Binary Marginal Invariance and the Quadratic Probability Rule in a Cost-Motivated Scalar Amplitude Model

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Abstract

We prove a conditional uniqueness theorem for the quadratic probability rule in a scalar product-amplitude model. In a standard scalar route, $\boldsymbol{P(c) = |c|^2}$ follows once modulus multiplicativity, $\boldsymbol{P(c_1c_2) = P(c_1)P(c_2)}$, is assumed. We show that this primitive multiplicativity assumption can instead be replaced by product-amplitude composition together with binary marginal invariance on a second subsystem. This is a model-specific marginal-invariance condition, not an operational no-signaling axiom for generalized probabilistic theories. Under scalar reduction and the adopted **2**-norm normalization convention, the resulting marginal invariance equation forces $\boldsymbol{f(r) = r^2}$ on the normalized amplitude domain $\boldsymbol{0 \leq r \leq 1}$, and hence $\boldsymbol{P(c) = |c|^2}$ for normalized outcome amplitudes. Because the proof proceeds through additivity of $\boldsymbol{h(x) = f(\sqrt{x})}$ on the unit interval, the same argument yields a finite normalized amplitude-list extension; an all-positive $\boldsymbol{r > 1}$ formula is treated only as an optional homogeneous intensity extension, not as a probability statement outside normalized contexts. The result is motivated by a cost-functional amplitude framework, but it does not derive quantum probability from cost axioms alone and does not reconstruct Hilbert-space structure. We identify where the quadratic rule is forced, how

binary marginal invariance can replace primitive scalar multiplicativity in this model, and which assumptions remain independent or open.

Keywords: Born rule, quantum foundations, marginal invariance, functional equations, scalar amplitudes, cost-functional models

MSC Classification: 81P10, 81P15, 39B22

1 Introduction

The Born rule assigns to each outcome amplitude c the probability $P(c) = |c|^2$. Derivations of this rule differ less in their algebraic endpoint than in where they place the probabilistic work: in Hilbert-space measure structure, operational composition principles, envariance, decision-theoretic rationality, dynamical constraints, or operational no-signaling constraints. The aim of this paper is narrower. We ask whether, inside a scalar product-amplitude model motivated by a cost-functional amplitude framework, primitive modulus multiplicativity must be assumed as a scalar probability axiom, or whether it can be replaced by binary marginal invariance. By *binary marginal invariance* we mean that subsystem-1 marginal probabilities are independent of the choice of binary measurement on subsystem 2 in the stated scalar product-amplitude model; this is a model-specific condition, not a reconstruction-level no-signaling principle formulated over a full operational state space of a generalized probabilistic theory (GPT). The precise statement appears in Definition 4.10.

The answer is conditional, though precise within its scope. Given scalar reduction $P(c) = f(|c|)$, the adopted 2-norm normalization convention, and binary marginal invariance in a product-amplitude subsystem model, the resulting functional equation yields $f(r) = r^2$ on the normalized domain $0 \le r \le 1$. Modulus multiplicativity is therefore not an input in Route B; it is recovered afterward as an algebraic corollary on normalized moduli. The same additive structure delivers a finite normalized amplitude-list extension. The result does not reconstruct quantum probability in general, nor does it derive scalar reduction, 2-norm normalization, or Hilbert-space structure from the cost axioms alone. Its content is the localization of premises: binary marginal invariance can replace primitive scalar multiplicativity in this scalar product-amplitude setting, while the stronger reconstruction questions remain open. Although the cost-functional framework motivates the model class, the scalar uniqueness result itself—Proposition 5.6—can be read independently as a functional-equation theorem about scalar probability assignments satisfying product-amplitude marginal invariance.

The Recognition Science (RS) framework starts from a different primitive: a cost functional on positive ratios together with ledger bookkeeping, path actions, and a d'Alembert-type functional-equation uniqueness chain for the cost itself [19, 27–30]. It contrasts with Gleason-type approaches, which presuppose Hilbert-space measure structure, and with Hardy- or Chiribella–D'Ariano–Perinotti-style reconstructions, which begin with operational axioms about states, composites, and transformations. The RS setup begins with neither. Instead it derives a unique cost geometry from a

reciprocal composition law and builds path weights and complex amplitudes from that geometry. Those ingredients fix single-path amplitude moduli via the $C = 2A$ bridge and later support multiplicative modulus structure under subsystem composition, but they do not yet determine the probability law for outcomes reached by multiple paths. The remaining question is which additional assumptions, beyond cost and amplitude bookkeeping, force the quadratic rule.

The argument proceeds in four stages. First, Route A is a benchmark calculation: if one assumes modulus multiplicativity (MA), then the quadratic rule follows by a standard scalar functional-equation argument. Second, Route B replaces (MA) with product-amplitude composition (PA) and binary marginal invariance $(\mathrm{P5})_2$ (the binary restriction of Definition 4.10), deriving $f(r) = r^2$ on $0 \le r \le 1$ without taking scalar multiplicativity as primitive. Third, the resulting additive structure extends to finite normalized amplitude lists. Fourth, a thinner scalar binary chart T_{bin} is shown not to determine Hilbert-space structure among its completions (Proposition 6.11); this limited comparison result should not be read as a proof that the full geometric package (G1)–(G4) is strictly weaker than Hilbert-space structure (Remark 6.5).

In short: Route A supplies the functional constraint through primitive scalar multiplicativity (MA), and Route B supplies it through product-amplitude composition (PA) plus binary marginal invariance $(\mathrm{P5})_2$. The assumptions needed for this replacement are identified, the resulting scalar uniqueness theorem (Proposition 5.6) is proved, and the stronger reconstruction questions that remain open are recorded in Section 8.2. Appendix A records Lean 4 status for upstream RS ingredients; Proposition 5.6, Corollary 5.11, Theorem 6.6, and Proposition 6.11 are proved in the present text.

Structure of the paper. Section 2 fixes the empirical target by recalling Born's scattering rule. Section 3 gives the cost-functional motivation for the scalar amplitude model. Section 4 separates the assumptions used in the two scalar routes. Section 5 proves the benchmark multiplicativity route, the Route B marginal invariance route, and the finite normalized amplitude-list extension. Section 6 packages the result with the adopted two-outcome geometry and records Proposition 6.11, a scope limitation for the reduced binary chart. Section 7 compares the result with other Born-rule programs. Section 8 discusses the empirical scope of the result and lists the open problems that mark the boundary between the conditional theorem proved here and a possible unconditional derivation. Appendix A records the Lean 4 formalization status for the relevant upstream RS material. Figure 1 summarizes this architecture, with Route A appearing as a benchmark side branch.

2 Born's 1926 Scattering Rule as the Empirical Target

Born's rule first appeared in particle scattering [5, 6]. For an incident plane wave e^{ikz} scattered by a rotationally symmetric potential, the wave function at large radius $\rho = |\mathbf{r}|$ takes the form

$$\psi(\mathbf{r}) \xrightarrow{\rho\to\infty} e^{ikz} + f_{\mathrm{sc}}(\theta)\,\frac{e^{ik\rho}}{\rho}, \tag{1}$$

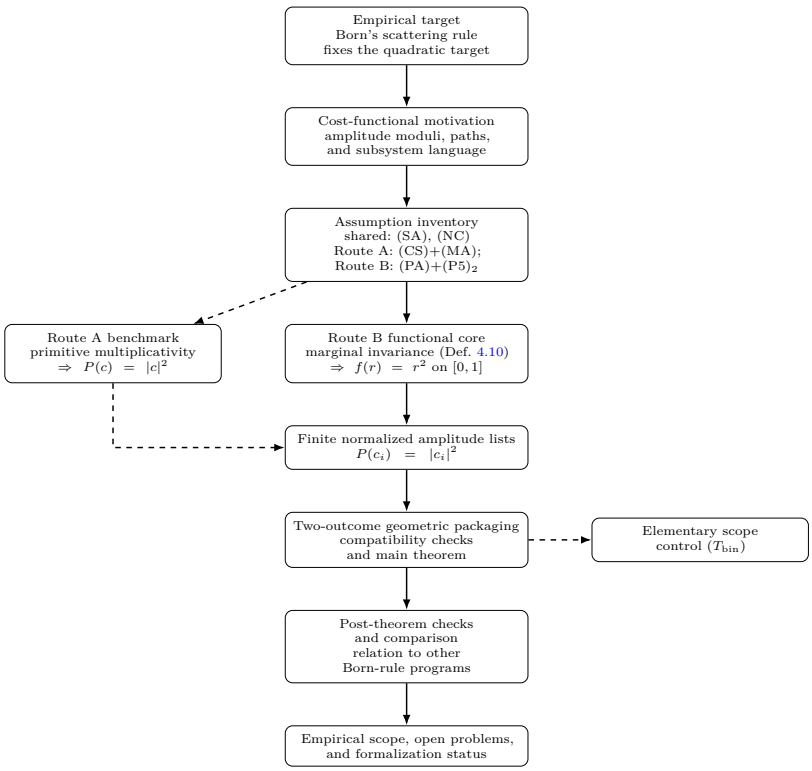

Fig. 1 Roadmap of the argument. The central path tracks the conditional uniqueness result: cost-functional motivation, assumption inventory, Route B marginal-invariance functional core (Definition 4.10), finite normalized amplitude-list extension, and two-outcome geometric packaging. Side branches record the Route A multiplicativity benchmark and Proposition 6.11 (scope limitation for T_{bin}).

where $f_{\mathrm{sc}}(\theta)$ is the scattering amplitude. The differential cross-section satisfies

$$\frac{d\sigma}{d\Omega} = |f_{\mathrm{sc}}(\theta)|^2, \tag{2}$$

so the detection probability in direction θ is proportional to $|f_{\mathrm{sc}}(\theta)|^2\, d\Omega$. This square-modulus dependence is the *quadratic rule* [7, 15, 18, 23].

In the RS framework, each path carries a positive weight $w = e^{-C}$ that is blind to phase, so the passage from path weights to outcome probabilities requires additional assumptions. Sections 3–6 identify which assumptions—scalar ansatz (SA), 2-norm convention (NC), product-amplitude composition (PA), and binary marginal invariance $(\mathrm{P5})_2$ as in Definition 4.10—force the quadratic dependence on normalized outcome amplitudes in Route B, and then separate that scalar result from the two-outcome geometric packaging used in Section 6. Continuity (CS) is retained only for the Route A benchmark.

3 The RS Framework: Cost Functional, Paths, and Amplitudes

The results of Sections 5 and 6 require only a limited part of the RS framework: a cost-motivated amplitude modulus, subsystem language, and motivation for product-amplitude composition. The cost-functional material below explains the origin of that model. Readers interested only in the scalar uniqueness theorem may proceed from Section 4 directly to Proposition 5.6.

Section 3.1 fixes the cost functional J and records its uniqueness theorem. Section 3.2 defines path actions $C[\gamma]$, positive weights $w(\gamma) = e^{-C}$, and complex amplitudes $\psi(\gamma) = e^{-C/2}e^{i\phi}$. Neither component by itself determines a probability law; together they supply the objects used in Section 5.

3.1 Cost Functional and Uniqueness

This subsection states the two RS axioms on the cost functional $J : (0, \infty) \to \mathbb{R}$, which assigns a mismatch cost to each positive ratio, and records the resulting uniqueness theorem. It does not yet supply a probability law or a measurement model; the phase variable introduced in Section 3.2 is additional structure, not a consequence of the cost axioms.

Definition 3.1 (Ledger) A ledger is the RS bookkeeping device for recognition events. Formally it is a collection of triples $(t_i, r_i, J(r_i))$, where t_i indexes the event, $r_i > 0$ is the compared ratio, and $J(r_i)$ is its assigned mismatch cost. The specific form of J is derived below from axioms (A1)–(A2) in Theorem 3.4. A ledger region is any subset of these events. Later sections use disjoint ledger regions to represent independent subsystems [19, 26]; that subsystem interpretation is part of the adopted RS setup.

Remark 3.2 (Discrete ledger entries and continuous path actions) The ledger picture is discrete, whereas the path action introduced in Definition 3.6 is written as a continuum integral. Here a continuum idealization is adopted: when the event spacing is sufficiently fine relative to the scale of the path description, the discrete sum $\sum_i J(r_i)\,\Delta(\ln r_i)$ is replaced by $\int J(r)\,d(\ln r)$. All later path-action manipulations assume this regime. A genuinely discrete treatment is not developed here.

The guiding principle is reciprocal symmetry: exchanging overshoot and undershoot sends ratio x to $1/x$, so mismatch in opposite directions should be treated symmetrically. In the present framework that principle is encoded in two axioms [27–30]:

(A1) Recognition Composition Law (RCL):

$$J(xy) + J(x/y) = 2J(x)J(y) + 2J(x) + 2J(y) \quad \text{for all } x, y > 0, \tag{3}$$

(A2) Calibration:

$$\frac{d^2}{dt^2} J(e^t)\Big|_{t=0} = 1. \tag{4}$$

Lemma 3.3 (Normalization from the RCL) *Any solution satisfying (A1)–(A2) obeys $J(1) = 0$. Continuity is not required for this conclusion; it enters only in Theorem 3.4 for the d'Alembert classification.*

Proof Setting $x = y = 1$ in (A1) gives $J(1) + J(1) = 2J(1)^2 + 4J(1)$, i.e. $2J(1)^2 + 2J(1) = 0$, so $J(1) \in \{0, -1\}$. Now set $y = 1$ in (A1). Then

$$J(x) + J(x) = 2J(x)J(1) + 2J(x) + 2J(1),$$

hence $J(1)(J(x) + 1) = 0$ for every $x > 0$. If $J(1) = -1$, it follows that $J(x) = -1$ for all $x > 0$, so J is constant. But a constant function has $\frac{d^2}{dt^2} J(e^t)\Big|_{t=0} = 0 \neq 1$, contradicting (A2). Therefore $J(1) = 0$. □

Theorem 3.4 (Cost Uniqueness) *The unique continuous real-valued solution to (A1)–(A2) is*

$$J(x) = \tfrac{1}{2}(x + x^{-1}) - 1, \tag{5}$$

satisfying $J(x) \geq 0$ with equality iff $x = 1$.

Proof By Lemma 3.3, $J(1) = 0$. Setting $H(t) = 1 + J(e^t)$ reduces the RCL to the d'Alembert equation

$$H(s + t) + H(s - t) = 2H(s)H(t). \tag{6}$$

Continuity of J gives continuity of H. Standard classification results for continuous real solutions of d'Alembert's equation [1] then leave only the cosh-family, the cos-family, and the constant/zero degeneracies. Here $H(0) = 1$ and calibration gives $H''(0) = 1 > 0$, so the oscillatory, zero, and constant branches are excluded and one obtains uniquely $H(t) = \cosh t$. Hence

$$J(x) = \tfrac{1}{2}(x + x^{-1}) - 1.$$

The full derivation is given in [28]; a Lean 4 formalization of this uniqueness result appears in [10, 29]. □

Axiom (A1) encodes the composition rule for mismatch costs, Axiom (A2) fixes the local calibration at exact balance, and continuity then forces the unique closed form (5). This fixes the ratio-cost geometry used in Section 3.2, but not yet a probability law, phase structure, or measurement model.

Remark 3.5 (Immediate structural consequences of the closed form) Equation (5) yields three structural consequences, not additional postulates. First, $J(x) = J(1/x)$ (reciprocal symmetry): overshoot and undershoot carry the same cost. Second, $J(x) \geq 0$ with equality only at $x = 1$: exact balance is the unique zero-cost state; together with the positive path-action measure fixed in Definition 3.6, this ensures $C[\gamma] \geq 0$ and hence $w(\gamma) = e^{-C[\gamma]} \in (0, 1]$. Third, $J(x) \to \infty$ as $x \to 0^+$ or $x \to \infty$: extreme mismatch is infinitely costly. The first and third are structural background; the second is directly used in Section 3.2.

3.2 Path Weights and Amplitudes

This subsection defines the path-level objects used throughout the paper: the path action $C[\gamma]$, the path weight $w(\gamma)$, and the complex amplitude $\psi(\gamma)$. The construction assigns a cost-derived modulus and a free phase to each path, but it remains per-path bookkeeping only: it does not yet provide a normalized path measure, an outcome probability rule, or an outcome-level sum-over-paths postulate in the sense needed for Born-rule claims [12, 22]. Those questions are deferred to Sections 4–6.

Definition 3.6 (Path Action) A recognition path γ is a continuous curve in the space of positive ratios $(0, \infty)$, parameterizing a sequence of scale-ratio comparisons on a ledger region. For a parameterization $u \mapsto r(u)$ of such a path, the path action is

$$C[\gamma] = \int J(r(u)) \left| \frac{d}{du} \ln r(u) \right| du, \tag{7}$$

where $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ is the unique cost functional of Theorem 3.4. Equivalently, this is integration of $J(r)$ against the positive logarithmic path-length measure $|d(\ln r)|$. On the monotone branch paths used later, this reduces to the oriented integral over the increasing log-ratio coordinate.

Remark 3.7 (Conventions on the path-action measure) The logarithmic path-length measure $|d(\ln r)|$ is part of the RS formulation adopted here [28, 30]. Because positive ratios compose multiplicatively, $d(\ln r) = dr/r$ is the natural left-invariant differential on $(\mathbb{R}_{>0}, \cdot)$; equivalently, it is the Haar measure on that group up to scale. The absolute path-length convention fixes the non-negative action used in Definition 3.8. The choice matters later because $\int J(r)\, d(\ln r)$ along the increasing log-ratio branch diverges on the semi-infinite measurement branches of Section 6; there the finite quantity is the induced half-action rate instead (Remark 6.3).

Definition 3.8 (Path Weight and Amplitude) Each path is assigned a positive weight $w(\gamma) = e^{-C[\gamma]}$ and a complex amplitude

$$\psi(\gamma) = e^{-C[\gamma]/2} \cdot e^{i\phi(\gamma)}, \tag{8}$$

where $\phi(\gamma) \in \mathbb{R}/2\pi\mathbb{Z}$ is an additional phase variable not fixed by the cost alone. With this definition, the single-path modulus is tied identically to the cost:

$$|\psi(\gamma)|^2 = e^{-C[\gamma]} = w(\gamma).$$

Remark 3.9 (The $C = 2A$ bridge) Definition 3.8 gives $|\psi(\gamma)| = e^{-C[\gamma]/2}$ directly. Writing $A(\gamma) := -\ln|\psi(\gamma)|$ for the *half-action* of path γ, this is equivalent to $C[\gamma] = 2A(\gamma)$. This relation—referred to throughout Section 6 as the $C = 2A$ bridge—is an algebraic consequence of Definition 3.8 alone; it requires no additional assumption. In particular, it does not presuppose the Born rule: the modulus $|\psi|$ is assigned by the cost, not by an outcome probability. Section 6 later uses the bridge to convert the convergent half-action integral of Proposition 6.2 into the adopted branch modulus.

Remark 3.10 (Why the single-path identity is not yet the Born rule) Definition 3.8 gives $|\psi(\gamma)|^2 = w(\gamma)$ by algebra, so no Born-rule theorem has yet been obtained. The crucial

distinction is between a single-path amplitude and the amplitude for an outcome. If several paths lead to the same outcome and one writes $\psi_i = \sum_{\gamma \to i} \psi(\gamma)$, then in general

$$\left| \sum_{\gamma \to i} \psi(\gamma) \right|^2 \neq \sum_{\gamma \to i} |\psi(\gamma)|^2$$

because interference produces cross-terms. Probabilities therefore become an outcome-level question, not a path-by-path one. The positive path weight $w = e^{-C}$ is blind to the phase $\phi(\gamma)$, since the cost functional depends only on positive ratios; that phase blindness is the structural motivation for the later phase-independence assumption on probabilities. The actual Born-rule-facing question is therefore postponed to Sections 4–6, where outcome amplitudes, subsystem composition, and scalar probability assignments are constrained by additional assumptions.

4 Assumptions and Routes

This section presents and separates the assumptions underlying the conditional uniqueness claims, distinguishing three layers: the framework conventions (P1)–(P7), the shared scalar-reduction premises (SA) and (NC), and the two route-specific packages. Both routes require the scalar ansatz (SA) and the 2-norm normalization convention (NC). Route A additionally assumes continuity (CS) and modulus multiplicativity (MA). Route B instead adopts product-amplitude composition (PA) and binary marginal invariance $(\mathrm{P5})_2$ (Definition 4.10), without taking either (MA) or (CS) as a primitive axiom. The dependencies of each result on these conventions are summarized in Table 1.

We do not claim to derive the 2-norm convention (NC), product-amplitude composition (PA), or scalar reduction (SA) from the cost axioms (A1)–(A2); those inputs are adopted as modeling choices before the functional comparison begins. The question isolated here is which *further* scalar hypothesis closes the uniqueness argument for the map f in $P(c) = f(|c|)$. Route A takes probability-level modulus multiplicativity (MA), together with continuity (CS), as its classical scalar package. Route B instead takes binary marginal invariance $(\mathrm{P5})_2$ from Definition 4.10, imposed jointly with (PA) at the amplitude-composition level. The substantive content is not that (NC) or (PA) are without physical content—readers who regard them as already "quantum-like" should see Remarks 4.1 and 5.10—but that, given (SA)+(NC)+(PA), marginal invariance can replace primitive scalar multiplicativity (MA) as the hypothesis that forces $f(r) = r^2$ on normalized moduli.

The analysis uses the following labeled assumptions and conventions:

(P1) Cost-functional package: axioms (A1)–(A2), together with the continuity hypothesis on J used in Theorem 3.4, fix the cost functional.

(P2) Phase degree of freedom: each recognition path carries a phase $\phi(\gamma) \in \mathbb{R}/2\pi\mathbb{Z}$ independent of the cost magnitude. This is not derived from (A1)–(A2) here. The broader RS forcing chain offers structural motivation via the 8-tick periodicity (Remark A.1), but that derivation is not reproduced here.

(P3) Log-ratio integration measure $d(\ln r)$ on path actions: the specific integration convention adopted in Definition 3.6.

(P4) Geometric definitions (G1)–(G4): the adopted binary-split measurement model, including the angular parametrization, branch-modulus identification, and angular projection used in Section 6.

(P5) Marginal invariance for disjoint ledger regions (Definition 4.10). In the proofs, the binary restriction of this condition is used and denoted $\mathbf{(P5)}_2$.

(P6) Continuity of the scalar probability function: a regularity assumption used only in the Route A benchmark. This postulate is assigned the short label **(CS)** in Theorem 5.1 and in Table 1.

(P7) Product-amplitude composition: for independent subsystems, the joint outcome amplitude is modeled by the product rule of Remark 4.9. Corollary 4.8 supplies only the modulus part; additive phase is stipulated. This postulate is assigned the short label **(PA)** in the theorems and in Table 1.

Postulate (P2) is adopted independently of the cost axioms, and the 2-norm convention (NC) enters separately as a modeling choice rather than as an item of the (P1)–(P7) list. Whether the cost-functional package together with the geometric package (G1)–(G4) is strictly weaker than Hilbert-space-equivalent structure remains unresolved (Remark 6.5). Table 1 records the exact premise dependencies of each result.

The theorem statements use six short labels. Two of them are aliases for items already listed: **(CS)**≡**(P6)** and **(PA)**≡**(P7)**. Three—**(SA)**, **(NC)**, and **(MA)**—are modeling choices not covered by (P1)–(P7). The last, $\mathbf{(P5)}_2$, is the binary restriction of postulate (P5).

(SA) *Scalar Ansatz.* $P(c_i) = f(|c_i|)$ for a non-negative function of the amplitude modulus. Route A uses an all-positive scalar function $f : [0, \infty) \to [0, \infty)$; Route B requires only its normalized-domain restriction $f : [0, 1] \to [0, \infty)$. Probability depends only on the amplitude modulus. This is adopted as a modeling hypothesis; it is not derived from (A1)–(A2).

(NC) *2-Norm Convention.* Outcome amplitudes satisfy $\sum_i |c_i|^2 = 1$. This normalization is adopted independently of the cost axioms; Section 6 records an internal two-outcome compatibility check with the geometric packaging, not a derivation of (NC) from (A1)–(A2).

(PA)≡(P7) *Product-Amplitude Composition.* For independent subsystems, the joint outcome (i, j) is assigned amplitude $c_{1i} \cdot c_{2j}$. The modulus rule $|\psi_{12}| = |\psi_1| \cdot |\psi_2|$ follows from Corollary 4.8; the phase rule $\phi_{ij} = \phi_i + \phi_j$ is stipulated separately and is not derived from (A1)–(A2).

(CS)≡(P6) *Continuity of Scalar Rule.* The function f in (SA) is continuous on $[0, \infty)$. This is the regularity assumption used in the Route A benchmark theorem. Route B does not assume (CS); non-negativity of the probability codomain supplies the monotonicity needed in Proposition 5.6.

Remark 4.1 (Three roles of $|c|^2$ under the 2-norm convention (NC)) Readers may worry about circularity: (NC) requires $\sum_i |c_i|^2 = 1$, while the conclusion is $P(c_i) = |c_i|^2$. It helps to separate three distinct roles of the squared modulus in the adopted model. *(1) State bookkeeping:* $|c_i|^2$ fixes the normalization weights in (NC) on amplitude lists, independently of any probability law. *(2) Marginal sums in Definition 4.10:* that condition sums joint probabilities over subsystem-2 outcomes whose squared moduli form a partition of unity; the same exponent 2 enters that sum. *(3) Target probability law:* the uniqueness theorems identify

f such that $P(c_i) = f(|c_i|) = |c_i|^2$. The proof does *not* derive the exponent 2 in (NC) from the cost axioms; (NC) is an explicit modeling bridge to standard quantum amplitude normalization. A variant model that replaced (NC) by a different p-norm convention $\sum_i |c_i|^p = 1$ would require rephrasing both the marginal sums of Definition 4.10 and the uniqueness step; the Cauchy-equation route used here is tied to the adopted Euclidean ($p = 2$) convention. We do not pursue such variants in this paper.

Route A additionally requires:

(MA) *Modulus Multiplicativity.* $P(c_1 c_2) = P(c_1)P(c_2)$ for all independent amplitude pairs. This is the decisive assumption in Route A: once it is in place, the quadratic rule $P(c) = |c|^2$ follows by a purely classical functional-equation argument without any RS-specific geometric input (Theorem 5.1).

Remark 4.2 (Two-layer role of the RS framework) The RS framework supplies the objects used here—the cost functional J, ledger bookkeeping, the path-action convention, and disjoint-region subsystem language—and motivates several scalar probability choices (phase independence, modulus-based laws) without deriving them from (A1)–(A2). Once those choices are adopted, Routes A and B proceed by standard scalar functional-equation reasoning; the paper's contribution is to isolate which assumptions force the quadratic law and which remain open (Section 1).

Route A is the classical scalar uniqueness route: given (SA), (NC), (CS), and (MA), the quadratic rule $P = |c|^2$ follows by the standard Cauchy-equation argument (Theorem 5.1). Route B does not use (MA) or (CS) as primitive scalar axioms; it replaces them with the binary subsystem package (PA)+$(\mathrm{P5})_2$ to derive $f(r) = r^2$ on normalized moduli (Proposition 5.6). The structural work is thereby relocated from a primitive scalar product law to subsystem composition and binary marginal invariance (Definition 4.10).

The scalar probability function is formalized below, followed by the Route A requirements and the Route B subsystem package.

Definition 4.3 (Probability Function) A probability function P assigns to each outcome amplitude c_i in a fixed measurement context a real number $P(c_i)$ representing the probability of outcome i. The notation suppresses dependence on the full normalized state and measurement basis; the structural requirements below concern the functional dependence on the single amplitude argument.

This suppression carries a strong scalar-reduction assumption: $P(c_i)$ depends on c_i alone, not on the full normalized state, the measurement basis, or which other outcomes are jointly measurable. The setup is therefore narrower than Gleason-type frame-function analyses, which formulate probability assignments on full measurement contexts under a noncontextuality condition. Within the present RS framework the reduced form $P(c_i) = f(|c_i|)$ is adopted because the cost mechanism is defined on positive scale ratios and is blind to phase (Remark 3.10). That alignment is motivational only. The paper therefore analyzes this restricted scalar class, not general contextual or state-dependent probability assignments.

Definition 4.4 (Scalar probability requirements used in Route A) A scalar probability function in the class (SA) is required to satisfy all five of the following conditions in Route A. Conditions (i)–(iii) are shared with Route B, while conditions (iv)–(v) are Route-A-specific. Condition (i) follows immediately from (iii)/(SA), since $f : [0, \infty) \to [0, \infty)$ forces $P(c_i) = f(|c_i|) \geq 0$; it is listed explicitly for direct comparison with standard Born-rule axiom sets.

(i) *Non-negativity:* $P(c_i) \geq 0$ for all c_i (subsumed by (SA)).

(ii) *Normalization* [(NC)]: $\sum_i P(c_i) = 1$ whenever $\sum_i |c_i|^2 = 1$.

(iii) *Phase independence* [(SA)]: $P(c_i) = f(|c_i|)$ for some $f : [0, \infty) \to [0, \infty)$.

(iv) *Modulus multiplicativity* [(MA)]: $P(c_1 c_2) = P(c_1)P(c_2)$ for independent amplitude pairs.

(v) *Continuity* [(CS)]: f is continuous on $[0, \infty)$.

Requirements (i)–(iii) are shared by both routes; multiplicativity (iv) and continuity (v) are Route A only. Item (i) is subsumed by (SA). Item (ii) is the adopted (NC); Section 6 records a two-outcome internal check, not a derivation of (NC) from the cost axioms. Item (iii)/(SA) is motivated by phase-blind costs but remains an adopted modeling restriction, not a general noncontextuality theorem.

Route B replaces multiplicativity (iv)/(MA) by the binary subsystem package. The *axioms* in that package are: (PA) (product-amplitude composition, Remark 4.9) and $(P5)_2$ (binary marginal invariance, Definition 4.10). Cost additivity over disjoint regions (Definition 4.5) then yields weight multiplicativity as a *theorem* (Proposition 4.7), which in turn yields modulus multiplicativity (Corollary 4.8). Route B therefore does not posit weight multiplicativity as an axiom; it derives it from the disjoint-ledger structure and uses marginal invariance to close the functional-equation argument. The premise load is thus relocated rather than simply reduced.

Definition 4.5 (Composite path in the composite continuum regime) For two independent subsystems with paths γ_1 and γ_2 over disjoint ledger regions Λ_1 and Λ_2 (Definition 3.1), assume that each subsystem and their union $\Lambda_1 \cup \Lambda_2$ lie in the continuum regime of Remark 3.2. The composite path γ_{12} is then defined with integration domain $\Lambda_1 \cup \Lambda_2$. Since $\Lambda_1 \cap \Lambda_2 = \varnothing$, the continuum path-action integral over $\Lambda_1 \cup \Lambda_2$ splits by additivity of integration, yielding $C[\gamma_{12}] = C[\gamma_1] + C[\gamma_2]$.

Remark 4.6 (Composite continuum regime) The assumption that the composite system $\Lambda_1 \cup \Lambda_2$ lies in the continuum regime does not follow automatically from the individual regime conditions on Λ_1 and Λ_2 separately; it is an additional hypothesis of the product-amplitude composition model. In practice, it requires that the two subsystems' ledger regions do not interleave at a granularity that would invalidate the continuum approximation for the union. For subsystems that are spatially or temporally well-separated—the standard setting in which no-signaling is physically motivated—this condition is typically satisfied, because the disjoint regions can be ordered into two contiguous blocks, each already in the continuum regime. The case of subsystems whose ledger regions are finely interleaved at the inter-event scale falls outside the continuum regime assumed here and would require a discrete-sum treatment not developed in this paper.

Proposition 4.7 (Composite path-action weight multiplicativity) *If independent subsystems occupy disjoint ledger regions in the composite continuum regime of Definition 4.5, then the path weight $w = e^{-C}$ is multiplicative: $w_{12} = w_1 \cdot w_2$.*

Proof By Definition 4.5, $C_{12} = C_1 + C_2$. Exponentiating: $w_{12} = e^{-C_{12}} = e^{-C_1} \cdot e^{-C_2} = w_1 w_2$. □

Corollary 4.8 (Composite-path modulus multiplicativity) *For independent subsystems, the joint amplitude modulus is the product of the individual moduli: $|\psi_{12}| = |\psi_1| \cdot |\psi_2|$.*

Proof By the $C = 2A$ bridge (Remark 3.9), $|\psi| = e^{-C/2}$. Weight multiplicativity (Proposition 4.7) gives $|\psi_{12}| = e^{-C_{12}/2} = e^{-C_1/2} \cdot e^{-C_2/2} = |\psi_1| \cdot |\psi_2|$. □

Remark 4.9 (Operational composition of amplitudes) For independent subsystems we model joint outcomes by product amplitudes: if subsystem 1 has outcome amplitude c_{1i} and subsystem 2 has outcome amplitude c_{2j}, then the joint outcome (i, j) is assigned amplitude $c_{1i}c_{2j}$. Corollary 4.8 supplies the modulus part of this rule; the additive phase convention $\phi_{ij} = \phi_i + \phi_j$ is adopted and is not derived from (A1)–(A2). The product-amplitude rule for independent subsystems is the RS analogue of the tensor-product postulate in standard quantum mechanics; it is a substantive structural assumption. Proposition 5.6 uses this product-amplitude representation as part of the subsystem model.

Definition 4.10 (Binary marginal invariance under the adopted 2-norm convention) For subsystems occupying disjoint ledger regions and composed according to the product-amplitude model of Remark 4.9, the marginal probability of outcome i for subsystem 1 is invariant under changes of the measurement performed on subsystem 2. This condition is imposed only on normalized subsystem-1 outcome moduli $0 \le |c_{1i}| \le 1$, which is the physical probability domain for normalized amplitude lists. The subsystem-2 outcomes $\{j\}$ have amplitudes $\{c_{2j}\}$ satisfying the adopted normalization convention

$$\sum_j |c_{2j}|^2 = 1. \tag{9}$$

Thus

$$\sum_j P_{\text{joint}}(c_{1i} \cdot c_{2j}) = P(c_{1i}) \tag{10}$$

for every choice of measurement basis for subsystem 2 and every normalized subsystem-1 outcome modulus $|c_{1i}| = r \in [0, 1]$. Here $P_{\text{joint}}(c_{1i} \cdot c_{2j}) := f(|c_{1i} \cdot c_{2j}|) = f(|c_{1i}||c_{2j}|)$ denotes the scalar rule (SA) applied to the product amplitude; this application of (SA) to joint outcomes is an explicit hypothesis of the model, not a consequence of (SA) alone. The condition expresses marginal invariance of subsystem-1 probabilities under subsystem-2 measurement choices, specialized to the present scalar product-amplitude model; it is not a reconstruction-level no-signaling principle over a full operational state space, and contextual or nonlinear probability assignments are outside its scope. For the uniqueness proof below, it suffices to impose this condition for binary measurements on subsystem 2; this binary restriction is denoted $(\text{P5})_2$.

Remark 4.11 (Binary versus larger finite measurements on subsystem 2) Lemma 5.5 uses only two outcomes on subsystem 2. Once Proposition 5.6 has identified $f(r) = r^2$ on $0 \le r \le 1$, the full marginal sum in (10) holds for *every* finite subsystem-2 measurement satisfying $\sum_j |c_{2j}|^2 = 1$, because $\sum_j |c_{1i}c_{2j}|^2 = |c_{1i}|^2 \sum_j |c_{2j}|^2$ and probabilities add to $|c_{1i}|^2$. Hence, within the concluded quadratic scalar class on normalized moduli, binary marginal invariance is not weaker than the finite-outcome restriction of Definition 4.10. If some of the other hypotheses of Proposition 5.6 are weakened—for example scalar reduction or normalization—then the relation between binary marginal invariance and higher-outcome marginal invariance becomes a separate question. Under the full hypotheses of Proposition 5.6, however, binary marginal invariance already forces $f(r) = r^2$ on the normalized domain, and hence finite-outcome marginal invariance follows.

This completes the assumption inventory. The two scalar derivations—Route A as benchmark and Route B as the marginal invariance route—are given in Section 5.

Table 1 summarizes the premise load of each main result. An entry "✓" means the assumption is directly invoked in the proof; "—" means it is not used.

Table 1 Assumption dependency map. (A1)–(A2): RS cost axioms. (SA): scalar ansatz. (NC): 2-norm convention. (CS): continuity of scalar rule f, used only in Route A and distinct from the continuity of J used in Thm. 3.4. (MA): modulus multiplicativity (Route A only). (PA): product-amplitude composition. $(P5)_2$: binary marginal invariance (Definition 4.10). (G1)–(G4): binary-split geometry. Corollary 5.11 uses the Route B functional result to extend the scalar probability rule to finite normalized amplitude lists; it does not invoke the geometric package (G1)–(G4).

Result	**(A1)–(A2)**	**(SA)**	**(NC)**	**(CS)**	**(MA)**	**(PA)**	**$(P5)_2$**	**(G1)–(G4)**
Lem. 3.3 / Thm. 3.4	✓	—	—	—*	—	—	—	—
Prop. 4.7 (path-action mult.)	—‡	—	—	—	—	—	—	—
Cor. 4.8 (modulus mult.)	—‡	—	—	—	—	—†	—	—
Thm. 5.1 (Route A)	—	✓	✓	✓	✓	—	—	—
Prop. 5.6 (Route B core)	—	✓	✓	—	—	✓	✓	—
Cor. 5.11 (finite norm. amp.-list ext.)	—	✓	✓	—	—	✓	✓	—
Rem. 6.1 (normalization check)	—	—	—	—	—	—	—	✓
Prop. 6.2 (path consistency)	✓	—	—	—	—	—	—	✓
Thm. 6.6 (packaged result)	—	✓	✓	—	—	✓	✓	✓
Prop. 6.11 (T_{bin} scope)	—	—	—	—	—	—	—	✓

* Thm. 3.4 uses continuity of the cost functional $J : (0, \infty) \to \mathbb{R}$; the later non-negativity of J is a conclusion of that theorem, not part of its hypothesis. This continuity assumption is logically distinct from (CS), the continuity of the scalar probability function $f : [0, \infty) \to [0, \infty)$.

† Cor. 4.8 establishes $|\psi_{12}| = |\psi_1| \cdot |\psi_2|$ for composite *path* amplitudes using cost additivity and the $C = 2A$ bridge alone; it does not use (PA). (PA) is the outcome amplitude product rule; the corollary provides motivation for (PA) but does not use it as input.

‡ Prop. 4.7 and Cor. 4.8 use the composite path-action additivity convention of Definition 4.5; they do not use the uniqueness axioms (A1)–(A2) as functional-equation hypotheses.

Route A (Thm. 5.1) and Route B (Prop. 5.6) differ in route-specific hypotheses—(CS)+(MA) versus (PA)+$(P5)_2$—while sharing (SA) and (NC). The geometric package (G1)–(G4) enters only in Section 6 and in Thm. 6.6, not in the abstract scalar argument Prop. 5.6.

5 Conditional Scalar Uniqueness for the Quadratic Rule

This section gives two independent scalar routes to $P(c) = |c|^2$ within the adopted model. Section 5.2 records the benchmark calculation with primitive multiplicativity (MA): the substitution $g(x) = \ln f(e^x)$ turns $f(r_1 r_2) = f(r_1)f(r_2)$ into the additive Cauchy equation, and normalization then fixes the quadratic exponent. Section 5.3 replaces (MA) by the marginal invariance subsystem package of Section 4; there the substitution $h(x) = f(\sqrt{x})$ reaches interval additivity directly from the binary marginal invariance constraint.

5.1 Logical core of the scalar argument

The cost axioms (A1)–(A2), Theorem 3.4, and the ledger–path material of Section 3 motivate the scalar amplitude picture and the product rule for composite path moduli (Corollary 4.8), but they are not used in the derivation of the quadratic scalar law in Route B. The dependencies are recorded in Table 1: Theorem 5.1 (Route A) invokes (SA), (NC), (CS), and (MA), while Proposition 5.6 (Route B) invokes (SA), (NC), (PA), and $(P5)_2$. Corollary 5.11 likewise uses only the Route B scalar hypotheses. The driver of $f(r) = r^2$ in Route B is therefore the scalar subsystem package together with non-negativity of probabilities, not the reciprocal cost functional itself. The RS narrative in Sections 3–4 explains where (PA) and disjoint-subsystem language come from and aligns this work with the broader program; a reader interested only in the functional proof may treat Proposition 5.6 as self-contained once (a)–(c) of its hypotheses are accepted. Neither the continuum path-action idealization (Remarks 3.2 and 4.6) nor the closed form of J enters the proof of Proposition 5.6.

5.2 Route A: Benchmark Under Primitive Multiplicativity

Route A is the benchmark calculation: once one adopts the scalar ansatz (SA), the 2-norm normalization convention (NC), continuity (CS), and multiplicativity (MA), the quadratic rule follows by a standard functional-equation argument. This makes the classical premise load explicit before Proposition 5.6 (Route B) replaces (MA) by the binary subsystem package of Section 4.

Theorem 5.1 (Route A benchmark: scalar uniqueness under primitive multiplicativity) *Let P be a scalar probability function satisfying the Route A requirements (i)–(v) of Definition 4.4. Then $P(c) = |c|^2$ is the unique scalar probability function satisfying those conditions.*

Remark 5.2 (Independent conventions in Route A) The five Route A requirements (i)–(v) invoke exactly four independent adopted conventions: (SA), (NC), (CS), and (MA). Condition (i) (non-negativity $P(c) \geq 0$) is subsumed by condition (iii)/(SA), since $f : [0, \infty) \to [0, \infty)$ is non-negative by domain. No subsystem-composition argument, marginal invariance condition, or RS-specific geometric input enters the derivation beyond the scalar-reduction class.

Proof Step 1: Scalar reduction and regularity. By (SA), write $P(c) = f(|c|)$ for some $f : [0,\infty) \to [0,\infty)$. By (CS), f is continuous. For $r_1, r_2 > 0$, multiplicativity (MA) gives

$$f(r_1 r_2) = f(r_1) f(r_2) \qquad (r_1, r_2 > 0).$$

Step 2: Establish $f(1) = 1$ and $f(0) = 0$. Applying (MA) with $r_1 = r_2 = 1$ gives $f(1) = f(1)^2$, so $f(1) \in \{0, 1\}$. Normalization (NC) on the balanced two-outcome state with $|\alpha|^2 + |\beta|^2 = 1$ and $|\alpha| = |\beta| = 1/\sqrt{2}$ gives $2f(1/\sqrt{2}) = 1$, hence $f(1/\sqrt{2}) = 1/2 > 0$. Since $f(1)f(1/\sqrt{2}) = f(1/\sqrt{2}) > 0$, we must have $f(1) = 1$. Normalization (NC) on the degenerate state $(1, 0)$ with $1^2 + 0^2 = 1$ then gives $f(1) + f(0) = 1$, hence $f(0) = 0$.

Step 3: Positivity on $(0,\infty)$. If $f(r_0) = 0$ for some $r_0 > 0$, then $f(1) = f(r_0) f(r_0^{-1}) = 0$, contradicting $f(1) = 1$. Hence $f(r) > 0$ for all $r > 0$.

Step 4: Reduce to the additive Cauchy equation. Since $f(r) > 0$ for all $r > 0$ (Step 3), the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) := \ln f(e^x)$ is well-defined. Continuity and positivity of f on $(0,\infty)$ together imply continuity of g. Multiplicativity of f becomes

$$g(x + y) = g(x) + g(y).$$

By the standard classification of continuous additive functions [1, 3], $g(x) = kx$ for some $k \in \mathbb{R}$, so $f(r) = r^k$ for $r > 0$.

Step 5: Pin $k = 2$ by normalization. Apply (NC) to a two-outcome state with amplitudes α, β satisfying $|\alpha|^2 + |\beta|^2 = 1$. Setting $p = |\alpha|^2 \in (0, 1)$ gives $p^{k/2} + (1 - p)^{k/2} = 1$ for all $p \in (0, 1)$. At $p = 1/2$: $2(1/2)^{k/2} = 1$, so $(k/2)\ln(1/2) = \ln(1/2)$. Since $\ln(1/2) \neq 0$, $k = 2$. The resulting form $f(r) = r^2$ then satisfies $p+(1-p) = 1$ for every $p \in (0, 1)$, confirming consistency with (NC) on the full balanced-amplitude family. Therefore $f(r) = r^2$ and $P(c) = |c|^2$. □

Remark 5.3 (Scope limitation of Route A) The decisive scalar product law (MA) is adopted rather than derived in Route A, so the route is diagnostic rather than foundational. What Route A clarifies is the classical premise set—(SA), (NC), (CS), (MA)—and thereby makes visible the additional work that Proposition 5.6 (Route B) performs: replacing (MA) by the binary marginal invariance subsystem package (PA)+$(\mathrm{P5})_2$.

5.3 Route B: Scalar Uniqueness from Binary Marginal Invariance

Route B works on the normalized amplitude domain. If $|c_i|$ is an outcome modulus in a normalized list, then $0 \leq |c_i| \leq 1$, and products with normalized subsystem-2 moduli remain in the same interval. The proof therefore needs only a scalar rule $f : [0, 1] \to [0,\infty)$ on physical probability inputs. Once binary marginal invariance gives additivity for $h(x) = f(\sqrt{x})$ on the unit interval, non-negativity of probabilities implies monotonicity and rules out the usual Cauchy pathologies. Thus Route B does not require continuity (CS).

Remark 5.4 (Why continuity is unnecessary in Route B) Continuity remains part of Route A as written because the logarithmic reduction of multiplicativity reaches an additive equation on all of $\mathbb{R}$ and then appeals to the standard continuous Cauchy classification. Route B has a stronger positivity structure: h maps the unit interval into $[0,\infty)$, and additivity on subinterval sums gives $h(y) = h(x) + h(y - x) \geq h(x)$ whenever $0 \leq x \leq y \leq 1$. This monotonicity is derived, not assumed. A monotone additive function on an interval is linear, so no separate continuity, measurability, or boundedness postulate is needed for Proposition 5.6.

Lemma 5.5 (Binary subsystem model implies the scalar functional equation) *Under the adopted binary subsystem model—product-amplitude composition (PA), scalar ansatz (SA), 2-norm convention (NC), and binary marginal invariance (P5)$_2$ from Definition 4.10—the scalar probability function f satisfies*

$$f(r \cdot s) + f\left(r\sqrt{1-s^2}\right) = f(r) \tag{11}$$

for all $0 < r \leq 1$ and $s \in (0,1)$. This restricted marginal invariance requirement is sufficient for Proposition 5.6: only binary measurements on subsystem 2 are used. The full condition of Definition 4.10 implies (11), so the proposition remains valid under the stronger condition as well.

Proof By (PA), the joint outcome (i,j) has amplitude $c_{1i} \cdot c_{2j}$. By (SA), $P_{\text{joint}}(c_{1i} \cdot c_{2j}) = f(|c_{1i}||c_{2j}|)$. Set $r = |c_{1i}| \in (0,1]$, and write the two subsystem-2 binary outcome moduli as $|c_{21}| = s$ and $|c_{22}| = \sqrt{1-s^2}$, so that $s^2 + (1-s^2) = 1$ satisfies the (NC) constraint on subsystem 2. The marginal invariance equation (10) of Definition 4.10 then becomes exactly $f(rs) + f(r\sqrt{1-s^2}) = f(r)$. Each of (PA), (SA), (NC), (P5)$_2$ is needed; none alone suffices. □

The functional equation (11) is precisely where the Route B premise load enters. Proposition 5.6 then gives a scalar uniqueness argument from that equation, normalization, and non-negativity of probabilities.

Proposition 5.6 (Conditional scalar uniqueness in the adopted binary subsystem model) *Let $P(c) = f(|c|)$ be a scalar probability law on normalized amplitude moduli, with $f : [0,1] \to [0,\infty)$, and assume that*

(a) Scalar reduction *(SA): $P(c) = f(|c|)$,*
(b) Normalization *(NC): $\sum_i f(|c_i|) = 1$ whenever $\sum_i |c_i|^2 = 1$,*
(c) Functional equation*: $f(r \cdot s) + f\left(r\sqrt{1-s^2}\right) = f(r)$ for all $0 < r \leq 1$, $s \in (0,1)$.*

Then $f(r) = r^2$ for all $0 \leq r \leq 1$, and hence $P(c) = |c|^2$ for normalized outcome amplitudes, uniquely within this scalar class. Condition (c) is supplied physically by Lemma 5.5 via the binary marginal invariance subsystem model.

Proof Step 0: Normalize the endpoints. Apply (NC) to the balanced binary state with moduli $(1/\sqrt{2}, 1/\sqrt{2})$. This gives $2f(1/\sqrt{2}) = 1$, so $f(1/\sqrt{2}) = 1/2$. Applying the functional equation (c) with $r = 1$ and $s = 1/\sqrt{2}$ gives

$$f(1/\sqrt{2}) + f(1/\sqrt{2}) = f(1),$$

hence $f(1) = 1$. Applying (NC) to the degenerate binary state with moduli $(1,0)$ then gives $f(1) + f(0) = 1$, so $f(0) = 0$.

Define $h : [0,1] \to [0,\infty)$ by $h(x) := f(\sqrt{x})$. By Step 0, $h(0) = 0$ and $h(1) = 1$.

Step 1: Reduce to additivity on the unit interval. For any $0 < r \leq 1$ and $s \in (0,1)$, condition (c) gives $f(rs) + f(r\sqrt{1-s^2}) = f(r)$. Setting $R := r^2 \in (0,1]$ and $p := s^2 \in (0,1)$, so that $rs = \sqrt{Rp}$ and $r\sqrt{1-s^2} = \sqrt{R(1-p)}$, this becomes

$$h(Rp) + h(R(1-p)) = h(R).$$

Step 2: All positive pairs with normalized sum. For any $x, y > 0$ with $x + y \leq 1$, set $R := x + y$ and $p := x/(x + y) \in (0, 1)$. Then $Rp = x$ and $R(1 - p) = y$, so the identity from Step 1 gives

$$h(x) + h(y) = h(x + y), \qquad x, y > 0, \quad x + y \leq 1. \tag{12}$$

Together with $h(0) = 0$, this also covers the cases $x = 0$ or $y = 0$.

Step 3: Non-negativity gives monotonicity. If $0 \leq x \leq y \leq 1$, then $y = x + (y - x)$ and Step 2 gives

$$h(y) = h(x) + h(y - x).$$

Since h takes values in $[0, \infty)$, $h(y) \geq h(x)$. Thus h is non-decreasing on $[0, 1]$.

Step 4: Identify h. Additivity and $h(1) = 1$ first fix rational points. For $n \in \mathbb{N}$, repeated use of Step 2 gives $n\,h(1/n) = h(1) = 1$, so $h(1/n) = 1/n$; similarly $h(m/n) = m/n$ for integers $0 \leq m \leq n$. For an arbitrary $x \in [0, 1]$, choose rational sequences $q_k \uparrow x$ and $r_k \downarrow x$. Monotonicity gives

$$q_k = h(q_k) \leq h(x) \leq h(r_k) = r_k,$$

and letting $k \to \infty$ yields $h(x) = x$.

Step 5: Conclude on normalized amplitudes. For $0 \leq r \leq 1$, $f(r) = h(r^2) = r^2$. Hence $P(c) = f(|c|) = |c|^2$ for every normalized outcome amplitude. Uniqueness follows from the interval-additivity and monotonicity argument above. □

Remark 5.7 (Partial minimality and the role of non-negativity) The conditions in Proposition 5.6 separate normalization, non-negativity, and the marginal-invariance functional equation within the adopted scalar subsystem class.

Necessity of (NC) [condition (b)]. Without the normalization condition, every function $f(r) = c \cdot r^2$ with constant $c > 0$ satisfies scalar reduction and the functional equation: the equation is verified by $c \cdot r^2(s^2 + (1 - s^2)) = c \cdot r^2$. Only (NC) fixes $f(1) = 1$, thereby pinning $c = 1$ and uniquely selecting $f(r) = r^2$.

Role of non-negativity. The codomain $f : [0, 1] \to [0, \infty)$ is doing real mathematical work in Route B. If one allowed signed real-valued functions after Step 2, additive non-linear Cauchy-type pathologies could survive on rationally generated substructures. Non-negativity turns interval additivity into monotonicity, and monotonicity forces linearity. Thus Route B replaces an explicit continuity postulate with the ordinary positivity requirement for probabilities.

Necessity of (c) [the functional equation]. Without the marginal invariance functional equation, scalar reduction and normalization alone place no constraint on the functional form of f beyond non-negativity and normalization on lists. No uniqueness conclusion is obtainable from (a)–(b) alone; the functional equation is the structural link that forces interval additivity.

Status of (SA) [condition (a)]. Whether (SA) is necessary—i.e., whether any phase-dependent probability function could satisfy (b)–(c) and yet differ from $|c|^2$—depends on extending condition (c) to phase-dependent probability assignments, which lies outside the scalar class of this paper. This question is related to open problem 1 of Section 8.2.

Together, these observations show that the normalization condition, non-negativity, and the marginal-invariance functional equation do independent work. Full minimality of the package (SA)+(NC)+(PA)+$(P5)_2$ in the sense that no proper subset suffices remains open; these remarks establish only the indicated independence claims within the specific scalar subsystem class.

Remark 5.8 (Optional homogeneous intensity extension) Proposition 5.6 proves the probability rule only on the normalized domain $0 \leq r \leq 1$, which is the domain needed for normalized amplitude lists. If one wants an all-positive formula, it should be read as an unnormalized

intensity extension rather than as a probability outside normalized contexts. Define $H : [0,\infty) \to [0,\infty)$ from h by choosing any integer n with $x/n \le 1$ and setting

$$H(x) := n\, h(x/n).$$

Since Proposition 5.6 gives $h(t) = t$ on $[0,1]$, this extension is well-defined and gives $H(x) = x$ for all $x \ge 0$. The corresponding homogeneous intensity law is $F(r) := H(r^2) = r^2$ for $r \ge 0$. This extension is useful algebraically, but only the restriction $0 \le r \le 1$ is used as a probability law in the normalized measurement setting.

Remark 5.9 (Multiplicativity as a corollary of Route B) Route B (Proposition 5.6) derives $f(r) = r^2$ on $0 \le r \le 1$ without ever invoking (MA). Once $f = r^2$ is established on normalized moduli, modulus-multiplicativity on that domain is immediately verified: if $0 \le r_1, r_2 \le 1$, then $f(r_1 r_2) = (r_1 r_2)^2 = r_1^2 r_2^2 = f(r_1) f(r_2)$. Thus the normalized-domain form of (MA) is a *verified corollary of the conclusion* of Route B, not an intermediate step within its proof. The all-positive form belongs to the optional intensity extension of Remark 5.8. This contrasts with Route A, in which (MA) is an input axiom.

The full constructive chain that motivates and connects the Route B hypotheses to each other spans Sections 3–5:

$$\underbrace{\text{disjoint ledger regions} \to \text{additive path actions} \to \text{mult. weights (Prop. 4.7)}}_{\text{Sec. 3}}$$

$$\to \underbrace{\text{product moduli (Cor. 4.8)} \xrightarrow{+\text{phase convention}} \text{product amplitudes (Rem. 4.9)}}_{\text{Sec. 4}}$$

$$\to \underbrace{\text{(PA)} + (\text{P5})_2 + \text{(SA)} + \text{(NC)} \to f(r) = r^2 \text{ on } [0,1] \text{ (Prop. 5.6)}}_{\text{Sec. 5}}.$$

The "+phase convention" label indicates that (PA) requires adopting $\phi_{ij} = \phi_i + \phi_j$ in addition to the modulus rule from Cor. 4.8; this phase stipulation is not derived from the cost axioms. What Route B establishes is that, within this premise package, (MA) need not be posited separately as a scalar probability axiom.

Remark 5.10 ((MA) versus (PA)+$(P5)_2$: packages and logical independence claims) Product-amplitude composition (PA) is a substantive *amplitude-level* tensor rule (including the stipulated additive phase law; Remark 4.9). Modulus multiplicativity (MA) is a *probability-level* constraint $f(r_1 r_2) = f(r_1) f(r_2)$ on the scalar law. These are not the same postulate: (PA) does not mention P, and (MA) does not determine joint amplitudes. Nor are the two route packages interchangeable without the rest of the axioms. On the one hand, (PA)+$(P5)_2$ together with (SA) and (NC) do *not* assume (MA) as an input; Proposition 5.6 derives $f = r^2$ on normalized moduli and only then recovers normalized-domain (MA) as a corollary (Remark 5.9). On the other hand, (MA)+(SA)+(NC)+(CS) alone impose marginal-invariance-type constraints only insofar as they force the same quadratic f; they do not encode the full joint-amplitude modeling content of (PA).

What $(P5)_2$ adds on top of (PA), within the adopted scalar class, is not "more tensor product" but a *consistency constraint on marginals*: it forbids any non-negative scalar rule that might otherwise be compatible with product joint *amplitudes* yet yield subsystem-1 marginals that vary with the choice of binary measurement on subsystem 2. Once f is known to be quadratic on normalized moduli, binary and finite-outcome marginal invariance coincide in the finite normalized setting described in Remark 4.11. A reader who regards (PA) as already "quantum-like" should read Route B as relocating the *probability* work from a primitive scalar

product law to this marginal consistency requirement, rather than as deriving tensor structure from marginal invariance alone.

Corollary 5.11 (Finite normalized amplitude-list extension) *Let $f : [0,1] \to [0,\infty)$ satisfy the hypotheses of Proposition 5.6. Then for every finite amplitude list $(c_1, \ldots, c_n)$ satisfying*

$$\sum_{i=1}^{n} |c_i|^2 = 1,$$

the unique scalar probability assignment is

$$P(c_i) = f(|c_i|) = |c_i|^2, \qquad i = 1, \ldots, n,$$

and hence

$$\sum_{i=1}^{n} P(c_i) = 1.$$

Proof Proposition 5.6 gives $f(r) = r^2$ for all $0 \le r \le 1$, so the conclusion follows immediately because every entry in a normalized amplitude list has modulus at most 1. Equivalently, the proof of Proposition 5.6 defines $h(x) = f(\sqrt{x})$ and derives interval additivity $h(x+y) = h(x)+h(y)$ for $x, y \ge 0$ with $x+y \le 1$, together with $h(1) = 1$. By induction, $h(x_1+\cdots+x_n) = h(x_1)+\cdots+h(x_n)$ for every finite list of nonnegative x_i with sum at most 1. Setting $x_i = |c_i|^2$ and using $\sum_i |c_i|^2 = 1$ gives $\sum_i f(|c_i|) = h(1) = 1$ and $f(|c_i|) = |c_i|^2$ for each i. □

Remark 5.12 (Scope of the finite normalized amplitude-list extension) Corollary 5.11 extends the scalar functional result to finite normalized amplitude lists. Section 6 keeps the geometric packaging two-outcome; broader measurement geometry is outside the paper's stated scope (Section 1).

6 Two-Outcome Geometry and the Main Theorem

This section records consistency checks for the adopted binary-split geometry (G1)–(G4), then states the packaging theorem (Theorem 6.6) and Proposition 6.11. The functional content is Proposition 5.6; what follows fixes (G4) among the geometry clauses and records a scope limitation for T_{bin}.

6.1 Consistency Checks for the Two-Outcome Geometry

This subsection records two consistency checks for the adopted two-outcome geometry (G1)–(G4). Remark 6.1 gives the normalization identity from (G4) alone, and Proposition 6.2 checks compatibility with a finite auxiliary half-action accumulation (the raw path-action integral diverges on the semi-infinite branches). Both assume (NC) as in Proposition 5.6. The comparison of (G1)–(G4) with Hilbert-space structure is Remark 6.5.

The compatibility calculation uses four additional model clauses beyond the cost axioms (A1)–(A2) and their consequence $J(1) = 0$ (Lemma 3.3). These clauses define the adopted two-outcome measurement geometry; they are not consequences of the cost axioms alone:

(G1) **Angular chart.** Positive scale ratios $r > 0$ are parametrized by an angle $\theta \in (0, \pi/2)$ via $r = \tan\theta$. The balance point $r = 1$ corresponds to $\theta = \pi/4$, while $r \to 0$ and $r \to \infty$ correspond to $\theta \to 0$ and $\theta \to \pi/2$. The normalized ratio vector $(r, 1)/\sqrt{1 + r^2}$ then has Cartesian components $(\sin\theta, \cos\theta)$; these are used to define the branch moduli in clause (G4).

(G2) **Two-branch structure.** A binary measurement partitions outcomes into two branches. The branch point is parametrized by an angle $\theta_s \in (0, \pi/2)$. The complementary branch is assigned the angle $\pi/2 - \theta_s$, motivated by the reciprocal symmetry $J(x) = J(1/x)$: since $\tan(\pi/2 - \theta) = 1/\tan\theta$, the two branches correspond to reciprocally related ratios.

(G3) **Path parametrization.** For branch 1, the path from the branch point to the boundary is described by the relative ratio $r(\theta) = \tan\theta/\tan\theta_s$ as θ ranges from θ_s to $\pi/2$. This is the monotone path used in the proof of Proposition 6.2.

(G4) **Angular projection.** The branch moduli are defined as the Cartesian components of the unit vector at angle θ_s in the angular chart (G1): $|\psi_1| = \sin\theta_s$ and $|\psi_2| = \cos\theta_s$. In the auxiliary path-compatibility calculation below, this endpoint assignment is extended along the branch by the natural continuation $|\psi(\theta)| = \sin\theta$. Whether the endpoint assignment, or its path-continuation version, is a logical consequence of (G1)–(G3) alone or independently encodes Hilbert-space content is left open.

Definitions (G1)–(G4) belong to the specified measurement model; they are not consequences of (A1)–(A2) alone. No Riemannian metric, geodesic extremality, or variational uniqueness claim is used here.

Remark 6.1 (Two-outcome normalization under (G4)) Under the adopted angular-projection clause (G4), $|\psi_1| = \sin\theta_s$ and $|\psi_2| = \cos\theta_s$. Branch-modulus normalization $|\psi_1|^2 + |\psi_2|^2 = 1$ is then immediate from $\sin^2\theta_s + \cos^2\theta_s = 1$. The clauses (G1)–(G3) play no role; this is a direct consequence of (G4) alone.

Proposition 6.2 (Auxiliary half-action compatibility with the adopted branch moduli) *Under (G1)–(G4), the path-continuation convention stated in (G4), and the $C = 2A$ bridge (Remark 3.9), the finite auxiliary half-action accumulation integral $A_i = \int q(u)\,du$ is compatible with the trigonometric moduli $|\psi_1| = \sin\theta_s$ and $|\psi_2| = \cos\theta_s$ defined by (G4). This proposition is a model-internal compatibility calculation, not a derivation of (G4) from the cost axioms: the raw path-action integral $C_i = \int J(r)\,d(\ln r)$ diverges on the semi-infinite branches, so the finite quantity established here is the auxiliary accumulation density $q(u) = \cos^2\theta(u)$ rather than the cost functional J directly.*

Proof Step 1: Branch 1. By (G3), the relative ratio along branch 1 is $r(\theta) = \tan\theta/\tan\theta_s$ for $\theta \in [\theta_s, \pi/2)$. Pass to the log-ratio coordinate

$$u = \ln r = \ln(\tan\theta/\tan\theta_s),$$

so that $u = 0$ at $\theta = \theta_s$ and $u \to \infty$ as $\theta \to \pi/2$. Differentiating gives

$$\frac{d\theta}{du} = \sin\theta\cos\theta.$$

Using the path-continuation convention in (G4), along branch 1 the modulus is $|\psi(\theta)| = \sin\theta$. Hence

$$A(\theta) = -\ln(\sin\theta), \qquad \frac{dA}{d\theta} = -\cot\theta.$$

Therefore the half-action accumulation rate

$$q(u) := -\frac{dA}{du}$$

is

$$q(u) = \cot\theta\,\frac{d\theta}{du} = \cos^2\theta(u).$$

To express q explicitly in u: from $u = \ln(\tan\theta/\tan\theta_s)$ one has $\tan\theta = e^u\tan\theta_s$, so

$$q(u) = \cos^2\theta = \frac{1}{1+\tan^2\theta} = \frac{1}{1+e^{2u}\tan^2\theta_s}.$$

The half-action itself is then computed by changing variable from u back to θ. Since $u = \ln(\tan\theta/\tan\theta_s)$, differentiating gives $du = d\theta/(\sin\theta\cos\theta)$, so

$$q(u)\,du = \cos^2\theta\cdot\frac{d\theta}{\sin\theta\cos\theta} = \cot\theta\,d\theta.$$

Therefore

$$\begin{aligned} A_1 &= \int_0^\infty q(u)\,du \\ &= \int_{\theta_s}^{\pi/2}\cot\theta\,d\theta \\ &= \big[\ln(\sin\theta)\big]_{\theta_s}^{\pi/2} \\ &= -\ln(\sin\theta_s). \end{aligned} \tag{13}$$

Thus the $C = 2A$ bridge gives $|\psi_1| = e^{-A_1} = \sin\theta_s$, confirming compatibility with the endpoint modulus specified in (G4).

Step 2: Complementary branch. For branch 2, the complementary angle is $\widetilde{\theta}_s = \pi/2 - \theta_s$ by (G2). The same calculation gives

$$A_2 = -\ln(\sin(\pi/2-\theta_s)) = -\ln(\cos\theta_s),$$

so that $|\psi_2| = \cos\theta_s$, again confirming compatibility with the endpoint modulus specified in (G4). □

Remark 6.3 (Convergent vs. divergent integral) The quantity evaluated in Proposition 6.2 is $A_1 = -\ln(\sin\theta_s)$, not $\frac{1}{2}C_1 = \frac{1}{2}\int J(r)\,d(\ln r)$, which diverges as $\theta\to\pi/2$; the two coincide only in the finite-interval regime. Specifically, $J(e^u) = \cosh(u)-1$ gives $C_1 = \int_0^\infty(\cosh u - 1)\,du \to \infty$, while the auxiliary accumulation integral $A_1 = \int_0^\infty q(u)\,du$ with $q(u) = \cos^2\theta(u)$ is finite and equals $-\ln(\sin\theta_s)$ by the change of variable $\theta \mapsto u = \ln(\tan\theta/\tan\theta_s)$.

Remark 6.4 (Logistic structure of the accumulation density) A brief calculation shows that the accumulation density $q(u) = \cos^2\theta(u)$ satisfies $q'(u) = -2q(u)(1-q(u))$—the logistic equation with fixed points at $q = 0$ (boundary) and $q = 1$ (branch point). This is a derived side relation of the adopted angular parametrization; it is not an independent dynamical postulate, and the proof of Proposition 6.2 does not invoke it.

Remark 6.5 (Relationship to Hilbert-space structure) The normalization analysis leaves one foundational question open: do the adopted geometric clauses (G1)–(G4) already encode Hilbert-space-equivalent content? Three observations matter.

Structural differences. The space of scale ratios $(\mathbb{R}_{>0}, \cdot)$ is a multiplicative group, not a vector space. At the level of the cost functional and ledger geometry, there is no primitive notion of state addition or scalar multiplication: the angular variable θ_s parametrizes cost comparisons rather than superpositions. Complex amplitudes and superposition enter only through the additional product-amplitude convention (PA) and phase postulate (P2) adopted in Sections 4–5, which lie beyond the cost-functional layer. The RS cost functional $J(e^t) = \cosh(t) - 1$ is also not a quadratic form in the natural coordinate $t = \ln x$: its Taylor expansion $t^2/2 + t^4/24 + \cdots$ contains higher-order terms. These differences show that the RS presentation is not trivially identical to Hilbert-space language, but by themselves they do not prove strict weakness.

Shared output. What the RS model shares with Hilbert-space geometry is the two-outcome trigonometric modulus chart and, once the scalar rule is packaged with it, the corresponding quadratic probability chart. Under the adopted geometry, (G4) alone assigns $|\psi_1| = \sin\theta_s$ and $|\psi_2| = \cos\theta_s$, and hence gives the normalization identity $\sum_i |\psi_i|^2 = 1$ by the Pythagorean identity. The probability identification $p_i = |\psi_i|^2$, however, is not supplied by (G4) alone; it enters through the conditional scalar uniqueness result packaged in Theorem 6.6. This shared output is precisely why the comparison question between the RS framework and Hilbert-space quantum mechanics remains open.

Open question: three scenarios. The relationship between (A1)–(A2) together with (G1)–(G4) and Hilbert-space structure admits three possibilities:

(i) (G1)–(G4) are equivalent to the relevant Hilbert-space fragment. In this case, the paper is a reformulation in cost-functional language rather than an independent reconstruction.

(ii) (G1)–(G4) are strictly weaker than Hilbert-space structure. In this case, the paper is a genuine reconstruction from weaker assumptions. Establishing this would require a proof, which is absent.

(iii) (G1)–(G4) are *incomparable* with Hilbert-space structure: the two frameworks share some consequences (e.g., the two-outcome trigonometric modulus chart and the packaged quadratic probability chart) but neither implies the other. In this case, the paper derives the Born rule from a genuinely different axiomatic base, which is of foundational interest, but establishing incomparability requires explicit counter-models in both directions.

The structural differences above provide grounds for distinguishing the two frameworks but do not rule out a deeper equivalence. Resolving which scenario holds remains a central open problem for the RS program. Proposition 6.11 below records a limited comparison fact: strict weakness is established only for the reduced chart T_{bin}, not for the original geometric package (G1)–(G4).

One piece of partial evidence is provided by the Lean 4 theorem `born_weight_forced` in `RS.Foundation.BornRuleForcing` (Appendix A), which proves within the Lean formalization that under the explicit two-branch calibration $w(\cos\theta) = \cos^2\theta$, the weight function is forced to be $w(r) = r^2$. This shows that the RS formalism can carry the scalar r^2 forcing step independently of Hilbert-space language, which is circumstantial evidence for scenario (ii) or (iii), but falls short of resolving which scenario holds.

6.2 Packaging: Main Theorem and Scope Limitation

Theorem 6.6 (Packaged conditional result in the adopted two-outcome model) *Within the adopted two-outcome RS model, let $P(c) = f(|c|)$ be a phase-independent probability assignment with $f : [0,1] \to [0,\infty)$. Assume:*

(i) Scalar ansatz *(SA): $P(c) = f(|c|)$ as above,*
(ii) Normalization *(NC): $\sum_i f(|c_i|) = 1$ whenever $\sum_i |c_i|^2 = 1$,*
(iii) Product-amplitude composition *(PA): Remark 4.9,*
(iv) Binary marginal invariance *(P5)$_2$: Definition 4.10 restricted to binary subsystem-2 measurements,*
(v) Angular projection *(G4): the branch moduli are the Cartesian components of the angular chart at θ_s.*

Then $P(c) = |c|^2$ is the unique scalar rule satisfying these conditions on normalized outcome amplitudes, and the adopted branch moduli satisfy $|\psi_1|^2 + |\psi_2|^2 = 1$. Clauses (G1)–(G3) fix the narrative binary-split model elsewhere in Section 6 but are not used in the proof.

Proof Assumptions (i)–(iv) supply the hypotheses of Proposition 5.6 (with the functional equation from Lemma 5.5), which yields the unique normalized-domain result $f(r) = r^2$ for $0 \le r \le 1$, hence $P(c) = |c|^2$ for normalized outcome amplitudes. The normalization identity $|\psi_1|^2 + |\psi_2|^2 = 1$ is immediate from (v)/(G4) by Remark 6.1. Together, these give the packaged two-outcome result. □

Remark 6.7 (Finite normalized amplitude-list extension versus two-outcome geometry) Theorem 6.6 packages two ingredients that should be kept distinct. The scalar functional result of Proposition 5.6, together with Corollary 5.11, extends to finite normalized amplitude lists: once $f(r) = r^2$ is forced, the assignment $P(c_i) = |c_i|^2$ is fixed for every finite list satisfying $\sum_i |c_i|^2 = 1$. By contrast, the geometric packaging via (G1)–(G4), the angular branch moduli, and the normalization identity of Remark 6.1 remain specific to the adopted two-outcome binary-split model. Thus a finite normalized amplitude-list extension is obtained without committing to a general n-outcome cost-functional measurement geometry.

Scope limitation via a reduced scalar chart

The remainder of this subsection passes to a thinner scalar binary chart T_{bin}, which retains only the two-outcome probability assignments and the complement involution. Proposition 6.11 is a scope limitation for that reduced chart; the comparison of the full package (G1)–(G4) with Hilbert-space structure remains the open question of Remark 6.5.

Definition 6.8 (Weak binary chart) Let T_{bin} denote the weak binary chart: a family of two-outcome probability assignments indexed by a positive parameter $r \in (0, \infty)$, with

$$p_1(r) = \frac{r^2}{1 + r^2}, \qquad p_2(r) = \frac{1}{1 + r^2}, \tag{14}$$

together with the complement involution $\kappa(r) = 1/r$ exchanging the two outcomes. The structure T_{bin} contains only this scalar binary probability chart; it does not include amplitude vectors, superposition, phase, incompatible measurements, or tensor-product composition.

Definition 6.9 (Completion of the binary chart) A completion of T_{bin} is an operational theory [4, 8] T with state space Ω_T, a distinguished binary measurement $E_T = \{e_T, u_T - e_T\}$, and an embedding $\iota_T : (0, \infty) \to \Omega_T$ such that $e_T(\iota_T(r)) = p_1(r)$ and $(u_T - e_T)(\iota_T(r)) = p_2(r)$ for every $r > 0$.

Two completions are compared here with their standard full effect and measurement sets. An operational equivalence is understood to preserve the convex state-space structure, effects, and measurement-compatibility relations relevant to the embedded binary chart.

Definition 6.10 (Hilbert and classical completions) T_H is the qubit theory with state space $\{\rho \in \mathbb{C}^{2\times 2} : \rho \geq 0, \operatorname{tr}\rho = 1\}$, measurement $E_Z = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, and embedding $\iota_H(r) = |\psi_r\rangle\langle\psi_r|$ where $|\psi_r\rangle = (r|0\rangle + |1\rangle)/\sqrt{1+r^2}$. A direct calculation gives $\langle\psi_r|0\rangle\langle 0|\psi_r\rangle = r^2/(1+r^2)$, so T_H is a completion of T_{bin}.

T_C is the classical bit theory with state space $\Delta_2 = \{(p, 1-p) : p \in [0,1]\}$ and the embedding $\iota_C(r) = (r^2/(1+r^2),\ 1/(1+r^2))$. By construction T_C is also a completion of T_{bin}.

Proposition 6.11 (Scope limitation: T_{bin} admits inequivalent completions) *The completions T_H and T_C of T_{bin} are inequivalent as operational theories. Consequently, the reduced scalar chart T_{bin} alone does not uniquely select Hilbert-space structure among its completions. This is a scope result for the reduced chart, not a claim about the full geometric package (G1)–(G4). The result is elementary, but it is included to prevent overinterpretation of the reduced binary chart.*

Proof Inequivalence follows because simplex state-space structure and measurement compatibility are operational invariants [4]. T_C has simplex state space Δ_2, so all its measurements are jointly measurable [8]. T_H admits incompatible measurements (E_Z and $E_X = \{|+\rangle\langle +|, |-\rangle\langle -|\}$); these are not jointly measurable in qubit theory. Since T_H and T_C differ on an operational invariant, they are inequivalent. Hence T_{bin} does not uniquely select Hilbert-space structure. □

Remark 6.12 (Scope of Proposition 6.11) This result concerns T_{bin} alone: a family of binary outcome probabilities and a complement involution, stripped of tensor-product composition, phase, incompatible measurements, and the rest of the operational apparatus assumed in Hardy- or Chiribella–D'Ariano–Perinotti-style reconstructions [9, 16]. Full reconstructions therefore prove substantially more than Proposition 6.11 refutes; the proposition only shows that *this* thin data cannot by itself distinguish a qubit completion from a classical bit completion. It prevents overinterpretation of the reduced chart as a surrogate for those richer programs, but it does not settle whether the full geometric package (G1)–(G4) is weaker than, equivalent to, or incomparable with Hilbert-space structure (Remark 6.5).

6.3 Reading of the Conditional Result

The Route B uniqueness theorem is conditional on the scalar product-amplitude model together with its normalization, product-amplitude composition, and marginal-invariance assumptions (Section 1); it is not a derivation from binary marginal invariance alone, nor a general reconstruction of quantum theory. The two routes supply the same scalar functional constraint by different means: Route A through primitive multiplicativity (MA) plus continuity (CS), Route B through (PA) together

Table 2 Results and scope limitations by object. Proposition 6.11 addresses only the reduced chart T_{bin}; Remark 6.5 concerns the distinct (and unresolved) comparison between the full geometric package (G1)–(G4) and Hilbert-space structure.

Object	**What is (and is not) proved in this paper**
T_{bin}	Proposition 6.11: the chart of $(p_1(r), p_2(r))$ with $\kappa(r) = 1/r$ admits inequivalent quantum and classical operational completions. No statement here implies that the RS geometry (G1)–(G4) is strictly weaker than a Hilbert-space fragment.
(G1)–(G4)	Internal consistency checks (Remark 6.1, Proposition 6.2) and packaging in Theorem 6.6. Remark 6.5: equivalence, strict weakness, or incomparability with Hilbert-space structure is *open*.
Cost + RS narrative (Sec. 3)	Uniqueness of J (Theorem 3.4); path weights and Corollary 4.8 for composite *paths*. Motivation for scalar modeling; not a logical prerequisite for Proposition 5.6 (Section 5.1).

with binary marginal invariance $(\text{P5})_2$. Route A encodes the constraint in the scalar product law $P(c_1c_2) = P(c_1)P(c_2)$; Route B does not assume that law as primitive but derives the functional equation forcing $f(r) = r^2$ on normalized moduli from (PA) and $(\text{P5})_2$, with normalized-domain scalar multiplicativity emerging as a corollary rather than as an axiom. Primitive scalar multiplicativity is thereby separated from product-amplitude composition in the premise list.

Proposition 6.11 records a scope limitation for the thinner scalar binary chart: it admits inequivalent quantum and classical completions, and so by itself does not select Hilbert-space structure. Read together, Proposition 5.6, Corollary 5.11, and Proposition 6.11 furnish a comparison point among Born-rule derivations: one precise way in which binary marginal invariance can replace primitive scalar multiplicativity, with the geometric strength of the remaining package left as an open question (Remark 6.5).

7 Post-Theorem Checks and Comparison

This section records two downstream checks of the adopted model and then compares the conditional RS result with other Born-rule programs. Table 2 above summarized the scope of each result; Table 3 provides a schematic program comparison, and Remark 7.2 expands the premise placement and scope contrast.

7.1 Post-Theorem Checks

The standard interference identity becomes physically relevant in the RS path picture only after adding the phase postulate (P2) and the adopted linear sum-over-paths representation, neither of which follows from (A1)–(A2).

Given the additional phase postulate (P2) of Section 4, the path-weight construction of Section 3.2 assigns a complex amplitude $\psi(\gamma)$ to each path γ. Passing from single-path amplitudes to an outcome amplitude requires an additional representation choice not forced by the cost axioms: if paths $\gamma_1, \gamma_2, \ldots$ reach the same outcome i, one

Table 3 Comparison of selected routes to the quadratic probability rule (schematic; orientation only). The final row records this paper's conditional uniqueness theorem for the scalar quadratic rule in an adopted scalar product-amplitude model, together with a finite normalized amplitude-list extension and a scope proposition for the reduced binary chart (scope: Section 1).

Program	**Core premise placement and scope**
Gleason	Hilbert-space projection structure is assumed from the outset; the result constrains probability measures on that structure. For projection-valued measures (PVMs) the standard theorem requires dimension ≥ 3.
Hardy	Five operational axioms on states, composites, and continuous reversible transformations reconstruct the full quantum formalism, with the Born rule following inside that broader reconstruction.
Zurek	The decisive work is done by envariance and Everettian branching inside standard quantum mechanics, yielding the Born rule from entanglement symmetry arguments.
Chiribella–D'Ariano–Perinotti	Informational and operational axioms reconstruct the full theory; the Born rule is part of that larger state-space and transformation-structure result.
Aharonov–Shushi	A dynamical postulate (property fixity over short measurement intervals) outside the standard quantum axioms yields $P = \|c\|^2$; the mechanism is dynamical stability rather than structural reconstruction.
Torres Alegre	In finite-dimensional GPTs with purification, no-signaling causal consistency is the decisive condition selecting $P = \|c\|^2$.
Galley–Masanes	Studies alternatives to the Born rule in terms of informational properties [13]. The present paper is narrower: it fixes a scalar product-amplitude model and asks whether binary marginal invariance can replace primitive scalar multiplicativity.
RS / this paper	A cost-functional amplitude framework motivates the scalar product-amplitude model. Route A assumes primitive scalar multiplicativity; Route B replaces it with product-amplitude composition plus binary marginal invariance (Definition 4.10). The result is a conditional uniqueness theorem that localizes the premises forcing the scalar quadratic rule, with a finite normalized amplitude-list extension and a scope limitation for the reduced chart T_{bin} (Proposition 6.11).

adopts the linear sum $\psi_i = \sum_{\gamma \to i} \psi(\gamma)$ (see Section 3.2). For two paths γ_1, γ_2 reaching outcome i, this becomes $\psi_i = \psi(\gamma_1) + \psi(\gamma_2)$. Writing $a = \psi(\gamma_1)$ and $b = \psi(\gamma_2)$, one has $\psi_i = a + b$ and hence $P(i) = |\psi_i|^2 = |a + b|^2$. The remark below records the corresponding interference expansion.

Remark 7.1 (Standard interference identity) Let $a, b \in \mathbb{C}$ be individual path amplitudes reaching the same outcome (where z^* denotes complex conjugate of $z \in \mathbb{C}$). Then

$$|a + b|^2 = |a|^2 + |b|^2 + 2\,\Re(a^* b).$$

The terms $|a|^2$ and $|b|^2$ are the diagonal (same-path) contributions to $P(i) = |a + b|^2$, and $2\,\Re(a^* b)$ is the standard off-diagonal interference cross-term. The single-path quantities $|a|^2$ and $|b|^2$ are not individually observable; only the full sum $P(i) = |a + b|^2$ is the observable outcome probability. Once complex phase and the adopted linear sum-over-paths representation are in place, the RS path picture is consistent with this identity.

7.2 Comparison with Other Born-Rule Derivations

This comparison concerns premise placement rather than theorem strength. Table 3 records where each program locates the probabilistic work, not a ranking across

different structural questions. The standing limitations of the present route are exactly those stated in Section 1: the result is conditional on the scalar product-amplitude model and its normalization, composition, and marginal-invariance assumptions, rather than being a global reconstruction of state space or transformations. Within that scope, the observation is that binary marginal invariance can replace primitive scalar multiplicativity as the functional input, and that the resulting rule extends to finite normalized amplitude lists.

Remark 7.2 (Literature comparison: programs, premises, and scope) Table 3 compares programs at different structural levels; it orients premise placement rather than ranking theorem strength. Gleason [14] uses *frame-function* regularity on the lattice of closed subspaces (dimension ≥ 3 for PVMs). Zurek [31] places the decisive work in entanglement symmetry (envariance) and branching. Deutsch–Wallace [11, 25] relocate it to decision-theoretic rationality constraints. Aharonov–Shushi [2] reach the quadratic rule via a dynamical fixity postulate outside the standard axioms, paralleling the present strategy of supplementing non-standard primitives to force $P = |c|^2$; the two approaches differ in mechanism—dynamical stability versus cost-functional geometry. Torres Alegre [24] uses no-signaling causal consistency in finite-dimensional GPTs with purification. Galley–Masanes [13] classify alternatives to the measurement/Born postulates while keeping standard pure-state dynamics fixed, relating them to informational properties (e.g. bit symmetry); the present work instead isolates a scalar product-amplitude route in which binary marginal invariance can replace primitive multiplicativity.

Hardy [16] and Chiribella–D'Ariano–Perinotti [9] take operational axioms on states, composites, and transformations and *derive* Hilbert-space structure and the Born rule; Zurek assumes quantum mechanics including entanglement. The present argument takes a ratio-cost functional as primitive, then adds the adopted scalar and subsystem package and, in Section 6, the two-outcome geometry. Path weights and amplitudes are organized from there rather than from a globally reconstructed state space. The payoff in Route B is that modulus multiplicativity (MA) need not be a primitive scalar axiom; the adopted scalar inputs before Proposition 5.6 are listed in Section 1, and Section 6 supplies packaging beyond that abstract scalar step.

Hardy's five axioms constrain global state space, composite structure, and continuous reversibility, and yield the full formalism; this paper constrains a scalar probability law inside an adopted scalar product-amplitude model and extends the resulting assignment to finite normalized amplitude lists, so a direct axiom-count comparison with full reconstruction programs would be misleading. In the conditional Route B scalar result here, the probabilistic work sits in (SA), (NC), (PA), and $(P5)_2$, together with non-negativity of probabilities; the two-outcome geometry enters in Theorem 6.6 and related checks, not in Proposition 5.6 itself.

8 Empirical Scope and Conclusion

Section 8.1 identifies which assumptions would be called into question if the quadratic rule were empirically violated, and Section 8.2 states the final conclusion and four principal open problems.

8.1 Empirical Scope of the Conditional Result

Since the same quadratic rule as ordinary quantum mechanics is adopted, the present framework entails no empirical deviation from standard quantum theory and introduces no RS-specific discriminator. A conditional uniqueness theorem does not by itself yield a distinguishing experiment: any such test would have to target an adopted premise directly, and RS-specific phenomenology lies elsewhere in the program [19, 27–30]. Evidence against the adopted quadratic phenomenology would show joint inadequacy of the premise package without identifying which individual premise fails.

Given postulate (P2) and the adopted complex path-sum representation, the quadratic rule reproduces ordinary second-order interference. Sorkin's third-order parameter [22] vanishes under linear superposition with $P(i) = |\psi_i|^2$, consistently with triple-slit bounds on higher-order interference [17, 20, 21]. This is a compatibility check for the broader quadratic package, not a new prediction of the uniqueness argument itself.

An empirical violation of that broader package would imply that at least one adopted ingredient fails. The candidate fault lines correspond to the separately identified premise groups:

- *Scalar reduction and phase independence.* Equivalently, $P(c) = f(|c|)$ or $P(ce^{i\varphi}) = P(c)$; a failure would signal dependence on fuller state or measurement context.
- *Adopted* 2*-norm convention.* A failure would break the normalization hypothesis used in both Route A and Route B.
- *Product-amplitude subsystem model or binary marginal invariance.* A failure of either would invalidate the Route B functional argument leading to Proposition 5.6.
- *Probability positivity.* Route B uses the non-negative codomain of probabilities to turn interval additivity into monotonicity. A failure of ordinary probability non-negativity would reopen nonquadratic signed additive possibilities, but such a failure would already take the model outside standard probability theory.
- *Adopted two-outcome geometry.* A failure here would undermine packaging tied to Section 6 but would not overturn the geometry-free scalar argument of Proposition 5.6.

The empirical force of this decomposition is asymmetric: a violation of the quadratic rule would show that the present premise package is not jointly adequate, but agreement with experiment would not by itself identify which premises are fundamentally right or which broader framework deserves credit for the observed quadratic law.

8.2 Conclusion and Open Problems

Within the adopted scalar product-amplitude model, Proposition 5.6 shows that scalar reduction (SA), 2-norm normalization (NC), product-amplitude composition (PA), and binary marginal invariance $(P5)_2$ (Definition 4.10) uniquely select the quadratic scalar rule $P(c) = |c|^2$ on normalized amplitudes; continuity (CS) is not a Route B premise. Modulus multiplicativity (MA) is likewise not among the Route B premises and follows algebraically afterward on normalized moduli. Corollary 5.11 extends the assignment to finite normalized amplitude lists, Theorem 6.6 packages the scalar result

with the two-outcome normalization from (G4), and Proposition 6.11 records that the reduced chart T_{bin} alone does not fix Hilbert-space structure among its completions.

The result is a foundational theorem with a definite scope: it is not a cost-only derivation of the Born rule, not a derivation of scalar reduction, and not a reconstruction of quantum theory. Its content is the localization of premises identified in Section 4: marginal invariance can replace primitive scalar multiplicativity (MA) once (SA), (NC), and (PA) are already adopted.

The conditional character of the result is intentional: the theorem isolates assumptions sufficient for the quadratic scalar rule, but several of those assumptions remain independent within this paper. The following four open problems mark the gap between the conditional result proved here and a possible unconditional derivation within the broader RS program.

1. *Scalar reduction.* The ansatz $P(c) = f(|c|)$ is adopted, not derived from the RS cost axioms. A stronger derivation would require an operational invariance argument showing that phase changes of a single outcome amplitude cannot affect its assigned probability except through the modulus. Deriving (SA) from an RS-native symmetry or invariance argument is the most consequential open task.
2. *Geometric strength.* Strict weakness is proved only for the reduced chart T_{bin}, not for the full geometric package (G1)–(G4). Whether the adopted geometry is equivalent to, strictly weaker than, or incomparable with Hilbert-space structure remains unresolved (Remark 6.5).
3. *Full multi-outcome geometry.* Corollary 5.11 extends the scalar functional result to finite normalized amplitude lists. What remains open is stronger: whether the binary-split geometry (G1)–(G4) has a natural cost-functional n-outcome analogue, and whether such an analogue can be derived rather than adopted.
4. *Unconditional chain.* A stronger RS-native derivation would require independent proofs of phase structure, marginal invariance as in Definition 4.10, and the 2-norm convention. The broader RS program contains norm-preserving structures (e.g., DFT-8 and Parseval-type identities), but the measurement normalization convention is not derived here from those ingredients; binary marginal invariance would in turn need to follow from disjoint ledger regions plus an appropriate localization premise. These open links do not weaken the theorem proved here, since none is used implicitly.

A Formalization status (Lean 4)

This appendix records formalization status only. Some upstream cost-functional ingredients motivating the scalar amplitude model have Lean 4 counterparts in the public repository https://github.com/jonwashburn/recognition-science (audited commit `26851086105c105a9fd74098b2d39c25756d3624`). The central results—Proposition 5.6, Corollary 5.11, Theorem 6.6, and Proposition 6.11—are proved in this manuscript; the Lean material supports reproducibility of the related RS background as of the audited revision.

For orientation, the following build-verified targets are closest to the narrative of Sections 3–6:

- Unique cost functional J on $(0, \infty)$ from the RS axioms: `IndisputableMonolith.CostUniqueness.T5_uniqueness_complete`.
- Born weight r^2 under a two-branch calibration (formally adjacent to Route B forcing): `RS.Foundation.BornRuleForcing.born_weight_forced`.
- Two-outcome Born probabilities ($\cos^2\theta$, $\sin^2\theta$) via the $C = 2A$ bridge: `IM.Verification.TwoOutcomeBornCert.P_cos_eq`; `P_sin_eq`.
- DFT-8 2-norm preservation (Parseval-type background for broader RS norm discussions): `dft8_preserves_norm`; `dft8_preserves_inner`.

Remark A.1 (8-tick periodicity and complex structure) Within the broader RS forcing chain [29], the period-8 shift operator forces complexification and supplies structural motivation for postulate (P2). That forcing chain is not derived here; it enters only as external motivation. Related Lean modules exist in the same repository but are omitted here because they are not logical prerequisites for Proposition 5.6.

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Statements and Declarations

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Competing interests

The authors declare that they have no competing financial or non-financial interests directly or indirectly related to the work submitted for publication.

Author contributions

J.W.: Conceptualization; Formal analysis; Methodology; Writing – original draft; Writing – review and editing. M.S.: Formal analysis; Writing – review and editing; Project administration. E.A.: Formal analysis; Methodology; Writing – review and editing.

Data availability

This work is theoretical and contains no experimental data.

Code availability

The Lean 4 source code referenced in Appendix A is publicly available at https://github.com/jonwashburn/recognition-science. The audited revision is commit

`26851086105c105a9fd74098b2d39c25756d3624`. The appendix lists a few build-verified upstream modules; the main results of this paper are proved in the manuscript.

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