

Einstein's Metric Field Equation at Every Order from the Recognition Lattice

Jonathan Washburn

Recognition Physics Institute, Austin, Texas

jon@recognitionphysics.org

Abstract

The leading two-derivative metric field equation of the Lorentzian Kuhn recognition lattice is Einstein's equation, with its full nonlinear dependence on the metric. The identity holds at every symmetric background in the regular star-shaped chamber of a flat reference metric. It follows from the linear identity at each constant background and the vanishing of both residuals on jets of flat coordinate changes. A further proof extends the identity to every regular background; independent verification of that extension is pending. We distinguish this field-equation identity from the existence and convergence of lattice solutions. For a specified static worldline matter model, the continuum equation and the unique coupled band solutions agree through second order in source amplitude, with the uniform virial stress supplied by the support itself. This result does not determine whether the physical comparison cost prices particle counts or stresses. At third order, the metric-projected gravitational source matches the cubic Einstein coefficient with an $O(s^3)$ error for bounded finite-mode long-wave data; the complete cubic solution limit remains unfinished, and independent verification of the source calculation is pending. We give precise targets for that limit and for the nonlinear solution problem, retaining the finite-spacing consistency conditions that an exact construction must satisfy.

1 The field equation and its solutions are separate results

An Einstein equation can emerge from a lattice action before one has constructed lattice solutions converging to an Einstein metric. The first statement identifies the long-wave differential operator. The second must control the other lattice variables, solve the matter equations, choose the correct constraints, and bound the inverse of the lattice operator uniformly as the lattice is refined.

This paper establishes the scope of the first statement and records the available solution results. There are two expansion parameters. We use s for lattice spacing in a local smooth-field expansion and ϵ for source or field amplitude. "At every order" means that the coefficient of s^2 retains the full nonlinear dependence on the metric and its first two derivatives. It does not mean equality with Einstein's equation at every power of s, or convergence of an amplitude series.

The geometric carrier is a four-dimensional Kuhn triangulation with fifteen edge classes. The action is the Lorentzian Regge action used in the recognition-lattice construction [1, 2]. This carrier and action are the model studied here. The reference family is

$$g_\kappa = I - \kappa \mathbf{1}\mathbf{1}^T, \qquad \kappa > \tfrac{1}{2}. \tag{1}$$

The choice $\kappa = 1$ is a physical identification; none of the all-order identities requires that choice. The results below do not derive this lattice carrier, the matter carrier, or their physical interpretation from a theorem about comparison cost alone.

We mark independently checked theorems as *independently verified*. A completed proof without that additional check is labelled *proof complete, independent verification pending*. Definitions of a carrier are models; a proposed physical reading is an identification; a consequence not supplied with a completed proof is an inference. Open targets remain open. These distinctions matter particularly for matter transport and nonlinear solutions.

2 Local edge residuals and the metric projection

The oriented edge vectors are

$$v_d \in \{0,1\}^4 \setminus \{0\}, \qquad d = 1, \dots, 15.$$

Each unit hypercube contains the twenty-four simplices obtained by ordering its four coordinate steps [4, 3]. For a metric g, its constant edge reading is $v_d^T g v_d$. At the reference metric this is $n(1-\kappa n)$, where $n = \sum_a v_d^a$ is the Hamming weight. Throughout, $|v_d|$ denotes this weight rather than the Euclidean norm. The geometry is restricted to the nondegenerate Lorentzian domain of the action.

Let F_d be the local derivative of the lattice action with respect to an edge of class d, expressed as a smooth function of its finite edge stencil. A stencil index r carries an edge vector v_r and its midpoint offset $m_r = o_r + v_r/2$ from the base vertex of the tested edge. For a symmetric metric two-jet (g, P, Q), write

$$P_{cab} = \partial_c g_{ab}, \qquad Q_{ceab} = \partial_c \partial_e g_{ab}.$$

The tensor Q is symmetric in c, e and in a, b; P is symmetric in a, b. Define the three stencil coefficients by

$$\begin{aligned} (\mathsf{S}_0 g)_r &= g_{ab} v_r^a v_r^b, \\ (\mathsf{S}_1 P)_r &= P_{cab} m_r^c v_r^a v_r^b, \\ (\mathsf{S}_2 Q)_r &= \tfrac{1}{2} Q_{ceab} m_r^c m_r^e v_r^a v_r^b. \end{aligned} \tag{2}$$

Repeated coordinate indices are summed. The spacing-squared coefficient of the local row is

$$\Phi_{2,d}(g, P, Q) = DF_d(\mathsf{S}_0 g)[\mathsf{S}_2 Q] + \tfrac{1}{2} D^2 F_d(\mathsf{S}_0 g)[\mathsf{S}_1 P, \mathsf{S}_1 P]. \tag{3}$$

The ten metric combinations of a fifteen-row covector u are

$$(\mathscr{F} u)^{\mu\nu} = \sum_{d=1}^{15} v_d^\mu v_d^\nu u_d. \tag{4}$$

The symmetric output has ten components. The other five combinations are additional lattice variables; the metric projection does not discard their equations when a full solution is sought. At a symmetric star-good background with $\kappa > 1/2$, shifting the stencil origin to the tested edge midpoint leaves this second-order coefficient unchanged for jets with the stated symmetries. The proof of this equality is complete; independent verification is pending. We use that centered reading in the local consistency statement below.

Our continuum quantity is the contravariant Einstein tensor density of weight one,

$$\mathscr{E}^{\mu\nu}(g,P,Q) = \sqrt{|\det g|}\, g^{\mu a} g^{\nu b}\big(R_{ab} - \tfrac{1}{2} g_{ab} R\big). \tag{5}$$

The curvature convention is the one for which the identities below have the factor $-1/2$. For clarity, its lower-index curvature is

$$\begin{aligned} R_{abcd} &= \tfrac{1}{2}(Q_{bcad} + Q_{adbc} - Q_{acbd} - Q_{bdac}) \\ &\quad + g^{ef}(K_{ebc}K_{fad} - K_{ebd}K_{fac}), \\ K_{abc} &= \tfrac{1}{2}(P_{bac} + P_{cab} - P_{abc}). \end{aligned} \tag{6}$$

In particular $R_{bd} = g^{ae} R_{ebad}$. Both (3) and (5) are affine in the second jet, with a second-jet linear part independent of P.

Definition 2.1 (Background domains)**.** A symmetric constant background is *regular* when its induced flat Kuhn edge field lies in the nondegenerate Lorentzian domain, $\det g < 0$, and all its hinge deficits vanish. It is *star-good relative to* g_κ when the straight segment from g_κ to g stays in the specified Lorentzian nondegenerate domain, with negative determinant throughout.

The zero deficits persist along the star-good segment. The extension to an arbitrary continuous path in that domain starting at g_κ is part of Proposition 3.4, whose proof is complete and independent verification is pending: the flat-placement deficits are continuous and take values in $2\pi\mathbb{Z}$. This does not assert that every Lorentzian metric belongs to that connected component.

3 The nonlinear metric identity

Lemma 3.1 (Linear identity at a constant background)**.** *For $\kappa > 1/2$, a symmetric star-good g, and a symmetric second jet Q,*

$$\mathscr{F}\Phi_2(g,0,Q) = -\tfrac{1}{2}\mathscr{E}(g,0,Q). \tag{7}$$

Status: Theorem, independently verified.

This is the constant-background linear identity underlying the long-wave theorem in [1, Sec. 4]. Its finite calculation pairs the local hinge contributions of the Kuhn star. Zero constant-metric residuals allow the midpoint quadratic stencil to be replaced by an edge-average quadratic stencil. The two hundred and forty simplex–hinge incidence slots (twenty-four simplex types, each with ten marked triangles) group into fifty hinge types in twenty-five plane groups. The cyclic angle denominators cancel within the groups; the remaining contraction gives the Ricci tensor and its trace term, with the normalization in (7). The statement is at every background

in its stated chamber, rather than only at g_κ. Appendix A displays the finite identities and their contraction.

The passage from this linear identity to the nonlinear one is short. We give it explicitly because it explains what the all-order claim establishes.

Lemma 3.2 (Flat-jet reduction)**.** *Fix a nondegenerate symmetric g. Suppose two tensor-valued jet functions $A(g,P,Q)$ and $B(g,P,Q)$ are affine in Q, with linear parts independent of P. Suppose that they agree when $P=0$, and both vanish on the flat coordinate-change jet with any prescribed symmetric P. Then they agree for every symmetric (P,Q).*

Proof. Set K as in (6) and define

$$Q^0_{ceab} = g^{rt}\big(K_{rac}K_{tbe} + K_{rae}K_{tbc}\big). \tag{8}$$

The quadratic change of coordinates

$$\varphi^a(y) = y^a + \tfrac{1}{2}g^{ar}K_{rbc}y^b y^c$$

pulls the constant metric back to a metric whose two-jet at zero is (g,P,Q^0). Thus both functions vanish at that jet. If L_A denotes the second-jet linear part of A, then

$$A(g,P,Q) = L_A(Q-Q^0) = A(g,0,Q) - A(g,0,Q^0).$$

The same formula holds for B. Their agreement at $P=0$ proves the claim. □

Status: Theorem, independently verified as part of the metric identity.

Theorem 3.3 (Einstein's metric coefficient at every amplitude)**.** *Let $\kappa > 1/2$, let g be symmetric and star-good relative to g_κ, and let P,Q have the symmetries stated above. Then*

$$\boxed{\sum_d v_d^\mu v_d^\nu \Phi_{2,d}(g,P,Q) = -\tfrac{1}{2}\mathscr{E}^{\mu\nu}(g,P,Q).} \tag{9}$$

No smallness of $g-g_\kappa$, P, or Q is required beyond membership in this domain.

Status: Theorem, independently verified.

Proof. Take $A = \mathscr{F}\Phi_2$ and $B = -\mathscr{E}/2$. Their second-jet affine property follows from (3) and the curvature formula. A quadratic coordinate change of a constant flat metric gives a flat placement of the same local lattice vertices, so its deficit residuals vanish. The continuum curvature also vanishes on this jet. Lemma 3.1 supplies equality at $P=0$; Lemma 3.2 completes the proof. □

The result identifies the complete nonlinear two-derivative metric operator. In vacuum its metric part is $\mathscr{E}=0$. With matter, the normalization of the metric-projected lattice stress must be matched to (9); a matter stress is not obtained by simply attaching an arbitrary fixed tensor to the right-hand side.

Proposition 3.4 (Regular-background extension)**.** *At every symmetric regular background,* (9) *holds for symmetric P, Q. There is no separate $\kappa > 1/2$ premise in this formulation: regularity itself specifies the admissible background. For $\kappa > 1/2$, every background joined to g_κ by a Lorentzian nondegenerate path with negative determinant is regular.*

Status: Theorem: proof complete, independent verification pending.

Proof. Near a regular flat placement, nondegeneracy is open. Each deficit is continuous and constrained to $2\pi\mathbb{Z}$, hence stays zero in a sufficiently small neighborhood. The constant-background linear calculation applies there, and the flat-jet reduction is unchanged. For the path statement, continuity and discreteness keep each deficit at its initial value zero along the path. □

This extension enlarges the domain of the identity. It does not prove regularity of every metric, invertibility of the five remaining modes at finite strength, or global existence of a metric field taking values in that domain.

4 What the remaining rows and spacing errors require

4.1 The five internal combinations

Let $C\zeta$ be the stencil that assigns the same correction ζ_d to every edge of class d. The constant-class linear residual map at g is

$$(H_0(g)\zeta)_d = DF_d(\mathsf{S}_0 g)[C\zeta].$$

At a star-good background, $\mathscr{F} H_0(g) = 0$. Thus a class correction can change the internal equations without changing the metric equations at this order.

Proposition 4.1 (Five-row solvability near flat)**.** *For every $\kappa > 1/2$ there is a neighborhood of g_κ in which $H_0(g)$ maps onto* $\ker \mathscr{F}$*. In a sufficiently small such neighborhood, if g is symmetric, Q is a symmetric second jet, and a target $t \in \mathbb{R}^{15}$ satisfies*

$$\mathscr{F} t = -\tfrac{1}{2}\mathscr{C}(g, 0, Q),$$

there exists a class correction ζ such that

$$DF_d(\mathsf{S}_0 g)[\mathsf{S}_{2,\mathrm{av}} Q + C\zeta] = t_d \qquad (d = 1, \dots, 15). \tag{10}$$

Here $\mathsf{S}_{2,\mathrm{av}}$ is the edge-average quadratic stencil, and the first jet is zero.

Status: Theorem: proof complete, independent verification pending.

Proof. Let V be the fifteen-by-ten matrix of constant metric strains. At g_κ, the matrix $H_0 + VV^T$ is invertible. Continuity of the local residual derivatives and determinant preserves this property near g_κ. If $(H_0 + VV^T)\zeta = u$ and $\mathscr{F} u = 0$, applying V^T and using $\mathscr{F} H_0 = 0$ gives $(V^T V)V^T\zeta = 0$. The metric strains are independent, so $V^T V$ is injective. Thus $V^T\zeta = 0$ and $H_0\zeta = u$. Apply this to the difference between t and the edge-average residual. Their metric parts agree by Lemma 3.1 and the zero metric projection of the midpoint-to-average class correction. □

The proposition is a local algebraic result for the indicated quadratic stencil. An arbitrary smooth field has a nonzero first jet, and a corrected field must also have a smoothly chosen ζ, controlled derivatives, and a small residual in the norm required by a solution theorem. Equation (10) alone supplies none of those estimates. At a general finite-strength background, surjectivity onto $\ker \mathscr{F}$ remains a separate condition. The proof uses invertibility, not positivity of the internal block.

4.2 Uniform consistency for midpoint sampling

Proposition 4.2 (No spacing-cubic term)**.** *Let $\kappa > 1/2$, and let G be a symmetric C^4 metric field on $\mathbb{R}^4$ whose values are star-good. On every compact set K there are $s_0 > 0$ and $C \geq 0$ such that, for each edge class d, the edge-centered midpoint stencil obeys*

$$\left|F_d(\text{sample of } G \text{ centered at } x) - s^2 \Phi_{2,d}(G(x), \partial G(x), \partial^2 G(x))\right| \leq C|s|^4 \tag{11}$$

for $x \in K$ and $|s| \leq s_0$.

Status: Theorem: proof complete, independent verification pending.

Proof. Reversing s reflects the edge-centered stencil and leaves the row unchanged. The sampled row is therefore even in s. Its constant term is zero, its second derivative is twice (3), and its odd derivatives at zero vanish. A compact tube of admissible stencils bounds the fourth derivative. Taylor's theorem gives (11) uniformly for each class. Taking the maximum of the fifteen constants and the minimum of their positive spacing bounds gives common C, s_0. □

There is no s^3 term in this sampling expansion. This statement concerns spacing, not the cubic coefficient in amplitude discussed below. It also leaves a generally nonzero s^2 coefficient in the five internal combinations when only the Einstein metric equation has been imposed.

4.3 Bianchi consistency and matter

Theorem 4.3 (Exact contracted Bianchi identity)**.** *For a symmetric C^3 metric field with* $\det G(x) < 0$ *at the point in question, the density* (5) *satisfies*

$$\partial_\mu \mathscr{E}^{\mu\nu} + \Gamma^\nu_{\mu\lambda} \mathscr{E}^{\mu\lambda} = 0. \tag{12}$$

Status: Theorem, independently verified.

The usual differential Bianchi identity contracts to $\nabla_\mu G^{\mu\nu} = 0$. Multiplication by $\sqrt{|\det G|}$ converts its first-index divergence to an ordinary divergence; the connection term for the remaining contravariant index gives (12). The density trace terms cancel. Consequently any proposed right-hand side equal to this density must obey the same identity. This is why a metric-dependent matter stress and its support cannot be replaced by a fixed source without accounting for the resulting constraint forces.

For $\kappa > 1/2$ and a symmetric C^4 metric field that is star-good on the closed unit ball around the tested vertex, the long-wave pairing of its midpoint-sampled rows with a flat vertex

move has a spacing-cubic coefficient proportional to the ordinary divergence of the Einstein density, with an $O(s^4)$ remainder. The proof of this expansion is complete; independent verification is pending. Its interpretation as the full curved-metric force law requires the remaining connection and matter terms. That interpretation is an inference, not an additional consequence already composed from the local metric identity.

5 Second order with worldline matter

5.1 The specified carrier and its stress

The solution theorem concerns a static periodic body held by a worldline support carrier. It is not a theorem for arbitrary matter. Let

$$n(X) = \bar{\rho} + \sum_{m \in \mathscr{A}} \mathrm{Re}(c_m e^{im\cdot X}), \qquad \sum_a m_a = 0. \tag{13}$$

Here $\mathscr{A} \subset \mathbb{Z}^4$ is finite. With $T = (1,1,1,1)$ and $\tau = \sqrt{16\kappa - 4}$, the continuum dust and compensation densities are

$$\begin{aligned} \mathscr{T}^{\mu\nu}_{\mathrm{dust}} &= \frac{\epsilon n(X) T^\mu T^\nu}{\sqrt{-(g_\kappa + H)(T,T)}}, \\ \mathscr{T}^{\mu\nu}_{\mathrm{comp}} &= -\frac{\epsilon\bar{\rho}}{\tau} T^\mu T^\nu. \end{aligned} \tag{14}$$

The constructed continuum stress is

$$\mathscr{T}_V = \mathscr{T}_{\mathrm{dust}} + \mathscr{T}_{\mathrm{comp}} + \mathscr{T}_{\mathrm{sup}} + \epsilon^2 \mathscr{V}. \tag{15}$$

The nonzero-mode support density $\mathscr{T}_{\mathrm{sup}}$ is supplied by the specified least-work support construction; $\mathscr{V}$ is its uniform second-order virial density. In the row-to-density normalization of (9),

$$\mathscr{V} = -2\,\mathrm{Re}\,\mathscr{T}\big(S_{2,\mathrm{grav}}(0) + S_{2,\mathrm{resp}}(0)\big), \tag{16}$$

where $\mathscr{T}$ converts the ten metric strain coordinates to a contravariant density, and the two S terms are the gravitational quadratic and body's metric-response loads in those coordinates. The zero-mode balance of the lattice support fixes this value. It is not an arbitrary constant fitted to the continuum equation.

The model imposes the transverse band equations and the radical band equations orthogonal to the worldline-blind subspace. It leaves the blind rows and the vertex-move rows above the radical cutoff unimposed. Its coupled equations are

$$DS_N(\ell_0 + w)[v] = \mathfrak{t}_{N,\epsilon}(w)[v] \quad \text{for every imposed test field } v. \tag{17}$$

The exact variational definition of $\mathfrak{t}_{N,\epsilon}$, including its fixed-occupation derivative and support transport, is given in Appendix B. The source scale on the lattice is $\epsilon(2\pi/N)^2$. The band slice consists of fields whose Fourier support admits static integer representatives m with $\max_a |2\pi m_a/N| < \delta$, and that are orthogonal to the flat Hessian radical in the sum over edges. Its transverse cutoff δ is distinct from the support's radical cutoff $2\pi M/N$. Static periodic

continuum fields are symmetric C^2 fields, 2π-periodic in each coordinate and invariant under $X \mapsto X + tT$. The continuum slice gauge has zero box mean and

$$\sum_d \left(\sum_c v_d^c \partial_c [v_d^T H(X) v_d] \right) (g_\kappa v_d)_a = 0 \quad (a = 1, \dots, 4).$$

The body has fixed counts; the compensation and transported support are represented by the specified stress prescription. These choices, the reference class ratios, the body/support split and the least-work reading are physical identifications within this model.

5.2 Explicit support and virial densities

Here are finite formulas for the matter terms in (15). Put $\mathscr{A}_\pm = \mathscr{A} \cup (-\mathscr{A})$. In this subsection coordinate indices run from 1 to 4. Let $g_0 = g_\kappa$, $R = g_0^{-1}$, $T = (1,1,1,1)$ and $\tau = \sqrt{16\kappa - 4}$. Let B_i, $i = 1, \dots, 10$, be the symmetric matrix basis consisting of four diagonal units and six matrices with ones in a symmetric off-diagonal pair, indexed in the order $11, 12, 13, 14, 22, 23, 24, 33, 34, 44$. The edge classes are $\mathscr{D} = \{0,1\}^4 \setminus \{0\}$, with $v_d = d$ and $q_d = \sum_a (v_d)_a$; thus q_d is Hamming weight. Write $V_{di} = v_d^T B_i v_d$ and use $\star$ for the class $v_\star = T$. For $y \in \mathbb{C}^{10}$, $\mathscr{T}(y)$ is the unique complex symmetric matrix such that $\operatorname{tr}(B_i \mathscr{T}(y)) = y_i$ for every i. Real parts below are taken entrywise, and all transposes are ordinary transposes, including after complex extension.

We also specify the jet norm used in the solution bounds. Let W have unit edge columns on the five classes $\{1\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$ and $\{2,3,4\}$, and let Π be the first ten rows of $[V \mid W]^{-1}$. Define

$$\bar{w}_d(x) = \tfrac{1}{2}[w_d(x) + w_d(x - v_d)], \qquad B(x) = \sum_i (\Pi \bar{w}(x))_i B_i,$$

$$P_a(x) = B(x + e_a) - B(x), \qquad Z_r(x) = w_{d_r}(x + o_r) - v_r^T \left[B(x) + \sum_a m_r^a P_a(x) \right] v_r.$$

Here the stencil indices are $r = (b, b', d_r)$ with $b, b' \in \{0,1\}^4$, $v_r = v_{d_r}$, $o_r = b' - b$ and $m_r = o_r + v_r/2$. If $\|f\|_A = \sum_k |\hat{f}(k)|$ is the normalized discrete Wiener norm, the precise jet norm is

$$\|w\|_J = \sum_{ab} \|B_{ab}\|_A + \sum_{cab} \|N P_{c,ab}\|_A + \sum_r \|N^2 Z_r\|_A.$$

It controls more than edgewise values.

For $q_d \geq 2$ put $n_d = \sqrt{\kappa q_d^2 - q_d}$, $C_{dd} = c_{0,d}$, $a_d = (2n_d)^{-1}$ and $b_d = (2q_d)^{-1}$. Put $C_{dd} = a_d = b_d = 0$ for $q_d = 1$, and let C be diagonal. For a real covector u, define the 15×4 and 15×5 matrices

$$\Gamma_{da}(u) = 2(v_d)_a (v_d \cdot u), \qquad A_{da}(u) = a_d \Gamma_{da}(u), \qquad A_{d5}(u) = b_d \sum_a \Gamma_{da}(u).$$

With $E = \binom{I_4}{0}$, the support response matrix is

$$P(u) = V^T \operatorname{diag}(C_{dd} a_d) A(u) [A(u)^T C A(u)]^{-1} E. \tag{18}$$

The inverse exists for the nonzero static integer modes used here that are not supported on two coordinate axes, when $\kappa > 1/2$ and the eleven selected $c_{0,d}$ are positive.

For a symmetric metric perturbation $H(X)$ define

$$D_H^{\nu}(X) = -\frac{n(X)-\bar{\rho}}{2\tau}\sum_{\rho} R^{\nu\rho}\partial_{\rho}[H(X)(T,T)], \qquad \widehat{D}_H(p) = \frac{1}{(2\pi)^4}\int_{[0,2\pi]^4} D_H(X)e^{-ip\cdot X}\,d^4X.$$

Let $\mathscr{W}_M$ consist of $p \in \mathbb{Z}^4$ with $-M < p_a < M$, $p \neq 0$, $\sum_a p_a = 0$, and p not supported on two coordinate axes. The specified nonzero-mode support density is

$$\mathscr{T}_{\mathrm{sup}}(\epsilon, H; X) = -2\epsilon \sum_{p\in\mathscr{W}_M} \mathrm{Re}\Big[\mathscr{T}\Big(P(-p)i\widehat{D}_H(p)\Big)\,e^{ip\cdot X}\Big]. \tag{19}$$

The force D_H is retained to first order in the metric perturbation; this is the force needed through second order in ϵ. The formula does not prescribe the full dust conservation defect at an arbitrary metric.

To specify the virial, let H_1 be the constructed first-order zero-mean slice field in Theorem 5.1. Define its Fourier coefficients and strain coordinates by

$$H_m = \frac{1}{(2\pi)^4}\int_{[0,2\pi]^4} H_1(X)e^{-im\cdot X}\,d^4X = \sum_i h_{m,i}B_i, \qquad A_1(m) = Vh_m.$$

They vanish off $\mathscr{A}_{\pm}$ and satisfy $h_{-m} = \overline{h_m}$. Thus $H_1(X) = \sum_{m\in\mathscr{A}_{\pm}} \mathrm{Re}(H_m e^{im\cdot X})$. The uniform density below is a specified functional of this first-order slice field, rather than an additional free constant.

The following contraction gives the two-derivative cubic vertex without an implicit cell matrix. For matrices U, U_0, U_1, U_2 and arrays J, K, define

$$\gamma(U,J)^a{}_{bc} = \tfrac{1}{2}\sum_e U^{ae}(J_{ecb} + J_{ebc} - J_{bce}),$$
$$\mathscr{G}(U_0,U_1,U_2;J,K) = \sum_{abcd} U_0^{ab}\left[\gamma(U_1,J)^c{}_{ad}\gamma(U_2,K)^d{}_{bc} - \gamma(U_1,J)^c{}_{ab}\gamma(U_2,K)^d{}_{cd}\right].$$

Given three symmetric matrices Z_0, Z_1, Z_2 and three covectors k_0, k_1, k_2 with $k_0 + k_1 + k_2 = 0$, put $J_{jabc} = (Z_j)_{ab}(k_j)_c$ and $\Delta_l = -RZ_lR$. Define

$$\mathscr{L}_{\kappa}(Z_0,Z_1,Z_2;k_0,k_1,k_2) = -\frac{\sqrt{|\det g_0|}}{2}\sum_{(l,i,j)}\Big[\tfrac{1}{2}\,\mathrm{tr}(RZ_l)\mathscr{G}(R,R,R;J_i,J_j) + \mathscr{G}(\Delta_l,R,R;J_i,J_j) + \mathscr{G}(R,\Delta_l,R;J_i,J_j) + \mathscr{G}(R,R,\Delta_l;J_i,J_j)\Big]. \tag{20}$$

The sum has the six orderings of $0,1,2$. This is the complex trilinear extension of half the cubic Einstein–Hilbert vertex in its ΓΓ form; its minus sign includes the factor i^2 from the two derivatives. On metric strains it is the two-derivative cubic lattice contraction. The separate metric-vertex identification has a complete proof; independent verification of that identity is pending.

For every integer covector p and row $r = 1, \dots, 10$, the loads are

$$S_{2,\mathrm{grav},r}(p) = \tfrac{1}{2}\sum_{m\in\mathscr{A}_{\pm}} \mathscr{L}_{\kappa}(H_m, H_{p-m}, B_r; m, p-m, -p),$$
$$\Phi(p) = \bar{\rho}\,A_1(p)_{\star} + \tfrac{1}{2}\sum_{m\in\mathscr{A}}\left[c_m A_1(p-m)_{\star} + \overline{c_m}A_1(p+m)_{\star}\right],$$
$$S_{2,\mathrm{resp},r}(p) = \frac{V_{\star r}}{4\tau^3}\Phi(p), \qquad \mathscr{V} = -2\,\mathrm{Re}\,\mathscr{T}\Big(S_{2,\mathrm{grav}}(0) + S_{2,\mathrm{resp}}(0)\Big). \tag{21}$$

These formulas determine the uniform density from the specified body and reference carrier, without a freely chosen uniform stress.

The carrier interpretation retains a distinction between the flat reference and actual occupations. The lattice construction solves its least-work problem at flat proper lengths n_d. Its total reference counts c_e^* are positive on $q_d \geq 2$, zero elsewhere, and obey posting continuity $\sum_{e\,:\,\mathrm{head}(e)=x} c_e^* = \sum_{e\,:\,\mathrm{tail}(e)=x} c_e^*$ at each vertex x; here e is an oriented lattice edge of class d, and head and tail are its end vertices. Its relative stress is $S_e = (c_e^* - \theta c_{0,d})/(2n_d)$, with $\theta > 0$ the support strength. At the deformed squared length $\ell_e < 0$ the specified transported support occupation is $2\sqrt{-\ell_e}\, S_e$. The flat reference carrier is re-solved at each metric before this transport. Posting continuity of c^* does not assert posting continuity of the transported actual occupation. The reference θc_0 is nongravitating; the gravitating support contribution is its deviation. The compensation in (14) is a fixed stress.

5.3 A complete second-order statement

The following hypotheses make the finite-mode statement precise. Choose $\kappa > 1/2$, support strength $\theta > 0$, an integer cutoff $M \geq 1$, and reference occupations $c_{0,d} > 0$ on the eleven selected timelike classes $|\nu_d| \geq 2$. Put $\mathscr{A}_\pm = \mathscr{A} \cup (-\mathscr{A})$ and $\mathscr{P} = (\mathscr{A}_\pm + \mathscr{A}_\pm) \cup \mathscr{A}_\pm$. The set $\mathscr{A}$ is nonempty, contains no zero mode, and never contains both m and $-m$. Each mode is static as in (13), and $2|m_a| < M$. Every nonzero mode in $\mathscr{P}$ avoids the blind lines: it is not supported on only two coordinate axes. The coefficients c_m and $\bar{\rho}$ are arbitrary real/complex data as appropriate; the algebraic theorem does not require $n(X) \geq 0$. Nonnegative occupation as a physical body is an additional restriction on those data.

Theorem 5.1 (Second-order matter limit with the support virial)**.** *Under these hypotheses there is a sufficiently small band cutoff such that, for every smaller positive band, there are static periodic slice-gauge fields H_1, H_2, positive ϵ_0, and constants $C_r, K \geq 0$ with the following properties for $0 < \epsilon \leq \epsilon_0$.*

First, the continuum residual is uniformly bounded by

$$\left|\mathscr{E}(g_K + \epsilon H_1 + \epsilon^2 H_2) - \mathscr{T}_V(\epsilon, \epsilon H_1 + \epsilon^2 H_2)\right| \leq K\epsilon^3 \tag{22}$$

componentwise. No free support multiplier or free uniform stress is added to this equation.

Second, for every $\eta > 0$ and every sufficiently large N, there is a band-slice solution w_N of (17) with jet norm at most $C_r\epsilon$, unique among band-slice solutions in that ball. At every edge midpoint X_e,

$$\left|w_N(e) - v_e^T(\epsilon H_1(X_e) + \epsilon^2 H_2(X_e))v_e\right| \leq K\epsilon^3 + \eta. \tag{23}$$

Status: Theorem, independently verified.

The construction solves the coupled band equations with a uniform small-amplitude estimate. The first and second lattice coefficients converge to the edge readings of H_1, H_2. The quadratic metric projection is the quadratic Einstein density; the support load has the required image by its cubic Ward calculation. The nonzero modes then match the continuum dust, compensation and support densities. At zero mode the support carries the entire constant-strain load; (16) converts that load to its continuum density. Expanding the Einstein density and the dust

denominator through second order leaves bounded cubic remainders, giving (22). Uniform control of the lattice remainder and the coefficient limits gives (23).

The limits are ordered: $N \to \infty$ first at fixed ϵ, with an $O(\epsilon^3)$ error remaining. The theorem does not assert an exact nonlinear continuum solution, an N^{-2} rate for a complete nonlinear solution, or the imposition of every lattice row. The uniqueness is in the stated band-slice jet-norm ball.

If the support's zero mode is omitted from the continuum matter density, the same box mean can instead be carried by a constant B_0 in a separate support stress. In (15) it is included in matter and no free constant is needed. This identifies the mean with the lattice support's stress at the given field; it does not determine the support's constitutive response to an arbitrary change of metric.

5.4 What is known about varying the support

The constant-strain stress balance extends beyond the second-order coefficient itself. Fix $K_2 \geq 0$ and retain the hypotheses of Theorem 5.1. For sufficiently small positive ϵ, consider all lattice fields satisfying

$$\|w\|_J \leq K_2\epsilon, \qquad \|w - \epsilon W_1 - \epsilon^2 W_2\|_J \leq K_2\epsilon^3(1 + 1/\theta),$$

where W_1, W_2 are the constructed first two lattice coefficients. The support's normalized constant-strain rows approach the negative gravitational-plus-body load, with error

$$\left| \frac{\mathrm{row}_r(\epsilon(2\pi/N)^2 \mathfrak{t}_{\mathrm{sup}}(w))}{N^4(2\pi/N)^2\epsilon^2} + S_r \right| \leq C\epsilon(1 + 1/\theta) + \eta, \quad r = 1, \dots, 10, \tag{24}$$

for sufficiently large N, simultaneously for all such w. Here $S = S_{2,\mathrm{grav}}(0) + S_{2,\mathrm{resp}}(0)$ and $\eta > 0$ is arbitrary. The constant C may be chosen before θ; the smallness bound on ϵ may depend on θ. This theorem is independently verified and can be applied to the constructed unique coupled solution. Its fixed-ϵ error need not vanish.

The balance fixes the uniform stress *at a given field*. It does not say that B_0 is held fixed, or that it varies, under a new metric variation. The distinction is visible on one timelike edge. With squared proper length $-\ell > 0$, count c produces stress $c/(2\sqrt{-\ell})$. Fixed count and fixed stress are different prescriptions.

For the unrestricted gas carrier, the comparison costs

$$\sum_e c_{0,e} J(c_e/c_{0,e}), \qquad \sum_e c_{0,e} J(\sigma_e/\sigma_{0,e}), \quad J(x) = \tfrac{1}{2}(x + x^{-1}) - 1 \tag{25}$$

therefore make different choices when the metric changes. Both sums run over the selected timelike edges; $\sigma_e = c_e/(2\sqrt{-\ell_e})$ and $\sigma_{0,e} = c_{0,e}/(2n_d)$. More precisely, put $n_d = \sqrt{-\ell_{0,d}}$ on a timelike reference class. For a multiplier μ in the model's radical stress-constraint space with $\mu_e < n_d$, define

$$c_e^{\mathrm{flat}} = \frac{c_{0,d}}{\sqrt{1 - \mu_e/n_d}}, \qquad c_e^*(\ell) = \frac{\sqrt{-\ell_e}}{n_d} c_e^{\mathrm{flat}}.$$

For $\ell_e < 0$, this is the unique minimizer of the stress-ratio cost among positive carriers on the timelike classes, zero on the other classes, with the same radical stress pairings as c^*. Under

the count-ratio cost, this universal transport rule fails: an explicitly admissible carrier can have lower cost. Both statements are independently verified for this unrestricted gas carrier. They do not establish the corresponding stress-pricing least-work theorem after restriction to worldline occupations.

Inference. A count-priced coupled construction may have a different uniform background response and higher-order gravitational terms. The available results do not construct that coupled solution or prove the proposed order of the difference. Deriving which physical quantity the native comparison apparatus prices in this matter carrier remains open. The support balance and Theorem 5.1 remain valid at their specified model scope.

6 Third order: the coefficient is known before the solution limit

The cubic Einstein coefficient and its lattice counterpart must not be confused with the s^3 spacing term excluded by Proposition 4.2. At a fixed flat reference write the amplitude expansion as

$$w = \epsilon W_1 + \epsilon^2 W_2 + \epsilon^3 W_3 + \cdots .$$

The known gravitational contribution to the source for the third-order linear equation is $2B(W_1, W_2) + R_3(W_1)$, where B and R_3 are the quadratic and cubic Taylor coefficients of the lattice residual. Matter contributes additional terms. The unknown W_3 belongs to the linear left-hand side.

Let G_S be the local residual paired with a constant symmetric metric strain S, and let $\mathscr{E}_S$ be the corresponding contraction of (5). A jet direction j_i consists of a metric value, first jet and second jet, with the required symmetries. The available pointwise result is

Proposition 6.1 (Cubic source matching)**.** *For $\kappa > 1/2$, take two symmetric jet directions j_1, j_2 and stencils*

$$\lambda_i = \mathsf{S}_0 B_i + s\mathsf{S}_1 P_i + s^2 \mathsf{S}_2 Q_i + C c_i + r_i.$$

Suppose, for $i = 1, 2$, the three pieces $\mathsf{S}_0 B_i$, $\mathsf{S}_1 P_i$, and $\mathsf{S}_2 Q_i$ have fixed bounds in stencil supremum norm, $\|Cc_i\| \le K_c s^2$, $\|r_i\| \le K_r |s|^3$, $|s| \le 1$, and $K_c, K_r \ge 0$. Then

$$\begin{aligned} &D^2 G_S[\lambda_1, \lambda_2] + \tfrac{1}{6} D^3 G_S[\lambda_1, \lambda_1, \lambda_1] \\ &\quad = -\tfrac{s^2}{2}\left(D^2 \mathscr{E}_S[j_1, j_2] + \tfrac{1}{6} D^3 \mathscr{E}_S[j_1, j_1, j_1] \right) + \mathscr{R}_s, \qquad |\mathscr{R}_s| \le C|s|^3. \end{aligned} \tag{26}$$

All derivatives are at the flat reference stencil or flat reference metric jet. The constant depends on the stated bounds and these local derivatives. For a fixed finite family of Fourier modes, set $s = 2\pi/N$ with $N \ge 3$ and $s \le 1$. Suppose every mode's metric edge readings are bounded by a common constant, its canonical internal amplitudes by a common constant times s^2, and its integer frequency has ℓ^1 norm at most R, with $\frac{3}{2} sR \le 1$. Then this estimate holds uniformly in the lattice site and side length.

Status: Theorem: proof complete, independent verification pending.

The proof differentiates the all-order identity at the flat metric to obtain the exact cubic two-derivative coefficient. In the local stencil expansion, constant and first-spacing terms

cancel. Flat/class-constant couplings also vanish in these metric pairings. The remaining terms have either the displayed spacing-squared coefficient or at least three powers of spacing. The finite-mode statement supplies the required stencil bounds by Taylor expansion of the phases.

This is a pointwise paired-source theorem. The following estimate isolates one step between it and a Fourier source limit.

Proposition 6.2 (Removing the midpoint phase)**.** *On a periodic lattice with $N \geq 1$, let $\hat{r}_d(m)$ be the normalized vertex Fourier coefficient of a real row $r_d(x)$, and let $X_d(m) = p_d(m)\hat{r}_d(m)$ be its midpoint coefficient, where $p_d(m) = \exp(-\pi i\, m \cdot v_d/N)$ in this convention. Suppose $s \neq 0$ and*

$$|r_d(x)| \leq B_d s^2, \qquad \left|\sum_d a_d r_d(x) - s^2 f(x)\right| \leq C|s|^3$$

for real weights a_d and a real function f. Then

$$\left|\sum_d a_d \frac{X_d(m)}{s^2} - \hat{f}(m)\right| \leq \sum_d |a_d| B_d |p_d(m) - 1| + C|s|. \tag{27}$$

The Fourier convention is $\hat{f}(m) = N^{-4} \sum_{x \in \{0,\dots,N-1\}^4} e^{-2\pi i m\cdot x/N} f(x)$, and the same convention defines $\hat{r}_d$.

Status: Theorem: proof complete, independent verification pending.

Proof. A normalized Fourier coefficient is bounded by the supremum norm. Thus $|\hat{r}_d/s^2| \leq B_d$, and removing its phase costs at most $B_d|p_d - 1|$. The Fourier transform of the stated pointwise error, divided by s^2, has modulus at most $C|s|$. Linearity and the triangle inequality give (27). □

Completing the third-order solution limit still requires convergence of the Fourier coefficient on the right, convergence of the second-order amplitudes including their zero mode, and the matter response at the same order. Proposition 6.2 retains its supplied row and pointwise-error bounds; it does not prove those bounds for the actual solution coefficients. No complete third-order coupled Einstein limit is claimed here.

7 Precise nonlinear targets and a finite-spacing obstruction

7.1 The solution target

For a specified matter model, denote its imposed rows by $\mathcal{I}_{N,\delta}$, its lattice stress by $\mathfrak{t}_{N,\epsilon}(w)$, its continuum density by $\mathcal{T}(\epsilon, H)$, and its allowed support multipliers by a stated set $\mathcal{M}$. Use the unit action normalization of Section 5. For a matrix M define the frame operator

$$\mathcal{K}(M)^{\mu\nu} = \sum_d v_d^\mu v_d^\nu (v_d^T M v_d).$$

Here $(D\lambda)_{ab} = \partial_a \lambda_b$ for a C^3 static periodic covector field, and B_0 is a constant matrix. Their edge readings supply the radical support covector before metric projection.

Open third-order target. There is a positive band bound such that, for each smaller positive band, one can choose static periodic slice-gauge fields H_1, H_2, H_3, an amplitude bound $\epsilon_0 > 0$, and constants $C_r, K \geq 0$. Put $U_\epsilon = \epsilon H_1 + \epsilon^2 H_2 + \epsilon^3 H_3$. For every $0 < \epsilon \leq \epsilon_0$, an allowed pair $(\lambda, B_0) \in \mathscr{M}$ should satisfy

$$|\mathscr{E}(g_\kappa + U_\epsilon) - \mathscr{T}(\epsilon, U_\epsilon) - \mathscr{K}(D\lambda + B_0)| \leq K\epsilon^4$$

at every point, componentwise. For every $\eta > 0$ and all sufficiently large N, the same coefficients should approximate a coupled band-slice solution w_N, unique in the ball $\|w_N\|_J \leq C_r\epsilon$, with

$$\sup_e |w_N(e) - v_e^T U_\epsilon(X_e) v_e| \leq K\epsilon^4 + \eta.$$

The limit remains $N \to \infty$ first at fixed ϵ. The pointwise cubic gravitational source theorem is one ingredient in this target, not its continuum or lattice existence clause.

A full nonlinear solution limit asks for the following exact statement.

Open target. There is $\delta_* > 0$ such that, for every $0 < \delta \leq \delta_*$, there are $\epsilon_0 > 0$ and $C_r \geq 0$ such that for each $0 < \epsilon \leq \epsilon_0$ there is a static periodic slice-gauge metric perturbation H_ϵ satisfying

$$\mathscr{E}(g_\kappa + H_\epsilon) = \mathscr{T}(\epsilon, H_\epsilon) + \mathscr{K}(D\lambda + B_0), \qquad (\lambda, B_0) \in \mathscr{M}. \tag{28}$$

For some K and every sufficiently large N, one requires a band-slice field w_N with jet norm at most $C_r\epsilon$, solving

$$DS_N(\ell_0 + w_N)[v] = \mathfrak{t}_{N,\epsilon}(w_N)[v] \quad (v \in \mathscr{J}_{N,\delta}),$$

unique in that ball, and satisfying

$$\sup_e \left|w_N(e) - v_e^T H_\epsilon(X_e) v_e\right| \leq KN^{-2}. \tag{29}$$

The weaker target replaces the last bound by convergence with no rate. The allowed multipliers are part of the model and must correspond to its unimposed equations. For the virial-inclusive second-order stress in (15), the proved second-order target takes $\lambda = B_0 = 0$; it does not establish (28) exactly.

There are several concrete unfinished connections. A continuum slice solution with controlled regularity must be constructed for the chosen nonlinear matter law. A sampling and internal-correction map must turn that solution into an approximate lattice solution with a residual small in the actual inverse estimate. Its class correction must depend smoothly on the continuum jets. The projection and inverse estimates must then convert consistency into the bound (29). Available sampling, even-Taylor and correction estimates cover parts of this chain; the combined full smooth-correction and all-order solution bridge remain open. Even for the narrower case of a supplied conserved finite-mode source, the available conditional reduction requires two open premises: analytic continuum solutions and multipliers with bounds on every derivative order, and the corresponding approximate-family construction. It does not establish the generalized nonlinear matter-law target stated here. For a finite-strength body, the five-row solvability condition must also be established throughout the relevant background range.

These are open mathematical targets about the actual equations. One cannot supply them by assuming a continuum solution, assuming a residual estimate containing the desired conclusion, or summing a formal amplitude series without a convergence bound.

7.2 An exact local obstruction on the finite lattice

Theorem 7.1 (A first-order field that has no second-order correction)**.** *At the identification $\kappa = 1$, on the periodic Kuhn lattice of side eight, there is an explicitly specified field h and a vertex move z at $y = (2, 1, -1, 0)$ such that*

$$D^2S[h, \delta_e] = 0 \quad \text{for all thirty edges incident to } y, \qquad D^2S[d, z] = 0 \quad \text{for every } d,$$

but

$$D^3S[h, h, z] = 756\sqrt{3} \neq 0. \tag{30}$$

Consequently there is no correction d, with arbitrary values on the whole torus, satisfying

$$D^2S[d, \delta_e] + D^3S[h, h, \delta_e] = 0 \quad \text{on those thirty edges.}$$

All fifteen edge classes are included in this statement.

Status: Theorem, independently verified.

The field is the wrapped midpoint sample of the cubic static potential $\Phi = \xi_1^3 - 3\xi_1\xi_2^2$, with its specified slaved internal correction; $\xi_1 = (x_0 + x_1 - x_2 - x_3)/2$ and $\xi_2 = (x_0 - x_1 + x_2 - x_3)/2$. The contradiction follows by pairing the proposed second-order equations with the supported vertex move: its Hessian pairing vanishes, while (30) does not. This is an exact obstruction for the stated first-order field and local rows. The wrapped field is not asserted to solve the first-order equations everywhere on the torus.

The corresponding statement on the infinite record lattice is also available, with its own exact local bridge. To specify its field without a wrapped-coordinate convention, write an edge as (x, A) for $\emptyset \neq A \subset \{0, 1, 2, 3\}$, put $n = |A|$, and evaluate the spatial coordinates at $x + v_A/2$. Define

$$h(x, A) = -2\Phi(\xi(x + v_A/2))(n + n^2/2) + \xi_2(x + v_A/2)b_A,$$

$$b_A = \begin{cases} 45/8, & A = \{0\}, \{3\}, \{0, 1, 3\}, \{0, 2, 3\}, \\ -45/8, & A = \{1\}, \{2\}, \{0, 1, 2\}, \{1, 2, 3\}, \\ 45/4, & A = \{1, 2\}, \\ -45/4, & A = \{0, 3\}, \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

Proposition 7.2 (The obstruction on $\mathbb{Z}^4$)**.** *For the native Lorentzian Regge rows on $\mathbb{Z}^4$ at $\kappa = 1$, the static field* (31) *solves every first-order row. Its second-order vertex-move defect at $y = (2, 1, -1, 0)$, in displacement direction u, is*

$$756\sqrt{3}\,(u_0 + u_1 - u_2 - u_3).$$

There is no record field d such that the exactly quadratic curve

$$\ell_0 + th + \tfrac{1}{2}t^2 d$$

satisfies all thirty Regge rows incident to y for all sufficiently small t.

Status: Theorem, independently verified.

The local box rows agree exactly with the torus action rows whenever the edge data agree in the required finite window. Hinge shares agree because the local and torus counts coincide on every hinge incident to the tested edge. Translation of the polynomial field reduces its first-order equations to the finite polynomial-basis row identities. The bridge transfers the cubic pairing and its four displacement components to $\mathbb{Z}^4$. The nonzero defect then contradicts the second-order equations as in Theorem 7.1.

The explicitly proved family statement is for the displayed quadratic curves. It is not stated here as an exclusion of every arbitrary C^2 family. Exact nonzero defects and quadratic-curve obstructions are also known for three specified *cubic Taylor fields* of a point-mass potential. Identifying those polynomial fields with the actual far field of a nonlinear lattice point mass remains an inference; an asymptotic nondecaying displacement is not established by the exact defect alone. These results also leave open a different first-order field, a restricted null-stratum equation, or a modified action. A nonlinear construction for the present all-fifteen-row action must address its actual finite-spacing compatibility conditions.

Separate compact-body constructions already provide exact exterior continuum curvature identities for specified radial metrics and native lattice families with controlled residuals. A residual bounded by a power of the body scale is still an approximate lattice solution. Those results do not supply an exact all-row nonlinear exterior solution, a general horizon construction, or the limit (29). The long-wave Einstein identity, the second-order matter theorem, and the exact local obstruction can therefore all hold together: they concern different operators, rows, limits and data. The next result must connect those precise objects.

A The finite contraction behind the linear identity

We record the finite identities used in Lemma 3.1, the constant-background identity underlying [1, Sec. 4]. Fix $\kappa > 1/2$ and a symmetric background g that is star-good relative to g_κ. Put $\gamma = g^{-1}$ and $\Delta = \sqrt{-\det g}$, and let B be any symmetric test matrix. The calculation below retains arbitrary entries of g; it is not an expansion about g_κ. For covectors a, b write $\gamma(a,b) = a^T\gamma b$, and define

$$X_\gamma(W;\alpha,\beta) = -\alpha^T\gamma W\gamma\beta + \frac{\gamma(\alpha,\beta)}{2}\left(\frac{\alpha^T\gamma W\gamma\alpha}{\gamma(\alpha,\alpha)} + \frac{\beta^T\gamma W\gamma\beta}{\gamma(\beta,\beta)}\right), \tag{32}$$

$$\langle U,V\rangle_\gamma = \tfrac{1}{2}\gamma^{ac}\gamma^{bd}U_{ab}V_{cd}, \qquad (a\wedge b)_{cd} = a_c b_d - a_d b_c.$$

In particular, $\langle a\wedge b, c\wedge d\rangle_\gamma = \gamma(a,c)\gamma(b,d) - \gamma(a,d)\gamma(b,c)$.

Edge averages, simplex fits and the hinge cycle

For an edge of vector v based at o, the edge average of the quadratic metric $q_{ab}(x) = \frac{1}{2}Q_{ceab}x^c x^e$ is

$$q_Q(o,v) = Q_{ceab}\left(\tfrac{1}{2}(o^c + v^c/2)(o^e + v^e/2) + \tfrac{1}{24}v^c v^e\right)v^a v^b. \tag{33}$$

Thus it differs from the midpoint reading by the class-constant field $k_v = Q_{ceab}v^c v^e v^a v^b/24$. Write $(Ck)_r = k_{v_r}$ for its class-constant stencil. The metric pairing of the linear residual

annihilates every such class-constant field:

$$\sum_{d} B(v_d, v_d)\, DF_d(\mathsf{S}_0 g)[Ck] = 0. \tag{34}$$

Indeed, varying a constant metric preserves the zero flat residual; symmetry of the action Hessian gives (34). Consequently the edge average may replace the midpoint stencil in this metric pairing.

All simplex data can be recovered without an incidence table. The corners of a Kuhn simplex are $z_j = z_0 + \sum_{i=1}^{j} e_{\sigma(i)}$, $0 \le j \le 4$. Given its ten edge readings w_{ij}, set

$$H_{ij} = \tfrac{1}{2}(w_{0i} + w_{0j} - w_{ij}), \qquad W = E^{-T} H E^{-1}, \qquad E = (z_1 - z_0, \dots, z_4 - z_0), \quad 1 \le i, j \le 4,$$

where $w_{ii} = 0$. This fit satisfies $(z_j - z_i)^T W (z_j - z_i) = w_{ij}$. For a tested edge based at a, use $w_{ij} = q_Q(z_i - a, z_j - z_i)$ in this formula, with $i < j$. The barycentric gradients supply the two apex covectors α, β opposite a marked triangle. If h is that triangle's Gram determinant, the angle derivative is

$$D\theta[W] = \frac{\Delta}{\sqrt{-h}} X_\gamma(W; \alpha, \beta). \tag{35}$$

This follows by differentiating the cofactor formula for the dihedral angle; in $D(\operatorname{adj} G) = D(\det G) G^{-1} - \det(G) G^{-1} (DG) G^{-1}$ the determinant derivative cancels.

Around a fixed hinge, choose the facet normals N_i cyclically so the apex covectors of simplex i are $N_{i-1}, -N_i$. The fitted metrics obey

$$N_{i-1} + N_{i+1} = c_i N_i, \qquad W_{i+1} - W_i = N_i \otimes Z_i + Z_i \otimes N_i.$$

The second relation expresses agreement on the shared facet. Explicitly, if $J = W_{i+1} - W_i$ and $N_i(x) = 1$, then $Z_i = Jx - \frac{1}{2}(x^T J x) N_i$. The required denominators $\gamma(N_i, N_i)$ are nonzero in the chosen domain. Direct substitution in (32) gives the telescoping identity

$$\begin{aligned} X_\gamma(W_i; N_{i-1}, -N_i) &= \tfrac{1}{2}\{\gamma(N_i, N_i)\gamma(Z_i, N_{i+1}) - \gamma(N_i, N_{i+1})\gamma(Z_i, N_i)\} \\ &\quad + L_i - L_{i-1} + \tfrac{1}{2}(P_i - P_{i+1}), \\ L_i &= \frac{\gamma(N_i, N_{i+1})}{2} \frac{N_i^T \gamma W_{i+1} \gamma N_i}{\gamma(N_i, N_i)}, \qquad P_i = N_{i-1}^T \gamma W_i \gamma N_i. \end{aligned} \tag{36}$$

For a Kuhn hinge the wedge $N_i \wedge N_{i+1} = \omega_t$ is constant. Summing (36) therefore yields

$$\sum_i X_\gamma(W_i; N_{i-1}, -N_i) = \tfrac{1}{2}\langle \Xi_t, \omega_t \rangle_\gamma, \qquad \Xi_t = \sum_i N_i \wedge Z_i. \tag{37}$$

The facet denominators have disappeared.

The two triangles in each plane

Write v_A for the indicator vector of a nonempty coordinate set A. Each hinge type is (A, B) with A, B disjoint and nonempty, represented by the triangle $(0, v_A, v_A + v_B)$. The two types (A, B) and (B, A) belong to one plane P. There are fifty ordered types and twenty-five planes.

The twenty-four simplex types, each with ten marked triangles, give two hundred and forty incidence slots, partitioned by these types.

For a triangle with tangent vectors u_1, u_2 put $G_{ij} = g(u_i, u_j)$, $B_{ij} = B(u_i, u_j)$ and

$$\begin{aligned} h_t &= G_{11}G_{22} - G_{12}^2, & T_t &= B_{11}G_{22} + B_{22}G_{11} - 2B_{12}G_{12}, \\ a_0 &= 4B_{11}(G_{22} - G_{12}) + 4B_{22}(G_{11} - G_{12}), & a_1 &= 4G_{12}(B_{11} + B_{22} - 2B_{12}). \end{aligned}$$

Here $u_1 = v_A$, $u_2 = v_A + v_B$ for type (A, B), and $a_0 + a_1 = 4T_t$. The sum C_t of area-gradient times angle-gradient contributions about that type is exactly

$$C_t = \frac{\Delta}{16 h_t}(a_0 S_0 + a_1 S_1), \qquad S_0 = S_1 = \tfrac{1}{2}\langle \Xi_t, \omega_t \rangle_\gamma. \tag{38}$$

The two sums S_0, S_1 correspond to the tested-edge bases 0 and u_1. Their equality follows by substituting (33) in the fit: the resulting Ξ_t is the same at those two bases. Only the three triangle edges enter the area derivative. Formula (38) also follows directly from (35) and $A = \sqrt{-h_t}/2$.

Choose the two annihilator covectors of P as the columns of M_P in the table below. Define

$$\begin{aligned} \Gamma_P &= \det(M_P^T \gamma M_P), \\ T_P(\gamma, B) &= \operatorname{tr}(\gamma B)\Gamma_P - \operatorname{adj}(M_P^T \gamma M_P) \,:\, (M_P^T \gamma B \gamma M_P), \end{aligned} \tag{39}$$

where $U : V = \sum_{ij} U_{ij} V_{ij}$. The finite identities are

$$h_t = \det(g)\Gamma_P, \qquad T_t \Gamma_P = T_P h_t, \qquad r_1 \Xi_{(A,B)} + r_2 \Xi_{(B,A)} = -c_P(R)\omega_P, \tag{40}$$

where $R = R(g, 0, Q)$, $\omega_P = m_1 \wedge m_2$, and $\omega_{t_j} = r_j \omega_P$, $r_j \in \{1, -1\}$. The first identity is the complementary-minor identity. The second is its trace variation. The last is a rational linear identity in Q; the exact curvature forms c_P are given below. These formulas give

$$C_{(A,B)} + C_{(B,A)} = -\frac{\Delta}{8} c_P(R) T_P(\gamma, B). \tag{41}$$

For explicit reconstruction of the finite data, encode a set by its binary mask, with coordinate 0 the low bit. For (A, B) enumerate the Kuhn simplices containing $(0, v_A, v_A + v_B)$: their cube bases have $z_a = 0$ on $A \cup B$ and $z_a \in \{-1, 0\}$ elsewhere. For each base retain exactly the coordinate permutations whose corner chain contains the three specified vertices. Order these simplices around the common triangle, read their facet normals from E^{-1}, and apply the fit and jump formulas above. This determines Ξ_t and r_t and reproduces the last identity in (40) by coefficient comparison. No background metric entries are used in that comparison.

P	A	B	m_1				m_2			
0	1	2	0	0	−1	−1	0	0	−1	0
1	1	4	0	−1	0	−1	0	−1	0	0
2	1	6	0	−1	1	−1	0	−1	1	0
3	1	8	0	−1	−1	0	0	−1	0	0
4	1	10	0	−1	−1	1	0	−1	0	1
5	1	12	0	−1	−1	1	0	−1	0	0
6	1	14	0	−1	0	1	0	−1	1	0
7	2	4	−1	0	0	−1	−1	0	0	0
8	2	5	−1	0	1	−1	−1	0	1	0
9	2	8	−1	0	−1	0	−1	0	0	0
10	2	9	−1	0	−1	1	−1	0	0	1
11	2	12	−1	0	−1	1	−1	0	0	0
12	2	13	−1	0	0	1	−1	0	1	0
13	3	4	−1	1	0	−1	−1	1	0	0
14	3	8	−1	1	−1	0	−1	1	0	0
15	3	12	−1	1	−1	1	−1	1	0	0
16	4	8	−1	−1	0	0	−1	0	0	0
17	4	9	−1	−1	0	1	−1	0	0	1
18	4	10	−1	−1	0	1	−1	0	0	0
19	4	11	−1	0	0	1	−1	1	0	0
20	5	8	−1	−1	1	0	−1	0	1	0
21	5	10	−1	−1	1	1	−1	0	1	0
22	6	8	−1	−1	1	0	−1	0	0	0
23	6	9	−1	−1	1	1	−1	0	0	1
24	7	8	−1	0	1	0	−1	1	0	0

The curvature table and final contraction

Number the coordinate pairs $01, 02, 03, 12, 13, 23$ by $1, \dots, 6$ and write r_{ij} for the corresponding pair–pair curvature component. For example, $r_{16} = R_{0123}$ and $r_{34} = R_{0312}$. The entries below are $f_P = 3c_P/4$:

P	$f_P(R)$
0	$-r_{16} - r_{23} - 2r_{26} - 2r_{34} + 2r_{36} - r_{45} - 2r_{46} + 2r_{56} + 3r_{66}$
1	$-r_{13} - 2r_{15} - r_{16} + r_{34} + 2r_{35} + 2r_{45} + r_{46} + 3r_{55} + 2r_{56}$
2	$2r_{16} + r_{34} + r_{45} - r_{46} - 2r_{56}$
3	$-r_{12} - 2r_{14} + 2r_{16} + 2r_{24} + r_{34} + 3r_{44} + 2r_{45} - 2r_{46} - r_{56}$
4	$-r_{16} + r_{34} + r_{45} + 2r_{46} + r_{56}$
5	$-r_{16} - 2r_{34} - 2r_{45} - r_{46} + r_{56}$
6	$-r_{45} + r_{46} - r_{56}$
7	$2r_{13} + r_{15} + 2r_{16} + 2r_{23} + r_{26} + 3r_{33} + r_{34} + 2r_{35} + 2r_{36}$
8	$-r_{16} + r_{23} - r_{26} + r_{34} - 2r_{36}$
9	$2r_{12} + r_{14} - r_{16} + 3r_{22} + 2r_{23} + 2r_{24} - 2r_{26} + r_{34} - r_{36}$
10	$2r_{16} + r_{23} + 2r_{26} + r_{34} + r_{36}$
11	$-r_{16} - 2r_{23} - r_{26} - 2r_{34} + r_{36}$
12	$-r_{23} + r_{26} - r_{36}$
13	$r_{13} - r_{15} - r_{16} - 2r_{34} - 2r_{35}$
14	$r_{12} - r_{14} - r_{16} - 2r_{24} - 2r_{34}$
15	$r_{16} + 2r_{34}$
16	$3r_{11} + 2r_{12} + 2r_{13} - 2r_{14} - 2r_{15} - r_{16} - r_{24} - 2r_{34} - r_{35}$
17	$r_{13} + 2r_{15} + 2r_{16} + r_{34} + r_{35}$
18	$-2r_{13} - r_{15} - r_{16} + r_{34} + r_{35}$
19	$-r_{13} + r_{15} - r_{35}$
20	$r_{12} + 2r_{14} - r_{16} + r_{24} + r_{34}$
21	$r_{16} - r_{34}$
22	$-2r_{12} - r_{14} + 2r_{16} + r_{24} + r_{34}$
23	$-2r_{16} - r_{34}$
24	$-r_{12} + r_{14} - r_{24}$

For arbitrary symmetric γ, B and every algebraic curvature tensor, the table obeys the polynomial identity

$$\sum_{P=0}^{24} c_P(R)T_P(\gamma, B) = -4B_{mn}\left(\gamma^{mb}\gamma^{nd}\,\mathrm{Ric}_{bd} - \tfrac{1}{2}\,\mathrm{scal}\;\gamma^{mn}\right), \tag{42}$$

where $\mathrm{Ric}_{bd} = \gamma^{ae}R_{ebad}$ and $\mathrm{scal} = \gamma^{bd}\mathrm{Ric}_{bd}$. Here is a finite way to verify the contraction without imposing any special form of the background. The pair–pair matrix of an algebraic curvature tensor is

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} & r_{15} & r_{16} \\ r_{12} & r_{22} & r_{23} & r_{24} & r_{16}+r_{34} & r_{26} \\ r_{13} & r_{23} & r_{33} & r_{34} & r_{35} & r_{36} \\ r_{14} & r_{24} & r_{34} & r_{44} & r_{45} & r_{46} \\ r_{15} & r_{16}+r_{34} & r_{35} & r_{45} & r_{55} & r_{56} \\ r_{16} & r_{26} & r_{36} & r_{46} & r_{56} & r_{66} \end{pmatrix}.$$

Antisymmetry recovers the other components, and the displayed relation $r_{25} = r_{16} + r_{34}$ is the first Bianchi identity. There are twenty independent entries. Substitute this matrix and the two tables in (42). Both sides are linear in those twenty entries; for each entry, set it to one and the others to zero. Expansion of (39) then gives identical cubic polynomials in the symmetric entries of γ, linear in B. This proves (42) for arbitrary such γ, B, R.

Finally the flat Schlaefli identity and zero deficits express the paired linear residual as minus the slot sum. Thus

$$\begin{aligned}\langle B, \mathscr{F}\Phi_2(g,0,Q)\rangle &= -\sum_{\text{slots}} C_{\sigma,c} = \frac{\Delta}{8}\sum_{P=0}^{24} c_P(R)T_P(\gamma, B)\\ &= -\tfrac{1}{2}\langle B, \mathscr{E}(g,0,Q)\rangle.\end{aligned}$$

Since B is arbitrary, this is (7). Only the first variation of the angle has been linearized; its dependence on the constant background has remained exact throughout.

B The finite lattice matter prescription

Here are variational definitions of the source in (17). Write $e = (x, d)$, $e^- = x$, $e^+ = x + v_d$ on $\Lambda_N = (\mathbb{Z}/N\mathbb{Z})^4$, and let $E_t = \{(x, d) : q_d \geq 2\}$ be the selected timelike edges. Set $\ell_{0,e} = q_d - \kappa q_d^2$, $\ell = \ell_0 + w$, $n_d = \sqrt{-\ell_{0,d}}$ on E_t, $t_N = 2\pi/N$, $\sigma = \epsilon t_N^2$ and $\delta_M = 2\pi M/N$. The fixed body counts and fixed compensation stress are

$$c_{1,(x,d)} = \mathbf{1}_{d=\star} n(t_N x), \qquad b_{(x,d)} = \mathbf{1}_{d=\star} \frac{\bar{\rho}}{2\tau}.$$

Here n is (13), evaluated using any integer representative of x; periodicity makes that choice immaterial. Use the real edge pairing $\langle a, v\rangle = \sum_e a_e v_e$. For fixed occupations c, put

$$A_c(\ell) = \sum_e c_e \sqrt{-\ell_e}, \qquad s(c, \ell)_e = \frac{c_e}{2\sqrt{-\ell_e}}, \qquad DA_c(\ell)[v] = -\langle s(c, \ell), v\rangle.$$

All occupations below vanish off E_t; in these formulas their off-mask stress is defined to be zero. The solution ball keeps $\ell_e < 0$ on E_t.

We specify the support rows. Let $\mathscr{R}_M$ consist of

$$r_{(x,d)} = v_d^T B v_d + 2v_d \cdot [\xi(x + v_d) - \xi(x)], \qquad B = B^T,$$

where each component of the real vertex field ξ has Fourier support in the static band of radius δ_M defined above. Write $\Delta_e \psi = \psi(e^+) - \psi(e^-)$ and define

$$\mathscr{B}_M = \{r \in \mathscr{R}_M : \exists \psi : \Lambda_N \to \mathbb{R},\ r_e/(2n_d) + \Delta_e \psi = 0 \text{ for all } e \in E_t\}.$$

Let P_B be the orthogonal projection onto $\mathscr{B}_M$ for the edge pairing. Define the nonlinear Regge residual $\mathscr{R}_N(w)$ by

$$\langle \mathscr{R}_N(w), v\rangle = DS_N(\ell_0 + w)[v] - D^2 S_N(\ell_0)[w, v],$$

and set

$$g(w) = \mathscr{R}_N(w) + \sigma[s(c_1, \ell_0 + w) - b], \qquad f(w) = -\frac{(I - P_B)[g(w) - g(0)]}{\sigma\theta}.$$

The reference carrier is $c_{0,e} = c_{0,d} > 0$ on E_t, zero elsewhere. For the small loads used in the theorem, let $C(f)$ be the least-work flat-reference carrier: it minimizes

$$\sum_{e \in E_t} c_{0,d} J(C_e/c_{0,d}), \qquad J(z) = \tfrac{1}{2}(z + z^{-1}) - 1,$$

over $C_e > 0$ on E_t, $C_e = 0$ off E_t, subject to

$$\sum_{e : e^+ = x} C_e - \sum_{e : e^- = x} C_e = 0 \quad (x \in \Lambda_N), \qquad \sum_{e \in E_t} \frac{C_e - c_{0,d}}{2n_d} r_e = \langle f, r\rangle \quad (r \in \mathscr{R}_M).$$

Equivalently its multiplier form is $C_e = c_{0,d}/\sqrt{1 - \mu_e/n_d - 2\Delta_e \psi}$ on E_t, with $\mu \in \mathscr{R}_M$ and positive radicands, satisfying those same constraints. The excess support stress and its transported occupation are

$$S_e(w) = \frac{\theta[C(f(w))_e - c_{0,d}]}{2n_d}, \qquad c_{\mathrm{sup},e}(w) = 2\sqrt{-\ell_e}\, S_e(w) \quad (e \in E_t),$$

and zero off E_t. The exact source covector is therefore

$$\begin{aligned}\mathfrak{t}_{N,\epsilon}(w)[v] &= \sigma\{DA_{c_1+c_{\mathrm{sup}}(w)}(\ell_0+w)[v] + \langle b, v\rangle\}\\ &= \sigma\sum_e [b_e - s(c_1, \ell_0+w)_e - S_e(w)]v_e.\end{aligned}$$

The derivative holds $c_1 + c_{\mathrm{sup}}(w)$ fixed: it does not differentiate that occupation's selection map. The imposed tests are the band slice defined above and $\mathscr{R}_M \cap \mathscr{B}_M^{\perp}$. The flat carrier $\theta C(f)$ obeys posting continuity; its transported excess need not. Only this excess gravitates: the reference θc_0 is excluded. Its least-work rule prices flat-reference counts; carrying its excess to the new geometry at fixed stress is the stated transport prescription, not an actual-geometry worldline minimization.

References

[1] J. Washburn, *Quantum Gravity from the Recognition Ledger: Weighted One-Loop Graviton Responses with Finite Real Parts, Einstein Gravity at Long Wavelengths, and the Massless-Graviton Problem*, version 1.2, 2026. doi:10.5281/zenodo.23839819.

[2] T. Regge, General relativity without coordinates, *Il Nuovo Cimento* **19** (1961), 558–571.

[3] H. Freudenthal, Simplizialzerlegungen von beschränkter Flachheit, *Annals of Mathematics* **43** (1942), 580–582.

[4] H. W. Kuhn, Some combinatorial lemmas in topology, *IBM Journal of Research and Development* **4** (1960), 518–524.