
A finite gravitational path sum on the recognition lattice

The Lorentzian Regge sum on the fixed Kuhn graph, with a stated regulator, a model measure and a conditional sign

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Abstract

A sum over Lorentzian geometries needs a space of geometries, an action, a measure and a reason the sum exists. We give all four on the recognition lattice, a periodic Kuhn triangulation with real squared edge lengths and time on the body diagonal, and we state the status of each. The admissible geometries form an open cone of signature $(3, 1)$ simplices. The action is the Regge action. Its Lorentzian sign is conditional: the tick order fixes the time axis and the causal range, and the sign enters through a causality premise. The measure is a model measure pushed from ledger states; its account family, box radius and weight change the sum. With a stated regulator the sum is a finite integral whose normalization is an entire function of the phase scale, so expectations exist off a discrete set. The parameter κ is the posting speed, and one open premise decides it.

1 What a path sum over geometries needs

Quantum gravity as a sum over histories assigns an amplitude e^{iaS} to every spacetime geometry and adds them up. To make that a mathematical object four things have to be settled. Which geometries are summed over? Which action weighs them, and why that one? Which measure adds them? And does the sum exist: either it converges, or a regulator is stated. Each of the four has a history of failures. Euclidean sums lose positivity through the conformal factor [8]. Regge sums can be dominated by spikes [2]. Lorentzian Regge geometries develop complex angles where hinges turn null [3, 16, 17]. And the measure is hard to fix from first principles [9].

In Recognition Science spacetime is a lattice of recognition records. A companion paper derived the linearized sourced Einstein equations on that lattice at fixed spacing [18]. Here we go beyond linear order and set up the full sum. The lattice is the periodic Kuhn triangulation of $(\mathbb{Z}/N)^4$. Every edge carries a real squared length, and time runs along the body diagonal $(1, 1, 1, 1)$. The flat lattice has squared length $n(1 - \kappa n)$ on an edge that flips n coordinates.

What κ is. The parameter κ is not a free constant. It is the speed v of one posting, measured against the light cone of the cell's own interval:

$$\kappa = \frac{3 + v^2}{4v^2}.$$

No history of postings moves faster than one posting, and the causal range is $\kappa \geq 1$, $v \leq c$. For $1/2 < \kappa < 1$ a posting would be faster than light and no history is causal; we keep that range in some statements only because the geometry is still Lorentzian there. Two readings are on the table. If light crosses one posting per tick, $v = c$ and $\kappa = 1$; then the posting edges are null. If light crosses one record-cube edge per tick, $v = \frac{\sqrt{3}}{2}c$ and $\kappa = 5/4$; this is the reading in which the ledger's cost Hessian sets the spatial unit, and the companion paper works there. The premises the cell supplies leave $\kappa \in [1, \infty)$ free, and the two readings exclude each other. One statement decides between them: *light moves at the ledger's top speed*, so that some history of postings is light-like. It holds exactly at $\kappa = 1$. In Recognition Science the value is to be derived, not chosen, and this statement is the derivation target ([Problem 2.14](#)). The main results hold at both readings: on the open cone for $\kappa > 1$, and on the null stratum at $\kappa = 1$. [Section 6](#) lists the few that are proved only off the stratum.

The result. *On the fixed Kuhn graph, the Lorentzian Regge sum over the cone of admissible geometries is a finite integral once a regulator is stated, and its normalization is an entire function of the phase scale that vanishes only on a discrete set.* The regulator is a bound on squared lengths, or on the ledger box. It is stated, not derived. The first branch of the either/or is not closed either: the uncut sum is not absolutely convergent, and whether its Abel regularization converges is open.

[Table 1](#) gives the status of the four requirements. [Sections 2 to 5](#) take them in order, and [Section 6](#) collects what is open and what is chosen.

Contributions.

- 1 The admissible set is an open cone cut out by explicit polynomial inequalities, with an explicit ball around the flat lattice, and every admissible simplex has signature $(3, 1)$ ([Theorems 2.2 to 2.5](#)).
- 2 The parameter κ is the posting speed; the causal range is $\kappa \geq 1$; the cell's premises leave κ free; a field carried by postings cannot fill a causal light cone; and the statement "light moves at the ledger's top speed" holds exactly at $\kappa = 1$ ([Theorems 2.11 to 2.13](#)).
- 3 One Kuhn diagonal in every cell follows from a global clock without folds. Of the 384 cube symmetries the cut keeps 48, and central inversion keeps the admissible set and the action ([Theorems 2.6 to 2.10](#)).
- 4 The Regge action is smooth and homogeneous on the cone, with first variation area times deficit by Schläfli, and it is the hinge sum $\sum_t A_t \varepsilon_t$ for $N \geq 3$ ([Theorems 3.1 to 3.3](#)).
- 5 The tick order fixes the time axis. Given name-blindness and a causal reading it fixes $\kappa \geq 1$; the minus sign is part of that premise. Inside the admissible set only the Lorentzian form sees the geometry ([Theorem 3.4](#)).

Table 1. Status of the four requirements. *Proved*: a theorem with all hypotheses stated. *Conditional*: proved from a premise that is not derived. *Model*: a stated construction. *Open*: a precise problem stated in the text.

Requirement	Proved	Conditional or model	Open
1. Configuration space	Open semialgebraic cone for $\kappa \neq 1$; explicit ball; every simplex of signature $(3, 1)$; boundary on null and degenerate loci; null stratum at $\kappa = 1$; κ is the posting speed, $\kappa \geq 1$ causal	Model: fixed Kuhn graph, real squared lengths, body-diagonal time. Conditional: one diagonal per cell, given a clock without folds	The value of κ (Problem 2.14)
2. Action	Regge action, smooth and homogeneous; first variation by Schläfli; slot sum equals hinge sum for $N \geq 3$; continuity up to the boundary	Conditional: the Lorentzian sign and $\kappa \geq 1$, given name-blindness and a causal reading of the cell; weight past a null hinge, given a normal form	Native phase scale; continued action
3. Measure	Finite, nonzero, density for every admissible family; exact kernel at balance; no Faddeev–Popov factor; the family, the radius and the weight change the measure; a stratum measure at $\kappa = 1$	Model: the ledger chart, the account family, the box radius, Lebesgue (flat) bare weight	Whether the bare weight is flat; refinement
4. Finiteness	Finite integral at a stated regulator; normalization entire in the phase scale; expectations defined off a discrete set; finite with nonzero normalization at $\kappa = 1$	Model: the regulator L or R	Abel limit of the uncut sum; reflection positivity; a derived cutoff

- 6 The action is continuous up to the boundary, the boundary has measure zero, and under a normal-form assumption the weight past a null hinge is suppressed, enhanced or unimodular ([Theorems 3.7 to 3.9](#)).
- 7 The ledger measure is a model measure. We prove how it depends on its account family, box radius and weight, compute the kernel of the chart at balance exactly, and show that no Faddeev–Popov factor can be pulled out ([Theorems 4.3, 4.4, 4.6, 4.7 and 4.10](#)).
- 8 At a stated regulator the sum is a finite integral, and its normalization is entire in the phase scale; at $\kappa = 1$ the sum on the null stratum is finite with nonzero normalization ([Theorems 5.2, 5.4 and 5.13](#)).

Table 2. Terms used throughout.

Term	Object	Meaning
Edge class d	$d \in \{0, 1\}^4 \setminus \{0\}$; $ d $ its number of ones	The 15 edge directions; an edge joins x to $x + d$
Posting edge	An edge with $ d = 1$	One step of one record coordinate
Configuration	$\ell \in \mathbb{R}^{\mathcal{E}_N}$, one real squared length per edge	A geometry on the fixed graph; $\ell_e < 0$ timelike, $= 0$ null, > 0 spacelike
Flat field ℓ_κ^0	$\ell_\kappa^0(x, d) = d (1 - \kappa d)$	The flat lattice with metric $I - \kappa J$
Posting speed v	$\kappa = (3 + v^2)/(4v^2)$	How fast one posting moves; $v \leq c$
Admissible set Adm_κ	Definition 2.1	The configuration space
Null stratum	Adm_1	Admissible fields with every posting edge null
Slot	A pair (4-simplex, one of its triangles)	Where an area meets an angle
Ledger chart Φ_κ	Definition 4.1	Reads a squared length on every edge from a ledger state
Regulator	$ \ell_e \leq L$, or ledger box radius R	The stated bound that makes the sum finite
Phase scale a	The weight is e^{iaS}	Plays the role of $1/\hbar$ in lattice units

2 The configuration space

A geometry is a squared length on every edge of one fixed triangulation. The configuration space is the set of those assignments for which every simplex is a genuine Lorentzian simplex and the edges have the causal character of the lattice. We first describe it at the flat lattice, where every number can be written down, then everywhere.

2.1 The lattice and its configurations

Vertices are the points x of the torus $(\mathbb{Z}/N)^4$. An edge is a pair $e = (x, d)$ with $d \in \{0, 1\}^4 \setminus \{0\}$; it joins x to $x + d$, so there are $15N^4$ edges, and we write $\mathcal{E}_N$ for the set of them. Kuhn's rule [7, 11] cuts each unit cube into 24 four-simplices, one for each ordering (a_1, a_2, a_3, a_4) of the axes, with vertices

$$v_0 = x, \quad v_1 = x + e_{a_1}, \quad v_2 = v_1 + e_{a_2}, \quad v_3 = v_2 + e_{a_3}, \quad v_4 = x + (1, 1, 1, 1).$$

All 24 share the body diagonal from v_0 to v_4 , and the edge $v_i v_j$ ($i < j$) flips $j - i$ coordinates. A *configuration* is any $\ell \in \mathbb{R}^{\mathcal{E}_N}$, read as the squared length of each edge, with no sign restriction. The *flat field* at parameter κ is $\ell_\kappa^0(x, d) = |d|(1 - \kappa|d|)$, the squared length of the vector d in the metric $g_\kappa = I - \kappa J$ (J the all-ones matrix), with $\det g_\kappa = 1 - 4\kappa$.

Every statement about a single simplex is a statement about the Gram matrix of its edge vectors, and the Gram matrix is a function of squared lengths. For a simplex with vertices $v_0, \dots, v_k$ and squared lengths ℓ_{ij} , put

$$M_{ij} = \ell_{0i} + \ell_{0j} - \ell_{ij} \quad (1 \leq i, j \leq k), \quad (1)$$

which is twice the Gram matrix of the edge vectors at v_0 . We use three determinants, each a positive multiple of the Gram determinant: for a triangle with squared sides a, b at v_0 and c opposite, $H = \det M = 4ab - (a + b - c)^2$; for a tetrahedron, $P = \frac{1}{2} \det M$; for a four-simplex, $G = \det M$. A triangle with $H < 0$ is timelike, its Gram matrix having one negative and one positive eigenvalue, and its area is $A = \frac{1}{4} \sqrt{-H}$.

Definition 2.1 (Admissible configurations). A four-simplex is *Lorentzian nondegenerate* when $H < 0$ on all ten of its triangles, $P < 0$ on all five of its tetrahedra and $G < 0$. The admissible set $\text{Adm}_\kappa \subset \mathbb{R}^{\mathcal{E}_N}$ consists of the configurations ℓ such that every Kuhn simplex is Lorentzian nondegenerate, every edge with $|d| \geq 2$ is timelike ($\ell_e < 0$), and every posting edge ($|d| = 1$) has the sign of $1 - \kappa$: spacelike for $\kappa < 1$, null for $\kappa = 1$, timelike for $\kappa > 1$.

The flat lattice, concretely. At ℓ_κ^0 every Kuhn simplex has the same numbers. Name a triangle $v_i v_j v_k$ of a simplex by its span $k - i$. Then

	triangles $v_i v_j v_k$, by span $k - i$				simplex
	2	3	4: $v_0 v_2 v_4$	4: $v_0 v_1 v_4, v_0 v_3 v_4$	
number per simplex	3	4	1	2	1
determinant H or G	$-4(2\kappa - 1)$	$-8(3\kappa - 1)$	$-16(4\kappa - 1)$	$-12(4\kappa - 1)$	$16(1 - 4\kappa)$

and the five tetrahedra have $P = -4(3\kappa - 1)$ (omitting v_0 or v_4) and $P = -8(4\kappa - 1)$ (omitting v_1, v_2 or v_3). All are negative exactly when $\kappa > 1/2$, and the first to fail as κ decreases is the span-2 triangle, which turns null at $\kappa = 1/2$. The posting edges have $\ell_\kappa^0 = 1 - \kappa$. So $\ell_\kappa^0 \in \text{Adm}_\kappa$ for every $\kappa > 1/2$, the case $\kappa = 1$ included.

2.2 The shape of the configuration space

Theorem 2.2 (Shape). Let $N \geq 1$, $\kappa > 1/2$ and $\kappa \neq 1$.

- (i) $\ell_\kappa^0 \in \text{Adm}_\kappa$, the set Adm_κ is open, and the zero field is not in it.
- (ii) (Cone.) If $\ell \in \text{Adm}_\kappa$ and $c > 0$ then $c\ell \in \text{Adm}_\kappa$.
- (iii) (Semialgebraic.) Adm_κ is the set where the finitely many polynomials $-H, -P, -G$ (every triangle, tetrahedron and Kuhn simplex), $-\ell_e$ (every edge with $|d| \geq 2$) and $(1 - \kappa)\ell_e$ (every posting edge) are strictly positive. They have degrees 2, 3, 4, 1, 1 in the edge variables.
- (iv) (Explicit ball.) Put

$$\rho(\kappa) = \frac{2\kappa - 1}{144(48\kappa - 11)^3}, \quad r(\kappa) = \min\{|1 - \kappa|, \rho(\kappa)\} > 0.$$

Every ℓ with $\max_e |\ell_e - \ell_\kappa^0(e)| < r(\kappa)$ lies in Adm_κ .

- (v) (Independence of κ .) For $\kappa > 1$, Adm_κ is the set of fields at which every Kuhn simplex is Lorentzian nondegenerate and every edge is timelike; it is the same set for all $\kappa > 1$. Likewise Adm_κ is the same set for all $\kappa < 1$.

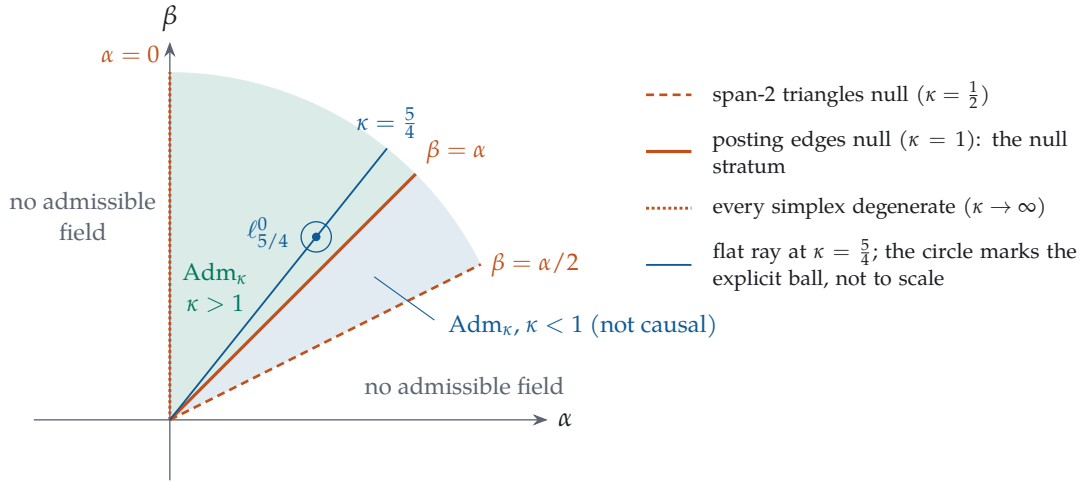


Figure 1. An exact two-parameter slice of the configuration space: the uniform fields $\ell(x, d) = \alpha|d| - \beta|d|^2$, which are the flat lattices with metric $\alpha I - \beta J$. By the cone property each ray is either inside or outside. The ray through $(1, \kappa)$ is the flat field ℓ_κ^0 and its multiples, admissible for every $\kappa > 1/2$ (Theorem 2.2); the two shaded sectors are Adm_κ for $\kappa > 1$ and for $\kappa < 1$, and they meet along the null stratum $\beta = \alpha$. The lower sector is Lorentzian but not causal: there a posting would outrun light (Theorem 2.11). The boundary rays carry the null and degenerate loci of Theorem 2.3. That no other ray of the slice is admissible was checked by evaluating the determinants on 3600 directions; it is a computation, not a theorem. The circle marks the ball of Theorem 2.2(iv), drawn far larger than its true radius.

Mechanism. (iii) is the definition rewritten: each clause is the sign of a polynomial in the edge variables, and for $\kappa \neq 1$ the posting clause is the strict inequality $(1 - \kappa)\ell_e > 0$. Openness follows because a finite intersection of strict polynomial inequalities is open. (ii) holds because H, P, G are homogeneous of degrees 2, 3, 4 in the squared lengths, so multiplying all of them by $c > 0$ multiplies every determinant by a positive number and keeps every sign. For (iv), write each determinant as $\det(A + E)$ with A its value at ℓ_κ^0 . The Leibniz expansion gives $|\det(A + E) - \det A| \leq k! ((\alpha + \varepsilon)^k - \alpha^k)$ when $|A_{ij}| \leq \alpha$ and $|E_{ij}| \leq \varepsilon$. Flat squared lengths are at most $4(4\kappa - 1)$ in size, so the entries of (1) obey $\alpha = 12(4\kappa - 1)$, and a perturbation of size $r \leq 1/3$ moves each entry by at most $3r$. For $k \leq 4$ this bounds every change of H, P or G by $288r(48\kappa - 11)^3$, which at $r = \rho(\kappa)$ is $2(2\kappa - 1)$, half the smallest flat margin $4(2\kappa - 1)$ of the table. The multi-posting edges have flat margin at least $2(2\kappa - 1) > \rho(\kappa)$, and the posting edges have margin $|1 - \kappa|$. (v) is (iii) with the sign of $1 - \kappa$ fixed. $\square$

Reading. The configuration space is a full-dimensional region of the $15N^4$ -dimensional space of squared lengths, so there is room for a measure with a density. The cone property is the freedom of the overall length unit, which nothing in the sum fixes; it returns in Section 5 as the reason a cutoff is needed. The ball is crude: at $\kappa = 5/4$ it has radius $\min\{1/4, (3/2)/(144 \cdot 49^3)\} \approx 8.9 \times 10^{-8}$. Its role is to give an explicit neighbourhood, not a sharp one. Figure 1 shows an exact two-dimensional slice.

2.3 Inside, and the boundary

Theorem 2.3 (Causal classification and boundary loci). *Let $\kappa \in \mathbb{R}$ and $\ell \in \text{Adm}_\kappa$. Every triangle of every Kuhn simplex is timelike ($H < 0$), every tetrahedron and simplex has negative determinant, and at every slot the dihedral angle θ has $\cos \theta \in (-1, 1)$: it is a real angle in $(0, \pi)$, and no branch point of the angle function lies inside Adm_κ . Here θ is the angle between the components of the two remaining edges v_0v_3, v_0v_4 orthogonal to the plane of the triangle $v_0v_1v_2$, in the metric of the simplex. For $\kappa \neq 1$, every boundary point of Adm_κ is a zero of one of the defining polynomials of Theorem 2.2(iii): a null triangle, a degenerate tetrahedron or simplex, or a null edge. Both kinds of zero occur: for every $\kappa < 1$ the flat field $\ell_{1/2}^0$ is a boundary point of Adm_κ at which the span-2 triangles are null, and for every $\kappa > 1$ the flat field ℓ_1^0 is a boundary point of Adm_κ at which the posting edges are null.*

Mechanism. The signs are the definition. On a Lorentzian nondegenerate simplex the two apex tetrahedra have $P < 0$ and the triangle has $H < 0$, so the squared norms of the two orthogonal components, which are P/H up to a positive factor, are positive and the product under the square root in the cosine is positive. The cosine cannot reach ± 1 , because by the Desnanot–Jacobi identity the gap $1 - \cos^2 \theta$ is proportional to $HG > 0$ (see Theorem 3.8). On the boundary of an open set cut out by strict inequalities, the closed inequalities hold and at least one is an equality. The two examples are limits of the flat family $s \mapsto \ell_s^0$, inside Adm_κ for $s \in (1/2, 1)$ when $\kappa < 1$ and for $s > 1$ when $\kappa > 1$, by Theorem 2.2(v). $\square$

Theorem 2.4 (Every admissible simplex has signature $(3, 1)$). *Let M be the doubled Gram matrix (1) of a four-simplex. If the triangle $v_0v_1v_2$, the tetrahedron $v_0v_1v_2v_3$ and the simplex all have negative determinant, there is an invertible Q with $Q^T M Q = \text{diag}(d_0, d_1, d_2, d_3)$, $d_0 < 0 < d_1, d_2, d_3$. In particular every simplex of every admissible configuration, and every simplex of ℓ_κ^0 for $\kappa > 1/2$, has one time and three space directions.*

Mechanism. The triangle block has negative determinant, so it is a plane of signature $(1, 1)$. Its Schur complement is a 2×2 matrix with corner entry D_3/D_2 and determinant D_4/D_2 , where D_k is the k -th leading principal minor. Both are positive when $D_2, D_3, D_4 < 0$, so the complement is positive definite. Sylvester’s law of inertia finishes the count. $\square$

Reading. “Lorentzian nondegenerate” is a signature, not only a name. The signature is a property of the configuration space we chose; it is not derived from the ledger here, and Section 3.3 says exactly where the sign enters.

The value $\kappa = 1$ is special. There the posting edges are null: a posting moves at the speed of light (Section 2.7), and the posting clause becomes an equation.

Theorem 2.5 (The null stratum). *Let $V_0 = \{\ell : \ell_e = 0 \text{ for every posting edge}\}$, a proper linear subspace. Then $\text{Adm}_1 = U \cap V_0$, where U is the open set of fields with every Kuhn simplex Lorentzian nondegenerate and every edge with $|d| \geq 2$ timelike. The null stratum Adm_1 contains ℓ_1^0 , is relatively open in V_0 , has empty interior in $\mathbb{R}^{\mathcal{E}_N}$ and is not open; it contains the relative ball*

of radius $\rho(1) = 1/(144 \cdot 37^3)$ about ℓ_1^0 inside V_0 ; and it has Lebesgue measure zero. If a curve γ is differentiable at 0 and $\gamma(t) \in \text{Adm}_1$ for all t near 0, then $\gamma'(0) \in V_0$.

Reading. At the light-speed reading the configuration space loses a dimension per posting edge. For $\kappa \neq 1$ we have an open set with room for a density. At $\kappa = 1$ we have a lower-dimensional stratum that every measure with a density on $\mathbb{R}^{\mathcal{E}_N}$ misses ([Theorem 4.4](#)). The stratum carries Haar measure of V_0 instead, and the sum is built on it in [Theorems 4.12](#) and [5.13](#).

2.4 One diagonal in every cell

Every cell of the lattice carries its Kuhn cut along the same diagonal, pointing the same way. The alternative, a cut that alternates from cell to cell, is just as good a triangulation. The choice is made by the record order. Read a cell as the lattice of records of four postings: a corner $v \in \{0, 1\}^4$ is the record of the postings it contains, and a *history* is a chain $v_0 < v_1 < v_2 < v_3 < v_4$ of records, each strictly containing the one before.

Theorem 2.6 (One cell). *The histories of a cell are exactly the 24 Kuhn simplices, each in its vertex order, and no chain of six strictly increasing records exists. No simplex of the cut reflected in the first axis is a history. A tick clock on the cell (a function T with $T(0) = 0$ that increases by one at every step of every history) is the rank, the number of postings in the record.*

To compare cells we need one clock for all of them. Let $T : \mathbb{Z}^4 \rightarrow \mathbb{Z}$ be a global clock, and read the cell at c from a corner $c + m(c)$, $m(c) \in \{0, 1\}^4$: the record v sits at the corner obtained from v by reflecting the axes set in $m(c)$. The premise is that T , read this way, is a tick clock on every cell. The cell then carries the Kuhn cut exactly when $m(c)$ is 0000 or 1111 ([Theorem 2.9](#)).

Theorem 2.7 (Coherent diagonals, conditional). *Suppose T is a tick clock on every cell for the readings m . Then the clock step along axis i at x is $T(x + e_i) - T(x) = -1$ if bit i of $m(x)$ is set and $+1$ otherwise, and two cells sharing a facet normal to axis j have the same bit on every axis $i \neq j$. All cells have the same reading if and only if the clock has no fold: each axis step $T(x + e_i) - T(x)$ is the same at every x . If moreover every axis step advances the clock, then every cell is read from its lowest corner, every cell carries the Kuhn cut with its diagonal along $(1, 1, 1, 1)$, and $T(x) = T(0) + x_0 + x_1 + x_2 + x_3$.*

Mechanism. On one cell a tick clock is the rank ([Theorem 2.6](#)), so moving one step along axis i changes the clock by $+1$ or -1 according to whether axis i is reflected in the cell's reading. The mask is therefore read off the clock, and equal masks everywhere means equal steps everywhere. Monotone steps force every bit to zero; a function on $\mathbb{Z}^4$ whose steps are all 1 is $T(0) + \sum x_i$. $\square$

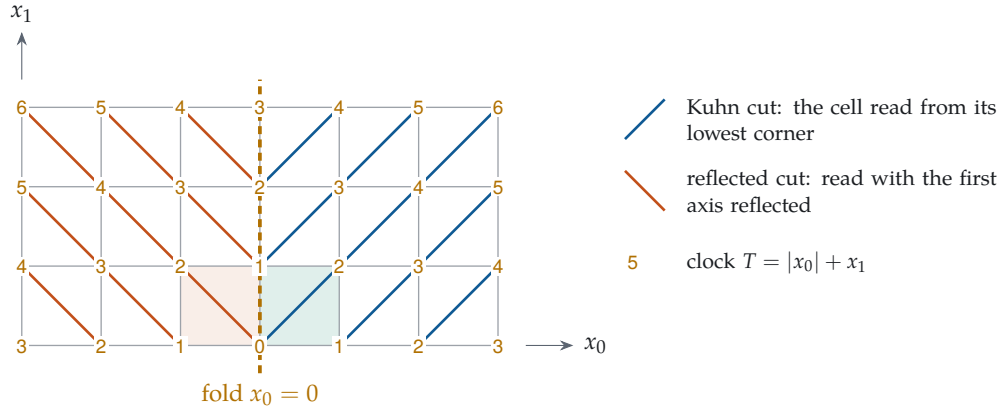


Figure 2. The two-dimensional analogue of Theorem 2.8(i). Vertex labels are the clock $T = |x_0| + x_1$. Each square is read from its lowest-clock corner and cut along the diagonal from that corner. Right of the fold every cell carries the Kuhn cut (blue); left of it every cell carries the reflected cut (orange). The two shaded cells across the fold read the same clock correctly and disagree on the diagonal. In four dimensions the same construction gives the counterexample of the theorem.

Theorem 2.8 (What a clock alone does not give). Let $T : \mathbb{Z}^4 \rightarrow \mathbb{Z}$ be a clock and m a choice of readings, as above.

- (i) (A fold.) The clock $T(x) = |x_0| + x_1 + x_2 + x_3$ is a tick clock on every cell when the cells with $c_0 < 0$ are read with the first axis reflected. The cells at $c = -e_0$ and $c = 0$ then have different readings: one carries the Kuhn cut and the other its one-axis reflection, which is not a Kuhn cut. This clock has a fold.
- (ii) (Alternating cuts.) If cell c is reflected in every axis on which c is odd, there is a global tick clock, and every such clock is $T(0) + \sum_i (x_i \bmod 2)$: it never runs more than four ticks ahead of the origin, and no such clock advances along the axes.
- (iii) (The torus.) If T is a tick clock on every cell and is periodic with period $N > 0$ along the first axis, then T has a fold.

Reading. A single global clock does not fix the diagonals; one without folds does, and a clock that advances along the axes fixes them to be the standard Kuhn cut with time on the body diagonal. Alternating cuts are incompatible with any clock that advances through more than one cell. On the torus an integer clock must fold, so coherent diagonals there come from the cyclic clock $x_0 + x_1 + x_2 + x_3 \bmod N$ or from the lift to $\mathbb{Z}^4$. This is a conditional theorem: the record reading of a cell, the global clock and the absence of folds are premises. Figure 2 draws the fold in two dimensions.

2.5 Symmetries of the cut, and time reversal

The unit four-cube has $2^4 \cdot 4! = 384$ symmetries: permute the axes, then reflect some of them.

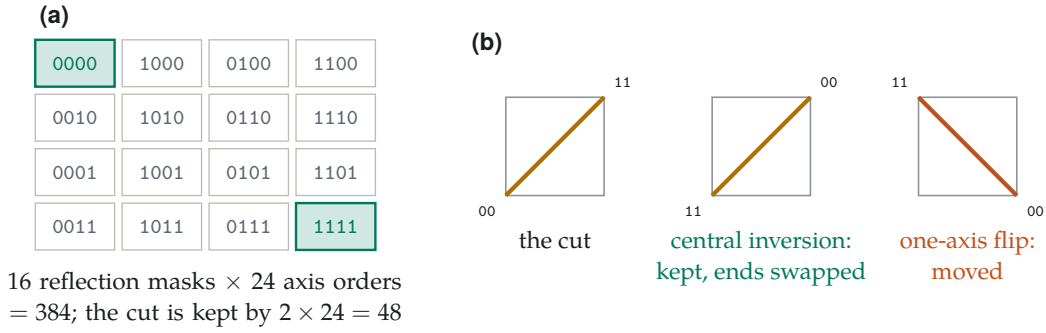


Figure 3. (a) A cube symmetry is an axis order and a reflection mask. The cut is kept exactly by the masks 0000 and 1111 (green), with any of the 24 axis orders. (b) In the square, where the Kuhn cut is the diagonal from 00 to 11, central inversion keeps the diagonal and swaps its ends, which reverses the clock; a one-axis flip moves the cut to the other diagonal.

Theorem 2.9 (The symmetries that keep the cut). *Exactly 48 of the 384 cube symmetries send every Kuhn simplex onto the vertex set of a Kuhn simplex. They are the symmetries that reflect no axis or all four, equivalently those that map the body diagonal $\{0000, 1111\}$ to itself, equivalently those that keep the rank clock ($\text{rank}(gv) = \text{rank}(v)$ for all v) or reverse it ($\text{rank}(gv) = 4 - \text{rank}(v)$ for all v). No reflection of a single axis is among them, and none keeps or reverses the clock. Central inversion, $v \mapsto 1111 - v$, is among them and reverses the clock.*

Reading. The symmetry group of the cut is the group of the record clock up to time reversal. A time reflection along one axis is not a symmetry of the Kuhn diagonal, and with time on the body diagonal it is not a time reversal either. The time reversal is central inversion (Fig. 3), and it is an exact symmetry of the whole lattice theory.

Theorem 2.10 (Central inversion). *On the torus let $\vartheta(x, d) = (-(x + d), d)$, which runs the edge from $-(x + d)$ to $-x$, and $(\vartheta\ell)(e) = \ell(\vartheta e)$. Then ϑ is an involution; it maps the Kuhn simplex of the cell at c to a Kuhn simplex of the cell at $-c - (1, 1, 1, 1)$, its corner k to minus corner $4 - k$, so the clock $x_0 + x_1 + x_2 + x_3$ is reversed along every simplex. For every κ it maps Adm_κ onto itself and fixes ℓ_κ^0 , and for every configuration ℓ it leaves both the Lorentzian Regge action of Section 3 and its Euclidean counterpart unchanged.*

2.6 Why the fixed graph

There were two ways to set up the sum: one fixed triangulation with varying lengths, or varying triangulations with fixed lengths, as in dynamical triangulations [1]. We take the fixed Kuhn graph, and the theorems of this section and of Section 4 are the reasons. The records of one cell fix its combinatorics (Theorem 2.6). A clock without folds fixes one diagonal in every cell (Theorem 2.7). The lengths are not frozen: the ledger reaches every direction of edge space with one fixed finite family of accounts, and its measure has a density (Theorems 4.2 and 4.4). On carriers that change the graph, by contrast, the two candidate

weights on bounded simplicial complexes, the inverse automorphism count and the gauge-volume weight, are both changed by the stellar $1 \rightarrow 5$ subdivision once the complex bound is at least 19, while constant weights pass (Theorem 4.9); refinement invariance there holds only for constants. On the fixed graph the only change of scale is the change of length unit, and no nonzero finite measure without an atom at zero is invariant under it, so some scale has to be supplied. The refinement behaviour of the fixed-graph measure itself is not established.

None of this derives the triangulation, the length family, κ or the time direction. The fixed Kuhn graph, real squared lengths, $n(1 - \kappa n)$ and body-diagonal time are the model, chosen for the reasons above. What is proved is what the model's premises force.

2.7 What fixes κ : the posting speed

The flat lattice depends on one number, κ . We now say what it is. Take one record cell and an interval $q_G(x) = x^T G x$ on record displacements $x \in \mathbb{R}^4$, of the form $q_G = \lambda q_\kappa$ with $\lambda > 0$ and $q_\kappa(x) = \sum_a x_a^2 - \kappa(\sum_a x_a)^2$. This is the interval of the lattice: an edge flipping n coordinates has $q_\kappa = n - \kappa n^2$. One posting on account a is the displacement e_a . Split it into a time leg and a space leg,

$$e_a = \tau + (e_a - \tau), \quad \tau = \frac{1}{4}(1, 1, 1, 1).$$

The time leg adds one tick of the rank clock and the space leg adds none. The two legs are orthogonal for every κ , with $q_\kappa(\tau) = \frac{1}{4} - \kappa$ and $q_\kappa(e_a - \tau) = \frac{3}{4}$. The posting's squared speed is space over minus time.

Theorem 2.11 (κ is the posting speed). *For every interval λq_κ with $\lambda \neq 0$ the squared speed of one posting is*

$$v^2 = \frac{3}{4\kappa - 1}, \quad \text{equivalently} \quad \kappa = \frac{3 + v^2}{4v^2}.$$

So $\kappa = 1$ exactly when $v = c = 1$, and $\kappa = 5/4$ is $v^2 = 3/4$. A history that posts $n_a \geq 0$ times on account a (not all zero) moves no faster than one posting, for $\kappa > 1/4$, and a single posting attains the bound. The family member q_κ makes every history causal exactly when $\kappa \geq 1$. For $1/4 < \kappa < 1$ a posting is faster than light and no history is causal.

Reading. The parameter of the lattice is a speed. The causal range is $\kappa \geq 1$, which is $v \leq c$. The theorems of this paper stated for $\kappa > 1/2$ include the range $1/2 < \kappa < 1$, where the simplices are still Lorentzian but a posting would outrun light; that range is not physical, and we keep it only because some geometric statements do not need causality. Figure 4 plots the relation.

Which speed does the ledger have? Two readings appear in the work on this lattice. In one, a posting reaches one spatial unit per tick, so it moves at c . In the other, the spatial unit is set by the ledger's cost Hessian at unit coefficient and the time leg lasts one tick; we call that the unit-Hessian reading. The record lattice helps to see the difference. The three moves $e_0 + e_i - 2\tau$ ($i = 1, 2, 3$) are orthogonal and of equal length for every κ ; they span a cube of which each posting's space leg is half a body diagonal; and crossing one edge of that cube takes two postings.

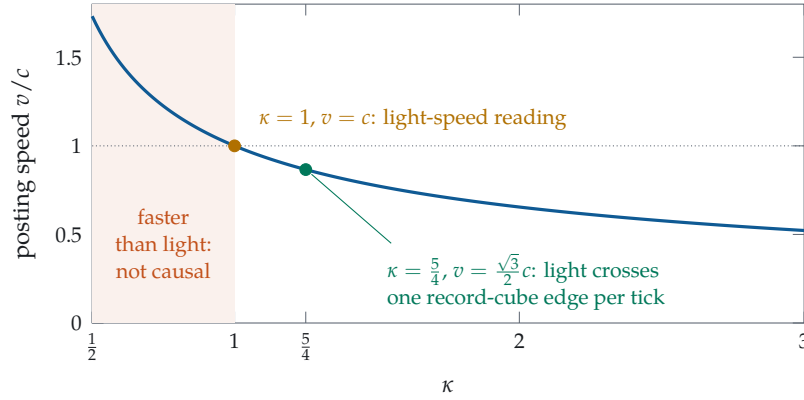


Figure 4. The posting speed v against κ , from Theorem 2.11: $v^2 = 3/(4\kappa - 1)$. The causal range is $\kappa \geq 1$. Below it (shaded) a posting would be faster than light. The two readings of Theorem 2.12 are marked: $\kappa = 1$, where a posting moves at c , and $\kappa = 5/4$, where light crosses one record-cube edge per tick and a posting moves at $\frac{\sqrt{3}}{2}c$. The cell's own premises allow every $\kappa \geq 1$.

Theorem 2.12 (Two readings, and what the cell leaves free). *Let G be name-blind, so $q_G = \lambda q_\kappa$ for some λ, κ .*

- (i) *If each posting reaches one voxel per tick (the posting is null in native units), then $\kappa = 1$.*
- (ii) *If space is the cost Hessian at unit coefficient and the time leg lasts one tick, then $\kappa = 5/4$ and $\lambda = 1$. Equivalently, light crosses one record-cube edge per tick. The two readings exclude each other.*
- (iii) *The premises the cell supplies without a light clause or a Hessian unit (name-blindness, a causal reading with the rank as time, space proportional to the cost Hessian, a time leg of one tick, the accounts at the corners of a regular tetrahedron) are met by a reading at every $\kappa \geq 1$, and by exactly one $\kappa \geq 1$ for each such interval. Readings at different κ are not proportional. So these premises fix everything except the posting speed, which ranges over $0 < v \leq c$.*

A natural way to settle the speed is to put light on the cell as a field whose signals travel by postings. That cannot work for a causal cell.

Theorem 2.13 (Light on the ledger cell). *Call a field law posting-local when the field at a record is computed from the field at records that precede it by at least one posting. For the response of such a law to a point source with a quiet past, every direction the field reaches has nonnegative counts, so it moves no faster than one posting. In a family reading the light cone lies inside the cone of histories exactly when $\kappa \leq 1/3$, and then a posting moves at $v \geq 3c$. Hence no name-blind causal cell has its light cone inside the cone of histories, and no posting-local field reaches every light direction of such a cell. The narrowest name-blind cone that contains every history is the one at $\kappa = 1$, and for a reading with $\kappa \geq 1$, some history is light-like exactly when $\kappa = 1$. Finally, if every posting is sent to one elementary move of a cubic carrier whose time is the rank, the map erases all of space; and no single posting crosses a record-cube edge.*

Reading. A photon of a causal cell is not a field carried by postings; postings carry light only along the four posting rays, and only when $\kappa = 1$. What decides κ is a statement about light and the ledger together: whether the speed of light is the ledger's own top speed. Both the light-speed reading and the unit-Hessian reading satisfy all the other cell premises and the one-step cone bound, so neither the premises nor the bound decide it.

Problem 2.14 (Light moves at the ledger's top speed). Derive from ledger dynamics that some history of postings is light-like, $q_G(n) = 0$ for some nonzero n with $n_a \geq 0$. For a causal reading, by [Theorems 2.11](#) and [2.13](#), this is equivalent to $\kappa = 1$ and to $v = c$.

In Recognition Science the posting speed is to be derived; it is not a dial. Until it is, we prove the main results at both readings. The open cone $\kappa > 1$ contains $\kappa = 5/4$. The light-speed reading $\kappa = 1$ is the null stratum of [Theorem 2.5](#), and it has its own measure ([Theorem 4.12](#)) and its own finite sum ([Theorem 5.13](#)).

3 The action

The action is the standard Lorentzian Regge action [[13](#), [16](#)], written on the fixed graph. This section proves what it is on the configuration space, why it and not a sinh-deficit variant, where its Lorentzian sign comes from, and what happens to it at the boundary of the configuration space.

3.1 The Regge action and its first variation

A slot $s = (\sigma, t)$ is a Kuhn simplex σ with one of its ten triangles t marked; there are $240N^4$ of them. Let n_s be the number of slots whose triangle has the same three vertices as t , $A_s = \frac{1}{4}\sqrt{-H_t}$ the area of t , and θ_s the dihedral angle of σ at t ([Theorem 2.3](#)). The action is

$$S(\ell) = \sum_s A_s(\ell) \left(\frac{2\pi}{n_s} - \theta_s(\ell) \right). \quad (2)$$

The bracket is the slot's share of the deficit: the shares of the slots on a triangle add up to its deficit $\varepsilon_t = 2\pi - \sum \theta_s$. On a torus of side $N \geq 3$ a pair of vertices determines its edge, so all slots of a triangle read the same three edges and one area, and the sum is exactly Regge's $\sum_t A_t \varepsilon_t$ ([Theorem 3.1\(vi\)](#)). On the side-1 torus two slots of one simplex name the same corners and read different areas, and the regrouping fails. We use (2) on all of $\mathbb{R}^{\mathcal{E}_N}$, with the conventions $\sqrt{x} = 0$ for $x < 0$, $x/0 = 0$ and arccos valued in $[0, \pi]$; they matter only on the boundary and outside the configuration space, and every theorem below says where it holds.

Theorem 3.1 (The action on the configuration space). For every N and every κ , on Adm_κ :

- (i) S is of class C^n for every n , including C^∞ and real analyticity.
- (ii) $S(c\ell) = cS(\ell)$ for every $c > 0$ and every field ℓ , and Adm_κ is a cone.
- (iii) Every area is a positive real number, $A_s = \frac{1}{4}\sqrt{-H_t} > 0$.
- (iv) (Schläfli.) $\sum_s A_s d\theta_s(\ell) \cdot h = 0$ for every direction h .

- (v) (First variation.) $dS(\ell) \cdot h = \sum_s (2\pi/n_s - \theta_s(\ell)) dA_s(\ell) \cdot h$, and S is stationary at ℓ exactly when this vanishes for every $h \in \mathbb{R}^{\mathcal{E}_N}$.
- (vi) (Hinge sum.) For $N \geq 3$ and every field, $S = \sum_t A_t \varepsilon_t$, the sum over the triangles of the lattice, with A_t the area of t and ε_t its deficit.

Mechanism. On Adm_κ every square root, quotient and arccosine in (2) is evaluated strictly inside its analytic domain (Theorem 2.3), so (i) holds. Areas scale as $\sqrt{c^2} = c$ and angles are invariant, which gives (ii). The product rule splits dS into an area term and an angle term $\sum_s A_s d(2\pi/n_s - \theta_s)$. The angle term vanishes by the Lorentzian Schläfli identity [10, 15], applied in each simplex and summed, since every slot area of a Lorentzian nondegenerate simplex is real. $\square$

Reading. The field equations of the sum are the Regge equations $\sum_s (\text{share})_s dA_s = 0$, area variation times deficit, at every admissible field and not only near flat space. The Schläfli cancellation is what makes this so. That is the reason for the choice of action, and the next theorem makes the reason sharp.

3.2 Why not a sinh deficit

Replace the deficit by a function of it: $S_f = \sum_t A_t f(\varepsilon_t)$. The variant $f = \sinh$ has been considered as a ledger-motivated alternative, and it is the test case.

Theorem 3.2 (Deficit functions). Let A_t, ε_t be differentiable functions on a normed space, indexed by a finite set, and f differentiable. Then

$$dS_f = \sum_t f(\varepsilon_t) dA_t + \sum_t A_t f'(\varepsilon_t) d\varepsilon_t.$$

If the Schläfli relation $\sum_t A_t d\varepsilon_t \cdot v = 0$ holds in a direction v , the extra term is unchanged when a constant is subtracted from f' , and it vanishes when f' is constant on the deficits. Conversely, the extra term $\sum_i A_i f'(\delta_i) v_i$ vanishes for every two-hinge datum with $A_1, A_2 > 0$ and $A_1 v_1 + A_2 v_2 = 0$ if and only if f' is constant. For $f = \sinh$ it does not vanish in general, since $\cosh$ is not constant.

On the lattice the obstruction is realized at an explicit configuration. Count each hinge once by weighting slot s with $1/n_s$: $S_f^{\text{lat}}(\ell) = \sum_s (A_s/n_s) f(\varepsilon_{t(s)})$, where $\varepsilon_{t(s)}$ is the deficit of the triangle of s .

Theorem 3.3 (The sinh action on the lattice). On the torus of side $N = 4$ there is a configuration $\ell_\star$ in the null stratum Adm_1 , with a positive deficit at one hinge, and a direction $v \in V_0$ along an affine family of null-stratum configurations through $\ell_\star$, such that

$$dS_{\sinh}^{\text{lat}}(\ell_\star) \cdot v - \sum_s \sinh(\varepsilon_{t(s)}) d\left(\frac{A_s}{n_s}\right)(\ell_\star) \cdot v < 0,$$

so the first variation of the sinh action is not the area form. For $f = \text{id}$ the same difference is zero:

the Regge member has the area form at the same point.

Mechanism. By [Theorem 3.2](#) the difference is the extra term $\sum_s (A_s/n_s) \cosh(\varepsilon_{t(s)}) d\varepsilon_{t(s)} \cdot v$. The configuration is one member of an explicit family of curved fields on the side-4 torus, all of whose simplices are Lorentzian nondegenerate, whose multi-posting edges are timelike and whose posting edges are null; the deficits, areas and their rates along the family are known in closed form from companion work, where the extra term is computed exactly and found negative. Its Schläfli counterpart, the same sum with $\cosh$ replaced by 1, is zero. $\square$

Reading. The $\sinh$ action's field equations are not “area times $\sinh$ of the deficit” on genuine curved Kuhn data. The criterion is the selection rule itself: by [Theorem 3.2](#) only an f with constant f' passes it, so the test selects the Regge action rather than supplying independent physics. The lattice instance shows that the rule has force on curved data. Its witness lies on the null stratum $\kappa = 1$, in a direction tangent to it; a witness inside the open sector $\kappa > 1$ has not been constructed.

3.3 Where the Lorentzian sign comes from

The action is evaluated on the timelike-hinge branch $\sqrt{-H}$. That presupposes an interval whose time direction is negative. Three different things go into that, and we prove each at its own scope. Return to one record cell. An interval is a symmetric matrix G acting on record displacements through $q_G(x) = x^T G x$, and a time functional is a linear map τ on displacements. Call (G, τ) a *causal reading* of the cell when every history step x has $\tau(x) > 0$ and $q_G(x) \leq 0$, when $q_G(x) > 0$ for every nonzero x with $\tau(x) = 0$, and when $q_G(x) < 0$ for some x . Call G *name-blind* when $G_{\pi a \pi b} = G_{ab}$ for every permutation π of the four accounts.

Theorem 3.4 (What the tick order fixes, and what it does not). *Let T be a tick clock on the cell and τ a linear time functional with $\tau(v) = T(v)$ for every record v .*

- (i) *(The time axis.) $\tau(x) = x_0 + x_1 + x_2 + x_3$: time runs along the body diagonal. No interval enters.*
- (ii) *(The causal range, conditional.) If moreover G is name-blind and (G, τ) is a causal reading, then $q_G = \lambda q_\kappa$ with $\lambda > 0$ and $\kappa \geq 1$; the reversed interval $-G$ is not a causal reading; the Kuhn edge class with m postings has squared length $\lambda m(1 - \kappa m)$; and $\lambda \ell_\kappa^0$ lies in Adm_κ with $S(\lambda \ell_\kappa^0) = \lambda S(\ell_\kappa^0)$. The member q_κ is a causal reading exactly when $\kappa \geq 1$, and the rank clock with q_κ , $\kappa \geq 1$, meets every premise.*
- (iii) *(The sign is in the premise.) A causal reading contains, as two of its clauses, a timelike direction and a positive form on the kernel of τ . Name-blindness and the tick order alone admit the Euclidean interval q_0 , which is positive definite; only the clause that history steps are causal excludes it.*
- (iv) *(Below $\kappa = 1$ the premise fails.) For $\kappa < 1$ every posting edge of every admissible configuration is spacelike, and no interval that reads the background lattice is a causal reading.*
- (v) *(Inside Adm_κ the form is forced.) For every κ and every admissible ℓ : every simplex has*

signature $(3, 1)$ ([Theorem 2.4](#)), the time edge of four postings is timelike, every Lorentzian area $\frac{1}{4}\sqrt{-H}$ is positive, and every Euclidean area $\frac{1}{4}\sqrt{H}$ is zero, so the Euclidean Regge action vanishes identically on Adm_κ .

Reading. The tick order derives the time axis. Given two premises, name-blindness and a causal reading of the cell, it derives the causal range $\kappa \geq 1$ and fixes the overall sign. The minus sign itself is supplied by the premise that history steps are causal; it is not derived from the ledger. Once the admissible set is chosen, the Lorentzian form is the only real Regge form that sees the geometry. So the sign is conditional, the range is $\kappa \geq 1$, and the signature on the configuration space is proved.

3.4 The phase

The weight is $w_a(\ell) = e^{iaS(\ell)}$ with a real phase scale a , which plays the role of $1/\hbar$ in lattice units.

Proposition 3.5 (The action does not fix the phase scale). For $c > 0$, $w_a(c\ell) = w_{ac}(\ell)$. For any two positive phase scales a, a' the rescaling $c = a'/a$ maps Adm_κ into itself and satisfies $w_{a'}(\ell) = w_a(c\ell)$ for every ℓ . The native candidate is $a = s/\hbar$ with s the posting action scale of the companion paper, and with squared lengths in units of λ_{rec}^2 it is $a \lambda_{\text{rec}}^2 = \ln 2 / (2\pi)$ per unit of Regge action.

Problem 3.6 (Native phase scale). Construct the path-sum amplitude from the ledger and prove that on Adm_κ it equals $e^{i(s/\hbar)S(\ell)}$.

A change of phase scale is a change of length unit, so the action alone cannot fix a . The candidate value is stated; that the ledger produces it is open.

3.5 The action at the boundary

Inside Adm_κ every angle is real. At the boundary a hinge can turn null, and in Lorentzian Regge calculus that is where complex angles appear [3, 6, 17]. We first show what the real action does as a configuration reaches the boundary from inside.

Theorem 3.7 (Continuity up to the boundary). Let ℓ_0 be a point of the closure of Adm_κ at which every Kuhn simplex has $G \neq 0$ or nonzero determinants on all five tetrahedra. Then S is continuous at ℓ_0 , as a function on all of $\mathbb{R}^{\mathcal{E}_N}$. This covers any number of null hinges, degenerate simplices with nondegenerate tetrahedra, and null edges. For every κ the boundary of Adm_κ has Lebesgue measure zero, so the closure exceeds Adm_κ by a null set and the cutoff integrals of [Section 5](#) are the same over Adm_κ and over its closure.

Mechanism. A slot angle is an arccosine, so it lies in $[0, \pi]$ at every field and never diverges. At a slot whose triangle is non-null at ℓ_0 the angle is continuous, because there the two apex determinants have positive product (from $G \neq 0$ by Desnanot–Jacobi, or directly from the

tetrahedra). At a slot whose triangle is null the area $\frac{1}{4}\sqrt{-H}$ tends to zero and multiplies a share bounded by 2π , so that slot's term tends to zero whatever value in $[0, \pi]$ the angle takes. The boundary lies in a finite union of zero sets of polynomials, each nonzero at ℓ_2^0 , and such zero sets are Lebesgue null; at $\kappa = 1$ the whole closure lies in a hyperplane. $\square$

The angle itself is not continuous at a null hinge. The next theorem says how it behaves, using the identity of Desnanot and Jacobi for the triangle t of a simplex: with P_3, P_4 the determinants of the two tetrahedra on t and D their mixed determinant, $4P_3P_4 - D^2 = HG$.

Theorem 3.8 (Angles at a null hinge). *On a Lorentzian nondegenerate simplex the dihedral angle at t is $\theta = \pi - \arccos(D/(2\sqrt{P_3P_4}))$; the arccosine form does not divide by H .*

- (i) *At a null triangle ($H = 0$) whose two tetrahedra have $P_3P_4 > 0$, the arccosine form gives exactly $\theta = 0$ or $\theta = \pi$.*
- (ii) *On a Lorentzian nondegenerate simplex, $\sin^2 \theta \cdot P_3P_4 = 4A^2(-G)$, so near a null hinge the angle leaves its limit 0 or π at the rate $\sqrt{-H}$ of the area.*
- (iii) *If at ℓ_0 every slot has $P_3P_4 > 0$, then approaching ℓ_0 from inside Adm_κ every deficit converges; at a null hinge each limiting angle is 0 or π , so the limiting deficit is $\delta_0 = (2 - k)\pi$, with k the number of slot angles tending to π .*
- (iv) *In the closure of Adm_κ a tetrahedron cannot flatten alone: if a tetrahedron on the triangle t has $P = 0$, then $H_tG = 0$, so t is null or the simplex is degenerate.*
- (v) *(The point $\ell_{1/2}^0$.) Along the flat family $s \downarrow 1/2$, inside Adm_κ for every $\kappa < 1$, the span-2 triangles have area $\frac{1}{2}\sqrt{2s-1}$. The dihedral angle $\alpha(s)$ at the end triangles $v_0v_1v_2$ and $v_2v_3v_4$ satisfies $\sin^2 \alpha(s) = (2s-1)/(2(3s-1))$ and tends to 0; the angle $\beta(s) = \arccos(-s/(3s-1))$ at the middle triangle $v_1v_2v_3$ tends to π . The slot angle at an end triangle has no continuous extension to $\ell_{1/2}^0$, while S is continuous there. For $N \geq 3$ every limiting deficit at $\ell_{1/2}^0$ is zero, exactly two angles of each span-2 hinge tending to π , and $S(\ell_{1/2}^0) = 0$.*
- (vi) *(The point ℓ_1^0 .) For $\kappa > 1$, ℓ_1^0 is a boundary point of Adm_κ at which only edges are null: every simplex is still Lorentzian nondegenerate, the flat family crosses the boundary there, and S is C^∞ on an open set containing both sides.*

Reading. A null hinge is a square-root branch point of the angle, but the real action does not see it: the angle leaves 0 or π at the same rate at which the area that multiplies it vanishes. A null edge is not a branch point at all. The explicit null hinges of $\ell_{1/2}^0$ are regular light cones, with zero limiting deficit. Figure 5(a) shows the approach.

3.6 Crossing a null hinge

Past a null hinge the triangle is spacelike ($H > 0$) and the action has to be continued. Let H travel from $-r$ to r ($r > 0$) along a half circle in the half-plane $\varepsilon \text{Im } H > 0$, $\varepsilon = \pm 1$. This is the $i\varepsilon$ prescription.

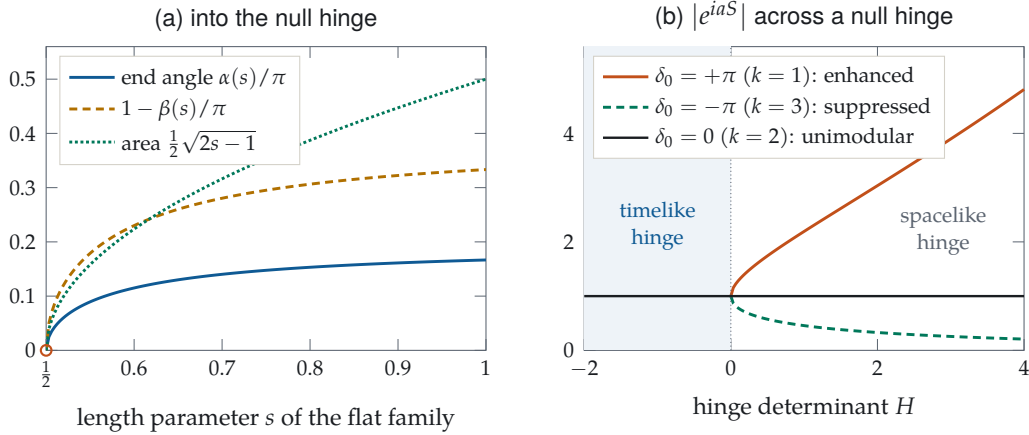


Figure 5. (a) The flat family ℓ_s^0 as $s \downarrow 1/2$, from the closed forms of [Theorem 3.8\(v\)](#): the end angle $\alpha(s)$ of a span-2 hinge and its area both vanish like $\sqrt{s - 1/2}$, and the middle angle $\beta(s)$ tends to π . At $s = 1/2$ (circle) the angle formula is $0/0$. (b) The modulus of the weight across a null hinge from the normal form of [Theorem 3.9](#), for $a = 1$ and prescription $\varepsilon = +1$, as a function of the hinge determinant H . On the timelike side ($H < 0$, shaded) the weight is unimodular. Past the hinge it is $e^{\varepsilon a \delta_0 \sqrt{H}/4}$; for $\varepsilon = -1$ the two non-flat curves exchange.

Theorem 3.9 (Continuation and the sign of the weight). (i) Every continuous function g on the path with $(4g)^2 = -H$ that starts at the real area $\sqrt{r}/4$ ends at $-\varepsilon i \sqrt{r}/4$. The principal branch of $\sqrt{-H}$ at $H = r$ gives $+i \sqrt{r}$, the $\varepsilon = -1$ continuation.

(ii) On a Lorentzian nondegenerate simplex the supplement $\pi - \theta$ of the dihedral angle is $\arcsin(4Ag)$ when it is at most $\pi/2$ and $\pi - \arcsin(4Ag)$ when it is at least $\pi/2$, with $g = \sqrt{-G}/(2\sqrt{P_3 P_4})$. So near a null hinge a slot angle is $\theta^0 \pm \arcsin(ug)$, $\theta^0 \in \{0, \pi\}$, $u = 4A = \sqrt{-H}$. Continued to $u = -\varepsilon i \sqrt{r}$, the deviation becomes $-\varepsilon i \operatorname{arsinh}(\sqrt{r} g)$, whose sine is the continued ug and whose cosine $\sqrt{1 + rg^2}$ exceeds 1. Its product with the continued area is real.

(iii) (Normal form.) Suppose a hinge term continues as $F(u) = (\delta_0/4)u + E(u)$ with E even and real on the real axis. Then at the far end $\operatorname{Im} F = -\varepsilon \delta_0 \sqrt{r}/4$ and $|e^{iaF}| = e^{\varepsilon a \delta_0 \sqrt{r}/4}$. For $a > 0$: if $\delta_0 = 0$ the weight is unimodular for both prescriptions; for $\varepsilon = +1$ it is suppressed exactly when $\delta_0 < 0$ and enhanced exactly when $\delta_0 > 0$; for $\varepsilon = -1$ the roles are exchanged.

Reading. Imaginary areas and imaginary angles cancel in the angle part of each slot term, which past the hinge is the real boost-angle term of a spacelike hinge. Only the deficit term contributes an imaginary part, so whether the weight is suppressed or enhanced past a null hinge is decided by the sign of $\varepsilon a \delta_0$: an irregular light cone ($k \neq 2$) is suppressed for one prescription and enhanced for the other. The action does not choose ε ; the prescription is a stated parameter, not a derived one. The regular light cones of $\ell_{1/2}^0$ produce neither effect.

Problem 3.10 (The continued action). Construct one complex action S_c equal to S on Adm_κ such that near every field whose tetrahedra and simplices all have negative determinants, S_c is a real-analytic real part plus fixed coefficients times the ε branches of the slot areas.

Theorem 3.9(iii) is proved hinge by hinge in one variable; Problem 3.10 is its lattice form. Its consequence is already proved: in any normal form of that kind the weight has modulus $\exp(\varepsilon a \sum_s c_s \sqrt{H_s}/4)$, the sum running over spacelike slots. Its hypotheses are met, for instance at ℓ_2^0 .

4 The measure

A measure on squared lengths could simply be chosen: Lebesgue measure, a DeWitt measure, a power of the simplex volumes. We build one from ledger states instead, and we say plainly what kind of object it is. It is a *model measure*: the chart that reads lengths from ledger states, the family of accounts, the size of the box of states and the weight on states are stated constructions. We prove how each of them changes the measure, so the reader can see which features are forced and which are chosen.

4.1 The ledger chart

The ledger has accounts: four axis accounts, one per posting direction, and a finite family X of further accounts. A ledger state says how the accounts move.

Definition 4.1 (Ledger states and the chart). Let $K = \{1, 2, 3, 4\} \sqcup X$. A *ledger state* is a pair $\sigma = (P, \zeta)$, where $P_a \in \mathbb{R}^K$ is the move of the accounts made by one posting on axis a , and $\zeta_x \in \mathbb{R}^K$ is a shift of the accounts at vertex x . The account move along the edge $e = (x, d)$ is

$$v_e(\sigma)_k = \sum_a d_a P_{a,k} + \zeta_{x+d,k} - \zeta_{x,k} \quad (k \in K),$$

and the *ledger chart* reads the squared length

$$\Phi_\kappa(\sigma)_e = \sum_{k \in K} v_e(\sigma)_k^2 - \kappa \left(\sum_{k \in K} v_e(\sigma)_k \right)^2.$$

The *balance* σ_{bal} has $P_{a,k} = 1$ if k is the axis account a and 0 otherwise, and $\zeta = 0$. The *ledger box* of radius R is $B_R = \{\sigma : \|\sigma - \sigma_{\text{bal}}\|_\infty \leq R\}$, and the *ledger measure* is

$$\mu_{\kappa, X, R} = (\Phi_\kappa)_* (\text{Leb}|_{B_R \cap \Phi_\kappa^{-1}(\text{Adm}_\kappa)}).$$

The first term of Φ_κ is the second derivative of the ledger's cost along the account move, since the cost $J(e^u) = \cosh u - 1$ has second derivative 1 at balance. The second is κ times the square of the total amount moved, which at balance is the number of postings. At balance the chart gives the flat lattice: $v_e(\sigma_{\text{bal}}) = d$, so $\Phi_\kappa(\sigma_{\text{bal}})_e = |d| - \kappa|d|^2 = \ell_\kappa^0(e)$. The chart is a modelling reading; it is not derived here.

Theorem 4.2 (The chart). $\Phi_\kappa(\sigma) = q(\sigma, \sigma)$ for an explicit symmetric bilinear form q , so Φ_κ is smooth and $d\Phi_\kappa(\sigma) = 2q(\sigma, \cdot)$ is linear in σ ; and $\Phi_\kappa(\sigma_{\text{bal}}) = \ell_\kappa^0$. Let $N \geq 4$ and $\kappa \neq 1/4$. There is a finite account family $X_\star$, assembled from one construction for each of the $15N^4$ single-edge directions in two copies shifted oppositely, and a state $\sigma_\star(\epsilon) = \sigma_{\text{bal}} + \epsilon \zeta_\star$, such that for every $\epsilon \neq 0$ the derivative $d\Phi_\kappa(\sigma_\star(\epsilon))$ is onto $\mathbb{R}^{\mathcal{E}_N}$. If also $\kappa > 1/2$ and $\kappa \neq 1$, then for every $R > 0$ some such state lies within R of σ_{bal} and reads an admissible configuration.

Reading. The family $X_\star$ is one witness of an existence theorem. It depends on κ , and nothing singles it out. What matters is the property it has: the chart is a submersion somewhere. Call a finite family *admissible* when $d\Phi_\kappa$ is onto $\mathbb{R}^{\mathcal{E}_N}$ at some ledger state. The next theorems are stated for that class, so that nothing depends on the particular witness.

4.2 What the measure depends on

Theorem 4.3 (The ledger measure, and its dependence on the family and the box). Let $\kappa > 1/2$, $\kappa \neq 1$.

- (i) For every family X and every R , $\mu_{\kappa, X, R}$ is finite and gives no mass outside Adm_κ ; for $R > 0$ it is not zero.
- (ii) For every admissible family and every R , $\mu_{\kappa, X, R}$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^{\mathcal{E}_N}$. The family $X_\star$ is admissible.
- (iii) An open set has positive mass exactly when it contains the reading of an admissible state of the box, and a measurable set that no admissible reading of the box meets has none.

Mechanism. Nonzero: the balance reads the flat field, which is admissible, and nearby states of the box form an open set. For absolute continuity pick a state where $d\Phi_\kappa$ is onto and write $\Phi_\kappa = T \circ G$, with T that derivative, ι a linear right inverse, and $G(\sigma) = \iota\Phi_\kappa(\sigma) + \sigma - \iota T\sigma$. Because $d\Phi_\kappa$ is linear in its base point, along every ray from the regular state the derivative of G has the form $1 + tM$. Its determinant is a polynomial in t with constant term 1, so it vanishes at finitely many t , and in polar coordinates the critical set of G is Lebesgue null. Off it the inverse function theorem and the change of variables formula show that G pulls null sets back to null sets, and so does the surjection T . $\square$

Theorem 4.4 (The choices change the measure). Let $\kappa > 1/2$, $\kappa \neq 1$.

- (i) (Too few accounts.) If the space of ledger states has smaller dimension than $\mathbb{R}^{\mathcal{E}_N}$, the image of Φ_κ is Lebesgue null. With no further accounts this happens on every torus of side $N \geq 2$, since $16 + 4N^4 < 15N^4$: that measure is nonzero, carried by a null set, and so not absolutely continuous with respect to the measure of $X_\star$ ($N \geq 4$, $R > 0$).
- (ii) (The box radius.) The measure grows with R . For every family there is $R_0 > 0$ such that for $R \leq R_0$ and $R' > 1$ the radius- R' measure is not absolutely continuous with respect to the radius- R measure: the small box reads only fields near the flat lattice, the large box reads fields far from it.

- (iii) (Scale.) $\Phi_\kappa(c\sigma) = c^2\Phi_\kappa(\sigma)$, and dilating edge space by c^2 carries the measure of the box of radius R about σ_0 to c^{-d} times the measure of the box of radius cR about $c\sigma_0$, with d the dimension of ledger space. The box about the balance has no exact scaling law in R .

Problem 4.5 (Admissible families). Show that any two admissible families give mutually absolutely continuous ledger measures at the same κ and R , or find two that do not.

Reading. The density is a property of a class of families, not of the witness. Within that class whether the measures agree up to null sets is open, and outside it they provably do not. The box radius is not a normalization convention: it changes which geometries are summed over.

4.3 The kernel at the balance, and no Faddeev–Popov factor

Regge calculus has a gauge redundancy near flat space: moving the vertices of a flat lattice inside flat space changes the edge lengths and not the geometry [14]. Away from flat space the redundancy is only approximate [4]. A vertex move displaces each vertex x by ξ_x inside the flat metric g_κ ; to first order it changes the squared length of (x, d) by $\delta(\xi^b)_{(x,d)} = 2\sum_a d_a(\xi_{x+d,a}^b - \xi_{x,a}^b)$, with $\xi^b = g_\kappa \xi$. The vertex moves span a subspace $\mathcal{V}$, and the constant strains, $\ell_{(x,d)} = d^T B d$, span another. The ledger has a translation for every vertex displacement: $g(\xi)$ shifts the four axis accounts at vertex x by ξ_x .

Theorem 4.6 (The derivative of the chart at the balance). Let $\kappa \neq 1/4$ and let X be any family.

- (i) (Range.) The further accounts enter $d\Phi_\kappa(\sigma_{\text{bal}})$ only through their totals. Its range is the span of the constant strains and the vertex moves, and the ledger translations are carried onto the vertex moves exactly: $d\Phi_\kappa(\sigma_{\text{bal}}) g(\xi) = \delta(\xi^b)$.
- (ii) (Rigidity.) A vertex move of the flat periodic lattice changes no squared length to first order only when it is a constant translation, so the vertex moves have a four-dimensional kernel.
- (iii) (Kernel.) $\dim \ker d\Phi_\kappa(\sigma_{\text{bal}}) = (4 + N^4) |X| + 10$: the further-account directions with their axis parts adjusted, six antisymmetric strain directions, and the four constant translations. For $N \geq 3$ the ledger translation that moves a single vertex is not in the kernel; it moves the reading to a displaced flat lattice.
- (iv) (Exact symmetries.) The chart is unchanged by every linear map of account space that preserves the sum of squares and the total, applied to every move and every shift; by a uniform shift of every account; and by $\sigma \mapsto -\sigma$. None of these is a vertex move.
- (v) (The orbit of the balance.) $\Phi_\kappa(\sigma_{\text{bal}} + g(\xi))$ is exactly the displaced flat lattice: on the edge (x, d) it is the squared length of $d + \xi_{x+d} - \xi_x$ in the metric g_κ .

The redundancy of the chart is its kernel and its fibres, and (iii) gives the kernel exactly. The vertex moves are not part of it. They are the expected redundancy of the action near flat space, not a redundancy of the chart, so the question is whether the integrand is constant

along the ledger translations.

Theorem 4.7 (No Faddeev–Popov factor). *Let $N \geq 3$ and $\kappa > 1/2$.*

- (i) *The flat field is a critical point of the action, $dS(\ell_\kappa^0) = 0$.*
- (ii) *Along every line $t \mapsto \sigma_{\text{bal}} + t g(\xi)$ the function $S(\Phi_\kappa(\cdot))$ has vanishing first and second derivatives at $t = 0$; the second uses the lattice Bianchi identity, that vertex moves lie in the radical of the flat Hessian.*
- (iii) *For $\kappa \neq 1$ and $R \geq 1$ the admissibility cut is not invariant under ledger translations: moving one vertex onto its axis neighbour takes the balance to a state in the box whose reading has a null posting edge. So the integrand $1_{B_R} 1_{\text{Adm}_\kappa}(\Phi_\kappa) e^{iaS(\Phi_\kappa)}$ is not constant on the translation orbits that meet the box, and no orbit volume factors out.*
- (iv) *Lebesgue measure on ledger states is invariant under every translation, and it disintegrates along the orbits: a property holds almost everywhere exactly when it holds at almost every point of almost every orbit. This is Fubini’s theorem; it makes no claim about the integrand.*
- (v) *Away from the balance, $\|d\Phi_\kappa(\sigma)g(\xi) - \delta(\xi^b)\| \leq \|d^2\Phi_\kappa\| \|\sigma - \sigma_{\text{bal}}\| \|g(\xi)\|$ in sup norms.*

Problem 4.8 (The action on the displaced flat lattice). Show that S is constant on the displaced flat lattices $\Phi_\kappa(\sigma_{\text{bal}} + g(\xi))$ for ξ near zero. Theorem 4.7(ii) is its infinitesimal form.

Reading. The ledger measure has no Faddeev–Popov factor. Its exact redundancy is the kernel of the chart, of known dimension. Its integrand is invariant along ledger translations only to second order at the balance, and the cut breaks invariance outright for boxes of radius at least one. Figure 6 sketches the picture.

4.4 Refinement and restriction

Theorem 4.9 (Refinement and restriction). (i) (Changing graphs.) *For every complex bound $B \geq 19$ a stellar $1 \rightarrow 5$ subdivision changes both the inverse-automorphism weight and the gauge-volume weight on bounded simplicial complexes, so neither is refinement invariant; constant weights are.*

- (ii) (Scale on the fixed graph.) *A finite measure on $\mathbb{R}^{\mathcal{E}_N}$ with no atom at zero that is invariant under every dilation $\ell \mapsto c\ell$ is zero; for $N \geq 4$, $\kappa > 1/2$, $\kappa \neq 1$, $R > 0$ the ledger measure of $X_\star$ is not dilation invariant.*
- (iii) (Restriction before the cut.) *Lebesgue measure on the box is a product over vertices, and an edge inside a region reads only the moves and the region’s shifts. So the law of the edges inside a region, under the box measure, is the region’s own box measure pushed through the region’s chart, times the box volume of the vertices outside.*

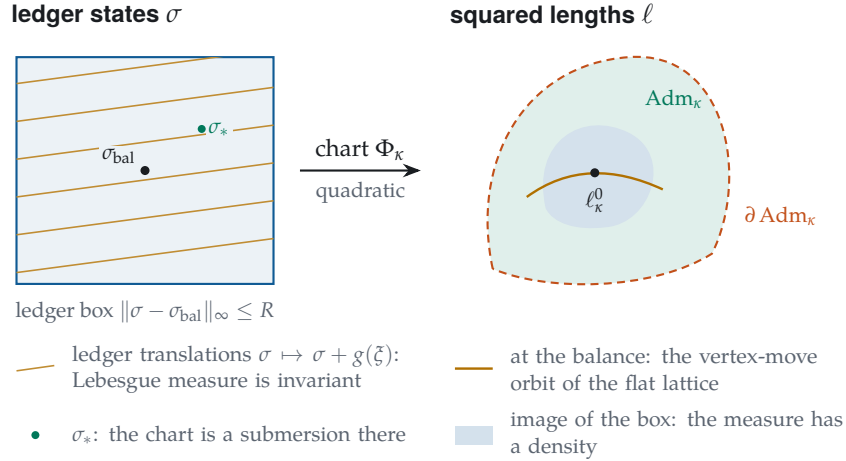


Figure 6. Schematic of Definition 4.1 and Theorems 4.2, 4.3, 4.6 and 4.7. Lebesgue measure on the ledger box is pushed through the quadratic chart. Ledger translations (amber lines) leave Lebesgue measure invariant. At the balance they map onto the vertex-move orbit of the flat lattice, which is not in the kernel of the chart. At an admissible regular state the chart is a submersion, which is why the image measure has a density. Shapes are illustrative.

- (iv) (Restriction after the cut.) Let $\kappa > 1/2$, $\kappa \neq 1$. The law of the region's edges under the cut measure is at most the region's own cut measure, the one that checks only the region's simplices and edges. For small boxes the two are equal. For $N \geq 3$, $R > 1$ and every region that misses a vertex they differ: the cut couples a region to the vertices outside it.

Reading. The refinement behaviour of the fixed-graph measure is not established. What replaces it is restriction to regions, which holds before the cut, holds after the cut for small boxes, and fails for large ones. Whether the two cut measures agree up to a constant factor is open.

4.5 The bare weight, and why it matters

Lebesgue measure on ledger states is the flat weight. The ledger has its own candidate: the J-cost Gibbs weight $w_\beta(\sigma) = \exp(-\beta \sum_{x,k} J(e^{\zeta_{x,k}}))$ of the record shifts. Write $\mu_{\kappa,X,R}^\beta$ for its path-sum measure, the pushforward of $w_\beta \cdot \text{Leb}$ cut to the box and to Adm_κ .

Theorem 4.10 (The weight matters). (i) (Haar selection.) Every measure on ledger states invariant under translation of the account amounts and finite on compact sets is $c \cdot \text{Leb}$, and its pushforward is $c \mu_{\kappa,X,R}$; after normalization c disappears. A nonzero finite measure on $\mathbb{R}^{\mathcal{E}_N}$ is never translation invariant, so the selecting symmetry cannot act on squared lengths.

(ii) (The ledger's own weight.) w_β is translation invariant exactly when $\beta = 0$.

(iii) (Same null sets.) For every β , family and box, $\mu_{\kappa,X,R}^\beta$ and $\mu_{\kappa,X,R}$ are mutually absolutely continuous. For $\beta \geq 0$, $\kappa > 1/2$, $\kappa \neq 1$ and $R > 0$ the Gibbs measure is finite and nonzero,

and for every bound on the phase scale its normalization is nonzero on small boxes.

- (iv) (Different expectations.) For $N \geq 2$, $\kappa > 1/2$, $\kappa \neq 1$, every admissible family and every $R > 0$, the normalized Gibbs measure is not independent of $\beta \geq 0$. At large β the weight concentrates on states with small shifts, which read fields near a proper subspace that the Lebesgue ledger measure barely sees.

Problem 4.11 (The bare weight). Determine from ledger dynamics which bare weight the path sum carries; in particular, whether $\beta = 0$.

Reading. The choice between the flat weight and the ledger's J-cost weight changes path-sum expectations, so it is a physical input, not a convention. Lebesgue measure on ledger states is the claim $\beta = 0$. It is selected by translation invariance, and translation invariance excludes the ledger's own cost weight. Which weight the ledger carries is open.

4.6 The light-speed reading

At $\kappa = 1$ the absolutely continuous ledger measure of X_* is zero, because the null stratum is Lebesgue null. The stratum needs its own measure.

Theorem 4.12 (A measure on the null stratum). Let $N \geq 4$ and $R > 0$. There is a finite account family, built by averaging X_* over the translations of the torus, and a finite measure ν on its ledger box whose reading $(\Phi_1)_*\nu$ is nonzero, gives no mass outside Adm_1 , and is absolutely continuous with respect to Haar measure of V_0 ; on a small ball near the flat field it equals that Haar measure exactly. With X_* itself the absolutely continuous ledger measure at $\kappa = 1$ is zero.

Mechanism. Averaging over translations makes the squared shifts along an edge a constant of its class, and a constant strain cancels that constant, so a curve of states reads null postings and leaves the flat field. On one copy the chart's derivative is the derivative of X_* at a shifted state, and the composite with a right inverse is $1 + \epsilon^2 M$, invertible for small ϵ off a finite set. The implicit function theorem gives a local section of the chart on the stratum, and ν is Haar measure of V_0 on a small ball pushed through it. $\square$

Reading. The measure at $\kappa = 1$ uses a constructed family and is one measure with the required properties, not a selected one. Whether X_* without averaging carries such a measure is open.

5 Finiteness at a stated regulator

The test is an either/or: the sum converges, or it is cut off at a stated scale. We take the second branch. The regulator is a bound on squared lengths, or the radius of the ledger box. It is stated, not derived, and we call it a regulator for that reason. The first branch is not closed either; we say exactly what is known about it in [Section 5.2](#).

5.1 Finite measures and the normalization

For a finite measure μ on $\mathbb{R}^{\mathcal{E}_N}$ and a phase scale a , the partition function and the expectation of an observable O are

$$Z_\mu(a) = \int e^{iaS(\ell)} d\mu(\ell), \quad \langle O \rangle_\mu(a) = \frac{1}{Z_\mu(a)} \int O(\ell) e^{iaS(\ell)} d\mu(\ell).$$

Since S is a Borel function on all of $\mathbb{R}^{\mathcal{E}_N}$ and the weight has modulus one, for every finite μ the weight and every bounded measurable observable times the weight are integrable, $|Z_\mu(a)| \leq \mu(\mathbb{R}^{\mathcal{E}_N})$, $Z_\mu(0) = \mu(\mathbb{R}^{\mathcal{E}_N})$, and Z_μ is continuous in a . Two regulated measures are used:

$$\mu_L = \text{Leb}|_{\text{Adm}_\kappa \cap \{\ell: |\ell_e| \leq L \text{ for all } e\}} \quad \text{and} \quad \mu_{\kappa, X_\star, R} \quad (\text{Definition 4.1}),$$

with the constant $C_N = 120\sqrt{13} \pi N^4$.

Theorem 5.1 (Action bound on the box). *For $L \geq 0$ and every field with $|\ell_e| \leq L$ for all e , degenerate simplices included, $|S(\ell)| \leq C_N L$. The same bound holds for the Euclidean Regge action.*

Proof. For a triangle with squared sides at most L in size, $-H = (a+b-c)^2 - 4ab \leq 9L^2 + 4L^2$, so its area is at most $\sqrt{13}L/4$. Every share $2\pi/n_s - \theta_s$ lies in $[-\pi, 2\pi]$. There are $240N^4$ slots, so $|S| \leq 240N^4 \cdot (\sqrt{13}L/4) \cdot 2\pi = C_N L$. $\square$

Reading. This is a bound on the box, not a solution of the spike problem: spikes in Regge sums [2] concern the measure as the cutoff is removed, which a fixed regulator does not address. On the box the weight cannot become singular.

Theorem 5.2 (The ledger measure normalizes). *Let $N \geq 4$, $\kappa > 1/2$, $\kappa \neq 1$.*

- (i) *For every phase scale a there is $R_0 > 0$ such that for every $0 < R \leq R_0$ the ledger measure has $Z(a) \neq 0$ and $\langle 1 \rangle = 1$; for every bound A one R_0 serves all $|a| \leq A$.*
- (ii) *For every $R > 0$, every state of the box reads squared lengths at most*

$$L(R) = (n + |\kappa|n^2)(4(1+R) + 2R)^2, \quad n = 4 + |X_\star|,$$

in size, and $Z(a) \neq 0$ throughout $|a| C_N L(R) < \pi/2$.

- (iii) *For every $R > 0$ the measure is finite and nonzero, the weight and bounded observables are integrable, and Z is continuous in a .*

Mechanism. For (i): the chart is continuous and sends the balance to ℓ_κ^0 , so a small box is read into a small ball around the flat field, and on that ball the action stays within an angle less than $\pi/4$ of its flat value. For (ii): on the box every frame posting is at most $1 + R$ and every shift at most R in size, so every account move along an edge is at most $4(1+R) + 2R$, and the interval of a move of size P is at most $(n + |\kappa|n^2)P^2$. Then Theorem 5.1 and the cosine argument below apply. $\square$

For the Lebesgue regulator the same cosine argument gives the window directly. It is the core of every window in this section.

Theorem 5.3 (The Lebesgue regulator). *Let $\kappa > 1/2$, $\kappa \neq 1$ and $L > 0$. Then μ_L is finite and nonzero, and for every a with $|a| C_N L < \pi/2$ we have $Z_{\mu_L}(a) \neq 0$ and $\langle 1 \rangle_{\mu_L}(a) = 1$.*

Proof. Nonzero: Adm_κ is an open cone containing ℓ_κ^0 , so it contains small multiples of a ball around the flat field, and those lie in the box. On the box $|S| \leq C_N L$, so $|aS| \leq \theta := |a| C_N L < \pi/2$ almost everywhere, and

$$\text{Re } Z_{\mu_L}(a) = \int \cos(aS(\ell)) d\mu_L(\ell) \geq \cos \theta \mu_L(\mathbb{R}^{\mathcal{E}_N}) > 0.$$

So $Z_{\mu_L}(a) \neq 0$ and $\langle 1 \rangle = Z/Z = 1$. $\square$

The windows of Theorems 5.2 and 5.3 are the regime in which the phase never wraps. They are too small for the physics. At the native phase scale of Proposition 3.5, $a = \ln 2 / (2\pi) \approx 0.110$ in units of $\lambda_{\text{rec}}^{-2}$, every cutoff box that contains the flat field lies outside the window: $|a| C_N L \geq \pi/2$ whenever L exceeds the size $4(4\kappa - 1)$ of the flat field. The window is not where the normalization stops making sense, and the next theorem shows why.

Theorem 5.4 (The normalization is entire in the phase scale). *Let μ be a nonzero finite measure with $|S| \leq M$ almost everywhere.*

- (i) $w \mapsto Z_\mu(w) = \int e^{iws} d\mu$ is complex differentiable at every $w \in \mathbb{C}$, it equals the partition function at real w , and $Z_\mu(0) = \mu(\mathbb{R}^{\mathcal{E}_N}) > 0$.
- (ii) Its real zeros form a closed discrete set, finite on every compact interval. Off that set $\langle O \rangle_\mu(a)$ is defined for every bounded measurable O , and $\langle 1 \rangle = 1$.
- (iii) Both regulated measures qualify: μ_L for every $L > 0$, and the ledger measure for every $R > 0$ ($N \geq 4$, $\kappa > 1/2$, $\kappa \neq 1$).
- (iv) (Scaling.) $Z_{\mu_L}(a) = L^D Z_{\mu_1}(aL)$ with $D = 15N^4$: only the product aL matters.
- (v) (The native scale.) The ledger measure has $Z \neq 0$ at $a = \ln 2 / (2\pi)$ for every small enough box; μ_L has $Z \neq 0$ there for every $L < \pi / (2aC_N)$; for every L , $Z_{\mu_L} \neq 0$ at every phase scale in a punctured neighbourhood of the native one; and the sizes $L > 0$ with $Z_{\mu_L} = 0$ at the native scale form a countable set with no accumulation point in $(0, \infty)$.

Mechanism. Differentiate under the integral sign: on the unit disc around w_0 the derivative $\int iS e^{iws} d\mu$ is dominated by $Me^{(|\text{Im } w_0| + 1)M}$, a constant, and a constant is integrable against a finite measure. An entire function that is not identically zero has isolated zeros, and $Z_\mu(0) > 0$. The scaling law is homogeneity of S and the cone property of Adm_κ . $\square$

Reading. At a stated regulator the sum is a finite-dimensional, absolutely convergent oscillatory integral, and its expectations are defined at every phase scale except a discrete set, the native scale included up to that set. Whether the normalization is nonzero at the native

scale for one particular cutoff above the flat field is not decided. The physical content of the regulator is the single number aL ; with the phase scale at its native value the remaining input is L in units of λ_{rec}^2 .

Problem 5.5 (A derived cutoff). Find a frame-posting bound F and a shift-cost budget B for the set of ledger states the physical ledger realizes. Every such state then reads squared lengths at most $(n + |\kappa|n^2)(4F + 2\sqrt{4B})^2$ in size, because $J(e^t) = \cosh t - 1 \geq t^2/4$ bounds each shift by $\sqrt{4B}$, and the regulator becomes a derived scale.

5.2 What is known without a regulator

Theorem 5.6 (The uncut sum). Let $\kappa > 1/2$, $\kappa \neq 1$, and let $D = 15N^4$.

- (i) Adm_κ has infinite Lebesgue measure, so e^{iaS} is not absolutely integrable on it for any phase scale.
- (ii) In polar coordinates Lebesgue measure is (sphere measure) $\times c^{D-1}dc$. For every field θ , phase scale a and power m , the truncated ray integral $\int_0^R c^m e^{iaS(c\theta)} dc$ has no limit as $R \rightarrow \infty$.
- (iii) (Abel regularization on rays.) For $\epsilon > 0$, $\int_0^\infty c^m e^{iaS(c\theta)} e^{-\epsilon c} dc = m! / (\epsilon - iaS(\theta))^{m+1}$ exactly. As $\epsilon \rightarrow 0^+$ it converges when $aS(\theta) \neq 0$ and diverges when $aS(\theta) = 0$. The flat ray has $S = 0$ for $N \geq 3$, so its Abel integral diverges.
- (iv) (The cone.) The damped cone integral $\int_{\text{Adm}_\kappa} e^{iaS(u)} e^{-\epsilon \|u\|} du$ equals $\epsilon^{-D} J(a/\epsilon)$ with $J(b) = \int_{\text{Adm}_\kappa} e^{ibS(u)} e^{-\|u\|} du$. So its limit as $\epsilon \rightarrow 0^+$ exists exactly when $\epsilon^{-D} J(a/\epsilon)$ converges. At $a = 0$ it does not.

Mechanism for (ii) and (iii). By homogeneity $S(c\theta) = cS(\theta)$ for $c \geq 0$, so with $\lambda = aS(\theta)$ the ray integral is $\int_0^R c^m e^{i\lambda c} dc$. When $\lambda = 0$ its real part is $R^{m+1}/(m+1)$. When $\lambda \neq 0$ its imaginary part gains at least $2/|\lambda|$ on every half-period past $c = 1$. With damping the integral is the Laplace transform of c^m at $\epsilon - i\lambda$. $\square$

Problem 5.7 (The Abel limit of the uncut sum). Decide whether the Abel-regularized cone integral converges as $\epsilon \rightarrow 0^+$ at a nonzero phase scale.

Reading. The uncut sum is not absolutely convergent, and its ray truncations do not converge for any power density. That does not close the first branch. Along every ray where the action is nonzero the Abel-regularized integral converges; any divergence must come from the angular integral near the zero set of S , which contains the flat ray. We expect divergence there, since S vanishes on the vertex-move orbit of the flat field and is quadratic transverse to it, but that is an inference and not a theorem. Figure 7 shows the plain ray integral.

5.3 The conformal factor

In Euclidean gravity the conformal mode makes the action unbounded below [8]. Its algebraic root is the DeWitt supermetric [5], $\mathcal{G}_g(h, k) = \text{tr}(g^{-1}hg^{-1}k) - \frac{1}{2} \text{tr}(g^{-1}h) \text{tr}(g^{-1}k)$ on

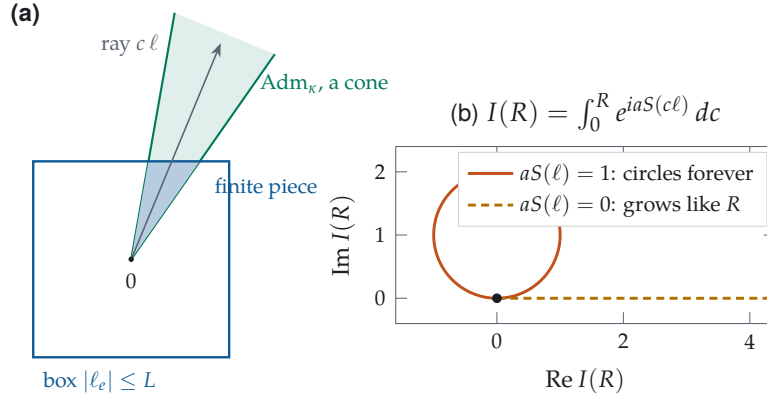


Figure 7. (a) Schematic: the configuration space is an open cone (Theorem 2.2); the regulator keeps the finite piece inside the box $|\ell_e| \leq L$. (b) Exact: the plain ray integral $\int_0^R e^{iaS(c\ell)} dc$ in the complex plane. For $aS(\ell) = 1$ it runs around the unit circle about i forever; for $aS(\ell) = 0$ it grows like R . With the true radial density c^{D-1} the truncations diverge too, while Abel damping makes the rays with $aS \neq 0$ converge (Theorem 5.6).

symmetric strains, which is the coefficient form of the gauge-fixed long-wavelength symbol of the Regge action: on the lattice that symbol is $-\frac{1}{4}\sqrt{|\det g|} q(u) \mathcal{G}_g$, with $q(u)$ the wave symbol [18]. Its sign structure can be computed exactly on the lattice backgrounds $g = I - \kappa J$.

Theorem 5.8 (The DeWitt index). *On $g = I - \kappa J$ the conformal strain $h = g$ has $\mathcal{G}_g(g, g) = -4$. For every $\kappa > 1/4$, which includes every Lorentzian Kuhn background $\kappa > 1/2$, $\mathcal{G}_g$ is negative definite on a four-dimensional space of strains and on no larger one. For every $\kappa < 1/4$, which includes the Euclidean cell $\kappa = 0$, it is negative definite on a line and on no two-dimensional space.*

Proof. In the frame $T = (1, 1, 1, 1)$, $w_1 = (1, -1, 0, 0)$, $w_2 = (1, 1, -2, 0)$, $w_3 = (1, 1, 1, -3)$, the inverse background is diagonal: $\Phi g^{-1} \Phi^T = \text{diag}(n, 2, 6, 12)$ with $n = 4/(1 - 4\kappa)$. The DeWitt form then has ten mutually orthogonal modes: the conformal mode, norm -4 ; the three time-space shears, norms $4n, 12n, 24n$; three space-space shears, norms $24, 48, 144$; and three trace-free diagonal modes, norms $2, 6, 12$. A form with an orthogonal basis of nonzero norms is negative definite on the span of its negative modes and on no space of larger dimension, by a dimension count against the span of the positive ones. For $\kappa > 1/4$, $n < 0$ and four norms are negative; for $\kappa < 1/4$, $n > 0$ and only the conformal norm is. $\square$

Reading. On Lorentzian backgrounds the DeWitt form has index four, the classical signature $(6, 4)$ of the DeWitt metric on Lorentzian metrics [5]; on Riemannian backgrounds its index is one, the conformal mode. Theorem 5.8 confirms both on the lattice backgrounds, exactly (Fig. 8). The three time-space shears turn negative only in Lorentzian signature. Its bearing on the Lorentzian sum is indirect, because the Lorentzian weight has unit modulus; it matters for the Euclidean continuation and for reflection positivity below.

	conformal $h = g$	time–space shears $T \otimes w_i + w_i \otimes T$			space–space shears $w_i \otimes w_j + w_j \otimes w_i$			trace-free diagonal modes		
$\kappa = \frac{5}{4}$ index 4	−4	−4	−12	−24	24	48	144	2	6	12
$\kappa = 0$ index 1	−4	16	48	96	24	48	144	2	6	12

Figure 8. The DeWitt norms of the ten orthogonal strain modes of [Theorem 5.8](#), exactly, with $n = 4/(1 - 4\kappa)$. Top: the Lorentzian background $\kappa = 5/4$ ($n = -1$), index four. Bottom: the Euclidean cell $\kappa = 0$ ($n = 4$), index one. The three time–space shears change sign between the two.

Proposition 5.9 (The conformal branch). *Let $\kappa > 1/2$.*

- (i) *On every nonzero static covector ($\sum_a u_a = 0$) the gauge-fixed long-wavelength symbol is positive on the conformal strain and negative on a space–space shear.*
- (ii) *For every torus side, the flat field ℓ_κ^0 , which is the edge reading of the conformal strain, lies in the kernel of the Regge Hessian at ℓ_κ^0 , from both sides. The reason is homogeneity: $dS(c\ell) = dS(\ell)$ for $c > 0$.*
- (iii) *There is a nonzero static covector at which the gauge-fixed static matrix $\sqrt{4\kappa - 1} B_\kappa(z) + \sum_a \gamma_a(z) \gamma_a(z)^*$ has a nonzero kernel vector, where $B_\kappa(z)$ is the Bloch matrix of the Regge Hessian and $\gamma_a(z)$ are the gauge columns of the companion paper.*

The lattice has a static resonance, a non-gauge zero mode at one static wave for every $\kappa > 1/2$ [18]. A numerical survey reads the branch that crosses zero there as the conformal mode: negative near $k = 0$ and turning positive at the resonance. That reading is evidence, not a theorem, and [Proposition 5.9](#) proves the two exact facts that bracket it.

Problem 5.10 (The resonance is the conformal branch). Find a nonzero static covector k and a continuous eigen-branch $(\lambda(s), v(s))$, $s \in [0, 2]$, of the gauge-fixed static matrix along $s \mapsto sk$, with $v(s) \neq 0$, $v(0)$ the conformal edge vector $(|d|(1 - \kappa|d|))_d$, $\lambda < 0$ on $(0, 1)$, $\lambda(1) = 0$ and $\lambda > 0$ just after $s = 1$.

5.4 Reflection positivity

Osterwalder–Schrader reflection positivity [12] is the Euclidean route to a positive Hilbert space. It needs a time reflection. With time on the body diagonal the time reflection is central inversion ϑ , an exact symmetry of the lattice, of the configuration space and of both actions ([Theorem 2.10](#)). The analytic half is a statement about the cutoff measure. For a set A of configurations, an observable F depends on positive times if $F(\ell)$ depends only on edges both of whose endpoints have clock value $x_0 + x_1 + x_2 + x_3 \bmod N$ in $\{1, \dots, h\}$, and the OS form is

$$\Theta_A(F) = \int \overline{F(\vartheta\ell)} F(\ell) e^{-S_E(\ell)} d\text{Leb}|_{A \cap \{|\ell_e| \leq L\}}(\ell),$$

with S_E the Euclidean Regge action of the Kuhn lattice.

Proposition 5.11 (What reflection positivity would need). *For every set A , $L \geq 0$ and bounded measurable F , the integrand of $\Theta_A(F)$ is integrable. For $N \geq 4$ and $h \geq 2$ there is a finite ϑ -invariant measure, two point masses exchanged by ϑ , and a bounded measurable F depending on positive times, with values ± 1 , whose OS form against that measure has negative real part.*

Problem 5.12 (Reflection positivity at the cutoff). Show that for every ϑ -invariant A , every L , every h and every bounded measurable F depending on positive times, $\text{Re } \Theta_A(F) \geq 0$ and $\text{Im } \Theta_A(F) = 0$.

Reading. The problem has content: the form is finite, and reflection symmetry alone provably does not give positivity, so a proof must use the product structure of Lebesgue measure and the locality of the action. It may turn out false, since the Euclidean action is unbounded below through the conformal mode ([Theorem 5.8](#)); a counterexample would also be a result. Unitarity of the Lorentzian sum is not addressed.

5.5 The light-speed reading

At $\kappa = 1$ the configuration space is the null stratum, and Lebesgue measure on $\mathbb{R}^{\mathcal{E}_N}$ does not see it. Replace it by Haar measure of the subspace V_0 .

Theorem 5.13 (The path sum on the null stratum). *Let μ_L^0 be Haar measure of V_0 restricted to $\text{Adm}_1 \cap \{|\ell_e| \leq L\}$.*

- (i) *For every L above the size of the flat field ℓ_1^0 , μ_L^0 is finite and nonzero, the weight and bounded observables are integrable, Z is continuous, and $Z \neq 0$, $\langle 1 \rangle = 1$ throughout $|a| C_N L < \pi/2$.*
- (ii) *For every bound A there is a ball around ℓ_1^0 , whose points in V_0 are all in Adm_1 , such that every nonzero finite measure carried by it has $Z(a) \neq 0$ for $|a| \leq A$.*
- (iii) *For $N \geq 4$, every $R > 0$ and every bound A , the stratum measure of [Theorem 4.12](#) has $Z(a) \neq 0$ and $\langle 1 \rangle = 1$ for $|a| \leq A$.*
- (iv) *Adm_1 is a cone, relatively open in V_0 , so it has infinite Haar measure there and the weight is not integrable over it: a regulator is needed on the stratum too.*
- (v) *Every clause of the action item holds at $\kappa = 1$: the action of [Theorem 3.1](#), the stationarity equations, continuity up to the boundary, a null frontier, the unfixed phase scale, and the conditional sign of [Theorem 3.4](#), whose premises are met at $\kappa = 1$ itself by the rank clock and q_1 .*

Reading. The light-speed reading carries the same finite sum as the open cone, with Haar measure of the null subspace in place of Lebesgue measure. What does not carry over is listed in [Section 6](#).

Proved

2 Configuration space: an open semialgebraic cone, explicit ball, every simplex of signature $(3,1)$, null and degenerate loci on the boundary; null stratum at $\kappa = 1$. κ is the posting speed; causal range $\kappa \geq 1$.

3 Action: Regge, smooth and homogeneous; first variation by Schläfli; hinge sum for $N \geq 3$; continuous to the boundary. Sign: the tick order fixes the time axis; $\kappa \geq 1$ and the sign given a causal reading.

4 Model measure: finite, nonzero, density for every admissible family; exact kernel at balance; no Faddeev–Popov factor; family, radius and weight change it; a stratum measure at $\kappa = 1$.

5 Finite at a stated regulator; Z entire in the phase scale, expectations defined off a discrete set; same finite sum at $\kappa = 1$; DeWitt index 4 and 1.

Chosen, not derived. The fixed Kuhn graph with real squared lengths; the family $n(1 - \kappa n)$ and the value of κ until derived; time along the body diagonal; the ledger chart, its account family and the flat bare weight; the regulator L or R ; the phase scale a .

Open

2 Derive the posting speed: light moves at the ledger's top speed ($\kappa = 1$) or not ($\kappa = 5/4$).

3 The native phase scale; one continued action past the null loci and the choice of $i\varepsilon$.

4 The bare weight ($\beta = 0?$); admissible families up to null sets; the action on displaced flat lattices; refinement.

5 A derived regulator; the Abel limit of the uncut sum; the resonance as conformal branch; reflection positivity.

Figure 9. What is proved and what remains, by requirement. Solid boxes are proved, each with the number of its section; conditional theorems keep their premises. Dashed boxes are open problems stated in the text. The grey bar lists the inputs that are chosen, not derived.

6 Scope and what remains

The four requirements hold together on the open cone. For every torus side N and every $\kappa > 1/2$ with $\kappa \neq 1$, the configuration-space, action, measure and regulator statements hold at once, each conditional statement keeping its premises; the ledger statements need $N \geq 4$. At $N = 4$, $\kappa = 5/4$ every hypothesis and premise is met, and the ledger measure has a nonzero normalization at the native phase scale for a small box. The physical range is $\kappa \geq 1$: for $1/2 < \kappa < 1$ the simplices are Lorentzian but a posting outruns light. At $\kappa = 1$ the configuration space, action, measure and finite sum of [Theorems 2.5](#), [4.12](#) and [5.13](#) hold on the null stratum. [Figure 9](#) sets this against what remains.

- *The model.* The fixed Kuhn graph, real squared lengths, the family $n(1 - \kappa n)$, time on the body diagonal, the posting clause of [Definition 2.1](#), the ledger chart, the account family and the flat bare weight are chosen. [Section 2](#) gives the reasons for the graph.
- *The posting speed.* κ is the posting speed, and the cell's premises leave it free on $[1, \infty)$. Deriving it is [Problem 2.14](#); the two candidate values are 1 and $5/4$.

- *The sign.* Conditional on name-blindness and a causal reading of the cell. The tick order fixes the time axis; the causal premise supplies the minus sign; the signature $(3, 1)$ of every admissible simplex is proved.
- *The measure.* A model measure. Its density holds for every admissible family; its dependence on the box radius and on the weight is proved; whether admissible families agree up to null sets, whether the bare weight is flat, and the refinement behaviour of the fixed-graph measure are open ([Problems 4.5](#) and [4.11](#)). There is no Faddeev–Popov factor ([Theorem 4.7](#)), and whether the action is constant on displaced flat lattices is [Problem 4.8](#).
- *The regulator.* L or R is stated, not derived ([Problem 5.5](#)). The explicit windows exclude the native phase scale; the entire-function theorem covers it up to a discrete set. The phase scale is a candidate, not derived ([Problem 3.6](#)).
- *Without a regulator.* Not absolutely convergent, and ray truncations diverge for every power density; the Abel limit is [Problem 5.7](#).
- *At $\kappa = 1$.* Proved only off the stratum: the explicit ball, the frontier classification, the why-fixed-graph package, the gauge and kernel statements, the dilation and Haar statements, and restriction consistency. The stratum measure uses a constructed family; whether X_* itself carries one is open.
- *The action.* The sinh obstruction is realized on the null stratum. The continued action past the null loci is [Problem 3.10](#), and the $i\varepsilon$ prescription is a parameter. The cubic part of the action as Einstein–Hilbert, and the classical limit, are separate targets.
- *Positivity.* The conformal branch is [Problem 5.10](#); reflection positivity is [Problem 5.12](#); unitarity is not addressed. No UV completion is claimed.

Three open problems decide the next step. Deriving the posting speed fixes κ and with it the reading on which every theorem is to be applied. Deriving the bare weight and the regulator from ledger dynamics turns the model measure into a derived one. And reflection positivity at the regulator decides whether the lattice sum has a positive transfer matrix. Underneath all of them is the derivation of the lattice itself: the triangulation, the length family and the time direction.

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