

The Long-Wave Regge Graviton on the $\kappa = 1$ Kuhn Lattice and the Recognition Cost's Quadratic Form

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Abstract

On the Kuhn lattice, a triangulated four-torus with flat Lorentzian metric $I - \mathbf{1}\mathbf{1}^{\top}$ and time along $\mathbf{1} = (1, 1, 1, 1)$ ($\kappa = 1$), we take Regge's action as the gravitational action. On a torus of side N, N^{-2} times its flat second variation on a plane wave of fixed integer wave vector tends to $-\sqrt{3}\pi^2/2$ times the Fierz–Pauli form: a preprint result, confirmed here by an independent exact computation over $\mathbb{Q}(\sqrt{3})$, Schläfli's identity included (inputs printed, script deposited). On the graviton's two polarizations, transverse-traceless strains H orthogonal to $\mathbf{1}$, Fierz–Pauli is the squared wave covector times $\operatorname{tr} H^2$, twice the quadratic part of the recognition cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$, an identity that uses J only through $J''(1) = 1$. In the Euclidean hypercubic model, not on the lattice, cube symmetry and gauge invariance fix Fierz–Pauli up to one factor. The ledger's charge, a quadratic form on spatial strains fixed by premises on how strained lattice steps are recorded, is, given the lattice's symmetries, $\frac{1}{2}\operatorname{tr} H^2$ on every polarization exactly when it is so on the plus and cross strains of the face wave $(1, 1, -1, -1)$ along the face axes $(1, -1, 1, -1)$ and $(1, -1, -1, 1)$; at a three-fold-axis wave, or for a rotated pair, two such numbers do not suffice. Three premise routes, none derived, give both numbers; two writing rules that the premises exclude fail only by birefringence, the premises' sensitive test inside the paper. The thesis is stated, not tested: *The graviton is not a field added to the recognition ledger. It is the cost the ledger charges for comparing a geometry with flat space.* These premises make it precise on the polarizations; nothing here about the graviton or the ledger's charge uses more of J than $J''(1) = 1$ and $J > 0$ away from 1. The scope is linear gravity around flat space.

1 The long-wave Regge graviton and the recognition cost

This paper proves three things about linear gravity on the $\kappa = 1$ Kuhn lattice, the triangulated four-torus whose counter coordinates carry the flat metric $g_1 = I - \mathbf{1}\mathbf{1}^{\top}$, with time along the diagonal $\mathbf{1}$. First, with Regge's action as the gravitational action, the scaled flat Hessian

$N^{-2}\mathscr{H}_N$ on a plane wave of fixed integer wave m tends to $-\frac{\sqrt{3}\pi^2}{2}$ times Fierz–Pauli, a result of [4] that an exact computation here confirms (Theorem 6.1, Section 6.3, Appendix A). Second, on the graviton's two polarizations Fierz–Pauli is the squared wave covector times $\operatorname{tr} H^2$, twice the long-wave form of the recognition cost (Theorems 3.1 and 5.1). Third, a charge of spatial strains with the lattice's symmetries is that long-wave form on every polarization exactly when it charges the plus and cross strains H'_+, $H'_\times$ of the face wave, diagonal and off-diagonal in the frame of the face axes, as that form does; three premise routes supply these two numbers, and birefringence is their sensitive test (Lemma 3.5, Section 3.4).

Comparing two positive quantities a and b on the recognition ledger costs

$$J(x) = \tfrac{1}{2}\big(x + x^{-1}\big) - 1, \qquad x = a/b, \tag{1}$$

the one continuous cost, zero at $x = 1$, that obeys the composition law $J(xy) + J(x/y) = 2J(x)J(y) + 2J(x) + 2J(y)$ and the calibration $J(1+\varepsilon) = \frac{1}{2}\varepsilon^2 + O(\varepsilon^3)$ [1, 2]: for $J + 1$ the law is d'Alembert's equation, whose continuous solutions [20] leave only (1) (Section 2). This paper's thesis, an identification and not a theorem, is the following. *The graviton is not a field added to the recognition ledger. It is the cost the ledger charges for comparing a geometry with flat space, read on its two physical polarizations, the transverse-traceless strains.* A geometry here is a metric matrix in the counter coordinates. The charge $J(g_1^{-1}g)$ of that comparison (Section 2.2) is not gauge invariant (Theorem 3.1(iii)), so the thesis is meant on the helicity-± 2 strains in the lattice's rest frame, the transverse-traceless strains with $H\mathbf{1} = 0$. It is stated here, not tested: Proposition 6.2 makes it precise on the polarizations under named premises, and no observation in this paper tells it from its negation. The gravitational action is an input: Regge's action on the $\kappa = 1$ Kuhn lattice, a model, which two stated premises, not derived, single out among hinge charges (Sections 6 and 7).

What is shown. On a transverse-traceless strain H at a wave m with $\eta(m) \neq 0$ the *strain factor* $S(H)$ is defined by $\lim_{N\to\infty} N^{-2}\mathscr{H}_N(u_{m,H}, u_{m,H}) = -\frac{\sqrt{3}\pi^2}{2}\,\eta(m)\,S(H)$, with $\mathscr{H}_N$ the flat Hessian of Section 6.2 and $u_{m,H}$ the plane wave (14): the long-wave Regge action with the wave factor and the lattice's constant taken out. Theorem 6.1 gives $S(H) = \operatorname{tr} H^2 = 2J_{\mathrm{lw}}(H)$, through the identity $\mathrm{FP}_{g_1} = \eta \cdot 2J_{\mathrm{lw}}$ on these strains (Theorems 3.1 and 5.1). The identity uses J only through its calibration: every cost that charges each principal stretch through an f with $f(1) = f'(1) = 0$ and $f''(1) = 1$ has the quadratic part $\frac{1}{2}\operatorname{tr} X^2$. The *ledger's charge* Q is a quadratic form on spatial strains, symmetric H with $H\mathbf{1} = 0$, fixed by premises on how the ledger records strained lattice steps (Section 3.4). The normalization statement, Proposition 6.2, is the thesis made precise on the polarizations: if Q has the lattice's symmetries and charges the plus and cross strains H'_+ and $H'_\times$ of the face wave $p_1 = (1, 1, -1, -1)$, diagonal and off-diagonal in the frame of the face axes, as J_{lw} does, $Q(H'_+) = Q(H'_\times) = 16$, then $Q = J_{\mathrm{lw}}$ on every transverse-traceless strain, and $S = 2Q$ there wherever $\eta(m) \neq 0$. Two numbers at a wave along a three-fold axis do not suffice: there every polarization carries the same mixture of the two parts (the paragraph after Lemma 3.5).

The two numbers are not derived. We reduce them to either of two named premise sets, at the record charge (8), an IDENTIFICATION that charges J on the comparisons a record's events change and J's quadratic part on each posting's share of the strain: *least-cost writing*, the ledger writes each strained step by its least-charged record (Proposition 3.6); or *one-comparison*

writing with share locality, each step is one comparison event with local shares (Proposition 3.7). Rotation invariance with J's form on the face axes also gives them (Section 3.4). These routes, too, use J only through $J''(1) = 1$ and a positive price per posting, so every statement here about the graviton or the ledger's charge holds for every such J that is positive away from 1, and none tests J's composition law or the thesis. Among hinge charges, two stated premises make the charge exactly Regge's (Proposition 7.3), and in the Euclidean model locality fixes the trace part (Fig. 1).

What is new. We have not found these in the literature: two numbers at the face wave, on H'_+ and $H'_\times$, fix a lattice-symmetric charge on the polarizations of every wave, and two numbers at a three-fold-axis wave do not (Lemma 3.5 and the paragraph after it); equal step weights are birefringent, charging one polarization twice the other; the least count of postings with a free tick gives both numbers, and least-cost writing gives that count (Proposition 3.6), as does one-comparison writing with share locality, with no minimization (Proposition 3.7); in the Euclidean hypercubic model, not on the $\kappa = 1$ lattice, cube symmetry and gauge invariance fix Fierz–Pauli up to one factor, with an explicit test table (Lemma 3.2 and Theorem 3.3); the exact check of the $\kappa = 1$ long-wave limit, Schläfli's identity at the flat simplices included (Section 6.3, Appendix A); and Table 4, which records what fixes each feature of the graviton's quadratic action. The rest is standard linearized gravity, the result of [4], or identifications.

What this unifies, and what is put in. One ledger carries three links, each at its status. First, the calibration $J''(1) = 1$ sets the unit in which the graviton's strain factor on its polarizations has coefficient one in the Fierz–Pauli normalization (Theorem 6.1): a match of units. Second, under either premise set of the count route, or the rotation route, twice the ledger's charge is that strain factor (Proposition 6.2); it is the strain factor, not the action. Third, through the posting identification the strength fixes the lattice length: $G\hbar = \lambda_{\mathrm{rec}}^2/(4 \ln 2)$, one bit of Bekenstein–Hawking entropy per area λ_{rec}^2 (Sections 7 and 8). What is put in: that gravity's action is a sum of hinge charges, a MODEL; the two premises of Proposition 7.3 that make it Regge's, the content reading, not derived, and the record clause, the posting identification extended to credits; for the ledger's charge, at the record charge (8), an IDENTIFICATION, least-cost writing or one-comparison writing with share locality, or else the rotation route, none derived; $\kappa = 1$ and the lattice's readings, IDENTIFICATIONS; and the field of squared edge lengths. The decisive test is still open: no number of a second sector, computed from the same J, has yet been matched to data.

Setting and notation. The recognition ledger keeps the record of events [4]. On it four counters advance one at a time. Each advance of one counter by one is a *posting*, and a *credit* takes one away; a lattice *step* $d \in \{0,1\}^4 \setminus \{0\}$ is made of n_d postings, the number of ones in d. The lattice is the torus of counter readings, cut into Kuhn simplices, with the flat metric $g_\kappa = I - \kappa \mathbf{1}\mathbf{1}^\top$, $\mathbf{1} = (1,1,1,1)$ (Section 6.1). On g_1 a single posting is a null step, so at $\kappa = 1$ the counters count postings along four light rays and time runs along their diagonal $\mathbf{1}$. A wave has covector k and squared covector $\eta(k) = k^\top g_1^{-1} k$ ($|k|^2$ in the Euclidean Sections 3 and 4); on the torus of side N, $k = 2\pi m/N$ with $m \in \mathbb{Z}^4$, and a wave is *static* when $m \cdot \mathbf{1} = 0$. The *spatial slice* is the set of directions orthogonal to $\mathbf{1}$; a *rotation of space* is an orthogonal map that fixes

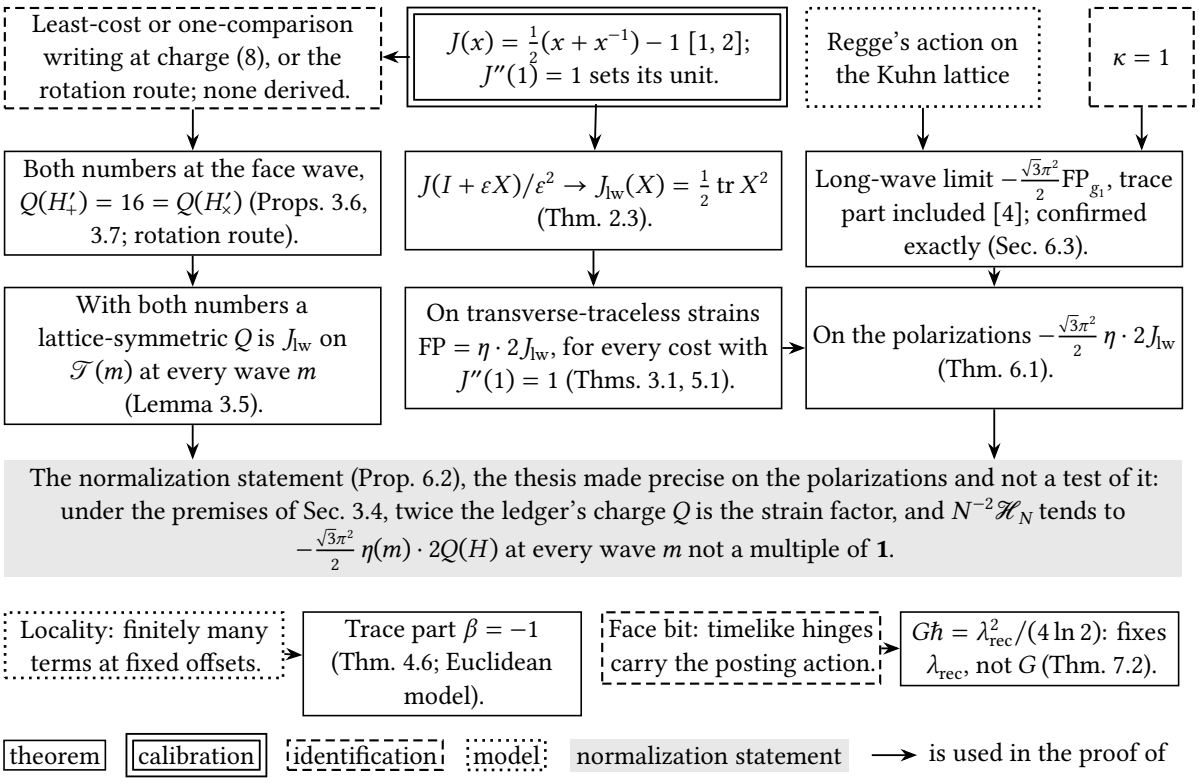


Figure 1. What is used where; an arrow means "is used in the proof of". Left: the ledger's charge, from premises not derived. Centre: J and the identity FP $= \eta \cdot 2J_{\mathrm{lw}}$. Right: Regge's action at $\kappa = 1$ gives the whole long-wave form. Neither the identity nor the left column uses J beyond $J''(1) = 1$ and $J > 0$ away from 1: every cost with that calibration gives the same figure, and nothing in it tests the thesis. η is the squared wave covector, FP_{g_1} Fierz–Pauli on the $\kappa = 1$ background.

$\mathbf{1}$, so it keeps g_1, and it is *proper* when its determinant is 1. Lattice units measure squared lengths by g_κ in the counter coordinates, and the *recognition length* λ_{rec} is their unit of length in physical units. A *lattice face* is a hinge, a triangle of the triangulation (Section 6.1); the *face bit* is the one bit that a full turn of twelve postings counts on a face of unit lattice area, of area λ_{rec}^2 (Section 7). Table 1 lists the ledger's terms, the object each names, and where it is defined.

Scope and status words. This is linear gravity: the quadratic form around flat space at long wavelengths, at integer waves m that are not multiples of $\mathbf{1}$; at a nonzero multiple of $\mathbf{1}$ the transverse-traceless strains form a five-dimensional space (Section 3.3). Theorem 6.1 rests on Regge's action as the lattice's gravitational action, a MODEL, with $\kappa = 1$ an IDENTIFICATION, and on [4, Thm. 3.1(ii)], a preprint, not peer reviewed, which Section 6.3 confirms by an exact computation. Proposition 6.2, on the ledger's charge, rests in addition on the premise sets of Propositions 3.6 and 3.7, at the record charge (8), or on the rotation route (Section 3.4), none derived. The reading $J(g_1^{-1}g)$ of Section 2.2, an IDENTIFICATION, meets the rotation route's premises. The trace part rests on locality, in the Euclidean model (Section 4), and the strength on the posting identification (Section 7). The identity of Theorems 3.1 and 5.1 needs no lattice. Only Regge's action (Section 6) and the strength (Section 7) use $\kappa = 1$. The dimension of space, three, we take as given. A *theorem* is proved as stated, with every hypothesis beside it; an IDENTIFICATION is a physical reading that the definitions do not force; a MODEL is a choice of mathematical object; a CALIBRATION fixes a unit; a *principle* is a premise on how the ledger

Table 1. The ledger's terms: what each names, and where it is defined.

term	mathematical object	defined in
posting, credit; tick	a change of one counter by $+1$, by -1; the change $\mathbf{1}$ of all four	Sections 1 and 3.4
record of a step d	postings, credits and $t \in \mathbb{Z}$ ticks that add up to $d \in \{0,1\}^4 \setminus \{0\}$	Section 3.4
comparison charge	$\sum_{a,b} J(e^{\Lambda_a - \Lambda_b})$ for a change $\Delta \in \mathbb{Z}^4$ of the counters	Section 3.4
record charge; least-cost writing	the charge (8) of a record, an identification; each strained step written by a record of least charge, a principle, not derived	Section 3.4
one-comparison writing; share locality	each step written by at most two postings, on distinct counters; equivariant shares, each a function of s; not derived	Section 3.4
the ledger's charge Q	a quadratic form on spatial strains, symmetric H with $H\mathbf{1} = 0$	Section 3.4
count route, rotation route	the routes that give $Q(H'_+) = 16$ and $Q(H'_\times) = Q(H'_+)$ at the face wave; the count route has two premise sets	Section 3.4
face bit; posting action	one bit counted by twelve postings on a face of area λ_{rec}^2; the action $a(A)$ of one posting	Theorem 7.2
content reading, record clause	premises (B) and (C) on a hinge charge $C(A, \delta)$	Section 7

writes, stated and not derived; a cited result is stated, with its status, in the work cited.

The chain in formulas. For a small strain, $g = g_1 + \varepsilon H$, the comparison $g_1^{-1} g$ is $I + \varepsilon X$ with the metric strain $X = g_1^{-1} H$, and J extends to matrices as $J(A) = \frac{1}{2}(\operatorname{tr} A + \operatorname{tr} A^{-1}) - n$ (Definition 2.2). We prove that for a strain h that is transverse-traceless at a wave covector k ($h\mathbf{1} = 0$, $hk = 0$, $\operatorname{tr} h = 0$)

$$\mathrm{FP}_{g_1}(h, h; k) = \eta(k) \cdot 2\, J_{\mathrm{lw}}(g_1^{-1} h), \qquad J_{\mathrm{lw}}(X) = \lim_{\varepsilon \to 0} \frac{J(I + \varepsilon X)}{\varepsilon^2} = \frac{1}{2} \operatorname{tr} X^2, \tag{2}$$

where FP_{g_1} is the Fierz–Pauli form on the $\kappa = 1$ background, the kinetic term of linearized Einstein gravity [5, 6], in the normalization of [4, Sec. 2.4], stated exactly in Sections 3.1 and 5; the Euclidean form (6) obeys the same with $|k|^2$ for $\eta(k)$ (Theorem 3.1(ii)). A form takes strains before the semicolon and the wave covector after it, and $\mathrm{FP}(h; k) = \mathrm{FP}(h, h; k)$. For $\eta(k) \neq 0$ the *coefficient*, the ratio $\mathrm{FP}_{g_1}/(\eta(k) \cdot 2J_{\mathrm{lw}})$ on these strains, is one.

What is physical, and what is a unit. J has no derivatives: it charges the strain at each point. Added to an action, such a derivative-free quadratic form would be a mass term, and J_{lw}, which lacks the $-(\operatorname{tr} h)^2$ of Fierz–Pauli's mass combination, would propagate the ghost of [6, 16]. The ledger's charge is never added to the action here: it matches only the metric $\operatorname{tr} H^2$ on the polarizations that the kinetic term carries. On the two polarizations every quadratic form invariant under the rotations about the wave is a multiple of $\operatorname{tr} h^2$, since those rotations turn the polarizations into each other through twice the angle. $J''(1) = 1$ sets the one free number in J's unit, and the coefficient one is the ratio of that unit to Fierz–Pauli's normalization (6); against the $\varepsilon_0\varepsilon_1$ coefficient of $\sqrt{|g|}\, R$ itself, per unit proper volume (Section 3.1), the same ratio is $-\frac{1}{2}$; with $J''(1) = c$ the coefficient would be $1/c$ in the Fierz–Pauli normalization and $-1/(2c)$ against Einstein–Hilbert. It is a match of units, and it says nothing about the strength (Section 7). Three numbers do not depend on units. First, in the long-wave limit the two polarizations cost the same under Regge's action and under the reading $J(g_1^{-1} g)$, but

not under equal step weights (Theorem 6.1 and Section 3.4); at finite N Regge's action splits them by about $2.78/N^2$ (Section 6.3). Second, the volume change enters with weight exactly -1, which in the Euclidean model locality forces (Corollary 4.7). Third, through the posting identification, $G\hbar/\lambda_{\mathrm{rec}}^2 = 1/(4\ln 2)$, which fixes λ_{rec} (Section 7).

2 The cost of a comparison

A small comparison costs half the square of its strain, with an exact correction (Theorem 2.3).

2.1 The recognition cost

Comparing two positive numbers costs J of their ratio. J vanishes only when the numbers agree and grows as half the square of a small change; that $\frac{1}{2}$, the calibration $J''(1) = 1$, sets J's unit, the unit in which the graviton's strain factor has coefficient one in the Fierz–Pauli normalization (Section 1).

Lemma 2.1 (Properties of J). *For $x, y > 0$, every real t and every $\varepsilon > -1$:*

(a) $J(x) = \dfrac{(x-1)^2}{2x}$, *so $J(x) \geq 0$, with equality only at $x = 1$;*

(b) $J(x) = J(x^{-1})$;

(c) $J(e^t) = \cosh t - 1$;

(d) $J(xy) + J(x/y) = 2J(x)J(y) + 2J(x) + 2J(y)$;

(e) $J(1+\varepsilon) = \dfrac{\varepsilon^2}{2(1+\varepsilon)}$, *so* $J(1+\varepsilon) - \frac{1}{2}\varepsilon^2 = -\dfrac{\varepsilon^3}{2(1+\varepsilon)}$.

Proof. (a) Put the definition over the common denominator $2x$. (b) The definition does not change under $x \mapsto x^{-1}$. (c) $\frac{1}{2}(e^t + e^{-t}) = \cosh t$. (d) $J(xy) + J(x/y) + 2 = \frac{1}{2}(x + x^{-1})(y + y^{-1}) = 2(J(x) + 1)(J(y) + 1)$. (e) is (a) at $x = 1 + \varepsilon$. □

Why this cost. Put $F = J + 1$ and $\phi(t) = F(e^t)$. With $x = e^t$ and $y = e^u$ the composition law (d) reads $\phi(t+u) + \phi(t-u) = 2\phi(t)\phi(u)$, d'Alembert's functional equation. Its continuous solutions on the real line are $\phi = 0$, $\phi(t) = \cos at$ and $\phi(t) = \cosh at$ with $a \geq 0$ [20]. So a cost J that is continuous on the positive reals, obeys the composition law and vanishes at 1 is $\cos(a \ln x) - 1$ or $\cosh(a \ln x) - 1$: $J(1) = 0$ excludes $\phi = 0$. The first has $J''(1) = -a^2$ and the second $J''(1) = a^2$, so $J''(1) = 1$ leaves $\cosh(\ln x) - 1$, which is (1). This is the uniqueness used in Section 1; for the composition law on the ledger see [1, 2].

2.2 Comparing two metrics

A *metric* here is a real symmetric invertible matrix g on the coordinate space of the lattice, the counter coordinates; it gives every edge d its squared length $d^{\top} g d$. Two metrics g_1 and g are compared through the matrix ratio $g_1^{-1} g$, which is I when they agree, and the comparison costs J of that matrix.

Definition 2.2 (The cost of a matrix comparison)**.** For an invertible real $n \times n$ matrix A, $J(A) = \frac{1}{2}(\operatorname{tr} A + \operatorname{tr} A^{-1}) - n$. The *long-wave form* of a strain X is $J_{\mathrm{lw}}(X) = \frac{1}{2}\operatorname{tr} X^2$.

When A is symmetric with positive eigenvalues $\lambda_1, \dots, \lambda_n$, $J(A) = \sum_i J(\lambda_i)$: the cost charges each principal stretch its own recognition cost; for $n = 1$ it is (1). For the Lorentzian g_1 of Sections 5 and 6 the eigenvalues of $g_1^{-1}g$ can be complex, and the trace formula needs none. There J is not a nonnegative cost: for $H = \mathbf{1}p_1^{\top} + p_1\mathbf{1}^{\top}$ with $p_1 = (1, 1, -1, -1)$, the matrix $g_1^{-1}(g_1 + \varepsilon H)$ has the eigenvalues 1, 1 and $1 \pm \frac{4}{3}\sqrt{3}\, i\varepsilon$, and $J\big(g_1^{-1}(g_1 + \varepsilon H)\big) = -16\varepsilon^2/(16\varepsilon^2 + 3) < 0$, negative already in its quadratic part $-\frac{16}{3}\varepsilon^2$. On spatial strains ($H\mathbf{1} = 0$), where $g_1^{-1}H = H$ is real symmetric, it is a cost as in Lemma 2.1.

A *strain* H changes the flat metric to $g = g_1 + \varepsilon H$ and every edge's squared length by $\varepsilon\, d^{\top}Hd$. The comparison of the strained metric with the flat one is

$$g_1^{-1}g = I + \varepsilon X, \qquad X = g_1^{-1}H, \tag{3}$$

and we call X the *metric strain*.

IDENTIFICATION. The ledger compares a strained metric with the flat one through $g_1^{-1}g$ and charges $J(g_1^{-1}g)$ of Definition 2.2. The theorems hold without this reading, which the definitions do not force. $J(g_1^{-1}g)$ is not gauge invariant (Theorem 3.1(iii)), so the identification is meant on the helicity-± 2 strains in the lattice's rest frame, the transverse-traceless strains with $H\mathbf{1} = 0$. It serves only Proposition 6.2: on spatial strains it gives the charge J_{lw}, which meets the rotation route's premises (Section 3.4).

A known object. For positive-definite g_0 and g, $J(g_0^{-1}g) = \frac{1}{2}(\operatorname{tr} g_0^{-1}g + \operatorname{tr} g^{-1}g_0) - n$ is the Jeffreys divergence, the symmetrized Kullback–Leibler divergence, between the centred Gaussian distributions with covariances g_0 and g [23, 24]; on spatial strains, where $g_1^{-1}g = I + \varepsilon H$, the reading $J(g_1^{-1}g)$ is this divergence between the covariances I and $I + \varepsilon H$, for $|\varepsilon|$ small enough that $I + \varepsilon H$ is positive definite. Its quadratic part $\frac{1}{2}\operatorname{tr}(g_0^{-1}h\, g_0^{-1}h)$ is the Fisher–Rao metric of that family [25], which is, pointwise, half the member with $\lambda = 0$ of the DeWitt family $G_\lambda(h, h) = \operatorname{tr}(g^{-1}hg^{-1}h) - \lambda(\operatorname{tr} g^{-1}h)^2$ on the space of metrics, the ultralocal L^2 metric [26, 27]. The Einstein–Hilbert kinetic term carries $\lambda = 1$, whose negative conformal direction is the sign of Theorem 3.1(iv). On transverse strains the completions C_β of Section 4 are $|k|^2 G_{-\beta}(h_\perp)$, so Fierz–Pauli, $\beta = -1$, is the DeWitt metric with $\lambda = 1$ on the transverse strain (Theorem 3.1(i)).

Theorem 2.3 (The cost of a small comparison, exactly)**.** *Let X be a real $n \times n$ matrix and ε a real number with $I + \varepsilon X$ invertible, and put $R = (I + \varepsilon X)^{-1}$. Then*

$$J(I + \varepsilon X) = \frac{\varepsilon^2}{2}\operatorname{tr}(X^2 R), \tag{4}$$

and $J(I + \varepsilon X)/\varepsilon^2 \to J_{\mathrm{lw}}(X)$ as $\varepsilon \to 0$ with $\varepsilon \neq 0$.

Proof. From $(I + \varepsilon X)R = I$ we get $R = I - \varepsilon XR$. Taking traces, $\operatorname{tr} R = n - \varepsilon \operatorname{tr}(XR)$. Multiplying $R = I - \varepsilon XR$ on the left by X gives $XR = X - \varepsilon X^2 R$, so $\operatorname{tr}(XR) = \operatorname{tr} X - \varepsilon \operatorname{tr}(X^2 R)$. Hence

$$\operatorname{tr} R = n - \varepsilon \operatorname{tr} X + \varepsilon^2 \operatorname{tr}(X^2 R), \qquad \operatorname{tr}(I + \varepsilon X) = n + \varepsilon \operatorname{tr} X.$$

Adding, halving and subtracting n gives (4). The same step once more, $X^2R = X^2 - \varepsilon X^3 R$, gives $\frac{\varepsilon^2}{2}\operatorname{tr}(X^2R) = \varepsilon^2 J_{\mathrm{lw}}(X) - \frac{\varepsilon^3}{2}\operatorname{tr}(X^3R)$. Only ring operations and the halving were used, so both identities hold over every commutative ring in which 2 is invertible, for any R with $(I+\varepsilon X)R = I$. For the limit, $I + \varepsilon X$ is invertible for all small ε, and $R \to I$ by the continuity of the inverse, so $\operatorname{tr}(X^2R) \to \operatorname{tr} X^2$. □

Example 2.4 (The wave carried through the paper)**.** Take the wave $m = (1,-1,0,0)$ and the unit vectors $\hat{p} = \frac{1}{2}(1,1,-1,-1)$ and $\hat{q} = \frac{1}{\sqrt{2}}(0,0,1,-1)$. They are orthogonal to each other, to m and to $\mathbf{1}$. The *plus* and *cross* strains are

$$H_+ = \hat{p}\hat{p}^\top - \hat{q}\hat{q}^\top, \qquad H_\times = \hat{p}\hat{q}^\top + \hat{q}\hat{p}^\top. \tag{5}$$

H_+ has the eigenvalues 1 and -1 on $\hat{p}$ and $\hat{q}$, and 0 on the two directions orthogonal to both. So for $|\varepsilon| < 1$, by Lemma 2.1(e), $J(I+\varepsilon H_+) = J(1+\varepsilon) + J(1-\varepsilon) = \varepsilon^2/(1-\varepsilon^2)$; the long-wave form is $J_{\mathrm{lw}}(H_+) = 1$, and indeed $J(I+\varepsilon H_+)/\varepsilon^2 = 1/(1-\varepsilon^2) \to 1$; $H_\times$ has the same eigenvalues, on $(\hat{p} \pm \hat{q})/\sqrt{2}$. We carry this wave and these two strains through the paper.

3 Fierz–Pauli on transverse-traceless strains

On transverse-traceless strains the Fierz–Pauli form is $|k|^2 \operatorname{tr} h^2$, $|k|^2$ times twice J's long-wave form; off them the two differ in the two ways of Theorem 3.1. These are standard facts of linearized gravity [6, 16]; what is new here is their reading through J, and in Section 3.2 the cube version with Table 2. We work in four dimensions with the Euclidean dot product, here and in Section 4; Section 5 gives the Lorentzian form.

3.1 The transverse decomposition

Fierz–Pauli sees a strain only through its part transverse to the wave (Theorem 3.1). A strain is a real symmetric 4×4 matrix h and a wave covector is $k \in \mathbb{R}^4$, with $|k|^2 = k\cdot k$. The *Fierz–Pauli form* is

$$\mathrm{FP}(h;k) = |k|^2 \operatorname{tr} h^2 - 2|hk|^2 + 2\operatorname{tr} h\; k^\top h k - |k|^2(\operatorname{tr} h)^2, \tag{6}$$

the quadratic form of linearized Einstein gravity on a plane wave [5], in the normalization of [4, Sec. 2.4]: for a constant metric g and $G = g + \varepsilon_0 h\, e^{ik\cdot y} + \varepsilon_1 h' e^{-ik\cdot y}$, the coefficient of $\varepsilon_0\varepsilon_1$ in $\sqrt{|\det G|}\,R$ is $-\frac{1}{2}\sqrt{|\det g|}\,\mathrm{FP}_g(h,h';k)$ up to total derivatives, with FP_g the form (12) with indices raised by g^{-1} and $\mathrm{FP} = \mathrm{FP}_I$. For a real wave $g + \varepsilon H\cos(k\cdot y)$ the mean ε^2 coefficient is therefore $-\frac{1}{8}\sqrt{|\det g|}\,\mathrm{FP}_g(H,H;k)$. A vertex displacement, a change of coordinates with amplitude ξ, changes the strain by a *gauge mode* $\gamma_{k,\xi} = k\xi^\top + \xi k^\top$. For $k \neq 0$ let $\Pi_k = I - kk^\top/|k|^2$ be the projector transverse to k, and call $h_\perp = \Pi_k h \Pi_k$ the *transverse strain*. A strain is *transverse-traceless* at k when $hk = 0$ and $\operatorname{tr} h = 0$; at a fixed $k \neq 0$ these strains form a five-dimensional space, the traceless symmetric forms on $k^\perp$. Asking also $h\mathbf{1} = 0$, orthogonality to the time direction $\mathbf{1}$, leaves, for k not a multiple of $\mathbf{1}$, a two-dimensional space, the graviton's two polarizations $\mathscr{T}(m)$ of Section 3.3. The *conformal mode* is the uniform rescaling $h = I$. Since $I - \Pi_k = kk^\top/|k|^2$ is the gauge mode with $\xi = k/(2|k|^2)$, Fierz–Pauli sees the conformal mode as Π_k, a uniform scaling of the transverse directions.

Theorem 3.1 (Fierz–Pauli through the transverse strain)**.** *Let h be symmetric and $k \neq 0$.*

(i) $\mathrm{FP}(h;k) = |k|^2\big(2J_{\mathrm{lw}}(h_\perp) - (\operatorname{tr} h_\perp)^2\big)$.

(ii) If h is transverse-traceless at k, then $\mathrm{FP}(h;k) = |k|^2 \cdot 2J_{\mathrm{lw}}(h) = |k|^2 \operatorname{tr} h^2$.

(iii) For every ξ, $\mathrm{FP}(h + \gamma_{k,\xi};k) = \mathrm{FP}(h;k)$ *and* $\mathrm{FP}(\gamma_{k,\xi};k) = 0$, *while* $J_{\mathrm{lw}}(\gamma_{k,\xi}) = |k|^2|\xi|^2 + (k\cdot\xi)^2$, *which is positive when $\xi \neq 0$.*

(iv) $\mathrm{FP}(\Pi_k;k) = -6|k|^2$, *while* $J_{\mathrm{lw}}(\Pi_k) > 0$.

Proof. Write $a = k^\top hk$ and $b = |hk|^2$. The projector satisfies $\Pi_k^2 = \Pi_k$, $\Pi_k k = 0$ and $\operatorname{tr}\Pi_k = 3$. So $\operatorname{tr} h_\perp = \operatorname{tr}(\Pi_k h) = \operatorname{tr} h - a/|k|^2$ and $\operatorname{tr} h_\perp^2 = \operatorname{tr}(\Pi_k h \Pi_k h) = \operatorname{tr} h^2 - 2b/|k|^2 + a^2/|k|^4$. Then

$$|k|^2\big(\operatorname{tr} h_\perp^2 - (\operatorname{tr} h_\perp)^2\big) = |k|^2 \operatorname{tr} h^2 - 2b + \frac{a^2}{|k|^2} - |k|^2(\operatorname{tr} h)^2 + 2a \operatorname{tr} h - \frac{a^2}{|k|^2} = \mathrm{FP}(h;k),$$

which is (i). For (ii), $hk = 0$ gives $a = b = 0$ and $h_\perp = h$, and $\operatorname{tr} h = 0$ removes the trace term. For (iii), $\Pi_k \gamma_{k,\xi} \Pi_k = 0$ because $\Pi_k k = 0$, so a gauge mode does not change $h_\perp$, and (i) gives both statements. Its long-wave form is $\frac{1}{2}\operatorname{tr}\gamma_{k,\xi}^2 = |k|^2|\xi|^2 + (k\cdot\xi)^2$. For (iv), $(\Pi_k)_\perp = \Pi_k$ has trace 3 and $\operatorname{tr}\Pi_k^2 = 3$, so (i) gives $|k|^2(3-9) = -6|k|^2$, while $J_{\mathrm{lw}}(\Pi_k) = \frac{3}{2}$. □

A gauge mode is a real change of the strain matrix, and J charges it; Einstein's form does not, because a change of coordinates changes the metric matrix but not the spacetime it describes. Gauge and conformal modes are not gravitons; Section 4 supplies the trace part. At the wave of Example 2.4, take $k = m$: H_+ is transverse-traceless, $\mathrm{FP}(H_+;m) = |m|^2 \operatorname{tr} H_+^2 = 4$, and the conformal mode has $\mathrm{FP}(\Pi_m;m) = -12$ and $J_{\mathrm{lw}}(\Pi_m) = \frac{3}{2}$.

3.2 Normalization of the gauge-invariant form

Gauge invariance and the symmetries of the cube fix the graviton's form up to one factor; normalizing it to $|k|^2 \cdot 2J_{\mathrm{lw}}$ on transverse-traceless strains gives Fierz–Pauli's normalization (6) (Lemma 3.2 and Theorem 3.3). For rotation-invariant forms this is classical [6, 16]; we have not found the cube version and Table 2 in the literature. This subsection's model, the Euclidean hypercubic model, treats the four axes alike; it is not the $\kappa = 1$ lattice. Its long-wave forms are invariant under the 384 *cube symmetries*, which relabel the four axes and flip any of their signs: $h \mapsto OhO^\top$ and $k \mapsto Ok$ for a signed permutation matrix O. The forms $Q(h;k)$, quadratic in h and in k, that these fix are, on symmetric strains, exactly the combinations of the nine

$$I_1 = (\operatorname{tr} h)^2|k|^2, \quad I_2 = \operatorname{tr}(h^2)|k|^2, \quad I_3 = \operatorname{tr} h\; k^\top hk, \quad I_4 = |hk|^2,$$
$$A_1 = \operatorname{tr} h \sum_x h_{xx}k_x^2, \quad A_2 = \sum_{x,y} h_{xy}^2 k_x^2, \quad A_3 = \sum_x h_{xx}k_x(hk)_x,$$
$$A_4 = |k|^2 \sum_x h_{xx}^2, \quad A_5 = \sum_x h_{xx}^2 k_x^2.$$

They are all the invariants. An invariant form is a combination of orbit sums of monomials $h_{ab}h_{cd}k_e k_f$, and a monomial survives the sign flips only if every axis occurs an even number of times among its six indices. Up to relabelling there are nine such monomials, $h_{xx}^2k_x^2$, $h_{yy}^2k_x^2$,

$h_{xx}h_{zz}k_x^2$, $h_{yy}h_{zz}k_x^2$, $h_{xy}^2k_x^2$, $h_{yz}^2k_x^2$, $h_{xy}h_{xx}k_xk_y$, $h_{xy}h_{zz}k_xk_y$ and $h_{xz}h_{yz}k_xk_y$ (distinct letters for distinct axes), and the sums of the distinct monomials in their orbits are A_5, $A_4 - A_5$, $A_1 - A_5$, $\frac{1}{2}I_1 - A_1 - \frac{1}{2}A_4 + A_5$, $A_2 - A_5$, $\frac{1}{2}I_2 - A_2 - \frac{1}{2}A_4 + A_5$, $A_3 - A_5$, $\frac{1}{2}I_3 - \frac{1}{2}A_1 - A_3 + A_5$ and $\frac{1}{2}I_4 - \frac{1}{2}A_2 - A_3 + A_5$, a triangular change of basis. Of the nine forms, $I_1, \dots, I_4$ are built from traces and dot products alone, and $A_1, \dots, A_5$ single out the axes. All nine are polynomials in k, and each is the long-wave symbol of a local lattice cost: the construction in the proof of Theorem 4.6(ii) realizes every form $\sum_{a,b} F_{ab}(h)k_ak_b$. Write $Q = \sum_{i=1}^{4} \theta_i I_i + \sum_{j=1}^{5} \alpha_j A_j$. Fierz–Pauli is Q with $\theta = (-1, 1, 2, -2)$ and $\alpha = 0$.

Lemma 3.2 (Cube symmetry and gauge invariance fix the form up to one factor)**.** *If $Q(h + \gamma_{k,\xi}; k) = Q(h; k)$ for every symmetric h and all $k, \xi \in \mathbb{R}^4$, then $\theta_1 = -\theta_2$, $\theta_3 = 2\theta_2$, $\theta_4 = -2\theta_2$ and $\alpha_1 = \cdots = \alpha_5 = 0$. So $Q = \theta_2$FP.*

Proof. The gauge difference $Q(h+\gamma_{k,\xi}; k) - Q(h; k)$ is linear in the nine coefficients and vanishes at every point (h, k, ξ). Table 2 lists eight points and the gauge difference at each, as a linear form in $(\theta_1, \theta_2, \theta_3, \theta_4, \alpha_1, \dots, \alpha_5)$, with $E_{ab} = e_ae_b^\top + e_be_a^\top$ for $a \neq b$ and $E_{aa} = e_ae_a^\top$. Each row is computed from the definitions of the nine forms, and at every point the Fierz–Pauli difference is zero. Rows 1 and 2 give $\theta_4 = -2\theta_2$ and $\alpha_2 = 0$; rows 3 to 8 are then a 6×6 linear system, with right side θ_2 times a fixed vector, in $(\theta_1, \theta_3, \alpha_1, \alpha_3, \alpha_4, \alpha_5)$ with determinant -8192 and the unique solution $\theta_2(-1, 2, 0, 0, 0, 0)$. So the solutions are exactly the multiples of $(\theta, \alpha) = (-1, 1, 2, -2, 0, 0, 0, 0, 0)$. Row 1 in full: $h = E_{01} + E_{03}$, $k = e_2$, $\xi = e_3$, so $\gamma_{k,\xi} = E_{23}$ and $|k|^2 = 1$. Both h and $h + E_{23}$ are traceless with zero diagonal, so only I_2 ($\operatorname{tr} h^2$ goes from 4 to 6), I_4 ($hk = 0$, $(h + E_{23})k = e_3$) and A_2 (row 2 of h goes from 0 to e_3) change, by 2, 1 and 1: the row is $(0, 2, 0, 1 \mid 0, 1, 0, 0, 0)$. □

Table 2. Test points for Lemma 3.2: the gauge differences of the nine forms in $(\theta_1, \dots, \theta_4 \mid \alpha_1, \dots, \alpha_5)$.

	h	k	ξ	coefficients		h	k	ξ	coefficients
1	$E_{01} + E_{03}$	e_2	e_3	$(0, 2, 0, 1 \mid 0, 1, 0, 0, 0)$	5	E_{22}	$e_0 + e_2$	e_0	$(16, 12, 14, 12 \mid 8, 6, 7, 8, 4)$
2	$E_{01} + E_{02}$	$e_2 + e_3$	e_0	$(0, 16, 0, 8 \mid 0, 4, 0, 0, 0)$	6	E_{03}	$e_1 + e_2 + e_3$	e_3	$(12, 24, 12, 18 \mid 4, 8, 8, 12, 4)$
3	$E_{01} + E_{13}$	e_2	e_2	$(4, 4, 4, 4 \mid 4, 4, 4, 4, 4)$	7	$E_{22} + E_{23}$	$\mathbf{1}$	e_0	$(32, 40, 30, 34 \mid 8, 10, 11, 16, 4)$
4	E_{02}	$e_0 + e_2 + e_3$	e_3	$(12, 24, 16, 22 \mid 4, 8, 8, 12, 4)$	8	E_{23}	$e_1 + e_2 + e_3$	e_2	$(12, 36, 16, 28 \mid 4, 12, 10, 12, 4)$

Theorem 3.3 (Normalization of the gauge-invariant form)**.** *Let Q be a member of the nine-term family that is gauge invariant. Then $Q(h; k) = |k|^2 \cdot 2J_{\mathrm{lw}}(h)$ for every strain h that is transverse-traceless at k if and only if $Q =$ FP.*

Proof. By Lemma 3.2, $Q = \theta_2$FP. Take $k = e_0$ and $h = e_1e_2^\top + e_2e_1^\top$, which is transverse-traceless at e_0. Then $\mathrm{FP}(h; e_0) = \operatorname{tr} h^2 = 2$ and $|e_0|^2 \cdot 2J_{\mathrm{lw}}(h) = 2$, so $\theta_2 = 1$. The converse is Theorem 3.1(ii); at $k = 0$ both sides vanish. □

The $\kappa = 1$ lattice has only the 24 relabellings of the axes, alone and with the reversal of all four; a single sign flip moves $\mathbf{1}$, so it does not keep g_1. So Lemma 3.2 does not apply there; Regge's action fixes the form (Theorem 6.1).

3.3 How many polarizations

A graviton exists only when space has at least three dimensions; it then has $(D+1)(D-2)/2$ polarizations, two when $D = 3$ (Theorem 3.4). Let space have D dimensions. The lattice has $D+1$ coordinates, and time runs along the diagonal, the all-ones vector $\mathbf{1} = (1, \dots, 1)$, as on the $\kappa = 1$ lattice [4]. A wave $m \in \mathbb{R}^{D+1}$ has the space of *transverse-traceless strains*

$$\mathcal{T}(m) = \{H = H^{\top} \,:\, H\mathbf{1} = 0,\ Hm = 0,\ \operatorname{tr} H = 0\}. \tag{7}$$

For $D = 3$ and the $\kappa = 1$ metric these are the covariant transverse-traceless strains with $H\mathbf{1} = 0$: with $g_1^{-1} = I - \frac{1}{3}\mathbf{1}\mathbf{1}^{\top}$ and $H\mathbf{1} = 0$, $Hg_1^{-1}m = Hm$ and $\operatorname{tr}(g_1^{-1}H) = \operatorname{tr} H$. Off shell, at a wave m that is not a multiple of $\mathbf{1}$, this space depends on the time direction $\mathbf{1}$: it is the helicity-± 2 part, in the lattice's rest frame, of the five-dimensional space $\{h \,:\, hg_1^{-1}m = 0,\ \operatorname{tr}(g_1^{-1}h) = 0\}$, on all of which, null or not, $\mathrm{FP}_{g_1}(h, h; m) = \eta(m) \operatorname{tr}(g_1^{-1}hg_1^{-1}h)$, since the two conditions remove every term of (12) but the first. At a nonzero multiple m of $\mathbf{1}$, $Hm = 0$ follows from $H\mathbf{1} = 0$, and $\mathcal{T}(m)$ is the five-dimensional space of traceless spatial strains.

Theorem 3.4 (The graviton needs three dimensions)**.** *Let $D \geq 1$.*

(i) For every wave m that is not a multiple of $\mathbf{1}$, $\dim \mathcal{T}(m) = (D+1)(D-2)/2$ when $D \geq 2$, and $\mathcal{T}(m) = 0$ when $D \leq 2$.

(ii) $J_{\mathrm{lw}}(H) > 0$ for every nonzero $H \in \mathcal{T}(m)$.

(iii) Some wave m that is not a multiple of $\mathbf{1}$ carries a nonzero transverse-traceless strain if and only if $D \geq 3$.

With $d = D + 1$ spacetime dimensions, (i) is the classical count $d(d-3)/2$ [16].

Proof. Let W be the plane spanned by $\mathbf{1}$ and m, and $W^{\perp}$ its orthogonal complement, of dimension $D-1$. A symmetric H with $H\mathbf{1} = Hm = 0$ has W in its kernel and, being symmetric, its range in $W^{\perp}$, so it is a symmetric form on $W^{\perp}$. These form a space of dimension $D(D-1)/2$. For $D \geq 2$ the trace is a nonzero linear functional on it, so the traceless ones have dimension $D(D-1)/2 - 1 = (D+1)(D-2)/2$; this is 0 at $D = 2$, and for $D = 1$ the space $W^{\perp}$ is 0. That proves (i). For (ii), $J_{\mathrm{lw}}(H) = \frac{1}{2}\operatorname{tr}(H^{\top}H) = \frac{1}{2}\sum_{a,b} H_{ab}^2$. Part (iii) follows from (i). □

For the wave of Example 2.4, $\mathcal{T}(m)$ is spanned by H_+ and $H_\times$. Figure 2 shows where the count comes from and draws the two polarizations.

3.4 What the two polarizations need from the identification

Proposition 6.2, on the ledger's charge, needs only two numbers from it, both at one wave, the face wave p_1 below, on its two polarizations H'_+ and $H'_\times$, diagonal and off-diagonal in the frame of the face axes: its value on H'_+, and an equal value on $H'_\times$. The wave and the pair matter (the paragraph after Lemma 3.5), and the lattice's own symmetries give neither number. Least-cost writing gives both (Proposition 3.6), and so do two other premise sets: one-comparison writing with share locality (Proposition 3.7) and the rotation route. Here a *cost*, the ledger's charge, is a quadratic form $Q(H)$ on spatial strains (symmetric H with $H\mathbf{1} = 0$); J's form is $Q = J_{\mathrm{lw}}$.

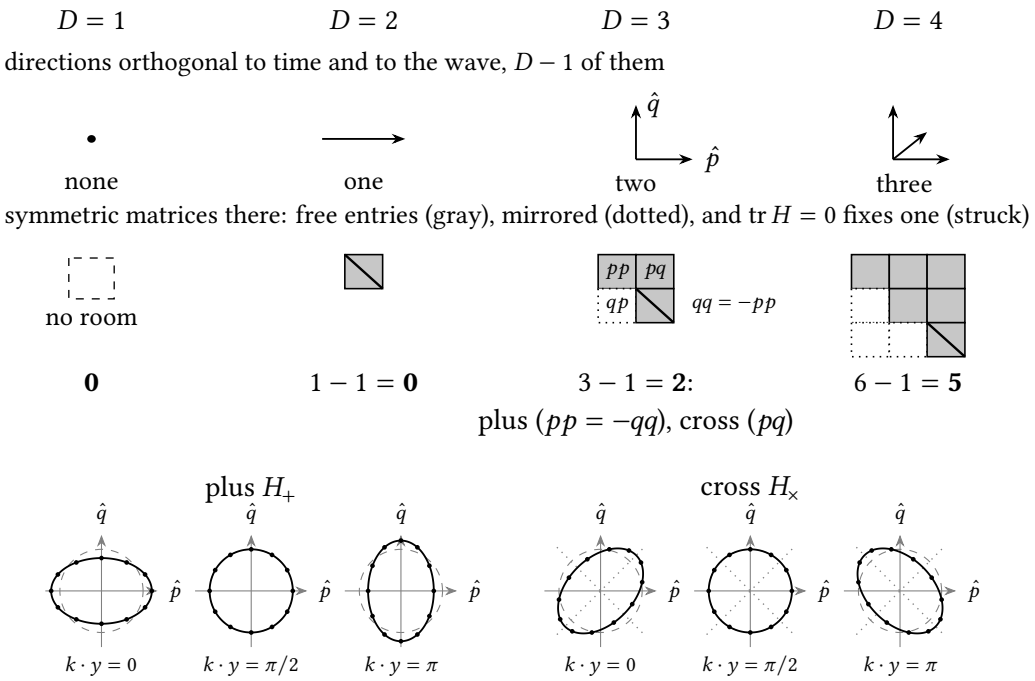


Figure 2. Why a graviton needs three dimensions, and its two polarizations. Top: the room a transverse-traceless strain has at each D. Bottom: plus and cross of Example 2.4 on a ring of points at fixed lattice coordinates, drawn at proper distances (dashed: flat; amplitude exaggerated); plus stretches along $\hat{p}$ and shrinks along $\hat{q}$ at $k \cdot y = 0$, cross along the dotted diagonals.

In the spatial slice the projections of the four one-posting steps $e_0, \dots, e_3$ point to the corners of a regular tetrahedron. The directions $p_1 = (1, 1, -1, -1)$, $p_2 = (1, -1, 1, -1)$ and $p_3 = (1, -1, -1, 1)$ are orthogonal to each other and to $\mathbf{1}$: they are the *face axes*, normal to the faces of the cube around that tetrahedron, and $p_1 = 2\hat{p}$ is the polarization axis of Example 2.4. Take the static wave p_1, the *face wave*, named for these cube faces and not for the lattice faces of Section 7. Its two polarizations are $H'_+ = p_2 p_2^\top - p_3 p_3^\top$ and $H'_\times = p_2 p_3^\top + p_3 p_2^\top$, with $J_{\mathrm{lw}}(H'_+) = J_{\mathrm{lw}}(H'_\times) = 16$.

Lemma 3.5 (Two numbers at the face wave suffice)**.** *A cost with the lattice's symmetries equals J_{lw} on $\mathcal{T}(m)$ for every wave m exactly when two numbers hold at the face wave: the* face calibration $Q(H'_+) = 16$*, and* no birefringence, $Q(H'_\times) = Q(H'_+)$.

Proof. By Schur's lemma. The 24 relabellings of the counters act on the five-dimensional space of spatial traceless strains as the symmetries of the tetrahedron, and each permutes p_1, p_2, p_3 up to sign. They split them into two parts: the strains diagonal in the frame p_1, p_2, p_3, a plane that contains H'_+, and the off-diagonal strains, a three-dimensional space that contains $H'_\times$. The two parts carry inequivalent irreducible representations of the relabellings, of dimensions 2 and 3. Both stay irreducible over the complex numbers: on the classes of the identity, a swap, a 3-cycle, a double swap and a 4-cycle their characters are $(2, 0, -1, 2, 0)$ and $(3, 1, 0, -1, -1)$, each of norm one. So a cost invariant under the relabellings is one multiple of J_{lw} on each part, with no cross term; a term in $(\mathrm{tr}\, H)^2$ drops out on traceless strains. A transverse-traceless strain is spatial and traceless, so it is the sum of its diagonal and off-diagonal parts, and Q charges it aJ_{lw} of the first plus bJ_{lw} of the second. At the face wave H'_+ is diagonal, $H'_\times$ is off-diagonal and both have $J_{\mathrm{lw}} = 16$, so $Q(H'_+) = 16a$ and $Q(H'_\times) = 16b$, and $Q = J_{\mathrm{lw}}$ on all

these strains exactly when $a = b = 1$. □

Two numbers at a three-fold-axis wave do not suffice. The wave and the pair in Lemma 3.5 matter. Write H_E and H_V for the diagonal and off-diagonal parts of a traceless spatial strain H in the frame of the unit face axes $\hat{p}_a = p_a/2$. Take the static wave $m = (3, -1, -1, -1)$, along a three-fold axis. The directions orthogonal to $\mathbf{1}$ and m are spanned by $v_1 = \frac{1}{\sqrt{2}}(0, 1, -1, 0)$ and $v_2 = \frac{1}{\sqrt{6}}(0, 1, 1, -2)$, and $\mathcal{T}(m)$ by $v_1 v_1^\top - v_2 v_2^\top$ and $v_1 v_2^\top + v_2 v_1^\top$. For $H = v_1 v_1^\top - v_2 v_2^\top$ the diagonal entries $\hat{p}_a^\top H \hat{p}_a$ are $\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}$, so $J_{\mathrm{lw}}(H_E) = \frac{1}{3} = \frac{1}{3} J_{\mathrm{lw}}(H)$. The three-cycle of the counters $1, 2, 3$ fixes m, permutes the face axes cyclically, and acts on $\mathcal{T}(m)$ and on the diagonal plane as rotations of order three; the projection of $\mathcal{T}(m)$ onto that plane commutes with them, so it is a multiple of an isometry, and $J_{\mathrm{lw}}(H_E) = \frac{1}{3} J_{\mathrm{lw}}(H)$ on all of $\mathcal{T}(m)$. Hence $Q(H) = 2 J_{\mathrm{lw}}(H_E) + \frac{1}{2} J_{\mathrm{lw}}(H_V)$ on traceless spatial strains, extended by any multiple of $(\operatorname{tr} H)^2$, has the lattice's symmetries and is J_{lw} on every polarization of m, yet $Q(H'_+) = 32$ and $Q(H'_\times) = 8$. At the face wave itself the pair $(H'_+ \pm H'_\times)/\sqrt{2}$ does not suffice either: every $\mu J_{\mathrm{lw}}(H_E) + \nu J_{\mathrm{lw}}(H_V)$ with $\mu + \nu = 2$ charges both 16.

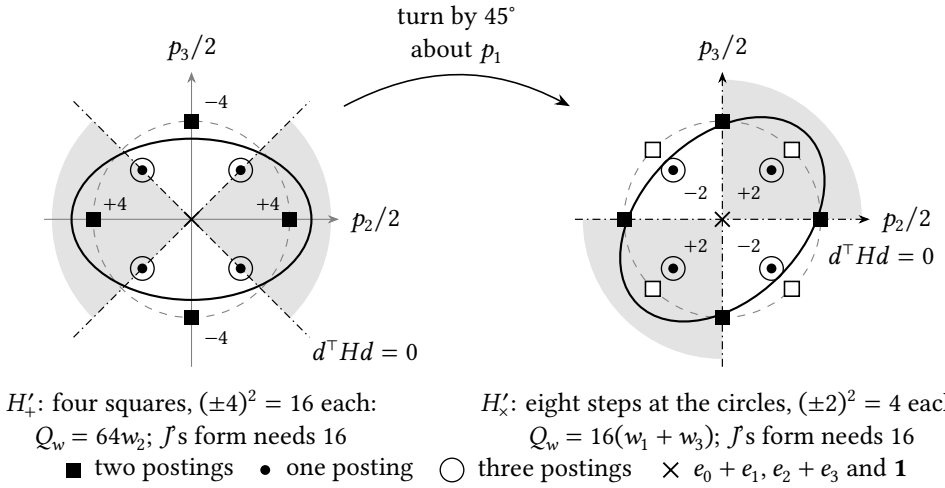


Figure 3. The face wave $p_1 = (1, 1, -1, -1)$, seen along the wave. Each lattice step d sits at its projection on the plane of the face axes $p_2/2$ and $p_3/2$, labelled with its reading $d^\top H d$ when nonzero (positive in the shaded lobes, zero on dash-dotted lines); solid, the strained ring; dashed, the flat one. Left: H'_+ vanishes on the diagonals, where the circles sit. Right: $H'_\times$ vanishes on the axes, where the squares sit; the turn takes them to the open squares, where no step lies.

The face calibration is a statement about the ledger's charge, not J's calibration $J''(1) = 1$, which sets J's unit. A step cost shows the two numbers at work (Fig. 3). Charge each lattice step d the square of the strain it reads, with a weight set by its number n_d of postings: $Q_w(H) = \sum_d w_{n_d} (d^\top H d)^2$. Only the four two-posting steps $e_0 + e_2$, $e_0 + e_3$, $e_1 + e_2$ and $e_1 + e_3$ read H'_+, each with $(d^\top H'_+ d)^2 = 16$, and only the four one-posting and the four three-posting steps read $H'_\times$, each with 4; a triple step reads the strain of its complement, $(\mathbf{1} - d)^\top H (\mathbf{1} - d) = d^\top H d$, since $H\mathbf{1} = 0$. So $Q_w(H'_+) = 64 w_2$ and $Q_w(H'_\times) = 16(w_1 + w_3)$, and Q_w carries J's form on both polarizations exactly when $w_2 = \frac{1}{4}$ and $w_1 + w_3 = 1$. Equal weights charge H'_+ twice as much as $H'_\times$: they are birefringent.

The count route: least-cost writing. A charge per posting fixes these weights by how the ledger writes a step. A *record* of a step d is a list of postings and of $t \in \mathbb{Z}$ *ticks* that adds up to d. In a record a posting adds one to a counter or, as a *credit*, takes one away; a tick advances all four counters, the diagonal step $\mathbf{1}$, which reads no spatial strain, $\mathbf{1}^{\top} H \mathbf{1} = 0$. Each posting moves one counter by one, so a record with t ticks has at least $\sum_i |d_i - t|$ postings, and some record has exactly that many; and since the postings on counter i add up to $d_i - t$, their number has the parity of $\sum_i (d_i - t)$, that is, of n_d. As $\sum_i |d_i - t|$ is $n_d - 4t$ for $t \le 0$ and $4t - n_d$ for $t \ge 1$, every record of d has at least $c_d = \min(n_d, 4 - n_d)$ postings: a single step is one posting, a double step two, and a triple step, $\mathbf{1}$ less one counter, a tick and one credit.

Two counters are compared through the ratio $e^{n_a - n_b}$ of their readings, so an event that changes the readings by $\Delta \in \mathbb{Z}^4$ changes that comparison by the factor $e^{\Delta_a - \Delta_b}$. Its *comparison charge* is J of these factors over the ordered pairs of counters, $\sum_{a,b} J(e^{\Delta_a - \Delta_b})$. A posting or a credit, $\Delta = \pm e_a$, changes the six comparisons of its counter by $e^{\pm 1}$ and is charged $\rho = 6J(e) = 6(\cosh 1 - 1) > 0$ (Lemma 2.1(b), (c)); the tick, $\Delta = \mathbf{1}$, changes no comparison and is charged nothing. A posting that carries the share u of the step's strain $s = d^{\top} H d$ is charged, in addition, J's quadratic part $\frac{1}{2}u^2$, as $J(e^u) = \frac{1}{2}u^2 + O(u^4)$. So a record with p postings and shares $u_1, \dots, u_p$, $\sum_j u_j = s$, is charged

$$R = p\,\rho + \sum_{j=1}^{p} \tfrac{1}{2} u_j^2. \tag{8}$$

The record charge (8) is an IDENTIFICATION: the shares are any real numbers with $\sum_j u_j = s$, a tick carries none, the comparison charge is paid per event, and the ledger's charge of a spatial strain is the sum of the *strain charges* of the fourteen steps other than 0 and $\mathbf{1}$, each counted once, a step's strain charge being the share part $\sum_j \frac{1}{2} u_j^2$ of the record the ledger writes for it. *Least-cost writing* is the principle that the ledger writes each strained step by a record of least charge R. Neither is derived here.

Proposition 3.6 (Least-cost writing gives both numbers)**.** *Let d be a step other than 0 and $\mathbf{1}$, and $c = c_d$.*

(i) For $p \ge 1$ postings, $\sum_j \frac{1}{2} u_j^2 \ge s^2/(2p)$, with equality exactly for the even shares $u_j = s/p$.

(ii) If $s^2 < 2\rho c(c+2)$, every record of d is charged at least $c\rho + s^2/(2c)$, with equality exactly for the records with c postings and even shares; so under least-cost writing the strain charge of d is $s^2/(2c)$.

(iii) Under least-cost writing the ledger's charge of a small spatial strain εH is $\varepsilon^2 Q_w(H)$, the step cost with $w_n = 1/\big(2\min(n, 4-n)\big)$, $(w_1, w_2, w_3) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{2})$. It has both numbers, $Q_w(H'_+) = 16 = Q_w(H'_\times)$, so $Q_w = J_{\mathrm{lw}}$ on $\mathscr{T}(m)$ for every wave m.

Proof. (i) When $\sum_j u_j = s$, $\sum_j \frac{1}{2} u_j^2 - s^2/(2p) = \frac{1}{2}\sum_j (u_j - s/p)^2$. (ii) A record of d has $p \ge c$ postings with $p \equiv n_d \equiv c \pmod 2$, so $p = c$ or $p \ge c + 2$. By (i) it is charged at least $f(p) = p\rho + s^2/(2p)$, with equality only for even shares, and

$$2cp\big(f(p) - f(c)\big) = (p - c)\big(2\rho cp - s^2\big),$$

which is positive for $p \ge c+2$ when $s^2 < 2\rho c(c+2)$. At $s = 0$ the least charge is $c\rho$. (iii) For small $|\varepsilon|$ every step's strain $\varepsilon\, d^{\top} H d$ lies in the window of (ii), so the charge is $\varepsilon^2 \sum_d (d^{\top} H d)^2/(2c_d)$,

with $c_d = 1, 2, 1$ for $n_d = 1, 2, 3$. Relabellings permute the steps with equal n_d, so Q_w has the lattice's symmetries; $Q_w(H'_+) = 64w_2 = 16$ and $Q_w(H'_\times) = 16(w_1 + w_3) = 16$, and Lemma 3.5 gives the rest. □

The window of (ii) is sharp: every step has records with $c + 2$ postings, and at $s^2 = 2\rho c(c + 2)$ the even one costs the same.

A second premise set: one comparison per step. The count and the shares also follow from two structural premises, with no minimization. *One-comparison writing*: the ledger writes each strained step by one comparison event, a record with at most two postings, on distinct counters. *Share locality*: the shares of a written step's postings add up to its strain s, commute with the relabellings, and each depends on the strain only through s. Neither is derived here.

Proposition 3.7 (One-comparison writing with share locality gives both numbers)**.** *Let d be a step other than* 0 *and* **1**.

(i) A record of d with at most two postings, on distinct counters, has exactly $c_d = \min(n_d, 4 - n_d)$ postings, and one exists.

(ii) Under share locality the shares of such a record are even, s/c_d each.

(iii) Under one-comparison writing and share locality the ledger's charge of a spatial strain is the step cost Q_w of Proposition 3.6(iii): it has both numbers, and $Q_w = J_{\mathrm{lw}}$ on $\mathcal{T}(m)$ for every wave m.

Proof. (i) In a record with t ticks the postings on counter i add up to $d_i - t$. With at most one posting per counter every $|d_i - t| \le 1$, and d has an entry 0 and an entry 1, so $t = 0$, with n_d postings, or $t = 1$, with $4 - n_d$ credits. At most two postings allow $t = 0$ for a single or double step and $t = 1$ for a double or triple step: c_d postings in each case. (ii) Two postings sit on counters $a \neq b$ with $d_a = d_b$, so the swap of a and b fixes d, the record and s. It takes the share on a to the share on b for the swapped strain, which by locality is the share on b for the strain itself; so the two shares are equal, $s/2$. A single posting carries s. (iii) A step with c even shares is charged $c \cdot \frac{1}{2}(s/c)^2 = s^2/(2c)$, and the proof of Proposition 3.6(iii) goes through without its window. □

Neither premise set is derived from the ledger's foundation, and two rules show what each excludes. The *forward rule* writes every step by its n_d postings, with no tick and even shares: it meets the record charge and share locality, but its triple steps have three postings, so it is neither least-cost nor one-comparison writing, and $w_1 + w_3 = \frac{2}{3}$. The shares $u_a = s/c + \chi\big(cH_{aa} - \sum_b H_{bb}\big)$, the sum over the counters b of the step's postings, keep the one-comparison counts, add up to s and commute with the relabellings, but for $\chi \neq 0$ they are neither local nor least-cost; they charge a double step $s^2/4 + \chi^2(H_{aa} - H_{bb})^2$. Both rules keep the face calibration, $Q(H'_+) = 16$, as only the double steps read H'_+ and its diagonal is zero, and both fail only as birefringence: $Q(H'_\times) = \frac{32}{3}$ for the forward rule and $16 + 64\chi^2$ for the other, as $H'_\times$ has the diagonal $(2, 2, -2, -2)$. So the face calibration misses both rules and the no-birefringence number catches both; against them it is the sensitive test of either premise set. Whole counts give both numbers only for $c_1 = c_3 = 1$ and $c_2 = 2$.

The rotation route. No birefringence also follows from a symmetry: the turn of the spatial slice by 45° about p_1, a proper rotation of space that carries H'_+ to $H'_\times$. It moves the squares of Fig. 3 to the open squares, where no step lies, so the lattice does not supply it. The premise is that the charge is unchanged by every proper rotation of space, the *rotation reading*. Each relabelling acts on the slice as a proper rotation or its negative, which turn a spatial strain alike, so the rotation reading gives the lattice's symmetries. It gives no birefringence, but not the face calibration.

Two more premises give the face calibration. With $U_a = \frac{1}{4} p_a p_a^\top$ the unit stretch along the face axis p_a, they are *J's calibration on each face axis*, $Q(\mu U_a) = \frac{1}{2}\mu^2 = J_{\mathrm{lw}}(\mu U_a)$, and *independent face axes*, $Q(U_a + U_b) = Q(U_a) + Q(U_b)$ for $a \neq b$, so that the bilinear form B of Q vanishes on U_a, U_b. As $H'_+ = 4U_2 - 4U_3$, $Q(H'_+) = 16Q(U_2) + 16Q(U_3) - 32B(U_2, U_3) = 16$.

The two routes compared. Neither route is derived: the count route rests on least-cost writing or on one-comparison writing with share locality, the rotation route on its three premises. The routes agree on every traceless spatial strain, the polarizations among them, and differ on the trace, so they are alternatives, not parts of one cost. For the relabellings the trace is a third part, of dimension one, so by Schur's lemma, as in Lemma 3.5, a cost with the lattice's symmetries and both numbers is $J_{\mathrm{lw}} + \lambda(\operatorname{tr} H)^2$. As $\operatorname{tr} U_a = 1$ and $\operatorname{tr}(U_2 U_3) = 0$, $Q(U_a) = \frac{1}{2} + \lambda$ and $B(U_2, U_3) = \lambda$: given both numbers, either face-axis premise alone forces $\lambda = 0$, so it gives J's form on every spatial strain, uniquely, and the rotation route leaves only J_{lw}. Relabellings permute the steps with the same n_d, so a step cost has the lattice's symmetries. At $P = I - \frac{1}{4}\mathbf{1}\mathbf{1}^\top$ the single, double and triple steps read $\frac{3}{4}$, 1 and $\frac{3}{4}$, so $Q_w(P) = \frac{9}{4}(w_1 + w_3) + 6w_2 = \frac{15}{4}$ against $J_{\mathrm{lw}}(P) = \frac{3}{2}$: with both numbers $\lambda = \frac{1}{4}$. So no step cost with both numbers meets the face-axis premises: it charges a unit face stretch $\frac{3}{4}$, not $\frac{1}{2}$. The reading of Section 2.2 gives $Q = J_{\mathrm{lw}}$ on spatial strains, which meets the rotation route's premises: one way to meet the premises of Proposition 6.2, not a further hypothesis.

4 The trace direction is fixed by locality

In the Euclidean model, locality fixes the trace part at Fierz–Pauli's value $\beta = -1$. On transverse-traceless strains Fierz–Pauli is $|k|^2$ times twice J's long-wave form (Theorem 3.1); what remains is one number, the coefficient β with which the trace of the transverse strain is charged. J does not fix it: every β gives a completion of J's form, the small-strain limit of the transverse cost $J(I + \varepsilon h_\perp)$ plus β times the volume cost $J(\det(I + \varepsilon h_\perp))$ (Theorem 4.3). This section works in the Euclidean model of Section 3, whose forms are invariant under all of $O(4)$ (Definition 4.2(b)). The $\kappa = 1$ lattice is Lorentzian and keeps only the relabellings and the reversal of all four counters, so for the lattice itself $\beta = -1$ comes from Regge's action through Theorem 6.1(i) [4]. That locality singles out Fierz–Pauli among the completions is classical [6, 16].

4.1 The completions of J's form

Definition 4.1 (The β-completions, the β-line)**.** For real β, $C_\beta(h; k) = |k|^2\big(\operatorname{tr} h_\perp^2 + \beta(\operatorname{tr} h_\perp)^2\big)$ for $k \neq 0$, and $C_\beta(h; 0) = 0$.

By Theorem 3.1(i), $C_{-1} = \mathrm{FP}$ on symmetric strains. And $C_0 = |k|^2 \cdot 2J_{\mathrm{lw}}(h_\perp)$ is J's own long-wave form of the transverse comparison.

Definition 4.2 (A completion of J's form)**.** A function $C(h;k)$ of a symmetric strain h and a wave covector $k \in \mathbb{R}^4$ is a *completion of J's form* when it is (a) gauge invariant, $C(h + \gamma_{k,\xi}; k) = C(h;k)$; (b) invariant under every rotation and reflection, $C(OhO^\top; Ok) = C(h;k)$ whenever $O^\top O = I$; (c) of two derivatives, $C(h;tk) = t^2 C(h;k)$; (d) quadratic in h for each k; and (e) J's form on transverse-traceless strains, $C(h;k) = |k|^2 \cdot 2J_{\mathrm{lw}}(h)$ when $hk = 0$ and $\operatorname{tr} h = 0$.

Theorem 4.3 (The cost leaves β free)**.** *For every real β:*

(i) C_β is a completion of J's form;

(ii) for $k \neq 0$, C_β is $2|k|^2$ times the small-strain limit of a cost built from J,

$$C_\beta(h;k) = \lim_{\varepsilon \to 0} 2|k|^2 \, \frac{J(I + \varepsilon h_\perp) + \beta\, J\big(\det(I + \varepsilon h_\perp)\big)}{\varepsilon^2},$$

whose volume cost $J(\det M)$ obeys the composition law in M: for real $n \times n$ matrices M, N with $\det M, \det N > 0$, $J(\det MN) + J(\det MN^{-1}) = 2J(\det M)J(\det N) + 2J(\det M) + 2J(\det N)$;

(iii) $C_{-1} = \mathrm{FP}$ on symmetric strains, and $C_{-1} \neq C_0$.

Proof. (i) A gauge mode does not change $h_\perp$ (proof of Theorem 3.1). For a rotation, $\Pi_{Ok} = O\Pi_k O^\top$, so $h_\perp$ becomes $Oh_\perp O^\top$ and both traces are unchanged. $\Pi_{tk} = \Pi_k$ for $t \neq 0$, which gives (c); (d) is clear; and a transverse-traceless h has $h_\perp = h$ and $\operatorname{tr} h = 0$, which gives (e). (ii) By Theorem 2.3, $J(I + \varepsilon h_\perp)/\varepsilon^2 \to \frac{1}{2} \operatorname{tr} h_\perp^2$. Since $\det(I + \varepsilon h_\perp) = 1 + \varepsilon \operatorname{tr} h_\perp + O(\varepsilon^2)$ and $J(1 + y) = y^2/(2(1 + y))$ by Lemma 2.1(e), $J(\det(I + \varepsilon h_\perp))/\varepsilon^2 \to \frac{1}{2}(\operatorname{tr} h_\perp)^2$; and since the determinant is multiplicative, $\det(MN^{-1}) = \det M / \det N$, so Lemma 2.1(d) at $x = \det M$, $y = \det N$ is the composition law. (iii) The first statement is Theorem 3.1(i). At $h = e_0 e_0^\top$ and $k = e_1$, $h_\perp = h$, so $C_{-1} = 0$ and $C_0 = 1$. □

At the wave of Example 2.4, $C_\beta(H_+; m) = 4$ for every β while $C_\beta(I; m) = 2(3 + 9\beta)$ (Fig. 4).

Theorem 4.4 (The completions are the β-line)**.** *If C is a completion of J's form, then $C(h;k) = C_\beta(h;k)$ for every symmetric h and every k, with $\beta = \big(C(\Pi_{e_0}; e_0) - 3\big)/9$. Every C_β is a completion.*

Proof. At $k = 0$, (c) with $t = 0$ gives $C(h;0) = 0 = C_\beta(h;0)$. At $k = e_0$, adding a gauge mode $e_0\xi^\top + \xi e_0^\top$ removes every entry h_{0a}, so $C(h;e_0)$ is a quadratic form in the spatial block h_s of h, a symmetric 3×3 matrix. Write $h_s = h_{s,0} + \frac{1}{3}(\operatorname{tr} h_s) I_3$ with $h_{s,0}$ traceless. A bilinear term between $h_{s,0}$ and $\operatorname{tr} h_s$ is a linear functional of $h_{s,0}$ invariant under the flips and swaps of the three spatial axes (orthogonal maps that fix e_0, so (b) applies). Invariance under the flips removes its off-diagonal part, and invariance under the swaps makes its diagonal weights equal, so it is a multiple of the trace and vanishes on $h_{s,0}$. On the traceless part, (e) gives $C = \operatorname{tr} h_{s,0}^2$. So $C(h;e_0) = \operatorname{tr} h_{s,0}^2 + c(\operatorname{tr} h_s)^2 = \operatorname{tr} h_s^2 + \beta(\operatorname{tr} h_s)^2$ with $\beta = c - \frac{1}{3}$, which is $C_\beta(h;e_0)$; evaluating at $h = \Pi_{e_0} = \operatorname{diag}(0,1,1,1)$ gives $3 + 9\beta$. A reflection takes any $k \neq 0$ to $|k|e_0$, and (b) and (c) carry the identity from e_0 to k. The last sentence is Theorem 4.3(i). □

4.2 Locality selects $\beta = -1$

A local lattice cost has a long-wave symbol that is a quadratic form in k, a polynomial. Among the completions only Fierz–Pauli is one. By Theorem 3.1(i) and $\operatorname{tr} h_\perp = \operatorname{tr} h - k^\top h k/|k|^2$,

$$C_\beta(h;k) = \mathrm{FP}(h;k) + (1+\beta)\,\frac{\big(|k|^2 \operatorname{tr} h - k^\top h k\big)^2}{|k|^2}, \tag{9}$$

and for $\beta \neq -1$ and most strains the last term is not a polynomial in k: it is the inverse wave operator $1/|k|^2$ applied to a local quartic term, a nonlocal kernel. At $h = I$ it is $9(1+\beta)|k|^2$, a polynomial: the conformal mode alone cannot see locality. The plus strain H_+ of Example 2.4 sees it, so locality forces $\beta = -1$ (Theorem 4.6, Fig. 4). Locality is also the hypothesis of the classical uniqueness [6].

Definition 4.5 (A local lattice cost)**.** A *local lattice cost* L has finitely many terms, indexed by j. Term j couples the strain at $x + p_j$ to the strain at $x + q_j$, with fixed offsets $p_j, q_j \in \mathbb{R}^4$ and a real coefficient $c_j(H,K)$, of either sign, for each pair of strain amplitudes, bilinear for an action quadratic in the strain. On the plane-wave pair $He^{ik\cdot x}$, $Ke^{-ik\cdot x}$ its cost at the site x is $\sum_j c_j(H,K)\, e^{ik\cdot d_j}$ with $d_j = p_j - q_j$, whose real part is the *symbol* $\Sigma_L(H,K;k) = \sum_j c_j \cos(k\cdot d_j)$. If $\sum_j c_j = 0$, then $\Sigma_L(\varepsilon k)/\varepsilon^2$ tends to the *long-wave symbol* $\Lambda_L(H,K;k) = -\frac{1}{2}\sum_j c_j (k\cdot d_j)^2$ as $\varepsilon \to 0$, since $\cos y = 1 - \frac{1}{2}y^2 + O(y^4)$. Each $(k\cdot d_j)^2$ is a quadratic form in k, so Λ_L obeys the *parallelogram law*

$$q(k+k') + q(k-k') = 2q(k) + 2q(k'). \tag{10}$$

Theorem 4.6 (Locality forces $\beta = -1$)**.** *(i) If for every symmetric h some local lattice cost has the long-wave limit $\Sigma_L(h,h;\varepsilon k)/\varepsilon^2 \to C_\beta(h;k)$ as $\varepsilon \to 0$ for every k, or has $\Lambda_L(h,h;\cdot) = C_\beta(h;\cdot)$, then $\beta = -1$.*

(ii) Some local lattice cost, with coefficients bilinear in the two amplitudes and $\sum_j c_j(h,h) = 0$, has $\Lambda_L(h,h;k) = \mathrm{FP}(h;k)$ for every symmetric h and every k.

Proof. (i) The limit at a fixed $k \neq 0$ exists only if $\sum_j c_j(h,h) = 0$, since $\Sigma_L(h,h;\varepsilon k) \to \sum_j c_j(h,h)$ as $\varepsilon \to 0$; then it is $\Lambda_L(h,h;k)$ (Definition 4.5). So in both cases $C_\beta(h;\cdot) = \Lambda_L(h,h;\cdot)$, which obeys the parallelogram law. By Theorem 3.1(i), $C_\beta = \mathrm{FP} + (1+\beta)|k|^2(\operatorname{tr} h_\perp)^2$, and FP is a quadratic form in k. The other term is not: at the traceless plus strain $h = H_+$ of Example 2.4, $|k|^2(\operatorname{tr} h_\perp)^2 = (k^\top H_+ k)^2/|k|^2$ takes the values $1, 1, 0, 0$ at $k = \hat{p}$, $\hat{q}$, $\hat{p}+\hat{q}$, $\hat{p}-\hat{q}$. So the parallelogram defect of C_β at $k = \hat{p}$, $k' = \hat{q}$ is $(1+\beta)(0+0-2\cdot 1-2\cdot 1) = -4(1+\beta)$, which vanishes only at $\beta = -1$. (ii) Write $\mathrm{FP}(h;k) = \sum_{a,b} F_{ab}(h) k_a k_b$ with $F_{ab}(h) = [a{=}b]\big(\operatorname{tr} h^2 - (\operatorname{tr} h)^2\big) - 2(h^2)_{ab} + 2\operatorname{tr} h\, h_{ab}$, where $[a{=}b]$ is 1 when $a = b$ and 0 otherwise, and polarize F_{ab} to a bilinear form in (H,K). For $a < b$ take the coefficient $-F_{ab}$ at the offset $e_a + e_b$ and $+F_{ab}$ at $e_a - e_b$: this pair contributes $\frac{1}{2}F_{ab}\big((k_a+k_b)^2 - (k_a-k_b)^2\big) = 2F_{ab}k_a k_b$ to the long-wave symbol. For each a take $-\frac{1}{2}F_{aa}$ at $2e_a$ and $+\frac{1}{2}F_{aa}$ at 0, which contribute $F_{aa}k_a^2$. The coefficients cancel in pairs, so $\sum_j c_j = 0$, and the long-wave symbol is $\sum_{a<b} 2F_{ab}k_a k_b + \sum_a F_{aa}k_a^2 = \mathrm{FP}(h;k)$. □

Corollary 4.7 (Fierz–Pauli is the transverse cost minus the volume cost)**.** *The only completion of J's form that is the long-wave symbol of local lattice costs is Fierz–Pauli, and for symmetric h and $k \neq 0$*

$$\mathrm{FP}(h;k) = \lim_{\varepsilon\to 0} 2|k|^2\, \frac{J(I+\varepsilon h_\perp) - J\big(\det(I+\varepsilon h_\perp)\big)}{\varepsilon^2}. \tag{11}$$

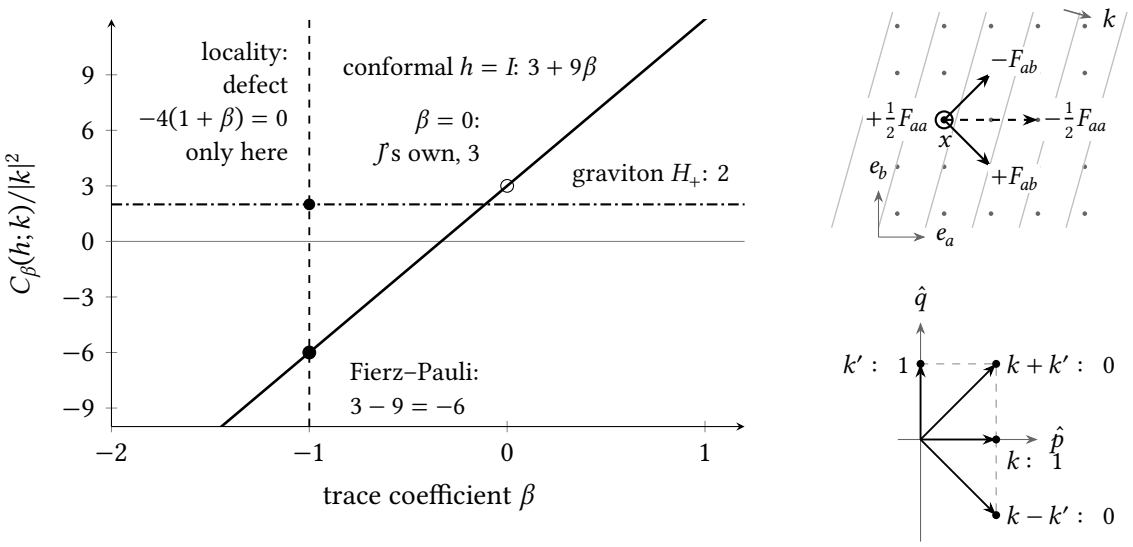


Figure 4. Locality fixes the trace part. Left: the β-line; the completions agree on the graviton (flat) and differ on the conformal mode (sloped); Fierz–Pauli is J's own 3 minus the volume cost 9. Top right: the local cost of Theorem 4.6(ii) (ring: offset 0; gray: wave fronts of k). Bottom right: the test at $h = H_+$, $k = \hat{p}$, $k' = \hat{q}$: the nonlocal term of (9) over $1 + \beta$ has defect -4, so C_β has $-4(1 + \beta)$, zero only at $\beta = -1$.

Proof. Theorem 4.4 puts such a completion on the β-line, and Theorem 4.6(i) gives $\beta = -1$. Fierz–Pauli is one, by Theorem 4.6(ii) and Theorem 4.3(i). Theorem 4.3(ii) at $\beta = -1$ is (11). □

Figure 4 draws the mechanism: subtracting the cost of the volume change, $J(\det(I + \varepsilon h_\perp))$, from that of the transverse comparison, $J(I + \varepsilon h_\perp)$, leaves Einstein's form, as locality demands, and gives the negative sign on the conformal mode.

4.3 Where the sign comes from

Fierz–Pauli's negative trace sector needs signed costs that couple different edge directions, as Regge's deficit angles do.

Theorem 4.8 (What cannot give the graviton)**.** *Neither single edges nor a nonnegative cost give Fierz–Pauli:*

(i) For any finitely many directions $d_j \in \mathbb{R}^4$ and any functions q_j of the wave covector, the same-direction edge form $E(h;k) = \sum_j (d_j^\top h d_j)^2 q_j(k)$ *is not c*FP *for any $c \neq 0$.*

*(ii) No function of (h,k) that is nonnegative on symmetric strains is c*FP *for any $c \neq 0$.*

Proof. (i) Let $\Phi(Q) = Q(E_{01}) - 2\big(Q(E_{00} + E_{11}) - Q(E_{00}) - Q(E_{11})\big)$ for a quadratic form Q in h, where $E_{01} = e_0 e_1^\top + e_1 e_0^\top$ and $E_{aa} = e_a e_a^\top$. For $Q(h) = (d^\top h d)^2$ it gives $4d_0^2 d_1^2 - 2\big((d_0^2 + d_1^2)^2 - d_0^4 - d_1^4\big) = 0$, so Φ vanishes on every same-direction edge form. On Fierz–Pauli at $k = e_2$ it gives $2 - 2(-2 - 0 - 0) = 6$. (ii) $\mathrm{FP}(E_{12}; e_0) = 2$ and $\mathrm{FP}(I; e_0) = -6$, so cFP takes both signs. □

So the trace sector needs such costs; Regge's action has them, and its long-wave form is Fierz–Pauli (Section 6). The negative conformal sector of (ii) and of Theorem 6.1(iii) is the

classical conformal-factor problem of Euclidean gravity [19]. On the polarizations, too, a single-edge cost with equal weights is wrong: it charges H'_+ twice as much as $H'_\times$ at the face wave (Section 3.4).

5 The Lorentzian form, an identity of polynomials

The identity of Theorem 3.1 does not depend on the signature or on the real numbers; (i) below is the Lorentzian projector form of Fierz–Pauli [6, 16]. It is an identity of polynomials: (i) and (ii) below hold over every commutative ring, and (iii) and (iv), which use $g_1^{-1} = I - \frac{1}{3}\mathbf{1}\mathbf{1}^\top$, over every commutative ring in which 3 is invertible. So they hold, as Theorem 2.3's identities do, on the real line of The δ Foundation, a commutative ring built with no choice principle [3, Prop. 12.19]; the limits in Theorems 2.3 and 6.1 are not such identities.

Let $\mathscr{R}$ be a commutative ring. Let S be a symmetric 4×4 matrix over $\mathscr{R}$, the inverse metric with which indices are raised; let $k \in \mathscr{R}^4$ be a wave covector with squared covector $\eta = k^\top S k$; and let $w \in \mathscr{R}$ satisfy $w\eta = 1$. Put $\Pi = I - w(Sk)k^\top$ and, for a strain X, let $X_\perp = \Pi^\top X \Pi$ be its part transverse to the raised wave Sk. In this section and the next, X and Y are strains with lower indices, like H in (3). The Fierz–Pauli form with indices raised by S [4] is

$$\mathrm{FP}_S(X, Y; k) = \eta\,\mathrm{tr}(SXSY) - 2k^\top SXSYSk + \mathrm{tr}(SY)\,k^\top SXSk + \mathrm{tr}(SX)\,k^\top SYSk - \eta\,\mathrm{tr}(SX)\,\mathrm{tr}(SY). \quad (12)$$

For a metric g we write FP_g for FP_S with $S = g^{-1}$, as in [4]; (6) is $\mathrm{FP}_I(h, h; k)$. With $S = g^{-1}$, $\frac{1}{2}\,\mathrm{tr}(SXSX) = J_{\mathrm{lw}}(g^{-1}X)$ is J's long-wave form of the comparison $g^{-1}(g + \varepsilon X) = I + \varepsilon g^{-1}X$.

Theorem 5.1 (The Lorentzian Fierz–Pauli form on transverse-traceless strains, an identity of polynomials)**.** *Let S, k and η be as above, with $S = g_1^{-1}$ in (iii) and (iv); only (i) uses w.*

(i) For symmetric S, X, Y and $w\eta = 1$: $\mathrm{FP}_S(X, Y; k) = \eta\big(\mathrm{tr}(SX_\perp SY_\perp) - \mathrm{tr}(SX_\perp)\,\mathrm{tr}(SY_\perp)\big)$.

(ii) For symmetric S, Y and every k, ξ: $\mathrm{FP}_S(k\xi^\top + \xi k^\top, Y; k) = 0$.

(iii) If X and Y are symmetric with $X\mathbf{1} = Y\mathbf{1} = 0$ and $\mathrm{tr}\,X = \mathrm{tr}\,Y = 0$, then $\mathrm{tr}(g_1^{-1}Xg_1^{-1}Y) = \mathrm{tr}(XY)$ and $\mathrm{tr}(g_1^{-1}X) = 0$; if moreover $Xk = Yk = 0$, which for $X\mathbf{1} = 0$ is the covariant condition $Xg_1^{-1}k = 0$, then $\mathrm{FP}_{g_1}(X, Y; k) = \eta\,\mathrm{tr}(XY)$.

(iv) $\mathrm{FP}_{g_1}(g_1, g_1; k) = -6\eta$ *and* $\mathrm{tr}(g_1^{-1}g_1g_1^{-1}g_1) = 4$.

Proof. (i) Put $\bar{k} = Sk$ and $S_\perp = S - w\bar{k}\bar{k}^\top$. Then $S\Pi^\top = S_\perp$ and $\Pi S_\perp = S - 2w\bar{k}\bar{k}^\top + w^2\eta\,\bar{k}\bar{k}^\top = S_\perp$, so $\Pi S\Pi^\top = S_\perp$, and by the cyclicity of the trace $\mathrm{tr}(SX_\perp SY_\perp) = \mathrm{tr}(S_\perp XS_\perp Y)$ and $\mathrm{tr}(SX_\perp) = \mathrm{tr}(S_\perp X)$. Expanding $S_\perp$,

$$\mathrm{tr}(S_\perp XS_\perp Y) = \mathrm{tr}(SXSY) - 2w\bar{k}^\top XSY\bar{k} + w^2(\bar{k}^\top X\bar{k})(\bar{k}^\top Y\bar{k}), \qquad \mathrm{tr}(S_\perp X) = \mathrm{tr}(SX) - w\bar{k}^\top X\bar{k}.$$

Multiplying by η and using $w\eta = 1$, the w^2 terms cancel and what remains is (12), since $\bar{k}^\top XSY\bar{k} = k^\top SXSYSk$ and $\bar{k}^\top X\bar{k} = k^\top SXSk$. (ii) With $X = k\xi^\top + \xi k^\top$ and $\zeta = S\xi$: $SXS = \bar{k}\zeta^\top + \zeta\bar{k}^\top$, $\mathrm{tr}(SX) = 2\,\bar{k}\cdot\xi$ and $\bar{k}^\top X\bar{k} = 2\eta\,\bar{k}\cdot\xi$. The first term of (12) is $2\eta\,\zeta^\top Y\bar{k}$. The second is $-2\eta\,\zeta^\top Y\bar{k} - 2(\bar{k}\cdot\xi)\,\bar{k}^\top Y\bar{k}$; its first piece cancels the first term and its second piece cancels the fourth, $2(\bar{k}\cdot\xi)\,\bar{k}^\top Y\bar{k}$. The third, $2\eta(\bar{k}\cdot\xi)\,\mathrm{tr}(SY)$, cancels the fifth. (iii) $X\mathbf{1} = 0$ gives

$g_1^{-1}X = X - \frac{1}{3}\mathbf{1}(\mathbf{1}^\top X) = X$ and likewise $Xg_1^{-1} = X$, so $g_1^{-1}Xg_1^{-1} = X$; with $Xk = Yk = 0$ and $\operatorname{tr} X = \operatorname{tr} Y = 0$ every term of (12) but the first vanishes. (iv) $g_1^{-1}g_1 = I$, so the five terms are $4\eta - 2\eta + 4\eta + 4\eta - 16\eta$. □

By (iii), on the lattice's transverse-traceless strains this is the Lorentzian form of (2), for every wave, null or not. At the wave of Example 2.4, $\eta(m) = 2$ and $\mathrm{FP}_{g_1}(H_+, H_+; m) = 4$.

6 Regge's action on the $\kappa = 1$ lattice at long wavelengths

We take Regge's action as the lattice's gravitational action and show, using [4, Thm. 3.1(ii)], that at long wavelengths its second variation is $-\sqrt{3}\pi^2/2$ times Fierz–Pauli. [4] is a preprint, not peer reviewed, so Section 6.3 confirms this constant by an exact computation and checks it numerically. On transverse-traceless strains Fierz–Pauli is the squared wave covector times twice J's long-wave form, with coefficient one (Theorem 5.1); so, with this action, that is the graviton's strain factor on its two polarizations at every wave that is not a multiple of $\mathbf{1}$ (Theorem 6.1). The long-wave result is not special to Regge's action: a *hinge charge* $C(A, \delta)$ is a function of area and deficit. Every hinge charge of class C^3 that vanishes at zero deficit and whose flat lattices with $\kappa > 1$ are stationary has this long-wave action, and through cubic order on plane waves it is $\frac{1}{2}\int \sqrt{|g|}\, R$, both up to one factor [4, Thm. 3.1, Remark 3.2]; that premise is equivalent, on the areas $A \geq \frac{1}{2}$ that it reads, to Regge's slope [4, Thm. 5.1(b)]. On this lattice the long-wave limit of Regge's action gives linearized Einstein gravity and Newton's law [11]; on a hypercubic lattice the weak-field limit of Regge calculus was found to be linearized Einstein gravity by Roček and Williams [13, 14], in Euclidean signature, and Cheeger, Müller and Schrader proved the convergence of Regge's action to Einstein–Hilbert's for Riemannian piecewise flat approximations under bounds on their fatness [15]. Barrett studied the convergence of the solutions of linearized Regge calculus to those of linearized Einstein gravity [21], and Dittrich, Freidel and Speziale the Hessian of the Regge action of one 4-simplex and the linearized dynamics it carries [22]. The result here is of the same kind, the long-wave limit of the Hessian on plane waves, for the Lorentzian Kuhn lattice at $\kappa = 1$, computed exactly.

6.1 The lattice

We use the setting of [4]. The vertices are the points of the torus $(\mathbb{Z}/N)^4$, and their coordinates are the readings of the four counters. Four counters advanced one step each from x to $x + \mathbf{1}$ have $4! = 24$ arrival orders, and these are the 24 simplices of Kuhn's triangulation of the unit cell [8]. Every edge joins x to $x + d$ for a step d, made of n_d postings. A *configuration* ℓ gives every edge a real squared length. The flat lattice of parameter κ is

$$\ell^{\kappa}_{(x,d)} = g_\kappa(d, d) = n_d(1 - \kappa n_d), \qquad g_\kappa = I - \kappa \mathbf{1}\mathbf{1}^\top. \tag{13}$$

g_κ has eigenvalue 1 on the hyperplane orthogonal to $\mathbf{1}$ and $1 - 4\kappa$ on $\mathbf{1}$, so for $\kappa > \frac{1}{4}$ it is Lorentzian with time along the diagonal $\mathbf{1}$. At $\kappa = 1$ a single posting is null and every longer edge is timelike, $\det g_1 = -3$ and $g_1^{-1} = I - \frac{1}{3}\mathbf{1}\mathbf{1}^\top$. The *hinges* are the triangles; a hinge has the area $A_h = \frac{1}{2}\sqrt{-\det G}$, with G the Gram matrix of two of its edges. For $\kappa > \frac{1}{2}$ every flat hinge is

timelike, and at $\kappa = 1$ the hinges with edges of $(1, 1, 2)$, $(1, 2, 3)$, $(1, 3, 4)$ and $(2, 2, 4)$ postings have the areas $\frac{1}{2}$, 1, $\frac{3}{2}$ and $\sqrt{3}$ [4, Sec. 2.2]. The plane normal to a timelike hinge is spacelike. So the *dihedral angle* of a simplex at a hinge, the angle between its two tetrahedra that contain the hinge, is an ordinary angle in that plane, and Regge's action is real. The *deficit angle* δ_h is 2π minus the sum of the dihedral angles of the simplices that share the hinge, and the *Regge action* is $S_{\mathrm{R}}(\ell) = \sum_h A_h(\ell)\,\delta_h(\ell)$ [7, 4]. A *plane wave* with integer wave covector $m \in \mathbb{Z}^4$ and symmetric polarization H is the edge field

$$u_{m,H}(x, d) = \big(d^{\top} H d\big) \cos\Big(\tfrac{2\pi}{N}\, m \cdot \big(x + \tfrac{d}{2}\big)\Big), \tag{14}$$

the metric strain $H\cos(k \cdot y)$, $k = 2\pi m/N$, sampled at the midpoint of each edge, with squared covector $\eta(m) = m^{\top} g_1^{-1} m = \sum_a m_a^2 - \frac{1}{3}(\sum_a m_a)^2$. Figure 5 draws one simplex and one hinge.

MODEL. The Kuhn lattice with real squared edge lengths and the midpoint plane-wave sampling of [4] is discretized spacetime, and Regge's action, computed from these Lorentzian areas and angles, is its gravitational action.

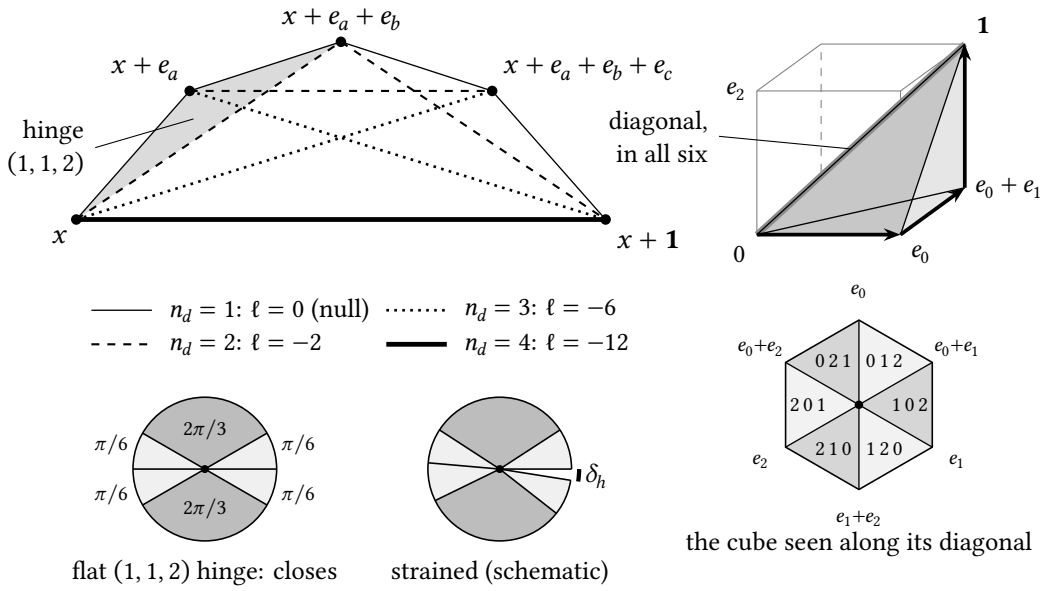


Figure 5. The $\kappa = 1$ Kuhn lattice. Top left: one simplex (schematic), edges styled by n_d, the number of postings; squared length $\ell = n_d(1 - n_d)$; shaded, a $(1, 1, 2)$ hinge of area $\frac{1}{2}$. Right: the three-counter cell, six tetrahedra around its diagonal; bold, the arrival order 0 1 2, whose tetrahedron is shaded. Bottom: hinge stars, one wedge per simplex; a flat $(1, 1, 2)$ star closes, a strained one leaves the deficit δ_h, charged $A_h\delta_h$ by Regge's action.

The $\kappa = 1$ identification. That the physical lattice has $\kappa = 1$ is an IDENTIFICATION [4]. On g_κ the step $n_1 e_p + n_2 e_q$ $(p \neq q)$ has the squared proper duration $\kappa(n_1 + n_2)^2 - n_1^2 - n_2^2$. It is a fixed multiple of the interval $4n_1n_2 = (n_1 + n_2)^2 - (n_1 - n_2)^2$ of n_1 postings along one light ray and n_2 along the other, for all $n_1, n_2 \geq 0$, exactly when $\kappa = 1$: the case $(1, 0)$ forces it, and then the squared duration is $2n_1n_2$. The case $(1, 0)$ says that one posting is null, which is the identification itself; so this restates the identification as two counters read as two light rays, and does not support it. The identification has a known cost: in the family g_κ with

$\kappa > \frac{1}{2}$, massless corner doublers occur exactly at $\kappa = 1$, and only at the four axis corners [12, Thm. 2.1], and through them the graviton limit diverges at one loop in the stated diagram model of [11].

6.2 The flat lattice and its second variation

Every flat lattice with $\kappa > \frac{1}{2}$ has zero deficit at every hinge [4]. At $\kappa = 1$ the angles are twelfths of π. For the arrival order 0123 the barycentric coordinates of the simplex $x = v_0, \dots, v_4 = x + \mathbf{1}$ have the gradients $-e_0$, $e_0 - e_1$, $e_1 - e_2$, $e_2 - e_3$, e_3. The exterior angle $\pi - \alpha_t$ at the triangle opposite the vertices a and b, with α_t its dihedral angle, is the angle between gradients a and b in g_1^{-1}. Their Gram matrix has the diagonal $\frac{2}{3}, 2, 2, 2, \frac{2}{3}$, the entry -1 between neighbours, $\frac{1}{3}$ between the first and the last, and 0 otherwise. Its cosines $-\sqrt{3}/2$, $-\frac{1}{2}$, $\frac{1}{2}$, 0 give 10, 8, 4, 6 twelfths of π. So the exterior angles at the triangles 012, 123 and 234 are 10, 4 and 10 twelfths of π. The relabellings of the counters carry this simplex to the other 23 and keep g_1, so every flat $\kappa = 1$ dihedral angle is a multiple of $\pi/6$. A $(1, 1, 2)$ hinge lies in six simplices, twice each at the positions 012, 123, 234, and its exterior angles add to $2(10 + 4 + 10)\pi/12 = (6 - 2)\pi$: its star closes (Fig. 5). At a flat lattice every deficit vanishes, so does the Regge action, and the lattice is a stationary point of it [4]. Its second derivative there in the squared lengths, $\mathscr{H}_N(u, v) = D^2 S_{\mathrm{R}}(\ell^1)[u, v]$ for edge fields u, v on the torus of side N, is the *flat Hessian*.

Theorem 6.1 (Regge's action at long wavelengths: Fierz–Pauli, and twice J's long-wave form on the polarizations)**.** *Let $u_{m,H}$ be the plane wave* (14) *on the torus of side N.*

(i) For every nonzero $m \in \mathbb{Z}^4$ and all symmetric H and K,

$$\lim_{N\to\infty} N^{-2}\, \mathscr{H}_N\big(u_{m,H}, u_{m,K}\big) = -\frac{\sqrt{3}\,\pi^2}{2}\, \mathrm{FP}_{g_1}(H, K; m).$$

(ii) If H is transverse-traceless at m (symmetric, $H\mathbf{1} = 0$, $Hm = 0$, $\operatorname{tr} H = 0$; for m a nonzero multiple of $\mathbf{1}$ these strains form a five-dimensional space), then for every m, null or not, $\mathrm{FP}_{g_1}(H, H; m) = \eta(m) \operatorname{tr} H^2 = \eta(m) \cdot 2 J_{\mathrm{lw}}(g_1^{-1} H)$: twice J's long-wave form, with coefficient one. Such strains with equal J_{lw} have equal limits in (i).

(iii) The conformal strain g_1 has the limit $3\sqrt{3}\pi^2\eta(m)$, of the sign opposite to that of every nonzero strain of (ii) when $\eta(m) \neq 0$.

Proof. (i) In [4] a hinge charge $C(A, \delta)$ has the action $\Phi_C = \hbar^{-1} \sum_h C(A_h, \delta_h)$ and the slope $\sigma(A) = \partial_\delta C(A, 0)$. By [4, Thm. 3.1(ii)], if C is of class C^3 with $C(A, 0) = 0$ and the flat lattices with $\kappa > 1$ on the torus of side 3 are stationary points of Φ_C, then

$$\lim_{N\to\infty} N^{-2} D^2\Phi_C[u_{m,H}, u_{m,K}] = \frac{\sigma(1)}{\hbar}\Big(-\frac{\sqrt{3}\,\pi^2}{2}\Big)\mathrm{FP}_{g_1}(H, K; m),$$

with D^2 taken in the squared lengths at the flat $\kappa = 1$ lattice. Regge's charge $sA\delta$ meets these premises [4, Sec. 3]; its action is $(s/\hbar)S_{\mathrm{R}}$ and its slope sA, so $\sigma(1) = s$ and $D^2\Phi_C = (s/\hbar)\mathscr{H}_N$. Dividing by $s/\hbar$ gives (i). (ii) is Theorem 5.1(iii). (iii) is (i) with Theorem 5.1(iv), as $\operatorname{tr} H^2 > 0$ for real $H \neq 0$. □

The factor in (i) reads one piece at a time [4, Remark 3.2]: $-\sqrt{3}\pi^2/2 = -\frac{1}{4} \cdot \sqrt{3} \cdot \frac{1}{2} \cdot (2\pi)^2$, with $-\frac{1}{4}$ the second derivative of $\frac{1}{2}\sqrt{|g|}\,R$ per unit proper volume, $\sqrt{3} = \sqrt{|\det g_1|}$ the cell's proper volume, $\frac{1}{2}$ the mean square of the cosine and $(2\pi)^2$ from $k = 2\pi m/N$. At the worked wave of Example 2.4, $\eta(m) = 2$ and both polarizations have $\mathrm{FP}_{g_1} = 4$, so they tend to $-2\sqrt{3}\pi^2 \approx -34.19$; the conformal mode g_1, with $\mathrm{FP}_{g_1}(g_1, g_1; m) = -12$, tends to $6\sqrt{3}\pi^2 \approx 102.57$.

At a null wave, $\eta(m) = 0$, both sides of (ii) vanish, so on shell the identity carries no information: it concerns the off-shell kinetic operator on the polarizations. "At every wave" means every fixed integer covector $m \neq 0$ in the limit $N \to \infty$, null or not.

Proposition 6.2 (The normalization statement: the ledger's charge and the strain factor)**.** *Let Q be a quadratic form on spatial strains (symmetric H with $H\mathbf{1} = 0$) with the lattice's symmetries and both numbers of Lemma 3.5, which least-cost writing (Proposition 3.6), one-comparison writing with share locality (Proposition 3.7) or the rotation route of Section 3.4 gives. Then $Q = J_{\mathrm{lw}}$ on $\mathcal{T}(m)$, and for every nonzero $m \in \mathbb{Z}^4$ and every $H \in \mathcal{T}(m)$*

$$\lim_{N\to\infty} N^{-2}\,\mathscr{H}_N\big(u_{m,H}, u_{m,H}\big) = -\frac{\sqrt{3}\,\pi^2}{2}\,\eta(m) \cdot 2\,Q(H),$$

so $2Q$ is the strain factor of Section 1 wherever $\eta(m) \neq 0$. For m not a multiple of $\mathbf{1}$, $\mathcal{T}(m)$ is the plane of the two polarizations. If Q also meets one of the two face-axis premises of Section 3.4, then $Q = J_{\mathrm{lw}}$ on every spatial strain, uniquely; there it is the quadratic part of the reading $J(g_1^{-1}g)$ of Section 2.2. The rotation route assumes both face-axis premises, and the charge Q_w of the count route meets neither.

Proof. By Lemma 3.5, both numbers give $Q = J_{\mathrm{lw}}$ on $\mathcal{T}(m)$ for every wave m, and Theorem 6.1(i)–(ii) give the limit, since $g_1^{-1}H = H$ when $H\mathbf{1} = 0$. With both numbers, either face-axis premise forces $\lambda = 0$ in $Q = J_{\mathrm{lw}} + \lambda(\mathrm{tr}\,H)^2$ (Section 3.4), so $Q = J_{\mathrm{lw}}$ on every spatial strain, and by Theorem 2.3 $J_{\mathrm{lw}}(H)$ is the quadratic part of $J(g_1^{-1}(g_1 + \varepsilon H)) = J(I + \varepsilon H)$. The last sentence is Section 3.4: a step cost with both numbers has $\lambda = \frac{1}{4}$. □

What Proposition 6.2 gives, and what it does not. It identifies the strain factor on the polarizations, under its premises, not an action. It is the thesis made precise there, and weaker than the thesis: on $\mathcal{T}(m)$ every form invariant under the rotations of space that fix m is a multiple of $\mathrm{tr}\,H^2$ (for m not a multiple of $\mathbf{1}$ by Section 1; for a multiple of $\mathbf{1}$ because the traceless spatial strains are irreducible under the rotations of space), so it fixes a normalization, and no observation in this paper tests the thesis. Q has no derivatives: the wave factor $\eta(m)$, the factor $-\sqrt{3}\pi^2/2$, the trace part and the coupling of the time-time strain all come from Regge's action (Theorem 6.1(i), Section 9). To be the action, the ledger's charge would need two derivatives, and as a term of the action a derivative-free form would be a mass term (Section 1); under premise (B) of Proposition 7.3 the action is a ledger charge in another sense, of each hinge's content, with its derivatives in the deficits. As written, $J(g_1^{-1}g)$ charges pure-gauge strains (Theorem 3.1(iii)) and is negative on some strains that are not spatial (Section 2.2).

6.3 Reproducing the constant

The constant of Theorem 6.1(i) is [4, Thm. 3.1(ii)], which [4] proves by a finite identity checked in exact arithmetic [4, Sec. 4]. For Regge's charge it can be recomputed from what is printed here (Appendix A). Two facts enter. Every flat deficit vanishes: at each of the hinge classes below, the exact interior angles, multiples of $\pi/6$, add to 2π. And Schläfli's identity holds at the flat lattice to first order. Regge's field equations rest on Schläfli's identity $\sum_t A_t\, d\theta_t = 0$ for a simplex, the sum over its ten triangles t [7]. For these Lorentzian simplices we use it only at the flat $\kappa = 1$ lattice and only in two forms: $\sum_t A_t\, \partial\theta_t/\partial\ell_e = 0$ for each edge e, and its derivative $\sum_t\big(\partial_f A_t\, \partial_e\theta_t + A_t\, \partial_e\partial_f\theta_t\big) = 0$ for each pair of edges, with $\partial_e = \partial/\partial\ell_e$. We check both exactly over $\mathbb{Q}(\sqrt{3})$ at all 24 simplices of the cell, in 240 and 2400 equations, from the formulas below, which hold where every $M_{aa} > 0$, every angle is real and every hinge is timelike, conditions that hold in a neighbourhood of the flat lattice. The first can also be checked by hand from Table 5. We also checked the identity numerically, to 10^{-15}, at perturbed simplices. At the flat lattice $\delta_h = 0$, so $D^2 S_{\mathrm{R}} = \sum_h (2\, dA_h\, d\delta_h + A_h\, d^2\delta_h)$, and the derivative identity summed over the simplices gives $\sum_h A_h\, d^2\delta_h = -\sum_h dA_h\, d\delta_h$; so $\mathscr{H}_N(u,u) = \sum_h dA_h[u]\, d\delta_h[u]$. A *hinge class* c is a triangle x, $x+a$, $x+a+b$ up to translation; there are 50, with 240 incidences on simplices. On the plane wave (14), $dA_h[u]$ and $d\delta_h[u]$ are the real parts of $e^{ik\cdot x}$ times the amplitudes

$$\alpha_c(k) = \sum_e \frac{\partial A_c}{\partial \ell_e}\, s_e\, e^{ik\cdot y_e}, \qquad \beta_c(k) = -\sum_\sigma \sum_e \frac{\partial \theta_{c,\sigma}}{\partial \ell_e}\, s_e\, e^{ik\cdot y_e},$$

summed over the three edges e of the hinge and over the ten edges of each simplex σ that contains it, with $s_e = d^{\top} H d$ the reading of the edge's step, y_e its midpoint relative to x, and $\theta_{c,\sigma}$ the dihedral angle of σ at the hinge. For $2m \not\equiv 0 \bmod N$ the sum over x gives

$$N^{-2}\mathscr{H}_N(u,u) = N^2 \sum_c \tfrac{1}{2}\,\mathrm{Re}\big(\alpha_c(k)\,\overline{\beta_c(k)}\big), \qquad k = 2\pi m/N. \tag{15}$$

The derivatives are taken at the flat lattice of Section 6.2: $\cos\theta = -M_{ab}/\sqrt{M_{aa}M_{bb}}$, with M the Gram matrix of the barycentric gradients, whose entries for $s,t = 1,\dots,4$ are those of the inverse of the Gram matrix $\frac{1}{2}(\ell_{0s} + \ell_{0t} - \ell_{st})$ of the four edges at vertex 0, with the row and column of vertex 0 minus the sums of the others; and $A = \frac{1}{2}\sqrt{-\det G}$, with G the Gram matrix of two edges of the hinge. At $\kappa = 1$ they lie in $\mathbb{Q}(\sqrt{3})$. Write $F^0 = \sum_e f_e$, $F^1 = \sum_e f_e\,(m\cdot y_e)$ and $F^2 = \frac{1}{2}\sum_e f_e\,(m\cdot y_e)^2$, so that an amplitude $F(k) = \sum_e f_e e^{ik\cdot y_e}$ is $F^0 + \frac{2\pi i}{N}F^1 - \frac{4\pi^2}{N^2}F^2 + O(N^{-3})$; as the f_e are real, $\mathrm{Re}(\alpha_c\overline{\beta_c}) = \alpha_c^0\beta_c^0 + \frac{4\pi^2}{N^2}(\alpha_c^1\beta_c^1 - \alpha_c^0\beta_c^2 - \alpha_c^2\beta_c^0) + O(N^{-4})$. In exact arithmetic over $\mathbb{Q}(\sqrt{3})$, $\beta_c^0 = 0$ for every class, so the limit of (15) is $2\pi^2\sum_c(\alpha_c^1\beta_c^1 - \alpha_c^0\beta_c^2)$; and this sum equals $-\frac{\sqrt{3}}{4}\mathrm{FP}_{g_1}(H,H;m)$ coefficient by coefficient, in all 550 coefficients of a form quadratic in the ten entries of H and in the four of m, 102 of them nonzero. By polarization this confirms (i) for Regge's charge without using [4]. The computation runs by machine in exact arithmetic. Its inputs are the formulas above and the flat derivatives of Appendix A; the script and its tables of $\alpha_c^0, \alpha_c^1, \alpha_c^2, \beta_c^1, \beta_c^2$ for all 50 classes are in the supplementary files of this record (Data availability). As numerical evidence of the convergence, Table 3 gives (15) at the wave of Example 2.4, and on eight random bilinear pairs (H,K,m) with $|m_a| \le 3$ the values of the bilinear form of (15) at $N = 128, 256, 512$, extrapolated in N^{-2} and N^{-4}, agree with $-\frac{\sqrt{3}\pi^2}{2}\mathrm{FP}_{g_1}(H,K;m)$ to a relative 10^{-9} (largest 9.7×10^{-10}). At finite N the two polarizations differ. Their relative splitting is 4.26%, 1.08%, 0.27% and 0.068% at $N = 8$, 16, 32 and 64, about

$2.78/N^2$: a birefringence of the lattice of order k^2 in lattice units, a property of Regge's action at finite N, not of the ledger's charge, which vanishes in the long-wave limit of Theorem 6.1.

Table 3. $N^{-2}\mathscr{H}_N(u,u)$ from (15) at the wave $m = (1,-1,0,0)$ of Example 2.4, for the plus and cross strains and the conformal strain g_1, with the limits of Theorem 6.1.

N	8	16	32	64	128	256	512	limit
H_+	−33.9128	−34.1205	−34.1721	−34.1850	−34.1882	−34.1890	−34.1892	$-2\sqrt{3}\pi^2 = -34.1893$
$H_\times$	−32.4676	−33.7522	−34.0796	−34.1619	−34.1824	−34.1876	−34.1889	$-2\sqrt{3}\pi^2$
g_1	106.3951	103.5485	102.8146	102.6297	102.5834	102.5718	102.5689	$6\sqrt{3}\pi^2 = 102.5679$

7 The face bit sets the strength

The strength of gravity is the overall scale s of the action, which Sections 3 to 6 leave free. In Regge's charge $sA\delta$ the scale s is the action per unit area per unit deficit, and reading the action as Einstein–Hilbert's gives Newton's constant $G = 1/(8\pi s)$ in lattice units, an inference in [4, Remark 3.2]. J does not fix s, and the coefficient one holds whatever s is (Corollary 7.1 and the paragraph after it). One identification ties s to the lattice's unit of length λ_{rec}: the lattice's timelike hinges carry the posting action of the face bit. Then $s = s_{\mathrm{p}} = \hbar \ln 2/(2\pi\lambda_{\mathrm{rec}}^2)$ and $G\hbar = \lambda_{\mathrm{rec}}^2/(4\ln 2)$ (Theorem 7.2). The paper determines λ_{rec} in no other way, so this fixes $\lambda_{\mathrm{rec}} = 2\sqrt{\ln 2}\,\ell_{\mathrm{P}} \approx 1.665\,\ell_{\mathrm{P}}$, with $\ell_{\mathrm{P}} = \sqrt{G\hbar}$ the Planck length ($c = 1$); it does not predict G.

Corollary 7.1 (The long-wave action of one mode)**.** *Let $s > 0$ be the Regge action's scale. At $\kappa = 1$, with no matter source, the long-wave quadratic action of one cosine mode with strain h and wave k, per unit coordinate volume, is $-\frac{s\sqrt{3}}{16}\,\mathrm{FP}_{g_1}(h,h;k)$ to leading order.*

Proof. Per unit coordinate volume the second-order part of sS_{R} on the mode $u = u_{m,h}$ is $\frac{s}{2}\mathscr{H}_N(u,u)/N^4$. By Theorem 6.1(i), N^2 times it tends to $\frac{s}{2}\cdot\left(-\frac{\sqrt{3}\pi^2}{2}\right)\mathrm{FP}_{g_1}(h,h;m)$. With $\mathrm{FP}_{g_1}(h,h;2\pi m/N) = (2\pi/N)^2\mathrm{FP}_{g_1}(h,h;m)$, to leading order it is $-\frac{s\sqrt{3}}{16}\mathrm{FP}_{g_1}(h,h;k)$, $k = 2\pi m/N$. Its factors are s, the $\frac{1}{2}$ of a second-order term and those read after Theorem 6.1. □

The calibration and the strength. s carries the units of action per area, $\hbar/\lambda_{\mathrm{rec}}^2$, while $J''(1) = 1$ is a pure number, so J's calibration does not fix s without a unit of action per area. Reading it as setting the weight $s\sqrt{3}/16$ to $\nu\hbar/\lambda_{\mathrm{rec}}^2$, with $\nu > 0$ a number, gives $s = 16\nu\hbar/(\sqrt{3}\lambda_{\mathrm{rec}}^2)$ and $G = 1/(8\pi s) = \sqrt{3}\lambda_{\mathrm{rec}}^2/(128\pi\nu\hbar)$. This is the face-bit value $\lambda_{\mathrm{rec}}^2/(4\hbar\ln 2)$ of Theorem 7.2 exactly when $\nu = \nu_* = \sqrt{3}\ln 2/(32\pi) = 0.011942\ldots$ At $\nu = 1$ the reading gives $G = \sqrt{3}\lambda_{\mathrm{rec}}^2/(128\pi\hbar)$, which is ν_* times the face-bit value, smaller by a factor of about 84. So J's calibration does not set the strength.

Theorem 7.2 (The face bit fixes the scale)**.** *Let $a(A)$ be the action of one posting on a lattice face of area A. Suppose that (a) a posting is a twelfth of a full turn of the face, a deficit of $\pi/6$; (b) a adds over area, $a(A + A') = a(A) + a(A')$, and never decreases as the area grows; and (c) a full turn of twelve postings on a face of area λ_{rec}^2 counts one bit, $\exp\bigl(12\, a(\lambda_{\mathrm{rec}}^2)/\hbar\bigr) = 2$. Then*

$$a(A) = s_{\mathrm{p}}\, A\, \frac{\pi}{6}, \qquad s_{\mathrm{p}} = \frac{\hbar \ln 2}{2\pi\lambda_{\mathrm{rec}}^2},$$

and one posting on one face of area λ_{rec}^2 carries $\hbar \ln 2/12$. The scale s of the timelike hinge's action is such a posting scale exactly when $s = s_{\mathrm{p}}$. Then $G = 1/(8\pi s_{\mathrm{p}}) = \lambda_{\mathrm{rec}}^2/(4\hbar \ln 2)$, and the weight of Corollary 7.1 is $\frac{\sqrt{3}}{16}\, s_{\mathrm{p}} = v_*\hbar/\lambda_{\mathrm{rec}}^2$, with $v_* = \sqrt{3}\ln 2/(32\pi)$ as in the paragraph after Corollary 7.1.

Proof. By (b), $a(nA) = n\,a(A)$ for whole n, so $a(qA) = q\,a(A)$ for rational $q > 0$; as a never decreases, $q\,a(\lambda_{\mathrm{rec}}^2) \le a(A) \le q'a(\lambda_{\mathrm{rec}}^2)$ for all rationals $q < A/\lambda_{\mathrm{rec}}^2 < q'$, so $a(A) = (A/\lambda_{\mathrm{rec}}^2)\, a(\lambda_{\mathrm{rec}}^2)$. By (a), Regge's charge $sA\delta$ gives one posting the action $sA\pi/6$, so s is a posting scale exactly when $sA\pi/6 = a(A)$ for every A. The bit (c) gives $12\, a(\lambda_{\mathrm{rec}}^2)/\hbar = \ln 2$, so $a(\lambda_{\mathrm{rec}}^2) = \hbar \ln 2/12$ and $a(A) = A\,\hbar \ln 2/(12\lambda_{\mathrm{rec}}^2) = s_{\mathrm{p}} A\pi/6$. The rest is arithmetic. □

IDENTIFICATION. The timelike hinge carries the posting action of the face bit. This includes (a)–(c) of Theorem 7.2; (a) fits the lattice's flat dihedral angles, multiples of $\pi/6$ (Section 6.2). It is stated, not derived. (c) reads an action as an entropy. The Euclidean conical-deficit derivations of horizon entropy [17, 18] do so too, but the analogy is only formal, in two ways. They concern the horizon's bifurcation surface, a spacelike surface of codimension two whose normal plane is Lorentzian and carries a rotation only after Euclidean continuation; the face bit sits on timelike hinges, whose spacelike normal plane carries a rotation deficit, as a cosmic string's does. And they read the entropy from the action's derivative at vanishing deficit, while the face bit evaluates Regge's charge, extended linearly, at a full turn 2π, where the hinge is degenerate. With Theorem 7.2's G, (c) says exactly that the Bekenstein–Hawking entropy of the area λ_{rec}^2 is $\ln 2$: it is equivalent to the choice $\lambda_{\mathrm{rec}}^2 = 4 \ln 2\, \ell_{\mathrm{P}}^2$, so it fixes λ_{rec}, not G.

Regge's charge from two premises. With one more premise, the posting identification, extended to credits, also fixes the action's form. A *hinge charge* $C(A, \delta)$, on areas $A > 0$ and real deficits, with $C(A, 0) = 0$, gives the action $\sum_h C(A_h, \delta_h)$. Two premises on it: (B), the *content reading*, $C(A, \delta) = F(A\delta)$ for some function F, so that a hinge is charged for its content $A\delta$ alone; and (C), the *record clause*, one posting on a face of area A, a deficit $\pi/6$, carries the posting action $a(A) = s_{\mathrm{p}} A\pi/6$ of Theorem 7.2, and a credit, $-\pi/6$, carries $-a(A)$.

Proposition 7.3 (Regge's charge is the content reading with the record)**.** *(B) and (C) hold together exactly when $C(A, \delta) = s_{\mathrm{p}} A\delta$, Regge's charge at the posting scale. Neither alone suffices: $s_{\mathrm{p}} A\delta + A^2(1 - \cos 12\delta)$, the charge $sA\delta + A^2(1 - \cos k\delta)$ of [4, Prop. 7.2] at $k = 12$ and $s = s_{\mathrm{p}}$, meets (C) and not (B), and the zero charge meets (B) and not (C).*

Proof. Regge's charge is $F(A\delta)$ with $F(x) = s_{\mathrm{p}} x$, and it meets (C). Conversely, for $x \neq 0$ put $A = 6|x|/\pi$; then $F(x) = C(A, \pm\pi/6) = \pm a(A) = s_{\mathrm{p}} x$, with the sign of x, and $F(0) = C(A, 0) = 0$;

so $C(A, \delta) = F(A\delta) = s_{\mathrm{p}} A\delta$. The first example equals Regge's charge at $\delta = \pm\pi/6$, where $\cos 12\delta = 1$, but at the equal contents of $(A, \delta) = (1, \pi/12)$ and $(2, \pi/24)$ it takes values that differ by 2; the zero charge gives a posting no action. □

Proposition 7.3 restates its premises: (C) fixes C on the lines $\delta = \pm\pi/6$, far outside the linear regime, and (B) carries those values along the curves $A\delta = \text{const}$; the long-wave physics reads only the jets of C at $\delta = 0$ (Section 8). So "gravity's action is a ledger charge" is premise (B), not derived here: the ledger charges each hinge for its content; with the record clause that charge is Regge's.

8 One cost, one ledger

The unification here is exact and narrow: the three links of Section 1, at their status.

One cost. Given the composition law and the calibration, the same J prices every comparison of two positive quantities on the ledger [1, 2]; the graviton's form uses only its calibration. The identity $\mathrm{FP} = \eta \cdot 2J_{\mathrm{lw}}$ holds for every cost with that calibration, so it does not test the thesis; nor do the premises of Proposition 6.2, which Propositions 3.6 and 3.7 reduce to either of two named premise sets at the record charge (8), an IDENTIFICATION, for they too use J only through $J''(1) = 1$ and a positive price per posting. Read through the identification of Section 2.2, gravity enters as one more comparison, a metric against the flat one, charged $J(g_1^{-1} g)$, and on spatial strains its quadratic part is the strain factor of Proposition 6.2. What else enters is listed in Section 1.

The face bit and the Planck area. With the posting identification, Theorem 7.2 gives $G = \lambda_{\mathrm{rec}}^2/(4\hbar \ln 2)$, so $G\hbar = \lambda_{\mathrm{rec}}^2/(4 \ln 2)$: the Planck area is $\lambda_{\mathrm{rec}}^2/(4 \ln 2)$. Equivalently, with this G and $c = k_{\mathrm{B}} = 1$ the Bekenstein–Hawking entropy [9, 10] of a horizon of area A is $A/(4G\hbar) = A \ln 2/\lambda_{\mathrm{rec}}^2$ nats, that is, $A/\lambda_{\mathrm{rec}}^2$ bits: one bit per area λ_{rec}^2. This is arithmetic from G, at the status of Theorem 7.2: it holds through the posting identification and restates its premise (c), so it fixes λ_{rec} and does not predict G (Section 7). It is not a derivation of black-hole entropy from a lattice horizon.

Not yet included. The premise sets of Propositions 3.6 and 3.7 and premise (B) of Proposition 7.3 are not derived from the ledger, and gauge consistency at cubic order cannot supply (B): a charge $\psi(A)\delta$, one ψ for every hinge, enters the action through cubic order only through ψ's jets at the flat areas [4, Sec. 4.3, Thm. 6.4]. The decisive test, a number of a second sector matched to data, the electron's mass scale for one, is still open (Section 1).

9 What fixes what, and what is not covered

Table 4 lists what fixes each feature of the graviton's quadratic action, and at what status; four readings carry the physics.

Table 4. What fixes each feature of the graviton's long-wave quadratic action.

feature	what fixes it	status	where
the strain factor on transverse-traceless strains	$\operatorname{tr} h^2 = 2J_{\mathrm{lw}}(h)$, proportional to every rotation-invariant form there; $\eta(m)$ and $-\sqrt{3}\pi^2/2$ from Regge's action; the ledger's charge under Prop. 6.2's premises	theorem, given Regge's action; Prop. 6.2 under its premises, not derived	Thms. 5.1, 6.1, Props. 3.6, 3.7, 6.2
its coefficient	one in the Fierz–Pauli normalization of [4]; $-\frac{1}{2}$ against the Einstein–Hilbert coefficient, where the relation reads $-\eta(m) \cdot J_{\mathrm{lw}}$; $J''(1) = 1$ sets the unit (Section 1)	convention (match of units); $J''(1) = 1$ a CALIBRATION	Thms. 3.3, 6.1(ii)
that a graviton exists, with two polarizations	three or more dimensions of space; space has three	theorem, at $D = 3$	Thm. 3.4
the trace coefficient $\beta = -1$, a negative trace sector	in the Euclidean $O(4)$-invariant model, locality of the cost (part of the model), with signed costs coupling edge directions; for the lattice, Regge's action at $\kappa = 1$	theorem (Euclidean model); for the lattice cited [4], recomputed in Sec. 6.3	Thms. 4.6, 4.8, 6.1
the lattice's own normalization	the Regge action at $\kappa = 1$: $-\frac{1}{4}$ per unit proper volume	cited [4]; exact computation (MODEL; $\kappa = 1$ an IDENTIFICATION)	Thm. 6.1, Sec. 6.3
the strength, Newton's constant	the face bit, through the posting identification: one bit per area λ_{rec}^2; it fixes λ_{rec}, not G	IDENTIFICATION	Thm. 7.2, Sec. 8

The identifications. (1) The ledger's charge: the premise sets of Propositions 3.6 and 3.7, at the record charge (8), an IDENTIFICATION, or the rotation route of Section 3.4, which the reading $J(g_1^{-1}g)$ of Section 2.2 meets. It serves only Proposition 6.2: two numbers at the face wave suffice, and with them one face-axis premise gives J's form on every spatial strain. Its intended content: a static strain of space costs what the records of its strained steps cost. Q enters no action here, so no observation tests it; inside the paper the sensitive test of its premises is the no-birefringence number at the face wave, $Q(H'_\times) = Q(H'_+)$ (Section 3.4). (2) The lattice's readings: $\kappa = 1$, squared edge lengths as a metric, and time along the diagonal $\mathbf{1}$ [4] (Section 6.1). Their content: a discretized spacetime with a rest frame, in which single postings are light rays. At long wavelengths they give linearized Einstein gravity (Theorem 6.1), with a polarization splitting of order k^2 in lattice units (Section 6.3); a frame dependence of long gravitational waves would falsify them, and at one loop, where $\kappa = 1$ carries the corner doublers, a massless graviton is open (Section 6.1). (3) The timelike hinge carries the posting action of the face bit, and a credit its negative (Section 7). Its content: one bit of entropy per area λ_{rec}^2 of horizon. It fixes $\lambda_{\mathrm{rec}} = 2\sqrt{\ln 2}\,\ell_{\mathrm{P}}$, so alone it cannot fail; a value of λ_{rec} fixed by another sector would test it. (4) The model of Section 6.1, Regge's action on the Kuhn lattice read as Einstein–Hilbert's; it is local, and Section 4.2 shows in the Euclidean model how locality fixes the trace part. Its content: gravity charges the deficit angles of a discretized spacetime. At long wavelengths it is linearized Einstein gravity, with Einstein–Hilbert's cubic vertex [4, Thm. 3.1(iii)], so the tests of weak-field gravity test it there. (2) and (4) make the graviton's form on the polarizations a theorem (Theorem 6.1). Everything else is a theorem proved here, with these exceptions: the results cited from [1, 2, 4, 11, 12] and the classical works, among

them the result of [4] that Theorem 6.1(i) cites, with the exact computation and the numerical check of Section 6.3 that confirm it; the record charge (8), an IDENTIFICATION; the paragraph on the calibration and the strength after Corollary 7.1; the calibration $J''(1) = 1$, which sets J's unit; the dimension of space, three, taken as given; the premise sets of Propositions 3.6 and 3.7 and premise (B) of Proposition 7.3, not derived; and the scope remarks of "Not covered" and "Not yet included".

No Newtonian limit from the ledger's charge. Proposition 6.2 gives no Newtonian limit, whose time-time coupling comes from Regge's action. Its costs are quadratic forms on spatial strains, so they are functions of the six readings $(e_p - e_q)^\top H(e_p - e_q)$, $p < q$, which fix a spatial strain: if all six vanish, $H_{pq} = \frac{1}{2}(H_{pp} + H_{qq})$, and $H\mathbf{1} = 0$ gives $H = 0$. These readings vanish on the time-time strain $\mathbf{1}\mathbf{1}^\top$, zero on the spatial slice, as $\mathbf{1}^\top(e_p - e_q) = 0$, so a cost read through them on every strain does not couple it. Fierz–Pauli, and with it Regge's action (Theorem 6.1(i)), couples it to the stretches U_2, U_3 across the face wave: only the last term of (12) survives, and as $\eta(p_1) = 4$, $\mathrm{FP}_{g_1}(\mathbf{1}\mathbf{1}^\top, U_a; p_1) = -\eta\,\mathrm{tr}(g_1^{-1}\mathbf{1}\mathbf{1}^\top)\,\mathrm{tr}(g_1^{-1}U_a) = \frac{16}{3}$ for $a = 2, 3$.

Not covered. This is linear gravity around flat space at long wavelengths: nonlinear gravity, the coupling to matter, the one-loop terms and the modes away from long wavelengths are not covered, among them the finite-wavelength splitting of the two polarizations, about $2.78/N^2$ at the wave of Example 2.4 (Section 6.3), and the massless doublers at the corners of the Brillouin zone, which in the family g_κ with $\kappa > \frac{1}{2}$ occur exactly at $\kappa = 1$, at the four axis corners [12, Thm. 2.1]. Off shell the polarizations themselves are defined in the lattice's rest frame, by $H\mathbf{1} = 0$. At one loop the author's own studies of this lattice find, in a stated diagram model, that the graviton limit diverges through those doublers [11], and that the native model graviton is not massless [12]; a massless graviton is open there. The nonlinear action is Regge's, whose long-wave cubic vertex on plane waves is Einstein–Hilbert's [4, Thm. 3.1(iii)]; the ledger's charge enters only as Proposition 6.2's strain factor.

Data availability

The script of the exact computation of Section 6.3 and Appendix A, in exact arithmetic over $\mathbb{Q}(\sqrt{3})$ with no external library, is in the supplementary files of this record, with a README that says how to run it and with its output: the checks of Section 6.3, the Taylor coefficients $\alpha_c^0, \alpha_c^1, \alpha_c^2, \beta_c^1, \beta_c^2$ of all 50 hinge classes, the flat derivatives, Table 3, the finite-N splitting and the eight random pairs.

A The flat derivatives of one simplex

The flat derivatives of the areas and dihedral angles that Section 6.3 reads all follow from one simplex. Take the arrival order 0123 (Section 6.2), with the vertices $v_0 = 0$, $v_1 = e_0$, $v_2 = e_0 + e_1$, $v_3 = e_0 + e_1 + e_2$ and $v_4 = \mathbf{1}$; write pq for the edge from v_p to v_q and ijk for a triangle. The flat squared lengths are $\ell = 0$ on 01, 12, 23, 34, $\ell = -2$ on 02, 13, 24, $\ell = -6$ on 03, 14, and $\ell = -12$ on 04. Table 5 gives, for each triangle t, the flat interior dihedral angle θ_t at t, the area A_t and the derivatives $\partial\theta_t/\partial\ell_e$; Table 6 gives the area derivatives, which depend only on the

hinge type, the numbers of postings on its edges. The relabellings of the counters carry this simplex to the other 23 and keep g_1, so they carry both tables to every simplex. The 50 hinge classes are 12 of type $(1,1,2)$, 24 of type $(1,2,3)$, 8 of type $(1,3,4)$ and 6 of type $(2,2,4)$. In each column of Table 5 the sum of $A_t\,\partial\theta_t/\partial\ell_e$ over the ten triangles is zero: that is Schläfli's identity at the flat simplex. The second derivatives and the class amplitudes α_c, β_c of (15) are computed by the script in the supplementary files of this record.

Table 5. The flat simplex 0123: for each triangle t its interior dihedral angle θ_t, its area A_t and the derivatives $\partial\theta_t/\partial\ell_e$ in the squared lengths of the ten edges $e = pq$.

t	θ_t	A_t	01	02	03	04	12	13	14	23	24	34
012	$\frac{\pi}{6}$	$\frac{1}{2}$	0	$-\frac{\sqrt{3}}{6}$	$\frac{\sqrt{3}}{12}$	0	0	0	0	0	$-\frac{\sqrt{3}}{12}$	$\frac{\sqrt{3}}{6}$
013	$\frac{\pi}{2}$	1	$-\frac{\sqrt{3}}{12}$	$\frac{\sqrt{3}}{6}$	$-\frac{\sqrt{3}}{12}$	0	0	$\frac{\sqrt{3}}{4}$	$-\frac{\sqrt{3}}{6}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$-\frac{\sqrt{3}}{6}$
014	$\frac{\pi}{3}$	$\frac{3}{2}$	0	0	0	0	0	$-\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{6}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{4}$	0
023	$\frac{\pi}{2}$	1	$\frac{\sqrt{3}}{6}$	$-\frac{\sqrt{3}}{12}$	$\frac{\sqrt{3}}{4}$	$-\frac{\sqrt{3}}{6}$	0	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\frac{\sqrt{3}}{4}$	$-\frac{\sqrt{3}}{6}$	0
024	$\frac{\pi}{2}$	$\sqrt{3}$	0	$\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{4}$	0
034	$\frac{\pi}{3}$	$\frac{3}{2}$	0	$-\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{6}$	0	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{4}$	0	0	0	0
123	$\frac{2\pi}{3}$	$\frac{1}{2}$	0	0	$-\frac{\sqrt{3}}{4}$	$\frac{\sqrt{3}}{6}$	0	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{4}$	0	0	0
124	$\frac{\pi}{2}$	1	0	$-\frac{\sqrt{3}}{6}$	$\frac{\sqrt{3}}{3}$	$-\frac{\sqrt{3}}{6}$	$\frac{\sqrt{3}}{4}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{4}$	0	$-\frac{\sqrt{3}}{12}$	$\frac{\sqrt{3}}{6}$
134	$\frac{\pi}{2}$	1	$-\frac{\sqrt{3}}{6}$	$\frac{\sqrt{3}}{3}$	$-\frac{\sqrt{3}}{6}$	0	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{4}$	$-\frac{\sqrt{3}}{12}$	0	$\frac{\sqrt{3}}{6}$	$-\frac{\sqrt{3}}{12}$
234	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{6}$	$-\frac{\sqrt{3}}{12}$	0	0	0	0	$\frac{\sqrt{3}}{12}$	0	$-\frac{\sqrt{3}}{6}$	0

Table 6. Areas and area derivatives of the four hinge types of the flat $\kappa = 1$ lattice, by the numbers of postings on the edges.

hinge type	triangles of 0123	A	$\partial A/\partial\ell_e$ on its three edges
$(1,1,2)$	012, 123, 234	$\frac{1}{2}$	$\frac{1}{4}$ on each single step, $-\frac{1}{4}$ on the double step
$(1,2,3)$	013, 023, 124, 134	1	$\frac{1}{2}$ on the single, $\frac{1}{4}$ on the double, $-\frac{1}{4}$ on the triple step
$(1,3,4)$	014, 034	$\frac{3}{2}$	$\frac{3}{4}$ on the single, $\frac{1}{4}$ on the triple step, $-\frac{1}{4}$ on **1**
$(2,2,4)$	024	$\sqrt{3}$	$\frac{\sqrt{3}}{4}$ on each double step, $-\frac{\sqrt{3}}{6}$ on **1**

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