The Recognition Ledger of a Black Hole: Horizon Records, Entropy Counts and Lattice Hawking Radiation

Jonathan Washburn

Recognition Physics Institute, Austin, Texas

jon@recognitionphysics.org

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Abstract

A black hole of horizon area A carries the entropy $A/(4\ell_P^2)$, radiates at the temperature $\hbar g_{\mathrm{H}}/(2\pi)$ set by its surface gravity g_{H}, and, if quantum mechanics holds, gives back the information that fell in. All three concern what a horizon records, and the recognition ledger's events are records. We state the four steps of a ledger account, a lattice horizon, its entropy, its radiation and the fate of its information, as precise targets; none is complete. On the ledger's lattice $G = 1/(8\pi s)$ is fixed by the action scale s and $\hbar = 1/a$ by the phase scale a, so at tree level the Bekenstein–Hawking quarter holds exactly when a horizon holds $2\pi as$ per unit area; the ledger's phase premise sets this to one bit per record area λ_{rec}^2, so a count that assumes it returns the quarter by construction. The lattice's order of postings holds no black hole, nor do the configurations the path sum integrates over, and in a geometry's interval order trapping needs a strong field. For the lattice action as it stands, the condition under which a weak vacuum field extends to second order fails one order past Einstein's equations, provably for a sampled field and for a point mass's Taylor fields, so a lattice black hole with a regular far field is not expected without an action exactly symmetric under vertex moves (an inference). A horizon hides the postings that cross it. One bit per crossing gives a coefficient that depends on the horizon's orientation, 0.108, 0.325 or 0.1875 at the identified spacing and tree level, none of them the quarter, so a quarter that is the same on every horizon needs a count that correlates the records, which is open. The lattice scalar disperses like a superluminal medium at short wavelengths. On test backgrounds that are not lattice solutions its computed in-out occupation is a Planck spectrum that tends to the Hawking temperature as the spacing shrinks: preliminary numerical evidence, read as emission only by inference, since record time has no stable vacuum. Where the information goes depends on a unitary evolution in record time, which the identified Regge path sum lacks; with the ledger's measure weight its record-time roots are only stable in the continuum limit. Where a proof is complete but its independent verification is not, the text says so.

1 What a horizon records

Take a black hole of one solar mass. In conventional units, with the measured values of G, $\hbar$, c and k_B, its horizon is a sphere of radius $2GM/c^2 = 2.95\,\mathrm{km}$ and area $A = 1.10 \times 10^8\ \mathrm{m}^2$. Bekenstein argued that such an object must carry an entropy proportional to that area [1], and Hawking's discovery that it radiates at the temperature

$$T_{\mathrm{H}} = \frac{\hbar g_{\mathrm{H}}}{2\pi c k_B} = \frac{\hbar c^3}{8\pi G M k_B} = 6.2 \times 10^{-8}\ \mathrm{K}$$

fixed the coefficient [2]:

$$\frac{S}{k_B} = \frac{A}{4\ell_P^2} = 1.05 \times 10^{77},$$

about 1.5×10^{77} bits. That is far more than the entropy of the star that collapsed, and it is set entirely by the area of a surface that nothing crosses outward. Three questions follow. What are the $\exp(10^{77})$ states this entropy counts, and why does each Planck area carry exactly a quarter of a unit of it? How does a classical horizon turn the vacuum of the fields around it into thermal radiation? And when the hole has radiated itself away, after about 10^{67} years, where has the information that fell in gone [3, 4]? This paper does not answer the three questions. It states the four steps a ledger account of them needs, a lattice horizon, its entropy, its radiation and the fate of its information, as precise targets, and none of the four is complete. From here on $c = k_B = 1$.

Why the ledger. Recognition Science describes events as records kept on a ledger. Four counters advance one step at a time, each step a *posting*, and the lattice of records they generate carries an order before it carries a geometry [5]. A real-time path sum over the geometries this lattice can carry, with the Lorentzian Regge action [6], is the candidate quantum theory of gravity of [7, 8]. If that theory is right, a horizon is a surface of the record lattice, its entropy counts the configurations of records on the surface that the outside can distinguish, its radiation is the response of lattice fields to a geometry that traps, and the fate of information is a question about whether records are kept. The ledger therefore offers concrete objects for each of the three questions. Whether it answers them is what this paper sets out to settle.

The quarter as a test. The first question has a sharp form on the lattice. There Newton's constant is fixed by the scale s of the action, $G = 1/(8\pi s)$, and $\hbar = 1/a$ is the inverse of the scale a of the path sum's phase [7, Sec. 3.5]. So

$$\frac{1}{4G\hbar} = 2\pi a s,$$

and at tree level the Bekenstein–Hawking law says that a horizon holds an entropy $2\pi a s$ per unit area: the action of a full turn, a deficit angle of 2π, around a unit of area, weighed in imaginary time. The ledger's phase premise fixes $as = \ln 2/(2\pi\lambda_{\mathrm{rec}}^2)$ by reading a full turn around one face of area λ_{rec}^2 as one bit, so with the premise the quarter becomes one bit per record area, and one bit per record area is what the premise put in. A count that assumes one bit per λ_{rec}^2 returns the quarter by construction. A derivation has to count the horizon's records independently and then compare with $2\pi a s$ (Section 4).

Table 1. The steps from the ledger to the three questions and their status. "Proved" means proved as stated, with the premises listed in the section named; "pending" marks a step whose proof is complete and whose independent verification is not yet finished; "identification" marks a physical premise that is not a consequence of the definitions; "open" marks a precise statement that is neither proved nor refuted.

step	status	where
the horizon	proved: the posting order holds no black hole, a trapped region needs spacelike postings, weak fields trap nothing, every trapped region leaks along postings, and configurations built from flat simplices carry exact Killing horizons, one with an inside; the path sum's own configurations hold no black hole; for the action as it stands a weak vacuum field does not extend past second order, one order beyond Einstein's equations (for a sampled field and for a point mass's Taylor fields). Open: a strong-field lattice black hole, which lies beyond the second-order solutions and outside those configurations, and with a regular far field needs an action exactly symmetric under vertex moves (inference)	Section 3
the entropy	proved: the quarter means $2\pi as$ per unit area; the phase premise is the quarter at one face; a horizon hides the postings that cross it; one bit per crossing gives 0.108, 0.325 or 0.1875 by orientation, none the quarter at the identified spacing and tree level, and no long-wave factor makes two of them the quarter; a field-like ledger hides one bit on an observer horizon. Open: the ledger's own count, with its balance conditions and one density on every kind of cut	Section 4
the radiation	proved: the lattice dispersion has the sign of a superluminal medium, its scattering conserves a lattice flux, and the regulated path sum selects Feynman's rule. Numerical, preliminary, on test backgrounds that are not lattice solutions, with the occupation read as an in-out coefficient (inference): a Planck spectrum on a test horizon, cooler by order $(g_{\mathrm{H}}b)^2$, and on the exact geometry of a lattice horizon with a test interior. Open: the theorem on a lattice solution; the radiation after a collapse, posed on the regulated path sum, which a fast interior is expected to decide through the scalar's unstable sector (numerical); and the graviton, for which the action goes blind near a horizon; restoring the term it drops keeps Schläfli's identity (both proved)	Section 5
the information	proved: at $\kappa = 1$ the Regge path sum has a growing mode in record time, so the record tick preserves no positive norm on its physical solutions; with the derived measure weight the record-time roots of its Gaussian equation are stable in the continuum limit at fixed weight strength, given the weight's identifications and an inference about modes off the unit circle, but not at the lattice scale; that is not yet a norm-preserving evolution. Open: a unitary evolution in record time at the lattice scale, and the evaporation dynamics	Section 6

What this paper establishes. Table 1 lists the four steps and their status. None of them is complete. The paper fixes what each step has to prove: the objects (the two causal orders of the lattice, the configurations a surface carries, the lattice fields on a background with a horizon), the target statements, and the results of [7, 8] each one rests on. Three steps have begun. On the horizon, the lattice settles where a horizon can form, it has exact Killing horizons on configurations built from flat simplices, one with an inside, and the configurations its path sum integrates over hold no black hole (Section 3); an action extended to spacelike triangles makes the lattice black hole a precise target (Sections 3 and 5), and for the action as it stands a weak vacuum field fails to extend past second order one order beyond Einstein's equations, so a black hole with a regular far field needs an action with an exact symmetry under vertex moves (Section 3; INFERENCE). On the entropy, it settles which counts test the quarter and which assume it, finds what a horizon of the lattice hides, and shows that a count giving each hidden posting a bit yields the quarter at the identified spacing on none of the three kinds of horizon computed (Section 4). On the radiation, it settles the sign of the lattice's dispersion, that of a superluminal medium, and the in-out occupations computed on a test horizon and on the exact geometry of a lattice horizon with a test interior are thermal to within corrections that vanish with the lattice spacing (preliminary numerical evidence, on backgrounds that are not lattice solutions, read as emission only by inference); after a collapse the question is posed exactly on the regulated path sum, and the numerics point to an answer decided by the scalar's unstable sector rather than by the surface gravity (NUMERICAL; Section 5).

Relation to other work. The entropy of quantum fields outside a horizon is proportional to the area, with a coefficient set by the short-distance cutoff [9, 10]. String theory counts the microstates of extremal black holes and finds the quarter [11]; loop quantum gravity counts the punctures of a horizon by spin-network edges and finds an area law whose coefficient depends on the Barbero–Immirzi parameter, which is then fixed to give the quarter [12, 13]. Jacobson derived Einstein's equations from the Clausius relation applied to local horizons with an entropy proportional to area, which fixes G in terms of the entropy per unit area [14], and Wald identified black-hole entropy with a Noether charge of the action [15]. On the radiation side, the thermal spectrum survives a modified dispersion at short distances [16, 17, 18, 19]. On information, the Page curve [4] and its recent derivation from quantum extremal surfaces [20, 21] describe what a unitary evaporation must look like. The ledger route differs in its objects: the states are configurations of records on a surface of a fixed lattice whose spacing is near the Planck length, and G is fixed by the action rather than by the entropy, so the quarter can come out of the count or fail to.

2 The lattice, its scales and what is known

Status: what the black-hole questions rest on

Proved in [7, 8]. On the Kuhn lattice with the Lorentzian Regge action: linearized Einstein gravity at long wavelengths with $G = 1/(8\pi s)$ for every $\kappa > 1/2$; at identified $\kappa = 1$, exact weak-field solutions with matter and Newton's potential; massless corner doublers exactly at $\kappa = 1$, gapped by a measure weight derived from the ledger's cost of curvature, under two identifications; the vertex symmetry of the flat lattice fails at second order in the field, one order past Einstein's

equations, for a sampled field and for the cubic Taylor fields of a point mass. Appendix A.2 lists each imported result with its source and hypotheses.

Definitions and identifications. $\hbar = 1/a$ and $c = 1$ are definitions; $\kappa = 1$ is an identification; the phase premise $as = \ln 2/(2\pi\lambda_{\mathrm{rec}}^2)$ is an identification; one record length per rank fixes the spacing $b = (2/\sqrt{3})\lambda_{\mathrm{rec}} \approx 1.92\,\ell_P$ at $\kappa = 1$ (identification).

Open in [8]. Strong-field lattice solutions; a massless graviton at one loop; a unitary evolution in record time.

We recall the objects the three questions are about; the proofs are in [7, 8], and [22] follows the chain of units in detail.

Records, postings and the lattice. Vertices are points of $\mathbb{Z}^4$ or of the side-N torus, with coordinates x_a that count the postings on four counters. The *record rank* of a vertex is $\sum_a x_a$, the number of postings it contains. Four counters advanced one step at a time from $(0, 0, 0, 0)$ to $(1, 1, 1, 1)$ have 24 arrival orders, the 24 simplices of Kuhn's triangulation of the unit cube [23, 24]. Every edge joins x to $x + d$ for one of the fifteen classes $d \in \{0, 1\}^4 \setminus \{0\}$, and an edge with $|d| = 1$ is a posting. A configuration is a squared length on every edge; the flat lattice has the metric $g_\kappa = I - \kappa \mathbf{1}\mathbf{1}^\top$ in record coordinates, with $g_\kappa(d, d) = |d|(1 - \kappa|d|)$. At $\kappa = 1$ every posting is null and every longer edge is timelike; that value is an identification [7, Sec. 2.2].

Two scales. The classical action is sS, with $S = \sum_h A_h \delta_h$ the Lorentzian Regge action, a sum over triangles h of area times deficit angle, and s the action scale. The path sum weighs a configuration by e^{iasS}, with the phase scale a. The lattice Newton clause holds for exactly one coupling, $G = 1/(8\pi s)$, and $\hbar = 1/a$ is a definition [7, Sec. 3.5]. The *phase premise* reads one face of area λ_{rec}^2 as holding one bit: a full turn around it, weighed in imaginary time, halves the weight, $e^{-2\pi as\lambda_{\mathrm{rec}}^2} = \frac{1}{2}$. With it $G\hbar = \lambda_{\mathrm{rec}}^2/(4 \ln 2)$, so $\ell_P = \lambda_{\mathrm{rec}}/(2\sqrt{\ln 2})$. That premise is the subject of Section 4.

What the classical lattice does, and what it does not yet do. At long wavelengths the lattice is Einstein gravity at linear order: for $\kappa > 1/2$ the linear lattice field of a conserved source is the linearized Einstein field in de Donder gauge [7, Thm. 4.6], and at identified $\kappa = 1$ the full Regge equations with a resting body have exact solutions whose weak-field limit gives Newton's potential [7, Thms. 4.8–4.10], with second-order corrections at fixed small strength [7, Thms. 4.12–4.16]. All of these are weak fields. A black hole is a strong field: near its horizon the metric differs from flat at order one. No strong-field lattice solution is known, and the nonlinear Einstein equations are not yet recovered as a long-wave limit [8, Sec. 4]. Two facts bear on a strong field. On a sampled smooth field, the metric part of the lattice field equation, ten of its fifteen combinations of edge classes, is at second order in the spacing exactly $-\frac{1}{2}$ times the Einstein density, every nonlinear term included, at every constant background joined to the flat metric by a segment of Lorentzian metrics with nondegenerate flat simplices [8, Thm. 4.2]; this concerns the field equation, not its solutions, and the other five combinations are not covered. And the condition under which a weak vacuum field extends to second order fails one order past Einstein's equations [8, Thm. 4.3] (Section 3).

What the quantum lattice does, and what it does not yet do. At $\kappa = 1$, and only there for $\kappa > 1/2$, the flat operator has exact massless doublers at the four axis corners of the Brillouin zone [8, Thm. 2.1]; with a chosen flat measure they make the zero-momentum anisotropy of the one-loop graviton self-energy diverge like $\log(1/\varepsilon)$ as the regulator ε is removed after the volume [7, Thms. 6.7 and 6.8]. A measure weight built from the ledger's cost of curvature gaps all four of them, under two identifications, and gives the real part of that ordered one-loop anisotropy a finite limit [8, Thms. 5.4, 5.5 and 7.4]. Whether a repaired action gives a massless graviton at one loop is open; for the path sum as it stands, in the one-loop model of [8] (a unit-weight gauge term, no ghost line, the Feynman regulator), the graviton is not massless [8, Thm. 3.3]. Whether the path sum has a unitary evolution in record time is open as well: the identified Regge path sum fails the test, and the derived weight passes only a weaker one, in the continuum limit (Section 6). Both matter here: Hawking radiation is computed for fields on a fixed background, but the information question is a question about the unitarity of the full theory.

3 A horizon on the ledger

Status: the horizon

Definitions. The posting order of the records; the interval order of a configuration; future-closed sets, causal pasts, event horizons and trapped regions for each order (Definition 3.1).

Proved in [7, Thm. 2.1]. On the flat lattice every posting-directed edge is causal exactly when $\kappa \geq 1$, so the two orders agree on flat space exactly when $\kappa \geq 1$.

Proved. The posting order has no black-hole region; an observer that posts on every counter has no event horizon, and one that stops posting on a counter has a coordinate facet as its horizon, timelike for $\kappa > 1/3$ (Proposition 3.2). If every posting is causal, an observer that posts on every counter has no event horizon, so such a horizon needs a spacelike posting; a Killing horizon needs a spacelike posting, a cell that is not oriented, or a cube of null edges (Proposition 3.3). Weak fields trap nothing (Proposition 3.4). Configurations built from simplices of flat spacetime carry exact Killing horizons: the Rindler wedge, and the river wedge, whose interior is a black-hole region of the interval order (Propositions 3.5 and 3.6). Every trapped region is left by spacelike postings (Proposition 3.7). No horizon arises through second order in the field in the second-order solutions the framework records, at $\kappa = 5/4$. The configurations the path sum integrates over hold no black hole: their diagonal is timelike everywhere, nothing is trapped, and for $\kappa \geq 1$ an observer that posts on every counter has no event horizon (Proposition 3.8); at $\kappa = 1$ a Killing norm near zero squeezes the whole cell (Proposition 3.9). The flat lattice meets every condition of Target 3.10 except strength (Proposition 5.13). The cut of a horizon by a slice of fixed rank is spacelike, with a density of records that depends on its orientation. For the action as it stands, a weak vacuum field does not always extend to second order: for a sampled cubic field, on a torus and on the infinite lattice, and for the cubic Taylor fields of a point mass's potential, the condition fails one order past Einstein's equations [8, Thm. 4.3].

Numerical. The river wedge's deficits vanish through its horizon for $\alpha b \leq 0.5$; it does not satisfy the Regge equations of the action of [7], which is blind to its spacelike triangles (Section 5). An explicit exterior that meets every condition of Target 3.10 but the Regge equation [8] misses that equation by the order of the local curvature; a correction in the edge combinations that are not metric lowers the miss by one power of the core's size. Read as the far field of a point mass,

the second-order defect is a force on the vertex placements of order $M^2 r^{-7}$.

Inference. A static admissible lattice approaches a horizon only by shrinking its cells; with cells of flat size its squared lapse stays at or above the value the Schwarzschild field reaches at $r = 3GM$. For the action as it stands, a static vacuum solution with a regular far field, expanded in its amplitude, stops one order past Einstein's equations, so a solution of Targets 3.10 and 3.11 with such a far field needs an action exactly symmetric under vertex moves, matter that carries the defect, or lattice-scale static modes in the far field.

Open. Target 3.10: a static exterior that stays strong as the lattice is refined. Target 3.11: a lattice black hole, static exteriors of cores of every size that solve the Regge equations of the action extended to spacelike triangles and have a Killing horizon outside the core; configurations that keep the causal clause have none.

Identifications and model. $\kappa = 1$; the posting order as the physical causal order where it is used; the causal clause of the admissible configurations (model, an input of the path sum).

A horizon is a statement about causal order: the boundary of the region from which no signal reaches the outside. The lattice has two causal orders, and the answer depends on which one is meant.

Definition 3.1 (Two orders)**.** The *posting order* relates records $x \preceq y$ when $y - x$ has every coordinate nonnegative: y can be reached from x by postings. It is fixed and does not depend on the geometry. For a configuration ℓ, the *interval order* is generated by the posting-directed edges $(x, x + d)$, $d \in \{0,1\}^4 \setminus \{0\}$, whose squared length satisfies $\ell_{(x,d)} \leq 0$: the edges that are causal in the geometry ℓ. In either order a set of records is *future-closed* if it contains every record above each of its members; the *causal past* of a set is everything below some member; the *event horizon* of a region that extends to infinity is the boundary of its causal past; a set is *spatially bounded* if the differences $x_a - x_b$ of the counts of its records are bounded, so that it stays within a bounded distance of a line of records along $\mathbf{1}$; and a *trapped region* is a nonempty, spatially bounded, future-closed set. For a configuration invariant under the time translation $T = \mathbf{1} = (1,1,1,1)$, itself an edge of the lattice, the *Killing norm* is the squared length of the edge $(x, x + T)$, and the *Killing horizon* is where it vanishes.

On the flat lattice an edge of n postings has squared length $n(1 - \kappa n)$, so every posting-directed edge is causal exactly when $\kappa \geq 1$, and then the interval order of flat space is the posting order [7, Thm. 2.1]. At $\kappa = 1$ the postings themselves are null. That makes the interval order sensitive in a way the posting order is not: an arbitrarily weak field decides, edge by edge, whether a posting is causal.

The posting order cannot hold a black hole. The posting order is the orthant order of $\mathbb{Z}^4$: a future-closed set contains, with each record, every record above it.

Proposition 3.2 (Observer horizons only)**.** *In the posting order no nonempty future-closed set is spatially bounded. An observer that posts on every counter without bound has the whole lattice in its past and no event horizon. An observer whose postings on one counter stop at a value M, while the others grow without bound, has the past $\{x : x_a \leq M\}$ and the event horizon $\{x : x_a = M\}$. In the lattice interval $Q_\kappa(v) = \sum_a v_a^2 - \kappa(\sum_a v_a)^2$, such a facet contains a future timelike vector if and only if $\kappa > 1/3$.*

So the posting order has no black-hole region at all, and the horizons it does have are not null surfaces of the geometry: at $\kappa = 1$ the facet $\{x_a = M\}$ is timelike, and the cone of the interval contains future timelike directions that cross it inward. A black hole, if the lattice has one, lives in the interval order of a curved configuration.

Where a horizon has to form. Call a configuration *faithful* if every posting in it is causal, and *oriented* if, in addition, the two legs of every lattice triangle, taken in the posting order, have a non-positive inner product, so that both point into the same half of the cone.

Proposition 3.3 (What a faithful configuration excludes)**.** *In a faithful configuration the interval order is the posting order, and an observer that posts on every counter has every record in its causal past: there is no event horizon. In a faithful, oriented configuration every edge is causal and the Killing norm is non-positive everywhere, and where it vanishes every edge of the cube from x to $x + T$ is null.*

So for an observer that posts on every counter, an event horizon needs the geometry to stop reading the posting order faithfully: some posting must turn spacelike. A Killing horizon needs that, or a cell whose triangles are not oriented, or a cube whose edges are all null. Trapping takes a strong field. Because the interval order only loses edges as edges turn spacelike, trapping needs edges that the flat lattice has as causal to become spacelike, and for edges of two or more postings that takes a change of order one in their squared length.

Proposition 3.4 (Weak fields trap nothing)**.** *In a trapped region every class of edges $d \neq \mathbf{1}$, the classes that move in space, is spacelike at some record of the region, that is, exceeds its flat value there by more than $\kappa n^2 - n$ for an edge of n postings (2 for two postings and 6 for three at $\kappa = 1$). A configuration whose squared lengths are all within $4\kappa - 2$ of their flat values has no trapped region. The flat lattice has a trapped region if and only if $\kappa < 1/3$.*

An exact lattice horizon. The simplest horizon is that of a uniformly accelerated observer, and the lattice has an exact one. Place the records on the orbits of a boost in Minkowski space, so that every simplex is a simplex of flat spacetime, and let α set the boost per unit of record time.

Proposition 3.5 (The lattice Rindler wedge)**.** *The resulting configuration is invariant under the time translation T, which acts as a boost of rapidity $2\sqrt{3}\,\alpha$. Its Killing norm is $-4\chi^2 \sinh^2(\sqrt{3}\,\alpha)$, where χ is the distance from the boost's fixed plane, so for $\alpha \neq 0$ its Killing horizon is $\chi = 0$, with the boost α per unit of record time, the time that advances by $\sqrt{3}/2$ per rank. Its postings have squared length $\frac{3}{4} - 4\chi\chi' \sinh^2(\sqrt{3}\,\alpha/4)$, which is spacelike wherever $\chi\chi' \leq 0$: the wedge is not faithful at its horizon, as Proposition 3.3 requires.*

The wedge is built from simplices of flat spacetime; whether they close up around every hinge through its horizon is not checked, and its spacelike triangles are ones the action of [7] cannot see (Proposition 5.9). An accelerated observer's horizon traps nothing, though. The lattice also has an exact horizon with an inside.

Proposition 3.6 (The lattice river wedge)**.** *Place the records in the flat spacetime $\mathrm{d}s^2 = -\mathrm{d}t^2 + (\mathrm{d}x + \sqrt{1 - 2\alpha x}\,\mathrm{d}t)^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, a uniformly accelerated frame in which space flows*

inward, with $\alpha > 0$ the surface gravity per unit time. The resulting configuration is static, with Killing norm $-4(1+2\alpha\xi)\sinh^2(\sqrt{3}\,\alpha)/\alpha^2$ in terms of a coordinate ξ along the flow, normal to the horizon: a regular Killing horizon at $\xi = -1/(2\alpha)$, static outside, with the diagonal edge spacelike inside. The interior $\xi \le -1/(2\alpha)$ is closed under causal steps and causal chains, a black-hole region of the interval order, and every edge from the interior to the outside is spacelike.

Its simplices are simplices of flat spacetime, and its deficits vanish: the dihedral angles close up to exactly 2π around every hinge, checked for $\alpha b \le 0.5$ in the ranges computed (NUMERICAL). It contains spacelike triangles (NUMERICAL), to which the action of [7] is blind, and computed in high precision it fails that action's Regge equations (NUMERICAL, Section 5). It is an exact geometry with a horizon, not a solution of that action.

A horizon leaks along postings. The river wedge exposes a tension between the two orders.

Proposition 3.7 (Leaks)**.** *If a region is left by no causal step of a configuration but is left by some posting, it is left through a spacelike posting. Every trapped region leaks in this sense; in the river wedge the interior is closed under causal steps but not under postings, and the postings that leave it are spacelike.*

So at a horizon of the interval order three readings compete: influence carried along postings is not trapped by the horizon; or the posting order is not the physical causal order in a strong field, which would withdraw an identification the programme uses; or the dynamics never makes the interval order unfaithful, and no horizon forms (INFERENCE). Which one holds is a decision for the programme, and it is the question of Section 6 in kinematic form.

A horizon is not perturbative. A Killing horizon needs the diagonal edge to change by its whole flat value: within $16\kappa - 4$ of flat, 12 at $\kappa = 1$, the Killing norm is negative everywhere. And the weak-field expansion does not reach one. A static field $g_{00} = -1 + 2U - 2\beta U^2$ correct to second order in the potential U vanishes somewhere if and only if $\beta \le 1/2$; the lattice's second-order solutions in the form the framework records them, at $\kappa = 5/4$, have $\beta = 1$ and $g_{00} \le -\frac{1}{2}$ for every mass, while the exact isotropic Schwarzschild field $-\big((1-U/2)/(1+U/2)\big)^2$ vanishes at $U = 2$. A lattice Schwarzschild horizon is therefore a non-perturbative object (INFERENCE, from an expansion truncated at second order, at $\kappa = 5/4$).

The path sum's configurations hold no black hole. The configurations the path sum of [7] integrates over are not arbitrary. A configuration is *admissible* if every simplex is timelike in all its faces, the main-diagonal edge and every edge of two or more postings are timelike, and every posting has squared length of the sign of $1 - \kappa$ [7, Sec. 3.1]. The clause that keeps the diagonal timelike places time on the main diagonal, and [7] states it as an input of the construction. It is also exactly what a horizon violates.

Proposition 3.8 (No black hole among admissible configurations)**.** *In every admissible configuration the Killing norm is negative at every record, so there is no Killing horizon and no static interior, and there is no trapped region, for every κ. For $\kappa \ge 1$ the interval order is the posting*

order, so an observer that posts on every counter has every record in its causal past: there is no event horizon. A configuration whose diagonal edge is somewhere not timelike, or which has a trapped region, is not admissible; the river wedge inside its horizon and the Rindler wedge on its horizon are not. The statements about the Killing norm and trapping are immediate from the causal clause; the one about the event horizon uses the posting clause.

So among the configurations of the path sum the third reading after Proposition 3.7 holds: no horizon forms. It holds because of the causal clause, an input of the construction, and not because a dynamics prevents collapse. At $\kappa = 1$ the clause also fixes how an admissible lattice can come close to a horizon.

Proposition 3.9 (The cells squeeze)**.** *Let $\kappa = 1$. In an admissible configuration every squared edge length of a cell, the cube from x to $x + T$, lies between the squared length of its diagonal and* 0*, and the diagonal's squared length is at most the sum of those of two of its edges of two postings. A Killing norm near* 0 *therefore forces every edge of the cell near* 0*.*

A static admissible lattice can come close to a horizon only by shrinking its cells, all their edges together: a frozen star rather than a black hole. Quantitatively, wherever $K/K_{\text{flat}} \le \rho$, with K the Killing norm and $K_{\text{flat}} = -12$, the two edges of two postings in each simplex have squared lengths summing to at least -12ρ, against -4 when flat, so for $\rho < 1/3$ the cells there are smaller than flat cells. Read as the squared lapse, $K/K_{\text{flat}} = 1/3$ is the value the Schwarzschild field reaches at $r = 3GM$, the radius of its photon sphere: a redshift deeper than the photon sphere's needs shrunken cells (INFERENCE: the reading of the lapse).

A strong static field. The first step toward a lattice black hole needs no horizon: a static exterior whose redshift stays strong as the lattice is refined.

Target 3.10 (A strong static field)**.** On the infinite record lattice at $\kappa = 1$ there is a number $\rho < 1$ such that for every R_0 there are $R \ge R_0$ and a configuration that is invariant under T and under the 24 permutations of the counters about a record line, tends to the flat lattice far away, satisfies the Regge equation at every edge at distance at least R from that line, has every simplex Lorentzian and every edge of two or more postings timelike there, and has a record at distance at least R where $K/K_{\text{flat}} \le \rho$.

The target is not trivially satisfiable: the flat lattice is strong nowhere, a constant multiple of the flat lattice tends to it far away only if the multiple is one, and the zero configuration does not tend to it. Its conditions are consistent: the flat lattice meets every one of them except strength (Proposition 5.13). It asks for a fixed compactness while R grows in lattice units, which is how a body of fixed GM/r looks as the lattice is refined; a variant asks for a mass, read from $K/K_{\text{flat}} = 1 - 2M/r + o(1/r)$, that grows in proportion to R. It is not a horizon target, since the causal clause keeps $K < 0$ outside the core; in the variant with null postings outside the core, a solution with $\rho < 1/3$ has cells smaller than flat ones there.

Every condition of the target except the Regge equation can also be met by construction, so its content is the two together, a strong field that solves the equation. For every large enough core, [8] gives an explicit static configuration, built by sampling a radial exterior profile at the edges, that is symmetric under the permutations of the counters, tends to the flat lattice

with a mass $M = R/16$ that grows in proportion to the core, keeps every simplex Lorentzian and every edge of two or more postings timelike outside the core, and has $K/K_{\text{flat}} \leq 7/8$ at some record there. As computed, it misses the Regge equations outside the core by the order of the local curvature: the residuals fall like R^{-2}, with no cancellation among the triangles (NUMERICAL). The cause is particular to the Kuhn lattice. A vacuum metric sampled at the edges needs a correction in the five combinations of edge classes that are not metric, linear in the tidal field, and that correction, added locally, lowers the residuals to order R^{-3} (NUMERICAL). For the action as it stands, an expansion of a lattice exterior about a sampled exterior metric, in corrections ordered in the spacing over the radius, meets an obstruction one order past Einstein's equations. Vertex moves are exact zero modes of the flat lattice, and a linearized vacuum solution h extends to second order only if the third variation $D^3S(h, h, \cdot)$ pairs to zero with every vertex move. At $\kappa = 1$ on the torus of side 8, the sampled static field of the cubic potential $\xi_1^3 - 3\xi_1\xi_2^2$, with its slaved correction in the edge combinations that are not metric, solves the linearized equations at every edge through a record y, and it pairs with the move of y to $756\sqrt{3}$, so no second-order correction exists there [8, Thm. 4.3]. The same holds on the infinite lattice, and for the cubic Taylor fields of a point mass's potential about records on an axis, a face diagonal and a body diagonal. At the order of Einstein's equations the same pairing vanishes in vacuum (INFERENCE), and read as the far field of a point mass of mass M the defect acts on the vertex placements as a force of order M^2r^{-7} with a cubic anisotropy (NUMERICAL) [8]. So a static vacuum solution with a regular far field, expanded in its amplitude, stops one order past Einstein's equations (INFERENCE). Three escapes remain open: an action exactly symmetric under vertex moves, the property a massless graviton needs as well [8, Target 6.8]; matter that carries the defect; and far fields with static modes at the lattice scale. None has been constructed.

Target 3.11 (A lattice black hole)**.** Take the action extended by the Lorentzian term on spacelike triangles (Proposition 5.10). For every R_0 there are $R \geq R_0$ and a configuration of the infinite record lattice that is invariant under T and under the 24 permutations of the counters about a record line, tends to the flat lattice at $\kappa = 1$ far from that line, satisfies every Regge equation of the extended action at every record at distance at least R from the line, has every simplex at distance at least $R - 2$ regular Lorentzian, with no null triangle or tetrahedron, and has a record at distance at least R where the Killing norm is not negative.

Distance here is the spread of a record's coordinates relative to the line, as in Target 3.10, and the permutations act as the tetrahedral group, the largest point group of the lattice about a line; an exact rotation group is not available on a lattice. A record outside the core whose Killing norm is not negative, in a field that is flat far away, puts a Killing horizon in the vacuum region, and for large cores the horizon is many lattice spacings across. Such a configuration is not admissible (Proposition 3.8): its diagonal edge turns null and then spacelike, and the extended action has to be derived from the ledger rather than chosen. The target does not ask that the horizon lie at $2GM$, with $G = 1/(8\pi s)$ and M read from the far field as in the variant of Target 3.10; a solution would have to show that as well to be the lattice Schwarzschild black hole.

The target is not trivially satisfiable. The flat lattice has Killing norm -12 everywhere; the exteriors of Target 3.10 keep every edge of two or more postings timelike, the diagonal included, so their Killing norm stays negative; and a flat exterior has a constant Killing norm

(argued, not proved), so a solution needs curvature. The second-order solutions have no horizon, and the river wedge, with its spacelike triangles, does not solve the Regge equations of the action of [7] (NUMERICAL). The target is settled either way, by such configurations or by a proof that none exists. Its far field meets the obstruction above unchanged: where every triangle is timelike the extended action has the equations of the action of [7] (Proposition 5.10), so a solution with a regular far field needs one of the three escapes, and the form of the target most likely to be met is the same statement for an action derived from the ledger that is exactly symmetric under vertex moves (INFERENCE). If the ledger keeps the causal clause, the question does not arise: its configurations hold no black hole (Proposition 3.8), a static lattice comes close to a horizon only by shrinking its cells (Proposition 3.9; INFERENCE), and the three questions of this paper are about that object.

A solution would also supply the two lattice quantities the other questions need: the area of a cross-section of the horizon and the surface gravity. The cross-section has a definite structure. The cut of a horizon by a slice of fixed rank is spacelike for every κ, and it contains no lattice faces at all, since every edge of the lattice raises the rank. Its records form a two-dimensional lattice whose density depends on the horizon's orientation: one record per $\sqrt{3}$ units of area on the facet of a posting-order horizon, one per 2 units on a cut normal to a cube axis, the orientation of the Rindler and river horizons. Regge calculus has a long history with the Schwarzschild geometry on its moment of time symmetry [25]; the question here is the full Lorentzian lattice with its causal order.

4 The entropy and the quarter

Status: the entropy

Proved. With $G = 1/(8\pi s)$ and $\hbar = 1/a$, the law $S = A/(4G\hbar)$ holds for every area exactly when the entropy per unit area is $2\pi as$; the phase premise is the law at a single face; and with a coupling read off from the entropy, every density satisfies the law (Proposition 4.1). At $\kappa = 1$ the posting order has null horizons ruled by postings (Proposition 4.2), and the past cone of a record is a closed causal boundary whose cut holds exactly the area term plus the Euler characteristic in records (Proposition 4.3). A horizon hides the postings that cross it (Proposition 4.4); if the ledger holds one state per record, an observer horizon hides one bit whatever its area, and any horizon at most a perimeter's worth (Proposition 4.5). With one bit per crossing, none of the three kinds of horizon cut computed gives the quarter under the identifications that fix the spacing, and on them no whole number of states per record does (Proposition 4.6); with the spacing equal to the record length, a spacing that conflicts with $\kappa = 1$ and one record length per rank, the inward crossings of an axis horizon give exactly the quarter, and no spacing gives it on two kinds of horizon (Proposition 4.7). Both targets can hold and can fail (Proposition 4.10). A reading that gives a cut one bit per unit of area b^2 meets the conditions of the count, which shows only that they are consistent; every reading computed so far fails them: the empty one, every whole number of states per record, one bit per crossing, and every reading held to a perimeter law. The phase premise at $b = \lambda_{\mathrm{rec}}$ meets the second target with one bit per b^2; a condition that leaves as free does not, and at the identified spacing the inward crossings of an axis horizon do not.

Model. What a crossing carries: one bit per crossing, or one ledger state per record; a fixed number of states per record, independently.

Open. Target 4.8: the ledger's own reading of a horizon's cross-section, its configurations, balance conditions and what the exterior reads of each, with one entropy density on tori of two orientations and on spheres. Target 4.9: a condition fixed without the count that determines as, with that density equal to $2\pi as$.

Identifications. The phase premise; $\kappa = 1$ and one record length per rank, for the conversion to Planck units; the posting order as the causal order at an observer horizon.

The quarter in the lattice's units. Let a horizon of area A carry the entropy $S = \eta A$, with η the entropy per unit area in natural units. With $G = 1/(8\pi s)$ and $\hbar = 1/a$,

$$\frac{A}{4G\hbar} = 2\pi as\, A,$$

so the Bekenstein–Hawking law is the statement $\eta = 2\pi as$. Here G is the coupling of the tree-level exchange; the coupling of the long-wave limit of the quantum path sum is ZG, with a factor Z that is not yet computed [22, Sec. 2], and with it the law reads $\eta = 2\pi as/Z$. The numbers of this section are at tree level, $Z = 1$; the comparisons between kinds of horizon do not depend on Z. The right side has a lattice meaning. Turn once around a hinge of area A: a deficit angle of 2π. Its Regge action is $2\pi sA$, and the weight e^{iasS} continued to imaginary time is $e^{-2\pi asA}$. The law therefore says that a horizon holds, per unit area, the imaginary-time action of a full turn. That is the lattice form of the observation that a horizon's entropy is the Euclidean action of a conical deficit at its bifurcation surface [26, 27], and of Wald's identification of the entropy with the Noether charge of the action [15].

When the quarter tests nothing. Two ways of fixing the couplings turn the law into a restatement.

Proposition 4.1 (The quarter and its circular forms)**.** *For $G, \hbar > 0$ the law $S = A/(4G\hbar)$ holds for every area at the density η if and only if $\eta = 1/(4G\hbar)$; with $G = 1/(8\pi s)$ and $\hbar = 1/a$ this is $\eta = 2\pi as$, and under the phase premise $\eta = \ln 2/\lambda_{\mathrm{rec}}^2$. Moreover:*

(i) for every density $\eta > 0$ there is exactly one coupling for which the law holds, the Clausius coupling $G = 1/(4\hbar\eta)$; so a coupling read off from the entropy satisfies the law whatever is counted;

(ii) a full turn around a cell of area α weighs 2^n states if and only if the law holds at n bits per area α: the phase premise, one bit per face of area λ_{rec}^2, is the law at a single face.

So a count that assumes one bit per λ_{rec}^2 and converts with $\ell_P^2 = \lambda_{\mathrm{rec}}^2/(4\ln 2)$ returns the quarter for every area, whatever the horizon and whatever its records; and taking G from the Clausius relation, as in [14], defines G so that the quarter holds. On the ledger lattice G is fixed independently, by the Newton clause, so η and as are two facts, and the quarter is a relation between them.

Two routes to the entropy. The lattice offers two independent ways to reach η, and the quarter is their agreement. The first is thermodynamic. If lattice fields on a lattice black

hole radiate at $T = \hbar g_{\mathrm{H}}/(2\pi)$ (Section 5) and the lattice solutions obey the classical first law $dM = g_{\mathrm{H}}\, dA/(8\pi G)$ with the G of the dynamics, then $dS = dM/T$ gives $S = A/(4G\hbar)$, that is $\eta_{\mathrm{thermo}} = 2\pi as$, with G and $\hbar$ from the action and the phase. The second is statistical: count what the horizon records. If the count gives η_{stat} in lattice units, the quarter is the equation $\eta_{\mathrm{stat}} = 2\pi as$. If as is fixed independently of the count and the two agree, the quarter is derived; if they disagree, the ledger's account of the horizon is wrong somewhere, and the disagreement says by how much; if as is not fixed independently, the equation fixes it, and a count that gave one bit per λ_{rec}^2 would derive the phase premise instead of assuming it [22].

The posting order's own horizons. A count needs a surface. At $\kappa = 1$ the posting order supplies null surfaces of its own. A posting has squared length $(1-\kappa)b^2$, null exactly at $\kappa = 1$, and at $\kappa = 1$ a combination of postings with nonnegative coefficients is null exactly when it runs along one counter's axis.

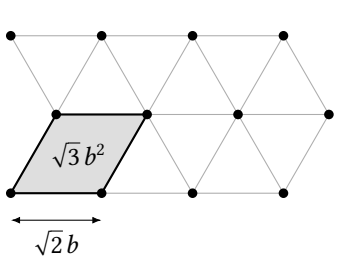
Proposition 4.2 (Null horizons of the posting order)**.** *Let $\kappa = 1$, and for a counter l let s_l be the sum of the other three coordinates. The hyperplanes $s_l = c$ are null, ruled by the postings of counter l, which they contain and to which they are orthogonal; s_l never decreases along the posting order, so $\{s_l \le c\}$ is a past set and $s_l = c$ a one-way boundary. Its cross-sections by slices of fixed rank are triangular lattices with one generator per $\sqrt{3}\,b^2$ of area, the same for every cut along the generators. On the side-N torus a cross-section carries exactly N^2 records in an area $\sqrt{3}N^2b^2$.*

These cross-sections are the same planes as the rank-slice cuts of the observer horizons of Proposition 3.2, $\{x_l = M\}$, although the two hypersurfaces differ: at $\kappa = 1$ the facet $x_l = M$ is timelike and $s_l = c$ is null. A third kind of cut is normal to a cube axis, the orientation of the Rindler and river horizons of Propositions 3.5 and 3.6: the planes $x_0 + x_1 - x_2 - x_3 = \text{const}$. On a slice of fixed rank it has one record per $2b^2$ for every κ; at each record two postings raise the level and two lower it; and a level held fixed in time meets records only at every other rank (Fig. 1). The faces that the phase premise reads as holding one bit are lattice triangles, all of them timelike for $\kappa > 1/2$ [7, Sec. 2.1], while a horizon's cross-section is a slice of fixed rank and contains none of them, since every edge raises the rank (Section 3); so "one bit per face" cannot be applied to a horizon directly, and the count has to say what a horizon holds.

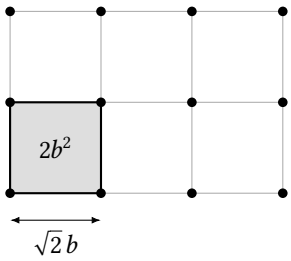
These horizons are planes. A closed causal boundary is the boundary of the past of a single record in the posting order, its past cone. It is not a black-hole horizon, but it shows what a count of records does on a closed surface.

Proposition 4.3 (The past of a record)**.** *The boundary of the past of a record p consists of the records below p that agree with p on at least one counter. Its cut at rank distance $m \ge 1$ below p is the surface of a tetrahedron with equilateral faces of side $\sqrt{2}\,mb$, for every κ; it holds exactly $2m^2 + 2$ records in an area $A = 2\sqrt{3}\,m^2b^2$, so the number of records is $A/(\sqrt{3}\,b^2) + 2$.*

The leading term is the density of a facet; the 2 is the Euler characteristic of the sphere, as the 0 for the torus of Proposition 4.2 is that of the torus. A count of records on such a closed surface is exactly an area term plus a topological one.



(a) observer facet or null plane



(b) cut normal to a cube axis

Figure 1. Records on two kinds of horizon cut, drawn at the same scale. (a) The cut of an observer horizon $\{x_a = M\}$ or of a null plane $s_l = c$ by a slice of fixed rank: a triangular lattice with one record per $\sqrt{3}\,b^2$. (b) A cut normal to a cube axis, the orientation of the Rindler and river horizons: a square lattice with one record per $2b^2$. Each shaded cell holds one record. Because the density depends on the orientation, a count that gives every record the same content is not a function of area alone.

What a horizon hides. Take the observer of Proposition 3.2 whose postings on counter a stop at M. Its past is $\{x_a \le M\}$, and the only way out of that past is a posting of counter a from a record with $x_a = M$. Once such a posting has landed, everything after it stays outside: the observer never receives it or anything that follows from it. What this horizon hides is a set of postings, one from each of its records. The same holds for every past set.

Proposition 4.4 (What a horizon hides)**.** *Let P be a past set of the posting order, and call a record $x \in P$ and a counter a a crossing of P if the posting of a from x lands outside P. No posting enters P from outside, and every record at or after the landing record of a crossing lies outside P. An observer that posts on every counter without bound has no crossings. For the observer whose postings on counter a stop at M, the crossings are the postings of counter a from the records of its horizon $\{x_a = M\}$, one per record. The cut of a horizon by a slice of fixed rank is an antichain of the posting order.*

What a crossing carries. A posting adds one unit to one account, as a debit or as a credit, and the record it leaves, the account's total of debits and credits, is the same either way. The side is the one datum of a posting that a reader of records cannot see, which makes it the natural content of a crossing. How much it holds depends on how the ledger is read (MODEL: each reading is stated, neither is derived here). In the *history reading* every posting is an event with a side of its own. In the *field reading* the ledger holds one state $L(x)$ at each record x, a debit total and a credit total for each of the four accounts, and each posting moves it by one posting of the ledger's model: $L(x + e_a)$ is $L(x)$ with one unit added to the debit or to the credit total of account a, according to the side of that posting (Appendix A.3).

Proposition 4.5 (What a crossing carries)**.** *The record a posting leaves does not depend on its side. In the history reading k crossings carry 2^k patterns of sides. In the field reading the two orders of two postings on different counters reach the same record and so must leave the same state, and this forces the side of a posting to depend only on its counter and that counter's count; so all crossings of an observer horizon $\{x_a = M\}$ carry the same side, both sides occur, and the horizon hides exactly one bit, whatever its area. For any set of crossings whose counts lie in a window of width N, the field reading realizes at most 2^{4N} patterns of sides.*

A ledger that is a field over the records therefore gives an observer horizon no area law at all,

and any horizon at most a perimeter law: $4N$ bits on a cut of linear size N, against the $k \propto N^2$ crossings of the history reading. An area law needs crossings that carry data of their own, as in the history reading, where each posting across a horizon is a separate event whose side the outside cannot read.

The coefficient of a crossing count. Give each crossing one bit. A cut with one record per $\sqrt{D}\,b^2$ and n crossings per record then holds $n \ln 2$ per $\sqrt{D}\,b^2$, and with $\ell_P^2 = \lambda_{\mathrm{rec}}^2/(4 \ln 2)$, the Planck length of the tree-level coupling, the entropy is $S = CA/\ell_P^2$ with

$$C = \frac{n\,\lambda_{\mathrm{rec}}^2}{4\sqrt{D}\,b^2}.$$

The three kinds of cut the lattice has so far give the values in Table 2.

Table 2. One bit per crossing posting, converted with the phase premise. The column at $b \approx 1.92\,\ell_P$ uses the spacing of [22], $b = (2/\sqrt{3})\,\lambda_{\mathrm{rec}}$ ($\kappa = 1$, one record length per rank); the last column uses $b = \lambda_{\mathrm{rec}}$.

cut	area per record	crossings per record	C at $b \approx 1.92\,\ell_P$	C at $b = \lambda_{\mathrm{rec}}$
observer facet $\{x_a = M\}$	$\sqrt{3}\,b^2$	1	$\sqrt{3}/16 \approx 0.108$	$1/(4\sqrt{3}) \approx 0.144$
null plane $s_l = c$	$\sqrt{3}\,b^2$	3	$3\sqrt{3}/16 \approx 0.325$	$\sqrt{3}/4 \approx 0.433$
axis cut, inward crossings	$2b^2$	2	$3/16 = 0.1875$	$1/4$
axis cut, averaged over ranks	$2b^2$	2, every other rank	$3/32 \approx 0.094$	$1/8$

Proposition 4.6 (Crossing counts at the identified spacing)**.** *Assume the phase premise, $\kappa = 1$ and one record length per rank, so that $\lambda_{\mathrm{rec}}^2 = \frac{3}{4}b^2$, and give each crossing one bit. Then $C = 3n/(16\sqrt{D})$: $\sqrt{3}/16$ on an observer facet, $3\sqrt{3}/16$ on a null plane, $3/16$ on an axis cut counting its two inward crossings per record, and $3/32$ on an axis cut averaged over ranks. None is $1/4$. Moreover:*

(i) no whole number ω of independent states per record gives $1/4$, on a facet or null cut, where it would need $\omega = 2^{4/\sqrt{3}}$, between 4 and 5, or on an axis cut, where it would need $\omega^3 = 2^8$, between 6^3 and 7^3;

(ii) on a facet or null cut no rational number of bits per record gives $1/4$, and none does for any rational value of $b^2/\lambda_{\mathrm{rec}}^2$ in place of $4/3$;

(iii) the same nonzero number of bits per record gives different coefficients on a facet and on an axis cut, so a count per record is not a function of area alone.

With the long-wave coupling ZG in place of G every coefficient is divided by Z. A single count could then give the quarter for a suitable Z, the axis cut's inward crossings for $Z = 3/4$; but the coefficients of the three kinds of cut keep their ratios, so no value of Z gives the quarter on two of them, and part (iii) is unchanged.

On facet and null cuts the obstruction in (i) and (ii) is the irrational area $\sqrt{3}$ of the cell of a rank-slice plane, measured in a unit that the identifications tie rationally to λ_{rec}^2. On an axis cut, whose cell has the rational area $2b^2$, it is arithmetic: the quarter needs $8/3$ bits per record, which no whole number of states supplies. The obstruction in (iii) is that the density of records on a cut depends on its orientation.

The other reading of the spacing. The coefficients above use the spacing that [22] derives from $\kappa = 1$ and one record length per rank. A second reading sets the spacing equal to the record length, $b = \lambda_{\mathrm{rec}}$, and the three readings $\kappa = 1$, $b = \lambda_{\mathrm{rec}}$ and one record length per rank cannot all hold [22]. At $b = \lambda_{\mathrm{rec}}$ one entry of Table 2 is exactly the quarter.

Proposition 4.7 (The quarter on an axis horizon)**.** *Assume the phase premise and one bit per crossing. For every spacing $b > 0$ the coefficients of the observer facet, the null plane and the axis cut with its inward crossings stand in the fixed ratio $1/\sqrt{3} : \sqrt{3} : 1$, so no spacing gives $1/4$ on two of them. The axis cut, counted at the ranks where it holds records, gives $1/4$ if and only if $b = \lambda_{\mathrm{rec}}$. With one record length per rank, $b = \lambda_{\mathrm{rec}}$ holds if and only if $\kappa = 5/4$, so it conflicts with the identification $\kappa = 1$, at which the axis cut does not give $1/4$.*

The reason is short. An axis cut carries two inward crossings per $2b^2$, one per b^2, and the phase premise is the quarter at one bit per λ_{rec}^2 (Proposition 4.1); the two agree exactly when $b = \lambda_{\mathrm{rec}}$. So the quarter on the axis horizon is the phase premise read on a surface whose crossings have the density of faces. It holds only at a spacing that conflicts with $\kappa = 1$ and one record length per rank, and it holds on one kind of horizon only. It is a consistency condition between the premise and the spacing, not a derivation of the quarter. Read the other way, it says what is at stake (INFERENCE): if the thermodynamic route gives the quarter and a Killing horizon's count is one bit per inward crossing, the spacing must be the record length, and one record length per rank fails at $\kappa = 1$; with one record length per rank, the same count needs $\kappa = 5/4$.

What the count has to count. The ledger data on a cross-section are the postings that cross it with their sides, the accounts of the hinges that lie in it with their net posting counts, the balance conditions those accounts obey, and the order of the crossing postings. Two configurations of these data that the exterior cannot tell apart are one state. In general terms, a *cut ledger* is a finite set of configurations, a balance condition on them, and a map sending each configuration to what the exterior reads of it; its count W is the number of balanced configurations the exterior can tell apart. The test surfaces are three families of closed cuts whose areas are known exactly in the lattice metric, in units of b^2: the *facet tori*, the cross-sections of Proposition 4.2 on the side-N torus, of area $\sqrt{3}\,N^2$; the *axis tori*, the cuts normal to a cube axis on the same torus, of area $2N^2$; and the *spheres*, the cuts of the past cone of a record at rank distance m, of area $2\sqrt{3}\,m^2$ (Proposition 4.3). They differ in orientation and in topology. A *reading* assigns a cut ledger to every cut of the three families, and it has *entropy density* η if $\log W = \eta A + o(A)$ along each family, with A the area in units of b^2.

Target 4.8 (The count)**.** Write down the ledger's own reading, whose configurations are the ledger data on the cut and whose balance conditions are the ledger's, and show that it has an entropy density $\eta_{\mathrm{lat}} > 0$: one density for the facet tori, the axis tori and the spheres.

Target 4.9 (The quarter)**.** Find a condition on the action scale as, in units of b^{-2}, that follows from the action or the ledger without the count and holds at exactly one value, and show that $\eta_{\mathrm{lat}} = 2\pi as$ at that value.

With $G = 1/(8\pi s)$ and $\hbar = 1/a$ the second target is the quarter law (Proposition 4.1). It says that the condition holds at, and only at, the value the count requires, $\eta_{\mathrm{lat}}/(2\pi)$. The phase premise is such a condition: with $b^2 = r\lambda_{\mathrm{rec}}^2$ it fixes $as = r\ln 2/(2\pi)$, and the target becomes $\eta_{\mathrm{lat}} = r\ln 2$, one bit per λ_{rec}^2. Both targets are stated so that they can fail.

Proposition 4.10 (The targets can hold and can fail)**.**

(i) A reading has at most one entropy density.

(ii) The reading that gives a cut of area A exactly $2^{\lfloor A \rfloor}$ configurations, with no balance condition and all of them distinguishable, has entropy density $\ln 2$.

(iii) None of the following readings has an entropy density $\eta > 0$: one configuration on every cut; ω independent states per record, for any whole number ω, whose densities are $\ln\omega/\sqrt{3}$ *on the facet tori and* $\ln\omega/2$ *on the axis tori; one bit per crossing, with one crossing per record on a facet torus and two inward crossings per record on an axis torus, whose densities are* $\ln 2/\sqrt{3}$ *and* $\ln 2$*; and any reading with* $\log W \le CN$ *on the facet torus of side N, for a constant C.*

(iv) The second target holds for the phase premise at $b = \lambda_{\mathrm{rec}}$, the condition $as = \ln 2/(2\pi)$, with $\eta_{\mathrm{lat}} = \ln 2$. It fails for a condition that every value of as meets; for the phase premise at $b^2 = \frac{4}{3}\lambda_{\mathrm{rec}}^2$, the spacing of $\kappa = 1$ with one record length per rank, with $\eta_{\mathrm{lat}} = \ln 2$; and for $\eta_{\mathrm{lat}} = 0$ under the phase premise at any spacing.

Part (ii) shows that the conditions of Target 4.8 can hold together. Its reading assigns bits to area instead of counting the ledger's data, so it is no candidate: the target asks for the ledger's own reading. Part (iii) excludes every count computed so far. The field reading's entropy on a facet torus is at most a perimeter's worth (Proposition 4.5), so it falls under the last case (INFERENCE: the field reading is not itself one of the readings of the proposition), and independent records and one bit per crossing see the orientation. The first case of part (iv), like part (ii), shows only that the conditions can hold together: its condition is chosen with the count in view, while the target asks for one that the ledger supplies without it. Part (iv) also shows that Target 4.9 fails when as is left free, as it is by [7] alone, and that at the identified spacing it fails for the one count that gave the quarter on an axis horizon at $b = \lambda_{\mathrm{rec}}$ (Proposition 4.7).

Relation to other counts. The entanglement entropy of a quantum field across a surface obeys an area law with a coefficient set by the cutoff [9, 10]; on the lattice the cutoff is the spacing, so such a count gives a pure number per b^2 that would have to come out right. In loop quantum gravity the area law's coefficient depends on the Barbero–Immirzi parameter, which is chosen so that the quarter holds [12, 13]; once as is fixed the ledger has no parameter to choose, and Propositions 4.6 and 4.7 show how rigid that makes the count. The count here is of the gravitational ledger itself, the records and postings of the lattice that carries the geometry, on a surface whose area is computed in that lattice's metric.

5 Hawking radiation from lattice fields

Status: the radiation

Known in the continuum [2, 28]. An exponential map of outgoing rays with rate g_{H}, and positive frequency defined by analyticity across the horizon, give the occupation $1/(e^{2\pi\omega/g_{\mathrm{H}}}-1)$: a thermal spectrum at $T=\hbar g_{\mathrm{H}}/(2\pi)$.

Proved. At $\kappa=1$ every axis edge of the lattice is null and every other edge timelike; for $\kappa>1/3$ the symbol of the lattice scalar field is positive on the light cone at short wavelengths in every direction, the sign of a superluminal medium (Proposition 5.1). On stationary Hermitian banded backgrounds the lattice field conserves a flux, and where the background is uniform outside a finite region scattering is a Bogoliubov map, $|\alpha|^2-|\beta|^2=1$ in lattice form (Proposition 5.2). On the river wedge the stationary modes obey such a band equation exactly, the band plus $i\varepsilon$ is invertible on every finite window, and on a uniform band the sign of ε selects Feynman's rule or its reverse (Proposition 5.3). Near a horizon the action of [7] is blind to a family of spacelike triangles (Proposition 5.9). The Lorentzian term on spacelike triangles leaves its equations unchanged where every triangle is timelike, keeps a nonzero term it drops, and satisfies Schläfli's identity on every regular Lorentzian simplex (Proposition 5.10). With that term each Regge equation at a record whose simplices are regular Lorentzian is a sum of area rates times deficits, and holds where the simplices close up around each triangle (Proposition 5.11); the simplices of the flat lattices of [7] close up around every triangle (Proposition 5.13). A static detector that reads the field once per step cannot tell excitation from de-excitation at the energy $\pi/(2\sqrt{3})$, so no state is empty at every energy; on the band $0<\omega<\omega_c\approx 0.385$, below that energy, an empty state exists, and after a collapse, given prepared states, thermal emission and emission that stops exclude each other (Proposition 5.8).

Proof complete, independent verification pending. Along the real waves of the flat lattice the derivative of the symbol in the frequency changes sign once, at the end of the long-wave branch (Proposition 5.8(ii)). On a collapse history the regulated two-point function exists on every finite window, with the two signs of the regulator related by conjugation, and its pair fraction lies between 0 and 1; Target 5.6 cannot hold vacuously.

Numerical, preliminary. On test backgrounds that are not solutions of the lattice equations, with the occupation read as an in-out coefficient (see Scope): on a test horizon a Planck spectrum at $0.986\,T_{\mathrm{H}}$ for $g_{\mathrm{H}}b=0.1$, with a correction of second order in $g_{\mathrm{H}}b$ that cools it; on the river wedge's geometry, exact near the horizon, with a test interior below the critical flow, a Planck spectrum that tends to T_{H} as the spacing shrinks; on the eternal river wedge the quanta are born on a critical surface inside the horizon, so the spectrum needs data from the horizon's formation. These support part (a) of Target 5.4 only. On the regulated path sum the pair fraction is thermal on a core below the critical flow and maximal on cores beyond it. Flat lattices with an inflow above 0.42 of the speed of light, and the river wedge, fail the Regge equations of the action of [7]; with the Lorentzian term on spacelike triangles added, every equation holds on them. Programs and logs: [29].

Scope. The lattice scalar has no stable evolution in record time, so no computed occupation is a number of particles in a record-time vacuum; reading it as the in-out coefficient $|\beta|^2$ of the regulated path sum, and that as a radiated flux, are inferences.

Open. Target 5.4: the thermal occupation in the long-wave limit as a theorem, on test backgrounds and on a lattice solution. Target 5.5: no outward waves beyond the critical flow. Target 5.6: after a collapse onto a fast interior, a late pair fraction of the regulated path sum that is thermal or zero; the numerics point to neither, a pair fraction near one, which would refute it. Target 5.7: the

same question in record time, where growth is a third outcome and the one expected (inference). Target 5.12: flat simplices close up beyond the flat lattices of [7], which makes every flat lattice a solution of the extended action. The graviton's radiation, which needs an action on spacelike triangles derived from the ledger: the action of [7] is blind to a family of triangles once the inflow exceeds 0.42 of the speed of light (Proposition 5.9).

Identifications and model. $\kappa = 1$, under which postings are null; the collapse histories are placed by hand (model), and the cores are test gluings.

The mechanism in the continuum. Near a horizon of surface gravity g_{H}, an outgoing ray that leaves the horizon region at the affine parameter U reaches a distant observer at the retarded time u with $U = -e^{-g_{\mathrm{H}}u}/g_{\mathrm{H}}$. A wave $e^{-i\omega u}$ at infinity, traced back, is $(-g_{\mathrm{H}}U)^{i\omega/g_{\mathrm{H}}}$ near the horizon. The state that is regular across the horizon is the one whose modes are of positive frequency in U, analytic in the lower half U-plane, and the wave $(-g_{\mathrm{H}}U)^{i\omega/g_{\mathrm{H}}}$ is not: its content at negative frequency is $e^{-\pi\omega/g_{\mathrm{H}}}$ times its content at positive frequency. So the Bogoliubov coefficients obey $|\beta_\omega/\alpha_\omega|^2 = e^{-2\pi\omega/g_{\mathrm{H}}}$, and with $|\alpha_\omega|^2 - |\beta_\omega|^2 = 1$ the mean number of quanta in the mode is

$$|\beta_\omega|^2 = \frac{1}{e^{2\pi\omega/g_{\mathrm{H}}} - 1},$$

a thermal spectrum at $T = \hbar g_{\mathrm{H}}/(2\pi)$ [2, 28]. Two inputs carry the result: the exponential map with rate g_{H}, and positive frequency defined by analyticity. The same two inputs give the Unruh effect for a uniformly accelerated observer [28, 30].

What a lattice changes. The exponential map means that a quantum reaching infinity at late time u left the horizon region with a frequency $e^{g_{\mathrm{H}}u}$ times larger. After a time of order $g_{\mathrm{H}}^{-1}\ln(1/(g_{\mathrm{H}}\ell_P))$ that frequency is beyond the Planck scale, and the continuum derivation relies on modes no one has seen [16]. A lattice with spacing b has no such modes, so the late quanta must come from somewhere else. In the analogue models of fluids and modified dispersion relations they come from short-wavelength modes whose dispersion departs from the light cone, and the thermal spectrum survives, up to corrections that vanish as the ratio of g_{H} to the dispersive scale goes to zero [17, 18, 19, 31]. Which case applies depends on the sign of the departure. The lattice settles it.

Proposition 5.1 (The sign of the lattice dispersion)**.** *At $\kappa = 1$ every axis edge of the lattice is null and every other edge is timelike. For every $\kappa > 1/3$ and in every direction, on the light cone of the continuum the quartic part of the symbol of the lattice scalar field of [7] is positive, with an explicit bound at finite wave number.*

A positive correction on the light cone means that short waves outrun light: read as dispersion (the dispersion surface itself is not constructed), the lattice is a superluminal medium, the case in which the late Hawking quanta come from short-wavelength modes inside the horizon that travel outward faster than light [19]. The scattering itself is exactly balanced on the lattice.

Proposition 5.2 (Scattering is a Bogoliubov map)**.** *For the lattice field on any stationary background whose operator is Hermitian and banded, a discrete flux is conserved between slices.*

Where the background is uniform outside a finite region, with distinct propagating roots there, evanescent modes carry none and the amplitudes of the propagating modes on the two sides balance: the lattice form of $|\alpha|^2 - |\beta|^2 = 1$, whatever the profile in between. The field equation of the lattice scalar of [7], taken cell by cell, on any configuration that depends on the one coordinate $x_0 + x_1 - x_2 - x_3$ reduces exactly to such a Hermitian band equation for its stationary modes.

A band equation fixes which waves exist, not which way they run. In the path sum that choice comes from the regulator.

Proposition 5.3 (The regulator selects Feynman's rule)**.** *The river wedge of Proposition 3.6 is invariant under every record shift along its horizon, so the stationary modes of the lattice scalar on it obey a Hermitian band equation exactly; the same holds when it is glued to the flat lattice beyond a surface outside its horizon. For every Hermitian band, every real frequency and every $\varepsilon \neq 0$, the band plus $i\varepsilon$ is invertible on every finite window with fixed ends: the regulated stationary path sum exists there, with the infinite window and the limit $\varepsilon \to 0$ not taken. For a uniform Hermitian band with symbol P, a propagating wave, a root z of P on the unit circle with $\operatorname{Im}\big(zP'(z)\big) \neq 0$, continues to a root of $P + i\varepsilon$ that leaves the circle at the rate $\mathrm{d}|z|^2/\mathrm{d}\varepsilon = -2/\operatorname{Im}\big(zP'(z)\big)$. As $\varepsilon \to 0^+$ waves of positive norm leave a source and waves of negative norm run toward it, Feynman's rule; as $\varepsilon \to 0^-$ both are reversed.*

What the spectrum counts. The lattice scalar of [7] has no stable evolution in record time: record time has growing modes at every $\kappa > 1/2$ (Section 6). So there is no record-time vacuum whose particles could be counted. The regulator $+i\varepsilon$ selects Feynman's rule (Proposition 5.3), and the occupation $|\beta|^2$ computed below is read as the in-out coefficient of the regulated path sum, the standard construction for scattering in a background field (INFERENCE). The radiated flux follows from the persistence of the regulated vacuum, whose probability falls by the factor $\exp\big(-\int \frac{\mathrm{d}\omega}{2\pi} \ln(1 + |\beta_\omega|^2)\big)$ per unit time, as for pair creation by an external field (INFERENCE).

A computed spectrum. With these facts an in-out occupation can be computed on a background with a horizon. Take the lattice scalar at $\kappa = 1$, in its sector of zero transverse momentum, on a Painlevé–Gullstrand flow sampled on the lattice edges: a test background with a horizon, not a solution of the lattice equations. With the incoming modes set by Feynman's rule, including the short-wavelength modes inside, the outgoing occupation is a Planck spectrum. At $g_{\mathrm{H}} b = 0.1$ the effective temperature is $0.986\,T_{\mathrm{H}}$ for $\omega/T_{\mathrm{H}} \leq 2$, and between $0.981\,T_{\mathrm{H}}$ and $0.986\,T_{\mathrm{H}}$ over $0.25 \leq \omega/T_{\mathrm{H}} \leq 8$, and the flux balance of Proposition 5.2 holds to 10^{-13}. The deviation follows

$$1 - \frac{T_{\mathrm{eff}}}{T_{\mathrm{H}}} = c\,(g_{\mathrm{H}} b)^2, \qquad c > 0,$$

with c between 0.44 and 3.09 for horizon regions of width $0.25 \leq D \leq 0.5$, depending on the profile of the flow, and growing as $1/D^2$ for narrower ones: the lattice spectrum is slightly cooler than the continuum one. Separating the width D of the horizon region, $c = c_\infty + c_p/D^2$, and the limit depends on the direction of the flow in the lattice: $c_\infty = 1.18$ along a pair of axes and 0.59 along one axis. The lattice correction is anisotropic. The quanta of the

computed occupation trace back to two lattice-scale modes inside the horizon, with incoming amplitudes set by Feynman's rule; no trans-Planckian frequency enters (NUMERICAL). These are preliminary numerical results on backgrounds that are not lattice solutions. Every number in this section, with the programs, operator tables and logs that produce it, is in the computations package of this version [29], whose README maps each statement to its program and log.

Target 5.4 (Thermal at long wavelengths)**.** (a) On the test backgrounds, the Painlevé–Gullstrand flows of every admissible profile and the river wedge with a smooth core below the critical flow: for every $x = \omega/T_{\mathrm{H}} > 0$ and every tolerance, at small enough $g_{\mathrm{H}}b$ the scattering problem of Proposition 5.2 has a solution, and every solution has $|\beta_\omega|^2$ within that tolerance of $1/(e^x - 1)$. (b) The same for a stationary lattice horizon with surface gravity g_{H} that solves the lattice equations, in the long-wave limit $b \to 0$ at fixed g_{H} and ω, taken after the torus size grows at fixed regulator and then the regulator is removed, with the correction of order $(g_{\mathrm{H}}b)^2$ given explicitly.

Target 5.4 is not trivially satisfiable: it fails for $\beta = 0$, for coefficients that do not depend on g_{H}, and on the flat background, and part (a) asks for a solution to exist. The computations above are preliminary numerical evidence for part (a) only, on test backgrounds; part (b) needs the lattice black hole of Target 3.11. The computation also found its own limit. Measure the flow in units of the speed of light, so that the horizon is where the inflow reaches 1. If the flow inside the horizon along a pair of axes exceeds a critical speed, $\sqrt{3} \approx 1.73$, the interior modes that feed the radiation stop propagating toward the horizon, and the stationary problem no longer fixes the outgoing modes from the incoming ones; along one axis the critical speed is about 1.18, beyond which the interior short modes move away from the horizon (NUMERICAL). The interior of a Schwarzschild black hole exceeds any such speed somewhere, so the radiation of an actual black hole needs a treatment in which the horizon forms (INFERENCE). The river wedge of Proposition 3.6 shows the structure exactly: its flow, along a pair of axes, grows without bound inside, and the critical surface lies inside the horizon, at $\xi = -3/(2\alpha)$, where the inflow reaches $\sqrt{3}$, against $-1/(2\alpha)$ for the horizon (NUMERICAL). Between the two, the short-wavelength modes of both norms run outward, toward the horizon; beyond the critical surface they are evanescent, and only the two long modes remain, both dragged inward (Fig. 2). The short pair is born on the critical surface, where its two roots meet at the edge of the zone. On the eternal river wedge, then, one incoming mode faces three outgoing ones, the asymptotic data do not fix the modes that become Hawking quanta, and the spectrum needs data from the formation of the horizon; the trail of ever shorter wavelengths that the continuum follows back to the horizon ends at the critical surface (NUMERICAL). With the interior replaced by a smooth core that stays below the critical flow, a test gluing, the computed occupation is a Planck spectrum: at $\omega/T_{\mathrm{H}} = 1$, $T_{\mathrm{eff}}/T_{\mathrm{H}} = 0.970$, 1.001, 0.9985 and 0.99994 for $\alpha b = 0.1$, 0.05, 0.025 and 0.0125, the same to 10^{-3} over $0.5 \le \omega/T_{\mathrm{H}} \le 4$ for $\alpha b \le 0.05$. The residual oscillates with $1/(\alpha b)$ through interference with the core, a property of the test core, so the lattice coefficient c_∞ found above stands (NUMERICAL). The region between the horizon and the critical surface is the configuration in which analogue black holes with superluminal dispersion amplify their radiation and lase [32]; whether the lattice black hole does so, through modes of complex frequency, bears on Targets 5.6 and 5.7, since emission that persists without being thermal refutes their thermal and vanishing outcomes alike. That nothing moves outward beyond the critical flow has a precise form.

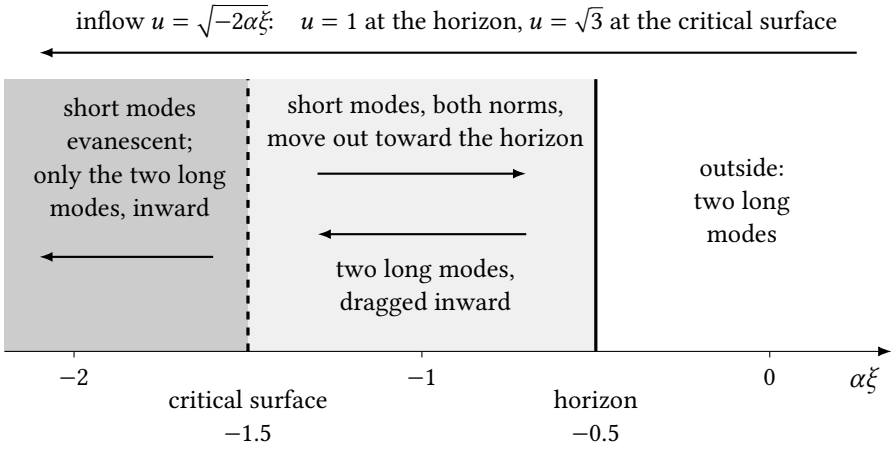


Figure 2. The river wedge along its flow, for the lattice scalar at $\kappa = 1$ (numerical). Between the horizon and the critical surface (light shading) the short-wavelength modes of both norms run outward, toward the horizon, while the long modes are dragged in; beyond the critical surface (dark shading) the short modes are evanescent. The short pair is born on the critical surface, so the modes that become Hawking quanta are not fixed by data at infinity.

Target 5.5 (No outward waves beyond the critical flow)**.** In a uniform flow along a pair of axes with inflow between $7/4$ and $5/2$, for every frequency $0 < W \leq 1/10$ per rank, the lattice scalar has a propagating wave, and every propagating wave moves inward.

It holds at all 640 points of a grid over that range, and it is not trivially satisfiable: it demands a propagating wave, and at an inflow of 3 an outward wave of frequency $W = 0.134$ exists, outside the range (NUMERICAL). Carrying it from a uniform flow to the river wedge's interior is a short-wavelength approximation (INFERENCE).

What a lattice would predict. In the computations the lattice correction to the spectrum is of second order, $(g_{\mathrm{H}}b)^2$, and it cools the spectrum (NUMERICAL). For a black hole of one solar mass $g_{\mathrm{H}}b = b/(2r_s) \approx 5 \times 10^{-39}$ (with $c = 1$), far below any measurement; the correction reaches order one only for holes near the Planck mass, at the end of evaporation, where it becomes part of the information question of Section 6. Analogue black holes have superluminal dispersion of the same kind, among them the Bose–Einstein condensate in which spontaneous Hawking radiation has been observed [33]. One lattice effect is specific to $\kappa = 1$: the corner doublers. Their cone shares no timelike or null direction with the graviton's, and record time is spacelike for them [7, Thm. 6.15]; they do not propagate as waves in the graviton's time, so a stationary analysis does not treat them as emission channels (INFERENCE), and the measure of [8] damps them.

Radiation after collapse. The eternal river wedge leaves the late quanta undetermined, so the question has to be posed on a horizon that forms. Take the simplest history: a configuration that depends on a record only through its rank and its coordinate along the flow, equals the flat lattice of [7] at $\kappa = 1$ before a rank T_0, and from a rank $T_1 \geq T_0$ on equals the river wedge of surface gravity α, glued to the flat lattice outside its horizon, with any configuration in between. It is a background placed by hand, not a solution of the lattice equations (MODEL).

The sudden collapse, $T_0 = T_1$, is such a history, and the flat lattice is not, since deep inside the river's horizon the diagonal edge is not timelike (Proposition 3.6). The lattice scalar has no vacuum of record time to start from (Section 6), but the path sum supplies a stable object: its regulated two-point function.

On such a history the scalar's sector equation has real, symmetric coefficients in space and record time, so on every finite window the operator B of the action is Hermitian, $B + i\varepsilon$ is invertible for $\varepsilon \neq 0$, and $G_{n,\varepsilon} = i(B + i\varepsilon)^{-1}$ is symmetric, with $G_{n,-\varepsilon} = -\overline{G_{n,\varepsilon}}$. The regulated two-point function G is the limit in which the window grows at fixed $\varepsilon > 0$ and then $\varepsilon \to 0^+$; the other sign of the regulator then gives $-\overline{G}$ (proof complete, independent verification pending). G is an in-out function. At late times its out end is the out vacuum, so a static detector reading G registers no emission: a question put through such a detector would be decided by the in-out structure rather than by the physics (INFERENCE). The Hawking content of an in-out function is the amplitude for a pair, an outgoing wave outside and its partner inside. For a two-mode Bogoliubov map its square is $|\beta|^2/|\alpha|^2 = n/(1+n)$, with the thermal value e^{-x} at $x = \omega/T_{\mathrm{H}}$. Its stationary form is the *pair fraction*. At a fixed frequency, take a solution on the river with a uniform exterior and a uniform core whose content in the core is the one the reversed regulator allows (Proposition 5.3); the pair fraction is the flux of its partner wave per unit flux of its Hawking wave. By the flux balance of Proposition 5.2 it lies between 0 and 1, and when the partner is the only propagating wave the core allows, it is one less the fraction of the exterior wave sent toward the horizon that comes back (proof complete, independent verification pending). That this solution is a column of G is the standard construction of a Green function, not written out here (INFERENCE). The river's own interior has no uniform region in which a partner can be read, so the target takes a test core.

Target 5.6 (Radiation after collapse, on the regulated path sum)**.** Let the river wedge of surface gravity α have, from some depth on, a uniform core flowing along the same pair of axes at a speed between $7/4$ and $5/2$, beyond the critical flow, and take the sudden collapse onto it from the flat lattice at $\kappa = 1$. (a) For every such core and all small enough α, the regulated two-point functions of the collapse and of the eternal cored river exist, and along record time the first approaches the second. (b) On every such core, as $\alpha \to 0$ at fixed $x = \omega/T_{\mathrm{H}}$, the pair fraction at the frequency ω either tends to e^{-x}, thermal emission, or tends to 0, emission that stops.

The target cannot hold vacuously: each outcome includes the existence of the pair data, the zero solution is excluded, and the two outcomes exclude each other (proof complete, independent verification pending). Computed, the answer is neither (NUMERICAL, [29]). With a core below the critical flow the pair fraction is e^{-x} to within 3×10^{-4} at $x = 0.5$, 1 and 2: thermal, where the standard picture holds. With cores beyond it, at speeds 2 and 3, it is 1 to within 10^{-5}. The Hawking wave there is fed by the core's evanescent wave, which carries no flux, so the partner's flux equals the Hawking flux. A pair fraction that tends to 1, with an in-out occupation that diverges, refutes the target. So on a fast interior the regulated path sum is expected to be decided by the scalar's unstable sector, here its stationary face, the evanescent feed, and not by the surface gravity (INFERENCE, from the numerics).

The same question in record time. A state of the scalar in its sector of zero transverse momentum is a two-point function $\mathscr{W}$ that is Hermitian and positive and solves the field

equation in each argument; there is no vacuum of record time to start from (Section 6). A detector at rest in the flat region outside the horizon, at the position d, reads the field once per diagonal step s, four ranks or $2\sqrt{3}$ units of its proper time, and over L steps from the T-th it responds at the energy ω with

$$R_{T,L}(\omega) = \frac{1}{L}\,\mathrm{Re}\sum_{k,k'=0}^{L-1} e^{-2\sqrt{3}\,i\omega(k-k')}\,\mathscr{W}\big(d+(T+k)s,\; d+(T+k')s\big).$$

It is non-degenerate at ω if its de-excitation response $R_{T,L}(-\omega)$ stays above a positive bound at all late T and long enough L, and its late ratio is $R_{T,L}(\omega)/R_{T,L}(-\omega)$ at late T and long L.

Read once per step, the detector cannot tell excitation from de-excitation at $\omega = \pi/(2\sqrt{3})$, so the empty state has to be defined on a band (Proposition 5.8 below), and the flat lattice supplies it. Its sector waves $e^{i(Wt+KX)}$, with t the rank, W the frequency per rank and X the coordinate along the flow, are real for $|W| \le 2/5$ along the curve $\cos K = (5+\cos^2 2W)/(6\cos W)$. Along that curve the derivative of the field's symbol in the frequency changes sign at $W_c \approx 0.333$, where $\cos 2W_c = (\sqrt{19}-2)/3$: the long-wave branch meets the optical branch there. In the detector's time the long-wave branch covers $0 < \omega < \omega_c = 2W_c/\sqrt{3} \approx 0.385$. Above ω_c the flat lattice's state of positive norm excites a static detector through its optical waves, so the empty state is defined below ω_c (INFERENCE). A state of the flat lattice is empty on the band if at every detector and every energy $0 < \omega < \omega_c$ it is non-degenerate with a late ratio tending to 0. A state on a collapse history is prepared if before the collapse it agrees with an empty state of the flat lattice. The condition reads only late responses, so an empty state may also carry modes that decay in record time, of any amplitude, and a prepared state need not be near the vacuum long before the collapse.

If the outgoing waves carry the Planck occupation $n = 1/(e^x - 1)$, with $x = \omega/T_{\mathrm{H}}$, and the incoming waves are empty, the late ratio is $n/(2+n) = 1/(2e^x - 1)$, since the flat lattice's spectral weight in the sector is even in the momentum (INFERENCE). The Hawking temperature is $T_{\mathrm{H}} = \alpha/(2\pi)$ in the exterior's proper time, the normalization under which the computation above finds the Planck spectrum (NUMERICAL).

Target 5.7 (Radiation after collapse, in record time)**.** (a) For every small enough surface gravity α and every collapse time, the sudden collapse has a prepared state that is non-degenerate at every detector in the flat region outside the horizon and every energy $0 < \omega < \omega_c$. (b) On the fixed lattice one of three outcomes holds. Either the emission is thermal: for every $x > 0$ and every tolerance, on every collapse history with small enough α, every prepared state that is non-degenerate at such a detector at $\omega = x\alpha/(2\pi)$ has there a late ratio within the tolerance of $1/(2e^x - 1)$. Or the emission stops: on every collapse history, every prepared state has a late ratio tending to 0 at every such detector and every energy $0 < \omega < \omega_c$ at which it is non-degenerate. Or the responses grow: for every small enough α and every collapse time, every prepared state of the sudden collapse has, at some such detector and some energy $0 < \omega < \omega_c$, a de-excitation response that exceeds every bound at late times.

Proposition 5.8 (The detector's band)**.** *(i) At $\omega = \pi/(2\sqrt{3})$ the excitation and de-excitation responses coincide, for every state on every background, so no state of the flat lattice is non-degenerate with a late ratio tending to 0 at every $\omega > 0$. (ii) Along the real sector waves of the flat lattice the derivative of the symbol in the frequency is negative for $0 < W < W_c$, zero at W_c and positive for $W_c < W \le 2/5$, and $\omega_c < \pi/(2\sqrt{3})$. (iii) The state $\mathscr{W}(a,b) = \int_{-W_c}^{0} f_W(a)\,\overline{f_W(b)}\,\mathrm{d}W$*

built from the long-branch waves f_W of frequency between $-W_c$ and 0 is empty on the band: at every detector and every $0 < \omega < \omega_c$ its excitation response is at most a constant over L, and its de-excitation response is at least $\pi/24$ once L is long enough. The zero state is not empty on the band. (iv) If part (a) of Target 5.7 holds, its thermal and stopping outcomes exclude each other.

So Target 5.7 cannot hold vacuously: part (a) demands prepared states, and the empty state they start from exists. Growth is not an empty outcome either: the flat lattice's sector equation has solutions, uniform in space, that grow in record time, the pure state of such a solution grows at every detector and every energy, and the zero state does not (proof complete, independent verification pending). It is the outcome to expect. Part (a) is the scalar's evolution across the collapse, a Cauchy problem in record time, and on the flat lattice that evolution has solutions that grow by a factor of about 2.65 per rank, some fifty per step of the detector (NUMERICAL; Section 6). Unless the collapse leaves them unexcited, the late responses grow (INFERENCE), and the question is decided by the instability of record time, the subject of Target 6.1, not by Hawking emission. The three outcomes are not exclusive as stated, since a growing state can have any late ratio; the first two still exclude each other (Proposition 5.8(iv)). In both forms of the question, then, the computed evidence points away from thermal emission and from emission that stops alike: on the regulated path sum to a maximal pair fraction, in record time to growth. Thermal emission at the Hawking temperature after a collapse needs a stable evolution, or an interior that stays below the critical flow.

The graviton near a horizon. The graviton's own radiation runs into the action itself. Take the flat lattice with a constant inflow $u \geq 0$ along a pair of axes, measured as above; at $u = 0$ it is the lattice of [7] at $\kappa = 1$.

Proposition 5.9 (The action goes blind near a horizon)**.** *In the flat lattice with constant inflow u, the triangle spanned by two postings and the edge of two postings that closes them has Gram determinant $2(3u^2 + 2\sqrt{3}\,u - 2)$: timelike at $u = 0$, spacelike exactly when the inflow exceeds $(3 - \sqrt{3})/3 \approx 0.42$, and spacelike at a horizon, where the determinant is $2 + 4\sqrt{3}$. On a spacelike triangle of a Lorentzian simplex the action of [7] assigns no area, and its dihedral angle there is 0, $\pi/2$ or π; for every inflow faster than $1/2$ its term on these triangles vanishes identically.*

So well before a horizon, at less than half the speed of light, the action stops seeing a family of triangles, and that breaks the Schläfli identity of each simplex. Computed in high precision, the flat lattices of constant inflow satisfy all fifteen Regge equations at a record while every triangle is timelike, and fail seven of them beyond the onset, by 0.013 at an inflow of 0.43, 0.14 at $1/2$ and 0.78 at 0.8 in lattice units; the river wedge fails them too, through its horizon (NUMERICAL). None of the flat backgrounds with a horizon computed is a solution of that action, and none is expected to be (INFERENCE). Give each spacelike triangle the Lorentzian term of [34] instead: its area times the boost between the two tetrahedra of a simplex that meet there, read from the same squared lengths as the angle at a timelike triangle, with the boost entering with the sign for which Schläfli's identity below holds and flat configurations are solutions (the opposite sign doubles their residual, NUMERICAL).

Proposition 5.10 (The Lorentzian term on spacelike triangles)**.** *With that term the action is unchanged on timelike and null triangles, and at a record whose surrounding triangles are all*

timelike its Regge equations are those of the action of [7]. On a spacelike triangle of a Lorentzian simplex whose two tetrahedra through the triangle are of one causal type the boost is not zero, so the action of [7] drops a nonzero term there; a simplex of the flat lattice of constant inflow $1/\sqrt{3}$ is regular Lorentzian, with no null triangle or tetrahedron, and has such a triangle. For every regular Lorentzian simplex and every straight line of squared edge lengths through it, Schläfli's identity holds with these boosts: the areas times the rates of change of the angles at the timelike triangles, less the areas times the rates of change of the boosts at the spacelike ones, sum to zero.

Schläfli's identity is what turns a field equation into a statement about deficits.

Proposition 5.11 (The equations are deficits)**.** *At a record whose surrounding simplices are all regular Lorentzian, each Regge equation of the extended action is a sum over the triangles through the record of the rate of change of the triangle's area times its deficit, 2π less the angles of the simplices around a timelike triangle, or minus the boosts around a spacelike one. Every equation there holds if those angles sum to 2π and those boosts to 0.*

Without the boosts the identity fails on these lattices, by 0.05 to 1.1 per simplex, and so do the equations of the action of [7]; with the extended action every equation holds to within 4×10^{-24} on every one of the backgrounds tested, and around every triangle the angles sum to 2π or the boosts to 0 (NUMERICAL). For a flat lattice the equations therefore come down to a statement about flat geometry: the simplices around a triangle fill the plane normal to it once.

Target 5.12 (Flat simplices close up)**.** Let a flat lattice have its records placed linearly in Minkowski space, with regular Lorentzian simplices around a record. Around every timelike triangle through the record the angles of the simplices sum to 2π, and around every spacelike one their boosts sum to 0.

By Proposition 5.11 the target makes every flat lattice of this kind a solution of the extended action at every such record. It is not empty: the lattices of constant inflow beyond 0.42 meet its hypothesis with spacelike triangles through every record, and with the action of [7] in place of the extended one their equations fail, by 0.139 in one equation at an inflow of $1/2$ (NUMERICAL). For the flat lattices of [7] themselves, where every triangle is timelike, the target holds.

Proposition 5.13 (The flat lattices close up)**.** *For every $\kappa > 1/2$ the simplices of the flat lattice of [7] close up around every triangle through every record of $\mathbb{Z}^4$: every triangle is timelike, and the angles of the simplices around it sum to 2π. So the flat lattice satisfies every Regge equation at every record, and at $\kappa = 1$ it meets every condition of Target 3.10 except strength: it is invariant under T and under the 24 permutations of the counters, tends to itself far away, satisfies every Regge equation, and keeps every simplex Lorentzian and every edge of two or more postings timelike.*

Every other uniform flat lattice, with timelike triangles only or with spacelike ones, remains within Target 5.12. The extended action is a candidate for the configurations a black hole needs (Target 3.11); it still has to be derived from the ledger. With such an action, and the river wedge as its vacuum, each long-wave polarization of the graviton obeys the scalar's wave equation on the river's flat geometry, so its spectrum follows the scalar's at leading order, with lattice corrections of its own (INFERENCE).

6 Where the information goes

Status: the information

Proved in [7, 8]. Ordering the path sum by record time does not remove the doublers; at the lattice scale record time has growing modes for the massless scalar at every $\kappa > 1/2$ and for the graviton at $\kappa = 1$ and $\kappa = 3/2$, so the Regge action fails the record-time test at both values: at $\kappa = 1$ no positive norm on its physical solutions is preserved by the record tick, nor for any action whose quadratic part is a multiple of Regge's; at $\kappa = 1$ the growth survives the restriction to solutions that keep every posting null; a real measure weight enters the record-time field equation as a damping. With the measure weight derived from the ledger's cost of curvature the path sum is not stable at the lattice scale, but at each fixed strength of the weight the record-time roots of its Gaussian equation are stable in the continuum limit, given the identifications of the weight and an inference about modes off the unit circle: a statement about roots, not a norm-preserving evolution.

Open. Target 6.1: a unitary or reflection-positive evolution of the path sum in record time. Target 6.2: under the ledger's unitary evolution in record time, the entropy of a lattice black hole's radiation rises from zero to a single maximum and returns to zero.

The question. Hawking's radiation is thermal, and its state depends on the hole's mass, charge and angular momentum alone. If the hole evaporates completely and the radiation is exactly thermal, a pure state has become a mixed one, and quantum mechanics has failed [3]. If instead the evolution is unitary, the entanglement entropy of the radiation must rise, turn over at about the time the hole has lost half its entropy, and return to zero: the Page curve [4]. In semiclassical gravity that curve has been obtained from quantum extremal surfaces [20, 21]. The question for the ledger is which of the two its dynamics produces, and if the second, which records carry the information out.

What the ledger says, and what it does not. The ledger's foundation contains the statement that a completed recognition is kept [35]. If that holds for the records of whatever falls into a black hole, the ledger erases nothing. That is a statement about records. The information question is a statement about the quantum state of the radiation, and the step from kept records to a pure final state needs the evolution of the path sum in record time to be unitary. That step is open, and what is known stands in its way at the lattice scale. Ordering the path sum by record time leaves the doublers in place; record time has modes that grow by a fixed factor per tick, $\sqrt{2}$ at $\kappa = 1$ and $(1 + \sqrt{10})/3$ at $\kappa = 3/2$ [7, Prop. 6.16]. At $\kappa = 1$ the growing mode solves the linearized Regge equation and is not a move of the vertices, so no positive norm on the physical solutions is preserved by the record tick, for the Regge action and for every action whose quadratic part is a multiple of it [8, Thm. 8.1]: they fail the test that a unitary evolution has to pass. The native path sum fixes the postings, and the growth survives that restriction: at $\kappa = 1$ the solutions that keep every posting null still contain a mode multiplied by $\sqrt{2}$ per tick that is not a move of the vertices [8, Prop. 8.2]. And a real measure weight enters the record-time field equation as a damping [8, Sec. 8.1]. For the weight derived from the ledger's cost of curvature the damping decides the continuum limit. Its path sum is not stable at the lattice scale, but no family of lattice-scale modes, the

doublers included, approaches the unit circle away from the zero wave, and the graviton's modes grow per tick by at most a constant times the cube of their spatial momentum; so the record-time roots are stable in the continuum limit, given the identifications of the weight and an inference about modes off the circle [8, Sec. 8.1]. That is weaker than a unitary evolution: no space of states with a norm that the tick preserves is constructed. The statement holds at each fixed strength of the weight, the ratio of its coefficient to the action's, with constants that depend on it; a limit in which the spacing itself shrinks changes the strength and would need the bound uniformly in it, which is not proved. A unitary evolution at the lattice scale still needs a repair of the quadratic action that keeps its long-wave limit. For a black hole the difference matters at the end: the radiation of a large hole is a long-wave process, but the last stage of evaporation, when the hole is a few lattice spacings across, is decided at the lattice scale (INFERENCE).

The question in kinematic form. The lattice already poses the question before any dynamics is solved. A trapped region of the interval order is always left by postings, and those postings are spacelike (Proposition 3.7). If postings carry the ledger's records, the records of whatever falls in are not trapped by the horizon: the posting order connects the inside to the outside even where the geometry does not. Then either information leaves along postings, or the posting order is not the physical causal order in a strong field, or no horizon forms (INFERENCE). Among the configurations the path sum integrates over, the third holds by construction (Proposition 3.8): nothing is trapped, and an observer that posts on every counter has every record in its past. Whether the ledger keeps that causal clause in a strong field is the same question in another form. The ledger has to say which order its quantum dynamics follows at a horizon.

Target 6.1 (Unitary record time)**.** An action and measure derived from the ledger whose path sum has a unitary or reflection-positive evolution in record time: for every spatial momentum, every physical record-time frequency real.

The Regge action with the derived weight does not meet Target 6.1, since its path sum is not stable at the lattice scale; it meets the weaker statement in which the growth per unit of physical time vanishes in the continuum limit, at fixed strength [8, Sec. 8.1].

Target 6.2 (Evaporation)**.** Let the ledger's record-time step with back-reaction, the evolution Target 6.1 asks for, act as a unitary evolution of a unit vector on the product of a carrier for the hole, the records of a cut of the horizon of the black hole of Target 3.11, and a carrier for the radiation, over record times 0 to N, by which the hole is gone. Show that the entanglement entropy of the radiation, the von Neumann entropy of its state traced over the hole, is 0 at record times 0 and N, never negative, and rises to a positive maximum at some record time strictly between them, then falls.

This is the shape of Page's curve, though not all of it: the target does not ask that the maximum come when the hole has lost about half its entropy. The hole's carrier and the step are identifications, named here and not derived; the target is for the history that Target 6.1 will supply. Its shape condition is not trivially met: an entropy that grows for ever, as in Hawking's original argument for information loss, fails it, and so does an entropy that stays zero, a radiation never entangled with the hole; a curve that rises and then falls meets it.

Table 3. The premises of each statement in the paper.

statement	premises
$G = 1/(8\pi s)$	every $\kappa > 1/2$ [7, Sec. 3.5]
$\hbar = 1/a$, $c = 1$	definitions
the quarter means $\eta = 2\pi as$	the two lines above
one bit per λ_{rec}^2; $\ell_P = \lambda_{\mathrm{rec}}/(2\sqrt{\ln 2})$	the phase premise (identification)
flat interval order equals posting order	$\kappa \geq 1$ [7, Thm. 2.1]
Propositions 3.2 to 3.7	as stated; the posting order as the physical causal order where it is used; the river wedge's deficits are computed (numerical)
Propositions 3.8 and 3.9	the admissible configurations of [7] (model); $\kappa \geq 1$ for the event horizon, $\kappa = 1$ (identification) for the squeeze; the redshift bound reads K/K_{flat} as the squared lapse (inference)
Proposition 4.1	$G = 1/(8\pi s)$, $\hbar = 1/a$; the phase premise where it is used
Proposition 4.2	$\kappa = 1$
Proposition 4.3	the posting order as the physical causal order (identification)
Proposition 4.4	as stated; the posting order as the observer's causal order (identification)
Proposition 4.5	the history or the field reading (model)
Proposition 4.6	the phase premise; $\kappa = 1$; one record length per rank (identifications); one bit per crossing, or independent records (model)
Proposition 4.7	the phase premise (identification); one bit per crossing (model)
Proposition 4.10	as stated; each reading is a model, and in part (iv) the phase premise is the condition tested

Two features of the lattice make the question concrete. First, the radiation after a collapse decides where the late quanta come from (Targets 5.6 and 5.7), and in both of its forms the computed evidence points to the scalar's instability rather than to the surface gravity (Section 5), so that question waits on Target 6.1 (INFERENCE). If a fixed lattice stops radiating and the emission needs new records near the horizon, then the quanta that carry the hole's mass away originate with those records, and the information question becomes a question about how the new records are correlated with the records inside. Second, the lattice has a smallest horizon (INFERENCE). With a spacing near the Planck length no horizon is smaller than a few lattice spacings, and the last stage of evaporation, where the corrections of Target 5.4 reach order one, is a finite lattice problem with a definite answer once the dynamics is known.

7 What remains

The paper has fixed what each of the three questions asks of the ledger. In order of consequence, the missing statements are these.

1. *The count* (Targets 4.8 and 4.9). It decides whether the quarter is derived. It needs no strong-field solution for the count itself: the posting order's horizons are exact, and what

Table 4. The premises of each statement in the paper (continued).

statement	premises
Propositions 5.1 and 5.2	$\kappa > 1/3$; a stationary Hermitian banded background
Proposition 5.3	the river wedge, an exact geometry that is not a solution of the action; a Hermitian band; $\varepsilon \neq 0$ before the limit
flux from the in-out coefficient	vacuum persistence of the regulated path sum (inference)
computed spectrum	a test background that is not a lattice solution; $\kappa = 1$; the occupation read as an in-out coefficient (inference); programs and logs in [29] (numerical)
the second-order obstruction	$\kappa = 1$ (identification); the action of [7]; the side-8 torus and the infinite lattice, local at one record, for a sampled cubic field and for a point mass's cubic Taylor fields [8, Thm. 4.3]; the point mass's far field numerical; its consequence for regular far fields an inference
Target 5.6	a collapse placed by hand (model); a uniform test core beyond the critical flow; the in-out reading of the regulated function and the Green-function link of the pair data (inference)
Target 5.7	a collapse placed by hand (model); the thermal ratio $1/(2e^x - 1)$ and the band's upper end (inference); $T_{\mathrm{H}} = \alpha/(2\pi)$ in the exterior's proper time (numerical)
Proposition 5.8	$\kappa = 1$; the sector of zero transverse momentum and a static detector read once per diagonal step
Proposition 5.9	the flat lattice with constant inflow; the action of [7]
Proposition 5.10	a regular Lorentzian simplex for the identity; the sign of the boost term, fixed by the identity and by flat solutions (numerical)
Proposition 5.11	regular Lorentzian simplices around the record
Proposition 5.13	$\kappa > 1/2$; $\kappa = 1$ (identification) for the comparison with Target 3.10
the strong field, the black hole, the count, thermality, collapse, the closing-up of flat simplices, unitarity	open (Targets 3.10, 3.11, 4.8, 4.9, 5.4 to 5.7, 5.12, 6.1 and 6.2)

they hide is known, the postings that cross them (Propositions 4.2 and 4.4). Their cross-sections contain no lattice faces and carry a density of records that depends on orientation. A field-like ledger hides one bit on an observer horizon and at most a perimeter law on any (Proposition 4.5); one bit per crossing gives a coefficient that depends on the orientation and on none of the three kinds of cut is the quarter at the identified spacing, and independent records cannot give it (Propositions 4.6 and 4.7). A count that gives the same density on every kind of cut has to correlate the records. The ledger's balance conditions are the candidate for that; whether they do it, with one density on the facet tori, the axis tori and the spheres, is Target 4.8, and whether that density is $2\pi as$ is Target 4.9. Both are open, and both are posed so that they can fail: every reading computed so far fails the first, and a free action scale fails the second (Proposition 4.10).

2. *A lattice black hole* (Targets 3.10 and 3.11). Without a strong-field solution there is no

lattice horizon to count on and no surface gravity. The posting order holds no black hole, a trapped region needs spacelike postings, weak fields trap nothing, and the expansion to second order has no horizon (Propositions 3.2 to 3.4); the exact lattice horizons known so far are Killing horizons of configurations built from flat simplices, the Rindler and river wedges (Propositions 3.5 and 3.6). Neither wedge, and no trapped region or Killing horizon at all, occurs among the configurations the path sum integrates over, whose causal clause keeps the time diagonal timelike (Proposition 3.8). A black hole needs that configuration space and the action extended, derived from the ledger. With the Lorentzian term on spacelike triangles (Proposition 5.10) the question is definite: static exteriors of cores of every size that solve the extended equations and have a Killing horizon outside the core (Target 3.11). Every condition of the strong-field target except the Regge equation can be met by an explicit exterior, which misses that equation by the order of the local curvature (numerical); a lattice exterior needs corrections in the edge combinations that are not metric. For the action as it stands a weak vacuum field does not extend past second order, one order beyond Einstein's equations (proved for a sampled field and for a point mass's cubic Taylor fields), so a black hole with a regular far field needs an action exactly symmetric under vertex moves, matter that carries the defect, or static modes at the lattice scale in the far field (inference); the first is also what a massless graviton needs [8]. If the ledger keeps the clause, collapse ends in shrinking cells instead (Proposition 3.9), and the count and the radiation have to be posed on that object.

3. *The radiation* (Targets 5.4 to 5.7 and 5.12). The lattice dispersion has the sign of a superluminal medium, its scattering conserves the lattice flux, and the regulated path sum selects Feynman's rule (Propositions 5.1 to 5.3). On a test horizon the computed occupation is a Planck spectrum with a cooling correction of order $(g_{\mathrm{H}}b)^2$, and on the exact river geometry with a test interior it tends to the Hawking temperature as the spacing shrinks: preliminary numerical evidence, on backgrounds that are not lattice solutions, for part (a) of Target 5.4 only. Since the lattice scalar has no stable evolution in record time, what is computed is read as the in-out coefficient of the regulated path sum, an inference. What remains is the theorem, a lattice solution as background, and a horizon formed by collapse, whose formation data the eternal river shows the spectrum needs. After a collapse placed by hand the question is posed twice: on the regulated path sum, through the pair fraction (Target 5.6), and in record time, with an empty state on a band below the end of the flat lattice's long-wave branch (Target 5.7 and Proposition 5.8). On a fast interior the first is expected to give a maximal pair fraction (numerical) and the second growth (inference), so neither form points to thermal emission or to emission that stops: both wait on the scalar's instability. The graviton's radiation needs an action on spacelike triangles first: the action of [7] goes blind to a family of triangles once the inflow exceeds 0.42 of the speed of light (Proposition 5.9), and the Lorentzian term on those triangles keeps its equations where every triangle is timelike and satisfies Schläfli's identity (Proposition 5.10); every equation of the extended action holds on the flat backgrounds tested (numerical), and on every flat lattice once its simplices close up around each triangle (Proposition 5.11), which they do for the flat lattices of [7] (Proposition 5.13) and which in general is Target 5.12. The thermal factor gives the thermodynamic entropy $2\pi as$ per unit area through the first law, the second route to the quarter.

4. *Unitarity and evaporation* (Targets 6.1 and 6.2). Shared with the quantum theory of [8]:

without a unitary evolution in record time, kept records do not by themselves give a pure final state, and the Regge path sum as it stands fails the test: at $\kappa = 1$ the record tick preserves no positive norm on its physical solutions (Section 6). With the derived measure weight the record-time roots are stable in the continuum limit, at fixed strength of the weight, which is weaker than a unitary evolution, but not at the lattice scale, and the last stage of evaporation is a lattice-scale problem (inference). The leak of Proposition 3.7 makes the first decision precise: which order the dynamics follows at a horizon.

The ledger's case for black holes rests on one structural fact: the lattice fixes Newton's constant by its action and the quantum of action by its phase, so the entropy per unit area of a horizon is a number the theory already contains, $2\pi as$. Counting what a horizon records is then a test, not a fit. The simplest counts do not pass it at the identified spacing, and that says what the full count must do; the radiation half of the test has begun.

A The lattice, the imported results and the proofs

This appendix makes the paper self-contained. It collects the definitions the text uses (Appendix A.1), lists each result taken from [7], [8] and [22] with its source and hypotheses (Appendix A.2), proves Propositions 3.4 and 4.5 (Appendix A.3), and proves the propositions of Section 5 that are not imported (Appendix A.4). The other propositions of Sections 3 and 4 are elementary statements about the two orders, the counts and the flat lattice, and their proofs are short arguments given or indicated where they are stated. The paper's statements about the dynamics beyond these proofs, the long-wave limits, the measure weight, the record-time norm and unitarity in the continuum limit, rest on the results of [7, 8, 22] listed in Appendix A.2.

A.1 Definitions

Records, postings and simplices. The vertices are the points x of $\mathbb{Z}^4$, or of the side-N torus, and are called *records*; the coordinate x_a counts the postings on counter $a \in \{0, 1, 2, 3\}$, and the *record rank* is $t_{\mathrm{r}}(x) = \sum_a x_a$. An edge joins x to $x + d$ for one of the fifteen classes $d \in \{0, 1\}^4 \setminus \{0\}$; it contains $|d| = \sum_a d_a$ postings, and an edge with $|d| = 1$ is a *posting*. For each ordering π of the four counters the records x, $x+e_{\pi(1)}$, $x+e_{\pi(1)}+e_{\pi(2)}$, $x+e_{\pi(1)}+e_{\pi(2)}+e_{\pi(3)}$ and $x+\mathbf{1}$ span a simplex, and these 24 simplices triangulate the unit cube at x (Kuhn's triangulation [23, 24]). The *posting order* relates $x \preceq y$ when every coordinate of $y - x$ is nonnegative; the edges $(x, x + d)$ are *posting-directed*.

Configurations and the flat lattice. A configuration ℓ assigns a real squared length $\ell_{(x,d)}$ to every edge. The flat lattice at κ is $\ell_{(x,d)} = g_\kappa(d, d) = |d|\,(1 - \kappa|d|)$, the squared length of d in the record-coordinate metric $g_\kappa = I - \kappa\,\mathbf{1}\mathbf{1}^\top$. For $\kappa > 1/4$ this metric is Lorentzian, $\det g_\kappa = 1 - 4\kappa < 0$, with its time direction along $\mathbf{1}$ [7, Sec. 2]. At $\kappa = 1$ every posting is null and every longer edge is timelike. The squared lengths of the three edges of a triangle determine

$$H = 4\ell_1\ell_2 - (\ell_1 + \ell_2 - \ell_3)^2,$$

four times the Gram determinant of the two edges ℓ_1, ℓ_2 at a common corner; the triangle is timelike for $H < 0$, null for $H = 0$ and spacelike for $H > 0$. A simplex is *regular Lorentzian* if the Gram determinants of its edge vectors, of each of its five tetrahedra and of each of its ten triangles are nonzero, the first negative.

The action, the weight and the admissible configurations. The action is Regge's in the slot form of [7, Sec. 3.1, eq. (3)],

$$S(\ell) = \sum_{(\sigma,h)} A_h\Big(\frac{2\pi}{n_h} - \theta_{\sigma,h}\Big), \qquad A_h = \tfrac{1}{4}(-H_h)^{1/2},$$

summed over the pairs of a simplex σ and one of its ten triangles h, with $\theta_{\sigma,h}$ the dihedral angle of σ at h and n_h the number of simplices that share h; summed over σ the bracket is the deficit angle δ_h, so $S = \sum_h A_h\delta_h$. The classical action is sS, with the *action scale* s, and the path sum weighs a configuration by e^{iasS}, with the *phase scale* a. A configuration is *admissible* if every simplex is timelike in all its faces (the Gram determinants of its ten triangles, its five tetrahedra and itself are negative), the main-diagonal edge and every edge of two or more postings are timelike, and every posting has squared length of the sign of $1-\kappa$ [7, Sec. 3.1]. The admissible configurations are the domain of the path sum of [7]; the clause is an input of that construction (MODEL).

Units. The lattice form of Newton's law holds for exactly one coupling, $G = 1/(8\pi s)$, for every $\kappa > 1/2$ and $s > 0$ [7, Sec. 3.5]; it is the coupling of the tree-level exchange, and the coupling of the long-wave limit of the quantum path sum is ZG with a factor Z that is not yet computed [22, Sec. 2]. $\hbar = 1/a$ and $c = 1$ are definitions. The *phase premise* $e^{-2\pi as\lambda_{\mathrm{rec}}^2} = \frac{1}{2}$ reads a face of area λ_{rec}^2 as holding one bit (IDENTIFICATION), so that $G\hbar = \lambda_{\mathrm{rec}}^2/(4\ln 2)$ and $\ell_P = \lambda_{\mathrm{rec}}/(2\sqrt{\ln 2})$. If light crosses one record length per record rank (IDENTIFICATION), the spacing is $b = \lambda_{\mathrm{rec}}/\sqrt{\kappa - 1/4}$, $b = (2/\sqrt{3})\,\lambda_{\mathrm{rec}} \approx 1.92\,\ell_P$ at $\kappa = 1$ [22, Secs. 3 and 4].

The lattice scalar. The scalar field φ of [7] lives on the records, with the action $-\frac{1}{48}\sqrt{|\det G_\sigma|}\,(\mathrm{d}\varphi)^{\top}G_\sigma^{-1}\mathrm{d}\varphi$ on each simplex σ, where G_σ is the Gram matrix of the simplex's edge vectors from its first record, computed from the squared lengths, and $\mathrm{d}\varphi$ the differences of φ along them [7, Sec. 6.1]. On the flat lattice at κ its field equation is, up to a positive factor,

$$(\mathscr{D}_\kappa f)(x) = \sum_a \big(2f(x) - f(x+e_a) - f(x-e_a)\big) - \beta\big(2f(x) - f(x+\mathbf{1}) - f(x-\mathbf{1})\big), \qquad \beta = \frac{\kappa}{4\kappa - 1},$$

[7, Prop. 6.16], with the symbol

$$\hat{\mathscr{D}}_\kappa(k) = \sum_a (2 - 2\cos k_a) - \beta\Big(2 - 2\cos\sum_a k_a\Big)$$

on waves $e^{ik\cdot x}$. Its quadratic part $\sum_a k_a^2 - \beta(\sum_a k_a)^2$ is the continuum wave operator $g_\kappa^{-1}(k,k)$, since $g_\kappa^{-1} = I - \beta\,\mathbf{1}\mathbf{1}^{\top}$. On a configuration that depends on a record only through $X = x_0 + x_1 - x_2 - x_3$, the *sector of zero transverse momentum* consists of the fields that depend on a record only through its rank t and X; its stationary modes are $e^{iWt}f(X)$, with W the frequency per rank. The regulated path sum weighs a field by $\exp\big(i\,S_\varphi - \varepsilon\sum|\varphi|^2\big)$, which for a stationary band B is $\exp\big(i\sum\bar{\varphi}(B + i\varepsilon)\varphi\big)$.

Lattices of constant inflow. The flat lattice at $\kappa = 1$ with a constant inflow u along the pair of axes of the first two counters, the Painlevé–Gullstrand flow of speed u sampled on the edges, gives an edge of n postings, m more of them on the first two counters than on the last two, the squared length

$$\ell(n, m) = n - n^2 + \frac{\sqrt{3}}{2}\, u\, nm + \frac{3}{4}\, u^2 n^2. \tag{1}$$

At $u = 0$ it is the flat lattice of [7] at $\kappa = 1$. The horizon of a flow along a pair of axes is where the inflow reaches 1, the speed of light in these units.

A.2 Imported results

Tables 5 and 6 list every result this paper takes from elsewhere, with the place it is proved and its hypotheses. Results of [8] are cited by the numbering of its version 1.2, which is that of version 1.1.

Table 5. Results imported from [7], with their sources and hypotheses.

result used here	source	hypotheses
every posting-directed edge of the flat lattice is causal iff $\kappa \geq 1$	[7, Thm. 2.1]	the flat lattice g_κ
every triangle of the flat lattice is timelike	[7, Sec. 2.1]	$\kappa > 1/2$
the admissible configurations; the slot form of the action	[7, Sec. 3.1]	definitions
$G = 1/(8\pi s)$, exactly one coupling	[7, Sec. 3.5]	$\kappa > 1/2$, $s > 0$
linearized Einstein equation in de Donder gauge	[7, Thm. 4.6]	$\kappa > 1/2$; a conserved source family off the light cone
exact solutions with a resting body; weak-field limit; Newton's potential	[7, Thms. 4.8–4.10]	$\kappa = 1$; a model medium; small coupling; odd sides not divisible by 105
second order at fixed small strength	[7, Thms. 4.12–4.16]	$\kappa = 1$; band equations; a model medium
the ordered one-loop anisotropy diverges like $\log(1/\varepsilon)$	[7, Thms. 6.7, 6.8]	$\kappa = 1$; the flat measure; the Feynman regulator; the diagram model
record time is spacelike for the doublers	[7, Thm. 6.15]	$\kappa = 1$
record time has growing modes: the scalar at every $\kappa > 1/2$, the graviton at $\kappa = 1$ and $3/2$	[7, Prop. 6.16]	as stated; for the graviton, the Bloch reading of kernel vectors
the lattice scalar and its symbol	[7, Sec. 6.1, Prop. 6.16]	the cell action above

A.3 Proofs of Propositions 3.4 and 4.5

Proof of Proposition 3.4. Let U be a trapped region of a configuration ℓ (Definition 3.1), let $d \neq \mathbf{1}$ be a class of edges, and suppose that $\ell_{(x,d)} \leq 0$ at every record x of U. Then every edge $(x, x + d)$ with $x \in U$ is causal, so $x + d \in U$, because U is future-closed, and by induction

Table 6. Results imported from [8] and [22], with their sources and hypotheses.

result used here	source	hypotheses
massless corner doublers exactly at $\kappa = 1$	[8, Thm. 2.1]	$\kappa > 1/2$
the derived weight gaps the doublers and gives the real part of the ordered anisotropy a finite limit	[8, Thms. 5.4, 5.5, 7.4]	two identifications of the weight; a channel count; $\kappa = 1$
the native model graviton is not massless	[8, Thm. 3.3]	$\kappa = 1$; the flat measure; the Feynman regulator; the one-loop model (graviton and scalar loops, a unit-weight gauge term, no ghost line)
the metric part of the long-wave field equation is Einstein's, every nonlinear term included	[8, Thm. 4.2]	$\kappa > 1/2$; a background joined to the flat metric by Lorentzian metrics with nondegenerate flat simplices; sampled smooth fields; the field equation, not its solutions
the vertex symmetry fails at second order	[8, Thm. 4.3]	$\kappa = 1$; the side-8 torus and the infinite lattice; a sampled cubic field, and the cubic Taylor fields of a point mass; local at one record
an explicit strong exterior that misses the Regge equations by the local curvature	[8, Sec. 4]	numerical
no positive norm preserved by the record tick	[8, Thm. 8.1]	$\kappa = 1$; the Regge action, or one whose quadratic part is a multiple of it
the growth survives the posting-null restriction	[8, Prop. 8.2]	$\kappa = 1$
a real weight is a damping; with the derived weight the record-time roots are stable in the continuum limit	[8, Sec. 8.1]	the weight's identifications; fixed strength; an inference about modes off the unit circle
the spacing $b = \lambda_{\mathrm{rec}}/\sqrt{\kappa - 1/4}$; the three readings of the spacing	[22, Sec. 4]	one record length per rank and the phase premise (identifications)
$G\hbar = \lambda_{\mathrm{rec}}^2/(4 \ln 2)$; G is the tree-level coupling	[22, Secs. 2 and 3]	the phase premise (identification)

$x + kd \in U$ for every $k \geq 0$. Since $d \neq \mathbf{1}$ there are counters a with $d_a = 1$ and c with $d_c = 0$, and along this sequence the difference $x_a - x_c$ grows by one at each step, so U is not spatially bounded, a contradiction. Hence $\ell_{(x,d)} > 0$ at some record x of U. The flat value of an edge of n postings is $n(1 - \kappa n)$, so there the edge exceeds its flat value by more than $\kappa n^2 - n$. For $n = 2$ this is $4\kappa - 2$, so a configuration whose two-posting edges all lie within $4\kappa - 2$ of their flat values has no trapped region. On the flat lattice with $\kappa \geq 1/3$ every edge of three postings has squared length $3(1 - 3\kappa) \leq 0$, so the flat lattice has no trapped region by the first part. With

$\kappa < 1/3$ every edge of one, two or three postings has squared length $n(1-\kappa n) > 0$, so only the edges along $\mathbf{1}$ can be causal. The record line $\{x + k\mathbf{1} : k \geq 0\}$ is then nonempty, contains every record that a causal step reaches from it, and is spatially bounded, since the differences of the counts of its records are those of x: it is a trapped region. □

Proof of Proposition 4.5. The ledger's state holds, for each of the four accounts, a debit total and a credit total. A posting on account a adds one to the debit total of a if its side is a debit and one to its credit total if its side is a credit, and changes nothing else. The record of an account is the sum of its two totals, so the record a posting leaves is the same for both sides. In the history reading every crossing carries a side of its own and every assignment of sides is a pattern, so k crossings carry 2^k patterns. In the field reading a state $L(x)$ sits at every record, and $L(x + e_a)$ is $L(x)$ after the posting on a with side $s(x, a)$. Take a counter $b \neq a$ and follow the debit total of account a from x to $x + e_a + e_b$ along the two orders. A posting on b leaves the totals of a unchanged, so posting on a first adds one to that total exactly when $s(x, a)$ is a debit, and posting on b first adds one exactly when $s(x + e_b, a)$ is a debit. Both orders reach the same record, and the state there is single-valued, so $s(x + e_b, a) = s(x, a)$: advancing another counter does not change the side of a posting on a. Applying this to each of the other three counters, forward and backward, gives $s(x, a) = g_a(x_a)$ for a function g_a, so the side depends only on the counter and its count. The crossings of the observer horizon $\{x_a = M\}$ are the postings on a from records with $x_a = M$, and they all carry the side $g_a(M)$. The field in which every posting is a debit, with debit totals x and credit totals 0 at the record x, and the field in which every posting is a credit both satisfy the definition, so both sides occur, and the horizon hides one bit whatever its area. Finally, a set of crossings whose counts lie in a window of width N uses at most $4N$ pairs of a counter and a count, and a field's pattern of sides on it is fixed by the values of the g_a at those pairs, so the fields realize at most 2^{4N} patterns. □

A.4 Proofs of the propositions of Section 5

Proof of Proposition 5.1. At $\kappa = 1$ an edge of n postings has squared length $n(1-n)$, zero for $n = 1$ and negative for $n \geq 2$. For the symbol write $S_j = \sum_a k_a^j$ and $S_1 = \sum_a k_a$. For every real t, $t^2 - t^4/12 \leq 2 - 2\cos t \leq t^2 - t^4/12 + t^6/360$, so

$$\hat{\mathscr{D}}_\kappa(k) \geq \big(S_2 - \beta S_1^2\big) - \tfrac{1}{12}\big(S_4 - \beta S_1^4\big) - \tfrac{\beta}{360} S_1^6.$$

On the continuum light cone $S_2 = \beta S_1^2$, and $k \neq 0$ there forces $S_1 \neq 0$. Since $S_4 \leq S_2^2 = \beta^2 S_1^4$, the quartic part $-\frac{1}{12}(S_4 - \beta S_1^4)$ is at least $\frac{1}{12}\beta(1-\beta)S_1^4$, which is positive exactly when $\beta < 1$, that is $\kappa > 1/3$. At finite wave number the same inequalities give

$$\hat{\mathscr{D}}_\kappa(k) \geq \frac{\beta S_1^4}{360}\big(30(1-\beta) - S_1^2\big) > 0 \qquad \text{on the cone, for } 0 < S_1^2 < 30(1-\beta),$$

the explicit bound; at $\kappa = 1$, $\beta = 1/3$ and the bound holds for $|S_1| < 2\sqrt{5}$. □

Proof of Proposition 5.2. Let $(Hf)(X) = \sum_{|j| \leq m} c_j(X) f(X + j)$ on sequences on $\mathbb{Z}$, Hermitian in the sense $c_{-j}(X + j) = \overline{c_j(X)}$. Summing by parts over $A \leq X \leq B$,

$$\sum_{X=A}^{B} \Big(\overline{g(X)}\,(Hf)(X) - \overline{(Hg)(X)}\, f(X) \Big) = J_B(g, f) - J_{A-1}(g, f),$$

where $J_Y(g, f)$ is the sum of $c_j(X)\,\overline{g(X)}\,f(X+j)$ over the couplings that cross the cut between Y and $Y+1$, each with the sign of j. For two solutions of $Hf = Hg = 0$ the flux $J_Y(g, f)$ is therefore the same at every cut. Where the coefficients are constant, a solution is a finite sum of modes z^X over the roots of the symbol $P(z) = \sum_j c_j z^j$, assumed distinct, and $J_Y(w^X, z^X)$ is a constant times $(\bar{w}z)^Y$; since it does not depend on Y, it vanishes unless $\bar{w}z = 1$. A propagating mode, $|z| = 1$, therefore pairs only with itself, with $J(z^X, z^X) = zP'(z)$, which is purely imaginary because P is real on the unit circle. An evanescent mode pairs only with $1/\bar{z}$, which grows where it decays; a solution that is bounded on a uniform half-line contains no growing mode there, so evanescent modes carry no flux. Equating the flux on the two uniform sides gives $\sum_{\text{left}} \operatorname{Im}\big(z_j P'(z_j)\big)|a_j|^2 = \sum_{\text{right}} \operatorname{Im}\big(z_j P'(z_j)\big)|a_j|^2$ for the amplitudes a_j of the propagating modes, whatever the coefficients in between; with the incoming wave normalized and the waves sorted by the signs of their norm and their direction, this is the lattice form of $|\alpha|^2 - |\beta|^2 = 1$. For the last claim, a configuration that depends on a record only through X is invariant under every record shift v with $v_0 + v_1 - v_2 - v_3 = 0$, the time translation $\mathbf{1}$ among them. The scalar's operator is assembled simplex by simplex from the configuration, so it commutes with these shifts, and on the sector of zero transverse momentum at frequency W it acts as a banded operator in X. Its coefficients come from the real symmetric quadratic form of the scalar's action, so the band is Hermitian. □

Proof of Proposition 5.3. The river wedge depends on a record only through its coordinate along the flow, a multiple of X, and so does its gluing to the flat lattice beyond a surface of constant X; by the last part of the previous proof its stationary modes obey a Hermitian band equation exactly. On a finite window with fixed ends the band is a Hermitian matrix B, and $B + i\varepsilon$ has the eigenvalues $\lambda + i\varepsilon$, none of them zero for $\varepsilon \neq 0$. In a uniform region the stationary modes of the regulated weight are the roots of $P(z) + i\varepsilon = 0$. A root z_0 of P on the unit circle with $\operatorname{Im}(z_0 P'(z_0)) \neq 0$ is simple, so it continues to a differentiable branch $z(\varepsilon)$ with $\mathrm{d}z/\mathrm{d}\varepsilon = -i/P'(z)$, and at $\varepsilon = 0$

$$\frac{\mathrm{d}|z|^2}{\mathrm{d}\varepsilon} = 2\operatorname{Re}\Big(\bar{z}\,\frac{\mathrm{d}z}{\mathrm{d}\varepsilon}\Big) = 2\operatorname{Re}\frac{-i}{z_0 P'(z_0)} = -\frac{2}{\operatorname{Im}\big(z_0 P'(z_0)\big)},$$

since $z_0 P'(z_0)$ is purely imaginary. For $\varepsilon \to 0^+$ the root moves inside the circle, so that the wave decays to the right of a source, exactly when $\operatorname{Im}(z_0 P'(z_0)) > 0$, and outside when it is negative; $\varepsilon \to 0^-$ reverses both. The norm of a stationary wave has the sign of $\partial_W P$, and its group velocity the sign of $\operatorname{Im}(z_0 P'(z_0))\,\partial_W P$, so $\operatorname{Im}(z_0 P'(z_0)) > 0$ means a wave of positive norm moving right or one of negative norm moving left. A bounded solution keeps on each side of a source the waves that decay away from it; for $\varepsilon \to 0^+$ waves of positive norm therefore leave the source and waves of negative norm run toward it. □

Proof of Proposition 5.9. By Eq. (1), the triangle with corners x, $x + e_0$ and $x + e_0 + e_1$ has the squared lengths $p = \frac{\sqrt{3}}{2}u + \frac{3}{4}u^2$ on its two postings and $q = -2 + 2\sqrt{3}\,u + 3u^2$ on its edge of two postings, so with the corner at x

$$H = 4pq - (p + q - p)^2 = q(4p - q) = 2\,(3u^2 + 2\sqrt{3}\,u - 2).$$

It is -4 at $u = 0$, its positive root is $u = 1 - 1/\sqrt{3} = (3 - \sqrt{3})/3 \approx 0.42$, and at the horizon, $u = 1$, it is $2 + 4\sqrt{3}$. The area $\frac{1}{4}(-H)^{1/2}$ of the action of [7] is real only for $H \leq 0$; read with

the real square root, as the action is evaluated beyond its admissible configurations, it is 0 on a spacelike triangle, and so is the triangle's term, whatever value its angle is given. The dihedral angle of [7] is the arccosine of the inner product of the two heights of the simplex over the triangle divided by the product of their lengths, all computed from cofactors of the simplex's Gram matrix. Evaluated as real functions, with the real square root, it is $\pi/2$ on a spacelike triangle whose two heights have mixed causal type, where the product of their lengths is read as 0 and the ratio with it, and 0 or π when the heights have one type, where the ratio has modulus above 1 by the identity in the next proof. For $u \geq 1/2$, $H \geq 2(\sqrt{3} - \frac{5}{4}) > 0$, so the term vanishes for every such inflow. □

Proof of Proposition 5.10. On a timelike or null triangle the term is that of [7] by definition, and the Regge equation at an edge involves only the triangles and simplices that contain the edge, so where all of them are timelike the two actions agree near the configuration and have the same equations. For a triangle h of a simplex σ, let a_3, a_4 and a_{34} be the cofactors of the simplex's Gram matrix that are proportional, with one positive factor, to the squared lengths of the two heights of σ over h (in the tetrahedra through h) and to twice their inner product. They satisfy

$$a_{34}^2 - 4a_3a_4 = -H_h\,\Delta_\sigma,$$

with $\Delta_\sigma < 0$ the Gram determinant of the Lorentzian simplex. On a spacelike triangle, $H_h > 0$, so $a_{34}^2 > 4a_3a_4$; when the heights are of one causal type, $a_3a_4 > 0$ and $|a_{34}|/(2\sqrt{a_3a_4}) > 1$, and the boost arcosh of that ratio is not zero. The simplex of the flat lattice of inflow $1/\sqrt{3}$ with this property is found by direct computation from Eq. (1). For Schläfli's identity we give the argument in outline. For a Euclidean simplex, $\sum_h A_h\,\mathrm{d}\theta_h = 0$ for every variation of the squared lengths, an identity between analytic functions of the squared lengths, with $A_h = \frac{1}{4}H_h^{1/2}$. On a regular Lorentzian simplex every triangle, every tetrahedron and the simplex itself has a nonzero Gram determinant, so the angles and areas continue analytically along a path of squared lengths that avoids their zeros. After the continuation the area of a timelike triangle is i times its Lorentzian area $\frac{1}{4}(-H_h)^{1/2}$ and its angle is real, while the area of a spacelike triangle is real and its angle is a constant plus i times its boost. Divided by i, the identity becomes the stated one, the boost entering with the sign fixed by the branch, which is the sign stated with the term [34]. The identity in this form, for triangles of every causal type, is also checked, to 10^{-22}, on every simplex at a record of the flat lattices of inflow $1/2$, 1.2 and 2 [29]. □

Proof of Proposition 5.11. Write each term of the extended action as $A_h\big(\phi_{\sigma,h} - \psi_{\sigma,h}\big)$, with $A_h = \frac{1}{4}|H_h|^{1/2}$; at a timelike triangle $\phi_{\sigma,h} = 2\pi/n_h$ and $\psi_{\sigma,h}$ is the dihedral angle, and at a spacelike one $\phi_{\sigma,h} = 0$ and $\psi_{\sigma,h}$ is the boost oriented so that the term is the area times minus $\psi_{\sigma,h}$. Summed over the simplices around h, $\sum(\phi - \psi)$ is the deficit ϵ_h of the statement, and in the variable ψ Schläfli's identity of Proposition 5.10 reads $\sum_{h\subset\sigma} A_h\,\partial\psi_{\sigma,h}/\partial\ell_e = 0$ for each simplex σ. At a record whose surrounding simplices are regular Lorentzian every term is differentiable in the squared length ℓ_e of an edge through the record, and

$$\frac{\partial S}{\partial \ell_e} = \sum_h \frac{\partial A_h}{\partial \ell_e}\,\epsilon_h - \sum_\sigma \sum_{h\subset\sigma} A_h \frac{\partial \psi_{\sigma,h}}{\partial \ell_e},$$

since ϕ does not depend on ℓ_e. The double sum vanishes simplex by simplex. So each Regge equation is $\sum_h (\partial A_h/\partial\ell_e)\,\epsilon_h = 0$, a sum of area rates times deficits, and it holds when every

ϵ_h vanishes, that is when the angles around each timelike triangle sum to 2π and the boosts around each spacelike one to 0. □

References

[1] J. D. Bekenstein, Black holes and entropy, *Phys. Rev. D* **7**, 2333–2346 (1973).

[2] S. W. Hawking, Particle creation by black holes, *Commun. Math. Phys.* **43**, 199–220 (1975).

[3] S. W. Hawking, Breakdown of predictability in gravitational collapse, *Phys. Rev. D* **14**, 2460–2473 (1976).

[4] D. N. Page, Information in black hole radiation, *Phys. Rev. Lett.* **71**, 3743–3746 (1993).

[5] J. Washburn, The record cell of the four-dimensional lattice: time, Lorentzian signature and the Regge action near flatness, in *The Formation of the Universe from Recognition: The Integrated Manuscript and Its Focused Papers*, Zenodo (2026), doi:10.5281/zenodo.23738724.

[6] T. Regge, General relativity without coordinates, *Nuovo Cimento* **19**, 558–571 (1961).

[7] J. Washburn, Linearized Einstein gravity from Lorentzian Regge calculus on the Kuhn lattice, and the divergence of its one-loop graviton limit, Zenodo (2026), doi:10.5281/zenodo.23764491; Pith preprint doi:10.67347/pith.preprint.2026.000017.

[8] J. Washburn, Quantum gravity from the recognition ledger: weighted one-loop graviton responses with finite real parts, Einstein gravity at long wavelengths, and the massless-graviton problem, Zenodo (2026), doi:10.5281/zenodo.23806703; Pith preprint doi:10.67347/pith.preprint.2026.000018.

[9] L. Bombelli, R. K. Koul, J. Lee and R. D. Sorkin, Quantum source of entropy for black holes, *Phys. Rev. D* **34**, 373–383 (1986).

[10] M. Srednicki, Entropy and area, *Phys. Rev. Lett.* **71**, 666–669 (1993).

[11] A. Strominger and C. Vafa, Microscopic origin of the Bekenstein–Hawking entropy, *Phys. Lett. B* **379**, 99–104 (1996).

[12] C. Rovelli, Black hole entropy from loop quantum gravity, *Phys. Rev. Lett.* **77**, 3288–3291 (1996).

[13] A. Ashtekar, J. Baez, A. Corichi and K. Krasnov, Quantum geometry and black hole entropy, *Phys. Rev. Lett.* **80**, 904–907 (1998).

[14] T. Jacobson, Thermodynamics of spacetime: the Einstein equation of state, *Phys. Rev. Lett.* **75**, 1260–1263 (1995).

[15] R. M. Wald, Black hole entropy is the Noether charge, *Phys. Rev. D* **48**, R3427–R3431 (1993).

[16] T. Jacobson, Black-hole evaporation and ultrashort distances, *Phys. Rev. D* **44**, 1731–1739 (1991).

[17] W. G. Unruh, Sonic analogue of black holes and the effects of high frequencies on black hole evaporation, *Phys. Rev. D* **51**, 2827–2838 (1995).

[18] R. Brout, S. Massar, R. Parentani and Ph. Spindel, Hawking radiation without trans-Planckian frequencies, *Phys. Rev. D* **52**, 4559–4568 (1995).

[19] S. Corley and T. Jacobson, Hawking spectrum and high frequency dispersion, *Phys. Rev. D* **54**, 1568–1586 (1996).

[20] A. Almheiri, N. Engelhardt, D. Marolf and H. Maxfield, The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole, *JHEP* **12** (2019) 063.

[21] G. Penington, Entanglement wedge reconstruction and the information paradox, *JHEP* **09** (2020) 002.

[22] J. Washburn, Newton's constant and the recognition ledger: Einstein's equations, the graviton, and the one number that fixes G, Zenodo (2026), doi:10.5281/zenodo.23806705; Pith preprint doi:10.67347/pith.preprint.2026.000019.

[23] H. Freudenthal, Simplizialzerlegungen von beschränkter Flachheit, *Ann. of Math.* **43**, 580–582 (1942).

[24] H. W. Kuhn, Some combinatorial lemmas in topology, *IBM J. Res. Dev.* **4**, 518–524 (1960).

[25] C.-Y. Wong, Application of Regge calculus to the Schwarzschild and Reissner–Nordström geometries at the moment of time symmetry, *J. Math. Phys.* **12**, 70–78 (1971).

[26] G. W. Gibbons and S. W. Hawking, Action integrals and partition functions in quantum gravity, *Phys. Rev. D* **15**, 2752–2756 (1977).

[27] M. Bañados, C. Teitelboim and J. Zanelli, Black hole entropy and the dimensional continuation of the Gauss–Bonnet theorem, *Phys. Rev. Lett.* **72**, 957–960 (1994).

[28] W. G. Unruh, Notes on black-hole evaporation, *Phys. Rev. D* **14**, 870–892 (1976).

[29] J. Washburn, Computations for Section 5 of *The recognition ledger of a black hole*, version 1.1: programs, operator tables and run logs, file BlackHoleRecognitionLedger_computations_v1.1.zip, SHA-256 b4b7a6d687717bdf4c36989148aac8a638eeca5c7fd653976c154f495879607e, in the Zenodo record doi:10.5281/zenodo.23813522 (2026).

[30] J. J. Bisognano and E. H. Wichmann, On the duality condition for quantum fields, *J. Math. Phys.* **17**, 303–321 (1976).

[31] A. Coutant, R. Parentani and S. Finazzi, Black hole radiation with short distance dispersion, an analytical S-matrix approach, *Phys. Rev. D* **85**, 024021 (2012), doi:10.1103/PhysRevD.85.024021; arXiv:1108.1821.

[32] S. Corley and T. Jacobson, Black hole lasers, *Phys. Rev. D* **59**, 124011 (1999).

[33] J. Steinhauer, Observation of quantum Hawking radiation and its entanglement in an analogue black hole, *Nature Phys.* **12**, 959–965 (2016).

[34] R. D. Sorkin, Lorentzian angles and trigonometry including lightlike vectors, arXiv:1908.10022 (2019).

[35] J. Washburn, *The Formation of the Universe from Recognition: The Integrated Manuscript and Its Focused Papers*, Zenodo (2026), doi:10.5281/zenodo.23738724.