Exact Control and Encoded Readout in a Native Recognition Register

Jonathan Washburn

Recognition Science Institute

jon@recognitionphysics.org

Abstract

Can fixed recognition interactions supply the control needed to write an exact record? We prove full unitary controllability on the 1,536-dimensional neutral sector of two source–reference pairs. The allowed controls are simultaneous sums of explicit local braid and paired-contact Hamiltonians, with adjustable positive strengths and durations; the original holding and positive scalar energy charges remain present. The main algebraic step converts effective control on a 1,024-dimensional matched subspace into control supported there, then uses a contact with an explicit right inverse to reach the remaining coordinates. Compact-group control gives actual finite words. An exact eight-label coherent readout follows, together with arbitrary encoded interventions and preparation of each label from one encoded blank. The four active cells sit in a fourteen-cell register: the other ten evolve by a retained unitary, with the same identity valid for every finite external reference. The result establishes exact existence within the prepared encoding, without a uniform synthesis-time or pulse-count bound.

1 A record is an operation

A record of a distinguished basis label can be written reversibly. For two labels, the desired operation is

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle \longmapsto \alpha|0,0\rangle + \beta|1,1\rangle. \tag{1}$$

The source label survives, and a second register carries the same label. The output is generally correlated. It is not two independent copies of an arbitrary unknown state. The physical question is whether the prescribed interactions can enact this map with the source, pointer and surrounding system all included.

We answer that question for eight labels in a finite native recognition register. Here *native* refers to the fixed eight-state clock, sign and flip matrices, and the local braid and contact generators specified below. The apparatus contains fourteen eight-state cells. Four cells form two source–reference pairs used by the reader. The other eight physical cells and the third source–reference pair remain in the state space throughout the construction.

The model comes from the construction of physical records in Recognition Science [1]. A source is paired with a reference so that its clock can act while the pair retains a fixed charge. The reference correction in Section 2 keeps that clock action and a positive holding. The outstanding control question for this register is whether its fixed local interactions supply the operations needed by a reader. Assuming a complete pair algebra would bypass that question. We derive the algebra from the stated clock, braids and contacts, including the sectors that the contacts leave accessible. The finite register and its allowed Hamiltonians are the physical premises of this theorem; their specification is not a claim that arbitrary laboratory matter already supplies the required controls.

There are two obstacles to a direct control argument. First, a pulse acts together with the holding Hamiltonian and its positive scalar energy charge. Removing either from the physical evolution would change the question. Second, the tensor product of two matched pairs is not invariant under the contact between them. That contact reaches two additional charge sectors. We retain those sectors, prove control on their direct sum, and only then choose the readout operation.

The general connection between a full dynamical Lie algebra and exact controllability on a compact unitary group is established control theory [2]. Encoded universal control also has a substantial history [3]. The contribution here is the derivation of the required algebra from these particular fixed interactions on the complete neutral sector, together with the exact embedding identity for the retained register and external references. No arbitrary entangling Hamiltonian is supplied as a control assumption.

Matrix-unit generation is a standard tool in finite-level controllability [4]. Network control gives related propagation arguments, including sufficient conditions based on zero forcing with selected local controls [5]. Here the relevant connection is a block map: after producing control supported on the matched subspace, we exhibit a right inverse for the contact from that subspace to its complement. This supplies every missing direction. We use established Lie-algebra methods to prove this model-specific statement; the subsequent basis-copying permutation is a consequence of that control.

2 The register and its allowed evolution

2.1 One cell and one matched pair

Write $\mathscr{H}_c = \mathbb{C}^8$, with orthonormal basis indexed by $0, \dots, 7$. All indices in a clock difference are taken modulo eight. Define

$$S = \operatorname{diag}(1, -1, -1, 1, -1, 1, 1, -1), \tag{2}$$

$$F|j\rangle = |j \operatorname{xor} 1\rangle, \qquad Q = \mathrm{i}SF, \qquad n = \tfrac{1}{2}(I - Q), \quad p = I - n. \tag{3}$$

Thus S and F are anticommuting Hermitian involutions; Q is a Hermitian involution and n is a projection of rank four. The native clock is the Hermitian matrix

$$C_{jk} = \frac{7\delta_{jk} - (1 - \delta_{jk})}{2} + \frac{\mathrm{i}}{2}(a_{j-k} + \sqrt{2}\, b_{j-k}), \tag{4}$$

where

$$a = (0, 1, 1, -1, 0, 1, -1, -1), \qquad b = (0, 1, 0, 1, 0, -1, 0, -1).$$

The discrete Fourier basis diagonalizes C, with eigenvalues $0, 1, \dots, 7$ in some order. In particular $0 \le C \le 7I$. Put

$$h = pCp + nCn, \qquad \overline{D} = \tfrac{1}{4}(C + SCS + FCF + QCQ). \tag{5}$$

The average $\overline{D}$ commutes with S, F, Q and is positive.

Let $d = (1, -1, -1, 1)$ and define the isometries $C_\pm : \mathbb{C}^4 \to \mathscr{H}_c$ by

$$C_+|j\rangle = \frac{|2j\rangle - \mathrm{i}d_j|2j+1\rangle}{\sqrt{2}}, \qquad C_- = FC_+. \tag{6}$$

Their ranges are the $+1$ and -1 eigenspaces of Q. They are orthogonal and complete: $C_+C_+^* = p$ and $C_-C_-^* = n$.

A source cell s and a reference cell r have pair charge

$$q = I \otimes n - n \otimes I. \tag{7}$$

Its zero-charge space is the *matched pair* $\mathscr{M} \simeq \mathbb{C}^8 \otimes \mathbb{C}^4$. An isometric inclusion is

$$J = p \otimes C_+ + n \otimes C_-, \qquad P = JJ^* = p \otimes p + n \otimes n. \tag{8}$$

The second factor in $\mathscr{M}$ is a retained four-state reference multiplicity. It is part of the physical pair, not a scalar reference label.

The complete pair holding is

$$\begin{aligned} H_{\text{pair}} &= L + I \otimes \overline{D} + 7I, \\ L &= h \otimes I + (nCp) \otimes (nFp) + (pCn) \otimes (pFn). \end{aligned} \tag{9}$$

This is the original reactivated clock $L + I \otimes h$ with the reference correction $I \otimes (7I + \overline{D} - h)$ retained. The correction is positive because $h \le 7I$ and $\overline{D} \ge 0$. Equivalently, L acts as $C \otimes I_4$ on the matched space and as $h \otimes I_4$ on the opposite-charge space, so $L \ge 0$ and $H_{\text{pair}} \ge 0$ directly.

The useful restriction is

$$\begin{aligned} H_{\text{pair}}J &= J(C \otimes I_4 + I_8 \otimes D + 7I_{32}), \\ D &= C_+^* \overline{D} C_+. \end{aligned} \tag{10}$$

The matrix D is displayed in Appendix A. Both reference charge blocks carry the same D. The source clock retains its off-diagonal charge transitions through the paired raising and lowering terms in L.

2.2 The full neutral sector

Use pairs (s_0, r_0) and (s_1, r_1) for the reader. The relevant conserved quantity is their *sum* $q_0 + q_1$. Define

$$\mathscr{N} = \ker(q_0 + q_1) = (\mathscr{M} \otimes \mathscr{M}) \oplus \mathscr{N}_{-+} \oplus \mathscr{N}_{+-}. \tag{11}$$

The pair charge has multiplicities $16, 32, 16$ at $-1, 0, 1$. Consequently

$$\dim \mathscr{N} = 32^2 + 16^2 + 16^2 = 1536. \tag{12}$$

The three inclusions, in the order in (11), are

$$N_0 = J \otimes J, \tag{13}$$
$$N_- = (C_- \otimes C_+) \otimes (C_+ \otimes C_-), \tag{14}$$
$$N_+ = (C_+ \otimes C_-) \otimes (C_- \otimes C_+). \tag{15}$$

Write $N = [N_0, N_-, N_+]$. Orthogonality of C_+ and C_- gives $N^*N = I$; the pair-charge eigenspace decomposition proves that these are all the neutral coordinates.

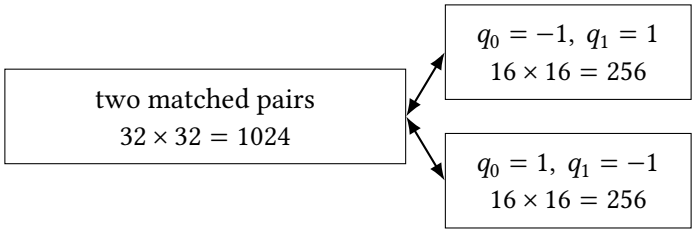

Figure 1. The complete neutral carrier. Restricting control to the matched 1024 coordinates would remove part of the allowed contact action. The proof keeps all 1536 coordinates.

2.3 The ten retained cells

Let $\mathscr{H}_B = (\mathbb{C}^8)^{\otimes 8}$ denote the eight physical cells and retain the third pair in full. Thus $\mathscr{R} = \mathscr{H}_B \otimes (\mathbb{C}^8 \otimes \mathbb{C}^8)$. Reordering tensor factors gives an isometry

$$E : \mathscr{R} \otimes \mathscr{N} \longrightarrow (\mathbb{C}^8)^{\otimes 14}. \tag{16}$$

The full positive holding $H_0(m)$ at fixed integer flux m consists of the eight-cell holding, the three complete pair holdings, and the boundary electric energy. The latter is a spectral function of total charge. Its dependence on the first two pairs is only through $q_0 + q_1$.

For precision, the structural form used here is

$$H_0(m) = H_B(m) + H_{\mathrm{pair},0} + H_{\mathrm{pair},1} + H_{\mathrm{pair},2} + e(Q_B - q_0 - q_1 - q_2), \tag{17}$$

with tensor identities suppressed. Here $H_B(m)$ is the retained positive eight-cell holding, Q_B its conserved charge, so $[H_B(m), Q_B] = 0$, and e is the fixed nonnegative boundary energy function, applied by spectral calculus. The theorem does not require their individual eigenvalues: it uses their retained action and the displayed charge dependence. All these terms remain present. On the code,

$$\begin{aligned} H_0(m)E &= E(H_R(m) \otimes I + I \otimes H_N), \\ H_R(m) &= H_B(m) + H_{\mathrm{pair},2} + e(Q_B - q_2), \\ H_N &= N^*(H_{\mathrm{pair}} \otimes I + I \otimes H_{\mathrm{pair}})N. \end{aligned} \tag{18}$$

This follows by $q_0 + q_1 = 0$ on $\mathscr{N}$. It also follows from the commutant calculation: subtracting $I \otimes H_N$ from the compressed holding leaves an operator commuting with every matrix on $\mathscr{N}$, hence an operator $H_R \otimes I$. In either derivation, H_R is a complete Hermitian matrix on the ten retained cells. It need not be diagonal and those cells need not begin in a product state.

2.4 Literal bundles and positive segments

For $j \neq k$, put $R_{jk} = |j\rangle\langle k| - |k\rangle\langle j|$, and set

$$B_{jkl} = \frac{1}{\sqrt{3}}(R_{jk} + R_{kl} + R_{lj}).$$

A braid token acts on one specified cell. The local bundles used below have tangents

$$\begin{aligned} A &= B_{035} + B_{124} \qquad (19)\\ &= B_{037} + B_{357} - B_{057} + B_{127} + B_{247} - B_{147},\\ X &= B_{017} + B_{167}. \qquad (20) \end{aligned}$$

The second line of (19) is its six-token realization. The signs are allowed braid orientations. Both A and X commute with Q, while $FAF = A$ and $FXF = -X$.

For a source s and a reference r, the two oppositely addressed entangling tokens have generators $i\pi S_s F_r/4$ and $i\pi F_s S_r/4$. Their bundle tangent is $(\pi/2)K_{sr}$, where

$$K_{sr} = \frac{\mathrm{i}}{2}(S_s F_r + F_s S_r). \qquad (21)$$

We use both within-pair contacts and the contacts $s_0 r_1$ and $s_1 r_0$. These bundles commute with $q_0 + q_1$. The normalization in (21) is for Lie-algebra calculations; the literal token angle $\pi/4$ is retained in every physical segment.

If a bundle b contains skew-Hermitian token generators $G_1, \dots, G_k$, write

$$A_b = \sum_j G_j, \qquad p_b = \sum_j \|G_j\|, \qquad C_b = \sum_j (\mathrm{i}G_j + \|G_j\| I). \qquad (22)$$

Each summand of C_b is positive semidefinite. For $\hbar > 0$, $\varepsilon > 0$ and $t > 0$, an allowed segment is

$$U(b, \varepsilon, t) = \exp\left[-\frac{\mathrm{i}t}{\hbar}\left(H_0(m) + \frac{\hbar}{\varepsilon} C_b\right)\right]. \qquad (23)$$

Only bundles commuting with total charge are used. A word is a finite chronological product of such segments; the empty word is allowed. The empty bundle supplies positive holding evolution. The segment energy is positive, and the holding and the scalar sum p_b have not been removed or replaced by freely chosen Hamiltonians.

The simultaneous sum C_b is an available Hamiltonian during a segment. Equation (23) does not mean a sequence of separate token gates or a product approximation to that sum. The strength $1/\varepsilon$ and duration t are externally chosen. The scalar terms make the control Hamiltonian positive and contribute phases retained in the evolution; positivity alone gives no bound on the energy or time required for a word.

3 Exact control with the rest retained

Theorem 3.1 (Exact neutral-sector action)**.** *Fix $m \in \mathbb{Z}$ and $\hbar > 0$. For every $V \in U(\mathcal{N})$ there are an allowed finite word w and a unitary W on $\mathscr{R}$ such that*

$$U_w E = E(W \otimes V). \qquad (24)$$

The same w and W satisfy, for every finite-dimensional external reference Z,

$$(U_w \otimes I_Z)(E \otimes I_Z) = (E \otimes I_Z)((W \otimes V) \otimes I_Z). \tag{25}$$

The existential choices precede the quantifier over Z. Thus the result does not select a new control word after inspecting the reference or the input state.

3.1 From actual words to directions

Let $\mathcal{S} \subseteq U(\mathcal{N})$ consist of the V for which (24) holds for some actual word and some unitary W. It is a monoid. It also contains every scalar phase: the empty word realizes $V = e^{i\theta}I$ with $W = e^{-i\theta}I$. This is a factorization of the same physical identity, not an additional phase actuator.

Suppose $A_b E = E(I \otimes B)$ and put $K_0 = -iH_N/\hbar$. From (18) and (23), taking $t = s\varepsilon$ gives

$$\exp\big(s(B + \varepsilon K_0)\big) \in \mathcal{S} \qquad (s, \varepsilon > 0). \tag{26}$$

Indeed the other factor is the actual retained holding evolution multiplied by $e^{-is p_b}$. It is absorbed into W without discarding it. Also $\exp(sK_0) \in \mathcal{S}$ for $s > 0$.

Let $\mathscr{L}$ be the real Lie algebra generated by these positive-flow directions and the scalar direction iI. It contains K_0 and $B = (B + K_0) - K_0$. Subtraction here takes place in a vector space of directions. It does not enact negative holding time. We will prove

$$\mathscr{L} = \mathfrak{u}(1536). \tag{27}$$

3.2 What the fixed local controls generate

We first record two finite calculations, with their inputs and elimination certificates in Appendix A.

Lemma 3.2 (Two four-state calculations)**.** *The least real Lie algebra containing A and closed under $Y \mapsto [ih, Y]$ contains*

$$C_+ \mathfrak{su}(4) C_+^* \;\oplus\; C_- \mathfrak{su}(4) C_-^*. \tag{28}$$

On a reference multiplicity, the least real Lie algebra containing $C_+^* A C_+$ and closed under $Y \mapsto [iD, Y]$ contains $\mathfrak{su}(4)$.

The holding in these statements is obtained from the full holding. For a cross contact $K = K_{s_0 r_1}$, direct multiplication gives

$$K^3 = -K, \qquad R = I + 2K^2 = -Q_{s_0} Q_{r_1}. \tag{29}$$

Conjugation by $R = \exp(\pi K)$ preserves any Lie algebra containing K. If a source operator Y commutes with Q, the paired clock formula yields

$$\tfrac{1}{2}\Big([iH_0, Y_{s_0}] + R[iH_0, Y_{s_0}]R\Big) = [ih, Y]_{s_0}. \tag{30}$$

The two paired raising/lowering terms are odd under this reflection and cancel. The reference clock, other pair holdings and boundary energy commute with Y_{s_0}. The reflected relation

therefore isolates precisely the drift action in Lemma 3.2. The other cross contact gives the same relation on s_1.

For reference matrices use

$$\mathscr{P}(Y) = C_+ Y C_+^* + C_- Y C_-^*. \tag{31}$$

This commutes with the reference flip and charge. Hence the paired raising/lowering terms commute with it, and its holding commutator is $\mathscr{P}([\mathrm{i}D, Y])$. The balanced reference bundle supplies the initial matrix in the second calculation of Lemma 3.2.

Within a matched pair, the contact K_{sr} acts on the source as iS, a skew-Hermitian unitary interchanging the two charge blocks, and acts as the identity on the four-state multiplicity. Together with the independently available block algebras in (28), this gives $\mathfrak{su}(8) \otimes I_4$. One can see this using matrix units: the off-diagonal block of that source action is invertible; commutators with the independent traceless diagonal and off-diagonal matrices in the two blocks isolate every inter-block edge. Their commutators supply the missing diagonal difference.

The reference directions give $I_8 \otimes \mathfrak{su}(4)$. The conditional bundle X has opposite reference blocks. If $T = C_+^* X C_+$, its action on the matched pair is $Q \otimes T$, where

$$T_{03} = \frac{\mathrm{i}}{\sqrt{3}} \neq 0. \tag{32}$$

It is a genuine coupling between the two factors. Local $\mathfrak{su}(8)$ and $\mathfrak{su}(4)$, together with this coupling, generate $\mathfrak{su}(32)$: commutators with the two local algebras span all products of traceless Hermitian factors, while the local terms supply products with an identity factor. These are exactly a basis of $\mathfrak{su}(32)$ after multiplication by i.

There is also a coupling between the two matched pairs. The Lie polynomial

$$\tfrac{1}{2}[K_{s_0 r_1}, [X_{r_1}, K_{s_0 r_1}]] - X_{r_1} = Q_{s_0} (QX)_{r_1} \tag{33}$$

is obtained by expanding the contact and using $FXF = -X$. It acts across the pairs. In matched coordinates it has the nonzero entry $-1/\sqrt{3}$ between

$$((0,0),(0,0)) \quad \text{and} \quad ((1,0),(0,3)).$$

The first and second pairs both change. The same local-algebra argument now gives all of $\mathfrak{su}(1024)$ as *effective* actions on $\mathscr{M} \otimes \mathscr{M}$. Each is represented by a full neutral-sector direction preserving that matched subspace. The direction can still act on the other two sectors, so this is not yet a supported $\mathfrak{su}(1024)$ block.

3.3 A supported block and a surjective contact

The distinction between an effective action and a supported action is essential. Let $a = K_{s_0 r_0}$ and $x = X_{r_0}$. Then

$$a^2 = -P_0, \qquad ax = -xa, \qquad [a,[a,x]] = -4P_0 x. \tag{34}$$

On the full neutral sector, P_0 is exactly the projection onto $\mathscr{M} \otimes \mathscr{M}$: if the first pair has charge zero, neutrality forces the second pair to have charge zero too. Equation (34) therefore

provides a nonzero traceless skew-Hermitian direction supported only on the matched block. Its nonzero entry follows from (32).

The supported directions form a real subspace invariant under commutators with the effective $\mathfrak{su}(1024)$ actions. The intersection with $\mathfrak{su}(1024)$ is thus a nonzero ideal in $\mathfrak{su}(1024)$. Simplicity gives the entire supported block. Equivalently, the usual matrix-unit proof of simplicity successively produces R_{jk}, $\mathrm{i}(E_{jk}+E_{kj})$ and $\mathrm{i}(E_{jj}-E_{kk})$ from a nonzero traceless skew-Hermitian element. No multiplication by a projection has been assumed to be an allowed control: (34) supplies it as a double commutator.

Now split $\mathcal{N} = \mathbb{C}^{1024} \oplus \mathbb{C}^{512}$ and compress the cross contact $K_{s_0 r_1}$ to $\mathcal{N}$. Its upper-left block is zero. Write its lower-left block as $C_\times$. Define the two isometries into the matched block

$$T_+ = (C_+ \otimes I_4) \otimes (C_+ \otimes I_4), \quad T_- = (C_- \otimes I_4) \otimes (C_- \otimes I_4). \tag{35}$$

Multiplying (21) by the displayed charge columns gives

$$K_{s_0 r_1} N_0 T_+ = -N_-, \qquad K_{s_0 r_1} N_0 T_- = N_+. \tag{36}$$

Consequently

$$C_\times [-T_+, T_-] = I_{512}. \tag{37}$$

Every additional neutral coordinate is reached. This is stronger than finding a single edge out of the matched block.

Lemma 3.3 (Completing a supported block)**.** *Let $a \geq 3$. Suppose a real Lie algebra of skew-Hermitian matrices on $\mathbb{C}^a \oplus \mathbb{C}^b$ contains the supported $\mathfrak{su}(a)$ block and a matrix*

$$B = \begin{pmatrix} 0 & -C^* \\ C & D_0 \end{pmatrix}$$

for which C has a right inverse. Then it contains $\mathfrak{su}(a+b)$.

Proof. Write $X_C = \left(\begin{smallmatrix} 0 & -C^* \\ C & 0 \end{smallmatrix}\right)$. Commuting B with $\mathrm{diag}(A, 0)$ for $A \in \mathfrak{su}(a)$ gives $-X_{CA}$. Further such commutators give right multiplication of C by products of $\mathfrak{su}(a)$ matrices. These products and their real linear combinations contain every complex $a \times a$ matrix. For example, for distinct u, v, w, the product

$$\mathrm{i}(E_{uu} - E_{vv})\,\mathrm{i}(E_{uu} - E_{ww}) = -E_{uu}$$

provides each diagonal matrix unit; multiplying by skew off-diagonal units gives the real and imaginary off-diagonal units. Summing the diagonal products first gives X_C itself. If $CT = I_b$, then $Y = C(TY)$ for every $b \times a$ matrix Y, so all X_Y belong to the Lie algebra. Their elementary real and imaginary edges, and commutators of those edges, generate $\mathfrak{su}(a+b)$. □

Applying Lemma 3.3 with (37) gives $\mathfrak{su}(1536)$. The scalar direction already belongs to $\mathscr{L}$, proving (27).

3.4 Why finite positive words suffice

For completeness, the compact-group control step is stated in the form needed here.

Lemma 3.4 (Positive-time compact control)**.** *Let G be a compact connected matrix Lie group. Let $\mathscr{T}$ be the monoid of finite products of positive-time flows $\exp(tA)$, $t > 0$, from a family of directions whose real Lie algebra is the Lie algebra of G. Then $\mathscr{T} = G$.*

Proof. The closure of a submonoid of a compact group is a subgroup: powers have a subsequence returning arbitrarily close to the identity, which supplies inverses in the closure. The closed subgroup containing the flows has the full Lie algebra and hence is G. Thus $\mathscr{T}$ is dense.

The Lie rank condition also supplies a finite product endpoint map with surjective differential and with every duration strictly positive. One formulation takes the span of the adjoint translates of the generating directions. That span is the full Lie algebra; finitely many translates suffice. Density permits their translating words to be chosen among positive words while preserving the nonzero rank minor. Vary the associated positive durations in a small open interval about positive base durations. The inverse function theorem then puts an open set O inside $\mathscr{T}$. This is the standard finite-dimensional rank argument; see [2] for the general, not necessarily bilinear, formulation.

Finally let $g \in G$. Density supplies $t \in \mathscr{T} \cap gO^{-1}$, so $t = go^{-1}$ for some $o \in O \subseteq \mathscr{T}$. Hence $g = to \in \mathscr{T}$. The conclusion is membership in the actual finite-word monoid, rather than only in its closure. □

Apply the lemma to the positive flows in (26), the holding flow and the scalar phases. Every product is in $\mathcal{S}$, and (27) supplies full rank. Hence $\mathcal{S} = U(\mathcal{N})$, proving (24). Tensoring this matrix identity with I_Z proves (25) and completes Theorem 3.1.

4 The exact eight-label reader

4.1 A concrete unitary to be enacted

Use matched coordinates $((a, p), (b, q))$, where $a, b \in \{0, \dots, 7\}$ and $p, q \in \{0, \dots, 3\}$. Let τ_a swap labels 0 and a, with τ_0 the identity. The permutation

$$((a, p), (b, q)) \longmapsto ((a, p), (\tau_a(b), q)) \tag{38}$$

is an involution on the matched block. Extend it by the identity on both other neutral sectors, obtaining a unitary $V_{\mathrm{read}} \in U(\mathcal{N})$.

Define the input and output isometries $j_{\mathrm{in}}, j_{\mathrm{out}} : \mathbb{C}^8 \to \mathcal{N}$ by

$$j_{\mathrm{in}}|a\rangle = |(a, 0), (0, 0)\rangle, \qquad j_{\mathrm{out}}|a\rangle = |(a, 0), (a, 0)\rangle. \tag{39}$$

Then $V_{\mathrm{read}} j_{\mathrm{in}} = j_{\mathrm{out}}$ exactly. The source label is preserved; the pointer receives it. The four-state coordinates have explicitly chosen blank values in these two isometries, while the full permutation preserves arbitrary p, q.

Put $E_{\mathrm{in}} = E(I_R \otimes j_{\mathrm{in}})$ and $E_{\mathrm{out}} = E(I_R \otimes j_{\mathrm{out}})$.

Corollary 4.1 (Coherent readout with every retained reference)**.** *There are an allowed finite word w and a unitary W on the retained ten cells such that, for every finite Z,*

$$(U_w \otimes I_Z)(E_{\text{in}} \otimes I_Z) = (E_{\text{out}} \otimes I_Z)((W \otimes I_8) \otimes I_Z). \tag{40}$$

Proof. Choose V_{read} in Theorem 3.1 and right-multiply its embedding identity by $I_R \otimes j_{\text{in}}$, then tensor with I_Z. □

For any density matrix ρ on $\mathscr{R} \otimes \mathbb{C}^8 \otimes Z$, (40) and its adjoint give the corresponding equality of full output density matrices. Thus ρ may contain arbitrary source–rest, source–reference and rest–reference correlations. No partial trace, reset or re-preparation of the retained ten cells occurs in the proof. The unitary W may change those cells; the guarantee is exact factorization, not their pointwise fixation.

For a normalized source $\psi = \sum_a \psi_a |a\rangle$, the encoded output is

$$j_{\text{out}}\psi = \sum_{a=0}^{7} \psi_a |(a,0),(a,0)\rangle. \tag{41}$$

If Π_S is the pointer projector for labels in $S \subseteq \{0,\dots,7\}$, orthogonality gives

$$\langle j_{\text{out}}\psi, \Pi_S j_{\text{out}}\psi\rangle = \sum_{a\in S} |\psi_a|^2. \tag{42}$$

The eight pointer labels therefore give the exact source-basis event weights.

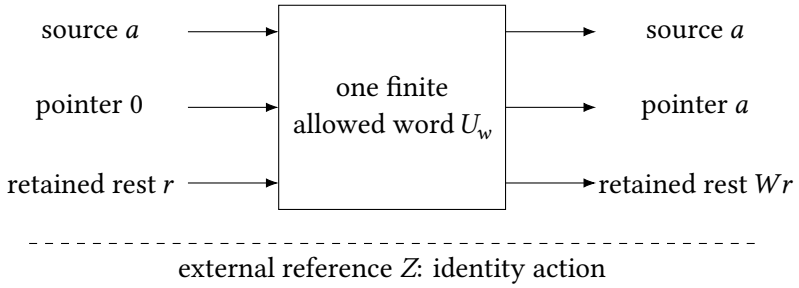

Figure 2. The read identity on basis inputs. By linearity it holds for superpositions and arbitrary correlations with the retained rest and the external reference. It does not map a general input to two independent copies.

4.2 Interventions and a common encoded blank

The read predicate includes operational access to the input encoding. Let $U \in U(8)$. Since j_{in} is an isometry,

$$\hat{U} = I + j_{\text{in}}(U - I_8)j_{\text{in}}^* \tag{43}$$

is unitary: it acts as U on ran j_{in} and as the identity on the orthogonal complement. Theorem 3.1 therefore enacts every encoded $U(8)$ intervention, with the same retained-reference quantifier order as before.

Take the common blank to be $j_{\text{in}}|0\rangle$. For each a, apply (43) to the source permutation swapping 0 and a. It prepares $j_{\text{in}}|a\rangle$ from that one blank. This does not prepare a pure blank from

an arbitrary mixed register: initialization in the encoded blank is an explicit resource. Nor does it capture an arbitrary external physical source into the code. The result supplies exact operations *within* the stated prepared encoding.

5 Prepared readout and the PPT criterion

Do coherent basis readout, arbitrary encoded unitary interventions and preparation of each basis label from a common blank characterize dimensions where the positive-partial-transpose (PPT) criterion is complete? The construction gives a precise negative answer. Index the source labels as $a = 4u + v$, with $u \in \{0, 1\}$ and $v \in \{0, 1, 2, 3\}$. All three operational properties hold on this prepared 2×4 encoding, with arbitrary retained references and an apparatus of four active eight-state cells. The factorization of the label space does not impose separate local controls on its two factors.

For a density matrix ρ on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, PPT means $\rho^{\Gamma_2} \geq 0$. Separability implies PPT in every dimension. The converse holds for 2×2 and 2×3, but fails already for 2×4 [6, 7]. These are statements about the cone of *all* bipartite states and its dual positive maps. Equation (40) is a statement about enacting a specified isometry on a prepared encoding. The two statements have different quantifiers and different objects.

Thus the three prepared-read properties do not imply PPT completeness or decomposability of all positive maps at these dimensions. They also cannot, by themselves, imply $d_1 d_2 \leq 6$. This is a consequence confined to the stated composite encoding. It makes no claim about readers subject to additional locality, indivisibility or interface restrictions.

6 Consequences and remaining construction work

The finite native register can implement an exact coherent eight-label read while retaining the original positive holding and the full surrounding register. The mechanism is specific: local holding commutators generate independent four-state charge-block actions; fixed contacts turn these into matched-pair control; a double commutator makes that control supported; and a contact with an explicit right inverse reaches the remaining neutral coordinates. Compact positive-time control then converts full rank into an actual finite word.

The theorem is an exact existence result for externally controlled strengths and durations. It gives no uniform word length, duration, energy budget or robustness margin, and the proof of compact controllability is not a pulse-synthesis algorithm. Deriving useful bounds and a constructive synthesis procedure is a separate control problem. An autonomous physical controller would additionally need to generate the segment choices and timings. Preparation of the encoded blank, coupling arbitrary incoming matter to the code, and performance in a physical device remain separate apparatus tasks. None is replaced by the exact unitary identity proved here.

A Exact finite matrix certificates

This appendix gives the small algebraic calculations used in Lemma 3.2. All entries lie in $\mathbb{Q}(\sqrt{2}, \mathrm{i})$. Splitting real and imaginary parts reduces the linear identities to arithmetic in $\mathbb{Q}(\sqrt{2})$. The normalization $1/\sqrt{3}$ in the balanced braid can be removed because the algebra is a real vector space.

A.1 Reference multiplicity

With $r = \sqrt{2}$, the matrices are

$$D = \frac{1}{2}\begin{pmatrix} 7 & r-\mathrm{i} & -\mathrm{i} & -1+\mathrm{i} \\ r+\mathrm{i} & 7 & -1-\mathrm{i} & \mathrm{i} \\ \mathrm{i} & -1+\mathrm{i} & 7 & -r+\mathrm{i} \\ -1-\mathrm{i} & -\mathrm{i} & -r-\mathrm{i} & 7 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 0 & \mathrm{i} & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 1 & 0 \\ -\mathrm{i} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{44}$$

Here $C_+^* A C_+ = A_4/\sqrt{3}$. Put $v_0 = \mathrm{i}D$, $v_1 = A_4$ and define the following bracket words:

j	v_j	j	v_j	j	v_j
2	$[v_1, v_0]$	3	$[v_0, v_2]$	4	$[v_1, v_2]$
5	$[v_0, v_3]$	6	$[v_1, v_3]$	7	$[v_0, v_5]$
8	$[v_1, v_5]$	9	$[v_0, v_6]$	10	$[v_1, v_6]$
11	$[v_0, v_7]$	12	$[v_1, v_7]$	13	$[v_0, v_8]$
14	$[v_0, v_{11}]$	15	$[v_0, v_{12}]$		

The required generating edges are $R_{01}, I_{01}, R_{12}, R_{23}$, where $I_{01} = \mathrm{i}(E_{01} + E_{10})$. Each is a linear combination of $v_1, \dots, v_{15}$; v_0 is used only through its commutator action. The table gives the identities $T = \sum_{j=1}^{15} c_j(T) v_j$. The omitted rows $j = 8, 15$ are zero. Substitution uses only $r^2 = 2$ and $\mathrm{i}^2 = -1$.

j	$c_j(R_{01})$	$c_j(I_{01})$	$c_j(R_{12})$	$c_j(R_{23})$
1	$-\frac{831}{119}-\frac{3181r}{714}$	$-\frac{349r}{357}-\frac{473}{357}$	$\frac{82}{51}+\frac{77r}{51}$	$\frac{2153}{357}+\frac{3589r}{714}$
2	$\frac{1515}{1288}+\frac{1499r}{1288}$	$\frac{3}{322}-\frac{94r}{161}$	$-\frac{94}{161}-\frac{79r}{322}$	$\frac{3125}{1288}+\frac{2465r}{1288}$
3	$\frac{76r}{7}+\frac{893}{56}$	$\frac{23}{7}+\frac{17r}{7}$	$-4-\frac{5r}{2}$	$-\frac{52r}{7}-\frac{513}{56}$
4	$-\frac{3883}{1428}-\frac{2461r}{1428}$	$-\frac{407r}{714}-\frac{155}{238}$	$\frac{19}{34}+\frac{67r}{102}$	$\frac{843r}{476}+\frac{3611}{1428}$
5	$\frac{2897}{1288}+\frac{2263r}{1288}$	$\frac{37r}{322}+\frac{150}{161}$	$\frac{37}{322}+\frac{75r}{161}$	$\frac{3229r}{1288}+\frac{5151}{1288}$
6	$-\frac{859}{322}-\frac{1121r}{644}$	$-\frac{283}{161}-\frac{219r}{322}$	$-\frac{283r}{322}-\frac{219}{322}$	$-\frac{799r}{644}-\frac{537}{322}$
7	$\frac{7057}{952}+\frac{2547r}{476}$	$\frac{151}{119}+\frac{125r}{119}$	$-\frac{32}{17}-\frac{29r}{34}$	$-\frac{2909}{952}-\frac{983r}{476}$
9	$-\frac{3}{4}-\frac{r}{2}$	0	0	$\frac{r}{2}+\frac{3}{4}$
10	$\frac{r}{238}+\frac{389}{1428}$	$\frac{4}{357}-\frac{29r}{357}$	$\frac{13}{51}-\frac{5r}{51}$	$\frac{19}{1428}-\frac{173r}{714}$
11	$\frac{661}{1288}+\frac{491r}{1288}$	$\frac{15r}{322}+\frac{26}{161}$	$\frac{15}{322}+\frac{13r}{161}$	$\frac{169r}{1288}+\frac{339}{1288}$
12	$\frac{1137}{644}+\frac{219r}{161}$	$\frac{65r}{322}+\frac{59}{161}$	$\frac{65}{322}+\frac{59r}{322}$	$-\frac{103r}{161}-\frac{473}{644}$
13	$-\frac{715r}{322}-\frac{488}{161}$	$-\frac{118}{161}-\frac{65r}{161}$	$-\frac{59r}{161}-\frac{65}{161}$	$\frac{156}{161}+\frac{251r}{322}$
14	$\frac{533}{952}+\frac{195r}{476}$	$\frac{11}{119}+\frac{19r}{238}$	$-\frac{5}{34}-\frac{r}{17}$	$-\frac{193}{952}-\frac{59r}{476}$

The three adjacent real edges generate every real skew edge by $[R_{ij}, R_{jk}] = R_{ik}$ for distinct indices. One imaginary symmetric edge then generates all the imaginary symmetric edges and diagonal differences. These matrices span $\mathfrak{su}(4)$.

A.2 Independent source charge blocks

Put $u_0 = \sqrt{3}A = R_{03} + R_{35} + R_{50} + R_{12} + R_{24} + R_{41}$ and $\delta Y = [\mathrm{i}h, Y]$, with h given by (5). Define

j	u_j	j	u_j	j	u_j
1	δu_0	2	δu_1	3	$[u_0, u_1]$
4	δu_2	5	$[u_0, u_2]$	6	$[u_0, u_3]$
7	δu_4	8	$[u_0, u_4]$	9	δu_5
10	$[u_0, u_5]$	11	δu_6	12	$[u_0, u_6]$
13	δu_7	14	$[u_0, u_7]$	15	δu_8
16	$[u_0, u_8]$	17	$[u_0, u_9]$	18	δu_{10}
19	$[u_0, u_{11}]$	20	δu_{13}	21	$[u_0, u_{13}]$
22	δu_{14}	23	$[u_0, u_{14}]$	24	δu_{15}
25	$[u_0, u_{15}]$	26	δu_{16}	27	$[u_0, u_{16}]$
28	δu_{17}	29	$[u_0, u_{17}]$		

For each of $T = R_{01}, \mathrm{i}(E_{01} + E_{10}), R_{12}, R_{23}$, the eight target matrices

$$C_+ T C_+^*, \qquad C_- T C_-^* \tag{45}$$

belong to the real span of $u_0, \dots, u_{29}$. A compact exact certificate is the elimination procedure below. It specifies the coefficient identities without printing eight columns of large rational numbers.

For a complex matrix Y, let $\mathrm{vec}_{\mathbb{R}}(Y)$ list its real entries in row order followed by its imaginary entries. Form M with columns $\mathrm{vec}_{\mathbb{R}}(u_j)$ and T with the eight columns in (45), in that order for each charge. Row-reduce $[M \mid T]$ over $\mathbb{Q}(\sqrt{2})$, scanning columns of M from left to right and choosing the first available nonzero row as pivot. Scale each pivot to one and clear its column in all other rows. The eight target columns have zero entries in every row in which the M block vanishes. Setting free unknowns to zero and reading the pivot rows therefore gives a matrix Z satisfying

$$MZ = T. \tag{46}$$

All inputs and the finite recurrence are specified above; the following exact-arithmetic listing supplies a directly reproducible version of the calculation. Arithmetic is in $\mathbb{Q}(\sqrt{2}, \mathrm{i})$, with real coordinates in $\mathbb{Q}(\sqrt{2})$. No numerical tolerance or floating-point rank decision enters the certificate.

Exact elimination algorithm.

1. Construct $S, F, Q, n, p, C, h, C_+, C_-$ from (2)–(6). Construct the thirty u_j from the displayed recurrence, using $[A, B] = AB - BA$.
2. Construct $M \in \mathbb{Q}(\sqrt{2})^{128\times 30}$ and $T \in \mathbb{Q}(\sqrt{2})^{128\times 8}$ by the specified real vectorization. Set $A = [M \mid T]$, $r = 0$, and initialize $Z = 0$ of size 30×8.
3. For $j = 0, \dots, 29$, choose the least $k \geq r$ with $A_{kj} \neq 0$, if one exists. Swap rows r, k, divide row r by A_{rj}, and for every $\ell \neq r$ subtract $A_{\ell j}$ times row r from row ℓ. Store the pivot pair (r, j) and increase r.
4. For each stored pair (r, j), set $Z_{j,t} = A_{r,30+t}$ for $t = 0, \dots, 7$. All nonpivot rows have zero entries in columns $30, \dots, 37$. Direct multiplication gives $MZ - T = 0$ entry by entry.

The arithmetic operations are exact. For $a, b \in \mathbb{Q}$, $(a + b\sqrt{2})^{-1} = (a - b\sqrt{2})/(a^2 - 2b^2)$ whenever the input is nonzero. Thus the procedure specifies a finite list of rational identities rather than a numerical rank test. The accompanying `verify_certificates.py` implements the same elimination and checks its residuals; the mathematical inputs and algorithm are all given here.

Each u_j belongs to the drift-invariant Lie algebra by its recurrence. Equation (46) supplies four generating edges independently in each charge block. Applying the preceding matrix-unit argument within each block proves (28) and Lemma 3.2.

References

[1] J. Washburn, *Recognition Structure from Responses and Records*, unpublished manuscript (18 September 2026).

[2] D. D'Alessandro, *The Lie Algebra Rank Condition for Non-Bilinear Quantum System*, arXiv:quant-ph/0301144 (2003). arXiv:quant-ph/0301144.

[3] L. Viola, E. Knill and S. Lloyd, Dynamical generation of noiseless quantum subsystems, *Physical Review Letters* **85**, 3520 (2000). doi:10.1103/PhysRevLett.85.3520.

[4] H. Fu, S. G. Schirmer and A. I. Solomon, Complete controllability of finite-level quantum systems, *Journal of Physics A: Mathematical and General* **34**, 1679–1690 (2001). doi:10.1088/0305-4470/34/8/313.

[5] D. Burgarth, D. D'Alessandro, L. Hogben, S. Severini and M. Young, Zero forcing, linear and quantum controllability for systems evolving on networks, *IEEE Transactions on Automatic Control* **58**, 2349–2354 (2013). doi:10.1109/TAC.2013.2250075.

[6] M. Horodecki, P. Horodecki and R. Horodecki, Separability of mixed states: necessary and sufficient conditions, *Physics Letters A* **223**, 1–8 (1996). doi:10.1016/S0375-9601(96)00706-2.

[7] P. Horodecki, Separability criterion and inseparable mixed states with positive partial transposition, *Physics Letters A* **232**, 333–339 (1997). arXiv:quant-ph/9703004.