

The Five Postulates of Quantum Mechanics on the Record

Theorems, premises, and what is owed

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Preprint, version 1.3, 14 November 2026. Not peer reviewed.

Abstract

Quantum mechanics is taught from five postulates: a state is a vector in a complex space; left alone it turns by a unitary step that an energy operator generates; a measurement returns an outcome with a probability equal to a squared length; the state afterwards is the part that matches the outcome; and two systems together live in the product of their spaces. The textbook does not say where the five come from. We read each of them as a statement about one object, the recognition ledger, a record of events that is written in postings and never erased, and we say for each how much of it is then a theorem. The guiding sentence is short: histories that leave no record add, and what is on the record is counted.

The record's clock makes a complex unit. The tick of the eight-position register has no complete set of real stationary states, and on the states that four ticks reverse, two ticks square to minus one. For a law on a finite-dimensional space whose factors all lie above zero, the unitary step is fixed up to its unit: a linear rule for the step that uses only the law between two records, keeps every length, regroups and is continuous is $\exp(-icH)$ with $H = -\log T$, where T is the law itself. The square follows, for reads of a register's positions, from two principal properties and three mild ones: a read that weighs each position by its own value and adds over positions, is silent on an empty position and is never less for a larger value, and whose total is blind to a common turn of the register, is the sum of squares. For reads along other splittings of a space the square is a hypothesis. Two acts can be read together exactly when they commute, and reading one first moves the other's weights by an amount we give in closed form, never more than one half. Where a crossing takes held states off the held part, a contact divides a held state into a part left exactly as it was and a part with one crossing applied, with squared lengths $\cos^2\theta$ and $\sin^2\theta$; where it keeps them held, the contact reads one sign of the register and nothing else. Two blocks of one cube compose by the product, their acts commute, and their correlations have three ceilings, 2, $2\sqrt{2}$ and 4, for acts that only read, acts that write, and tables that merely do not signal; the cube reaches $2\sqrt{2}$ exactly.

Each theorem is proved on the page from premises we name, and we say postulate by postulate what that gives. The second follows from four statements about a linear step. The third follows for reads of positions, given that a read is taken position by position, which is owed. For the first, the clock makes a complex unit with no premise, the one such unit that is a whole number of ticks; that states are complex rests on a premise about the clock that is close to the conclusion, that a held register is among the states the unit acts on is a working statement, and from the second postulate on the spaces are taken complex, as in the textbook. Of the fourth the algebra is proved; the weights and the state afterwards are the textbook's, and that one outcome occurs is owed. The fifth is proved for two blocks of one cube and owed for two laboratories, where published network experiments exclude a real product whose sources share nothing. Several of the theorems are known results of the textbook's own mathematics, and the ledger enters through what their objects are identified with. What is owed is named in four statements, and we say how exact each is. The paper closes with twelve paradoxes of quantum mechanics, from the double slit to Wigner's friend, each in three lines: the puzzle, what the record says, and what is owed. Eleven of the twelve wait on the same owed statement, and every number they return is the textbook's own.

1 The question

Open any textbook of quantum mechanics and the first chapter gives the rules [1, 2]. A system has a state, and the state is a vector in a complex space. While nothing looks at the system the vector turns, by a unitary step fixed by an energy operator. When something does look, one outcome appears out of a fixed list, and its probability is a squared length. After that the state is the part of the old state that matches the outcome. Two systems taken together have states in the product of their two spaces.

These five rules have passed every test for a century. They were also found by working backwards from spectra and scattering, and the textbook says so: they are postulates. Why a complex number and not a real one? Why does energy generate the step? Why a square and not a fourth power? What counts as a measurement? Why the product? Each question has its own literature and its own paradoxes: the double slit, Schrödinger's cat, Bell's inequality, Wigner's friend.

This paper reads all five as statements about one object. On that reading quantum mechanics is not a separate set of laws for small things: it is what the law of the recognition ledger does between two records. We say for each rule how much of the reading is a theorem.

The ledger. Recognition Science starts from a fact that can be checked in the act of doubting it: occurrences can be told apart, and telling two of them apart leaves a record. The record is a ledger. An event is a *posting*, one unit written in one column of one account, and a posting is never erased. From this starting point earlier work derives the cost of a comparison [3], the scale between levels [4], the three dimensions of space [5] and a clock of eight ticks [6]; the law that carries a record from one posting to the next is the subject of [7]. We use those results as given and add nothing to them except where a premise is named.

One sentence. Between two postings there are histories that leave no record. The ledger cannot tell them apart, so it does not count them one by one. They add. What is posted is counted. Each of the five rules is one of these two clauses read at a particular place, and

The textbook's rule	On the record	Where it stands
1. States are vectors in a complex space.	The clock makes the number i: two ticks, on the states that four ticks reverse.	Theorems 1, 3. Premise: every state is a sum of states the tick only multiplies. Working: a held register lies in the core.
2. The state turns by $e^{-i\hat{H}t/\hbar}$.	The step between records is $\exp(-icH)$, $H = -\log T$, with T the law itself.	Theorem 8. Premises: linearity and four statements about a step.
3. Probability is a squared length.	A read that weighs each position by its own value, with a total blind to a common turn, is the sum of squares.	Theorems 16, 18, for reads of positions. Premise: the total is blind to a common turn. Owed: the position read, **(O2)**.
4. A measurement gives one outcome and leaves the matching part.	At a contact a held state divides in two parts; acts read together exactly when they commute.	Theorems 22–25: the algebra. The weights and the state afterwards are the textbook's. Owed: that one outcome occurs, **(O1)**.
5. Two systems live in the product space.	Two blocks of one cube: product states, commuting acts, ceilings 2, $2\sqrt{2}$, 4.	Theorems 26–28, on one cube. Owed for two laboratories: **(O3)**.

Table 1. The five postulates and where each stands on the record. **(O1)** and **(O3)** are two of the four owed statements listed below.

Table 1 says where each reading stands.

What is proved and what is owed. Every theorem below is proved on the page from the hypotheses in its statement. Several are known results of the textbook's own mathematics, cited where they appear; the ledger enters through what their objects are identified with, and each identification is named as one. The hypotheses are premises, and we name them: one about the clock (Section 3), linearity and four statements about a step (Section 4), and two principal ones about a read (Section 5), the first of which is the owed statement **(O2)**. Where a result is proved elsewhere we say "proved in work not yet published" and do not rely on it. Beyond the premises, four statements are owed, and Section 8 shows that the twelve best-known paradoxes wait on these four:

(O1) *Attachment.* What a laboratory system, its apparatus and its outcome are on the ledger: which posting a click is, and that one is posted.

(O2) *The position read.* That a read weighs each position of a register by the value held there and by nothing else.

(O3) *Composition.* That two registers held by two recognizers are two blocks of one cube, each recognizer acting on its own block, with their values composed over one complex unit.

(O4) *Exchange.* That exchanging two identical records is one full turn of one about the other.

The four are not equally exact. **(O2)** is an exact proposition, (B1) of Section 5. **(O3)** is exact once "held by two recognizers" is. **(O1)** is a question, and **(O4)** uses a phrase, the turn of one record about another, that the ledger has yet to define. Each is stated as sharply as we can state it in the section where it enters.

The numbers this paper returns are the textbook's numbers. That is what a derivation of the postulates should return. What the ledger adds is a place for each rule to come from, a statement about a record and its clock in place of a statement about vectors and operators, and a short list of what is left. The theorems are about the register, its clock and operators on them. That the states, steps and outcomes of a laboratory are these objects is an identification, and it meets measurement only through what the law of [7] gives for a real system; this paper computes no spectrum.

Earlier derivations. Hardy [9] and Chiribella, D'Ariano and Perinotti [10] derive the whole finite-dimensional formalism, the product rule and the square on every splitting included, from axioms about information: tomography, purification, continuous reversibility. Gleason [11] derives the square from additivity over all orthogonal frames. Those axioms are about what an experimenter can do. This paper derives less than they do, and from a different place. Its premises are about a ledger and its clock, the same ledger that gives the cost, the scale and three dimensions; it leaves the square on other splittings as a hypothesis and the product for two laboratories as an owed statement; and three of its five results need no axiom about composite systems at all.

Every finite statement in the paper (the tick's spectrum, the identities for two acts, the three ceilings, the numbers quoted for the twelve) can be checked by a direct computation, and a short program that carries the computations out numerically accompanies the paper.

2 The record and its register

We fix the objects. Each is taken from the cited work; the words in brackets are the textbook's nearest terms, given once.

Posting, record, recognition. A *posting* writes one unit in one column of one account. The *record* is everything posted so far. A *recognition* is a posting together with its reflection: a debit and the credit that answers it. It is two ticks of the clock, and the ledger closes at the end of it [7].

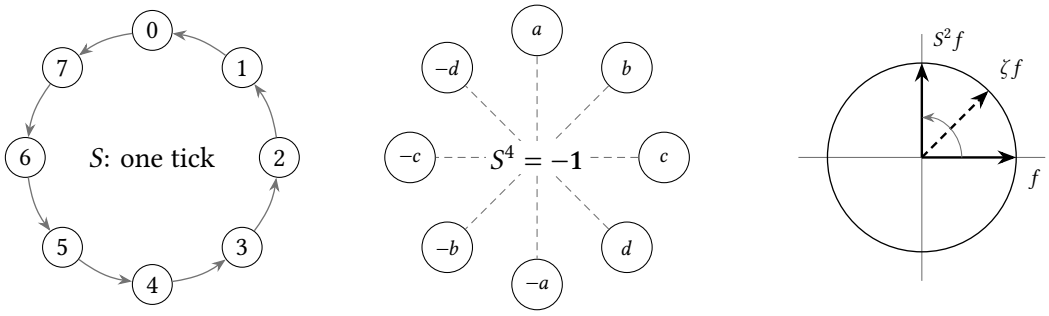
The register. Space has three dimensions [5], so one cell of the ledger has three binary axes and $2 \cdot 2 \cdot 2 = 8$ positions. The *register* of a cell is what the cell holds at its eight positions: a function $f : \mathbb{Z}_8 \to \mathbb{R}$, one real *value* per position [the textbook's amplitude]. We write $e_0, \dots, e_7$ for the register holding 1 at one position and 0 elsewhere.

The tick. One tick of the clock advances the positions by one:

$$(Sf)(t) = f(t+1), \qquad t \in \mathbb{Z}_8.$$

Eight ticks return the register to itself: $S^8 = \mathbf{1}$ [6].

The law between records. Let D be the operator one posting applies to what the neighbouring cells hold. The law over one recognition is the posting followed by its reflection, $T = D^\dagger D$



(a) the tick (b) the core (c) two ticks are i

Figure 1. The register and its clock. (a) One tick moves every value to the next position; eight ticks return the register. (b) The core is the set of registers that four ticks reverse: the value at each position is minus the value opposite it, so four free values remain. (c) On the core, two ticks are a quarter turn, the number i, and $\zeta = (\mathbf{1} + S^2)/\sqrt{2}$ is the eighth turn that squares to it. Drawn in the plane spanned by a core register f and $S^2 f$.

[7]. It is self-adjoint and not negative. On a state that T only multiplies, $Tv = fv$, the number f is the state's *factor*, and $E = -\log f$ is its *decay rate*: left to the law, the state shrinks by e^{-E} at each recognition.

Held, read, contact. A register is *held* while nothing is posted about what it holds. A *read* is the weight that a posting about a set of positions carries: a real number $\mu(f, A)$ for a register f and a set A of positions [a probability, once the total is set to one]. A *contact* is a posting that crosses from outside into a register [a measurement, an interaction with an environment].

One standing statement. Nothing is outside the record. A step between two records uses the law between them and nothing else; a read uses what the register holds and nothing else. Each premise named below is this statement read at one place, and we say which.

3 The first postulate: the clock makes the complex number

The textbook's states are complex. On the record the question is concrete: the register holds real values, and its clock is the tick S. A state the tick only multiplies, $Sv = \lambda v$, is a *stationary state* of the clock: it keeps its shape from tick to tick. Does the register have enough of them?

Theorem 1 (No real set of stationary states)**.** *Let S be the tick on $\mathbb{R}^8$. If $Sv = \lambda v$ with λ real and $v \neq 0$, then $\lambda = \pm 1$ and $v(t+2) = v(t)$ for every t. So the real stationary states span a subspace of dimension two, and no basis of $\mathbb{R}^8$ consists of them. Over $\mathbb{C}$ the eight registers*

$$d_k(t) = \omega^{kt}, \qquad \omega = e^{i\pi/4}, \quad k = 0, \dots, 7,$$

are a basis of stationary states, with $Sd_k = \omega^k d_k$. The multiplier of d_2 is i.

Proof. From $Sv = \lambda v$ we get $v(t+1) = \lambda v(t)$, so $v(t) = v(t+8) = \lambda^8 v(t)$ for every t. Since $v \neq 0$, $\lambda^8 = 1$, and a real number with that property is 1 or -1. Then $v(t+2) = \lambda^2 v(t) = v(t)$. A register with $v(t+2) = v(t)$ is fixed by $v(0)$ and $v(1)$, so such registers form a space of dimension two, and eight independent ones do not exist. For the second part, $(Sd_k)(t) = \omega^{k(t+1)} = \omega^k d_k(t)$; the eight multipliers ω^k are distinct, so the d_k are independent, and $\omega^2 = i$. □

So six of the register's eight dimensions have no real stationary state. The tick turns them, in three planes, by an eighth, a quarter and three eighths of a full turn. A turn in a plane is multiplication by a complex number, and the next theorem shows that the clock supplies that number itself.

Definition 2 (The core)**.** The *core* K is the set of registers that four ticks reverse:

$$K = \{f \in \mathbb{R}^8 \ : \ f(t+4) = -f(t) \text{ for every } t\}.$$

Theorem 3 (Two ticks are i)**.** *The core has dimension four, and the tick maps it onto itself. On the core $S^4 = -\mathbf{1}$, so*

$$i := S^2\big|_K \qquad \text{satisfies} \qquad i^2 = -\mathbf{1}.$$

With this i the core is a complex space of dimension two. The tick commutes with i, so it is complex-linear there; its two multipliers are ω and $-\omega$, so the tick is not itself a number. The operator

$$\zeta := \frac{\mathbf{1}+i}{\sqrt{2}} = \frac{\mathbf{1}+S^2}{\sqrt{2}}\Big|_K$$

is a number on the core: $\zeta^2 = i$ and $\zeta^8 = \mathbf{1}$. In the complex basis $u_2 = e_2 - e_6$, $u_3 = e_3 - e_7$ a core register has the coordinates

$$f = z_2\, u_2 + z_3\, u_3, \qquad z_2 = f(2) + i\, f(0), \quad z_3 = f(3) + i\, f(1),$$

so each complex value is carried by two positions two ticks apart. On the register $a(e_0-e_4)+b(e_2-e_6)$ the eighth turn acts as

$$(a,b) \ \longmapsto \ \Big(\frac{a+b}{\sqrt{2}},\ \frac{b-a}{\sqrt{2}}\Big). \tag{1}$$

Proof. A core register is fixed by its values at positions $0,1,2,3$, so $\dim K = 4$, with basis $u_k = e_k - e_{k+4}$, $k = 0,1,2,3$. If $f \in K$ then $(Sf)(t+4) = f(t+5) = -f(t+1) = -(Sf)(t)$, so $SK \subseteq K$, and S is invertible. On K, $(S^4 f)(t) = f(t+4) = -f(t)$. Hence $(S^2)^2 = -\mathbf{1}$ on K. S commutes with S^2. Direct evaluation gives $Su_k = u_{k-1}$ for $k = 1,2,3$ and $Su_0 = -u_3$; hence $iu_2 = u_0$ and $iu_3 = u_1$, so $\{u_2, u_3\}$ is a complex basis, with $f = f(0)u_0 + f(1)u_1 + f(2)u_2 + f(3)u_3 = (f(2) + if(0))u_2 + (f(3) + if(1))u_3$, and in it $Su_2 = u_1 = iu_3$ and $Su_3 = u_2$. The matrix of S is $\begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}$, with characteristic equation $\lambda^2 = i$ and roots $\pm\omega$. For ζ: $\zeta^2 = (\mathbf{1} + 2i + i^2)/2 = i$, and $\zeta^8 = i^4 = \mathbf{1}$. Finally $S^2\big(a\,u_0 + b\,u_2\big) = b\,u_0 - a\,u_2$, since $S^2 u_0 = -u_2$ and $S^2 u_2 = u_0$; adding $a\,u_0 + b\,u_2$ and dividing by $\sqrt{2}$ gives (1). □

Proposition 4 (The clock's i is the number i)**.** *Write a core register in the stationary states of Theorem 1: $f = c_1 d_1 + c_3 d_3 + c_5 d_5 + c_7 d_7$ with $c_7 = \bar{c}_1$ and $c_3 = \bar{c}_5$. The map $f \mapsto c_1 d_1 + c_5 d_5$ is a real-linear bijection of the core onto the complex span of d_1 and d_5, and it carries two ticks to multiplication by the number i.*

Proof. $S^4 d_k = \omega^{4k} d_k = (-1)^k d_k$, so the complexified core is the span of the four odd d_k. A register is real exactly when the coefficient of d_{8-k} is the conjugate of that of d_k, because $\bar{d}_k = d_{8-k}$. So c_1 and c_5 fix f, and the map is a bijection between spaces of real dimension four. $S^2 d_k = \omega^{2k} d_k$, and $\omega^2 = \omega^{10} = i$, so $S^2 f$ is sent to $ic_1 d_1 + ic_5 d_5$. □

Remark 5 (The other units, and why two ticks)**.** Two ticks are not the only operator on the core that squares to $-\mathbf{1}$ and commutes with the tick. Put $J' = (S + S^3)/\sqrt{2}$. On the core $S^4 = -\mathbf{1}$ and $S^6 = -S^2$, so $J'^2 = (S^2 + 2S^4 + S^6)/2 = -\mathbf{1}$, and J' is a polynomial in the tick. It is not $\pm S^2$: it multiplies d_1 and d_3 by i, where S^2 multiplies d_1 and d_5 by i. These are all there are. The tick turns the core in two planes, by an eighth and by three eighths of a turn, and a real operator that commutes with it and squares to $-\mathbf{1}$ is a quarter turn in each plane, in one sense or the other: four operators, $\pm S^2$ and $\pm J'$. Two things single out $\pm S^2$ among the four. They are whole numbers of ticks, S^2 and S^6, and no other power of the tick squares to $-\mathbf{1}$ on the core. And they carry each position to one position, so that a complex value is held by two positions; J' spreads one position over two, $J'u_2 = (u_1 - u_3)/\sqrt{2}$. The working identification below takes the unit to be a number of ticks. Between S^2 and S^6 the choice is the direction in which the clock is read.

What this says. We identify the imaginary unit of quantum mechanics with this one. It is then not an added structure: it is two ticks of the clock, acting on the registers that half a cycle reverses, and by Proposition 4 it is the same number i that multiplies the complex stationary states of Theorem 1. The theorem needs no premise; the identification is a working statement of the record. Every register has this unit, made the same way. What the unit of two registers taken together is, is a separate question, and Section 7 shows that it is not answered by the clock alone. Map (1) is worth a second look. It has the form of the balanced two-port of optics, the map a beam splitter applies to the values in its two arms. The likeness is one of form: here the two values are the two halves of one complex value, and the map is a common turn of the register by an eighth of a cycle. Section 5 uses the form to derive the square.

The premise, and what the theorems use. Theorem 1 excludes a complete set of *real* stationary states. That the register's states are complex follows once every state is required to be a sum of stationary states of the clock. This is the standing statement read at the clock: a register is exhausted by how it stands from tick to tick, with no remainder that the tick does not account for. It is a premise, and it is close to the conclusion, since it asks for numbers that contain the tick's multipliers. A description that keeps real planes and their turns, with no stationary state for six of the eight dimensions, is not contradicted by Theorem 1, and the premise alone does not single out the complex numbers among larger number systems that contain them. What the record supplies beyond the premise is Theorem 3: a unit made of two ticks that commutes with the tick, and by Remark 5 the one such unit that is a whole number of ticks, up to the direction of the clock. Theorem 3 and Proposition 4 need no premise.

Two further things should be said plainly. The construction uses the clock only through its period being a multiple of four: on the registers that half a cycle reverses, a quarter of the cycle squares to $-\mathbf{1}$. That the period is eight, so that a quarter of the cycle is two ticks and the core is as defined above, is taken from [6, 5]. That a held register lies in the core, each value answered by its opposite half a cycle later, is the ledger's closing read at one register: it is a working statement of the record and is not proved here. And from Section 4 on the spaces are complex inner-product spaces, as in the textbook: the unit of this section reaches them through the identification above and not through a theorem, and what the unit of two registers taken together is, is the owed statement **(O3)**.

4 The second postulate: the step between records

The textbook's second rule is that a state left alone turns by $e^{-i\hat{H}t/\hbar}$, with $\hat{H}$ the energy operator of the system. Which operator $\hat{H}$ is has to be supplied for each system. On the record one object is supplied, the law between two records, T, and the question is how a history that leaves no record steps under it.

Definition 6 (Laws and step rules)**.** A *law* is a self-adjoint operator T on a finite-dimensional complex inner-product space whose factors lie in $(0, 1]$. For a number $f \in (0, 1]$ we write (f) for the law on $\mathbb{C}$ that multiplies by f. A *step rule* R gives every law T on a space $\mathscr{H}$ an operator $R(T)$ on the same space: the step of a held history over one recognition. That the step is an operator, so that the step of a sum of two states is the sum of their steps, is part of what a step is taken to be here. It is a premise, and it is the first clause of the opening sentence: histories that leave no record add. We consider four statements about a step rule.

(S1) *It uses only the law.* If $V : \mathscr{H}_1 \to \mathscr{H}_2$ keeps lengths and $VT_1 = T_2V$, then $VR(T_1) = R(T_2)V$.

(S2) *It loses nothing and gains nothing.* $\|R(T)\psi\| = \|\psi\|$ for every ψ.

(S3) *It regroups.* $R(T^n) = R(T)^n$ for $n = 1, 2, 3, \ldots$: the law over n recognitions, read in one step, is n steps.

(S4) *It is continuous in the factor.* $f \mapsto R\big((f)\big)$ is continuous on $(0, 1]$.

(S1) is the standing statement read at a step: carry a law into a larger law, on more classes, and the step is carried with it, because nothing but the law was used. (S2) says that a history which leaves no record loses nothing and gains nothing: there is no record for a loss to go to, or for a gain to come from. (S3) says that what is not kept as a record leaves no mark: whether three recognitions are read as one stretch or as three makes no difference when nothing was written in between. Note that T^n is again a law.

Lemma 7 (A rule that uses only the law is a function of the law)**.** *Let R satisfy* (S1)*, and put $r(f) := R\big((f)\big) \in \mathbb{C}$. If $Tv = fv$ then $R(T)v = r(f)\,v$. So $R(T) = r(T)$: the step multiplies each stationary state of the law by one function of its factor, the same function for every law of every size.*

Proof. Let $Tv = fv$ with $\|v\| = 1$, and let $V : \mathbb{C} \to \mathscr{H}$ be $Vz = zv$. Then V keeps lengths and $V(f) = TV$, so by (S1) $R(T)v = R(T)V1 = VR\big((f)\big)1 = r(f)\,v$. A self-adjoint T has a basis of such v. □

Theorem 8 (The step in real time)**.** *Let R satisfy* (S1) *to* (S4)*. Then there is exactly one real number c such that, for every law T,*

$$R(T) = T^{\,ic} = \exp(-icH), \qquad H := -\log T.$$

H is self-adjoint and not negative, and it multiplies each stationary state of the law by its decay rate. Conversely, for every real c the rule $T \mapsto \exp(-icH)$ satisfies (S1) *to* (S4)*.*

Proof. By Lemma 7, $R(T) = r(T)$. Applying (S2) and (S3) to the laws (f) gives $|r(f)| = 1$ and $r(f^n) = r(f)^n$. Put $u(E) := r(e^{-E})$ for $E \geq 0$. Then u is continuous by (S4), $|u| = 1$, and $u(nE) = u(E)^n$. At $E = 0$ this reads $u(0) = u(0)^2$, so $u(0) = 1$: *rest does not turn.* A continuous function into the unit circle on $[0, \infty)$ has a unique continuous angle: $u(E) = e^{-i\theta(E)}$ with θ continuous and $\theta(0) = 0$. Fix n. The number

$$k_n(E) = \frac{\theta(nE) - n\,\theta(E)}{2\pi}$$

is a whole number for every E, because $u(nE) = u(E)^n$; it is continuous in E; and it is zero at $E = 0$. A continuous whole number is constant, so $k_n = 0$ and $\theta(nE) = n\,\theta(E)$ for all $n \geq 1$ and $E \geq 0$. Then $\theta(E) = \theta(n \cdot E/n) = n\,\theta(E/n)$, so $\theta(\frac{m}{n}E) = \frac{m}{n}\theta(E)$ for whole $m, n \geq 1$, and in particular $\theta(q) = q\,\theta(1)$ for every positive rational q. By continuity $\theta(E) = cE$ with $c := \theta(1)$. Hence $r(f) = e^{-icE} = f^{ic}$ with $E = -\log f$, and $R(T) = T^{ic} = \exp(-icH)$. The angle θ is unique, so c is. H is self-adjoint with eigenvalues $E = -\log f \geq 0$. The converse is direct: $\exp(-icH)$ is a function of T, which gives (S1); it is unitary, (S2); $-\log(T^n) = nH$, (S3); and $f \mapsto f^{ic}$ is continuous, (S4). □

Corollary 9 (The group of steps)**.** *Put $U(t) := \exp(-ictH)$ for real t. Then $U(s)U(t) = U(s+t)$, each $U(t)$ is unitary, $U(n) = R(T)^n$, and*

$$\frac{d}{dt}U(t)\psi = -ic\,H\,U(t)\psi.$$

This is the textbook's second postulate, with one difference: the generator is not a second object to be supplied. It is $-\log T$, so whatever fixes the law between records fixes the energies. *The energy of a stationary state is its decay rate under the law between records, read as a rate of turn.* A state at rest under the law, with factor 1, does not turn.

What the unit is. The theorem leaves one real number free, and nothing more. The textbook writes the step as $e^{-i\hat{H}t/\hbar}$ with $\hat{H}$ in joules and t in seconds; here the step over n recognitions is e^{-icHn} with H a pure number. So c is the unit of the tick: it converts a decay rate per recognition into a turn per recognition. The next proposition shows what counts see of it. Ratios of beat rates never see it: they are ratios of differences of decay rates. Counts taken at known numbers of recognitions do see it, and so does a ratio of two such counts. They fix its size, and with a prepared relative turn between the two states its sign too, but only relative to the convention that names that turn: reversing c and conjugating every prepared value together changes no count. So the sign of c is a convention, the direction in which the clock is read (Remark 5), and its size is what counts can fix. The size of one recognition in seconds and of one unit of rate in joules are not derived here, and the action constant $\hbar$ is not derived here. Nor does the theorem exclude $c = 0$, the rule under which nothing turns. That histories turn at all, $c \neq 0$, is not proved here. Through one identification, light as the long wave of the law, c is the speed of light in cells per recognition, and the law of motion adopts $c = 1$ as its unit [7]. That is an adopted unit and an identification, not a measured value of c.

Proposition 10 (Counts show beats)**.** *Let v_1, v_2 be orthonormal stationary states of a law with decay rates E_1, E_2, let $\psi = \alpha v_1 + \beta v_2$, and let $\chi = v_1 + v_2$. Then*

$$\left|\langle \chi,\, U(t)\psi\rangle\right|^2 = |\alpha|^2 + |\beta|^2 + 2\,\mathrm{Re}\big(\bar{\alpha}\beta\, e^{-ic(E_2 - E_1)t}\big).$$

The count repeats at c times the difference of the two decay rates. A turn common to every state changes no count, and every ratio of two beat rates is a ratio of differences of decay rates, whatever c is.

Proof. $U(t)\psi = \alpha e^{-icE_1 t}v_1 + \beta e^{-icE_2 t}v_2$, so $\langle \chi, U(t)\psi\rangle = \alpha e^{-icE_1 t} + \beta e^{-icE_2 t}$; take the squared modulus. Multiplying $U(t)$ by a common phase changes no modulus. Two beat rates $c(E_2 - E_1)$ and $c(E_4 - E_3)$ stand in the ratio $(E_2 - E_1)/(E_4 - E_3)$. □

Proposition 11 (Weights and turns are one complex rate)**.** *Let R satisfy* (S1)*,* (S3) *and* (S4)*, with $r(f) \neq 0$ for every f, but not necessarily* (S2)*. Then there are real numbers a and c with*

$$R(T) = \exp\big(-(a + ic)H\big) \qquad \text{for every law } T,$$

and R keeps lengths exactly when $a = 0$. The law itself is the case $a = 1$, $c = 0$.

Proof. Write $u(E) = r(e^{-E}) = \rho(E)\, w(E)$ with $\rho = |u| > 0$ and $|w| = 1$, both continuous, both satisfying the regrouping relation. The proof of Theorem 8 applies to w and gives $w(E) = e^{-icE}$. For ρ, the function $\ell = \log \rho$ is continuous with $\ell(nE) = n\ell(E)$, so by the same argument $\ell(E) = -aE$ with $a = -\ell(1)$. Then $|r(f)| = f^a$, which equals 1 for every $f \in (0, 1]$ exactly when $a = 0$. For the law, $r(f) = f = e^{-E}$. □

Proposition 11 is the sentence of Section 1 as a theorem at the level of one stationary state. There are two ways to step under one law. Along the real axis, e^{-aE}, a weight shrinks: this is the record being counted, and a world read this way only fades; nothing in it repeats. Along the imaginary axis, e^{-icE}, nothing shrinks and everything turns: this is a history that leaves no record. The two are the same law read on two axes of one complex rate, and (S2), nothing lost and nothing gained, is exactly the statement that picks the second.

Remark 12 (What is known already, and what is new)**.** That a positive transfer operator $T = e^{-H}$ has a unitary partner e^{-iHt} is the Wick rotation, and its rigorous form is the reconstruction theorem of Osterwalder and Schrader [12]. There the unitary group is reached by continuing an analytic function. Theorem 8 is a smaller statement in a simpler setting. The space and the law are given, and the step is taken to be linear and to keep every length; the four statements then leave nothing free but a unit, so that given the law the continuation is not a choice. The reconstruction theorem does more: it builds the space and the generator from the data of the other axis.

Remark 13 (Scope)**.** The theorem is stated for laws on finite-dimensional spaces with every factor above zero. In that setting (S4) cannot be dropped: without it the relation $\theta(nE) = n\theta(E)$ has discontinuous solutions. For laws on infinite lattices with factors in $[m, 1]$, $m > 0$, the same conclusion holds with (S4) no longer assumed, when (S1) is read on the maps between the algebras the laws generate and (S3) as the composition of any two stretches. Among rules of that kind, whose step lies in the algebra the law generates and is carried by the maps between such algebras, a rule required to step every law with factors in $[0, 1]$, including laws whose factors come down to zero continuously, can only be the rule of rest; a law whose zero factor stands apart from its others is stepped by the same $\exp(-icH)$ on its other classes, with its null class sent to zero, where the step no longer keeps lengths. These three statements are proved in [8] (its Theorems 11, 13 and 14) and are not used here. The first holds in the setting

of algebras only: on a Hilbert space the four statements do not fix the step of a law with continuous spectrum ([8], Remark 12). In the textbook's terms the theorem and its extension cover bounded energy operators: laws with a floor under their factors. Laws whose factors come down toward zero with no floor, which is how the textbook writes the oscillator and the atom, are outside both. For them a turning step needs a wider class of rules than the one treated here, functions of the law that are only measurable, and that class is not examined. What the law of [7] is for a given body, in this respect, is not examined here.

Statements (S1) to (S4), and the linearity of the step, are premises. Said of the record, they become four statements the record makes of itself, read on a step, and continuity: the step reads only the law between two records; it is the same function wherever on the record the stretch sits, time being the count of recognitions; nothing on the record is removed; and the ledger stays bounded. The last two rest on one identification: what a state keeps over a stretch is what the record keeps. This is a restatement of equal strength, each clause carried by an identification, and not a derivation. (When (S3) is asked of any two stretches, continuity is needed only at rest.) Deriving the four from the starting fact of the ledger is owed by the foundation [7], not by quantum mechanics separately.

5 The third postulate: the read is the square

The textbook's third rule is the Born rule: the probability of an outcome is the squared length of the part of the state that matches it. Nothing in the first two rules hints at a square. On the record the square follows from two properties of a read, and the argument is short.

Definition 14 (Reads of the register)**.** A *read* gives every register $f \in \mathbb{R}^8$ and every set A of positions a real number $\mu(f, A)$. We consider five statements.

(B1) *Position by position.* There is one function $\phi : \mathbb{R} \to \mathbb{R}$ with $\mu(f, \{t\}) = \phi\big(f(t)\big)$ for every register and every position.

(B2) *Additive.* $\mu(f, A) = \sum_{t \in A} \mu(f, \{t\})$.

(B3) *An empty position is silent.* $\phi(0) = 0$.

(B4) *More is not less.* If $|x| \le |y|$ then $\phi(x) \le \phi(y)$.

(B5) *Blind to a common turn.* For every core register f, $\mu(\zeta f, \mathbb{Z}_8) = \mu(f, \mathbb{Z}_8)$, with ζ the eighth turn of Theorem 3.

(B1) is the statement that a read of one position uses the value held there and nothing else: not the values elsewhere, not the label of the position. It is the principal premise of this section, and it is the owed statement **(O2)**. (B5) is the second principal premise. The eighth turn is not a tick: a tick permutes the positions, and every read of the form (B1), (B2) is unchanged by it. The eighth turn multiplies a core register by one number of the clock's own making, so it turns every value of the register together. (B5) says that the total weight does not change under that common turn. It is the standing statement read at the read: a turn of the whole register is a turn relative to nothing, because no second register is there to compare it with. In additive form it is the argument by which Maxwell obtained the law of velocities in a gas [13]: a weight that is taken coordinate by coordinate and is blind to turns is quadratic. The reason

reaches further than the total. If a common turn is a turn relative to nothing, then no count the record holds can depend on one, whichever positions it is a count of. (B5) states this for the total, which is all the theorem needs. A single position's term is moved by a common turn (Corollary 17), and the paragraph after that corollary draws the consequence.

Lemma 15 (The parallelogram law forces the square)**.** *Let $\phi : \mathbb{R} \to \mathbb{R}$ satisfy $\phi(x+y)+\phi(x-y) = 2\phi(x)+2\phi(y)$ for all x, y, and* (B4)*. Then $\phi(x) = \phi(1)\,x^2$.*

Proof. With $x = y = 0$, $\phi(0) = 0$. By (B4), $\phi(x) \le \phi(-x) \le \phi(x)$, so ϕ is even. By induction $\phi(nx) = n^2\phi(x)$: it holds for $n = 0, 1$, and the law with (nx, x) in place of (x, y) gives $\phi((n+1)x) = 2n^2\phi(x) + 2\phi(x) - (n-1)^2\phi(x) = (n+1)^2\phi(x)$. Hence $\phi(x) = n^2\phi(x/n)$ and $\phi(\frac{m}{n}) = \frac{m^2}{n^2}\phi(1)$. For real $x > 0$ and rationals $p \le x \le q$, (B4) gives $p^2\phi(1) \le \phi(x) \le q^2\phi(1)$; letting $p, q \to x$ gives $\phi(x) = x^2\phi(1)$, and evenness covers $x < 0$. □

Theorem 16 (The read is the square)**.** *Let μ satisfy* (B1) *to* (B5)*. Then for every register $f \in \mathbb{R}^8$ and every set A of positions*

$$\mu(f, A) = \phi(1) \sum_{t \in A} f(t)^2.$$

In particular the total read is $\phi(1)\|f\|^2$, and when $\phi(1) \neq 0$ the share of A in the total is

$$\frac{\mu(f, A)}{\mu(f, \mathbb{Z}_8)} = \frac{\sum_{t \in A} f(t)^2}{\sum_t f(t)^2}.$$

Proof. As in the lemma, (B4) makes ϕ even. Take the core register $f = a(e_0 - e_4) + b(e_2 - e_6)$. By (B1) to (B3) its total read is $\phi(a) + \phi(b) + \phi(-a) + \phi(-b) = 2\phi(a) + 2\phi(b)$. By (1), ζf is the register of the same form with (a, b) replaced by $\big((a+b)/\sqrt{2}, (b-a)/\sqrt{2}\big)$. So (B5) reads

$$\phi(a) + \phi(b) = \phi\Big(\frac{a+b}{\sqrt{2}}\Big) + \phi\Big(\frac{a-b}{\sqrt{2}}\Big) \qquad \text{for all real } a, b. \tag{2}$$

With $b = a$ this is $2\phi(a) = \phi(\sqrt{2}\,a)$, that is, $\phi(x/\sqrt{2}) = \phi(x)/2$. Substituting into (2) gives $2\phi(a) + 2\phi(b) = \phi(a+b) + \phi(a-b)$, the parallelogram law. By Lemma 15, $\phi(x) = \phi(1)x^2$, and (B1), (B2) give the formula for every register, on the core or off it. □

The step $b = a$ has the form of the dark port of an interferometer. Send equal values into both arms of a balanced two-port and one output is empty; whatever the read is, all of it must sit in the other output, whose value is $\sqrt{2}$ times larger. So the read doubles when the value grows by $\sqrt{2}$. Among powers only the square does that: the magnitude $|x|$ grows by $\sqrt{2}$ and the fourth power by 4. Among all reads the doubling alone is not enough, and it is the whole of (2) with (B4) that forces the square. Neither mild statement can be dropped. Without (B3), $\phi(x) = k + cx^2$ with $k \neq 0$ meets the rest. Without (B4), take an additive function A on the reals that is linear over the numbers $p + q\sqrt{2}$ with p, q rational and is not continuous (one exists, by the axiom of choice): $\phi = A^2$ meets (B1) to (B3) and (B5), and it is not a multiple of x^2.

Corollary 17 (The read of a complex value is its squared modulus)**.** *Let μ satisfy* (B1) *to* (B5)*, and let $z_2 = f(2) + if(0)$ and $z_3 = f(3) + if(1)$ be the complex coordinates of a core register (Theorem 3). Then*

$$\mu(f, \{0, 2, 4, 6\}) = 2\phi(1)\,|z_2|^2, \qquad \mu(f, \{1, 3, 5, 7\}) = 2\phi(1)\,|z_3|^2,$$

and these two reads are unchanged when the register is multiplied by any number $\cos\alpha + i\sin\alpha$. *The read of a single position is not:* $e_0 - e_4$ *reads* $\phi(1)$ *at position* 0, *and* $\zeta(e_0 - e_4)$ *reads* $\phi(1)/2$ *there.*

Proof. Positions $0, 2, 4, 6$ hold $f(0), f(2), -f(0), -f(2)$, so by Theorem 16 their read is $2\phi(1)(f(0)^2 + f(2)^2) = 2\phi(1)|z_2|^2$; likewise for z_3. Multiplying the register by $\cos\alpha + i\sin\alpha$ multiplies each coordinate by a number of modulus one. By (1), $\zeta(e_0 - e_4)$ holds $1/\sqrt{2}$ at position 0. □

So on the core the reads that a common turn cannot move are those of the complex coordinates, each carried by a pair of positions two ticks apart, and their weights are squared moduli. A single position is half of a complex value, and its read $\phi(f(t))$ is a term of the total and not an outcome. Were it posted alone, the record would hold a count that a common turn moves. The step of Section 4 turns a stationary state in common at its own rate, so, with the unit of Section 3, that count would rise and fall at twice the rate itself, where the counts of Proposition 10 show only differences of rates. So the outcomes a held register offers are sets of positions whose read no common turn moves, and the least of them are the complex coordinates. (B1) is to be read accordingly. It gives the form of a read: one function of each value, added up. It does not say that each term can be posted by itself. On the core, the content of **(O2)** is that the read of one complex value is the sum of one function of each of its two halves. The squared modulus is the form in which the textbook states the rule.

The second theorem starts from the other end, and it is about a different object. It is stated for eight positions that each hold one complex value, $\mathbb{C}^8$. That is not the core of one register, which has two complex coordinates; it is a system with several outcomes, each carrying a complex value, such as registers taken together. The proof uses three of the eight positions, so the statement holds for any number of positions from three up. It takes as part of its hypothesis that the total of a register is measured by squared moduli, as Corollary 17 gives on the core, and it drops the clock, monotonicity and every continuity assumption.

Theorem 18 (Unit total forces the weights)**.** *Call* $\psi \in \mathbb{C}^8$ normalized *when* $\sum_k |\psi_k|^2 = 1$. *Let* $\mu(\psi, A)$ *be defined for normalized* ψ *and sets* A *of positions, and suppose that for all normalized* ψ, χ*:*

(R1) the total read is one: $\mu(\psi, \mathbb{Z}_8) = 1$*;*

(R2) reads are not negative: $\mu(\psi, \{k\}) \geq 0$*;*

(R3) an empty position is silent: $\psi_k = 0$ *implies* $\mu(\psi, \{k\}) = 0$*;*

(R4) the read is taken position by position: $|\psi_k| = |\chi_l|$ *implies* $\mu(\psi, \{k\}) = \mu(\chi, \{l\})$*;*

(R5) reads add: $\mu(\psi, \varnothing) = 0$ *and* $\mu(\psi, A \cup B) = \mu(\psi, A) + \mu(\psi, B)$ *for disjoint* A, B.

Then $\mu(\psi, A) = \sum_{k \in A} |\psi_k|^2$.

Proof. By (R4) there is a function $g : [0, 1] \to \mathbb{R}$ with $\mu(\psi, \{k\}) = g(|\psi_k|^2)$, and by (R5) $\mu(\psi, A) = \sum_{k \in A} g(|\psi_k|^2)$. By (R3), $g(0) = 0$. For $x, y \geq 0$ with $x + y \leq 1$ the register $(\sqrt{x}, \sqrt{y}, \sqrt{1 - x - y}, 0, \ldots, 0)$ is normalized, so (R1) gives

$$g(x) + g(y) + g(1 - x - y) = 1.$$

With $y = 0$: $g(x) + g(1 - x) = 1$, so $g(1) = 1$, and $g(1 - x - y) = 1 - g(x + y)$. Substituting, $g(x + y) = g(x) + g(y)$ whenever $x, y \geq 0$ and $x + y \leq 1$. By (R2), $g \geq 0$, so g is not decreasing. An additive function with $g(1) = 1$ has $g(\frac{m}{n}) = \frac{m}{n}$ for whole $0 \leq m \leq n$, $n \geq 1$, and a function that does not decrease and agrees with x on the rationals of $[0, 1]$ is $g(x) = x$. □

Remark 19 (Three positions are needed)**.** The proof uses registers with three values, and the theorem is false with two. On two positions the hypotheses reduce to $g(0) = 0$, $g \geq 0$ and $g(x) + g(1 - x) = 1$, which $g(x) = \sin^2(\pi x/2)$ meets. So on the core alone, with its two complex coordinates, the square is Theorem 16 and Corollary 17. Theorem 18 is a statement about registers of complex values with three or more positions.

Proposition 20 (Each premise carries weight)**.** (a) *The magnitude read* $\mu(f, A) = \sum_{t \in A} |f(t)|$ *satisfies* (B1) *to* (B4) *and fails* (B5)*: on* $e_0 - e_4$ *its total is* 2 *before the eighth turn and* $2\sqrt{2}$ *after.*

(b) *The fourth-power read* $\mu(\psi, A) = \sum_{k \in A} |\psi_k|^4 \big/ \sum_k |\psi_k|^4$ *satisfies* (R1), (R2), (R3) *and* (R5) *and fails* (R4)*. On the register with squared values* $(\frac{1}{4}, \frac{3}{4}, 0, \dots)$ *it weighs the first position* $\frac{1}{10}$ *and not* $\frac{1}{4}$*; the same squared value* $\frac{1}{4}$ *in the register* $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0, \dots)$ *weighs* $\frac{1}{6}$*.*

(c) *Under the fourth-power read, splitting another position moves the weight here. In the register with squared values* $(\frac{1}{2}, \frac{1}{2}, 0, \dots)$ *the first position weighs* $\frac{1}{2}$*. Split the second position in two, giving* $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0, \dots)$*, and the first position weighs* $\frac{2}{3}$*. Under the square read it stays at* $\frac{1}{2}$*.*

Proof. Direct evaluation. (a) $\zeta(e_0 - e_4) = \big((e_0 - e_4) - (e_2 - e_6)\big)/\sqrt{2}$ by (1), with four values of magnitude $1/\sqrt{2}$. (b) $\frac{1}{16}\big/\big(\frac{1}{16} + \frac{9}{16}\big) = \frac{1}{10}$ and $\frac{1}{16}\big/\big(\frac{1}{16} + \frac{1}{16} + \frac{1}{4}\big) = \frac{1}{6}$. (c) $\frac{1}{4}\big/\big(\frac{1}{4} + \frac{1}{16} + \frac{1}{16}\big) = \frac{2}{3}$. □

Why the position read is physical. Part (c) is the reason (R4) is more than a convenience. If a read at one position depended on how the rest of the register is divided, then an act elsewhere that only splits a position, touching nothing here, would change what is read here. The square is the one read of this family for which a split elsewhere is invisible. That makes **(O2)** a statement with content: a read that is not taken position by position lets one place steer another without any posting passing between them.

What is proved and what is owed. Given the position read and blindness to a common turn, the square is a theorem: Theorem 16 derives it, and Theorem 18 derives the weights from a unit total. Neither uses the cost of a comparison, and the square is not a consequence of that cost. Owed is **(O2)**, the position read itself, as a consequence of the ledger rather than a premise; blindness to a common turn stands beside it as a premise. The reason given above for the position read, that a split elsewhere must not move a weight here, is a reason about outcomes, and so about whole complex values. It does not reach inside one complex value, where **(O2)** asks that the read be a sum over the two halves; for that the paper gives no reason beyond the form. Owed also, under **(O1)**, is that the outcomes of a laboratory apparatus *are* reads of a register's positions: the theorems are about those reads. For reads along other orthogonal splittings of a space the square is an additional hypothesis in this paper. Gleason's theorem [11] derives it, in dimension three and above, from additivity over every orthogonal splitting.

6 The fourth postulate: what posts

The textbook's fourth rule has two halves. A measurement returns one of the values of an operator, and afterwards the state is the part that matches. The first half hides the hardest question in the subject: what counts as a measurement? On the record the answer has a fixed form. A measurement is a posting. What is proved below is the form a posting has at a register. That one outcome occurs, and what a posting *is* for a body in a laboratory, is the owed statement **(O1)**.

6.1 Acts and their parts

Definition 21 (Act, part)**.** An *act* on a real or complex inner-product space is an operator A that is self-adjoint and is its own undoing: $A = A^\dagger$ and $A^2 = \mathbf{1}$. Its two outcomes are $s = +1$ and $s = -1$, and the *part* of a state ψ selected by outcome s is $P_s^A\psi$ with

$$P_s^A = \tfrac{1}{2}(\mathbf{1} + sA).$$

An act is the least that a two-valued recognition can be: it has two outcomes, and doing it twice returns what was there. The operators $P_\pm^A$ are orthogonal projectors with $P_+^A + P_-^A = \mathbf{1}$ and $P_+^A P_-^A = 0$. So an act read twice agrees with itself, and $\|P_+^A\psi\|^2 + \|P_-^A\psi\|^2 = \|\psi\|^2$. Throughout this section the weight of a part is its squared length, $\|P_s^A\psi\|^2$ for outcome s. Theorem 16 gives this when the parts are sets of positions; for other parts it is a hypothesis of the section. That the state after outcome s is the part $P_s^A\psi$ is the definition of a part, which is the textbook's rule; deriving it belongs to **(O1)**. For two acts A, B we read B first and then A; the part selected by the outcomes (s, u) is $P_s^A P_u^B\psi$.

Theorem 22 (The price of disturbance)**.** *Let A, B be acts and ψ a state. Parts with different outcomes of A never overlap. For parts with the same outcome s of A and different outcomes of B,*

$$\operatorname{Re}\langle P_s^A P_+^B\psi,\ P_s^A P_-^B\psi\rangle = -\frac{s}{8}\langle\psi,\ B(AB - BA)\psi\rangle,$$

and on a real space the left side is the overlap itself.

Proof. For different outcomes of A the overlap is $\langle P_u^B\psi,\ P_s^A P_{-s}^A P_{u'}^B\psi\rangle = 0$. For the same outcome, since P_s^A is a self-adjoint projector, the overlap is $\langle\psi,\ P_+^B P_s^A P_-^B\psi\rangle$. Now

$$8\,P_+^B P_s^A P_-^B = (\mathbf{1} + B)(\mathbf{1} + sA)(\mathbf{1} - B) = (\mathbf{1} - B^2) + s\,(\mathbf{1} + B)A(\mathbf{1} - B) = s\,(A - AB + BA - BAB).$$

The operator $BA - AB$ is anti-self-adjoint, so $\langle\psi,\ (BA - AB)\psi\rangle$ has no real part, and on a real space it is zero. What is left is $A - BAB = -B(AB - BA)$, using $B^2 = \mathbf{1}$; it is self-adjoint, so its quadratic form is real. □

Corollary 23 (What reading first costs)**.** *Reading B first changes the weight of outcome s of A by*

$$\sum_{u=\pm}\|P_s^A P_u^B\psi\|^2 - \|P_s^A\psi\|^2 = \frac{s}{4}\langle\psi,\ B(AB - BA)\psi\rangle,$$

which is minus twice the real part of the overlap of Theorem 22. Its size is at most $\frac{1}{2}\|\psi\|^2$, the real part of the overlap is at most $\frac{1}{4}\|\psi\|^2$, and both bounds are reached: when $AB = -BA$ and $A\psi = s\psi$ the overlap is exactly $\frac{1}{4}\|\psi\|^2$.

Proof. $P_s^A\psi = P_s^A P_+^B\psi + P_s^A P_-^B\psi$, so $\|P_s^A\psi\|^2$ is the sum of the two squared lengths plus twice the real part of the overlap. For the bound, $\langle\psi,\ B(AB-BA)\psi\rangle = \langle\psi,\ BAB\psi\rangle - \langle\psi,\ A\psi\rangle$, and A and BAB both keep lengths, so each term is at most $\|\psi\|^2$ in size. If $AB = -BA$ then $B(AB-BA) = 2BAB = -2A$, and with $A\psi = s\psi$ the real part of the overlap is $-\frac{s}{8}\cdot(-2s)\|\psi\|^2 = \frac{1}{4}\|\psi\|^2$; the imaginary part vanishes, because $\langle\psi,\ BA\psi\rangle = s\langle\psi,\ B\psi\rangle = \langle\psi,\ AB\psi\rangle$. □

Theorem 24 (Acts can be read together exactly when they commute)**.** *For two acts A, B the following are equivalent.*

(i) $AB = BA$.

(ii) For every state the four parts $P_s^A P_u^B\psi$ are mutually orthogonal.

(iii) For every state, reading B first leaves the weights of A unchanged.

Proof. If $AB = BA$ the four operators $P_s^A P_u^B$ are projectors with products zero, which gives (ii). (ii) implies (iii) by Corollary 23. If (iii) holds then $\langle\psi,\ B(AB-BA)\psi\rangle = 0$ for every ψ. The operator $B(AB-BA) = BAB - A$ is self-adjoint, and a self-adjoint operator whose quadratic form vanishes is zero, so $BAB = A$; multiplying by B on the left gives $AB = BA$. □

The same statement holds for any finite family of acts; that is proved in work not yet published. In the textbook's terms Theorem 24 is Lüders' theorem [14]: measurements of this kind leave each other's weights alone exactly when they commute. Corollary 23 puts a number on the disturbance when they do not.

What this says. The textbook teaches complementarity and the disturbance of one measurement by another as separate lessons. Here they are one identity. Two acts either commute, and then every state divides cleanly among their joint outcomes, or they do not, and then there is a state on which reading one first moves the weights of the other, by an amount fixed by their commutator, never more than a half. There is no third case, and no appeal to the clumsiness of an apparatus.

6.2 A contact at a register

Theorem 25 (A contact leaves the state or applies the crossing)**.** *Let X be a self-adjoint operator on a complex inner-product space with $X^2 = \mathbf{1}$, and for real θ let*

$$C(\theta) = \cos\theta\,\mathbf{1} + i\sin\theta\,X.$$

(a) *$C(\theta)$ is unitary and $C(\theta)C(\theta') = C(\theta+\theta')$.*

(b) *Let P be an orthogonal projector with $PXP = 0$. If $\psi = P\psi$, then*

$$C(\theta)\psi = \cos\theta\,\psi + i\sin\theta\,X\psi,$$

where the first term lies in the range of P and the second is orthogonal to it. The part in the range of P is ψ itself, unchanged, with weight $\cos^2\theta\,\|\psi\|^2$; the part outside is $X\psi$, with weight $\sin^2\theta\,\|\psi\|^2$.

(c) *$X^n = \mathbf{1}$ for even n and $X^n = X$ for odd n.*

(d) *If $X\psi = s\psi$ with $s = \pm 1$, then $C(\theta)\psi = e^{is\theta}\psi$ for every θ.*

(e) *Let P be an orthogonal projector with $PX = XP$, and let $\Pi_\pm = \frac{1}{2}(\mathbf{1} \pm X)$. If $\psi = P\psi$, then*

$$C(\theta)\psi = e^{i\theta}\,\Pi_+\psi + e^{-i\theta}\,\Pi_-\psi,$$

both terms lie in the range of P, and their lengths do not depend on θ.

Proof. (a) $C(\theta)C(\theta') = \cos\theta\cos\theta' - \sin\theta\sin\theta' X^2 + i(\sin\theta\cos\theta' + \cos\theta\sin\theta')X = C(\theta+\theta')$, and $C(\theta)^\dagger = C(-\theta)$. (b) $P\psi = \psi$ and $PX\psi = PXP\psi = 0$. X keeps lengths. (c) $X^2 = \mathbf{1}$. (d) $C(\theta)\psi = (\cos\theta + is\sin\theta)\psi$. (e) $\Pi_\pm$ are the projectors on the two signs of X, they commute with P, and by (d) $C(\theta)$ multiplies the range of $\Pi_\pm$ by $e^{\pm i\theta}$. □

At a register X is the *crossing*, the operator that one posting across the cut applies, and θ is the strength of the contact; a whole crossing is $\theta = \pi/2$. Parts (b) and (e) are the two things a crossing can do to a part P of the state: move it off itself, or keep it.

Part (b) has the form of the textbook's fourth postulate. Where the crossing moves the state off where it stood, a contact divides it into two parts. In one nothing was posted, and that part is *exactly* the state as it was. In the other the one crossing was applied. Their squared lengths are $\cos^2\theta$ and $\sin^2\theta$. Part (e) is the other case. Where the crossing keeps the state where it stood, nothing is moved at all: the contact turns the two parts of definite sign in opposite senses and changes the length of neither. It reads one two-valued quantity, the sign of the crossing, and nothing else; a state on which that sign is already definite is left alone by a contact of any strength (part d). What the theorem does not give, in either case, is the event: that one of the two parts is posted and the other is not. That is **(O1)**. Part (a) says that weak contacts add up to strong ones, and part (c) that a state crossed twice has not been crossed.

Which operator the crossing is. On the record's working statement the crossing acts on a register of three binary axes by the sign of the register's own moves, times a flip on the channel the posting came from. Take P to be the projector on the register's held states, its code. Whether the crossing takes the code off itself, case (b), or keeps it and reverses the register's own tick, case (e), depends on one count: the number of the register's own moves in a window of the clock. The count is not settled, and it is shared with the law of motion [7]. The record's working reading is one recognition per tick, and under it the case is (e): a held register stays held, and a contact reads the sign of its moves and nothing else. Case (b) for the register is the other reading. The register statements are proved in work not yet published and are not used elsewhere in this paper.

The projector in part (b) need not be the register's code. Write the crossing as $X = \Sigma \otimes F$, the sign Σ on the register and the flip F on the channel outside, and let that channel stand in a state e that the flip takes to an orthogonal one, $\langle e, Fe\rangle = 0$. Then $P' = \mathbf{1} \otimes |e\rangle\langle e|$ has $P'XP' = 0$, and part (b) applies to every state $\psi \otimes e$: with squared length $\cos^2\theta$ the channel stands where it stood and the register is exactly as it was, and with squared length $\sin^2\theta$ the channel is flipped and the register carries the sign. So the division of part (b), with the textbook's weights, holds under either reading of the register, on whichever side the crossing moves.

6.3 What is owed

A held register posts only at a contact: without a crossing, the step of Section 4 is unitary and nothing is written. What a contact is at the register is Theorem 25. What it is for a body on a laboratory bench is not yet a theorem:

> **(O1)** *Attachment.* State which postings of the ledger a laboratory system, its apparatus and its outcome are: which counted event takes an entry from a held history, how often, and what it does to the history.

It includes the event itself: Theorem 25 gives the two parts and their weights, and **(O1)** is what makes one of them the record.

Two things are on the record about **(O1)** today, both working statements and neither a theorem. The first: *only a contact resolves.* Histories of a body add until a posting tells them apart, and there is no rate of resolution that belongs to the body alone. The second is a candidate for the counted event when the body is massive: the ledger's own strain, which the textbook calls the gravitational field. Two histories of one body that are already apart stop being one history when the gravitational self-energy of the difference between their two mass distributions has built up one unit of action. The form of this rule is Penrose's [16] (see also Diósi [17]); what the record adds is the reason, that a record which is one and is counted can hold two strains only while they differ by less than one of its own entries. Its unit is open, and it is adopted as a working statement, not derived. Section 8 shows that eleven of the twelve paradoxes wait on **(O1)** and on nothing deeper.

7 The fifth postulate: two contacts on one record

The textbook's fifth rule is that the states of two systems together are the product of their two spaces, and that an operator of one system acts as itself on its own factor and as the identity on the other. From this rule come entanglement and the violation of Bell's inequality, and with them the oldest unease about the subject: how can a choice here be correlated with an outcome there?

On the record the unease starts from a false picture. There are not two worlds joined by a wire. There is one record. Two registers are two blocks of axes of one cube, and the product rule is what one cube looks like when its axes are sorted into two groups.

7.1 Two blocks of one cube

A cube with N binary axes has 2^N configurations $p \in \{0,1\}^N$, and its states are the functions on them, with real or with complex values. Every statement of this section holds for both, and Section 7.3 returns to the difference. Write X_j for the flip of axis j and Z_j for the sign of axis j, which multiplies a configuration by $(-1)^{p_j}$. *The cube's own generators* are

$$\gamma_j = Z_0 Z_1 \cdots Z_{j-1} X_j, \qquad j = 0, \dots, N-1 \ :$$

the flip of one axis, carrying the sign of every axis before it (the construction of Jordan and Wigner [15]). Sort the axes into a first block of m and a second block of n, $N = m + n$. An

operator *touches only the first block* when it has the form $a \otimes \mathbf{1}$ in the splitting $\{0,1\}^N = \{0,1\}^m \times \{0,1\}^n$, and only the second when it has the form $\mathbf{1} \otimes b$.

Theorem 26 (Blocks of one cube)**.** (a) *The states of the cube are the product of the states of its two blocks, over the real numbers or over the complex numbers according to the values, and every operator that touches only the first block commutes with every operator that touches only the second, with no condition on either.*

(b) *The cube's own generators are acts, and any two of them anticommute: $\gamma_j\gamma_k = -\gamma_k\gamma_j$ for $j \neq k$. In particular a generator of the first block and a generator of the second do not commute, and by Theorem 24 they cannot be read together.*

(c) *Let $\Gamma = Z_0 \cdots Z_{m-1}$ be the sign of the first block. For an axis $m + j$ of the second block, $\Gamma\gamma_{m+j} = \mathbf{1} \otimes \gamma'_j$, where γ'_j is the jth generator of the second block taken as a cube of its own. So with the first block's sign taken off, the second block's generators touch only the second block.*

Proof. (a) A configuration of the cube is a pair of configurations of the blocks, so functions on the cube are sums of products; $(a \otimes \mathbf{1})(\mathbf{1} \otimes b) = a \otimes b = (\mathbf{1} \otimes b)(a \otimes \mathbf{1})$. (b) X_j and Z_j are symmetric, square to $\mathbf{1}$ and anticommute; operators of different axes commute. So γ_j is symmetric with $\gamma_j^2 = \mathbf{1}$, and for $j < k$ the product $\gamma_j\gamma_k$ meets exactly one anticommuting pair, X_j against the Z_j inside γ_k. (c) $\gamma_{m+j} = \Gamma \cdot Z_m \cdots Z_{m+j-1}X_{m+j}$ and $\Gamma^2 = \mathbf{1}$. □

Part (a) is the fifth postulate, for two blocks of one cube, and it is nearly a tautology: the product is how a single configuration space divides. Parts (b) and (c) are where the content is. The cube has one set of generators, and they are not sorted into systems: each carries the sign of everything before it. Two parties whose acts are raw generators of different blocks would have no joint outcomes at all. Among the cube's own generators, the two parties' acts have joint outcomes exactly when the second party's carry the first block's sign as well, and then each party's acts touch its own block only.

7.2 Three ceilings

Give each of two parties two acts, A_0, A_1 and B_0, B_1, on one state ψ of unit length. For a pair of acts that can be read together, the correlation of their outcomes is

$$E(x,y) = \sum_{s,u=\pm} s\,u\,\|P_s^{A_x}P_u^{B_y}\psi\|^2,$$

with the squared lengths of Section 6 as weights, and the test of Clauser, Horne, Shimony and Holt [18] is the sum

$$\mathcal{S} = E(0,0) + E(0,1) + E(1,0) - E(1,1).$$

Theorem 27 (Three ceilings)**.** (a) Acts that only read. *If every act multiplies each configuration by a sign, $A_x = \mathrm{diag}(a_x)$ and $B_y = \mathrm{diag}(b_y)$ with $a_x, b_y = \pm 1$, then $|\mathcal{S}| \leq 2$.*

(b) Acts that write. *If A_x and B_y are any acts with $A_xB_y = B_yA_x$ for all x, y, then $E(x,y) = \langle A_x\psi,\, B_y\psi\rangle$, a real number, and $|\mathcal{S}| \leq 2\sqrt{2}$.*

(c) Tables that do not signal. *There is a table of probabilities $p(s,u\,|\,x,y)$ in which each party's own probabilities do not depend on the other's choice, with $\mathcal{S} = 4$.*

The three ceilings are strictly apart, $2 < 2\sqrt{2} < 4$*, and each is reached.*

Proof. (a) Diagonal acts commute, and $E(x,y) = \sum_p |\psi(p)|^2 a_x(p) b_y(p)$. For each configuration $a_0(b_0+b_1) + a_1(b_0-b_1) = \pm 2$, because one of $b_0 \pm b_1$ is zero and the other is ± 2. Averaging with the weights $|\psi(p)|^2$, which sum to one, gives $|\mathcal{S}| \le 2$; a configuration with all four signs $+1$ reaches 2.

(b) Commuting acts have projectors that commute, so each $P_s^A P_u^B$ is a projector, and

$$\sum_{s,u} su \, \|P_s^A P_u^B \psi\|^2 = \Big\langle \psi, \ \sum_s s P_s^A \sum_u u P_u^B \, \psi \Big\rangle = \langle \psi, \ AB\psi \rangle = \langle A\psi, \ B\psi \rangle.$$

Put $a_x = A_x\psi$ and $b_y = B_y\psi$; these are unit vectors, since acts keep lengths. Then

$$\mathcal{S} = \langle a_0, \, b_0 + b_1 \rangle + \langle a_1, \, b_0 - b_1 \rangle \le \|b_0 + b_1\| + \|b_0 - b_1\| \le \sqrt{2\big(\|b_0+b_1\|^2 + \|b_0-b_1\|^2\big)} = \sqrt{2 \cdot 4},$$

by the Cauchy–Schwarz inequality twice and the parallelogram law. The same holds for $-\mathcal{S}$. Theorem 28 reaches $2\sqrt{2}$.

(c) Take $p(s,u \,|\, x,y) = \frac{1}{4}\big(1 + su\,\varepsilon_{xy}\big)$ with $\varepsilon_{00} = \varepsilon_{01} = \varepsilon_{10} = 1$ and $\varepsilon_{11} = -1$. These are probabilities; summing over u gives $\frac{1}{2}$ whatever y is, and likewise over s; and $E(x,y) = \varepsilon_{xy}$, so $\mathcal{S} = 4$ [21]. □

The middle ceiling is Tsirelson's [20]. The proof shows where the $\sqrt{2}$ comes from: it is the diagonal of the unit square in the geometry of squared weights, and the ceiling $2\sqrt{2}$ is the square's two diagonals together, the sum and the difference of two unit vectors at a right angle. The dark port of Section 5 turns on the same parallelogram law.

Theorem 28 (One cube reaches the middle ceiling)**.** *Take a cube of* $n + n$ *axes,* $n \ge 2$*. Let* g_0, g_1 *be the cube's own generators on two axes of the first block, and* h_0, h_1 *the generators on the matching axes of the second block with the first block's sign taken off. Let* ψ *put equal weight on the configurations whose two blocks agree. Then*

$$\langle \psi, \ g_i h_j \psi \rangle = \delta_{ij},$$

and the acts $A_x = g_x$*,* $B_y = \big(h_0 + (-1)^y h_1\big)/\sqrt{2}$ *commute across the blocks and give* $\mathcal{S} = 2\sqrt{2}$ *exactly.*

Proof. By Theorem 26, $g_i = M_i \otimes \mathbf{1}$ and $h_j = \mathbf{1} \otimes M_j$, where M_0, M_1 are two generators of a cube of n axes: symmetric, squaring to $\mathbf{1}$, anticommuting. With $\psi = 2^{-n/2} \sum_a e_a \otimes e_a$,

$$\langle \psi, \ (M_i \otimes M_j)\psi \rangle = 2^{-n} \sum_{a,b} (M_i)_{ab} (M_j)_{ab} = 2^{-n} \operatorname{tr}(M_i M_j).$$

For $i = j$ the trace is $\operatorname{tr}\mathbf{1} = 2^n$. For $i \ne j$, $\operatorname{tr}(M_i M_j) = \operatorname{tr}(M_j M_i) = -\operatorname{tr}(M_i M_j)$, so it is zero. Each B_y is symmetric, and $B_y^2 = \frac{1}{2}\big(h_0^2 + h_1^2 \pm (h_0 h_1 + h_1 h_0)\big) = \mathbf{1}$, so B_y is an act; A_x and B_y commute by Theorem 26(a). Then $E(x,y) = \langle \psi, \ g_x B_y \psi \rangle = \big(\delta_{x0} + (-1)^y \delta_{x1}\big)/\sqrt{2}$, so $E(0,0) = E(0,1) = E(1,0) = 1/\sqrt{2}$ and $E(1,1) = -1/\sqrt{2}$, and $\mathcal{S} = 4/\sqrt{2} = 2\sqrt{2}$. □

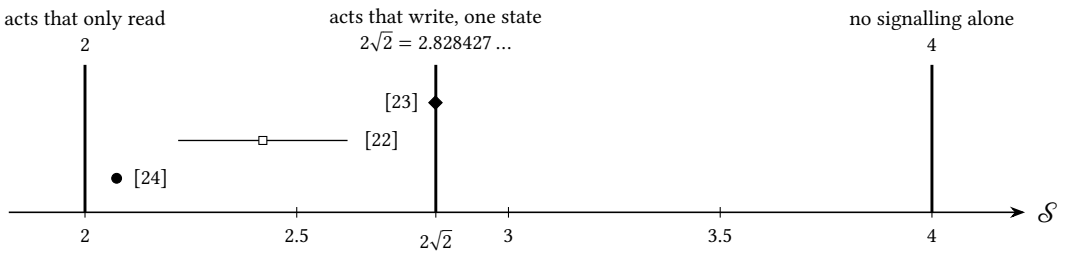


Figure 2. The three ceilings of the correlation test, and three measurements. The ceiling at 2 holds for acts that only read what a configuration already holds; the ceiling at $2\sqrt{2}$ for acts that write, read on one state with squared weights; the ceiling at 4 for any table of probabilities in which neither party's choice shows at the other. The middle value is computed on one cube of four axes with the acts of Theorem 28. Measured: 2.0747 ± 0.0033 (circle), 2.42 ± 0.20 (square, with its error bar), and 2.82759 ± 0.00051 (diamond), the closest approach on file. No measurement has passed $2\sqrt{2}$.

7.3 What the three ceilings say

Bell. Part (a) is Bell's theorem [19] in the record's terms: if each party only reads a value that the configuration already holds, the configuration itself is the hidden variable and the sum cannot pass 2. Experiments pass 2 (Figure 2). What the violation excludes is values held in advance. The record's reading of what is left is that the outcome is written by the act and not read off an earlier entry. That reading needs no influence travelling between the parties: by Theorem 26(a) each party's acts touch only its own block, and by Theorem 24 acts that commute do not move each other's weights.

Tsirelson. Part (c) shows that "probabilities, and no signalling" is not enough to explain why nature stops at $2\sqrt{2}$: that much allows 4. What stops the sum at $2\sqrt{2}$ is part (b): one state, acts that are their own undoing and commute, and weights that are squares.

What real values cannot do. Every statement of this section holds for real values and for complex ones, and the correlation test cannot tell them apart: its middle ceiling is $2\sqrt{2}$ for both. A test with three parties and two sources can, on one condition [29]. If the two sources share nothing, registers composed as real blocks reach at most about 7.66 in it, a bound found numerically and not known to be tight; composed over the complex numbers they reach $6\sqrt{2} \approx 8.49$. The measured values are 8.09(1) on one chip [30] and 7.83(3) with the sources and the measurements kept apart [31]. So real blocks whose sources share nothing are not the composition law of two laboratories. The condition matters: real blocks whose sources are allowed to share a suitable state reproduce the complex values. In this network one shared entangled bit is necessary and sufficient [32], and when the independence of the sources is defined by what can be observed, every correlation of the complex theory in a finite network is reached [33, 34]. The next proposition says what that state is on the record.

Proposition 29 (The complex product inside the real one)**.** *Let K be the core with its unit $i = S^2$ (Theorem 3), and on the real product $K \otimes K$ of two cores put $i_1 = i \otimes \mathbf{1}$ and $i_2 = \mathbf{1} \otimes i$. Two ticks of the joint clock $S \otimes S$ are $i \otimes i = i_1 i_2$, and $(i \otimes i)^2 = \mathbf{1}$. So the product splits into the part that two ticks reverse and the part that two ticks keep, each of real dimension eight. On the first, $i_1 = i_2$: with that one unit it is a complex space of dimension four, and it is the product of the two cores over the complex numbers. On the second, $i_1 = -i_2$.*

Proof. $(S \otimes S)^2 = S^2 \otimes S^2 = i \otimes i$, and $(i \otimes i)^2 = i^2 \otimes i^2 = \mathbf{1}$. The operators i_1, i_2 commute and each squares to $-\mathbf{1}$. If $i_1 i_2 v = -v$, apply i_2: $-i_1 v = -i_2 v$. If $i_1 i_2 v = v$, the same step gives $i_1 v = -i_2 v$. The trace of i on K is zero, because $i^2 = -\mathbf{1}$ on a real space, so the trace of $i \otimes i$ is zero and the two parts have equal dimension, eight. The map $(f, g) \mapsto \frac{1}{2}(f \otimes g - if \otimes ig)$ lands in the first part, is linear over the complex numbers in each argument for the unit $i_1 = i_2$, since $i_1 \frac{1}{2}(f \otimes g - if \otimes ig) = \frac{1}{2}(if \otimes g + f \otimes ig)$, and its values span the part; a complex space of dimension $2 \cdot 2$ spanned by such a map is the complex product. □

So two ticks of the one clock are the unit i on one register and the number -1 on a pair composed over the complex numbers: "one clock, therefore one i" is not an inference. The unit of a pair is one register's own two ticks with the other left alone, and the two choices agree exactly on the part that two ticks of the joint clock reverse. That part is what real blocks would have to share: both registers on the same sense of the two-tick turn. On the record's picture of two registers, the network measurements are therefore evidence that registers held in two laboratories agree on their unit. They do not by themselves tell the complex product from real blocks that share that state, and they do not measure the owed statement. This is where the first postulate meets the fifth.

7.4 What is owed

Theorems 26 to 28 are about two blocks of one cube. The fifth postulate, as the textbook uses it, is about two separate laboratories.

> **(O3)** *Composition.* Two registers held by two recognizers are two blocks of one cube, and their joint states lie in the part of the product that two ticks of the clock reverse, which is their product over one complex unit (Proposition 29). Each recognizer's acts are the generators of its own block with the other block's sign taken off, and the read is the one read of the whole cube.

Given **(O3)**, with **(O2)** for the read of the whole cube and the squared weights of Section 6 for parts that are not sets of positions, the fifth postulate and Tsirelson's ceiling are Theorems 26 and 27. Without it, what is left for two parties is a table of probabilities with no signalling, whose ceiling is 4 (Theorem 27c). The statement is not proved. The network tests above bear on it as evidence: they exclude real blocks whose sources share nothing. There is a working candidate on the record, in words only: two channels compose as one cube and not as two cubes side by side, so that the posts of one channel carry the other's sign. Which of the two carries it is the open order of the blocks: by Theorem 26(c) only the second block's generators carry the first block's sign. In that form the crossing of Theorem 25, a sign times a flip, is a generator of the second block, and the candidate's claim is that the contact law and the composition law are one statement. Three gaps in it are known and stand: with more than two channels the sign is that of every earlier channel; acts that anticommute fix their operators only up to signs and a choice of basis; and it is a single signed flip, not a product of two entries, that has this form.

8 Twelve paradoxes, three lines each

A paradox of quantum mechanics is a place where the five postulates, taken as bare rules, collide with a picture of what is "really" there. If the postulates are theorems about a record, each paradox should come apart into the theorem it uses and the owed statement it waits on. Here are the twelve best known. Each gets three lines: the puzzle; what the record says, with the theorem it uses; and what number comes back and what is owed.

1. The double slit and the delayed-choice eraser

The puzzle. One quantum sent at two slits builds fringes, as if it took both paths. Mark which path it took and the fringes vanish. Erase the mark later, even after the quantum has landed, and fringes return in sorted subsets of the counts [25].

On the record. Two paths that nothing tells apart are one held history with two parts, and their values add (Section 4). A mark is a value held in a second register that tells the parts apart. The parts then sit at different positions of the two registers taken together, and reads add over positions (Theorem 16), so the bands go. Nothing has to be posted about the mark for that. A landed dot is a posting, and a posting is never erased, so a later read of the marker cannot touch what has landed. Read blind to the path, the marker gives one of two outcomes, and sorting the landed dots by it shows bands and their complement. Had the mark been posted before that read, the sorted halves would show none.

Returns, and owed. The textbook's values: no fringes in the unsorted counts at any delay; fringes and their complement in the sorted halves, adding to the flat total; fringe contrast V and path contrast D with $V^2 + D^2 \leq 1$ (measured 0.97 ± 0.03 [26]). Owed: **(O1)**, **(O2)**, and **(O3)** for the two detectors.

2. The Born rule

The puzzle. Why is a probability the square of a value, and not the value or its fourth power?

On the record. A read taken position by position, with a total blind to a common turn, is the sum of squares: equal values into the balanced two-port leave one output empty, so the read must double when the value grows by $\sqrt{2}$ (Theorem 16). With a unit total the weights are $|\psi_k|^2$ (Theorem 18).

Returns, and owed. $p_k = |\psi_k|^2$ exactly. The read is a square, and the square of a sum of three values has terms in pairs only, so three paths interfere only in pairs: the third-order term is zero (measured below 10^{-2} of the pair term [27]). Owed: **(O2)**, **(O1)**.

3. Why complex numbers

The puzzle. The formalism needs i. A version over the real numbers was long thought to do the same work [28]; experiments on small networks now exclude it when the two sources share nothing [29, 30, 31].

On the record. The number i is two ticks of the clock, on the registers that four ticks reverse (Theorem 3), and the tick has no complete set of real stationary states (Theorem 1). Each register's own two ticks are its i; two ticks of the joint clock on a pair of registers are the number -1, not a unit (Proposition 29).

Returns, and owed. In the network test, real blocks whose sources share nothing reach at most about 7.66, and complex ones $6\sqrt{2} \approx 8.49$; measured 8.09(1) [30] and, with the sources and the measurements kept apart, 7.83(3) [31]. The record returns the complex value if two registers agree on their unit, which is **(O3)**; the measurements are evidence for that, and what they exclude is real blocks that share nothing. Owed: **(O3)**, **(O2)**, **(O1)**.

4. Schrödinger's cat and the measurement problem

The puzzle. The linear step never picks an outcome. A cat coupled to one atom should be left as alive plus dead [35]; every look finds one or the other.

On the record. The order is the other way round. Records are definite first: a posting is an entry, never a sum of entries. The adding of values is what the law does between postings (Theorem 8). A measurement is a posting. At a contact a state divides in two parts (Theorem 25): where the crossing moves it off where it stood, a part left exactly as it was and a part with one crossing applied, with weights $\cos^2\theta$ and $\sin^2\theta$; where the crossing keeps it, the two signs of the crossing, turned in opposite senses. A cat is in contact with its surroundings at every tick.

Returns, and owed. Weights are squares; a held register posts only at a contact; no number for a laboratory body. Owed: **(O1)**, of which this paradox is the plainest statement, and **(O2)**.

5. Where the quantum world ends

The puzzle. Molecules of two thousand atoms interfere [36]. Dust grains do not. Where is the line, and what draws it?

On the record. The line is not a size. The parts of a held history show bands for exactly as long as nothing tells them apart. A value held elsewhere that tells them apart takes the bands out of every read of the body alone; once that value is posted they are gone for good, because a posting is never erased. A large warm body is told apart constantly: each scattered molecule and each emitted photon is a contact that carries a mark away.

Returns, and owed. No number of its own for any laboratory body. Measured loss of contrast follows the count of scattering events [37]. Owed: **(O1)**, with the working candidate of Section 6 for massive bodies, and **(O2)**.

6. Wigner's friend

The puzzle. A friend measures inside a sealed laboratory and has a result. Wigner, outside, describes the whole laboratory as a sum of both results and can in principle make them interfere [38, 39]. Which description is right?

On the record. A written result is a posting, and a posting is never undone. Making the friend's two results interfere needs them to be one history with no record. Both cannot be true of the same friend. A friend that writes nothing is a held register and interferes; a friend that writes is a record and does not.

Returns, and owed. For friends that write nothing, which is every friend tested so far (photons), the textbook's correlations: measured 2.416 ± 0.075 against the bound 2 [40], and [41]. Owed: **(O1)** (which posting a friend's result is), **(O3)** and **(O2)**.

7. Bell's inequality

The puzzle. Outcomes at two distant stations are correlated more strongly than any set of values written in advance allows [42, 19].

On the record. Acts that only read values already in the configuration cannot pass 2 (Theorem 27a). Passing 2 means the outcome is written by the act. There is one record with two blocks; each party's acts touch its own block (Theorem 26), and acts that commute do not move each other's weights (Theorem 24). Nothing travels.

Returns, and owed. Up to $2\sqrt{2}$, with no signalling; measured 2.42 ± 0.20 with every loophole closed [22]. Owed: **(O3)**, **(O2)**, **(O1)**.

8. Tsirelson's bound

The puzzle. Why does the correlation stop at $2\sqrt{2}$, when the absence of signalling alone would allow 4 [21]?

On the record. One state, acts that are their own undoing and commute, weights that are squares: the sum is at most $2\sqrt{2}$, the two diagonals of the unit square together (Theorem 27b), and one cube reaches it (Theorem 28). Tables that merely do not signal reach 4 (Theorem 27c).

Returns, and owed. $2\sqrt{2} = 2.828427\ldots$; measured 2.82759 ± 0.00051 [23]; never passed. Not resolved until **(O3)** is a theorem: this paradox *is* the composition law. Owed: **(O3)**, **(O2)**, **(O1)**.

9. Contextuality

The puzzle. The outcomes of quantum reads cannot have been fixed in advance independently of which other reads are made with them [43, 44].

On the record. A read is a posting, not a look at an entry already there. Take two of the register's three binary axes and nine acts in a square, each row and each column commuting, so that each can be read together (Theorem 24). The three rows multiply to $+1$; the columns to $+1$, $+1$ and -1. No table of nine signs written in advance has both properties. It is a count.

Returns, and owed. The textbook's value 6 for the square's witness against 4 for signs written in advance; measured 5.46 ± 0.04 [45]. Owed: **(O1)**, **(O2)**, and **(O3)** where two ions carry the two axes.

10. Leggett and Garg: is it there when nobody looks?

The puzzle. Is a large object in one definite state between looks, and can a look leave it undisturbed [46]?

On the record. Definiteness belongs to records. Between postings a held register has values, not a settled state. A look is a posting, and Corollary 23 gives what looking first costs: a weight moves by a fixed amount, up to one half. A register that is only permuted and read stays under the bound; a turn between looks is what passes it.

Returns, and owed. The textbook's ceiling $\frac{3}{2}$ for three looks with two-valued records, reached at a turn of $\pi/3$ between looks; measured violation of the bound 1 by 84 standard deviations on a superconducting loop [47]. Owed: **(O1)**, **(O2)**.

11. Tunnelling time

The puzzle. How long does a particle spend inside a barrier it has too little energy to enter? Each definition gives a different answer, and some do not grow with the barrier's width [48].

On the record. Time is a count of postings. A value held inside a barrier posts nothing, so the crossing has no count of its own. Each "tunnelling time" is the read of a different apparatus making a different contact.

Returns, and owed. No number of its own. Two clocks, two answers: at most 1.8 attoseconds for hydrogen by angular streaking [49]; 0.61(7) milliseconds for cold atoms by a spin clock [50]. Owed: **(O1)**.

12. Spin and statistics

The puzzle. Particles of half-integer spin exclude one another; those of integer spin crowd together [51]. Why should spin decide?

On the record. The core is the set of registers whose sign four ticks, half the clock's cycle, reverse (Section 3). One full turn reverses the sign of a half-integer spin's state. That the two reversals are one is not proved. Given the odd sign under exchange, exclusion follows at once, since a state odd under exchange vanishes when two entries are equal.

Returns, and owed. Signs only, -1 and $+1$, with no violation; none has been found. Not resolved. Owed: **(O4)**, that exchanging two identical records is one full turn of one about the other, together with which turn of a body four ticks are.

8.1 The count

	Paradox	(O1)	(O2)	(O3)	(O4)	Theorems used
1	Double slit, eraser	•	•	•		8, 16
2	Born rule	•	•			16, 18
3	Complex numbers	•	•	•		1, 3, 29
4	The cat; measurement	•	•			8, 25
5	Where the quantum world ends	•	•			16, 25
6	Wigner's friend	•	•	•		25, 27
7	Bell's inequality	•	•	•		24, 26, 27
8	Tsirelson's bound	•	•	•		27, 28
9	Contextuality	•	•	•		24
10	Leggett and Garg	•	•			22, 24
11	Tunnelling time	•				25
12	Spin and statistics				•	3
	Count	11	10	6	1	

Table 2. The twelve paradoxes against the four owed statements, with the theorems each one uses. A paradox is marked **(O2)** when its line uses the square read, directly or through a correlation.

Table 2 gives the count. Eleven of the twelve wait on **(O1)**: what a laboratory system, its

apparatus and its outcome are on the ledger. That statement is wide by design. It is the single question of what a laboratory event is, and every paradox about an apparatus contains it. Ten use the square read and so wait on the position read; six wait on the composition of two registers. Spin and statistics is a puzzle about the states themselves and stands alone. The three-line accounts above call on the definition of the record (a posting is never erased), on the premises and hypotheses of the theorems they name, among them the squared weights that Section 6 takes for parts that are not sets of positions, and on these four statements. The textbook's values quoted for three looks, for the nine reads and for the two contrasts are the textbook's own computations with squared weights: the accompanying program recomputes them, and the theorems of this paper do not produce them.

Every number in the twelve is the textbook's number. That is as it should be for a derivation of the postulates, and it has one consequence that should be said plainly: none of these twelve experiments can tell the record from the textbook. What separates them is what each takes as given.

9 What follows

The five share their parts. Taken as postulates the five rules are independent, and a student learns them as a list. As theorems they share their parts, and the sharing can be checked line by line.

The two-port that gives the square in Section 5 is built from the unit i of Section 3 and from nothing else. The square of Section 5 and the middle ceiling of Section 7 both use the parallelogram law, in different roles: in the first it is what the premises come to, and in the second it is a property of the length the square gives. The second rule's unitary step is the first clause of the opening sentence, a history that leaves no record loses nothing, and Proposition 11 shows that dropping the clause gives the record's own way of stepping, a weight that fades. The fourth rule's contact and the fifth rule's composition are, on the working candidate of Section 7, one law seen from inside a register and from outside it. And on the record's picture the network tests tie the first rule to the fifth: two registers agree on their unit, or their sources share nothing and they contradict a measurement.

What would break it. Because the numbers are the textbook's, the tests are shared. An interference term among three paths that pairs do not account for would break the square read of Theorem 16. A correlation sum above $2\sqrt{2}$ would break Theorem 27(b). A weight at one place moved by a split at another would break the position read. Each would break the textbook too.

There are three places where the record says more than the five postulates do, and each is a place to look.

- *The energy operator is not a second choice.* It is minus the logarithm of the law between records (Theorem 8). A system's energies are its decay rates under that law. Once the law of motion is fixed [7], every spectrum within the theorem's scope, a law with a floor under its factors, is fixed with it. This paper computes no spectrum. The test is to take that law for a known system and compare ratios of differences of its decay rates with ratios of measured lines; by Proposition 10 those ratios do not depend on the unit c.

- *A posting is never undone.* A friend who keeps a record cannot be made to interfere, at any degree of isolation. Experiments so far have used friends that keep none, and what counts as keeping a record in a laboratory is **(O1)**, so this is not yet a test. It follows from the definition of the record and from no theorem of this paper, and it is the one place among the twelve where the record says something that the unitary step alone does not.
- *Something counted takes the entry.* The step of Section 4 never posts. What does is owed, **(O1)**, and the working candidate for massive bodies has the form of a threshold in action. Its unit is open; when it is fixed, it is a number the five postulates do not contain.

What is owed, in order. First **(O1)**, attachment: eleven of the twelve paradoxes are this one question asked of eleven kinds of apparatus, and it includes the event itself, that one outcome is posted. Then **(O3)**, composition over one complex unit, which on the working candidate is the same law as the contact. Then **(O2)**, the position read, for which Proposition 20(c) gives the reason and not yet the proof. Then **(O4)**. Beneath all four stand the premises of the theorems: that a register is exhausted by its stationary states (Section 3); the linearity of a step and the four statements about it (Section 4); and that the total of a read is blind to a common turn (Section 5). Each is the statement that nothing is outside the record, read at one place, and deriving them from the starting fact of the ledger is owed by the foundation. With them stand the working statements named in the text: that a held register lies in the core, and which operator the crossing is.

Quantum mechanics has been called the theory nobody understands. The difficulty was never in the five rules. It was that they were rules. Read as what one ledger does between two of its entries, their algebra is a set of short theorems with named premises, and what is left is on a list. The widest item on that list, the one eleven of the twelve paradoxes wait on, is a single question: what, in a laboratory, is an entry.

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