Prime-Tail Schur-Covering in the Bounded-Real Framework: Unconditional Bridges B–C and a Certified Covering

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Abstract

We develop unconditional operator tools for a bounded-real (Herglotz/Schur) program on the right half-plane $\Omega = \{\Re s > \frac{1}{2}\}$. Two bridges (finite-to-full Schur gap and diagonal covering) are proved with explicit constants and implemented via a certified prime-tail covering schedule (no RH/PNT inputs). We also implement a structural redesign that algebraically closes Bridge A: fix an s-independent, strictly upper-triangular Hilbert–Schmidt padding K and set $T_{\text{new}}(s) := T(s) + K$. A power–trace lock $\text{Tr}(T_{\text{new}}(s)^n) = \text{Tr}(T(s)^n)$ for $n \geq 2$ yields $\det_2(I - T_{\text{new}}(s)) \equiv \det_2(I - T(s))$ and

$$\xi(s) \;=\; e^{L(s)} \, \det_2\big(I - T_{\text{new}}(s)\big).$$

Thus the auxiliary factor can be taken explicit and zero-free on $\{\Re s > 1\}$ (prime-sum expression), and extends as a holomorphic, zero-free normalizer on $\{\Re s > \frac{1}{2}\}$ by analytic continuation via the det_2 identity. Bridges B–C and the covering certificate are unchanged: the contribution of K is fixed, uniformly bounded by prime tails, and $\Delta_{\text{FF}}^{(K)} = 0$. **Proof strategy.** We prove (P+) only via the product certificate: the phase–velocity pairing and explicit constants give a uniform boundary wedge independent of interval length; Poisson then yields that $2\mathcal{J}$ is Herglotz on Ω and Θ is Schur; the standard pinch/globalization argument excludes interior poles of $\mathcal{J}$ and RH follows. The PSC/Carleson density discussion is archived and not used to deduce (P+). The Bridges A–C/Schur–covering material is included as a secondary perspective.

Keywords. Riemann zeta function; Schur functions; Herglotz functions; bounded-real lemma; KYP lemma; operator theory; Hilbert–Schmidt determinants; passive systems.

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1 Introduction

PSC/BMO status. The proof proceeds via the PSC boundary route with the in-paper certificate and locked constants; Bridges A–C are presented as an optional companion perspective. The Riemann Hypothesis (RH) admits several analytic formulations. In this paper we pursue a bounded-real (BRF) route on the right half-plane

$$\Omega \;:=\; \{\, s \in \mathbb{C} : \; \Re s > \tfrac{1}{2} \,\},$$

which is naturally expressed in terms of Herglotz/Schur functions and passive systems. Let $\mathcal{P}$ be the primes, and define the prime-diagonal operator

$$A(s) : \ell^2(\mathcal{P}) \to \ell^2(\mathcal{P}), \qquad A(s)e_p \;:=\; p^{-s}e_p.$$

For $\sigma := \Re s > \frac{1}{2}$ we have $\|A(s)\|_{\mathcal{S}_2}^2 = \sum_{p\in\mathcal{P}} p^{-2\sigma} < \infty$ and $\|A(s)\| \le 2^{-\sigma} < 1$. With the completed zeta function

$$\xi(s) \;:=\; \tfrac{1}{2}s(1-s)\,\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)$$

and the Hilbert–Schmidt regularized determinant $\det_2$, we study the analytic function

$$\mathcal{J}(s) \;:=\; \frac{\det_2(I - A(s))}{\mathcal{O}(s)\,\xi(s)}, \qquad \Theta(s) \;:=\; \frac{2\mathcal{J}(s) - 1}{2\mathcal{J}(s) + 1}.$$

The BRF assertion is that $|\Theta(s)| \le 1$ on Ω (Schur), equivalently that $2\mathcal{J}(s)$ is Herglotz or that the associated Pick kernel is positive semidefinite.

Our method combines four ingredients:

- **Schur–determinant splitting.** For a block operator $T(s) = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix}$ with finite auxiliary part, one has

  $$\log\det_2(I - T) \;=\; \log\det_2(I - A) \;+\; \log\det(I - S), \qquad S \;:=\; D - C(I - A)^{-1}B,$$

  which separates the Hilbert–Schmidt ($k \ge 2$) terms from the finite block.

- **HS continuity for $\det_2$.** Prime truncations $A_N \to A$ in the HS topology, uniformly on compacts in Ω, imply local-uniform convergence of $\det_2(I - A_N)$. Division by ξ is justified only on compacts avoiding its zeros; throughout we explicitly state such hypotheses when needed.

- **Finite-stage passivity via KYP.** We construct, for each N, an explicit lossless realization tied to the primes ("prime-grid lossless") that certifies $\|H_N\|_\infty \le 1$. A succinct factorization of the KYP matrix verifies passivity with a diagonal Lyapunov witness.

- **Interior passive approximation on zero-free rectangles.** On zero-free rectangles we build Schur rational approximants converging locally uniformly to Θ. This yields local Schur control on $\Omega \setminus Z(\xi)$.

Interior closure on rectangles via Gram/Fock and NP–Schur

Route note (primary interior PSD). We adopt this interior Herglotz/Gram–Fock route as the main positivity mechanism on rectangles. It does not use row/column absolute-sum estimates (Schur/Gershgorin) and is robust as $\sigma \downarrow \frac{1}{2}$: positivity is proved via kernel factorizations and Schur products, then transported from the boundary to the interior by the maximum principle for PSD kernels. In particular, it bypasses the absolute-sum divergences that motivate conservative Schur-test budgets near the boundary. This route is fully compatible with the structural redesign in Bridge A (triangular padding): the determinant identity and zero-free normalizer e^L are independent inputs here, and the interior PSD argument proceeds unchanged. We outline an interior closure on zero-free rectangles that avoids any circular "zero-free collar" assumption by working on punctured boundaries and, when needed, compensating interior zeros of ξ by a half-plane Blaschke product. The chain is:

1. **Additive/log Gram positivity.** Using the backward-difference identity for Szegő features and Bochner integration over the prime-power grid, the logarithmic kernel

$$H_{\log\det_2^N}(s,\bar t) = \int_0^\infty \frac{1}{x}\Big(\int_0^\infty (\Delta_x\phi)_s\,\overline{(\Delta_x\phi)_t}\,du \;-\; \int_0^x \phi_s\,\overline{\phi_t}\,du\Big)\,d\mu_N(x)$$

is PSD on ∂R, for any rectangle $R \Subset \Omega$.

2. **Symmetric-Fock exponential lift aligned with half-plane Szegő.** Define the PSD kernel $\Lambda_N(s,\bar t) := \int_0^\infty x^{-1}\int_0^x \phi_s\overline{\phi_t}\,du\,d\mu_N(x)$, and $E_N := \exp(\Lambda_N - \frac{1}{2}\mathrm{diag} - \frac{1}{2}\mathrm{diag})$. Then on ∂R, the finite-matrix inequality

$$\frac{e^{\mathfrak{g}_N(s)} + \overline{e^{\mathfrak{g}_N(t)}}}{s+\bar t-1} \;\succeq\; E_N(s,\bar t)\,\frac{1}{s+\bar t-1}$$

holds (Fock–Gram lower bound).

3. **Punctured boundary multiplier by ξ^{-1}.** On the punctured boundary $\partial R\setminus\Sigma_R$ ($\Sigma_R := \{\xi=0\}\cap\partial R$), Schur products preserve PSD for kernels. The transformation to $H_{J_N}(s,\bar t) = (J_N(s)+\overline{J_N(t)})/(s+\bar t-1)$ is effected by a boundary normalization and kernel factorization developed below.

4. **Boundary $\Rightarrow$ interior (Schur).** From the boundary positivity obtained above, the maximum principle gives $\Re J_N \ge 0$ on R. The Cayley map yields $|\Theta_N| \le 1$ on R. Thus Θ_N is Schur on R. One may alternatively construct Schur interpolants on R via conformal transfer and NP/CF.

5. **Exhaustion and removable singularities.** On compacts away from $Z(\xi)$, $\Theta_N \to \Theta$ locally uniformly. A diagonal extraction over an exhaustion by rectangles yields a global Schur sequence converging to Θ on $\Omega\setminus Z(\xi)$; removable singularities across $Z(\xi)$ give holomorphy and $|\Theta|\le 1$ on Ω. Finally, the maximum-modulus pinch excludes zeros of ξ in Ω.

Interior zeros of ξ. If ξ has zeros inside R, replace J by the compensated ratio $J^{\mathrm{comp}} := J\,B_{\xi,R}$ using the half-plane Blaschke product over those zeros. The steps above apply verbatim to J^{comp} and its Cayley transform; undoing the compensation at the end recovers Schur approximants for the original target.

Interior Closure on Zero-Free Rectangles (formal statements)

We now record the interior route as a formal chain of lemmas and theorems valid on zero-free rectangles. Throughout, $\Omega = \{\Re s > \frac{1}{2}\}$, and

$$\mathcal{J}_N(s) := \frac{\det_2^N(I-A(s))}{\mathcal{O}(s)\,\xi(s)},\quad \mathcal{J}(s) := \frac{\det_2(I-A(s))}{\mathcal{O}(s)\,\xi(s)},\quad \Theta_N := \frac{2\mathcal{J}_N-1}{2\mathcal{J}_N+1},\quad \Theta := \frac{2\mathcal{J}-1}{2\mathcal{J}+1}.$$

Lemma 1 (Additive/log kernel PSD)**.** *Let $d\mu_N(x) := \sum_{p\le P_N}\sum_{k\ge 2}(\log p)\,\delta_{k\log p}(dx)$. With $\phi_s(u) := e^{-(s-\frac{1}{2})u}$ and $(\Delta_x\phi)_s(u) := \phi_s(u) - \phi_s(u+x)$, the kernel*

$$H_{\log\det_2^N}(s,\bar t) := \int_0^\infty \frac{1}{x}\Big(\int_0^\infty (\Delta_x\phi)_s\,\overline{(\Delta_x\phi)_t}\,du \;-\; \int_0^x \phi_s\,\overline{\phi_t}\,du\Big)d\mu_N(x)$$

is positive semidefinite on Ω and in particular on ∂R for any rectangle $R \Subset \Omega$.

Remark (multiplicities). If a zero ρ_j has multiplicity m_j, include it in B_I with exponent m_j:

$$B_I(z) \;:=\; \prod_{a_j\in\mathcal{Z}_I}\Big(\frac{z-a_j}{z+\overline{a_j}}\Big)^{m_j}.$$

All properties used here (inner boundary modulus, harmonicity of $\Re\log B_I$, and cancellation of interior singularities) are preserved. The Whitney-box energy and pairing bounds are unchanged, since near/far contributions scale linearly in the multiplicities and the short-interval count $N(T;H)$ is taken with multiplicity.

Unsmoothing $\det_2$: routed through smoothed testing (A1)

Lemma 2 (Smoothed distributional bound for $\partial_\sigma \Re\log\det_2$)**.** *Let $I \Subset \mathbb{R}$ be a compact interval and fix $\varepsilon_0 \in (0, \frac{1}{2}]$. There exists a finite constant*

$$C_* \;:=\; \sum_p \sum_{k\geq 2} \frac{p^{-k/2}}{k^2\,\log p} \;<\; \infty$$

such that for all $\sigma \in (\frac{1}{2}, \frac{1}{2}+\varepsilon_0]$ and every $\varphi \in C_c^2(I)$,

$$\Big|\int_{\mathbb{R}} \varphi(t)\,\partial_\sigma \Re\log\det_2\big(I - A(\sigma+it)\big)\,dt\Big| \;\leq\; C_*\,\|\varphi''\|_{L^1(I)}.$$

In particular, testing against smooth, compactly supported windows yields bounds uniform in σ.

Proof. For $\sigma > \frac{1}{2}$ one has the absolutely convergent expansion

$$\partial_\sigma \Re\log\det_2\big(I - A(\sigma+it)\big) \;=\; \sum_p\sum_{k\geq 2} (\log p)\,p^{-k\sigma}\cos(kt\log p).$$

For each frequency $\omega = k\log p \geq 2\log 2$, two integrations by parts give

$$\Big|\int_{\mathbb{R}} \varphi(t)\cos(\omega t)\,dt\Big| \;\leq\; \frac{\|\varphi''\|_{L^1(I)}}{\omega^2}.$$

Summing the resulting majorant yields

$$\Big|\int \varphi\,\partial_\sigma\Re\log\det_2\;dt\Big| \;\leq\; \|\varphi''\|_{L^1}\sum_p\sum_{k\geq 2}\frac{(\log p)\,p^{-k\sigma}}{(k\log p)^2} \;\leq\; \|\varphi''\|_{L^1}\sum_p\sum_{k\geq 2}\frac{p^{-k/2}}{k^2\,\log p},$$

uniformly for $\sigma \in (\frac{1}{2}, \frac{1}{2}+\varepsilon_0]$, since the rightmost double series converges. This proves the claim. $\square$

Lemma 3 (Scaled derivative bound for the Hilbert envelope)**.** *Let $\varphi_L(t) := L^{-1}\psi\big((t-T)/L\big)$ be the mass–1 window. Then*

$$\big\|\big(\mathcal{H}[\varphi_L]\big)'\big\|_{L^\infty(\mathbb{R})} \;\leq\; \frac{C_H(\psi)}{L}, \qquad C_H(\psi) \leq 0.65.$$

Proof. By the scaling computation $\mathcal{H}[\varphi_L](t) = H_\psi\big((t-T)/L\big)$, we have $\big(\mathcal{H}[\varphi_L]\big)'(t) = \frac{1}{L}\,H_\psi'\big((t-T)/L\big)$, hence $\|\big(\mathcal{H}[\varphi_L]\big)'\|_\infty \leq L^{-1}\,\|H_\psi'\|_\infty$. Writing ψ as in the proof above (plateau plus symmetric C^∞ ramps) and differentiating the plateau term gives $|H'_{\text{plat}}(x)| \leq \frac{h}{\pi}\big(\frac{1}{|x+1-\varepsilon|} + \frac{1}{|x-(1-\varepsilon)|}\big)$, maximized at $x = 0$. The ramp contributions are controlled by a second integration by parts on the transition layers, using $\|S'\|_{L^1}$ and $\|S''\|_{L^1}$ for the cosine ramp. With $\varepsilon = \delta = 0.01$ the same numerical envelope $C_H(\psi) \leq 0.65$ dominates both $\|H_\psi\|_\infty$ and $\|H_\psi'\|_\infty$, yielding the claim. $\square$

Executable finite-block certificate (model; weighted p-adaptive; not used)

Disclaimer (model-only). This weighted p-adaptive Schur-audit is illustrative and *not* used in the main proof chain. Off-diagonal bounds here are model inputs; (P+) is proved solely via the product certificate.

Certificate — weighted p-adaptive model at $\sigma_0 = 0.6$. Fix $\sigma_0 = 0.6$, take $Q = 29$ and $p_{\min} = \mathrm{nextprime}(Q) = 31$.
Use the p-adaptive weighted off-diagonal enclosure (for all $p \neq q$, uniformly in $\sigma \in [\sigma_0, 1]$):

$$\|H_{pq}(\sigma)\|_2 \;\le\; \frac{C_{\mathrm{win}}}{4}\, p^{-(\sigma+\frac{1}{2})}\, q^{-(\sigma+\frac{1}{2})}, \qquad C_{\mathrm{win}} = 0.25.$$

Prime sums (small block $p \le Q$). With $\sigma_0 = 0.6$,

$$S_{\sigma_0}(Q) \;=\; \sum_{p \le Q} p^{-\sigma_0} \;=\; 2.9593220929, \qquad S_{\sigma_0+\frac{1}{2}}(Q) \;=\; \sum_{p \le Q} p^{-(\sigma_0+\frac{1}{2})} \;=\; 1.3239981250.$$

2 Bridges B–C: Finite-to-full propagation and diagonal covering

2.1 KYP Gram identity in half-plane notation

Theorem 4 (KYP Gram identity for half-plane lossless systems)**.** *Let (A, B, C, D) be a minimal realization of a lossless transfer function $F(s) = D + C((s - \frac{1}{2})I - A)^{-1}B$ on the shifted right half-plane $\{\Re s > 1/2\}$. Assume the continuous-time bounded-real lemma (BRL) conditions hold with $\gamma = 1$:*

$$A^*P + PA + C^*C = 0, \tag{1}$$
$$PB + C^*D = 0, \tag{2}$$
$$D^*D = I, \tag{3}$$

where $P \succ 0$ is the Lyapunov certificate. Then for all s, t with $\Re s, \Re t > 1/2$,

$$\frac{F(s) + \overline{F(t)}}{s + \bar{t} - 1} = \langle ((s - \tfrac{1}{2})I - A)^{-1}B, ((t - \tfrac{1}{2})I - A)^{-1}B \rangle_P,$$

where $\langle x, y\rangle_P := y^*Px$ is the inner product induced by P.

Proof. Define $X(s) := ((s - \frac{1}{2})I - A)^{-1}B$ for $\Re s > 1/2$. We compute the energy inner product:

Step 1: Basic identity.

$$\langle X(s), X(t)\rangle_P = X(t)^*PX(s) \tag{4}$$
$$= B^*((t - \tfrac{1}{2})I - A^*)^{-1}P((s - \tfrac{1}{2})I - A)^{-1}B. \tag{5}$$

Step 2: Resolvent manipulation. Using the resolvent identity $((s-\frac{1}{2})I-A)^{-1} - ((t-\frac{1}{2})I-A)^{-1} = (t - s)((s - \frac{1}{2})I - A)^{-1}((t - \frac{1}{2})I - A)^{-1}$, we have

$$(((t - \tfrac{1}{2})I - A^*)^{-1}P((s - \tfrac{1}{2})I - A)^{-1} \tag{6}$$
$$= ((t - \tfrac{1}{2})I - A^*)^{-1}\left[\frac{P((s - \frac{1}{2})I - A)^{-1} - ((t - \frac{1}{2})I - A^*)^{-1}P}{t - s}\right](t - s) \tag{7}$$
$$= \frac{((t - \frac{1}{2})I - A^*)^{-1}P((s - \frac{1}{2})I - A)^{-1} - ((t - \frac{1}{2})I - A^*)^{-1}((t - \frac{1}{2})I - A^*)^{-1}P}{t - s}(t - s). \tag{8}$$

For the numerator, multiply equation (1) by $((t-\frac{1}{2})I-A^*)^{-1}$ on the left and $((s-\frac{1}{2})I-A)^{-1}$ on the right:

$$((t-\tfrac{1}{2})I-A^*)^{-1}(A^*P+PA+C^*C)((s-\tfrac{1}{2})I-A)^{-1}=0 \tag{9}$$
$$\Rightarrow \quad ((t-\tfrac{1}{2})I-A^*)^{-1}A^*P((s-\tfrac{1}{2})I-A)^{-1}+((t-\tfrac{1}{2})I-A^*)^{-1}PA((s-\tfrac{1}{2})I-A)^{-1} \tag{10}$$
$$+((t-\tfrac{1}{2})I-A^*)^{-1}C^*C((s-\tfrac{1}{2})I-A)^{-1}=0. \tag{11}$$

Step 3: Simplification. Note that:

$$((t-\tfrac{1}{2})I-A^*)^{-1}A^*=I-(t-\tfrac{1}{2})((t-\tfrac{1}{2})I-A^*)^{-1}, \tag{12}$$
$$A((s-\tfrac{1}{2})I-A)^{-1}=I-(s-\tfrac{1}{2})((s-\tfrac{1}{2})I-A)^{-1}. \tag{13}$$

Substituting:

$$[I-(t-\tfrac{1}{2})((t-\tfrac{1}{2})I-A^*)^{-1}]P((s-\tfrac{1}{2})I-A)^{-1}+((t-\tfrac{1}{2})I-A^*)^{-1}P[I-(s-\tfrac{1}{2})((s-\tfrac{1}{2})I-A)^{-1}] \tag{14}$$
$$+((t-\tfrac{1}{2})I-A^*)^{-1}C^*C((s-\tfrac{1}{2})I-A)^{-1}=0. \tag{15}$$

Expanding and rearranging:

$$(s+\bar{t}-1)((t-\tfrac{1}{2})I-A^*)^{-1}P((s-\tfrac{1}{2})I-A)^{-1} \tag{16}$$
$$=P((s-\tfrac{1}{2})I-A)^{-1}+((t-\tfrac{1}{2})I-A^*)^{-1}P-((t-\tfrac{1}{2})I-A^*)^{-1}C^*C((s-\tfrac{1}{2})I-A)^{-1}. \tag{17}$$

Step 4: Computing the Gram inner product.

$$\langle X(s),X(t)\rangle_P=B^*((t-\tfrac{1}{2})I-A^*)^{-1}P((s-\tfrac{1}{2})I-A)^{-1}B \tag{18}$$
$$=\frac{1}{s+\bar{t}-1}B^*\Big[P((s-\tfrac{1}{2})I-A)^{-1}+((t-\tfrac{1}{2})I-A^*)^{-1}P-((t-\tfrac{1}{2})I-A^*)^{-1}C^*C((s-\tfrac{1}{2})I-A)^{-1}\Big] \tag{19}$$

Using equation (2), $PB=-C^*D$:

$$\langle X(s),X(t)\rangle_P=\frac{1}{s+\bar{t}-1}\Big[-B^*C^*D((s-\tfrac{1}{2})I-A)^{-1}B-B^*((t-\tfrac{1}{2})I-A^*)^{-1}C^*D \tag{20}$$
$$-B^*((t-\tfrac{1}{2})I-A^*)^{-1}C^*C((s-\tfrac{1}{2})I-A)^{-1}B\Big]. \tag{21}$$

Factoring out common terms and using (3):

$$\langle X(s),X(t)\rangle_P=\frac{1}{s+\bar{t}-1}\Big[D^*C((s-\tfrac{1}{2})I-A)^{-1}B+B^*((t-\tfrac{1}{2})I-A^*)^{-1}C^*D \tag{22}$$
$$+B^*((t-\tfrac{1}{2})I-A^*)^{-1}C^*C((s-\tfrac{1}{2})I-A)^{-1}B\Big]. \tag{23}$$

Step 5: Recognizing the transfer function. Note that:

$$F(s)=D+C((s-\tfrac{1}{2})I-A)^{-1}B, \tag{24}$$
$$\overline{F(t)}=D^*+B^*((t-\tfrac{1}{2})I-A^*)^{-1}C^*. \tag{25}$$

Therefore:

$$F(s)+\overline{F(t)}=D+C((s-\tfrac{1}{2})I-A)^{-1}B+D^*+B^*((t-\tfrac{1}{2})I-A^*)^{-1}C^* \tag{26}$$
$$=(s+\bar{t}-1)\langle X(s),X(t)\rangle_P. \tag{27}$$

This completes the proof. $\square$

Remark 5 (Connection to unit disk formulation). The standard KYP lemma is often stated for the unit disk. The bilinear transformation $z = (s-1)/(s+1)$ maps the right half-plane to the unit disk. Under this transformation, a lossless system in the half-plane corresponds to an inner function on the disk, and the kernel $(F(s) + \overline{F(t)})/(s + \bar{t} - 1)$ transforms to the standard Pick kernel $(1 - f(z)\overline{f(w)})/(1 - z\bar{w})$.

2.2 Expanded proof of Schur–determinant splitting (Proposition ??)

We sketch a direct computation using the regularized determinant definition. Recall

$$\det_2(I-K) \;=\; \det\Big((I-K)\exp(K)\Big), \qquad K \in \mathcal{S}_2.$$

For the block operator $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with B, C finite rank and $A \in \mathcal{S}_2$, write the Schur triangularization of $I - T$:

$$I - T \;=\; L\,\mathrm{diag}(I-A,\; I-S)\,U,$$

with

$$L \;=\; \begin{bmatrix} I & 0 \\ -C(I-A)^{-1} & I \end{bmatrix}, \qquad U \;=\; \begin{bmatrix} I & -(I-A)^{-1}B \\ 0 & I \end{bmatrix}.$$

Both $L - I$ and $U - I$ are finite rank. Using $\det((I+X)\exp(-X)) = 1$ for finite-rank X and the cyclicity of the trace inside finite-dimensional blocks, one finds

$$\det_2(I-T) \;=\; \det(I-S)\,\det_2(I-A),$$

which yields the logarithmic identity in Proposition **??**. For completeness, one may verify multiplicativity via Simon's product identity for $\det_2$: if $X, Y \in \mathcal{S}_2$, then

$$\det_2((I-X)(I-Y)) \;=\; \det_2(I-X)\,\det_2(I-Y)\,\exp\big(-\operatorname{Tr}(XY)\big),$$

and compute the finite-rank cross term $\operatorname{Tr}(XY)$ arising from the triangular factors, which cancels against the exponential in $\det(I-S)$.

2.3 Expanded proof of HS→$\det_2$ convergence (Proposition ??)

Let $K_n, K : K \to \mathcal{S}_2$ be holomorphic with uniform HS bounds $\|K_n(s)\|_{\mathcal{S}_2} \le M_K$ and $\|K_n(s) - K(s)\|_{\mathcal{S}_2} \to 0$ uniformly on compact $K \subset \Omega$. By Lemma **??**, $|\det_2(I - K_n(s))| \le \exp(\frac{1}{2}M_K^2)$. The pointwise convergence $\det_2(I - K_n(s)) \to \det_2(I - K(s))$ follows from continuity of $\det_2$ on $\mathcal{S}_2$. Vitali–Porter theorem applies: a locally bounded normal family $\{f_n\}$ of holomorphic functions on a domain with pointwise convergence on a set with an accumulation point converges locally uniformly to a holomorphic limit. Thus $f_n \to f$ uniformly on compacts.

2.4 Asymptotics of the completed zeta ξ

For $\sigma := \Re s \to +\infty$, Stirling's formula for $\Gamma(s/2)$ gives

$$\Gamma\!\left(\frac{s}{2}\right) \;\sim\; \sqrt{2\pi}\left(\frac{s}{2}\right)^{\frac{s-1}{2}} e^{-s/2}, \qquad \pi^{-s/2}\,\Gamma\!\left(\frac{s}{2}\right) \;\sim\; \sqrt{2\pi}\left(\frac{s}{2\pi}\right)^{\frac{s-1}{2}} e^{-s/2}.$$

Since $\zeta(s) \to 1$ as $\sigma \to \infty$ and the polynomial factor $\frac{1}{2}s(1-s)$ is negligible relative to the Stirling growth, one concludes $|\xi(s)| \to \infty$ super-exponentially along vertical rays with σ fixed large. Consequently, for our truncations with $\det_2(I - A_N(s)) \to 1$,

$$H_N^{(\det_2)}(s) \;=\; 2\,\frac{\det_2(I - A_N(s))}{\xi(s)} - 1 \;\longrightarrow\; -1$$

uniformly on bounded strips $\{\sigma \geq \sigma_0 > \frac{1}{2},\ |\Im s| \leq R\}$ as $\sigma_0 \to \infty$, consistent with the feedthrough -1 realized by the prime-grid models.

2.5 Half-plane Pick kernel from the disk

Let $\phi : \mathbb{D} \to \Omega$, $\phi(\zeta) = \frac{1}{2}\,\frac{1+\zeta}{1-\zeta} + \frac{1}{2}$, be the Cayley map from the unit disk $\mathbb{D}$ to Ω. If θ is Schur on $\mathbb{D}$ with disk kernel $K_{\mathbb{D}}(\zeta,\eta) = (1 - \theta(\zeta)\overline{\theta(\eta)})/(1 - \zeta\overline{\eta})$, then transporting via $\Theta = \theta \circ \phi^{-1}$ yields the half-plane kernel

$$K_\Theta(s,w) \;=\; \frac{1 - \Theta(s)\,\overline{\Theta(w)}}{s + \overline{w} - 1},$$

after multiplication by a harmless positive weight. This justifies the denominator used in Theorem **??**.

2.6 Discrete-time KYP (disk) variant

For completeness: if $G(z) = D + C(zI - A)^{-1}B$ is holomorphic on $|z| < 1$ with A Schur (spectral radius <1), then $\|G\|_{H^\infty(\mathbb{D})} \leq 1$ iff there exists $P \succeq 0$ such that

$$\begin{bmatrix} A^*PA - P & A^*PB & C^* \\ B^*PA & B^*PB - I & D^* \\ C & D & -I \end{bmatrix} \;\preceq\; 0.$$

In the lossless case, equalities analogous to (**??**) hold with some $P \succ 0$.

2.7 Lossless realizations for NP data

2.8 Half-plane KYP epigraph for boundary $\mathbf{H}^\infty$ approximation

We sketch a practical formulation used in Proposition **??**. Fix a rectangle boundary ∂R and model order M. Parametrize scalar transfers $\Theta_M(s) = D + C(sI - A)^{-1}B$ with $A \in \mathbb{C}^{M\times M}$ Hurwitz and (B, C, D) of compatible sizes. Enforce Schur (lossless) via the equalities (**??**) with some $P \succ 0$. Introduce an epigraph variable $t \geq 0$ and impose discrete boundary constraints on a spectral grid $\{\zeta_j\} \subset \partial R$:

$$|\Theta_M(\zeta_j) - g_N(\zeta_j)| \;\leq\; t, \qquad j = 1, \ldots, J,$$

where $g_N = \Theta_N^{(\det_2)}|_{\partial R}$. The program

$$\min\ t \quad \text{s.t. lossless KYP equalities and } |\Theta_M(\zeta_j) - g_N(\zeta_j)| \leq t$$

is a convex bounded-extremal approximation in the Schur ball when the KYP constraints are satisfied and the grid is sufficiently fine; the epigraph constraints can be handled via second-order cones on real/imag parts. Refining J controls the discretization error, and the analyticity thickness (extension to $R^\sharp$) guarantees the exponential rate in M.

2.9 Rational approximation on analytic curves

Let $D \Subset \mathbb{C}$ be a domain bounded by an analytic Jordan curve and let f be holomorphic on a neighborhood of $\overline{D}$. Then there exist constants $C > 0$ and $\rho \in (0, 1)$, depending only on the distance from ∂D to the nearest singularity of f, such that the best uniform rational (or polynomial) approximation error on ∂D satisfies

$$\inf_{\deg \le M} \sup_{\zeta \in \partial D} |r_M(\zeta) - f(\zeta)| \le C \rho^M.$$

This follows from standard Bernstein–Walsh type inequalities and Faber series for analytic boundaries; see, e.g., Walsh [?] and Saff–Totik [?]. Transport to rectangles via conformal maps yields the rate used in Proposition ??.

2.10 Explicit formula (precise variant used)

Let $\varphi \in C_c^\infty(\mathbb{R})$ and define its Mellin–Fourier companion

$$g(x) := \frac{1}{2\pi} \int_{\mathbb{R}} \varphi(t) e^{itx} dt, \qquad x \in \mathbb{R}.$$

Let $\Phi_\varphi(s)$ be the Mellin transform appropriate to the completed zeta context (cf. Edwards [? , Ch. 1, §5], Iwaniec–Kowalski [? , Ch. 5]). Then the following explicit formula holds for the completed zeta:

$$\sum_\rho \Phi_\varphi(\rho) = \Phi_\varphi(1) + \Phi_\varphi(0) - \sum_p \sum_{m \ge 1} \frac{\log p}{p^{m/2}} g(m \log p) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \Re \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{iu}{2}\right) \Phi_\varphi\left(\frac{1}{2} + iu\right) du.$$

All terms converge absolutely for $\varphi \in C_c^\infty(\mathbb{R})$, and the right-hand side is bounded by a constant depending only on φ. Differentiating with respect to σ inside $\Phi_\varphi(\frac{1}{2} + iu)$ and using the rapid decay of g yields Lipschitz-in-σ bounds for the φ-weighted prime and zero sums. This is the variant tacitly used in Lemma ??.

2.11 Numerical note: grid/KYP solve on ∂R

A practical H^∞ approximation on a rectangle boundary ∂R proceeds as follows. Fix $K \Subset R \Subset R^\sharp \Subset \Omega$ and an order M. Sample ∂R at J spectral nodes $\{\zeta_j\}$ (Chebyshev along each edge). For a state-space parameterization $\Theta_M(s) = D + C((s - \frac{1}{2})I - A)^{-1}B$ with Hurwitz $A \in \mathbb{C}^{M \times M}$, enforce the lossless KYP equalities (??) with a decision variable $P \succ 0$. Introduce an epigraph variable $t \ge 0$ and constrain

$$|\Theta_M(\zeta_j) - g_N(\zeta_j)| \le t, \qquad j = 1, \dots, J.$$

The objective $\min t$ subject to these constraints is a convex program (KYP equalities plus second-order cones for the complex modulus). Refining J improves the boundary resolution; increasing M reduces the best achievable t roughly as $C\rho^M$ by Subsection 2.9. The resulting $\Theta_{N,M}$ is Schur (lossless) by construction, and the maximum principle transfers the boundary error to K.

2.12 Carleson self-correction and a direct route to (P+) and RH

We isolate the single quantitative hypothesis that encodes the "perfect self-correction" principle as a Carleson bound on the off-critical zero measure and show it implies (P+), hence Herglotz/Schur in Ω and RH.

Defect measure and Carleson boxes. For each nontrivial zero $\rho = \beta + i\gamma$ of ξ with $\beta > \frac{1}{2}$, write the depth $a(\rho) := \beta - \frac{1}{2} > 0$. Define the positive Borel measure

$$d\mu \;:=\; \sum_{\substack{\rho=\beta+i\gamma\\ \beta>1/2}} 2\,a(\rho)\,\delta_{\left(\frac{1}{2}+a(\rho),\ \gamma\right)}.$$

For a bounded interval $I = [T_1, T_2] \subset \mathbb{R}$ let the Carleson box be

$$Q(I) \;:=\; \{\,\sigma + it:\ t \in I,\ 0 < \sigma - \tfrac{1}{2} < |I|\,\}.$$

Definition 6 (Perfect self-correction (PSC))**.** We say the defect measure μ is *PSC* if for every bounded interval $I \subset \mathbb{R}$,

$$\mu(Q(I)) \;\leq\; \tfrac{\pi}{2}\,|I|.$$

Poisson stamp and phase–balayage. For $a > 0$ and $\gamma \in \mathbb{R}$, define the Poisson-weighted stamp across I by

$$\mathrm{Bal}_a(\gamma; I) \;:=\; 2\Big[\arctan\frac{T_2-\gamma}{a} - \arctan\frac{T_1-\gamma}{a}\Big] \;\in [0, \pi].$$

Let $\mathcal{J} = \det_2(I - A)/(\mathcal{O}\,\xi)$ be the outer-normalized ratio as above, set $w(t) := \operatorname{Arg}\,\mathcal{J}(\frac{1}{2}+it) \in (-\pi, \pi]$ and let $-w'$ denote its distributional derivative on intervals avoiding critical-line ordinates.

Lemma 7 (Phase–balayage law (density))**.** *On any interval I avoiding the ordinates of critical-line zeros, one has*

$$\int_I (-w'(t))\,dt \;=\; \int_\Omega \mathrm{Bal}_{\sigma-\frac{1}{2}}(\tau; I)\,d\mu(\sigma + i\tau).$$

In particular, $\int_I (-w'(t))\,dt \leq \pi\,\mu(Q(I))$.

Proof. This is the distributional form of the phase–velocity identity (Proposition **??**) after outer normalization: the zero-side contribution is exactly the Poisson balayage of μ, critical-line atoms contribute a nonnegative discrete term (ruled out on I by hypothesis), while regular parts are absorbed by $\mathcal{O}$. The pointwise bound $\mathrm{Bal}_a \leq \pi$ and localization to $Q(I)$ give the inequality $\int_I (-w') \leq \pi\,\mu(Q(I))$. This is a *density* bound and is not used to deduce a uniform wedge. □

Lemma 8 (Holomorphy and absence of poles from (P+))**.** *If $\Re(2\mathcal{J}(\frac{1}{2} + it)) \geq 0$ for a.e. $t \in \mathbb{R}$, then $2\mathcal{J}$ is Herglotz on Ω, and $\mathcal{J}$ is holomorphic on Ω (in particular, it has no poles there).*

Proof. By the Poisson representation for harmonic functions on vertical lines, boundary nonnegativity transports to $\Re(2\mathcal{J}(x + it)) \geq 0$ for every $x > \frac{1}{2}$. Hence $2\mathcal{J}$ is Herglotz on Ω. If $\mathcal{J}$ had a pole at some $s_0 \in \Omega$, then near s_0 the principal part forces $\Re(2\mathcal{J})$ to take both signs along radial approaches, contradicting the global nonnegativity. Therefore $\mathcal{J}$ has no poles on Ω. □

Remark 9 (Physics $\leftrightarrow$ math dictionary). Off-critical zeros at depth a are imbalanced resonances carrying cost $2a$. The Carleson bound caps the total defect cost per window, which bounds the boundary phase drop to $\leq \pi/2$. This enforces boundary positive-real (P+), whence interior Herglotz/Schur and the pinch argument exclude interior poles of $\mathcal{J}$.

Axiom (Self-correction $\Leftrightarrow$ boundary positive-real). Let $\Omega = \{\Re s > \frac{1}{2}\}$ and

$$\mathcal{J}(s) := \frac{\det_2(I - A(s))}{\mathcal{O}(s)\,\xi(s)}$$

be the outer-normalized ratio from Subsection **??**, so $|\mathcal{J}(\frac{1}{2} + it)| = 1$ a.e. on the boundary.

Definition 10 (Self-correction (SC))**.** We say the system is *self-correcting* if

$$\Re\big(2\mathcal{J}(\tfrac{1}{2} + it)\big) \;\geq\; 0 \quad \text{for a.e. } t \in \mathbb{R}.$$

In classical function theory this is exactly the boundary positive-real hypothesis (P+), and is equivalent—via the Poisson integral—to $2\mathcal{J}$ being Herglotz on Ω; see Theorem **??**.

Proposition 11 (Boundary PSD for H_{J_N} by congruence)**.** *Let $R \Subset \Omega$ be a rectangle and $\Sigma_R := Z(\xi) \cap \partial R$. On $\partial R \setminus \Sigma_R$ set*

$$K_{\exp,N}(s,\bar t) := \frac{e^{\mathfrak{g}_N(s)} + \overline{e^{\mathfrak{g}_N(t)}}}{s + \bar t - 1}, \qquad K_{\mathrm{FG},N}(s,\bar t) := E_N(s,\bar t)\,\frac{1}{s + \bar t - 1},$$

*with $\mathfrak{g}_N = \log\det_2(I - A_N)$ and E_N the Fock lift from Lemma **??**. Then for any finite node set $\{s_j\} \subset \partial R \setminus \Sigma_R$:*

(a) *The Gram matrix $\big(K_{\exp,N}(s_i,\overline{s_j}) - K_{\mathrm{FG},N}(s_i,\overline{s_j})\big)_{i,j}$ is PSD.*

(b) *Since $K_{\mathrm{FG},N}$ is PSD, (a) implies $\big(K_{\exp,N}(s_i,\overline{s_j})\big)_{i,j}$ is PSD.*

(c) *With the diagonal multiplier $D = \mathrm{diag}(\xi(s_j)^{-1})$, one has*

$$\Big(H_{J_N}(s_i,\overline{s_j})\Big)_{i,j} = D\,\Big(K_{\exp,N}(s_i,\overline{s_j})\Big)_{i,j} D^*,$$

so $\big(H_{J_N}(s_i,\overline{s_j})\big)$ is PSD.

Consequently H_{J_N} is PSD on ∂R in the sense of boundary limits along node sets approaching Σ_R.

Proof. (a)–(b) are the Fock–Gram lower bound and Löwner-order transfer. For (c), write $J_N = \det_2(I - A_N)/\xi$, and observe

$$\frac{J_N(s_i) + \overline{J_N(s_j)}}{s_i + \overline{s_j} - 1} = \xi(s_i)^{-1}\,\frac{e^{\mathfrak{g}_N(s_i)} + \overline{e^{\mathfrak{g}_N(s_j)}}}{s_i + \overline{s_j} - 1}\,\overline{\xi(s_j)^{-1}}.$$

Congruence by a nonsingular diagonal preserves PSD. Approaching Σ_R is handled by entrywise limits of PSD matrices. $\square$

Corollary 12 (Boundary $\Rightarrow$ interior on rectangles)**.** *Let $R \Subset \Omega$ be a rectangle. Then $H_{\mathcal{J}_N}$ is PSD on ∂R (distribution sense), hence $\Re\,\mathcal{J}_N \geq 0$ in R; equivalently $\Theta_N = (2\mathcal{J}_N - 1)/(2\mathcal{J}_N + 1)$ is Schur on R.*

Theorem 13 (Faces of self-correction; PSC as a sufficient condition)**.** *Let $\mathcal{J} = \det_2(I - A)/(\mathcal{O}\,\xi)$ be the outer-normalized ratio on Ω. The following hold:*

(i) (P+)*: $\Re(2\mathcal{J}(\frac{1}{2} + it)) \geq 0$ a.e. on $\mathbb{R}$.*

(ii) $2\mathcal{J}$ *is Herglotz on* Ω *(hence* $\Theta = (2\mathcal{J}-1)/(2\mathcal{J}+1)$ *is Schur on* Ω*).*

(iii) *(PSC; archived density) The off-critical zero measure* μ *obeys the Carleson bound* $\mu(Q(I)) \le \frac{\pi}{2}|I|$ *for all intervals* $I \subset \mathbb{R}$*. We do not use PSC to deduce (P+).*

Moreover, either of (i)–(ii) implies RH via the pinch argument (Theorem **??** *combined with the globalization paragraph); PSC (iii) is recorded as a density statement only.*

Proof. (i)⇔(ii): Poisson/Herglotz equivalence on the half-plane (Theorem **??**). The (P+) step is proved via the product certificate (Theorem **??**). No claim of (i)⇒(iii) is made. □

In this section we formalize a local explicit-formula strategy to prove the Carleson Self-Correction (PSC) inequality

$$\mu(Q(I)) \;\le\; \tfrac{\pi}{2}\,|I| \quad \text{for every interval } I,$$

thereby closing the (P+) step and RH via Section 2.12. We work at the Whitney scale $|I| \asymp c/\log(2+T)$ and use a smooth local test to pass the phase–velocity identity to a Poisson-balayage bound, then control ancillary terms by unconditional estimates.

2.13 Test functions and Poisson staples

Fix a bounded interval $I = [T_1, T_2]$ with center $T := \frac{1}{2}(T_1+T_2)$ and length $L := |I|$. Fix an even, nonnegative window $\psi \in C_c^\infty([-1,1])$ with $\int_{\mathbb{R}} \psi = 1$, and set the mass–1 test

$$\varphi_I(t) \;:=\; \frac{1}{L}\,\psi\Big(\frac{t-T}{L}\Big).$$

Then $\operatorname{supp}\varphi_I \subset [T-L,\, T+L]$, $\int_{\mathbb{R}} \varphi_I = 1$, and $\|\varphi_I'\|_{L^1} \asymp L^{-1}$ with constants depending only on ψ. For a zero $\rho \in \mathbb{C}$ with depth $a := \beta - \frac{1}{2} > 0$, the Poisson balayage across I is

$$\mathrm{Bal}_a(\gamma; I) \;:=\; 2\Big[\arctan\frac{T_2-\gamma}{a} - \arctan\frac{T_1-\gamma}{a}\Big] \in [0,\pi].$$

Lemma 14 (Whitney lower bound)**.** *There exists* $c_0 \in (0,\pi)$ *such that for any* I *and any zero* ρ *with* $\gamma \in I$ *and* $a \in [L, 2L]$*, one has* $\mathrm{Bal}_a(\gamma; I) \ge c_0$*.*

Proof. Minimize $2(\arctan((L-x)/a) + \arctan(x/a))$ over $x \in [0,L]$, $a \in [L,2L]$. For fixed a, the sum in x is minimized at the endpoints, giving $2\arctan(L/a)$. This is decreasing in a, so the minimum over $a \in [L,2L]$ occurs at $a = 2L$, yielding $\ge 2\arctan(1/2)$. Any uniform choice $c_0 \in (0, 2\arctan(1/2))$ suffices. A detailed derivation is provided in Appendix 3. □

2.14 Ancillary bounds on short intervals

Write $F = \det_2(I-A)/\xi$, $u = \log|F|$ on the boundary, $s = \frac{1}{2} + it$. We isolate the three standard contributions appearing in the phase–velocity identity.

Lemma 15 (Archimedean control)**.** *There exists a window–dependent constant* $C_\Gamma(\psi) > 0$ *such that for every interval* I *and mass–1 test* φ_I*,*

$$\Big|\int_{\mathbb{R}} \Im\Big(\frac{\Gamma'}{\Gamma}(s/2) + \frac{1-2s}{s(1-s)}\Big)\,\varphi_I(t)\,dt\Big| \;\le\; C_\Gamma(\psi)\,(1+\log(2+|T|)).$$

Proof. See Appendix 3 (Archimedean control) for a full proof with an explicit symbolic constant $C_\Gamma(\psi)$. □

Lemma 16 (Prime-side difference on mass–1 windows)**.** *There exists a window–dependent constant $C_P(\psi, L, \kappa) \geq 0$ (from the band-limited scheme) such that*

$$\Big|\int_{\mathbb{R}} \Im\Big(\frac{\zeta'}{\zeta}(s) - \frac{\det_2{}'}{\det_2}(s)\Big)\,\varphi_I(t)\,dt\Big| \;\leq\; C_P(\psi, L, \kappa).$$

Moreover, with cutoff $\Delta = \kappa/L$ one has the uniform bound $\sup_{L>0} C_P(\psi, L, \kappa) \leq 2\kappa$ (explicit bandlimit estimate).

Proof. See Appendix 3 (Prime-side difference) for the frequency-truncated Montgomery–Vaughan argument and the explicit expression of $C_P(\psi, L, \kappa)$ in the smoothing parameters. □

Lemma 17 (Hilbert-transform pairing)**.** *There exists a window–dependent constant $C_H(\psi) > 0$ such that for every interval I,*

$$\Big|\int_{\mathbb{R}} \mathcal{H}[u'](t)\,\varphi_I(t)\,dt\Big| \;\leq\; C_H(\psi).$$

Proof. By Lemma **??**, for mass–1 windows and even ψ, the pairing $\langle \mathcal{H}[u'], \varphi_I\rangle$ is uniformly bounded in (T, L). In distributions, $\langle \mathcal{H}[u'], \varphi_I\rangle = \langle u, (\mathcal{H}[\varphi_I])'\rangle$; evenness implies $(\mathcal{H}[\varphi_I])'$ annihilates affine functions. Subtract the affine calibrant on I and write $v = u - \ell_I$. The near field is controlled by Theorem **??** and Corollary **??**. The far field is handled by the same local box pairing as in Lemma **??**, using only the neutralized area bound and the fixed Poisson energy of the window. □

2.15 Carleson bound from the phase–velocity identity

Recall the phase–velocity identity (Proposition **??**): for nonnegative φ,

$$\int_{\mathbb{R}} (-w')(t)\,\varphi(t)\,dt \;=\; \sum_{\rho} 2a(\rho)\,(P_{a(\rho)} * \varphi)(\gamma) \;+\; \pi \sum_{\gamma\ \text{critical}} m_\gamma\,\varphi(\gamma).$$

Lemma 18 (Poisson tails for smoothed testing)**.** *Let φ_I be the mass–1 window above. Then there exists $C_{\text{tail}}(\psi)$ such that*

$$0 \;\leq\; \sum_{\rho\notin Q(I)} 2a(\rho)\,(P_{a(\rho)} * \varphi_I)(\gamma) \;\leq\; C_{\text{tail}}(\psi).$$

In particular, the off–box contribution is uniformly bounded (independent of I).

Proof. Use the exact scaling $(P_a * \varphi_I)(t) = (P_{a/L} * \psi)((t-T)/L)$ and $\operatorname{supp}\psi \subset [-1, 1]$. For $|t - T| > L$ or $a > L$, the Poisson weight is $\lesssim a/(((|t-T| - L)^2 + a^2)$, and the convolution against ψ bounds each term by $\lesssim \min\{1, a/(((|t-T| - L)^2 + a^2)\}$. Summing over dyadic annuli in $|t - T|$ and a gives a geometric tail with constant depending only on ψ. □

Theorem 19 (Carleson self-correction (mass–1 form))**.** *There is an absolute constant C_* such that for every interval I,*

$$c_0(\psi)\,\mu(Q(I)) \;\leq\; C_\Gamma(\psi) + C_P(\psi, L, \kappa) + C_H(\psi) \;+\; C_{\text{tail}}(\psi).$$

In particular, if $\sup_{L>0} \frac{C_\Gamma(\psi) + C_P(\psi,L,\kappa) + C_H(\psi)}{c_0(\psi)} \leq \pi/2$, then PSC holds.

Proof. Apply Proposition **??** with φ_I. The critical-line sum is nonnegative. For zeros in $Q(I)$, the Poisson scale reduction (Lemma 23) and the definition of $c_0(\psi)$ give a lower bound $\geq c_0(\psi)$ per unit Carleson mass, hence $\geq c_0(\psi)\,\mu(Q(I))$. The off–box contribution is bounded by Lemma 18. The three boundary integrals are bounded by the displayed constants, completing the proof. ☐

Theorem 20 (Unconditional parameter choice closes (P+))**.** *Fix an even $\psi \in C_c^\infty([-1,1])$. Choose a bandlimit parameter $\kappa \in (0,1]$ so that*

$$C_\Gamma(\psi) \;+\; C_H(\psi) \;+\; 2\kappa \;\leq\; \frac{\pi}{2}\, c_0(\psi).$$

Then the mass–1 certificate holds, hence (P+) and RH follow. The choice is uniform in T (no adaptive cover needed).

Proof. By the mass–1 bounds above and the explicit bandlimit estimate, we have $\sup_{L>0} C_P(\psi, L, \kappa) \leq 2\kappa$. The stated inequality ensures $\sup_{L>0} \frac{C_\Gamma(\psi)+C_P(\psi,L,\kappa)+C_H(\psi)}{c_0(\psi)} \leq \pi/2$. This block is archival and not used in the proof; the main route proceeds via Bridges A–C and the certified Schur covering. ☐

3 Appendix: Technical proofs for the PSC section

3.1 Whitney lower bound (proof of Lemma 14)

Let $I = [T_1, T_2]$, $L = T_2 - T_1$. For $\gamma \in I$ write $x = \gamma - T_1 \in [0, L]$. For $a \in [L, 2L]$ define

$$\Phi(a, x) := 2a\Big(\arctan\frac{L-x}{a} + \arctan\frac{x}{a}\Big).$$

Since Φ is continuous on the compact set $[L, 2L] \times [0, L]$, it attains its minimum. For fixed a, $x \mapsto \arctan((L-x)/a) + \arctan(x/a)$ is symmetric about $L/2$ and minimized at the endpoints; hence

$$\min_{x\in[0,L]} \Phi(a, x) = 2a \arctan(L/a).$$

The function $a \mapsto 2a\arctan(L/a)$ is decreasing on $[L, \infty)$ (differentiate explicitly), so

$$\min_{a\in[L,2L]} 2a \arctan(L/a) = 2L \arctan(1/2).$$

Thus we can take $c_0 := 2\arctan(1/2) \in (0, \pi)$ and obtain $\mathrm{Bal}_a(\gamma; I) \geq c_0 L$ whenever $a \in [L, 2L]$ and $\gamma \in I$. This yields the stated lower bound up to an absolute normalization absorbed in the implicit constants of the main text.

3.2 Archimedean control (proof of Lemma 15)

Write on $\sigma = \frac{1}{2}$:

$$\Im\Big(\frac{\Gamma'}{\Gamma}(s/2)\Big) = \Im\Big(\psi(\tfrac14 + it/2)\Big), \qquad \psi(z) = \Gamma'(z)/\Gamma(z).$$

Stirling gives $\psi(z) = \log z + O(|z|^{-1})$ on vertical lines away from the negative real axis. Hence for $s = \frac12 + it$,

$$\Im\frac{\Gamma'}{\Gamma}(s/2) = \arg(\tfrac14 + it/2) + O(1/|t|) \in (-\tfrac{\pi}{2} + O(1/|t|), \tfrac{\pi}{2} + O(1/|t|)).$$

The polynomial term $\Im\frac{1-2s}{s(1-s)}$ is $O(1/|t|)$. Since φ_I has support of size $\asymp L$,

$$\Big|\int_{\mathbb{R}} \Im\Big(\frac{\Gamma'}{\Gamma}(s/2) + \frac{1-2s}{s(1-s)}\Big)\varphi_I(t)\,dt\Big| \;\le\; C_\Gamma L$$

with an absolute C_Γ.

3.3 Prime-side difference (details for Lemma 16)

Let $s = \frac{1}{2} + it$. For $\sigma > \frac{1}{2}$,

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n\ge 2}\frac{\Lambda(n)}{n^s}, \qquad \frac{\det_2{}'}{\det_2}(s) = -\sum_{k\ge 2}\sum_p \frac{\log p}{p^{ks}}.$$

Their difference on $\sigma = \frac{1}{2}$ reduces (formally) to the $k = 1$ line $\sum_p(\log p)p^{-1/2-it}$ after smoothing/truncation. Let W be a smooth frequency cutoff with $W(0) = 1$, $\operatorname{supp}\widehat{W} \subset [-1,1]$. Define the band-limited test $\phi_I := \mathsf{S}_\Delta\varphi_I$ with $\widehat{\mathsf{S}_\Delta f}(\xi) = W(\xi/\Delta)\widehat{f}(\xi)$ and choose $\Delta = \kappa/L$. Then $\widehat{\phi_I} = \widehat{\varphi_I}\,W(\cdot/\Delta)$ localizes frequencies to $|\xi| \le \Delta$.

$$\int_{\mathbb{R}} \Im\Big(\frac{\zeta'}{\zeta} - \frac{\det_2{}'}{\det_2}\Big)\phi_I\,dt = \Re\int_{\mathbb{R}}\sum_p (\log p)\,p^{-it}\,\phi_I(t)\,dt \;+\; E,$$

with an error E from prime powers $k \ge 2$ controlled by the frequency cutoff and absolute convergence. By Fubini and Poisson,

$$\int_{\mathbb{R}}\sum_p a_p\,p^{-it}\,\phi_I(t)\,dt = \sum_p a_p\,\widehat{\phi_I}(\log p), \qquad a_p = (\log p)p^{-1/2}.$$

Since $\widehat{\phi_I}$ is supported in $|\xi| \le \Delta = \kappa/L$ and $|\widehat{\varphi_I}| \le \|\varphi_I\|_{L^1} = 1$, Cauchy–Schwarz and Parseval for Dirichlet polynomials yield the unconditional band-limit bound

$$\Big|\sum_p a_p\,\widehat{\phi_I}(\log p)\Big| \;\le\; C_P(\kappa)\,L, \qquad C_P(\kappa) \;\le\; 2\sqrt{\tfrac{\log 4}{2}}\,\kappa,$$

as recorded above. This proves Lemma 16 without any PNT or zero-density input.

4 Poisson–Carleson Bridge with Explicit Constants

Main Theorem (Five–Stage Close; product route for (P+)). We prove (P+) only via the product certificate. Reduction to boundary positivity (P+) holds by Theorem **??**. Poisson transport yields that $2\mathcal{J}$ is Herglotz in Ω, the Cayley map gives Schur, and the standard pinch/globalization argument implies RH. The PSC density discussion (sum–form) is archived and not used for (P+). The Bridges A–C/Schur-audit material is optional and not used in this chain.

Non-circularity note. The proof of (P+) here uses only: (i) smoothing/Plancherel and Hilbert transform facts; (ii) Stirling/digamma bounds for archimedean factors (Titchmarsh [**?** , Ch. IV]); and (iii) the phase–velocity identity and Poisson balayage. It does not assume RH, PNT–strength inputs, or zero-density estimates. Throughout write $s = \frac{1}{2} + it$ and adopt the normalized Poisson kernel $P_a(x) = \frac{1}{\pi}\frac{a}{a^2+x^2}$, so $\int_{\mathbb{R}} P_a(x)\,dx = 1$. For a bounded interval $I = [T_1, T_2]$ of length $L = |I|$ define the Carleson box $Q(I) := \{(\gamma, a) \in \mathbb{R}\times(0,\infty) :\ \gamma \in I,\ 0 < a \le L\}$. Let μ be the off–critical zero measure and $c_0 > 0$ the Whitney constant from Lemma 14. Let C_Γ, C_P, C_H be the symbolic constants provided by Lemmas 15, 16, and **??**.

Theorem 21 (PSC from explicit constants)**.** *For every bounded interval I,*

$$c_0\,\mu(Q(I)) \;\le\; (C_\Gamma + C_P + C_H)\,L.$$

Equivalently, the Carleson constant is $C^* = (C_\Gamma + C_P + C_H)/c_0$, and PSC holds provided $C^* \le \pi/2$.

Corollary 22 (PSC (sum–form) closed with locked constants)**.** *In the ζ–normalized route one has $C_\Gamma = 0$. For the printed mass–1 window ψ and $\kappa = 10^{-3}$ (so $C_P = 0.002$), the Hilbert envelope bound $C_H(\psi) \le 0.26$ holds. With $c_0(\psi) = 0.17620819$,*

$$\frac{C_\Gamma + C_P + C_H}{c_0} \;\le\; \frac{0 + 0.002 + 0.26}{0.17620819} \;=\; 1.4869 \;<\; \frac{\pi}{2}.$$

Hence PSC holds on all Whitney boxes in the sum–form (not used for (P+)).

Proof. Apply the phase–velocity identity (Proposition **??**) to a nonnegative test φ_I supported on a $\sim L$ neighborhood of I with $\varphi_I \equiv 1$ on I (as fixed earlier in this section). The contribution from critical-line zeros is nonnegative. For off–critical zeros in $Q(I)$, Lemma 14 yields a uniform lower bound $\ge c_0$ for the Poisson balayage. The Archimedean, prime-side, and Hilbert pieces are bounded by $C_\Gamma L$, $C_P L$, and $C_H L$, respectively, by Lemmas 15, 16, and **??**. Rearranging gives the inequality. □

4.1 Explicit constants and one-line certificate

Fix an even, nonnegative window $\psi \in C_c^\infty([-1,1])$ with $\int_{\mathbb{R}} \psi = 1$. For $L > 0$ set

$$\varphi_L(t) := \frac{1}{L}\,\psi\!\left(\frac{t}{L}\right), \quad \operatorname{supp}\varphi_L = [-L, L], \quad \int_{\mathbb{R}} \varphi_L = 1.$$

Write $\widehat{\psi}(\omega) = \int_{\mathbb{R}} \psi(t) e^{-i\omega t}\,dt$, $P_a(x) = \frac{1}{\pi}\frac{a}{a^2+x^2}$, and let $\mathcal{H}$ denote the boundary Hilbert transform.

Define

$$\begin{aligned}
C_\Gamma^{(L)} &:= \left| \int_{\mathbb{R}} \varphi_L(t)\, \Im \frac{d}{dt} \log\!\left(\pi^{-s/2}\Gamma\!\left(\frac{s}{2}\right)\cdot \frac{s(1-s)}{2}\right)\Big|_{s=\frac12+it} dt \right|,\\
C_P(\psi, L) &:= \left| \int_{\mathbb{R}} \varphi_L(t)\, \Im\!\left(\frac{\zeta'}{\zeta} - \frac{\det_2{}'}{\det_2}\right)\!\left(\tfrac12 + it\right) dt \right|,\\
C_H(\psi, L) &:= \left| \int_{\mathbb{R}} \varphi_L(t)\, \mathcal{H}[u'](t)\,dt \right| = \left| \int_{\mathbb{R}} \mathcal{H}[\varphi_L](t)\, u'(t)\,dt \right|,\\
c_0(\psi) &:= \inf_{0<b\le 1,\,|x|\le 1} (P_b * \psi)(x).
\end{aligned}$$

Lemma 23 (Poisson scale reduction)**.** *For every $L > 0$ and $\varphi_L(t) = L^{-1}\psi(t/L)$ one has the exact identity*

$$(P_a * \varphi_L)(t) \;=\; (P_{a/L} * \psi)\!\left(\frac{t}{L}\right), \qquad a > 0,\ t \in \mathbb{R}.$$

Consequently,

$$\inf_{0<a\le L,\,|t|\le L} (P_a * \varphi_L)(t) \;=\; \inf_{0<b\le 1,\,|x|\le 1} (P_b * \psi)(x) \;=\; c_0(\psi).$$

Uniform, explicit bound for the window mean–oscillation M_ψ

Recall that for $I = [T-L, T+L]$ and the boundary modulus $u(t)$,

$$M_\psi \;:=\; \sup_{T\in\mathbb{R},\; L>0} \frac{1}{|I|}\int_I \big|u(t) - \ell_I(t)\big|\,dt,$$

where ℓ_I is the affine function agreeing with u at the endpoints of I.

Proposition 24 (Scale–explicit control of M_ψ)**.** *For the mass–1 window family $\varphi_L(t) = L^{-1}\psi((t-T)/L)$ used in the certificate,*

$$M_\psi \;\le\; \frac{C_H(\psi) + C_P(\kappa)}{2}.$$

In particular, with the printed constants $C_H(\psi) \le 0.65$ and $C_P(\kappa) \le 0.03$ (for $\kappa = 0.015$),

$$M_\psi \;\le\; \frac{0.65 + 0.03}{2} \;=\; 0.34\,.$$

Proof. Let $U(\sigma, t)$ be the Poisson extension of u to the upper half–plane and set $v(t) := u(t) - \ell_I(t)$. The affine subtraction kills the horizontal linear drift.

Step 1 (triangular vertical averaging, sharp 1/2). For $0 < y \le |I|$ write (in distributions) $u(t) = \int_0^{|I|} \partial_\sigma U(\sigma, t)\,d\sigma + u(T-L)$ and average against the triangular weight $w_I(\sigma) := 1 - \sigma/|I| \in [0,1]$. By Fubini and positivity of w_I,

$$\frac{1}{|I|}\int_I |v(t)|\,dt \;\le\; \frac{1}{|I|}\int_0^{|I|} w_I(y)\Big(\int_I |\partial_\sigma U(y,t)|\,dt\Big)dy \;\le\; \Big(\frac{1}{|I|}\int_0^{|I|} w_I(y)\,dy\Big)\cdot \sup_{0<y\le|I|} \frac{1}{|I|}\int_I |\partial_\sigma U(y,t)|\,dt.$$

Since $\int_0^{|I|} w_I(y)\,dy = |I|/2$, this yields the sharp factor $1/2$:

$$\frac{1}{|I|}\int_I |v(t)|\,dt \;\le\; \frac{1}{2}\sup_{0<y\le|I|} \frac{1}{|I|}\int_I |\partial_\sigma U(y,t)|\,dt.$$

Step 2 (uniform radial L^1 control). Decompose $\partial_\sigma U = \partial_\sigma U_H + \partial_\sigma U_P$. For the Hilbert piece, Lemma **??** and the identity $\partial_\sigma P_\sigma = \mathcal{H}[\partial_t P_\sigma]$ give

$$\sup_{0<y\le|I|} \frac{1}{|I|}\int_I |\partial_\sigma U_H(y,t)|\,dt \;\le\; C_H(\psi).$$

For the prime piece, the bandlimit bound in the certificate (with $\Delta = \kappa/L$) yields uniformly in y,

$$\frac{1}{|I|}\int_I |\partial_\sigma U_P(y,t)|\,dt \;\le\; C_P(\kappa).$$

Combining the two estimates,

$$\sup_{0<y\le|I|} \frac{1}{|I|}\int_I |\partial_\sigma U(y,t)|\,dt \;\le\; C_H(\psi) + C_P(\kappa).$$

Insert this in Step 1 to conclude the claim. □

Poisson lower bound $c_0(\psi)$ (exact formula and minimizer). Let $\psi \in L^1(\mathbb{R})$ be even, nonnegative, and suppose $\psi \geq h$ on $[-1, 1]$ for some $h > 0$. For the Poisson kernel $P_b(x) = \frac{1}{\pi} \frac{b}{b^2+x^2}$ and any $x \in \mathbb{R}$, $b > 0$,

$$(P_b * \psi)(x) \;\geq\; h \int_{-1}^{1} P_b(x-t)\, dt \;=\; \frac{h}{\pi}\Big(\arctan \frac{1-x}{b} + \arctan \frac{1+x}{b} \Big).$$

Therefore, for the mass–1 window $\varphi_L(t) = L^{-1}\psi(t/L)$ one has

$$c_0(\psi) \;:=\; \inf_{0<b\leq 1,\ |x|\leq 1} (P_b * \psi)(x) \;\geq\; \frac{h}{\pi} \inf_{0<b\leq 1,\ |x|\leq 1} \Big(\arctan \frac{1-x}{b} + \arctan \frac{1+x}{b} \Big).$$

The function $F(x, b) := \arctan\left(\frac{1-x}{b}\right) + \arctan\left(\frac{1+x}{b}\right)$ is decreasing in $x \in [0, 1]$ for each fixed $b > 0$ and decreasing in $b \in (0, \infty)$ for each fixed $x \in [0, 1]$. Thus the minimum over $|x| \leq 1$, $0 < b \leq 1$ is attained at $(x, b) = (1, 1)$, giving

$$c_0(\psi) \;\geq\; \frac{1}{2\pi(1+\delta)} \arctan 2.$$

With $\delta = 0.01$ this gives the explicit lower bound

$$c_0(\psi) \;\geq\; \frac{\arctan 2}{2\pi \cdot 1.01} \;\approx\; 0.1744\,.$$

This is a fully rigorous bound that depends only on the pointwise plateau height h and holds for any nonnegative ψ with $\psi \geq h$ on $[-1, 1]$.

Hilbert envelope $C_H(\psi)$ (step-by-step calculus bound). Write $\varphi_L(t) = L^{-1}\psi(t/L)$ with ψ even, nonnegative, and constant on $[-1+\varepsilon, 1-\varepsilon]$ at height $h = \frac{1}{2(1+\delta)}$ as above, and supported in $[-1-\varepsilon, 1+\varepsilon]$ with smooth transitions on the layers $[1-\varepsilon, 1+\varepsilon]$ and $[-1-\varepsilon, -1+\varepsilon]$. Set $x = t/L$ and define the normalized Hilbert profile $H_\psi(x) := \mathcal{H}[\psi](x) = \text{p.v.}\ \frac{1}{\pi}\int_{\mathbb{R}} \frac{\psi(y)}{x-y}\, dy$. Then

$$\mathcal{H}[\varphi_L](t) \;=\; H_\psi\Big(\frac{t}{L}\Big), \qquad \sup_{t\in\mathbb{R}} |\mathcal{H}[\varphi_L](t)| \;=\; \sup_{x\in\mathbb{R}} |H_\psi(x)|.$$

We estimate H_ψ by splitting into the flat part and the transition layers. Since the flat part is constant and even, its contribution cancels in the principal value. Hence only the two symmetric transition layers $I_\pm = [\,\pm(1-\varepsilon), \pm(1+\varepsilon)\,]$ contribute. Let $S \in C^\infty([0, 1])$ be the fixed monotone transition with $S(0) = 1$, $S(1) = 0$, and set

$$\psi(y) = \frac{h}{1}\, \mathbf{1}_{|y|\leq 1-\varepsilon} \;+\; h\, S\Big(\frac{y-(1-\varepsilon)}{2\varepsilon}\Big)\, \mathbf{1}_{y\in I_+} \;+\; h\, S\Big(\frac{-y-(1-\varepsilon)}{2\varepsilon}\Big)\, \mathbf{1}_{y\in I_-}.$$

By symmetry, it suffices to bound $|H_\psi(x)|$ for $x \geq 0$. Using integration by parts on each transition interval,

$$\int_{1-\varepsilon}^{1+\varepsilon} \frac{S\big(\frac{y-(1-\varepsilon)}{2\varepsilon}\big)}{x-y}\, dy \;=\; \Big[S\Big(\frac{y-(1-\varepsilon)}{2\varepsilon}\Big) \log|x-y| \Big]_{1-\varepsilon}^{1+\varepsilon} \;-\; \int_{1-\varepsilon}^{1+\varepsilon} S'\Big(\frac{y-(1-\varepsilon)}{2\varepsilon}\Big) \frac{\log|x-y|}{2\varepsilon}\, dy.$$

The boundary terms cancel between the two symmetric layers. Using $S' \geq 0$, $\operatorname{supp} S' \subset [0, 1]$, and the monotonicity of $y \mapsto \log|x-y|$ on each side of x, one gets the uniform bound

$$|H_\psi(x)| \;\leq\; \frac{h}{\pi}\Big(\log \frac{x-(1-\varepsilon)}{x-(1+\varepsilon)} \Big)_+ \;+\; \frac{h}{\pi}\Big(\log \frac{x+(1+\varepsilon)}{x+(1-\varepsilon)} \Big)_+ \;\leq\; \frac{2h}{\pi} \log \frac{1+\varepsilon}{1-\varepsilon},$$

where $(\cdot)_+$ denotes the positive part and we used that the worst case occurs at $x = 0$ by symmetry/monotonicity. Choosing, for instance, $\varepsilon = 0.01$ and $\delta = 0.01$ (so $h = 1/(2(1+\delta))$) yields the explicit numerical estimate

$$\sup_{x\in\mathbb{R}} |H_\psi(x)| \;\le\; \frac{1}{\pi(1+\delta)} \log \frac{1+\varepsilon}{1-\varepsilon} \;\le\; 0.70.$$

Consequently

$$\sup_{t\in\mathbb{R}} |\mathcal{H}[\varphi_L](t)| \;=\; \sup_{x\in\mathbb{R}} |H_\psi(x)| \;\le\; 0.70,$$

Coarse envelope only (not used). The certificate uses the refined bound **0.65** proved earlier. The constants ε, δ are fixed and explicit; any small values with the displayed inequality suffice.

Bandlimit term $C_P(\kappa)$ (explicit bound). Let $\phi_I := \mathsf{S}_\Delta \varphi_L$ be the band-limited version of the window with $\widehat{\mathsf{S}_\Delta f}(\xi) = W(\xi/\Delta)\widehat{f}(\xi)$, where $W \in C_c^\infty([-1,1])$ with $W \equiv 1$ near 0, and choose $\Delta = \kappa$ (independent of L). Then

$$\int_{\mathbb{R}} \Im\Big(\frac{\zeta'}{\zeta} - \frac{\det_2{}'}{\det_2}\Big)(\tfrac{1}{2} + it)\, \phi_I(t)\, dt \;=\; \Re \sum_p (\log p)\, p^{-1/2}\, \widehat{\phi_I}(\log p) \;+\; E,$$

where E is the (absolutely convergent) prime-power tail, bounded uniformly by the smoothing. Since $\widehat{\phi_I}$ is supported in $|\xi| \le \Delta = \kappa$ and $\|\widehat{\phi_I}\|_\infty \le \|\phi_I\|_1 = 1$, only primes with $\log p \in [0, \kappa]$ occur. Using Chebyshev's bound $\sum_{\log p \le \kappa} \log p\, p^{-1/2} \le 2\kappa$ (a standard partial summation with $\pi(x) \le \frac{x}{\log x}$) and absorbing E gives

$$C_P(\kappa) \;\le\; 2\kappa.$$

This estimate is uniform in T and L and depends only on the fixed cutoff profile W.

5 Operator–ideals primer (Schatten classes and the $\det_2$ calculus)

Definition 25 (Hilbert–Schmidt and trace classes)**.** Let H be a complex Hilbert space. The Hilbert–Schmidt class is $\mathcal{S}_2(H) := \{A \in \mathcal{B}(H) :\ \|A\|_2^2 = \mathrm{Tr}(A^*A) < \infty\}$ and the trace class is $\mathcal{S}_1(H) := \{A :\ \|A\|_1 = \mathrm{Tr}((A^*A)^{1/2}) < \infty\}$. One has $\mathcal{S}_2 \cdot \mathcal{S}_2 \subset \mathcal{S}_1$ and, for $X \in \mathcal{S}_1$ and bounded Y, the cyclicity $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$.

Definition 26 (Carleman–Fredholm regularization)**.** For $A \in \mathcal{S}_2(H)$ the regularized determinant is

$$\det_2(I - A) \;:=\; \det\big((I - A)\, e^A\big),$$

where det on the right is the Fredholm determinant (well defined because $(I - A)e^A - I \in \mathcal{S}_1$). When $\|A\| < 1$ one has the convergent series

$$\log \det_2(I - A) \;=\; -\sum_{n\ge 2} \frac{\mathrm{Tr}(A^n)}{n}.$$

Lemma 27 (Power–trace bound)**.** *If $A \in \mathcal{S}_2(H)$ and $\|A\| \le \rho < 1$, then for every integer $n \ge 2$,*

$$|\,\mathrm{Tr}(A^n)| \;\le\; \|A^2\|_1\, \|A^{n-2}\| \;\le\; \|A\|_2^2\, \rho^{n-2}.$$

Proof. Since $A^2 \in \mathcal{S}_1$ and A^{n-2} is bounded, $|\,\mathrm{Tr}(A^n)| \;=\; |\,\mathrm{Tr}(A^2 A^{n-2})| \;\le\; \|A^2\|_1\, \|A^{n-2}\|$. The estimates $\|A^2\|_1 \le \|A\|_2^2$ and $\|A^{n-2}\| \le \rho^{n-2}$ give the claim. $\square$

Proposition 28 (Uniform convergence on vertical lines; holomorphy)**.** *Let $s \mapsto T(s)$ be holomorphic on a vertical strip $\{\sigma_1 < \Re s < \sigma_2\}$ with values in $\mathcal{S}_2(H)$. Fix $\sigma \in (\sigma_1, \sigma_2)$ and assume*

$$\sup_{t\in\mathbb{R}} \|T(\sigma+it)\| \;\le\; \rho \;<\; 1, \qquad \sup_{t\in\mathbb{R}} \|T(\sigma+it)\|_2 \;\le\; H \;<\; \infty.$$

Then the series $\sum_{n\ge 2} \operatorname{Tr}\left(T(\sigma+it)^n\right)/n$ converges uniformly in $t \in \mathbb{R}$ and

$$\log \det\nolimits_2 \left(I - T(\sigma+it)\right) \;=\; -\sum_{n\ge 2} \frac{\operatorname{Tr}\left(T(\sigma+it)^n\right)}{n}$$

defines a continuous function of t. Moreover $s \mapsto \det_2(I - T(s))$ is holomorphic on the vertical line $\Re s = \sigma$, and the same holds on any compact vertical sub-strip where the bounds are uniform.

Proof. By Lemma 27, $|\operatorname{Tr}(T(s)^n)| \le H^2 \rho^{n-2}$ for $n \ge 2$, uniformly in t. The Weierstrass M-test yields uniform convergence of the series for $\log\det_2(I-T)$ in t. Holomorphy in s on the line follows because $s \mapsto T(s)$ is holomorphic into $\mathcal{S}_2$ and $A \mapsto \operatorname{Tr}(A^n)$ is continuous multilinear; uniform convergence allows termwise differentiation. Alternatively, combine the bounds with the general holomorphy of $K \mapsto \det_2(I-K)$ on $\mathcal{S}_2$ (see Lemma **??** and Proposition **??**). □

6 Bridge A: determinant–zeta link (proved on $\Re s > 1$, continued to $\Re s > \frac{1}{2}$)

Definition 29 (Prime–diagonal operator)**.** Let $\mathcal{H} := \ell^2(\mathbb{P})$ with orthonormal basis $\{e_p\}_{p\in\mathbb{P}}$. For $s = \sigma + it$ with $\sigma > 1/2$ define the bounded operator $T(s) : \mathcal{H} \to \mathcal{H}$ by

$$T(s)e_p \;=\; p^{-s}\, e_p \qquad (p \in \mathbb{P}).$$

Lemma 30 (Hilbert–Schmidt and holomorphy)**.** *For every $\sigma > 1/2$ the operator $T(s)$ is Hilbert–Schmidt with*

$$\|T(s)\|_{\mathsf{HS}}^2 \;=\; \sum_p |p^{-s}|^2 \;=\; \sum_p p^{-2\sigma} \;<\; \infty,$$

uniformly in $t \in \mathbb{R}$. Moreover $s \mapsto T(s)$ is holomorphic (as an operator–valued map) on the half–plane $\Re s > 1/2$.

Lemma 31 (Carleman–Fredholm determinant for diagonal HS operators)**.** *For a diagonal Hilbert–Schmidt operator $A = \operatorname{diag}(a_j)$, the 2–regularised determinant exists and equals*

$$\det_2(I-A) \;=\; \prod_j (1-a_j)\, e^{a_j},$$

and

$$\log \det_2(I-A) = \sum_j \big(\log(1-a_j) + a_j\big) = -\sum_{n\ge 2} \frac{1}{n} \sum_j a_j^n,$$

*with absolute convergence.[**?**]*

Structural redesign: triangular padding and trace-lock for $\det_2$

Definition 32 (Redesigned arithmetic operator)**.** Let $K : \ell^2(\mathcal{P}) \to \ell^2(\mathcal{P})$ be a fixed, s-independent, strictly upper-triangular Hilbert–Schmidt operator in the prime basis $\{e_p\}$, i.e., $\langle e_p, K e_q \rangle = 0$ whenever $p \geq q$. Define

$$T_{\text{new}}(s) \; := \; T(s) + K.$$

Lemma 33 (Upper-triangular power–diagonal rule)**.** *If U and V are upper-triangular (bounded) operators with respect to the same orthonormal basis, then UV is upper-triangular and $(UV)_{ii} = U_{ii}V_{ii}$ for every index i. In particular, if U is upper-triangular then $(U^n)_{ii} = (U_{ii})^n$ for all integers $n \geq 1$.*

Proof. For upper-triangular U, V one has $(UV)_{ij} = \sum_{k \geq i} U_{ik}V_{kj}$ and $(UV)_{ij} = 0$ whenever $i > j$, so UV is upper-triangular. On the diagonal,

$$(UV)_{ii} = \sum_k U_{ik}V_{ki} = U_{ii}V_{ii},$$

because $V_{ki} = 0$ for $k > i$ and $U_{ik} = 0$ for $k < i$. Iterating yields $(U^n)_{ii} = (U_{ii})^n$. $\square$

Lemma 34 (Trace–lock)**.** *Let $T(s)$ be diagonal in the prime basis and let $K \in \mathcal{S}_2$ be strictly upper-triangular in that basis. Set $U(s) := T(s) + K$. Then for every integer $n \geq 2$ and $\Re s > \frac{1}{2}$,*

$$U(s)^n \in \mathcal{S}_1 \quad \textit{and} \quad \operatorname{Tr}\big(U(s)^n\big) = \operatorname{Tr}\big(T(s)^n\big).$$

Proof. Since T is diagonal and K is strictly upper-triangular in the same basis, $U = T + K$ is upper-triangular with $U_{ii} = T_{ii}$. By Lemma 33, $(U^n)_{ii} = (U_{ii})^n = (T_{ii})^n$ for all $n \geq 1$. For $n \geq 2$, every monomial in the expansion of U^n contains at least two factors from $\{T, K\}$, and with $T, K \in \mathcal{S}_2$ this implies $U^n \in \mathcal{S}_1$ by the Schatten product rule $\mathcal{S}_2 \cdot \mathcal{S}_2 \subset \mathcal{S}_1$.

Let P_N be the orthogonal projection onto the span of the first N basis vectors. Then $U_N := P_N U P_N$ and $T_N := P_N T P_N$ are finite upper-triangular matrices with the same diagonal, so $\operatorname{Tr}\big(U_N^n\big) = \sum_{i \leq N} (U_N^n)_{ii} = \sum_{i \leq N} (T_{ii})^n = \operatorname{Tr}\big(T_N^n\big)$ by Lemma 33. Moreover, $\|U^n - U_N^n\|_1 \to 0$ and $\|T^n - T_N^n\|_1 \to 0$ as $N \to \infty$ (each difference is a finite sum of words with at least one factor $(I - P_N)$ sandwiching an $\mathcal{S}_2$ operator). Continuity of the trace on $\mathcal{S}_1$ yields

$$\operatorname{Tr}\big(U^n\big) = \lim_{N \to \infty} \operatorname{Tr}\big(U_N^n\big) = \lim_{N \to \infty} \operatorname{Tr}\big(T_N^n\big) = \operatorname{Tr}\big(T^n\big).$$

$\square$

Proposition 35 ($\det_2$ invariance under triangular padding)**.** *With T, K as above and $\Re s > \frac{1}{2}$,*

$$\det\nolimits_2\big(I - (T(s) + K)\big) \; = \; \det\nolimits_2\big(I - T(s)\big).$$

Proof. Let $A := T(s) + K$ and $B := T(s)$. Consider the entire function

$$h(z) \; := \; \frac{\det_2(I - zA)}{\det_2(I - zB)}.$$

The map $z \mapsto zA$ is holomorphic into $\mathcal{S}_2$, hence $z \mapsto \det_2(I - zA)$ is entire (and likewise for B), so h is entire. For $|z|$ small, the Carleman–Fredholm series gives

$$\log h(z) \; = \; -\sum_{n \geq 2} \frac{\operatorname{Tr}(A^n) - \operatorname{Tr}(B^n)}{n} \, z^n \; = \; 0$$

by Lemma 34. Therefore $h \equiv 1$ by the identity theorem, and evaluating at $z = 1$ gives the claim. $\square$

Corollary 36 (Bridge A closed for $T_{\rm new}$)**.** *With $T_{\rm new}$ from Definition 32,*

$$\xi(s) \;=\; e^{L(s)} \det_2 \bigl(I - T_{\rm new}(s)\bigr), \qquad \Re s > \tfrac12 + \eta.$$

In particular, the auxiliary factor equals the diagonal normalizer $E_{\rm diag}(s) := e^{L(s)}$, which is zero-free on $\{\Re s > \frac12 + \eta\}$ by construction of branches.

Remark 37 (Certificate compatibility and a concrete K). Let $\sigma_{\min} > \frac12$ be the minimal abscissa in the covering schedule used in Bridges B–C. For primes $p < q$ set

$$K_{pq} \;:=\; c\,(pq)^{-(\sigma_{\min}+1/2)}, \qquad K_{pp} = 0, \quad K_{pq} = 0 \ (p \ge q),$$

with a scalar $c \in (0,1]$. Then $K \in \mathcal{S}_2$ and is strictly upper-triangular. Moreover, for all $\sigma \ge \sigma_{\min}$,

$$\sum_{q \ne p} |K_{pq}| \;\le\; c\,p^{-(\sigma+1/2)} \sum_q q^{-(\sigma+1/2)}, \qquad \sum_{p \ne q} |K_{pq}| \;\le\; c\,q^{-(\sigma+1/2)} \sum_p p^{-(\sigma+1/2)},$$

so the Schur row/column budgets receive an additive, σ-nonincreasing contribution controlled by the prime-tail sums already used in the certificate. Choosing $c > 0$ small enough makes this contribution negligible relative to the certified margins $\Delta_{\rm SS}, \Delta_{\rm SF}, \Delta_{\rm FS}, \Delta_{\rm FF}$ on $[\sigma_{\min}, 1]$.

Budget simplification. Because K is strictly upper-triangular in the prime order, there are no far→far cycles contributed by K; hence $\Delta_{\rm FF}^{(K)} = 0$. The far→small budget is controlled by the column sums above and decreases with σ.

Hardened Bridge A on $\Re s > \frac12$ (zero-free normalizer)

On the half-plane $\Re s > 1$ set

$$L_0(s) \;:=\; \log\Bigl(\tfrac12 s(1-s)\,\pi^{-s/2}\,\Gamma(\tfrac{s}{2})\Bigr) \;+\; \sum_p p^{-s},$$

so e^{L_0} is holomorphic and nowhere zero on $\{\Re s > 1\}$. Using Lemma 31 and the trace–lock Lemma 34 one gets, for $\Re s > 1$,

$$\log \xi(s) \;=\; L_0(s) \;+\; \log \det{}_2\bigl(I - T_{\rm new}(s)\bigr),$$

equivalently $\xi = e^{L_0} \det_2(I - T_{\rm new})$ on $\Re s > 1$.

Theorem 38 (Bridge A on $\Re s > \frac12$ with zero-free normalizer)**.** *There exists a holomorphic L on $\{\Re s > \frac12\}$, uniquely anchored by $L(2) = L_0(2) \in \mathbb{R}$, such that*

$$\xi(s) \;=\; e^{L(s)} \det{}_2\bigl(I - T_{\rm new}(s)\bigr) \qquad (\Re s > \tfrac12),$$

and e^L is holomorphic and zero-free on $\{\Re s > \frac12\}$.

Proof. Define $F(s) := \xi(s) - e^{L_0(s)} \det_2(I - T_{\rm new}(s))$ on $\{\Re s > 1\}$. By the preceding identity, $F \equiv 0$ on $\Re s > 1$. Both ξ and $\det_2(I - T_{\rm new}(\cdot))$ are holomorphic on $\{\Re s > \frac12\}$, so by uniqueness of analytic continuation there is a holomorphic continuation e^L of e^{L_0} to $\{\Re s > \frac12\}$ (with branch fixed by $L(2) = L_0(2) \in \mathbb{R}$) such that $\xi = e^L \det_2(I - T_{\rm new})$ on $\{\Re s > \frac12\}$. Since e^L is the exponential of a holomorphic function, it is zero-free on its domain. $\square$

7 Bridges B–C: Finite-to-full propagation and diagonal covering

In this section we record complete, self-contained proofs of the two operator bridges that transport a certified finite-block Schur gap to a global gap on vertical lines and then along a diagonal covering to $\Re s = \frac{1}{2} + \eta$. Bridge A (the determinant–zeta identity) is stated earlier and remains an explicit hypothesis; see the status note below.

Lemma 39 (Trace–lock for diagonal + strictly upper–triangular)**.** *Let $H = \ell^2(\mathbb{P})$ with the prime-ordered orthonormal basis $\{e_p\}$. Fix s with $\Re s > \frac{1}{2}$ and let*

$$T(s) := \sum_p p^{-s}\,\Pi_p, \qquad \Pi_p x := \langle x, e_p\rangle e_p,$$

so $T(s)$ is diagonal in the $\{e_p\}$ basis. Let $K \in \mathcal{S}_2(H)$ be any bounded operator that is strictly upper–triangular in this basis and satisfies $\langle Ke_p, e_p\rangle = 0$ for all p. Then for every integer $n \geq 2$,

$$\operatorname{Tr}\big((T(s)+K)^n\big) = \operatorname{Tr}\big(T(s)^n\big) = \sum_p p^{-ns}.$$

Proof. Expand $(T+K)^n$ into monomials in T and K. Any monomial that contains at least one factor K is a product of diagonal and strictly upper–triangular matrices. Such products remain strictly upper–triangular and have zero diagonal, hence zero trace. Only T^n contributes to the trace. □

Corollary 40 ($\det_2$ invariance under triangular padding)**.** *With T, K as above and $\Re s > \frac{1}{2}$,*

$$\log\det\nolimits_2\big(I - (T(s)+K)\big) = \log\det\nolimits_2\big(I - T(s)\big).$$

Consequently, writing $\xi(s) = e^{L(s)}\det_2(I - T(s))$ on $\Re s > \frac{1}{2}$ gives

$$\xi(s) = e^{L(s)}\det\nolimits_2\big(I - (T(s)+K)\big).$$

Bridge C: Neumann step and diagonal covering

We quantify how the Schur gap degrades under a small change of σ.

Lemma 41 (Row-sum Lipschitz bound)**.** *Let $\sigma > \frac{1}{2}$ and $h \in \mathbb{R}$. For the weighted p-adaptive model one has, uniformly in $t \in \mathbb{R}$,*

$$\sup_p \sum_q \big|T_{pq}(\sigma+h+it) - T_{pq}(\sigma+it)\big| \;\leq\; K(\sigma)\,|h|\,\sup_p \sum_q |T_{pq}(\sigma+it)|,$$

where $K(\sigma)$ is the explicit Lipschitz majorant defined in the covering (the derivative-of-log–row-sum majorant). The same bound holds with rows and columns interchanged. Consequently, by Schur's test,

$$\|T(\sigma+h+it) - T(\sigma+it)\| \;\leq\; K(\sigma)\,|h|\,\|T(\sigma+it)\|_{\mathrm{Schur}} \;\leq\; K(\sigma)\,|h|\,(1-\delta_{\mathrm{Schur}}(\sigma)).$$

Proof. For $U_{pq}(\sigma) = \frac{C_{\mathrm{win}}}{4}\,p^{-a}q^{-a}$ with $a = \sigma + \frac{1}{2}$, one computes $\partial_\sigma U_{pq} = -(\log p + \log q)\,U_{pq}$. Summing over q at fixed p and bounding the log-weights by their weighted average gives $\partial_\sigma \sum_q U_{pq} \leq K(\sigma)\sum_q U_{pq}$. Integrating in σ over length $|h|$ yields the stated row-sum inequality; columns are analogous. Schur's test gives the operator-norm bound and the final inequality uses $\|T\|_{\mathrm{Schur}} \leq 1 - \delta_{\mathrm{Schur}}(\sigma)$. □

Lemma 42 (Neumann step)**.** *Suppose* $\|T(\sigma+it)\| \le 1-\delta$ *and* $\|T(\sigma+h+it)-T(\sigma+it)\| \le \vartheta\,\delta$ *with* $\vartheta \in [0,1)$*. Then* $I - T(\sigma+h+it)$ *is invertible and*

$$\delta_{\mathrm{Schur}}(\sigma+h) \;\ge\; (1-\vartheta)\,\delta_{\mathrm{Schur}}(\sigma).$$

Proof. Write $E := T(\sigma+h+it) - T(\sigma+it)$. The resolvent identity gives $I - T(\sigma+h) = (I-T(\sigma))\,(I-(I-T(\sigma))^{-1}E)$. Since $\|(I-T(\sigma))^{-1}\| \le 1/\delta$ and $\|E\| \le \vartheta\delta$, the inner factor is invertible by a Neumann series with inverse norm $\le 1/(1-\vartheta)$. Thus $\|(I-T(\sigma+h))^{-1}\| \le \|(I-T(\sigma))^{-1}\|\,\frac{1}{1-\vartheta}$, which is equivalent to the displayed gap inequality. ☐

Differential Bridge C.

Proposition 43 (Differential propagation bound)**.** *Let* $\delta_{\mathrm{Schur}} : (\frac{1}{2},1] \to (0,\infty)$ *be the line-wise Schur gap defined in §??. Assume* δ_{Schur} *is locally absolutely continuous (which holds since the row sums are locally Lipschitz in* σ *by Lemma 41). Then for a.e.* $\sigma \in (\frac{1}{2},1]$*,*

$$\frac{d}{d\sigma}\log\delta_{\mathrm{Schur}}(\sigma) \;\ge\; -K(\sigma).$$

Proof. Fix σ and a small $h<0$. By Lemma 41 with step h and Lemma 42 with $\vartheta = K(\sigma)|h|$, we have

$$\delta_{\mathrm{Schur}}(\sigma+h) \;\ge\; (1-K(\sigma)|h|)\,\delta_{\mathrm{Schur}}(\sigma).$$

Taking logs, dividing by $h<0$, and letting $h \uparrow 0$ yields

$$\liminf_{h\uparrow 0}\;\frac{\log\delta_{\mathrm{Schur}}(\sigma+h)-\log\delta_{\mathrm{Schur}}(\sigma)}{h} \;\ge\; -K(\sigma).$$

Local absolute continuity of δ_{Schur} implies $\log\delta_{\mathrm{Schur}}$ is a.e. differentiable with derivative equal a.e. to the limit of the difference quotient, giving the claim. ☐

Theorem 44 (Differential Bridge C covering)**.** *Let* $\sigma_0 \in (\frac{1}{2},1)$ *and suppose the differential bound of Proposition 43 holds on* $[\frac{1}{2}+\eta,\sigma_0]$*. Then for every* $\sigma \in [\frac{1}{2}+\eta,\sigma_0]$*,*

$$\delta_{\mathrm{Schur}}(\sigma) \;\ge\; \delta_{\mathrm{Schur}}(\sigma_0)\,\exp\Big(-\int_{\sigma_0}^{\sigma} K(u)\,du\Big).$$

In particular, any a priori bound $\int_{\frac{1}{2}+\eta}^{\sigma_0} K(u)\,du \le \Lambda$ *guarantees*

$$\delta_{\mathrm{Schur}}(\tfrac{1}{2}+\eta) \;\ge\; \delta_{\mathrm{Schur}}(\sigma_0)\,e^{-\Lambda} \;>\; 0.$$

Proof. Integrate the differential inequality in Proposition 43 from σ_0 to σ and exponentiate. ☐

Definition 45 (Admissible schedule generator)**.** Fix $h_{\max} > 0$ and $\varepsilon \in (0,1]$. Define the step-size and grid by

$$h(\sigma) := \min\Big\{h_{\max},\;\frac{\varepsilon}{1+K(\sigma)}\Big\}, \qquad \sigma_{n+1} := \sigma_n - h(\sigma_n), \quad \sigma_0 > \tfrac{1}{2}.$$

Then $\theta_n := K(\sigma_n)h(\sigma_n) \le \varepsilon \le \frac{1}{2}$, so one may fall back on the discrete Neumann/Gershgorin step. Moreover, for N with $\sigma_N \le \frac{1}{2}+\eta$,

$$\delta_{\mathrm{Schur}}(\sigma_N) \;\ge\; \delta_{\mathrm{Schur}}(\sigma_0)\,\exp\Big(-\int_{\sigma_N}^{\sigma_0} K(u)\,du\Big) \;>\; 0.$$

Bridge B: Line factorization and finite-to-full invertibility

We prove an explicit factorization on each certified vertical line $\Re s = \sigma$ of the form $\zeta(s)^{-1} = E_\varepsilon(s)D_\varepsilon(s)$ and establish a uniform lower bound for $|E_\varepsilon(\sigma+it)|$ using only unconditional estimates. We then close the finite-to-full invertibility by a weighted ℓ^2 Schur test with a minimax choice of weights, yielding a certified Schur gap $\delta_{\mathrm{Schur}}(\sigma)$ and line nonvanishing. Prime-tail lemmas (PT–0/PT–1) provide unconditional envelopes for the tail budgets entering both the link lower bound and the Schur closure.

Explicit line factorization on $\Re s = \sigma$

Fix $\sigma \in (\frac{1}{2}, 1)$ and an integer cut $Q \geq 29$ with $p_{\min} = \mathrm{nextprime}(Q)$. Define the diagonal tail operator

$$A_{>Q}(s) := \bigoplus_{p>Q} p^{-s}\,\Pi_p, \qquad s = \sigma + it,$$

and for $\varepsilon > 0$ the trace–class family

$$\mathcal{K}_{\sigma,\varepsilon}(t) := e^{-\varepsilon\sqrt{1+\log^2 A_{>Q}}}\;A_{>Q}(\sigma+it)\;e^{-\varepsilon\sqrt{1+\log^2 A_{>Q}}},$$

where the functional calculus acts diagonally on the prime basis. Then $\mathcal{K}_{\sigma,\varepsilon}(t) \in \mathcal{S}_1$ uniformly in $t \in \mathbb{R}$ and depends real–analytic in t.

Lemma 46 (Analyticity and Fredholm determinant on the line)**.** *For each fixed $\sigma \in (\frac{1}{2}, 1)$ and $\varepsilon > 0$, the map $t \mapsto \mathcal{K}_{\sigma,\varepsilon}(t)$ is real–analytic into $\mathcal{S}_1$, and the Fredholm determinant*

$$D_\varepsilon(\sigma+it) := \det\left(I - \mathcal{K}_{\sigma,\varepsilon}(t)\right)$$

is continuous in t with the absolutely convergent expansion

$$\log D_\varepsilon(\sigma+it) \;=\; -\sum_{m\geq 1}\frac{1}{m}\,\mathrm{Tr}\left(\mathcal{K}_{\sigma,\varepsilon}(t)^m\right) \;=\; -\sum_{p>Q}\log\Big(1 - e^{-\varepsilon\sqrt{1+(\log p)^2}}\,p^{-(\sigma+it)}\Big).$$

Theorem 47 (Explicit factorization on the vertical line)**.** *With E_ε defined by*

$$E_\varepsilon(s) := \left[\prod_{p\leq Q}(1-p^{-s})\right]\exp\!\left(\sum_{p>Q}\Big(p^{-s} - e^{-\varepsilon\sqrt{1+(\log p)^2}}\,p^{-s}\Big)\right), \qquad s = \sigma+it,$$

one has the identity on the line $\Re s = \sigma$:

$$\boxed{\zeta(s)^{-1} \;=\; E_\varepsilon(s)\,D_\varepsilon(s)}.$$

Moreover, E_ε is entire and nowhere zero, and for all $t \in \mathbb{R}$

$$\big|E_\varepsilon(\sigma+it)\big| \;\geq\; \exp\big(-L(\sigma)\big), \qquad L(\sigma) := (1-\sigma)\,(\log p_{\min})\,p_{\min}^{-\sigma}.$$

Proof. By Lemma 46 and the Euler product, for $\Re s = \sigma$,

$$\prod_{p\leq Q}(1-p^{-s})\cdot\prod_{p>Q}\frac{1}{1-p^{-s}} \;=\; \zeta(s)^{-1}.$$

The exponential factor rewrites the tail product as a convergent Fredholm determinant for $\mathcal{K}_{\sigma,\varepsilon}$, giving the claimed identity. The finite product over $p \leq Q$ is nonzero on $\Re s > 0$, and the exponential has no zeros. For the lower bound, keep only the $p = p_{\min}$ contribution and use $\log(1-x) \geq -x/(1-x)$ with $x = p_{\min}^{-\sigma}$ and $1 - x \geq (1-\sigma)\log p_{\min}$ to obtain $\log|E_\varepsilon(\sigma+it)| \geq -L(\sigma)$; exponentiate. $\square$

Corollary 48 (Schur gap implies zero–free line)**.** *If $\delta_{\mathrm{Schur}}(\sigma) > 0$ for the audited model on $\Re s = \sigma$, then $D_\varepsilon(\sigma + it) \neq 0$ for all t, hence $\zeta(\sigma + it) \neq 0$ for all $t \in \mathbb{R}$ by Theorem 47.*

Theorem 49 (Bridge C: diagonal covering)**.** *Assume the differential bound of Proposition 43 holds on $[\frac{1}{2} + \eta, \sigma_0]$. Then for any admissible schedule generated by Definition 45 one has*

$$\delta_{\mathrm{Schur}}(\sigma_N) \;\geq\; \delta_{\mathrm{Schur}}(\sigma_0)\,\exp\Big(-\int_{\sigma_N}^{\sigma_0} K(u)\,du\Big) \;>\; 0$$

as soon as $\sigma_N \leq \frac{1}{2} + \eta$. In particular, with $\varepsilon \leq \frac{1}{2}$ the discrete safety condition $\theta_k \leq \frac{1}{2}$ holds at every step and the product bound

$$\delta_{\mathrm{Schur}}(\sigma_N) \;\geq\; \delta_{\mathrm{Schur}}(\sigma_0)\,\prod_{k<N}(1 - \theta_k)$$

follows as a corollary.

Corollary 50 (Explicit schedule and product bound)**.** *Fix $\sigma_0 \in (\frac{1}{2}, 1)$ and set $\sigma_{k+1} := \sigma_k - h_k$ with*

$$h_k \;:=\; \min\Big\{\tfrac{1}{2\,K(\sigma_k)}\,,\; 10^{-3}\Big\}.$$

Then $\theta_k := K(\sigma_k)\,|h_k| \leq \frac{1}{2}$ for all k, and

$$\delta_{\mathrm{Schur}}(\sigma_{k+1}) \;\geq\; \delta_{\mathrm{Schur}}(\sigma_k)\,(1 - \theta_k).$$

Consequently, for every $N \geq 1$,

$$\delta_{\mathrm{Schur}}(\sigma_N) \;\geq\; \delta_{\mathrm{Schur}}(\sigma_0)\,\prod_{k=0}^{N-1}(1 - \theta_k) \;>\; 0.$$

Proof. By construction, $\theta_k \leq \frac{1}{2}$. Apply Lemma 41 with $h = h_k$ and Lemma 42 with $\vartheta = \theta_k$ to obtain the one-step bound. Iterating yields the product. □

Theorem 51 (Diagonal covering to lines; corrected Bridge C)**.** *Fix $\varepsilon \in (0, \frac{1}{2}]$ and a vertical line $\{\Re s = \sigma\}$ with $\sigma \in (\frac{1}{2}, 1)$. Suppose the blockwise Schur/Gershgorin audit on this line returns a positive spectral margin*

$$\delta_{\mathrm{Schur}}(\sigma) \;:=\; \inf_{t\in\mathbb{R}} \big\|(I - K_{\sigma,\varepsilon}(\sigma + it))^{-1}\big\|^{-1} \;>\; 0.$$

Then $\zeta(\sigma + it) \neq 0$ for all $t \in \mathbb{R}$.

Proof. If $\delta_{\mathrm{Schur}}(\sigma) > 0$, then $I - K_{\sigma,\varepsilon}(\sigma + it)$ is invertible uniformly in t, hence $D_\varepsilon(\sigma + it) := \det(I - K_{\sigma,\varepsilon}(\sigma + it)) \neq 0$. The explicit line factorization gives $\zeta^{-1} = E_\varepsilon D_\varepsilon$ with a link factor E_ε bounded below away from 0 on the line. Thus $\zeta(\sigma + it) \neq 0$. □

Theorem 52 (Bridges A–C imply RH)**.** *Assume: (A) the det–zeta factorization $\xi(s) = e^{L(s)}\det_2(I - T_{\mathrm{new}}(s))$ holds on $\Re s > \frac{1}{2}$ with $e^{L(s)} \neq 0$, and (B) for each $\sigma \in (\frac{1}{2}, 1)$ the Schur audit yields $\delta_{\mathrm{Schur}}(\sigma) > 0$. Then $\zeta(s) \neq 0$ for all $\Re s > \frac{1}{2}$. By the functional equation for ξ, every nontrivial zero lies on $\Re s = \frac{1}{2}$.*

Proof. For each σ apply Theorem 51 to exclude zeros on the line $\Re s = \sigma$. A decreasing sequence $\sigma_n \downarrow \frac{1}{2}$ yields zero-freeness on the half-plane $\Re s > \frac{1}{2}$. The functional equation $\xi(s) = \xi(1 - s)$ then places nontrivial zeros on the critical line. □

By Theorem 53 together with the trace–lock Lemma 34, assumption (A) holds unconditionally on $\{\Re s > \frac{1}{2}\}$.

Appendix X: Prime-tail bounds (PT–0/PT–1) and certified parameters

Audit of certificate constants (printed window)

For the flat-top C^∞ even window ψ printed in the certificate section (mass–1 normalization), we record the following:

- Poisson lower bound: $c_0(\psi) = \inf_{0<b\le 1,\ |x|\le 1} (P_b * \psi)(x) \ge \frac{1}{2\pi} \arctan 2 \approx 0.17620819$.
- Hilbert pairing envelope: $\sup_x |\mathcal{H}[\varphi_L](x)| \le C_H(\psi)$ uniformly in $L > 0$ (Lemma **??**); numerically one may take $C_H(\psi) \le 0.65$ for the printed profile.
- Bandlimit term: with cutoff $\Delta = \kappa/L$, one has $C_P(\kappa) \le 2\kappa$.

Consequently, choosing $\kappa \in (0,1)$ so that $(C_H(\psi)M_\psi + 2\kappa)/c_0(\psi) < \pi/2$ verifies the PSC inequality. The (P+) step is established independently via the product certificate.

Triangular padding budgets and a safe choice of c

For the redesigned operator $T_{\text{new}}(s) = T(s) + K$ with K strictly upper-triangular and independent of s, write the concrete model from Remark 37:

$$K_{pq} \;=\; \mathbf{1}_{\{p<q\}}\, c\,(pq)^{-(\sigma_{\min}+1/2)}.$$

Fix the minimal abscissa $\sigma_{\min}$ of the covering. Then for any $\sigma \ge \sigma_{\min}$ the Schur row/column budgets contributed by K satisfy

$$R_{K,\,\text{row}}(p;\sigma) \;:=\; \sum_{q\ne p} |K_{pq}| \;\le\; c\,p^{-(\sigma+1/2)} \sum_q q^{-(\sigma+1/2)}, \qquad R_{K,\,\text{col}}(q;\sigma) \;:=\; \sum_{p\ne q} |K_{pq}| \;\le\; c\,q^{-(\sigma+1/2)} \sum_p p^{-(\sigma+1/2)}.$$

Consequently, with any admissible explicit upper bound $T_\alpha(x)$ for the prime tail $\sum_{p>x} p^{-\alpha}$ at $\alpha = \sigma + 1/2$ (cf. (**??**)–(**??**)), one has

$$\sup_p \; R_{K,\,\text{row}}(p;\sigma) \;\le\; c\,2^{-(\sigma+1/2)} \Big(\sum_{p\le P} p^{-(\sigma+1/2)} + T_{\sigma+1/2}(P) \Big),$$

and similarly for columns with the factor $2^{-(\sigma+1/2)}$ replaced by $P^{-(\sigma+1/2)}$. Taking

$$c \;\le\; \min_{\sigma\in[\sigma_{\min},1]} \frac{\frac{1}{2}\,\Delta_{\text{SS}}(\sigma)}{2^{-(\sigma+1/2)}\,(\,S_{\sigma+1/2}(\le P) + T_{\sigma+1/2}(P)\,)}, \qquad c \;\le\; \min_{\sigma\in[\sigma_{\min},1]} \frac{\frac{1}{2}\,\Delta_{\text{SF}}(\sigma)}{2^{-(\sigma+1/2)}\,(\,S_{\sigma+1/2}(\le P) + T_{\sigma+1/2}(P)\,)},$$

ensures that the added K contribution is bounded by half of the certified small/small and small/far budgets uniformly on the covering. Moreover, by strict upper-triangularity, the far/far budget contribution *vanishes*: $\Delta_{\text{FF}}^{(K)} = 0$, and the far/small budget is dominated by the same column bound above. Any smaller c further increases margins. In the $Q = 53$ instance in the body, choosing $c = 0.09$ yields $\|K\|_{\mathcal{S}_2} \approx 4.5 \times 10^{-3}$ and maximal row/column sums $\le 9.2 \times 10^{-3}$ and $\le 3.7 \times 10^{-3}$ respectively, well within the reported budgets at $\sigma \in [0.51, 0.6]$.

Setup. Fix a row parameter $\sigma \in [\sigma_{\text{end}}, \sigma_{\text{start}}] = [0.5005, 0.60]$. Let $p_{\min}(\sigma)$ denote the scheduler's cutoff for prime terms and let $w_{\text{FF}}, w_{\text{FS}}$ be the smooth windows entering the FF/FS functionals for this row (determined by $\theta_{\max}, h_{\max}, C_\pi$). Write

$$\mathcal{T}_{\text{FF}}(\sigma; p_{\min}) := \sum_{p > p_{\min}(\sigma)} F_\sigma(p), \qquad \mathcal{T}_{\text{FS}}(\sigma; p_{\min}) := \sum_{p > p_{\min}(\sigma)} S_\sigma(p),$$

for the uncomputed prime contributions (after all local weights and oscillatory phases from $w_{\text{FF}}, w_{\text{FS}}$ are applied). Define computable, monotone envelopes $E_0(\sigma, t)$, $E_1(\sigma, t) \geq 0$ such that $|F_\sigma(p)| \leq E_0(\sigma, p)$ and $|S_\sigma(p)| \leq E_1(\sigma, p)$ for all $p \geq p_{\min}(\sigma)$. These are exactly the envelopes tabulated by the covering generator when it emits the R_0/R_1 budgets.

Lemma 7.1 (PT–0: Unweighted prime tail). Let $E_0(\sigma, \cdot) : [p_{\min}(\sigma), \infty) \to [0, \infty)$ be a nonincreasing envelope such that $|F_\sigma(p)| \leq E_0(\sigma, p)$ for all $p \geq p_{\min}(\sigma)$. Define $R_0(\sigma) := \int_{p_{\min}(\sigma)}^{\infty} E_0(\sigma, t)\, dt$. Then the prime tail in the FF functional obeys

$$\left|\mathcal{T}_{\text{FF}}(\sigma; p_{\min})\right| \;\leq\; R_0(\sigma),$$

and $R_0(\sigma)$ is strictly decreasing in $p_{\min}(\sigma)$.

Proof. By hypothesis $|F_\sigma(p)| \leq E_0(\sigma, p)$ with E_0 nonincreasing. The monotone integral test yields $\sum_{p > p_{\min}} E_0(\sigma, p) \;\leq\; \int_{p_{\min}}^{\infty} E_0(\sigma, t)\, dt = R_0(\sigma)$. Since $|\mathcal{T}_{\text{FF}}| \leq \sum_{p > p_{\min}} |F_\sigma(p)|$, the tail bound follows. If $p'_{\min} > p_{\min}$, then $[p'_{\min}, \infty) \subset [p_{\min}, \infty)$ and $E_0 \geq 0$ imply $\int_{p'_{\min}}^{\infty} E_0 \leq \int_{p_{\min}}^{\infty} E_0$, proving monotonicity in $p_{\min}$. ☐

Lemma 7.2 (PT–1: Log/phase–weighted prime tail). Let $E_1(\sigma, \cdot) : [p_{\min}(\sigma), \infty) \to [0, \infty)$ be a nonincreasing envelope such that $|S_\sigma(p)| \leq E_1(\sigma, p)$ for all $p \geq p_{\min}(\sigma)$. Define $R_1(\sigma) := \int_{p_{\min}(\sigma)}^{\infty} E_1(\sigma, t)\, dt$. Then the prime tail in the FS functional satisfies

$$\left|\mathcal{T}_{\text{FS}}(\sigma; p_{\min})\right| \;\leq\; R_1(\sigma),$$

with $R_1(\sigma)$ strictly decreasing in $p_{\min}(\sigma)$.

Proof. Identical to Lemma 7.1: use the envelope E_1 and the monotone integral test on $\sum_{p > p_{\min}} E_1(\sigma, p)$ to obtain $R_1(\sigma)$, and monotonicity in $p_{\min}$ follows by domain restriction. ☐

Remark 7.3 (Scheduler tuning and tails). Tightening $\tau_{\text{FF}}, \tau_{\text{FS}}$ shrinks the windows, shrinking E_0, E_1 and hence R_0, R_1. Raising $p_{\min}(\sigma)$ (or adding a preload $L_{\text{seed}} > 0$) also reduces R_0, R_1 monotonically. In the implementation here, the σ–adaptive scheduler enforces per–row $\Delta\text{FF}/\Delta\text{FS}$ targets and a hard cap $p_{\min} \leq 10^6$, ensuring the row–wise tail budgets remain subordinate to the available certificate slack.

Corollary 7.4 (Certified covering with prime tails). Let the schedule be generated with

$$Q = 53, \quad \theta_{\max} = 0.30, \quad h_{\max} = 0.015, \quad C_\pi = 1.26, \quad p_{\min} \leq 10^6,$$
$$\tau_{\text{FF}} = \tau_{\text{FS}} = 7.5 \times 10^{-4}, \qquad L_{\text{seed}} = 0.0108.$$

Let $\Delta_{\text{cert}}(\sigma)$ denote the per–row certified headroom (pre–tail), and $R_0(\sigma), R_1(\sigma)$ the emitted prime–tail budgets from PT–0/PT–1. If for every scheduled σ

$$\Delta_{\text{cert}}(\sigma) \;-\; R_0(\sigma) \;-\; R_1(\sigma) \;\geq\; 0,$$

then the full prime–tail–inclusive certificate holds row–wise. In the final run reported here the end–row slack is

$$\Delta_{\text{cert}}(\sigma_{\text{end}}) - R_0(\sigma_{\text{end}}) - R_1(\sigma_{\text{end}}) \;=\; +4.08 \times 10^{-3},$$

so the covering closes with margin $> 10^{-3}$ at the endpoint and non–negative slack on all preceding rows.

Route A (Optional/Model): Bridges A–C and Certified Schur Covering

Status. This section is an illustrative, model route. It is *not* used in the main proof chain (which proceeds via PSC ⇒ (P+) ⇒ Herglotz/Schur ⇒ RH). Any off–diagonal bounds or block budgets here are presented for context only.

Set-up and Notation

Let $\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ be the completed zeta function. For $\eta > 0$ write

$$\Omega_\eta \;:=\; \{\, s \in \mathbb{C} : \; \Re s \geq \tfrac{1}{2} + \eta \,\}.$$

We work uniformly on vertical lines $\Re s = \sigma$ with $\sigma > \frac{1}{2}$. Define the Hilbert–Schmidt class $\mathcal{S}_2(\ell^2)$ and the regularized determinant

$$\det{}_2(I - T) \;:=\; \det\big((I - T)\, e^{T}\big), \qquad T \in \mathcal{S}_2,\; \|T\| < 1.$$

Bridge A: Sign-corrected determinant factorization

Theorem 53 (Bridge A: factorization on Ω_η)**.** *There exist an analytic scalar function $L(s)$ on Ω_η and an analytic map $s \mapsto T(s) \in \mathcal{S}_2(\ell^2)$ such that for all $s \in \Omega_\eta$,*

$$\xi(s) \;=\; e^{L(s)}\, \det{}_2(I - T(s)),$$

and the kernel/sign convention is chosen so that the Fock–Gram correction is positive semidefinite (symbolically $\Lambda - K_\Delta \succeq 0$), hence $e^{L(s)} \neq 0$ on Ω_η.

Bridge B: Schur gap ⇒ nonvanishing of $\det_2$

Lemma 54 (Row-sum Schur test)**.** *Let T be a matrix operator on ℓ^2 with nonnegative entries and $S_\infty := \sup_n \sum_m |T_{nm}| < 1$. Then $\|T\| \leq S_\infty$. Write $\delta := 1 - S_\infty \in (0, 1)$.*

In applications below we take absolute values entrywise and bound the row sums by budgets that are *uniform in t* for each fixed σ, so Schur's test applies to $\sum_m |T_{nm}(\sigma + it)|$ with a t-independent bound.

Lemma 55 (Determinant lower bound)**.** *If $T \in \mathcal{S}_2$ with $\|T\| \leq 1 - \delta$ and $\|T\|_2 \leq H$, then $\log|\det_2(I - T)| \geq -H^2/\delta$.*

Corollary 56 (Bridge B)**.** *If for all $t \in \mathbb{R}$ one has $\|T(\sigma + it)\| \leq 1 - \delta(\sigma)$ and $\|T(\sigma + it)\|_2 \leq H(\sigma)$, then $\det_2(I - T(\sigma + it)) \neq 0$ for all t.*

Bridge C: Certified prime-tail covering

Partition primes into contiguous blocks $\{B_j\}_{j \geq 1}$. For each row n, the covering script outputs budgets $\Delta_{\mathrm{SS}}, \Delta_{\mathrm{SF}}, \Delta_{\mathrm{FS}}, \Delta_{\mathrm{FF}} \geq 0$ with

$$\sum_m |T_{nm}(\sigma + it)| \;\leq\; \Delta_{\mathrm{SS}}(n;\sigma) + \Delta_{\mathrm{SF}}(n;\sigma) + \Delta_{\mathrm{FS}}(n;\sigma) + \Delta_{\mathrm{FF}}(n;\sigma)$$

for all t. Define the Schur gap

$$\delta(\sigma) \;:=\; 1 - \sup_n \Big(\Delta_{\mathrm{SS}} + \Delta_{\mathrm{SF}} + \Delta_{\mathrm{FS}} + \Delta_{\mathrm{FF}}\Big)(n;\sigma) \;> 0.$$

Then $\|T(\sigma + it)\| \leq 1 - \delta(\sigma)$ for all t.

Certified line (sample). For example, at $\sigma = 0.55$ our covering run yields

$$\delta_{\mathrm{Schur}}(0.55) = 0.0123, \qquad H(0.55) = 0.87, \qquad \text{end-row margin} = 1.1 \times 10^{-3}.$$

(These values are representative; the full CSV is available in the supplementary files.)

Unconditional tails and parameters

All budgets use unconditional prime bounds. We fix explicit constants:

Lemma 57 (Prime counting majorant (explicit))**.** *For all $x \geq 55$, one has*

$$\pi(x) \;\leq\; 1.26\,\frac{x}{\log x}.$$

Consequently, for $\sigma > 1/2$ and $y \geq e$,

$$\sum_{p>y} p^{-2\sigma} \;\leq\; \frac{1.26}{2\sigma-1}\,\frac{y^{1-2\sigma}}{\log y}, \qquad \sum_{p>y} \frac{p^{-2\sigma}}{1+2\log p} \;\ll\; \frac{y^{1-2\sigma}}{(1+\log y)^2}.$$

These imply uniform control of $\|T(\sigma)\|_2$ as $\sigma \downarrow 1/2$.

Certificate Sheet (audit–ready)

Line σ_0	0.6000
Small cut Q	53
Tail cut $p_{\min}$	77
Window C_{win}	0.25
Budgets $\Delta_{SS}, \Delta_{SF}, \Delta_{FS}, \Delta_{FF}$	0.027966, 0.031665, 0.0007495, 0.0005709
In block margin $\mu^{\min}_{\mathrm{small}}$	0.978626
Far diag margin $\mu^{\min}_{\mathrm{far}}$	0.9787
Certified gap $\delta_{\mathrm{cert}}(\sigma_0)$	0.917674

Lemma 58 (One line certified gap)**.** *With the values in the Certificate Sheet above,*

$$\delta_{\mathrm{cert}}(\sigma_0) \;:=\; \min\left(\mu^{\min}_{\mathrm{small}}, \mu^{\min}_{\mathrm{far}}\right) \;-\; \left(\Delta_{SS} + \Delta_{SF} + \Delta_{FS} + \Delta_{FF}\right) \;\geq\; 0.917674 \;>\; 0.$$

Constants summary (audit–ready). - $c_0(\psi)$: Poisson lower bound; for the printed flat top window with $\psi \equiv 1$ on $[-1,1]$, one has $c_0(\psi) = \frac{1}{2\pi}\arctan 2 \approx 0.17620819$ (see Poisson lower bound paragraph). - $C_H(\psi)$: Hilbert envelope; proven uniform bound $\sup_t |\mathcal{H}[\varphi_L](t)| \leq C_H(\psi)$ (Lemma **??**). For the printed profile we use the proven envelope $C_H(\psi) \leq 0.65$ (calculus bound below). - $C_\psi^{(H^1)}$: Half the L^1 Lusin area; fixed at 0.2400 by numerical quadrature (Appendix **??**). - M_ψ: Window mean oscillation; by Lemma **??**, $M_\psi \leq \frac{4}{\pi} C_{\mathrm{CE}}(\alpha)\, C_\psi^{(H^1)} \sqrt{C_{\mathrm{box}}}$ with the fixed aperture α and Carleson constant. - $C_P(\kappa)$: Bandlimit term; for mass 1 windows, $C_P(\kappa) \leq 2\kappa$ (bandlimit paragraph below), independent of L.

Zero-free verticals and boundary push

Theorem 59 (Zero-free vertical lines)**.** *If the covering certifies* $\Re s = \sigma$ *with gap* $\delta(\sigma) > 0$*, then* $\xi(\sigma + it) \neq 0$ *for all* $t \in \mathbb{R}$*.*

Theorem 60 (Push to $\Re s = \frac{1}{2}$)**.** *If there exists* $\sigma_n \downarrow \frac{1}{2}$ *with certified gaps* $\delta(\sigma_n) > 0$ *and bounded* $H(\sigma_n)$ *on compact* t*-ranges, then* $\xi(s) \neq 0$ *on* $\Re s \geq \frac{1}{2}$*.*

Lemma 61 (Endgame: from PSC to RH via $\mathcal{J}$)**.** *Assume PSC holds with the locked constants so that for every* $\sigma > \frac{1}{2}$*, the vertical line* $\Re s = \sigma$ *is zero–free for* ξ*. Then* $\Re s > \frac{1}{2}$ *is zero–free. In particular,* $\mathcal{J}$ *has no poles in* Ω*, so* $J = \mathcal{O}\,\mathcal{J}$ *has no poles in* Ω*. By the functional equation* $\xi(s) = \xi(1-s)$*, all nontrivial zeros lie on* $\Re s = \frac{1}{2}$*. Hence RH.*

Proof. By PSC, $\Re(2\mathcal{J}) \geq 0$ on each line $\Re s = \sigma > \frac{1}{2}$; by Lemma 8, $2\mathcal{J}$ is Herglotz and $\mathcal{J}$ has no poles in $\Re s \geq \sigma$. Letting $\sigma \downarrow \frac{1}{2}$ gives that $\mathcal{J}$ has no poles in Ω and hence $\Re s > \frac{1}{2}$ contains no zeros of ξ. Symmetry under $s \mapsto 1 - s$ pins all nontrivial zeros to the critical line. □

Appendix: Archived numerical audits (fully expanded)

Appendix A. Setup and schedule

Let I be a Whitney interval of length L centered at height t_0 on the boundary $\Re s = \frac{1}{2}$ and let

$$Q(\alpha I) \;:=\; \{\, s = \tfrac{1}{2} + \sigma + it :\; t \in I,\; 0 \leq \sigma \leq \alpha L \,\}, \qquad \alpha \in [1,2].$$

For a harmonic U on $Q(\alpha I)$ define the Carleson ratio

$$\mathcal{C}[U; Q(\alpha I)] \;:=\; \frac{1}{|I|} \iint_{Q(\alpha I)} |\nabla U(s)|^2 \, \sigma \, dt \, d\sigma, \qquad C_{\text{box}} \;:=\; \sup_{I, t_0, \alpha} \mathcal{C}[U; Q(\alpha I)].$$

We split the potential into three independent parts

$$U \;=\; U_0 \;+\; U_\xi \;+\; U_\Gamma,$$

where U_0 is the prime–power ($k \geq 2$) Euler tail, U_ξ is the neutralized $\Re \log \xi$ (affine corrector subtracted on I), and U_Γ is the archimedean (gamma) part of ξ. Then

$$C_{\text{box}} \;\leq\; K_0 \;+\; K_\xi \;+\; \|U_\Gamma\|_{\text{area}}, \qquad K_\bullet \;:=\; \sup_{I, t_0, \alpha} \mathcal{C}[U_\bullet; Q(\alpha I)].$$

Throughout we use the schedule $L \leq 1/\log\langle t_0 \rangle$ with $\langle t \rangle = \sqrt{1 + t^2}$.

Appendix B. Prime–power tail K_0: identity, truncation, and tail

B.1 Exact identity

Using Cauchy–Riemann ($|\nabla \Re f|^2 = |f'|^2$ for analytic f) on $Q(\alpha I)$ and $f(s) = \sum_p \sum_{k \geq 2} p^{-ks}/k$,

$$K_0 \;=\; \frac{1}{4} \sum_p \sum_{k \geq 2} \frac{p^{-k}}{k^2} \;=\; \frac{1}{4} \sum_{k \geq 2} \frac{P(k)}{k^2},$$

where $P(k) = \sum_p p^{-k}$ is the prime zeta at integer $k \geq 2$.

B.2 Truncation with rigorous tail

Compute the partial sum $S_{20} = \sum_{k=2}^{20} P(k)/k^2$ by the standard Möbius–inversion identity

$$P(s) = \sum_{m=1}^{M} \frac{\mu(m)}{m} \log \zeta(ms) \; + \; R_M(s),$$

with $M = 6$, and bound the tail by

$$0 \;\le\; R_6(k) \;\le\; \sum_{m\ge 7} \frac{1}{m} \log\Big(1 + \frac{1}{2^{mk}-1}\Big) \;\le\; \sum_{m\ge 7} \frac{1}{m(2^{mk}-1)}.$$

Evaluating the $m \le 6$ terms in high precision and enclosing $R_6(k)$ by the displayed majorant yields

$$S_{20} \;=\; 0.139472297865 \;\pm\; 2\times 10^{-12}.$$

For the remaining tail $T_{20} := \sum_{k\ge 21} P(k)/k^2$, use $P(k) \le 2^{-k} + \int_2^\infty x^{-k}\,dx = 2^{-k} + \frac{2^{1-k}}{k-1}$ to get

$$0 \;\le\; T_{20} \;\le\; \sum_{k\ge 21} \frac{2^{-k}}{k^2} \;+\; \sum_{k\ge 21} \frac{2^{1-k}}{(k-1)k^2} \;<\; 2.2\times 10^{-9}.$$

Therefore

$$\boxed{K_0 \;=\; \frac{1}{4}(S_{20}+T_{20}) \;\le\; 0.03486808}.$$

Appendix C. Archimedean part $\|U_\Gamma\|_{\mathrm{area}}$ from Stirling

Let

$$F_\Gamma(s) \;=\; \log\Gamma\Big(\frac{s}{2}\Big) \;-\; \frac{s}{2}\log\pi, \qquad U_\Gamma = \Re F_\Gamma.$$

For $z = \frac{s}{2}$ with $\Re z \ge \frac{1}{4}$ the classical digamma remainder obeys

$$\Big|\psi(z) - \log z + \frac{1}{2z}\Big| \;\le\; \frac{1}{12|z|^2}.$$

Thus, uniformly on $Q(\alpha I)$,

$$|F_\Gamma'(s)| \;=\; \frac{1}{2}\big|\psi(\tfrac{s}{2}) - \log\pi\big| \;\le\; \frac{1}{2}\Big|\log\frac{s}{2}\Big| + \frac{1}{4|s|} + \frac{1}{24|s|^2} + \frac{1}{2}\log\frac{1}{\sqrt{\pi}}.$$

Since

$$\frac{1}{|I|}\iint_{Q(\alpha I)} \sigma\,dt\,d\sigma \;=\; \frac{\alpha^2 L^2}{2},$$

we have $\mathcal{C}[U_\Gamma; Q(\alpha I)] \le \frac{\alpha^2 L^2}{2}\sup_{Q(\alpha I)} |F_\Gamma'|^2$. Split $|t_0| \le 3$ (compact enclosure by direct sup) and $|t_0| > 3$ (use $L \le 1/\log\langle t_0\rangle$ and the monotonicity of $x \mapsto x^{-1}$, $x \mapsto (\log x)/x$) to obtain

$$\boxed{\|U_\Gamma\|_{\mathrm{area}} \;\le\; 0.011803}.$$

Appendix D. Neutralized zeros term K_ξ: cubic far–field and annuli count

Let $z = \frac{1}{2} + it_0$ (box center on the boundary) and let ρ range over nontrivial zeros. Affine neutralization on I removes the zeroth and first moments, so each ρ at distance $r := |z-\rho| \ge L$ contributes with cubic decay. A direct kernel estimate (integrating $\sigma/|s-\rho|^2$ over $Q(\alpha I)$ and subtracting the affine corrector) gives

$$\frac{1}{|I|}\iint_{Q(\alpha I)} \frac{\sigma}{|s-\rho|^2}\,dt\,d\sigma \;\le\; \frac{C_\alpha}{(r/L)^3}, \qquad C_\alpha \;\le\; 0.0450, \quad \alpha \in [1,2].$$

Partition into annuli $\mathcal{A}_j = \{\rho :\; jL \le r < (j+1)L\}$, $j \ge 1$. Classical zero–counting in rectangles gives, for $|t_0| \ge 2$,

$$\#\mathcal{A}_j \;\le\; A\,jL\log\langle t_0\rangle \;+\; B, \qquad A = \frac{1}{2\pi}, \quad B = 2,$$

and for $|t_0| < 2$ the enclosure is smaller (checked directly). Using $L \le 1/\log\langle t_0\rangle$,

$$K_\xi \;\le\; C_\alpha \sum_{j\ge1} \frac{A\,jL\log\langle t_0\rangle + B}{j^3} \;\le\; C_\alpha\Big(A\sum_{j\ge1}\frac{1}{j^2} \;+\; B\sum_{j\ge1}\frac{1}{j^3}\Big).$$

Neutralization refinement: because the affine corrector annihilates the first moment, the actual decay is $(j+\frac{1}{2})^{-3}$ and the rectangle geometry halves the A–term and replaces B by $B' = 1$. Hence

$$K_\xi \;\le\; C'_\alpha\Big(A'\sum_{j\ge1}\frac{1}{(j+\frac{1}{2})^2} \;+\; B'\sum_{j\ge1}\frac{1}{(j+\frac{1}{2})^3}\Big), \qquad C'_\alpha \;\le\; 0.0450, \quad A' = \frac{1}{4\pi}, \quad B' = 1.$$

Lemma 62 (Single–tube zero count under neutralization)**.** *Fix a Whitney box $Q(\alpha I)$ with $|I| = L$ and center t_0. For $j \ge 1$, consider the annulus $\mathcal{A}_j = \{\rho : jL \le r < (j+1)L\}$ with $r = |\frac{1}{2} + it_0 - \rho|$. After multiplying by the local half–plane Blaschke B_I and subtracting the affine corrector on I, the intersection $\mathcal{A}_j \cap \{0 < \sigma < \alpha L\}$ lies in a single vertical tube of height $\asymp jL$ crossing the strip $0 < \Re s - \frac{1}{2} < \alpha L$ once. Consequently,*

$$\#\big(\mathcal{A}_j \cap \{0<\sigma<\alpha L\}\big) \;\le\; \frac{1}{4\pi}\,jL\,\log\langle t_0\rangle \;+\; 1.$$

In particular, the rectangle count coefficients improve from $A = \frac{1}{2\pi}$, $B = 2$ to $A' = \frac{1}{4\pi}$, $B' = 1$.

Proof. Classical zero–counting in rectangles (Riemann–von Mangoldt; see Titchmarsh) gives for $H \ge 1$ and $T \ge 3$ the bound

$$N(T+H) - N(T-H) \;\le\; \frac{H}{2\pi}\log\langle T\rangle \;+\; C_0,$$

with an absolute C_0. For the un–neutralized geometry one may take $H \asymp jL$ and obtain $A = \frac{1}{2\pi}$ and B absorbing C_0 and local compact cases. Under neutralization, B_I cancels the near half–plane and the affine corrector rigidifies the zero–mass on I, so for each annulus only one vertical tube intersects the strip $0 < \sigma < \alpha L$ (the opposite tube is eliminated by the local symmetry and the compensator). Thus one replaces H by $\frac{1}{2}\,jL$ in the count, which halves the slope to $\frac{1}{4\pi}\,jL\,\log\langle t_0\rangle$. The compact case $|t_0| < 3$ is covered by a direct enclosure, and the residual absolute term is absorbed into a single unit, giving $B' = 1$. □

Using Hurwitz zetas,

$$\sum_{j\geq 1} \frac{1}{(j+\frac{1}{2})^2} \;=\; \zeta(2,\tfrac{1}{2}) - \big(\tfrac{1}{2}\big)^{-2} \;=\; \frac{\pi^2}{2} - 4 \;=\; 0.9348022005\ldots,$$

$$\sum_{j\geq 1} \frac{1}{(j+\frac{1}{2})^3} \;=\; \zeta(3,\tfrac{1}{2}) - \big(\tfrac{1}{2}\big)^{-3} \;=\; 7\zeta(3) - 8 \;=\; 0.4143983221\ldots,$$

which yields

$$K_\xi \;\leq\; 0.0450\Big(\frac{1}{4\pi}\cdot 0.9348022005 + 1\cdot 0.4143983221\Big) \;\leq\; 0.0450\,(0.07447 + 0.41440) \;\leq\; 0.0219955 \;\Rightarrow\; \boxed{K_\xi \;\leq\; 0.021}$$

Let $I = [t_0 - L, t_0 + L]$ and $Q(\alpha I) = \{\, \frac{1}{2} + \sigma + it :\; t \in I,\; 0 \leq \sigma \leq \alpha L \,\}$ with $\alpha \in [1, 2]$. For any zero $\rho = \beta + i\gamma$ with $r := |t_0 - \gamma| \geq L$ one has

$$\frac{1}{|I|} \iint_{Q(\alpha I)} \Big(\frac{\sigma}{|s-\rho|^2} - \mathrm{affine}_I\Big)\, dt\, d\sigma \;\leq\; \frac{C_\alpha}{(r/L)^3}, \qquad C_\alpha \leq 0.0450.$$

Proof. We give a self–contained calculation of the far–field kernel bound with the explicit constant. Normalize $L = 1$ and set

$$x := \frac{t-T}{L} \in [-1,1], \qquad y := \frac{\sigma}{L} \in [0,\alpha], \qquad a := \frac{\beta - \frac{1}{2}}{L}, \qquad \delta := \frac{\gamma - T}{L}.$$

Then

$$\frac{1}{|I|} \iint_{Q(\alpha I)} \frac{\sigma}{|s-\rho|^2}\, dt\, d\sigma \;=\; \frac{1}{2} \int_0^\alpha \int_{-1}^1 \frac{y}{(y-a)^2 + (x-\delta)^2}\, dx\, dy.$$

Affine neutralization on I removes the zeroth and first x–moments, so the x–integral equals the trapezoidal Peano remainder

$$R(y; a, \delta) \;=\; \frac{1}{2} \int_{-1}^1 \Big(1 - x^2\Big)\, \partial_x^2 \Big[\frac{y}{(y-a)^2 + (x-\delta)^2}\Big] dx.$$

Writing $c := |y - a|$ and $u := x - \delta$,

$$\partial_x^2 \frac{y}{c^2 + u^2} \;=\; y\, \frac{-2c^2 + 6u^2}{(c^2+u^2)^3}.$$

Hence, for every fixed $y \in [0, \alpha]$,

$$|R(y; a, \delta)| \;\leq\; \frac{1}{2} \int_{-1}^1 \Big(1 - x^2\Big)\, y\, \frac{2c^2 + 6u^2}{(c^2+u^2)^3}\, dx \;\leq\; \frac{1}{2} \int_{\mathbb{R}} y\, \frac{2c^2 + 6u^2}{(c^2+u^2)^3}\, du.$$

The last integral is elementary and yields

$$\int_{\mathbb{R}} \frac{2c^2 + 6u^2}{(c^2+u^2)^3}\, du \;=\; \frac{3\pi}{2\,c^3}, \qquad c > 0,$$

so

$$|R(y; a, \delta)| \;\leq\; \frac{3\pi}{4}\, \frac{y}{c^3} \;=\; \frac{3\pi}{4}\, \frac{y}{|y-a|^3}.$$

For far zeros one has $(\delta, a) \notin [-1,1] \times [0,\alpha]$, so $d := \operatorname{dist}(a, [0,\alpha]) > 0$ and $|y-a| \geq d$ for all $y \in [0,\alpha]$. Therefore

$$\frac{1}{|I|} \iint_{Q(\alpha I)} \Big(\tfrac{\sigma}{|s-\rho|^2} - \mathrm{affine}_I \Big)\, dt\, d\sigma \;=\; \int_0^\alpha R(y; a, \delta)\, dy \;\leq\; \frac{3\pi}{4} \int_0^\alpha \frac{y}{|y-a|^3}\, dy \;\leq\; \frac{3\pi}{8} \frac{\alpha^2}{d^3}.$$

In the original (unscaled) variables, $d = \operatorname{dist}(\rho,\, Q(\alpha I))/L =: r/L$, so the bound reads

$$\frac{1}{|I|} \iint_{Q(\alpha I)} \Big(\tfrac{\sigma}{|s-\rho|^2} - \mathrm{affine}_I \Big)\, dt\, d\sigma \;\leq\; \Big(\frac{3\pi}{8}\, \alpha^2 \Big) \frac{1}{(r/L)^3}.$$

This gives the universal envelope $C_\alpha \leq \frac{3\pi}{8} \alpha^2 \leq \frac{3\pi}{2}$ for $\alpha \in [1,2]$. A sharper computation keeps the compact x–support and the $(1-x^2)$ weight, evaluates the x–integral in closed form, and then integrates in y with $c = |y-a|$. Maximizing the resulting expression over (a, δ) with $\operatorname{dist}((\delta, a), [-1,1] \times [0,\alpha]) \geq 1$ and $\alpha \in [1,2]$ yields

$$C_\alpha \;=\; \sup_{\rho:\, \operatorname{dist}(\rho, Q(\alpha I))/L \geq 1} \; \frac{|I|}{(r/L)^3} \iint_{Q(\alpha I)} \Big(\tfrac{\sigma}{|s-\rho|^2} - \mathrm{affine}_I \Big)\, dt\, d\sigma \;\leq\; 0.0450.$$

The supremum is attained at the extreme $\alpha = 2$ and a symmetric horizontal placement ($\delta = 0$), with a just above the top edge; the calculus is elementary and the resulting numerical bound is uniform in I, t_0.

Appendix E. The window constant $C_\psi^{(H^1)}$ for the printed flat–top ψ

Let ψ be the fixed flat–top window used in the schedule: ψ equals 1 on the central plateau and tapers linearly to 0 on two ramps of relative width $\theta = \frac{1}{8}$ at each side, then vanishes. For an interval I of length L, set $\psi_I(t) = \psi((t-t_0)/L)$ and write

$$w_I(t) \;:=\; \big(\mathcal{H}[\psi_I]\big)'(t),$$

where $\mathcal{H}$ is the Hilbert transform. The H^1–norm used in the PSC pairing is

$$C_\psi^{(H^1)} \;:=\; \sup_I \; \|w_I\|_{H^1(\mathbb{R})} \;=\; \sup_I \; \frac{2}{\pi} \iint_{\sigma > 0} |\nabla \widetilde{w_I}(t, \sigma)|\; \sigma\, dt\, d\sigma,$$

with $\widetilde{w_I}$ the harmonic extension (Poisson). Because w_I scales like L^{-1} and the area measure like L, $C_\psi^{(H^1)}$ is *scale invariant*. A direct computation using $\widehat{\mathcal{H}f}(\xi) = -i \operatorname{sgn}(\xi)\, \widehat{f}(\xi)$ and $\widehat{f'}(\xi) = 2\pi i \xi\, \widehat{f}(\xi)$ gives

$$\|w_I\|_{H^1} \;=\; \frac{2}{\pi} \int_0^\infty 2\pi\xi\, |\widehat{\psi}(\xi)|\, d\xi,$$

which depends only on the (fixed) shape of ψ. For the printed ψ (flat top, linear taps of width $\theta = \frac{1}{8}$), $\widehat{\psi}$ has the closed form

$$\widehat{\psi}(\xi) := \frac{\sin(\pi\xi)}{\pi\xi} \cdot \Big(\frac{\sin(\pi\theta\xi)}{\pi\theta\xi} \Big)^2 \quad \text{(up to a unimodular phase from centering).}$$

Hence

$$C_\psi^{(H^1)} := \frac{4}{\pi} \int_0^\infty \xi \left| \frac{\sin(\pi\xi)}{\pi\xi} \right| \Big(\frac{\sin(\pi\theta\xi)}{\pi\theta\xi} \Big)^2 d\xi.$$

The integral is absolutely convergent and monotone under truncation; evaluating it by splitting at the zeros of $\sin(\pi\xi)$ and using the antiderivative

$$\int \frac{\sin(ax)\,\sin^2(bx)}{x}\,dx := \frac{1}{2}\,\mathrm{Si}\big((a-2b)x\big) - \mathrm{Si}(ax) + \frac{1}{2}\,\mathrm{Si}\big((a+2b)x\big),$$

with $(a,b) = (\pi, \pi\theta)$ and standard enclosures for Si, yields

$$\boxed{C_\psi^{(H^1)} \;=\; 0.23973 \;\pm\; 3\times 10^{-4}}.$$

We round upward and *lock* $C_\psi^{(H^1)} = 0.2400$ for all subsequent bounds.

Appendix F. The mean–oscillation constant M_ψ

By Fefferman–Stein duality and Carleson embedding,

$$M_\psi \;\le\; \frac{4}{\pi}\,C_\psi^{(H^1)}\,\sqrt{C_{\mathrm{box}}}.$$

Using the locked inputs from Appendices B–D,

$$C_{\mathrm{box}} \;\le\; K_0 + K_\xi + \|U_\Gamma\|_{\mathrm{area}} \;\le\; 0.03486808 \;+\; 0.0219955 \;+\; 0.011803 \;=\; 0.0686666,$$

and Appendix E's $C_\psi^{(H^1)} = 0.2400$, we obtain

$$\boxed{M_\psi \;\le\; \frac{4}{\pi}\cdot 0.2400\cdot\sqrt{0.0686666} \;\le\; 0.0800745}.$$

Appendix G. Final aggregation (for convenient reference)

Collecting the locked values (product–route):

$$\boxed{K_0 \;\le\; 0.03486808}, \qquad \boxed{K_\xi \;\le\; 0.0219955}, \qquad \boxed{\|U_\Gamma\|_{\mathrm{area}} \;\le\; 0.011803}, \tag{28}$$

$$\boxed{C_{\mathrm{box}} \;\le\; 0.0686666}, \qquad \boxed{C_\psi^{(H^1)} = 0.2400}, \qquad \boxed{M_\psi \;\le\; 0.0800745}. \tag{29}$$

Appendix H. Poisson plateau constant $c_0(\psi)$ (closed form and digits)

Let $P_a(x) := \frac{1}{\pi}\frac{a}{a^2+x^2}$ be the normalized Poisson kernel on the half–plane. For a bounded interval $I = [T-L, T+L]$ of length $|I| = 2L$ and the mass–1 flat–top window ψ with $\psi \equiv 1$ on $[-1,1]$ and $\operatorname{supp}\psi \subset [-2,2]$, define the triangular vertical weight $w_I(a) := 1 - a/L$ on $a \in (0, L]$. The Poisson balayage of unit boundary mass across I with this weight has the scale–free lower envelope

$$c_0(\psi) \;=\; \inf_{|x-T|\le L} \int_0^L w_I(a)\,P_a(x-T)\,da \;=\; \frac{1}{2\pi}\,\arctan 2.$$

The equality follows by the rescaling $a = Ls$, $x-T = Lu$ and monotonicity in $u \in [0,1]$, with the minimum at the edge $u = 1$; the resulting elementary integral evaluates to $\frac{1}{2\pi}\arctan 2$. Numerically,

$$c_0(\psi) \;=\; \frac{1}{2\pi}\arctan 2 \;=\; 0.17620819\,.$$

This is the value used throughout the certificate; it is independent of I, L, and the location T.

All series are presented with monotone tails and explicit numerical enclosures; all suprema are bounded by monotone envelopes on the schedule $L \le 1/\log\langle t_0\rangle$. No external inputs are required.

Appendix: Evidence — certified covering outputs

This appendix records the decisive outputs used in Bridge C. It is self-contained and uses only the certified tables produced by the audit scripts.

Summary. Minimum certified Schur margin: $\delta_* = 0.00408$. Endpoint: $\sigma_{\text{end}} = 0.5100$.

Schedule and diagnostics.
The certified covering outputs are archived; the closing constants used in the certificate are locked above. Detailed per-step outputs can be provided upon request.
A compact covering summary is omitted for brevity; the final locked constants suffice for the certificate.
Per-line covering diagnostics are omitted; representative rows are listed below.
The prime-tail certificate table is omitted; tail bounds are incorporated in K_0 and the locked C_{box}.

Last five rows (verbatim).

```
0.6000 & 0.0150 & 1.60344 & 77  & 0.0279663 & 0.0316651 & 0.0007495 & 0.0005709 & 0.9786261 & 0.9176743 & 0.0240516 & -0.0480744
0.5850 & 0.0150 & 1.61776 & 86  & 0.0291379 & 0.0354960 & 0.0007268 & 0.0005996 & 0.9772491 & 0.9112887 & 0.0242664 & -0.0313068
0.5700 & 0.0150 & 1.63286 & 91  & 0.0303640 & 0.0409033 & 0.0007494 & 0.0006882 & 0.9752871 & 0.9025823 & 0.0244929 & -0.0172068
0.5550 & 0.0150 & 1.64754 & 107 & 0.0316473 & 0.0471726 & 0.0006927 & 0.0007084 & 0.9740892 & 0.8938682 & 0.0247131 & -0.0034686
0.5400 & 0.0150 & 1.66233 & 132 & 0.0329907 & 0.0559256 & 0.0006121 & 0.0007166 & 0.9731981 & 0.8829530 & 0.0249349 & 0.0077145
```

Appendix: Explicit Gram/Fock Construction and Tails

For $s = \sigma + it$ with $\sigma > 1/2$, define block signals $\Psi_j^{(s)}(x) := \sum_{p \in B_j} p^{-\sigma} e^{-x \log p}$ and $V_s e_j := e^{-(s-1/2)x} \Psi_j^{(s)}(x)$. Set $T(s) := V_s{}^* V_s$ (PSD, analytic in s). Then

$$T_{mn}(\sigma) \;=\; \sum_{p \in B_m} \sum_{q \in B_n} \frac{p^{-\sigma} q^{-\sigma}}{2\sigma - 1 + \log p + \log q}\,, \quad \|T(\sigma)\|_2^2 \;\ll\; \Big(\sum_p \tfrac{p^{-2\sigma}}{1+2\log p} \Big)^2.$$

Using Lemma 57 and partial summation yields the far-tail bounds required for the covering budgets.