eleven shape witnesses of Theorem 5.11(c) (Tables 5 and 6). Witness w has the momenta K_1, K_2 and $K_3 = -K_1 - K_2$. Its first two legs have the polarizations $\varepsilon_1\varepsilon_1^\top$ and $\varepsilon_2\varepsilon_2^\top$, and its third is the gauge leg $K_3 \otimes e_a + e_a \otimes K_3$ along the coordinate axis a listed, with coordinates numbered from zero as in Section 2. The cubic test at w asks that the long-wave cubic vertex of the charge's cubic share vanish on these three legs, as in Definition 5.1.

For every witness, with the Euclidean dot product and no complex conjugation,

$$K_1 \cdot K_1 = K_2 \cdot K_2 = 0, \qquad \varepsilon_1 \cdot K_1 = \varepsilon_2 \cdot K_2 = 0, \qquad \varepsilon_1 \cdot \varepsilon_1 = \varepsilon_2 \cdot \varepsilon_2 = 0.$$

So both momenta are null for the Euclidean form, and each polarization is transverse to its momentum and traceless.

Table 5. The eleven shape witnesses: the axis a of the gauge leg and the momenta K_1 and K_2.

w	a	K_1	K_2
1	1	$(-1-3i,\ 5-3i,\ -3-5i,\ -3+i)$	$(3-5i,\ -8+3i,\ 1-i,\ 5+8i)$
2	0	$(-6+\frac{13}{2}i,\ \frac{13}{2}-6i,\ -\frac{17}{2}-\frac{13}{2}i,\ -\frac{7}{2}-\frac{13}{2}i)$	$(-\frac{13}{2}i,\ -\frac{11}{2},\ -\frac{5}{2}-\frac{1}{2}i,\ \frac{5}{2}-\frac{1}{2}i)$
3	3	$(5-\frac{3}{2}i,\ \frac{5}{2}+2i,\ 1-\frac{7}{2}i,\ -\frac{3}{2}-4i)$	$(-\frac{11}{2}+\frac{3}{2}i,\ \frac{9}{2}-\frac{7}{2}i,\ -\frac{7}{2}-\frac{9}{2}i,\ \frac{3}{2}+\frac{11}{2}i)$
4	3	$(\frac{7}{2}-i,\ -5+\frac{7}{2}i,\ -8i,\ 7+3i)$	$(-6-\frac{3}{2}i,\ -\frac{9}{2}-3i,\ \frac{9}{2}-7i,\ 2+\frac{9}{2}i)$
5	3	$(\frac{9}{2}-\frac{3}{2}i,\ \frac{9}{2}+\frac{3}{2}i,\ -8,\ 10i)$	$(-6+i,\ -7+6i,\ 4+8i,\ -4-4i)$
6	0	$(-5-\frac{11}{2}i,\ \frac{7}{2}-2i,\ \frac{13}{2}-4i,\ 1+\frac{11}{2}i)$	$(-2-7i,\ 1-4i,\ -1+2i,\ 8-i)$
7	0	$(-1+\frac{17}{2}i,\ -\frac{13}{2}-2i,\ \frac{7}{2}-2i,\ 5+\frac{1}{2}i)$	$(4i,\ 7+3i,\ 8i,\ -7+3i)$
8	0	$(\frac{7}{2}+\frac{13}{2}i,\ \frac{5}{2}+\frac{1}{2}i,\ -2+5i,\ 7-2i)$	$(1+i,\ 2+2i,\ 5-\frac{1}{2}i,\ \frac{1}{2}-5i)$
9	1	$(-\frac{5}{2}+\frac{15}{2}i,\ \frac{7}{2}+\frac{1}{2}i,\ 5+\frac{5}{2}i,\ \frac{9}{2}+i)$	$(-4+2i,\ -2-2i,\ 1+\frac{7}{2}i,\ -\frac{1}{2}-i)$
10	1	$(\frac{5}{2}-\frac{17}{2}i,\ -\frac{7}{2}+\frac{5}{2}i,\ -2+3i,\ 9+4i)$	$(\frac{5}{2}-\frac{3}{2}i,\ \frac{7}{2}-\frac{15}{2}i,\ \frac{9}{2}+2i,\ -6-\frac{7}{2}i)$
11	3	$(-4+3i,\ -2+3i,\ 5,\ 3+6i)$	$(3+\frac{5}{2}i,\ -\frac{5}{2}-3i,\ \frac{7}{2}-4i,\ 2-\frac{1}{2}i)$

Table 6. The eleven shape witnesses, continued: the vectors ε_1 and ε_2 of the polarizations.

w	ε_1	ε_2
1	$(1-5i,\ -1+3i,\ 3+i,\ -5-i)$	$(\frac{5}{2}+5i,\ 3+\frac{3}{2}i,\ 5-\frac{7}{2}i,\ \frac{1}{2}+i)$
2	$(-9-\frac{7}{2}i,\ \frac{17}{2}+3i,\ 3-11i,\ 4-6i)$	$(-9+i,\ 2+6i,\ -\frac{9}{2}+4i,\ -2-\frac{15}{2}i)$
3	$(-\frac{13}{2}-\frac{1}{2}i,\ -\frac{15}{2}-\frac{5}{2}i,\ -2+9i,\ -1+4i)$	$(\frac{7}{2}-\frac{13}{2}i,\ \frac{5}{2}-\frac{1}{2}i,\ 6+4i,\ 2)$
4	$(\frac{9}{2}-\frac{21}{2}i,\ -\frac{3}{2}-\frac{3}{2}i,\ -\frac{23}{2}-\frac{5}{2}i,\ -\frac{5}{2}-\frac{13}{2}i)$	$(-\frac{5}{2}+\frac{23}{2}i,\ -\frac{13}{2}+\frac{5}{2}i,\ -\frac{21}{2}-\frac{9}{2}i,\ \frac{3}{2}-\frac{3}{2}i)$
5	$(\frac{1}{2}-\frac{19}{2}i,\ \frac{17}{2}-\frac{1}{2}i,\ \frac{3}{2}+\frac{3}{2}i,\ \frac{9}{2}+\frac{3}{2}i)$	$(-9,\ -3i,\ -7,\ -11i)$
6	$(-\frac{9}{2}-3i,\ 6-\frac{5}{2}i,\ -2-\frac{15}{2}i,\ \frac{9}{2}-3i)$	$(-2+8i,\ -2+2i,\ \frac{1}{2}-\frac{5}{2}i,\ -\frac{17}{2}-\frac{5}{2}i)$
7	$(6+3i,\ -3i,\ \frac{17}{2}-i,\ 1-\frac{19}{2}i)$	$(-\frac{15}{2}+i,\ 2+\frac{15}{2}i,\ -\frac{1}{2}i,\ -\frac{5}{2}+3i)$
8	$(-\frac{17}{2}-\frac{1}{2}i,\ \frac{7}{2}+\frac{13}{2}i,\ \frac{7}{2}-\frac{15}{2}i,\ -\frac{3}{2}+\frac{1}{2}i)$	$(5-\frac{5}{2}i,\ -\frac{9}{2}-4i,\ -\frac{1}{2}+5i,\ 2-\frac{3}{2}i)$
9	$(-\frac{5}{2}-i,\ -7-\frac{7}{2}i,\ -\frac{1}{2}+\frac{17}{2}i,\ \frac{13}{2}-\frac{7}{2}i)$	$(-\frac{7}{2}+2i,\ -4-\frac{5}{2}i,\ -\frac{1}{2}+\frac{5}{2}i,\ \frac{1}{2}-\frac{7}{2}i)$
10	$(-5+\frac{7}{2}i,\ \frac{11}{2}+4i,\ -\frac{3}{2}+3i,\ -\frac{9}{2}i)$	$(-3+\frac{7}{2}i,\ \frac{5}{2}+6i,\ -\frac{5}{2}+3i,\ 6+\frac{1}{2}i)$
11	$(\frac{1}{2}-\frac{3}{2}i,\ \frac{1}{2}+\frac{7}{2}i,\ -\frac{9}{2}-\frac{5}{2}i,\ \frac{7}{2}-\frac{7}{2}i)$	$(-2+\frac{1}{2}i,\ -\frac{1}{2}-5i,\ \frac{11}{2}+i,\ -2+\frac{7}{2}i)$

Data availability

Every theorem, lemma, proposition and corollary of this paper not marked with a citation is stated with its hypotheses and proved here, as Section 1 explains; the results recalled from the cited papers keep the status stated in those papers. Several of those proofs are exact computations on tables: the eight quadratic and twelve cubic tables of Section 5.1 at the Euclidean Kuhn simplex, their versions for hinge classes and shape charges (Sections 5.7 and 5.9), and the table $T(\kappa)$ of Section 5.6. Those tables are fixed by the closed forms of Section 5.1 and are not printed. What the paper prints is the data a reader needs to pose each test: the witnesses W_1, W_2 and W_3 (Definition 5.1 and Appendix A), the eleven shape witnesses (Appendix E), the five conditions of the quadratic test (proof of Theorem 5.3) and

the six equations of the cubic test (Table 1). Available from the author on request: the tables themselves and the exact computation of the determinant in Table 1.

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