The Formation of the Universe from Recognition

The First Record, Time, the First Interacting Dynamics and the Gap to the Hot Universe

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Abstract

What can be derived about how the universe formed, and about what came before the hot early universe, if we start from recognition rather than from a hot initial state? This book works forward from three operational facts: distinguishable occurrences exist, comparing two of them needs information about both, and a later examination is a further occurrence. It uses four accepted results at their stated conditions: the comparison cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$, golden scaling under the recognizing-join conditions, three dimensions under the linking conditions, and an eight-tick floor.

On an account ledger the start is balance, not nothing. The empty record has cost zero and is an origin of the order of records, and the beginning is itself a premise: without nonnegative counts every ledger has a predecessor, and no shift-invariant law forces a first moment. The cheapest change is one unit post. The first tick costs $2J(r)$ with the ratio r left free; under two adopted premises, on a carrier where the post's bond is a bridge of weight one, the first event with its reciprocal costs $\sqrt{5} - 2$. A complete pass of the parity bits needs eight ticks. Four adopted premises, with three adopted readings of the state, the tick and the weight, leave exactly one law for one recognizer, and under it the arrow of time sits in the record, not in the state. On a cell of sixteen records, the intervals that read the four postings alike, take rank as time, make every history step causal and are positive on the nonzero differences of records of equal rank are exactly the multiples $\lambda(\sum_a y_a^2 - \kappa(\sum_a y_a)^2)$ with $\lambda > 0$ and $\kappa \geq 1$, all of them Lorentzian. On a clock-photon lattice with supplied data we prove regional stability, a dynamics of the register observables of the infinite lattice at every clock strength, a reference state, and, with the clock off and the registers on edges that fill their directions, the dephasing of each register's coherences. For many recognizers the joint law is not unique and conservation is pairing. Under an adopted energy reading one temperature, $\varepsilon / \log \varphi$, is forced.

The proved results stop before any thermal or expanding phase is derived. We state that gap as an exact ledger, from the first thermal state to the dark sector, keeping values that hold only under named premises or measured inputs apart from predictions, and we state fifty-five open problems so that each can be proved or refuted. Results are proved on separate constructions. A chapter builds on another only on a shared construction or one built on it; no correspondence theorem yet joins the others.

How to read this book

The book follows one question in physical order, from the state before any event to the place where the derivations stop. It is not one chain of derivations. Its results are proved on separate constructions, and a chapter builds on another only where the two share a construction or one construction is built on the other; Table 1 and the paragraph after it say where that happens. Its claims also differ in kind, and the text marks the kind of every claim. Reading a result correctly means knowing which kind it is and on which construction it holds.

Kinds of statement.

Proved.
A numbered theorem, proposition, lemma or corollary names its carrier and states its hypotheses; its proof is on the page, in Appendix B when long, or, for a result taken from a cited work, in that work, which the statement cites. A finite computation is called a direct exact computation, with its mechanism and its table. A certified number is printed with its bracket, never rounded beyond it. A value called *computed* is the evaluation of a proved enclosure and is not itself a theorem.

Conditional.
A premise that nothing in the book derives is adopted openly, carries a label and is stated in full where it enters: the opening premise (Chapter 2), (R1) and (R2) (Chapter 3), (P1)–(P4) with three readings (Chapter 4), the record cell's interval and the Regge start (Chapter 5), the supplied lattice data (Chapters 6 and 7), the native forms of the photon field and the register (Chapters 2, 5, 6, 7 and 9), the touching premise (L) and the journal clause (Chapter 8), the thermal readings and the unit premises (N1)–(N3) (Chapter 9). A result that uses one keeps it in its statement.

Identification.
A stated reading of a mathematical object as a physical one, such as reading single postings as light. The text says "identification" where one enters and never presents it as derived.

Measured.
A number taken from observation carries its dataset, year and uncertainty, and every comparison gives the deviation recomputed. Appendix C collects the numbers the book derives or compares with data, each with its formula and conditions, and pins one datum per measured quantity.

Refuted.
A route shown not to work is stated as a result, with its counterexample or the two models that differ. Appendix D lists the refuted routes so that none returns under another name.

Open.
A question the book cannot settle is stated as a question, where it arises. Chapter 11 collects all of them, numbered, each with what would settle it.

Words with one meaning. "Nothing" has four distinct senses (the empty type, the ratio limit $x \to 0^+$, the void and the empty record), and the book never claims a beginning from nothing: the start is balance. Tick order and record rank are orderings, not seconds; elapsed time in seconds needs the unit premises of Chapter 9. Appendix A fixes every term and every symbol with more than one use.

The carriers. Results are proved on separate constructions: ten carriers, labelled (a) to (j) in Table 1.1 and defined in full in Appendix A, two lattices that Chapter 5 uses beside its record cell, and four small foils (Appendix A). A result on one carrier is used on another only where the second is built on the first, as carrier (c) is built on the ledger (a), and (d) on (a) and (c), by definitions and two adopted readings; where an adopted premise joins them, as (R1) and (R2) join the ledger to the field of the first event; or where a statement takes a property on the first carrier as an explicit hypothesis, as one negative statement of Chapter 8 does. No correspondence theorem joins two carriers in the positive direction yet. Otherwise two carriers' results may stand side by side and agree in form; they are never chained into one argument. When a sentence seems to need two carriers at once, that need is an open problem, not a step.

Map. Table 1 gives each chapter's question, its carriers and the premises that enter in it. The chapters are best read in order, since later ones reuse earlier notation and questions, but reading order is not mathematical dependence. The column on the right names only the chapters whose definitions or results enter a statement or proof, with the carrier the two share; a dash means that none enters. Two entries point forward: Chapter 2 uses a self-adjointness theorem of Chapter 6, and a proof for Chapter 5 uses a bound of Chapter 9. A statement that only names an object of another chapter and restates it in full is not counted. Chapter 1's results are theorems about real functions, levels, spheres and traces. Later chapters use its cost J as a definition and say where any other of its results enters, so the column leaves Chapter 1 out.

Which chapters are connected. Three groups of chapters are joined by definitions and theorems, each on its own carriers.

- *The ledger and its recognizers*: Chapters 2, 3, 4, 8 and 9, on carriers (a), (c) and (d). Chapter 3's ledger is the book of Chapter 2; Chapter 4 counts histories with Chapter 2's tallies and Chapter 3's unit posts; Chapter 8 starts from Chapter 4's native law K and books each post in Chapter 3's event journal; and in Chapter 9 the relaxation of each recognizer to its temperature uses Chapter 4's law with Chapter 8's theorem that each recognizer keeps it under every coupling that meets the native target. Two of these joins are adopted, not proved: the imbalance of carrier (c) is the ledger's balance by an adopted reading, and a post of carrier (d) is booked as a reciprocal pair of events by the adopted journal clause.
- *The source and recorder*: Chapters 2 and 6, on carrier (g). Chapter 2 defines it and proves what energy does not select, using the self-adjointness of its Hamiltonian that Chapter 6 proves; Chapter 6 studies its dynamics in Chapter 2's notation.
- *The clock-photon lattice*: Chapters 6 and 7, on carrier (i). Chapter 7's dynamics of the infinite lattice is built from Chapter 6's regional Hamiltonians, their self-adjointness and

the operator bounds of Theorem 6.35. The continuum light cone of Chapter 5 and the photon field as light in Chapter 9 use the photon field of this carrier.

Everything else is a separate construction: the field graphs of the first event (b, Chapter 3), joined to the ledger only by the premises (R1) and (R2); the record cell (e) and the two lattices beside it in Chapter 5; the history dilation (f, Chapter 2); the boundary-theory carrier (j, Chapters 2 and 4); the self-consistent photon lattice (h, Chapter 6); and the four foils of Appendix A. Carriers (g), (h) and (i) share the photon field, and (g) and (i) also share the register; a shared building block carries no result about the dynamics from one carrier to another. Statements that relate two carriers are of three kinds: the join of the ledger to the first event's field by (R1) and (R2), the negative comparisons of Table A.3, and one use of a separate carrier's result inside another carrier's argument, where Proposition 8.18(ii) takes as its hypothesis that each booked event's ratio gives a stationary field of carrier (b) and Proposition 3.59 then excludes the law. Apart from these, no result passes between the groups or to a separate construction: the book never chains the first event's price into the clock-photon dynamics, for example, or a recognizer's temperature into the photon field's. Table A.3 in Appendix A lists every relation proved between two carriers.

Table 1. The chapters, their carriers, and the chapters their proofs use.

Ch.	Question	Carriers	Premises entering	Proofs use
1	What do we start from, and which accepted results do we use?	none	none	–
2	What is the state before the first event, and is a beginning forced?	a, f, g, j	opening premise, native forms	6 (g)
3	What is the first event, and what does it cost?	a, b	(R1), (R2)	2 (a)
4	How does time begin as an order, how do branches weigh, where is the arrow?	a, c, d, j	(P1)–(P4)	2, 3 (a)
5	When is a cell of records a spacetime cell?	e, i	record-cell interval, Regge start, native forms	6, 9 (i)
6	What does the first interacting dynamics do on finite carriers?	g, h, i	supplied data, native forms	2 (g)
7	What does it do on the infinite lattice?	i	supplied data, native forms	6 (i)
8	What follows for many recognizers?	a, c, d	(L), journal clause	3 (a); 4 (c); 3 (b) as a hypothesis
9	What do heat and units add?	a, c, d, i	thermal readings, (N1)–(N3), native forms	2, 3 (a); 4 (c); 8 (c, d); 6 (i)
10	What separates the proved results from the hot, expanding universe?	all	measured inputs	2–9, each on its own carrier
11	Which problems are open, exactly?	all	none	states problems from 1–10

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Chapter 1

The question and the starting facts

Observations of the cosmic microwave background, of the abundances of the light elements and of the recession of galaxies tell one consistent story: the universe we see was once hot, dense and nearly uniform, and it has been expanding and cooling ever since. That story starts from a hot state. It does not say where the hot state came from. General relativity gives the gap a sharp form. Under conditions on energy and causality that ordinary matter is expected to satisfy, together with a trapped region, spacetime is geodesically incomplete [58]; and a spacetime whose average expansion rate along a past-directed geodesic is positive is incomplete in the past even when those energy conditions fail [6]. The classical theory ends before it reaches a beginning. Proposals for what lies there choose a first state by a boundary condition on a wave function of the universe, such as the no-boundary proposal [54] or creation by tunnelling [122].

This book asks the question from the other end. Instead of running the hot universe backward, we start from what every act of recognition already exhibits and derive forward. We ask what the first state of a record is, what the first event is and what it costs, how time begins as an order, what the first cell of spacetime looks like, what the first interacting dynamics does, and how far the derivation reaches toward a hot universe. The derivation reaches a long way and then stops: no thermal or expanding phase has yet been derived on any of the constructions of this book, and Chapter 10 states that gap item by item.

This chapter fixes the starting point. It states the three facts we begin from, the four accepted mathematical results the later chapters use, each with its exact conditions, what those results do not say, the vocabulary, the separate constructions on which results are proved, and the way results are reported.

1.1 A derivation, not a fit

A fit and a derivation can end at the same formula. They differ in what was free. A fit starts from a family of candidate laws and chooses the member that matches the data, so the data carry the weight of the choice. A derivation starts from conditions stated in advance and shows that they leave exactly one law, or a family whose remaining freedom is itself named. The difference can be checked: for a derived law, any alternative must violate one of the stated conditions, and a reader can find which one.

Here is the case the rest of the book rests on. Let F be a real function on the positive ratios, read as the cost of comparing two positive quantities whose ratio is x. Suppose the costs of the two composite comparisons, the product xy and the quotient x/y, are fixed by the costs of the parts through

$$F(xy) + F(x/y) = 2F(x)F(y) + 2F(x) + 2F(y) \qquad (x, y > 0). \tag{1.1}$$

We call (1.1) the composition law. It does not determine F. For every $c > 0$ the function

$$F_c(x) = \frac{x^c + x^{-c}}{2} - 1 \tag{1.2}$$

satisfies it, and so do the constant functions 0 and -1. Take $c = 2$ and $x = y = 2$: $F_2(2) = \frac{9}{8}$, $F_2(1) = 0$ and $F_2(4) = \frac{225}{32}$, and the two sides of (1.1) agree, since $\frac{225}{32} + 0 = 2 \cdot \frac{81}{64} + 4 \cdot \frac{9}{8}$. At $c = 1$ the same point gives $F_1(4) + F_1(1) = \frac{9}{8} = 2 \cdot \frac{1}{16} + 4 \cdot \frac{1}{4}$. Near ratio one, $F_c(e^t) = \cosh(ct) - 1 = \frac{1}{2}c^2t^2 + O(t^4)$: the members of the family differ by how sharply they curve at balance.

So the composition law leaves the scale of cost open, and one more condition closes it. Theorem 1.1 shows that unit curvature at balance leaves exactly one function, and Proposition 1.2 that curvature κ leaves exactly $F_{\sqrt{\kappa}}$. Nothing is fitted. The only remaining freedom, the unit in which cost is measured, is named as a unit. This is the standard the book holds itself to: every result says which conditions it uses, and a condition that a result needs but that nothing yet derives is called a premise and labelled where it enters.

1.2 Three facts about recognition

We start from three facts that any act of recognition exhibits.

(i) *Distinguishable occurrences exist.* Some states, events or outcomes can be told apart. A question, its negation and the answer to it are distinguishable representations. This is a claim that distinction exists somewhere, not that every pair of states can be told apart.

(ii) *Comparison needs information about both terms.* To compare two things, information representing both must be available to the comparison. Comparing an earlier event with a later one requires information about the earlier event to survive until it is used. This asks for sufficient availability, not for permanent or indestructible memory.

(iii) *A later examination is a further occurrence.* A record of events through some stage does not contain the later occurrence of examining it. If a record has cutoff n and the examination happens at a stage $m > n$, every event in the record has stage at most n, so the examination is not among them. The record may contain a prediction or description of the examination; that description is a different occurrence.

We meet these facts in the act of asking whether they hold, which is why we call them facts rather than axioms. They are operational: each says what recognition in this world does. They are not offered as a theorem about every possible world, and they carry no claim that there are infinitely many future events or that self-referential descriptions are impossible. Fact (iii) is the reason a record has an order. Every record has a cutoff, and later events lie beyond it.

The three facts do not by themselves imply the four results below. Each result states the further conditions it uses, and those conditions travel with it into every later chapter.

1.3 The comparison cost

For $x > 0$ put

$$J(x) = \frac{1}{2}\Big(x + \frac{1}{x}\Big) - 1 = \frac{(x-1)^2}{2x}, \qquad J_{\log}(t) = J(e^t) = \cosh t - 1. \tag{1.3}$$

The second form is the cost in the logarithmic coordinate $t = \log x$. In the notation of (1.2), $J = F_1$.

Theorem 1.1 (the comparison cost is forced)**.** *Let $F\colon (0,\infty) \to \mathbb{R}$ satisfy the composition law* (1.1) *for all $x, y > 0$, and suppose it has unit curvature at balance:*

$$\lim_{t\to 0,\ t\neq 0} \frac{2F(e^t)}{t^2} = 1. \tag{1.4}$$

Then $F(x) = J(x)$ for every $x > 0$. Conversely, J satisfies (1.1) *and* (1.4)*. No continuity, differentiability, sign or normalization of F is assumed.*

Proof. Write $H(t) = 1 + F(e^t)$ for real t. With $x = e^s$ and $y = e^t$, the law (1.1) becomes d'Alembert's equation

$$H(s+t) + H(s-t) = 2H(s)H(t) \qquad (s, t \in \mathbb{R}), \tag{1.5}$$

and (1.4) reads $2(H(t)-1)/t^2 \to 1$. For $F = J$ we have $H = \cosh$, which satisfies (1.5) by the addition formula and (1.4) because $\cosh t = 1 + \frac{1}{2}t^2 + O(t^4)$. This is the converse.

For the forward direction we use (1.5) only on the diagonal $s = t$, where it reads

$$H(2s) + H(0) = 2H(s)^2 \qquad (s \in \mathbb{R}). \tag{1.6}$$

At $s = 0$ this gives $H(0) \in \{0, 1\}$. The curvature condition gives $H(v) - 1 = \frac{1}{2}v^2 \cdot 2(H(v) - 1)/v^2 \to 0$, so $H(v) \to 1$ as $v \to 0$ with $v \neq 0$. If $H(0) = 0$, then (1.6) reads $H(2s) = 2H(s)^2$; along $s_n = 2^{-n}$ the left side $H(s_{n-1})$ tends to 1 and the right side $2H(s_n)^2$ tends to 2, which is impossible. Hence $H(0) = 1 = \cosh 0$, and (1.6) is the doubling relation

$$H(2s) = 2H(s)^2 - 1, \tag{1.7}$$

which cosh also satisfies.

Fix $t \neq 0$ and put $v_n = t/2^n$. For small $v \neq 0$ the quotient $2(H(v)-1)/v^2$ is close to 1, hence positive, so there is N with $H(v_n) > 1$ for $n \geq N$; write $H(v_n) = \cosh u_n$ with a unique $u_n > 0$. By induction on k, using (1.7) and $\cosh 2u = 2\cosh^2 u - 1$, we get $H(2^k v_n) = \cosh(2^k u_n)$ for $k = 0, \dots, n$, and at $k = n$

$$H(t) = \cosh\big(2^n u_n\big) \qquad (n \geq N). \tag{1.8}$$

As $n \to \infty$, $\cosh u_n = H(v_n) \to 1$, so $u_n \to 0$, and

$$\Big(\frac{2^n u_n}{|t|}\Big)^2 = \frac{u_n^2}{2(\cosh u_n - 1)} \cdot \frac{2(H(v_n)-1)}{v_n^2} \longrightarrow 1,$$

the first factor because $\cosh u - 1 = \frac{1}{2}u^2 + O(u^4)$ and the second by (1.4). So $2^n u_n \to |t|$, and the right side of (1.8), which does not depend on n, equals $\cosh t$. Thus $H = \cosh$ on $\mathbb{R}$, and $F(x) = H(\log x) - 1 = \cosh(\log x) - 1 = J(x)$. □

Proposition 1.2 (the curvature sets the unit)**.** *Let $\kappa > 0$. A function $F\colon (0,\infty) \to \mathbb{R}$ satisfies* (1.1) *and has curvature κ at balance, $2F(e^t)/t^2 \to \kappa$ as $t \to 0$ with $t \neq 0$, if and only if $F(x) = J\big(x^{\sqrt{\kappa}}\big)$ for every $x > 0$.*

Proof. Put $c = \sqrt{\kappa}$ and $\tilde{F}(x) = F(x^{1/c})$. Since $x \mapsto x^{1/c}$ preserves products and quotients, $\tilde{F}$ satisfies (1.1), and $2\tilde{F}(e^t)/t^2 = c^{-2} \cdot 2F(e^{t/c})/(t/c)^2 \to \kappa/c^2 = 1$. Theorem 1.1 gives $\tilde{F} = J$, so $F(x) = \tilde{F}(x^c) = J(x^c)$. Conversely $J(x^c) = F_c(x)$ corresponds to $H(t) = \cosh(ct)$, which satisfies (1.5) and curves at rate $c^2 = \kappa$. □

Remark 1.3 (what each condition does). The proof shows three things about the hypotheses. First, the composition law is used only on its diagonal $y = x$; at $x = 1$ the diagonal already decides the value at balance. Second, that value comes from the law and not from the curvature: the function equal to $\cosh t$ for $t \neq 0$ and to $-\frac{1}{2}$ at $t = 0$ satisfies the doubling relation (1.7) as written (since $-\frac{1}{2} = 2 \cdot \frac{1}{4} - 1$) and the curvature condition, which never reads $t = 0$; yet it violates (1.1) at $x = y = 1$, where $F(1) = -\frac{3}{2}$ gives -3 on the left and $-\frac{3}{2}$ on the right. Third, the curvature does two jobs: near balance it puts H above 1, where cosh can be inverted, and in the limit it fixes the scale. Normalization $F(1) = 0$ and nonnegativity are conclusions. For continuous solutions of d'Alembert's equation the result also follows from Aczél's classification [1]; the published uniqueness theorem for the canonical reciprocal cost is [131].

The consequences used throughout the book follow from (1.3) at once: $J \geq 0$; $J(x) = 0$ exactly when $x = 1$; $J(1/x) = J(x)$; and $J(x) \to \infty$ as $x \to 0^+$ and as $x \to \infty$. Balance, the ratio one, is the only free comparison, and the cost of any other comparison is positive. Two values recur: $J(2) = J(\frac{1}{2}) = \frac{1}{4}$ and, for the golden ratio of the next section, $J(\varphi) = \varphi - \frac{3}{2}$.

What is proved and what is adopted. Theorem 1.1 is a theorem about real functions. Its physical use is the accepted foundation's: the cost of a comparison in recognition obeys the composition law, and its unit is fixed by unit curvature at balance. That reading is the premise of every later use of J. It does not say that every quantity called a cost, or every physical energy, equals J; Chapter 9 shows in particular that J is not the thermal energy.

1.4 Golden scaling under the recognizing-join conditions

Take a sequence of positive levels $s_0, s_1, s_2, \ldots$. Suppose each step multiplies by the same factor λ, and the third level is the sum of the first two. Then $\lambda^2 s_0 = \lambda s_0 + s_0$, so $\lambda^2 = \lambda + 1$, and $\lambda > 0$ leaves

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.6180340.$$

The arithmetic is the easy part. The theorem below says where the two conditions come from: the uniform factor from an additive reading that preserves positive comparisons, and the sum from the first join that costs anything.

Definition 1.4 (framework, levels and the recognizing-join conditions)**.** A *framework* is a set S of states with a step $T\colon S \to S$ and a reading $r\colon S \to (0,\infty)$ that is not constant; the frameworks of the accepted foundation also have countably many states, admit no injective

map from the real line, and carry a charge conserved by T, properties that nothing below uses. For a base state $b \in S$ the *levels* are $s_n = r(T^n b)$, $n \geq 0$.

(a) An *additive reading* consists of an abelian group G, an additive map $A : G \to G$, a map $\rho : S \to G$ and an additive map $\nu : G \to \mathbb{R}$ with $\nu(\rho(s)) = r(s)$ and $\rho(Ts) = A\rho(s)$ for every s. It need not be injective, and A need not be invertible.
(b) The reading *preserves positive comparisons* if $\nu(g) > 0$ implies $\nu(Ag) > 0$ for every $g \in G$.
(c) The levels have an *initial costly join* if the first generated level is the sum of two earlier ones and that join posts a nonzero cost: there are $a \leq b < 2$ with $s_2 = s_a + s_b$ and $J(s_{b-a}/s_0) > 0$. The cost is that of the comparison between the two parts; when $a = b$ the parts coincide and it is $J(1) = 0$.

Lemma 1.5 (positive additive readings are proportional)**.** *Let G be an abelian group, let $L, M : G \to \mathbb{R}$ be additive, and let $e \in G$ have $L(e) > 0$. If $L(g) > 0$ implies $M(g) > 0$ for every $g \in G$, then $M(g) = \frac{M(e)}{L(e)} L(g)$ for every $g \in G$.*

Proof. We have $M(e) > 0$. Suppose $L(g)/L(e) < M(g)/M(e)$ for some g, and choose a rational $q = m/n$ with $n \geq 1$ strictly between the two ratios. The element $y = me - ng$ has $L(y) = n\big(qL(e) - L(g)\big) > 0$, so $M(y) > 0$; but $M(y) = n\big(qM(e) - M(g)\big) < 0$. If instead $M(g)/M(e) < L(g)/L(e)$, the same argument applies to $y = ng - me$. Hence $M(g)/M(e) = L(g)/L(e)$. No completion of G to the real line is used: the rational cut compares integer combinations of actual elements. □

Lemma 1.6 (the first costly join)**.** *If all levels are positive and have an initial costly join, then $s_2 = s_1 + s_0$.*

Proof. If $a = b$, the cost is $J(s_0/s_0) = J(1) = 0$, contrary to the hypothesis. So $a < b < 2$, which leaves $a = 0$, $b = 1$ and $s_2 = s_0 + s_1$. No uniformity of the levels is used. □

Theorem 1.7 (golden scaling)**.** *Let a framework with base state b carry an additive reading that preserves positive comparisons, and let its levels have an initial costly join. Then $r(Ts) = \varphi r(s)$ for every state s. Consequently the levels are $s_n = \varphi^n s_0$ and their relative weights are $s_0/s_n = \varphi^{-n}$.*

Proof. Apply Lemma 1.5 with $L = \nu$, $M = \nu \circ A$ and $e = \rho(b)$, for which $L(e) = r(b) > 0$: there is $\lambda = \nu(A\rho(b))/r(b) > 0$ with $\nu(Ag) = \lambda\nu(g)$ for all g. Then $r(Ts) = \nu(\rho(Ts)) = \nu(A\rho(s)) = \lambda r(s)$ for every s, so $s_n = \lambda^n s_0$. Lemma 1.6 gives $\lambda^2 s_0 = \lambda s_0 + s_0$, so $\lambda^2 = \lambda + 1$, whose only positive root is φ. □

Example 1.8 (the conditions can be met)**.** Let $G = \mathbb{Z}^2$ with the census update $A(a, b) = (a + b, a)$, and $\nu(a, b) = a + (\varphi - 1)b$. Since $\varphi^2 = \varphi + 1$, $\nu(A(a, b)) = \varphi a + b = \varphi\,\nu(a, b)$, so positive comparisons are preserved. Take as states the orbit $(1, 0), (1, 1), (2, 1), (3, 2), \ldots$ of A, with ρ the inclusion, $T = A$ and $r = \nu \circ \rho$. The levels are $1, \varphi, \varphi^2, \ldots$; the first join $s_2 = s_1 + s_0$ posts the cost $J(\varphi) = \varphi - \frac{3}{2} > 0$. The census update is one realization of the hypotheses; the theorem does not require it.

The hypotheses are exactly what the theorem uses. No rank, basis, invertibility, faithfulness or prior hierarchy enters, and no golden number is supplied. Two conditions carry physical content and are adopted with the theorem: that recognition's levels have an additive reading

preserving positive comparisons, and that the first generated level is a join that is a recognition, that is, one whose two parts are distinguishable, so that by Section 1.3 its cost is positive. Retaining information alone is not the same as preserving the order of comparisons. The golden ratio enters the later chapters only through named premises: the conditional first-event price of Chapter 3 and the weighting premise of one recognizer's law in Chapter 4.

1.5 Three dimensions and the eight-tick floor

Three binary coordinates give $2^3 = 8$ patterns, the vertices of the cube $\{0,1\}^3$. A run of ticks that visits every pattern needs at least eight ticks, one per pattern, and eight suffice: the cycle

$$000 \to 001 \to 011 \to 010 \to 110 \to 111 \to 101 \to 100 \to 000$$

flips one coordinate per tick and returns to its start. This section states where the three comes from and what the eight does and does not mean.

The three comes from a property of loops. A pair of loops that stay linked cannot be pulled apart by deformation, and linking is detected by the first homology of a loop's complement.

Theorem 1.9 (only the three-sphere detects linking)**.** *For an integer $D \geq 0$, some embedded circle in the sphere S^D has a complement with nonzero first homology if and only if $D = 3$.*

Proof. For an embedding $h : S^k \to S^n$ with $k < n$, the reduced homology of $S^n \setminus h(S^k)$ is $\mathbb{Z}$ in degree $n - k - 1$ and zero in every other degree [56, Proposition 2B.1]; the proof is by induction on k from the fact that the complement of an embedded arc is acyclic, and it needs no tameness of the embedding. With $k = 1$ and $n = D \geq 2$, the first homology is nonzero exactly when $D - 2 = 1$. For $D = 1$, an embedded circle in S^1 is all of S^1, since its image is compact and, by invariance of domain, open; its complement is empty. For $D = 0$ there is no embedded circle. At $D = 3$ a round circle has complement homotopy equivalent to a circle, with first homology $\mathbb{Z}$. □

Definition 1.10 (the linking conditions)**.** A *spatial realization in dimension D* of a persistent pair of loops consists of configurations of the pair, a deformation relation on them that is reflexive, symmetric and transitive, the realized pair, and a split configuration, together with a *pairing observable*: an integer function on configurations unchanged by deformation and zero on the split configuration. The realization has the *linking correspondence* if a pairing observable that is nonzero on the realized pair exists exactly when some embedded circle in S^D has a complement with nonzero first homology. The *deformation-erasure condition* holds if some pairing observable is nonzero on the realized pair.

Under the deformation-erasure condition no deformation carries the pair to the split configuration: the pairing would be unchanged along the deformation and zero at its end. A recognized distinction of the pair is then stable under deformation.

Theorem 1.11 (three dimensions under the linking conditions)**.** *If a spatial realization in dimension D has the linking correspondence and satisfies the deformation-erasure condition, then $D = 3$.*

Proof. The deformation-erasure condition gives a pairing observable nonzero on the pair; the linking correspondence turns it into an embedded circle in S^D with homologically nontrivial complement; Theorem 1.9 gives $D = 3$. □

The linking correspondence and the deformation-erasure condition are part of the implication; they are the physical premises of the result. At $D = 3$ a realization with both exists, built from the winding pairing of the golden hierarchy, and at $D = 4$ both sides of the correspondence fail. The theorem selects the dimension. It does not prove that an arbitrary physical carrier is such a realization. The same dimension is detected on the stage built from D independent binary distinctions: the one-point compactification of the open D-cube is homeomorphic to S^D, so the detector of Theorem 1.9 fires on that stage exactly when $D = 3$. The published accounts of the three-dimensional result and of the eight-unit loop are [130, 128].

Theorem 1.12 (the eight-tick floor)**.** *Let D be such that the stage built from D binary distinctions, the one-point compactification of the open D-cube, contains an embedded circle whose complement has nonzero first homology. Call a map from T ticks onto the 2^D patterns $\{0,1\}^D$ a covering of the patterns. Then every covering has $T \geq 8$, and a covering with $T = 8$ exists.*

Proof. Detection gives $D = 3$ by Theorem 1.9 and the homeomorphism of the stage with S^D. A map from T ticks onto the 8 patterns needs $T \geq 8$. The cycle displayed above is a covering of length eight. □

A covering need not move one bit per tick or return to its start. The complete passes of Chapter 4 (Definition 4.4) are the closed walks that flip one bit per tick and visit every pattern; each of them is a covering, so the floor applies to them, and the cycle displayed above is one.

Remark 1.13 (what the eight does not say). Two cautions keep the floor at its scope. First, a statement of the form "$P(3)$ implies that the period 2^3 is 8" is true for every predicate P, the false and the irrelevant included, so it carries no content; Theorem 1.12 is not of that form, because its hypothesis fails at $D = 4$, where the period would be 16. Second, the floor counts ticks of a covering of the patterns. It is not a theorem that every physical process has least period eight. Whether a covering returns to its start is a separate property, and the least period of a process that completes passes using only its current state is the theorem of Chapter 4, with its own premises. The eight-step cycle of the recognition foundation is derived in [126, 125].

1.6 The arithmetic of records

A record grows by appending entries. The arithmetic the book uses is the arithmetic of that growth, built from one act of distinction and nothing else.

Definition 1.14 (traces and orbit numbers)**.** A *trace* is either the empty trace $\varnothing$ or a trace extended by one act of distinction δ. The *orbit traces* are $o_0 = \varnothing$ and $o_{n+1} = o_n\delta$. The *length* of a trace counts its acts: $\ell(\varnothing) = 0$ and $\ell(t\delta) = \ell(t) + 1$. The *orbit numbers* are generated by a zero and a successor.

Proposition 1.15 (counting from the empty trace)**.** *(i) $\ell(o_n) = n$ for every n. (ii) Zero is not a successor, the successor is injective, and every property that holds at zero and passes from each*

orbit number to its successor holds for all of them. (iii) The maps sending zero to 0 *and successor to* +1*, and back, are inverse bijections between the orbit numbers and the natural numbers. (iv) The tick index of a record, with its first tick and its successor, has the universal property of a natural-number object: for every set X, point $x \in X$ and map $f\colon X \to X$ there is exactly one map from tick indices to X sending the first tick to x and the successor to f. Hence the tick index is identified with the orbit numbers by a unique bijection preserving zero and successor.*

Proof. Parts (i) to (iii) are inductions on the definitions. For (iv), the required map is iteration of f from x, and two maps with the two properties agree at the first tick and, by induction, at every later one. Two natural-number objects are related by the unique map between them given by the universal property of each, and the two composites are the identity by uniqueness. □

Zero is the empty trace, and the first number is one distinction. Chapter 2 meets the same shape physically: the empty record, with posts appended one at a time. Proposition 1.15(iv) says the tick count and the orbit of distinctions are the same iteration object. It says nothing about duration. The count of ticks is an order, and reading it as time is an identification, named wherever it is used. The arithmetic also establishes no claim that physical space is discrete or that every physical object is countable.

1.7 What the foundation does not say

The three facts and four results above are the whole of the foundation this book uses. Their silences matter as much as their content, because the questions of cosmology begin exactly where they stop.

Nothing about a first event, a beginning of time or initial conditions. None of the results selects a first state, a first event, a law of motion or a physical unit. The empty record will be shown to be an origin of the order of records (Chapter 2), but that the record of this universe opened empty is a premise, which Chapter 2 names and proves load-bearing. Which first post occurs is not selected by anything proved so far, and Chapters 2 and 3 state precisely what remains unselected.

The order of an argument is not elapsed time. The steps of the forcing argument are steps of an argument. They do not label successive eras of the universe. Likewise the tick count orders events; it is not a number of seconds. Elapsed time in seconds needs at least one dimensional input, and Chapter 9 states the unit premises that supply it.

Nothing from nothing. The book begins at distinction. It does not derive distinction from its absence, and the book never claims a beginning from nothing: the start is balance, the ratio of zero cost (Chapter 2).

No physics beyond the stated scope. J is the cost of a comparison, not every energy; the golden ratio scales a hierarchy under the conditions of Theorem 1.7; three dimensions follow from the linking conditions; the eight is a floor on coverings of the patterns. Each later use of these results says which of them it takes and which further premise it adds.

1.8 Vocabulary: four nothings and the words of the record

A few words carry most of the weight, and some of them are used loosely elsewhere. Appendix A fixes each of them for the whole book (Table A.2); four need a word here. "Nothing" has four senses, and they are four different objects: the *empty type*, with no element at all, the floor below which the derivation does not go; the *ratio limit* $x \to 0^+$, where J diverges; the *void*, a carrier whose elements are all equal and which hosts no distinction; and the *empty record*, the record with no posts, equivalently the unity point of cost 0, which is balanced. Chapter 2 defines each and keeps them apart. The book never claims a beginning from nothing; the start is balance.

A *record* is an append-only list of posts, and the *ledger* (or book) is the account counts the record tallies; a *post* is one unit entry on one account and one side; a *tick* is one step of a history, a post or a hold; a *pass* is a complete visit of the three parity bits. Tick order and rank are orderings, charts, not seconds: "time is the tick count" is an adopted identification wherever it is used, and Chapter 4 proves that no global serial clock exists. "Cost equals posting count" holds by definition on the carrier where cost is counted in posts, and nowhere else. "Record entropy" is an adopted name for record length, not a thermodynamic entropy, and "exists means zero cost" is an adopted identification, the law of existence.

1.9 The carriers are separate constructions

The results of this book are proved on several mathematical constructions, which we call carriers. Each is the natural home of one question. Table 1.1 lists them; Appendix A defines each in full, with its objects, its dynamics and its supplied data.

Table 1.1. The carriers and the chapters that use them.

	Carrier	Objects	Chapters
(a)	account ledger and its records	three accounts, two sides, integer counts; records as lists or tallies	2, 3, 4, 8, 9
(b)	field graphs of the first event	weighted graphs with the nonlinear Gauss law: the possibility graph, the record carrier, the torus	3
(c)	one recognizer's imbalance	$\mathbb{Z}^3$ under the native law	4, 8, 9
(d)	many recognizers	finite sets of voxels with joint laws	4, 8, 9
(e)	the record cell	the four-dimensional cell of sixteen records	5
(f)	the history dilation	recorded histories and their reversible extension	2
(g)	source and recorder	a native pair, its internal coordinates, the transverse photon field, record labels	2, 6
(h)	self-consistent photon lattice	a signed Gauss field with a photon field at each site, of finite period	6
(i)	clock-photon lattice	cells of $\mathbb{Z}^3$, four eight-state registers per cell, one shared transverse photon field	5, 6, 7, 9
(j)	boundary-theory carrier	clock cell, native step, history space of the boundary theory	2, 4

The no-combination rule. A result proved on one carrier is never combined with a result proved on another unless one carrier is built on the other, an adopted premise joins them, as (R1) and (R2) join the ledger to the field of the first event, or a theorem establishes a correspondence between the two carriers; a statement may also take a property on another carrier as an explicit hypothesis. No correspondence has been proved in the positive direction; the relations that are known are nestings, a join by adopted premises, shared building blocks, results of the same form and negative results, and Table A.3 lists them with their chapters. Two carriers' results may be set side by side, and we say when they agree in form. They are not chained into one argument. Within one carrier, and along the nestings of carriers (a), (c) and (d), chapters do build on one another, and "How to read this book" names those chapters. The results of this chapter are theorems about real functions, levels, spheres and traces; each later chapter says on which carrier it uses them.

1.10 How results are stated

Every statement in the book is of one of five kinds, and the text keeps them apart: *proved*, with its carrier and hypotheses in the statement and its proof on the page or in Appendix B; *conditional* on a premise that nothing in the book derives, adopted openly, labelled and stated in full where it enters; an *identification*, a stated reading of a mathematical object as a physical one, named once and never presented as derived; *measured*, with dataset, year and uncertainty, collected with every printed number in Appendix C; and *refuted*, a route shown not to work, stated as a result with its counterexample and listed in Appendix D. Questions the book cannot settle are stated as open where they arise and collected in Chapter 11. The front matter, "How to read this book", says how the text marks each kind. Table 1.2 previews the premise labels; Appendix A collects them.

Table 1.2. Adopted premises and supplied data, with the chapter where each enters and is stated in full.

Label	Content in brief	Enters
opening premise	the books opened empty: counts are nonnegative and every count was posted	2
(R1), (R2)	the first event is one post; a post is the unit source of the exact-cost Gauss field, and its strain is the drop across its own bond	3
(P1)–(P4)	one recognizer's law is Markov; each tick holds or posts one unit; detailed balance against $\varphi^{-\lvert s\rvert_1}$; hold-return balance; read with three readings (state is imbalance, a tick may hold, weight by level)	4
record-cell premises	a name-blind interval (all four postings read alike) and $\kappa \geq 1$	5
supplied lattice data	lattice constant a, energy scale C_E, clock strength ϵ, an injective placement of unit edges, zero background, the region, the initial state; the reference configuration is a choice of representation, not a selected first state	6, 7
thermal readings	the golden weights read as Boltzmann factors; the record cell with its per-tick refresh; a supplied Bose occupation	9
(N1)–(N3)	the unit premises: one voxel per tick as the speed of light; Planck's constant from the coherence energy (for masses only); Newton's constant from the recognition length; with the measured value of Newton's constant	9

1.11 The plan of the book

The chapters follow the physical order, from before the first event to the edge of what is derived.

Chapter 2, *Before the first event*, defines the four nothings, shows that the start is balance, and proves that the empty record is an origin of the order of records. It then shows that the beginning is a premise: inside the opening premise the empty ledger is the unique first moment, while without it every ledger has a predecessor and no shift-invariant law forces a beginning. The configuration of the cost is a fuel gauge, not a clock. On the native history carrier the empty record has a reversible predecessor outside the recorded histories, and on the source and recorder carrier reading out or recording the seed's coordinate raises the energy by exactly 7 native energy units, so preparation has a price. The chapter ends with what stays unselected, and with the genesis state of boundary theory on its own carrier.

Chapter 3, *The first event and its price*, proves that the cheapest change is one unit post and that the first tick costs $2J(r)$ with net flow zero, the ratio r left free. Under the premises (R1) and (R2), on a carrier where the post's bond is a bridge of weight one, the first event with its reciprocal costs $2J(\varphi) = \sqrt{5} - 2$. On every finite carrier the field exists, is unique and has one drop, bracketed by the conductance; this gives a test every carrier of a golden first post must pass, and a stronger carrier can raise the drop, and at unit strength, with the post's bond of weight at least one, never above $\log \varphi$. Which carrier holds the first post's field is open, and the second event is left undetermined.

Chapter 4, *Time: the eight-tick order, the native law and the arrow*, proves that a complete pass takes at least eight ticks and counts the histories exactly. It shows that (P1)–(P4) with their three readings leave exactly one law for one recognizer, the native law K, in which the golden ratio enters only through (P3); under K the imbalance relaxes to $\pi(s) = \varphi^{-9}\varphi^{-|s|_1}$, returns to balance after φ^9 ticks on average, and completes its first pass late. The arrow of time sits in the record, not in the state, and under K the information a record of the present retains falls at every tick. Rank is a chart, and no global serial clock exists.

Chapter 5, *The first spacetime cell*, reads the four-dimensional cell of sixteen records: its time is its rank, and an interval that reads the four postings alike is a spacetime interval exactly when it is $\lambda(\sum x^2 - \kappa(\sum x)^2)$ with $\lambda > 0$ and $\kappa \geq 1$. On the lattice of such cells with Euclidean lengths, flatness is a stationary saddle of Regge's action along a uniform stretch of the main diagonals. The cell's cone and the lattice photon do not match without an identification.

Chapters 6 and 7, *The first interacting dynamics*, treat sources and eight-state clocks coupled to photon fields. On finite carriers a self-consistent field evolves globally and produces photons with an exact energy account, and the energy of every finite region of the clock-photon lattice is bounded below while the reference vector has no infinite-volume limit. On the infinite lattice the dynamics of register observables exists at every clock strength, the reference state is a state of the whole algebra, and with the clock off and an edge assignment that fills the directions it uses, each register's coherences dephase while gradient flip patterns, which the field does not see, stay coherent; with the clock off the dynamics also has states it preserves, reached from the reference state along long-time averages.

Chapter 8, *Many recognizers*, proves that each recognizer keeps K under every coupling, that the joint law is not unique, that joint balance forces independence and that conservation of total imbalance is pairing. Under a supplied law for an isolated pair the light cone is exact and octahedral.

Chapter 9, *Toward the hot universe*, derives one temperature $\varepsilon/\log\varphi$ under the adopted energy reading, which reads the golden weights as Boltzmann factors (a finite-state Gibbs construction from a recognition cost is [127]); it shows that sharing a thermal cell selects no coupling, treats the native photon field as light at long wavelength with its blackbody coefficient under a supplied Bose occupation, and gives the tick in seconds under the unit premises and the measured value of Newton's constant.

Chapter 10, *What is not yet derived*, is the exact ledger of the gap to the hot, expanding universe: for each topic, what is derived and on which carrier, what is conditional, adopted or measured, and what is refuted. Chapter 11 states every open problem exactly. The appendices give the vocabulary and the carriers in full (Appendix A), the deferred proofs (Appendix B), the numbers with their comparison data (Appendix C) and the refuted routes (Appendix D).

The next chapter begins where the foundation stops: with the state before any event, and with the question of whether anything forces it to have been first.

Chapter 2

Before the first event

What was there before the first event? The everyday answer is "nothing", and in discussions of the origin of the universe that one word does the work of four different objects. This chapter separates them, identifies the one the derivation starts from, and then asks the two questions any account of a beginning has to answer. Can time be counted from the start, with nothing before it and no freedom to run the count backwards? And does anything force the universe to have started there, or in one particular state?

The answers are exact. The start is the empty record, a state of balance in a world that admits distinctions, not the void and not "nothing". On records it is an origin of time, and that is a theorem. That the books of the universe opened empty is a premise, which we prove carries the whole weight of a first moment and which no law blind to a common shift of the counts can supply. On the quantum carriers used later a boundary of the same form appears; there neither energy nor the reversible dynamics selects a blank record, and preparing the native ready state has a positive price.

2.1 Four things called nothing

Start with two small sets. On $K = \{0, 1\}$ the predicate "$z = 0$" holds at 0 and fails at 1; it picks something out. On a one-element set $\{*\}$ every predicate holds everywhere or nowhere, so no predicate separates anything and no comparison can even be stated. Neither set is empty. The second is closer to what people usually mean by "nothing", and the starting state of this book lives in a world like the first.

Definition 2.1 (Four nothings)**.** Four different objects go by the name "nothing".

(i) The *empty type*: a type with no element.
(ii) The *ratio limit* $x \to 0^+$ of the comparison cost J: a limit at which the cost diverges, not a state.
(iii) The *void*: a nonempty set K with $x = y$ for all $x, y \in K$.
(iv) The *empty record*: the record with no posts, together with the empty event ledger (Section 2.2). Its cost is 0, the cost $J(1)$ of a comparison at ratio one, and it is balanced; we also call it the *unity point*.

The four are never interchangeable in this book. Item (i) is the floor below which the derivation does not go; item (ii) is the end of the cost line that no state reaches; item (iii) is a carrier on which no distinction can be stated; item (iv) is the start. The book never claims a beginning from nothing: the start is balance.

2.1.1 The floor

Proposition 2.2 (The floor)**.** *A type with no element and a type with exactly one element are distinct. Hence there are two distinct types.*

Proof. If they were equal, the element of the one-element type would be an element of the empty type, which has none. □

Every statement, including the statement that nothing can be told apart, is made in a language whose true and false are two distinct propositions. A derivation of distinction from its absence therefore has nothing to start from. We read Proposition 2.2 accordingly: the derivation starts at distinction, and a beginning "from nothing" is excluded rather than derived. This is a reading of the floor, not a further theorem.

Definition 2.3 (Floor witness)**.** Let K be a nonempty set. A *nontrivial specification* on K is a predicate on K that some element satisfies and some element fails. A *floor witness* for K consists of two distinct propositions together with a nontrivial specification on K.

A recognition on K needs at least a floor witness: something it picks out, something it does not, and a language that can state the difference.

Theorem 2.4 (Floor witness and distinguishability)**.** *Let K be a nonempty set. Then K has a floor witness if and only if K has two distinct elements.*

Proof. If $P(x)$ holds and $P(y)$ fails, then $x \neq y$. Conversely, if $x \neq y$, the predicate $P(z) : z = x$ is satisfied by x and failed by y, and the propositions "true" and "false" are distinct. □

So the floor is a set with two distinct elements. It is the precondition for stating anything, not a physical postulate of the theory.

Theorem 2.5 (Void or distinction)**.** *For every nonempty set K exactly one of the following holds: K is void, or K has two distinct elements. A void set has no floor witness.*

Proof. "$x = y$ for all x, y" is the negation of "$x \neq y$ for some x, y". The last sentence follows from Theorem 2.4. □

The void hosts no distinction, hence no recognition and no state that a first event could change. The empty record is something else: one element of the set of records, which has many distinct members, singled out by having no posts.

2.1.2 The cost line and the law of existence

The comparison cost of a ratio $x > 0$ is

$$J(x) = \frac{x + x^{-1}}{2} - 1 = \frac{(x-1)^2}{2x}, \tag{2.1}$$

the only cost that obeys the composition law with unit curvature at ratio one (Chapter 1).

Proposition 2.6 (The cost trichotomy)**.** *For every $x > 0$: (i) $J(x) \geq 0$; (ii) $J(x) = 0$ if and only if $x = 1$; (iii) $J(x) > 0$ whenever $x \neq 1$; (iv) for every real C there is $\varepsilon > 0$ with $J(x) > C$ for all $0 < x < \varepsilon$.*

Proof. Items (i) to (iii) follow from the second form of (2.1): the numerator is a square that vanishes only at $x = 1$, and the denominator is positive. For (iv), $x/2 \geq 0$ gives $J(x) \geq \frac{1}{2x} - 1$; with $\varepsilon = 1/(2(|C| + 2))$, every $0 < x < \varepsilon$ has $\frac{1}{2x} > |C| + 2$, so $J(x) > |C| + 1 > C$. □

The cost line has three regions: cost without bound toward $x \to 0^+$, nothing in sense (ii); cost zero at exactly one point, $x = 1$; and finite positive cost at every other ratio. The start of this book sits at the zero, read as the empty record; the configuration of zero cost in Proposition 2.22 is another object, an end rather than a start.

The law of existence, an adopted identification. We read "the ratio x exists" as "$x > 0$ and $J(x) = 0$". This reading of a mathematical condition as a physical one is adopted here and named once; it is not derived. Under it, Proposition 2.6 says that exactly one ratio exists, $x = 1$, which is also the unique minimizer of J on $(0, \infty)$, and that the ratio limit $x \to 0^+$ does not exist in this sense: near it the cost exceeds every bound.

2.2 The empty ledger and the append-only record

Before the first event nothing has been compared and nothing has been posted. A state carries what has happened in two forms, and before the first event both are empty. The objects of this section and the next are those of [129].

2.2.1 Events and the event ledger

A *recognition event* is a triple $e = (s, t, r)$ with a source $s \in \mathbb{N}$, a target $t \in \mathbb{N}$ and a ratio $r > 0$. Its *reciprocal* is $e^* = (t, s, 1/r)$, the same comparison read from the other side; $e^{**} = e$. An *event ledger* is a finite list $L = (e_1, \dots, e_m)$ in which every event occurs exactly as often as its reciprocal (double entry). Its cost and its net flow at a site $a \in \mathbb{N}$ are

$$c(L) = \sum_{i=1}^{m} J(r_i), \qquad \sigma_a(L) = \sum_{i\,:\; a \in \{s_i, t_i\}} \log r_i. \tag{2.2}$$

Proposition 2.7 (The empty event ledger)**.** *(i) The empty list is an event ledger, with cost* 0 *and net flow* 0 *at every site.*
(ii) Every event ledger L has $\sigma_a(L) = 0$ at every site a.

Proof. (i) Every event occurs zero times in the empty list, and so does its reciprocal; both sums in (2.2) are empty. (ii) Let $f_a(e)$ be the contribution of $e = (s, t, r)$ to σ_a: $\log r$ if $a \in \{s, t\}$ and 0 otherwise. The reciprocal touches the same sites with $\log(1/r)$, so $f_a(e^*) = -f_a(e)$. By double entry an event g occurs among $e_1^*, \dots, e_m^*$ as often as g occurs in L, so the reciprocals are the events of L in another order, and $\sigma_a(L) = \sum_i f_a(e_i) = \sum_i f_a(e_i^*) = -\sigma_a(L)$. □

The two zeros have different standing. Zero flow is a conservation law: every state has it. Zero cost is a floor that lasts only until something is distinguished. Adding an event with its reciprocal raises the cost by $2J(r)$, which is positive exactly when $r \neq 1$ (Chapter 3). So the starting state costs nothing and owes nothing: that is balance, not the void.

2.2.2 Posts, records and books

Definition 2.8 (Posts, account ledgers, records, tallies)**.** Fix a number $d \geq 1$ of accounts; the book uses $d = 3$. A *post* is a pair $p = (k, \sigma)$ with $k \in \{1, \dots, d\}$ and $\sigma \in \{\mathrm{D}, \mathrm{C}\}$ (debit or credit); the set A_d of posts has $2d$ elements. An *account ledger*, or *book*, is a pair $L = (D, C)$ of integer vectors in $\mathbb{Z}^d \times \mathbb{Z}^d$. Posting adds one to one count:

$$\mathrm{post}(L, (k, \mathrm{D})) = (D + e_k, C), \qquad \mathrm{post}(L, (k, \mathrm{C})) = (D, C + e_k),$$

with e_k the kth unit vector; a *posting step* $L \to L'$ means $L' = \mathrm{post}(L, p)$ for some post p. The *empty ledger* is $0 = (0, 0)$. The *balance* of account k is $\varphi_k(L) = D_k - C_k$, and the *total* is $T(L) = \sum_k (D_k + C_k) \in \mathbb{Z}$, which every posting step raises by exactly one. A *record* is a finite list $\rho = (p_1, \dots, p_n)$ of posts, oldest first, of length $|\rho| = n$; the empty record is $(\,)$. A *record step* appends one post, $\rho \to \rho \cdot p$. The *tally* of a record is the book it adds up to, $\mathrm{tal}(\rho) = \mathrm{post}(\cdots \mathrm{post}(\mathrm{post}(0, p_1), p_2) \cdots, p_n)$, with $\mathrm{tal}(\,) = 0$.

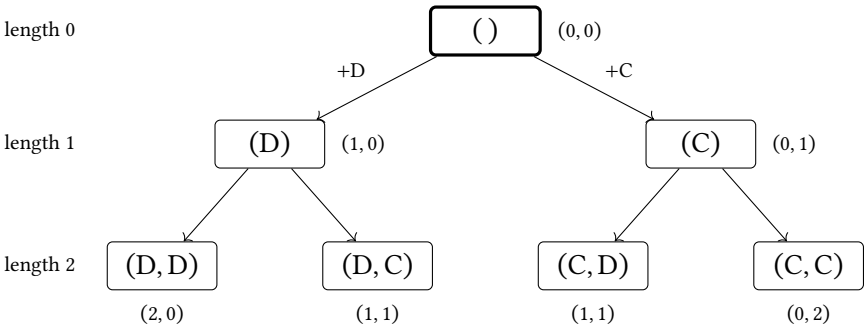


Figure 2.1. Records over one account ($d = 1$) up to length two, each with its tally (D, C). The empty record (thick box) is the root; every record has a successor one post longer, and nothing precedes the root. The records (D, C) and (C, D) differ and write the same book.

Figure 2.1 draws the case $d = 1$. The records form a tree rooted at the empty record, the tally forgets the order of posts, and the total of each tally equals the length of its record.

Proposition 2.9 (The record is append-only)**.** *Let ρ be a list over any alphabet and e an entry. Then the first $|\rho|$ entries of $\rho \cdot e$ are ρ; $|\rho \cdot e| = |\rho| + 1$; and every entry of ρ sits at the same position in $\rho \cdot e$.*

Proof. Appending writes e at position $|\rho| + 1$ and changes nothing before it. □

Writing never alters what was written.

2.3 The empty record is an origin of time

A clock needs a step relation and a quantity read along it. What should it take for a state to be the place where such a clock starts?

Definition 2.10 (Origin of time)**.** A *clock* on a set X is a step relation $\to$ on X with a potential $P\colon X \to V$ valued in a partially ordered set V. A state $o \in X$ is an *origin of time* for $(\to, P)$ when

(O1) nothing precedes it: $P(o) \le P(x)$ for every $x \in X$;
(O2) the clock must advance: every $x \in X$, including o, has a successor $x \to y$ with $P(x) < P(y)$;
(O3) the direction is rigid: no surjection $f\colon V \to V$ reverses strict order (that is, has $f(b) < f(a)$ whenever $a < b$).

Condition (O2) excludes a state the dynamics can rest in, since a state that cannot be left is an end, not a beginning. It asserts that an advancing successor exists, not which one occurs. Condition (O3) asks that no relabelling of the values turns the clock around.

Lemma 2.11 (Rigid and flippable orders)**.** *Let V be partially ordered with a least element b, and suppose every $v \in V$ has some $w > v$. Then V is rigid in the sense of (O3). In particular $\mathbb{N}$ and $[0, \infty)$ are rigid. The integers $\mathbb{Z}$ and the reals $\mathbb{R}$ are not: negation reverses each of them.*

Proof. Let f be a surjection reversing strict order. For each v, either $b < v$, and then $f(v) < f(b)$, or $v = b$; so $f(v) \le f(b)$ for all v. Choose $w > f(b)$ and u with $f(u) = w$; then $w = f(u) \le f(b) < w$, a contradiction. On $\mathbb{Z}$ and $\mathbb{R}$ the map $z \mapsto -z$ is a bijection with $-b < -a$ whenever $a < b$. □

Theorem 2.12 (The empty record is an origin of time)**.** *Let E be a nonempty set, X the set of finite lists over E, $\rho \to \rho'$ the relation $\rho' = \rho \cdot e$ for some $e \in E$, and $P(\rho) = |\rho| \in \mathbb{N}$. Then the empty list is an origin of time for $(\to, P)$. Along every committing history $\rho_0 \to \rho_1 \to \rho_2 \to \cdots$ the length satisfies $|\rho_n| = |\rho_0| + n$: it increases strictly at every step, by one per tick, and exceeds every bound. In particular this holds for records, $E = A_d$ with $d \ge 1$.*

Proof. (O1): $|(\,)| = 0 \le |\rho|$. (O2): pick $e \in E$; then $\rho \to \rho \cdot e$ and $|\rho \cdot e| = |\rho| + 1$. (O3): $\mathbb{N}$ is rigid by Lemma 2.11. Each step appends one entry, so the length rises by one at every step (Proposition 2.9). □

Corollary 2.13 (No signed clock is an origin)**.** *Let X be any set with any step relation. No potential $P\colon X \to \mathbb{R}$ and no potential $P\colon X \to \mathbb{Z}$ has an origin of time.*

Proof. Condition (O3) fails for $V = \mathbb{R}$ and $V = \mathbb{Z}$ by Lemma 2.11, whatever X, $\to$ and o are. □

The clock that counts posts has a floor and no ceiling, so no relabelling of the count runs it backwards; a clock read from a signed quantity (an energy, a cost difference, a signed total) can always be reversed by negation. Discussions of the arrow of time often postulate that the universe began in a special state. On records, the three conditions of Definition 2.10 are proved properties of the empty record, because a record is a list whose length is a natural number; choosing to keep time on an append-only record is the premise, and Section 2.4

measures what that premise is worth. We call the length of a record its *record entropy*, an adopted name for this count and not a thermodynamic entropy; reading the tick count as physical time is an adopted identification (Chapter 4).

2.3.1 The books a record can write

Later chapters also read the record through its books. On exactly the books that records write, the empty book keeps the role of the empty record.

Proposition 2.14 (Tallies)**.** *(i) For every record ρ, $T(\mathrm{tal}\,\rho) = |\rho|$; the tally has nonnegative counts and is reached from the empty book by posting steps.*
(ii) For every book L the following are equivalent: (a) $L = \mathrm{tal}\,\rho$ for some record ρ; (b) every count of L is nonnegative; (c) L is reached from the empty book by finitely many posting steps.

Proof. (i) By induction on ρ: $\mathrm{tal}(\rho \cdot p) = \mathrm{post}(\mathrm{tal}\,\rho, p)$, and a posting step raises T by one and one count by one. (ii) (a)⇒(c) is (i). (c)⇒(b): posting only increases counts, and the empty book's counts are 0. (b)⇒(a), by induction on $T(L) \geq 0$: if $T(L) = 0$ and all counts are nonnegative, all are zero and $L = \mathrm{tal}(\,)$; if $T(L) > 0$ some count is positive, and removing one unit from it gives L_0 with nonnegative counts, $T(L_0) = T(L) - 1$ and $L = \mathrm{post}(L_0, p)$ for the corresponding post; by induction $L_0 = \mathrm{tal}\,\rho_0$, so $L = \mathrm{tal}(\rho_0 \cdot p)$. □

Let $\mathscr{R}_d$ be the set of tallies. On it the total is a natural number, the *posting count* $N(L) = T(L)$, and posting steps keep a tally a tally.

Theorem 2.15 (The empty book is the unique least tally)**.** *Let $d \geq 1$.*

(i) The empty book is an origin of time for $\mathscr{R}_d$ with posting steps and the posting count N.
(ii) It is the only tally with $N = 0$, and hence the only least element: if $N(L) \leq N(L')$ for all $L' \in \mathscr{R}_d$, then $L = 0$.
(iii) Among books with nonnegative counts it is the only one with no predecessor of nonnegative counts.

Proof. (i) (O1): $N(L) \geq 0 = N(0)$. (O2): posting a debit on account 1 gives a tally with N larger by one; this uses $d \geq 1$. (O3): N is valued in $\mathbb{N}$. (ii) If $L = \mathrm{tal}\,\rho$ and $N(L) = 0$, then $|\rho| = 0$ by Proposition 2.14(i), so $L = 0$; a least L has $N(L) \leq N(0) = 0$. (iii) If $L \geq 0$ and $L \neq 0$, some count is positive, and removing one unit from it gives a predecessor with nonnegative counts. If $L_0 \to 0$ by a post on account k and side σ, that count of L_0 is -1. □

Many histories write the same book, so the passage from records to books forgets order, but it does not blur the beginning. Table 2.1 lists the state before the first event.

Table 2.1. The state before the first event, for $d = 3$ accounts.

Quantity	Value	Fixed by
event ledger	empty list	Section 2.2
record	( ), length 0	Section 2.2
debit and credit counts	$(0,0,0)$ and $(0,0,0)$	$\mathrm{tal}(\,) = 0$
balances φ_k	$(0,0,0)$	$\mathrm{tal}(\,) = 0$
posting count N	0	Proposition 2.14
cost of the event ledger	0 exactly	Proposition 2.7
net flow of the event ledger	0 at every site	Proposition 2.7
origin of time	yes, the unique least tally	Theorems 2.12, 2.15
possible first posts	$6 = 2 \cdot 3$	$\lvert A_3 \rvert = 2d$

2.4 The beginning is a premise

The books of the theory carry integer counts, so the full set of books is $\mathbb{Z}^d \times \mathbb{Z}^d$, which contains more than the tallies. Take one account. The book $(-1, 0)$ posts a debit and becomes the empty book $(0, 0)$; the book $(-1, -1)$, with every count equal to -1, reaches the empty book in two posts. On integer books the empty book has a past. Figure 2.2 draws the plane.

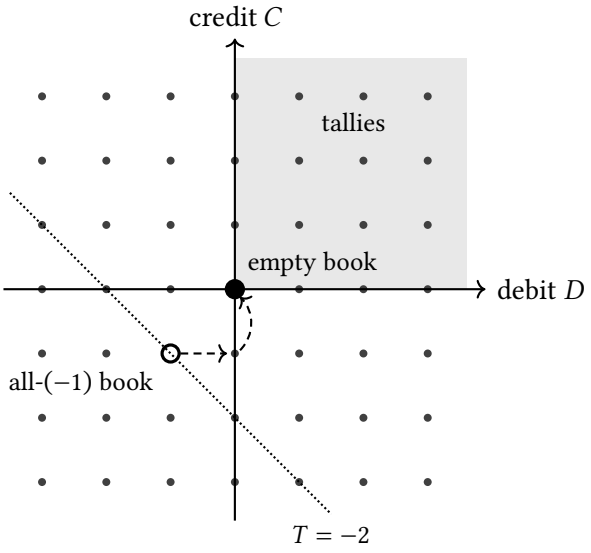


Figure 2.2. Books for one account as integer points (D, C). Posting steps move one unit right (debit) or up (credit), with the same shape everywhere. The shaded quadrant of nonnegative counts is the set of tallies. The shift by -1 carries the empty book (filled) to the all-(-1) book (open), which is the tally of no record, has total $T = -2$, and reaches the empty book in two posts (dashed).

Definition 2.16 (The opening premise)**.** A book L satisfies the *opening premise* when all its counts are nonnegative. We read it as "the books opened empty, and every count on them was posted"; by Proposition 2.14 the premise is equivalent to "L is reached from the empty book by posting steps" and to "L is the tally of a record".

We adopt the opening premise; nothing in the book derives it. For $c \in \mathbb{Z}$ write $L + c$ for the book with c added to every count, on both sides of every account.

Lemma 2.17 (Posting is blind to a common shift)**.** *For all books L, L', every $c \in \mathbb{Z}$, every post p and every account k:*

$$\operatorname{post}(L + c, p) = \operatorname{post}(L, p) + c, \qquad (L + c \to L' + c) \iff (L \to L'),$$
$$\varphi_k(L + c) = \varphi_k(L), \qquad T(L + c) = T(L) + 2dc.$$

Every cost of a transition $L \to L'$ that reads only $\Delta = L' - L$ is unchanged by the shift; in particular the unit cost $\sum_k(|\Delta D_k| + |\Delta C_k|)$ and the cost $\sum_k[(\cosh \Delta D_k - 1) + (\cosh \Delta C_k - 1)]$.

Proof. Adding one to a count and adding c to every count commute. A posting step is a statement about $L' - L$, which the shift does not change. Both terms of a balance move by c, and each of the $2d$ counts in T moves by c. □

Theorem 2.18 (The beginning is a premise)**.** *Let $d \geq 1$ and $L_- = 0 + (-1)$, the all-(-1) book.*

(i) Inside the opening premise the first moment is unique. *A book L with nonnegative counts has no predecessor with nonnegative counts if and only if $L = 0$; the empty book is the unique least tally and an origin of time for tallies (Theorem 2.15).*

(ii) Without it the past is unbounded. *Every book has a posting predecessor, and for every book L and every $n \in \mathbb{N}$ there is a book L_0 with $L_0 \to \cdots \to L$ and $T(L_0) = T(L) - n$. No book is an origin of time for posting steps with the total T, and the empty book is not least for T.*

(iii) The laws cannot see the premise. *The all-(-1) book has the balances of the empty book and its posting steps are those of the empty book shifted by -1, yet it is the tally of no record, is not reached from the empty book, and has $T(L_-) = -2d$. Hence "every book has nonnegative counts" and "every book is reached from the empty book" are both false on $\mathbb{Z}^d \times \mathbb{Z}^d$.*

(iv) No shift-invariant condition is the premise. *No predicate Q on books with $Q(L+c) \iff Q(L)$ for all L and c coincides with nonnegativity of the counts, and none coincides with reachability from the empty book.*

(v) Neither balance nor a zero total names the empty book. *Every shift $0 + c$ of the empty book is balanced, and a book with total 0 need not be empty: with a debit of 1 and a credit of -1 on one account and all other counts 0, $T = 0$.*

Proof. (i) Theorem 2.15(iii) and (i). (ii) Removing one unit from the debit count of account 1 gives a book one posting step before L with total $T(L) - 1$; repeat n times. Then no T-value is least, and (O3) fails for $\mathbb{Z}$ by Corollary 2.13. (iii) By Lemma 2.17, $\varphi_k(L_-) = \varphi_k(0)$, $L_- \to L'$ exactly when $0 \to L' + 1$, and $T(L_-) = T(0) - 2d$. Tallies and reachable books have nonnegative counts (Proposition 2.14), and $-1 < 0$. (iv) If Q coincided with nonnegativity, then $Q(0)$ would hold, so $Q(L_-)$ by invariance, so L_- would have nonnegative counts. Reachability coincides with nonnegativity by Proposition 2.14, so the same argument applies. (v) $\varphi_k(0+c) = \varphi_k(0) = 0$ by Lemma 2.17; the second book is displayed. □

The posting laws (allowed steps, balances, differences of totals, transition costs) read only changes and balances, so every one of them is blind to a common shift of the counts and none can say where the books opened. The all-(-1) book obeys all of them and has an endless past behind it. The eternal past is consistent with every posting law; the beginning is the opening premise, proved to carry the whole weight of a first moment and not to follow from any shift-invariant law. In particular no cost read through the balances, the cost minimum included, can single out the start.

The remaining identification. On the record carrier the start is unique by construction (Theorems 2.12 and 2.15). What connects the books to it is one identification: *every count on the books is a count of posts made*, that is, the books are the tally of a record. As a statement about every integer book it is false (part (iii)); as a statement about the actual books it is equivalent to the opening premise, and by part (iv) nothing built from posting steps, balances, differences of totals or transition costs derives it. Deriving it, an open problem, needs a law that reads an absolute count or a justified premise that the record is primitive and the books are its tally.

2.4.1 A finite record has a finite committed past

On lists the past is bounded by the present, whatever the alphabet.

Proposition 2.19 (The committed past of a finite record)**.** *Let E be any alphabet.*

(i) There is no sequence of lists $(\rho_n)_{n\geq 0}$ with $\rho_n = \rho_{n+1} \cdot e_n$ for every n: a finite record is not the end of an infinite chain of retained predecessors. Along any finite such chain, $|\rho_n| + n = |\rho_0|$.
(ii) The empty list is not $\rho \cdot e$ for any ρ and e.
(iii) A list has exactly one prefix of each length up to its own.
(iv) $\rho \cdot e = \rho \cdot f$ implies $e = f$. In particular two different entries give two different first records.

Proof. (i) Appending raises the length by one, so $|\rho_n| = |\rho_{n+1}| + 1$, and induction gives $|\rho_n| + n = |\rho_0|$; at $n = |\rho_0| + 1$ the length would be negative. (ii) $|\rho \cdot e| \geq 1$. (iii) The prefix of length m is the first m entries. (iv) Cancel the common initial segment. □

The scope is the record carrier: the proposition does not exclude an infinite past behind a finite aperture or behind a state that compresses its history, and it is no theorem about the whole universe. Beside Theorem 2.18(ii) it shows the premise from the other side: on records the past is finite by construction, on integer books it is unbounded.

2.5 The configuration is a fuel gauge and the record is a clock

A different candidate for a beginning reads time from the cost of a configuration rather than from a record, with the configuration of zero cost as its natural start. It fails in three ways.

Definition 2.20 (Configurations and the variational step)**.** Let $N \geq 1$. A *configuration* is a vector $c = (c_1, \dots, c_N)$ of positive reals; its *total defect* is $\Phi(c) = \sum_i J(c_i) \geq 0$ and its *log charge* is $\eta(c) = \sum_i \log c_i$. A configuration c' is a *variational successor* of c when it has the same log charge and minimizes Φ among configurations of that log charge; a *variational trajectory* is a sequence $(c_t)_{t\in\mathbb{N}}$ in which each c_{t+1} is a variational successor of c_t. The *burn* of step t is $\Phi(c_t) - \Phi(c_{t+1})$.

The update rule and the conservation of log charge are adopted, not derived. The successor exists and is unique: for log charge η it is the uniform configuration with every entry $e^{\eta/N}$, and Φ never rises along a trajectory.

Proposition 2.21 (The configuration clock is no origin)**.** *(i) For every step relation on configurations, the potential* Φ*, read as a real-valued potential, has no origin of time.*

(ii) It is exhaustible. *For every variational trajectory and every* $\delta > 0$*, it is false that every step burns at least* δ*.*

(iii) It halts at its minimum. *If* $\Phi(c_0) = 0$*, every step burns* 0*, and* c_t *is the configuration with all entries* 1 *for every* t*.*

Proof. (i) Corollary 2.13. (ii) If every step burned at least δ, then telescoping gives $\Phi(c_n) + n\delta \leq \Phi(c_0)$ for every n, and $\Phi \geq 0$; take $n > \Phi(c_0)/\delta$. (iii) Φ is nonincreasing and nonnegative, so it stays 0; $\Phi(c) = 0$ exactly when every entry is 1 (Proposition 2.6). □

A clock that runs down and stops is a fuel gauge. The configuration cannot tick forever at any fixed rate and stops at its minimum, while the record has an origin (Definition 2.10).

Proposition 2.22 (The cost minimum is an end, not a start)**.** *Let* $N \geq 1$*. The configuration* **1** *with all entries* 1 *is the unique configuration with* $\Phi = 0$*, and it minimizes* Φ*. The variational successor of every configuration of log charge* 0 *is* **1***, and a trajectory that reaches* **1** *never leaves it.*

Proof. Uniqueness and minimality follow from Proposition 2.6. The successor of a configuration of log charge η is uniform with entries $e^{\eta/N}$, which is **1** at $\eta = 0$; and Proposition 2.21(iii). □

So a "past hypothesis" placed at the cost minimum has the wrong direction: the variational dynamics descends to the minimum and stays there, and no result here makes it initial.

Proposition 2.23 (Defect is not the currency of time)**.** *Let* S *be a set of states,* $R : S \to \mathcal{Q}$ *a record map and* $\Phi : S \to W$ *a cost with values in a linearly ordered set. Read any ordered pair of states as a step* $s \to s'$*; it* burns *if* $\Phi(s') < \Phi(s)$ *and* posts *if* $R(s') \neq R(s)$*. Call* R *and* Φ coupled *if every burn posts.*

(i) R *and* Φ *are coupled if and only if* Φ *is a function of the record:* $R(s) = R(s')$ *implies* $\Phi(s) = \Phi(s')$*.*

(ii) Let $N \geq 1$ *and let* $\mathcal{Q}$ *be countable. For every record map from configurations to* $\mathcal{Q}$ *there are configurations* c, c' *with* $R(c) = R(c')$ *and* $\Phi(c') < \Phi(c)$*: a silent burn. So the total defect is never coupled to a countable record.*

(iii) No variational trajectory can post at every step while every posting is paid by a burn of at least some fixed $q > 0$*.*

(iv) Along every variational trajectory every burn after the first step is zero: $\Phi(c_{t+1}) = \Phi(c_{t+2})$ *for all* $t \geq 0$*.*

Proof. (i) If Φ is a function of the record, a burn changes Φ and so changes the record. Conversely, if $R(s) = R(s')$ and $\Phi(s) \neq \Phi(s')$, one of the two steps between them is a burn that does not post. (ii) The configurations $c_x = (1 + e^x, 1, \dots, 1)$, $x \in \mathbb{R}$, have $\Phi(c_x) = J(1 + e^x)$, injective in x because J increases strictly on $[1, \infty)$. A map from the uncountable set $\{c_x\}$ to a countable set is not injective, so $R(c_x) = R(c_y)$ for some $x < y$, and $\Phi(c_x) < \Phi(c_y)$. (iii) Proposition 2.21(ii). (iv) After one step the configuration is uniform with entries $e^{\eta/N}$, which is its own variational successor, and the successor is unique. □

A real-valued cost cannot be paid out in countable postings, unbounded posting cannot be funded by burning defect, and the variational dynamics does all its burning at once. Nothing the configuration burns pays for the record's growth.

2.6 The first state of the native history carrier

The record so far is a classical list. To ask whether a history can be continued backward past the empty record we need a carrier on which superpositions of histories are allowed and a reversible dynamics can be defined.

The carrier. The native cell is $\mathbb{C}^{64}$ with basis $|x, i\rangle$, where $x \in \mathcal{O} = \{00, 01, 10, 11\}$ is an occupation and $i \in (\mathbb{Z}/4)^2$ a pair of clock indices. The native step assigns to each input occupation x and output occupation y a 16×16 block M_{yx} acting on the clock indices and an integer current label $j(y, x) \in \mathbb{Z}^3$; there are fifteen labels, the label and the output determine the input, and the 64×64 matrix M with blocks M_{yx} is orthogonal. A recorded history with n postings carries an integer loop index $m \in \mathbb{Z}$, the cell state and the list of its n labels, newest first, so the sector of histories with n postings is

$$\mathscr{H}_n = \ell^2\big(\mathbb{Z};\, \mathbb{C}^{64\times 15^n}\big).$$

Posting the next current is the map $P_n : \mathscr{H}_n \to \mathscr{H}_{n+1}$,

$$(P_n v)(m)\big(y, i', (j, h)\big) = \sum_{x\,:\; j(y,x)=j} \sum_{i} (M_{yx})_{i'i}\, v(m - j_{\mathrm{m}})(x, i, h),$$

with j_{m} the middle component of j; it is an isometry because $M^{\top} M = I$. The space of all recorded histories is $\mathscr{H}_+ = \bigoplus_{n\geq 0} \mathscr{H}_n$ with injections $i_n : \mathscr{H}_n \to \mathscr{H}_+$, and posting acts on it as the isometry V with $V i_n x = i_{n+1} P_n x$. The empty record is the sector $\mathscr{H}_0$. A forward-only posting process is an isometry, never a unitary. Its minimal unitary extension [118] is built from the inductive limit $\mathscr{K}$ of the tower $\mathscr{H}_+ \xrightarrow{V} \mathscr{H}_+ \xrightarrow{V} \cdots$, with embeddings $\iota_n : \mathscr{H}_+ \to \mathscr{K}$ of the stages ($\iota_{n+1} V = \iota_n$, $\iota = \iota_0$) and the unique unitary U on $\mathscr{K}$ with $U\iota_n = \iota_n V$ for every n. This is the native history dilation; no amplitude, field profile, generator, energy or elapsed time is selected in its construction.

Theorem 2.24 (First state of the record)**.** *Let $x \in \mathscr{H}_0$ and $v \in \mathscr{H}_+$.*

(i) The empty-record state is orthogonal to every posted state: $\langle i_0 x, V v\rangle = 0$.
(ii) If $x \neq 0$, then $i_0 x$ is not a posted state: $V v \neq i_0 x$ for every $v \in \mathscr{H}_+$.
(iii) If $x \neq 0$, the reversible predecessor $U^{-1} i_0 x$ exists in $\mathscr{K}$ and lies outside $\iota\mathscr{H}_+$.
(iv) After $n+1$ postings the reversible predecessor stays inside the recorded histories:

$$U^{-1}\iota\, i_{n+1} P_n \cdots P_0 x = \iota\, i_n P_{n-1} \cdots P_0 x.$$

Proof. (i) Each $V i_n y$ lies in $\mathscr{H}_{n+1}$, which is orthogonal to $\mathscr{H}_0$; a general $v = \sum_n i_n v_n$ converges in norm and V is continuous. (ii) If $V v = i_0 x$, then $\|i_0 x\|^2 = \langle i_0 x, V v\rangle = 0$. (iii) U is unitary, so the predecessor exists. If $U^{-1} i_0 x = \iota v$, then $i_0 x = U\iota v = \iota V v$; ι is injective, so $i_0 x = V v$, which (ii) excludes. (iv) Apply $U\iota = \iota V$ to $i_n P_{n-1} \cdots P_0 x$. □

The theorem separates four things often run together when one asks whether the universe had a first event. *Logical priority* is the order of the record: the empty record comes before every posted state. *A boundary of the representation* is what parts (ii) and (iv) establish: no posting produces a nonzero empty-record state, and every history built by postings from the empty record runs back to it one posting at a time. *Mathematical backward extension* is part (iii): the reversible dynamics runs backward from the empty record to a well-defined vector, which is not a recorded history. *Actual physical history*, which of these vectors the universe occupies, is the remaining question; the theorem does not settle it, and a reversible mathematical extension does not establish a cosmic prehistory. The structure is that of an isometry and its wandering subspace [133, 123]: V is pure, and the empty record lies in its wandering subspace. Other proposals select a first state by a boundary condition on a wave function [54, 122] or find an edge of spacetime through geodesic incompleteness [58, 6]; the boundary here is of neither kind and belongs to the record, before any geometry is assigned. Theorems 2.24 and 2.18 have the same form on two different carriers, and no theorem connects the two.

Durations. The posting count orders histories; it does not measure the time between postings. A duration would come from a one-parameter unitary group $U(t)$ on $\mathscr{K}$ with $U(1) = U$. By the spectral theorem every unitary has a bounded self-adjoint logarithm, so such groups exist, with bounded generators, and they are not unique. None of them respects the recorded histories between postings.

Proposition 2.25 (No bounded continuous clock keeps the recorded histories)**.** *There is no bounded self-adjoint operator G on $\mathscr{K}$ with a unitary group $U(t) = e^{-itG}$, $U(1) = U$, such that $U(t)\iota\mathscr{H}_+ \subseteq \iota\mathscr{H}_+$ for every real $t \geq 0$.*

Proof. Suppose there were. The compressions $C(t) = \iota^* U(t)\iota$ on $\mathscr{H}_+$ then satisfy $\iota C(t) = U(t)\iota$ for $t \geq 0$, so $C(s+t) = C(s)C(t)$ and $C(1) = V$. Boundedness gives $\|U(t)\xi - \xi\| \leq \|G\|\, t\, \|\xi\|$, hence $\|C(\delta) - I\| \leq \|G\|\delta < 1$ for $\delta = 1/n$ with $n > \|G\|$, and $C(\delta)$ is invertible. Then $V = C(\delta)^n$ is surjective. But a nonzero empty-record state is not in the range of V (Theorem 2.24(ii)), and such states exist. □

The result leaves unbounded generators, intermediate states outside the recorded histories and other physical clocks open; Chapter 4 treats clocks, and Chapter 9 the unit premises that elapsed time in seconds needs.

2.7 The source and the recorder

A familiar way to select a first state is to take the state of lowest energy. To test that rule we need a system in which a record is written by a physical interaction with a definite energy. The source and recorder carrier is such a system. Chapter 6 studies its dynamics; here we define it and prove what it does not select.

The native objects. We use the photon Fock space $\mathscr{F}$ of the cubic lattice, with dispersion $\omega(k) = 2C_E \operatorname{arsinh}(\sqrt{s(k)}/2)$, $s(k) = 2\sum_d (1 - \cos k_d)$, one-photon energy h and vacuum Ω, and

the clock register $\mathbb{C}^8$ with clock $C = FDF^*$, $D = \mathrm{diag}(0, 1, \dots, 7)$ (Appendix A; Chapter 6 states them in full). The register's native Hamiltonian is $E_{\mathrm{nat}}C$; the *native energy unit* E_{nat} is the spacing of its levels, and converting it to joules needs the unit premises of Chapter 9. In the source Hamiltonian the registers enter with a supplied clock strength ϵ instead.

The source. One negative unit charge moves on $\mathbb{Z}^3$ while a positive unit charge is held at the origin. The moving charge carries the four eight-state registers of one cell, $8^4 = 4096$ basis codes, and the photon field: the source space is $\mathcal{S} = \ell^2(\mathbb{Z}^3; \mathbb{C}^{4096} \otimes \mathcal{F})$. At site r the source has the energy

$$H_r = \mathrm{d}\Gamma(h) + g \sum_{j=1}^{6} R(u_{r,j}, \ell_{r,j}) + \epsilon \sum_{k=1}^{4} C^{(k)} + U_a(r) \ :$$

the photon energy; a coupling of strength g to the field along each of the six lattice edges at r, where $u_{r,j}$ is the transverse profile of the unit current on the jth edge, $\ell_{r,j}$ its Gauss-law cross energy with the charge pair, and $R(u, \ell)$ the native price of the field mode along u at load ℓ (in the Gaussian representation of that mode, with coordinate ξ, it multiplies by $\mathrm{Y}(\ell + \|\sqrt{h}\, u\|\, \xi)$, where $\mathrm{Y}(u) = \frac{1}{2}(X(u) + X(u+1))$ and $X(u) = \frac{1}{2}(u + \sqrt{u^2+4})$); the clock energy of strength ϵ of the four registers; and the lattice Gauss-law energy $U_a(r) = \frac{a}{2} \int |1 - e^{ir\cdot k}|^2 / s(k)\ \mathrm{d}\mu(k)$ of the pair, with μ the normalized measure on the Brillouin zone. The source moves by hopping,

$$(\mathcal{T}\psi)(r) = -g \sum_{d} \big[U_d(r)\psi(r - e_d) + U_d(r + e_d)^* \psi(r + e_d) \big],$$

where the unitary $U_d(r)$ of the hop from $r - e_d$ to r advances the registers and displaces the field by the current profile of the edge it crosses, and $H_{\mathrm{src}} = \bigoplus_r H_r + \mathcal{T}$.

The recorder. A record is a label $q \in \mathbb{Z}^3$, with the blank value $q = 0$. The joint carrier is

$$\mathscr{H}_{\mathrm{SR}} = \ell^2\big(\mathbb{Z}^3_{\mathrm{source}} \times \mathbb{Z}^3_{\mathrm{label}};\ \mathbb{C}^{4096} \otimes \mathcal{F}\big),$$

with $\Psi(r, q)$ the amplitude for the source at r and the record reading q. The blank embedding $(B\psi)(r, q) = \delta_{q0}\psi(r)$ attaches a blank record to a source state; a state is *blank* when it lies in the range of B. The writer $(W\Psi)(r, q) = \Psi(r, \tau_{0r}(q))$, with τ_{0r} the transposition of the labels 0 and r, is a unitary involution that turns a blank record into the matching record $q = r$. The contact at rate $\kappa \in \mathbb{R}$ (a rate, not the record-cell parameter of Chapter 5) is $C_\kappa = \frac{\kappa}{2}(I - W)$, and the joint Hamiltonian is $H_\kappa = H_{\mathrm{src}} \otimes I + C_\kappa$. For $a > 0$, $C_E > 0$ and real g, ϵ, κ it is self-adjoint on a domain that does not depend on κ (Chapter 6); its energy is $E_\kappa(\Psi) = \mathrm{Re}\langle \Psi, H_\kappa \Psi \rangle$ on that domain. On its own the contact acts as $e^{-itC_\kappa} = \frac{1}{2}(I + W) + e^{-i\kappa t}\frac{1}{2}(I - W)$, so a pulse with $\kappa t = \pi$ is exactly the writer. The fixed charge sits at the blank label, so the results below take the source away from the origin.

The native pair. The last part of the carrier is a reciprocal pair of opposite charges whose internal state lies in $\mathbb{C}^{64}$, with basis $|ab; ij\rangle$: occupations $a, b \in \{0, 1\}$ of its two modes and clock indices $i, j \in \mathbb{Z}/4$. Its energy is the pair clock C_{pair}, a Hermitian 64×64 matrix built from the clock register, in units of E_{nat}. It is coupled to the photon field through one-photon profiles z_+

and z_- of its two charges: the coordinate $|ab;ij\rangle$ sources the profile $(1-b)z_+ - a\,z_-$, which is $z_+ - z_-$ at the *pair coordinate* $|10;00\rangle$. With the compensation terms of these sources and supplied longitudinal energies $\ell(x)$ on the 64 coordinates, this gives a self-adjoint pair-photon Hamiltonian H_{pair} on $\mathscr{F} \otimes \mathbb{C}^{64}$ for every $C_E \geq 0$. Two vectors matter here. The *coherent zero mode* ζ is a unit vector with $C_{\mathrm{pair}}\zeta = 0$ and weight $|\zeta_x|^2 = 1/32$ on each of the 32 coordinates with exactly one occupied mode ($a \neq b$), and 0 elsewhere; the *coherent seed* is $\Omega \otimes \zeta$. The *ready coordinate* is $|\mathrm{rd}\rangle = |01;00\rangle$, which sources no profile; it is the starting state of the pair's photon-producing flow. Direct exact computation gives the two diagonal facts used below:

$$\langle \mathrm{rd}, C_{\mathrm{pair}}\,\mathrm{rd}\rangle = 7E_{\mathrm{nat}}, \qquad \langle 01;ij, C_{\mathrm{pair}}\,01;ij\rangle + \langle 10;ij, C_{\mathrm{pair}}\,10;ij\rangle = 14E_{\mathrm{nat}} \quad (i,j \in \mathbb{Z}/4). \tag{2.3}$$

2.7.1 Energy does not select a blank record

Theorem 2.26 (An active blank is not an energy minimum)**.** *Let $a, C_E > 0$, $\kappa \neq 0$, a site $r \neq 0$ and a native code n. Let $x = B(\delta_r \otimes e_n \otimes \Omega)$ be the blank record with the source at r in code n and no photons, and y the same source with the matching record $q = r$.*

(i) x and y are orthogonal unit vectors in the domain, and $\langle y, H_\kappa x\rangle = -\kappa/2$.

(ii) For every real s the unit vector $z_s = (x + sy)/\sqrt{1+s^2}$ has

$$E_\kappa(z_s) = \frac{E_\kappa(x) - \kappa s + s^2 E_\kappa(y)}{1+s^2},$$

which is strictly below $E_\kappa(x)$ for small s of the sign of κ. So x does not minimize the energy among unit vectors.

(iii) For every source state ψ in the domain with $\psi(r) \neq 0$ at some $r \neq 0$, the blank state $B\psi$ is not an eigenvector of H_κ.

Proof. (i) The two vectors are supported on the labels 0 and r. The source part does not change the label, so only the contact connects them, and $Wx = y$ gives $\langle y, C_\kappa x\rangle = -\frac{\kappa}{2}$. (ii) Expand $\langle x+sy, H_\kappa(x+sy)\rangle$ using the symmetry of H_κ and (i); the cross terms give $-\kappa s$. Divide by $1+s^2$; the derivative at $s = 0$ is $-\kappa \neq 0$. (iii) If $H_\kappa B\psi = cB\psi$, compare components at source r and label r: $B\psi$ has none there, the source part keeps the label at 0, and $C_\kappa B\psi$ has the component $-\frac{\kappa}{2}\psi(r) \neq 0$. □

Once the contact is on, a blank record beside a source away from the blank label is not a resting state, and lowering the energy means beginning to write. A minimizer of the energy would be an eigenvector, so no such blank record minimizes it. The same holds for the coherent seed of the native pair.

Proposition 2.27 (The coherent seed is not selected)**.** *Let $C_E > 0$ and $z_+ \neq z_-$, with any longitudinal energies ℓ.*

(i) The seed $\Omega \otimes \zeta$ is not an eigenvector of H_{pair}, and some unit vector of the domain has strictly lower energy.

(ii) For every $\varepsilon > 0$ there are times $t \in (-\varepsilon, 0)$ and $t' \in (0, \varepsilon)$ at which the one-photon component of the evolved seed at the pair coordinate is nonzero. The photon-free seed is not a first onset within its own unitary group.

Proof. (i) At the pair coordinate, where the source profile is $z_+ - z_- \neq 0$ and the seed has weight $1/32$, the one-photon component of $H_{\mathrm{pair}}(\Omega \otimes \zeta)$ is nonzero, while that of $c\,\Omega \otimes \zeta$ is zero for every c. For a self-adjoint operator an attained minimum of the energy among unit vectors is an eigenvector: the form $\langle v, (H - E_{\min})v\rangle$ is nonnegative and vanishes at the minimizer, so by the Cauchy-Schwarz inequality for this form the minimizer lies in the kernel of $H - E_{\min}$. (ii) If the one-photon read Λ of the evolved seed vanished on $(-\varepsilon, 0)$, or on $(0, \varepsilon)$, it would vanish at 0 together with its one-sided derivative there, which is Λ applied to $-iH_{\mathrm{pair}}(\Omega \otimes \zeta)$ and is nonzero by (i). □

2.7.2 The reversible dynamics does not keep records

Theorem 2.28 (Readiness, return and reset)**.** *Let $a, C_E > 0$, $\kappa \neq 0$, $r \neq 0$ and any native code.*

(i) There are unit vectors Ψ and $\Phi = e^{-isH_\kappa}\Psi$, $s > 0$, on one orbit and of equal energy, with Ψ blank and Φ not.

(ii) For every time t there are unit vectors of equal energy, one of which is blank at time t and the other not.

(iii) For every $\delta > 0$ there are unit vectors Ψ and $\Phi = e^{-isH_\kappa}\Psi$ with $0 < s < \delta$ and equal energy, such that Ψ is not blank and Φ is.

(iv) No unitary operator on $\mathscr{H}_{\mathrm{SR}}$ maps every state into the blank states.

Proof. Energy is conserved along each orbit because the unitary group commutes with H_κ on its domain. (i) Start from the blank state x of Theorem 2.26; the derivative of its label-r component at $t = 0$ is $i\frac{\kappa}{2}e_n \otimes \Omega \neq 0$, so the orbit leaves the blank states at some small positive time. (ii) Evolve the pair of (i) backward by t. (iii) The same derivative gives $t \in (-\delta, 0)$ with $e^{-itH_\kappa}x$ not blank; put $\Psi = e^{-itH_\kappa}x$, $\Phi = x$ and $s = -t$. (iv) A unitary is onto, and nonblank unit vectors exist (a matching record $q = r \neq 0$). □

The reversible flow reaches a blank state from a written one as easily as the reverse, and part (ii) is a proved underdetermination: equal energy leaves blank readiness open at every time. Part (iv) excludes a closed reversible reset of the whole carrier into blanks; it does not exclude preparing a specified state, restricting to a sector, or a preparation that moves information elsewhere. Irreversible accumulation of records is a property of how records are retained (Chapter 6); the unitary group of one record does not supply it.

Proposition 2.29 (No homogeneous normalized source)**.** *No unit vector $\psi \in \mathcal{S}$ has $\|\psi(r)\|^2$ independent of $r \in \mathbb{Z}^3$, and no unit vector $\Psi \in \mathscr{H}_{\mathrm{SR}}$ has $\sum_q \|\Psi(r, q)\|^2$ independent of r, at any time of the evolution. No unit vector of $\mathcal{S}$ is carried by every lattice translation of the source into a phase multiple of itself.*

Proof. A square-summable family of nonnegative numbers on an infinite set that are all equal is zero, so the norm would be 0. The evolution preserves the norm. A vector carried by every translation into a phase multiple of itself has $\|\psi(r)\|$ independent of r. □

This restricts identifying this carrier with a homogeneous whole universe. It does not exclude homogeneous local observables, a finite volume, or states of a different representation.

2.7.3 Writing, posting and the price of preparation

The source and recorder writes records by a reversible contact, while the history dilation of Section 2.6 counts postings. Could a writing step simply be a posting? The next results answer for every candidate reading $\mathcal{J}\colon \mathscr{H}_{\mathrm{SR}} \to \mathscr{H}_{+}$ of source and recorder states as recorded histories. They are negative statements about correspondences; they do not combine results across the two carriers.

Proposition 2.30 (Writing is not posting)**.** *Let $a, C_E > 0$ and $\kappa \neq 0$.*

(i) Let $\mathcal{J}$ be any map, $n \in \mathbb{N}$ and $0 \neq x \in \mathscr{H}_n$ with $\mathcal{J}\Psi = i_n x$. Then $\mathcal{J}(W\Psi) = V\mathcal{J}\Psi$ and $\mathcal{J}(W^2\Psi) = V\mathcal{J}(W\Psi)$ cannot both hold. The same holds for two contact pulses with $\kappa t = \pi$. A reversible writer is not two successive postings.

(ii) In the return of Theorem 2.28(iii), for every map $\mathcal{J}$ with $\|\mathcal{J}\Phi\| = 1$ and $\mathcal{J}\Phi \in i_0\mathscr{H}_0$: $\mathcal{J}\Phi \neq V\mathcal{J}\Psi$. A return to blank is not one posting.

(iii) Let $\mathcal{J}$ be a linear isometry with $\mathcal{J}(Bv) \in i_0\mathscr{H}_0$ for the blank state Bv of a vacuum source at $r \neq 0$. For all small $t > 0$ the evolved state $\Psi_t = e^{-itH_\kappa}Bv$ has positive weight on the matching record, $\|\Psi_t(r,r)\|^2 > 0$, and yet $\mathcal{J}\Psi_t \neq V\mathcal{J}(Bv)$. An early signal is not a completed posting.

Proof. (i) $W^2 = I$ gives $V^2 i_n x = i_n x$; but $V^2 i_n x \in \mathscr{H}_{n+2}$, orthogonal to $\mathscr{H}_n$, so $i_n x = 0$. (ii) $\mathcal{J}\Phi$ is a nonzero empty-record state, which is not posted (Theorem 2.24(ii)). (iii) At $t = 0$, $\mathcal{J}(Bv) \neq V\mathcal{J}(Bv)$ by Theorem 2.24(ii); the map $t \mapsto \mathcal{J}\Psi_t$ is continuous, so the inequality persists for small $|t|$. The positive weight is the derivative computed in the proof of Theorem 2.28. □

Proposition 2.31 (Preparation has a price)**.** *Let $C_E \geq 0$, any profiles $z_\pm$ and any longitudinal energies ℓ.*

(i) Readout. *Replacing the seed $\Omega \otimes \zeta$ by the ensemble of its 64 coordinate branches raises the energy expectation of H_{pair} by exactly $7E_{\mathrm{nat}}$.*

(ii) The ready branch. *The branch at $|\mathrm{rd}\rangle$ has probability $1/32$, and after normalization it is $\Omega \otimes |\mathrm{rd}\rangle$. The pair-photon flow started there has energy $7E_{\mathrm{nat}} + \ell(\mathrm{rd})$ at every time, which is positive when $\ell = 0$.*

(iii) Recording. *Add a record register with one label for each of the 64 coordinates, one of which, b, serves as the blank; extend H_{pair} by the identity on the register, and write by exchanging the blank label with the coordinate, as the writer W does. Writing the seed's coordinate raises the energy by exactly $7E_{\mathrm{nat}}$, so this reversible write does not conserve the extended Hamiltonian.*

Proof. At the photon vacuum H_{pair} reads a pair vector v as $\langle v, C_{\mathrm{pair}} v\rangle$ plus a potential $\sum_x \Theta(x)|v_x|^2$ diagonal in the 64 coordinates (the source compensation and ℓ). A complete coordinate readout leaves every diagonal term unchanged and replaces $\langle \zeta, C_{\mathrm{pair}}\zeta\rangle = 0$ by

$$\sum_x |\zeta_x|^2 \langle x, C_{\mathrm{pair}} x\rangle = \frac{1}{32} \sum_{i,j \in \mathbb{Z}/4} \big[\langle 01; ij, C_{\mathrm{pair}}\, 01; ij\rangle + \langle 10; ij, C_{\mathrm{pair}}\, 10; ij\rangle\big] = \frac{16 \cdot 14}{32} E_{\mathrm{nat}} = 7E_{\mathrm{nat}}$$

by (2.3), which proves (i). The branch at $|\mathrm{rd}\rangle$ is $\zeta_{\mathrm{rd}}\, \Omega \otimes |\mathrm{rd}\rangle$ with $|\zeta_{\mathrm{rd}}|^2 = 1/32$; the energy of the flow is conserved and equals its initial value $\langle \mathrm{rd}, C_{\mathrm{pair}}\mathrm{rd}\rangle + \ell(\mathrm{rd}) = 7E_{\mathrm{nat}} + \ell(\mathrm{rd})$ by (2.3), since the ready coordinate sources no profile and so carries no compensation term; this proves (ii). In (iii) the blank state $\zeta \otimes |b\rangle$ reads $\langle \zeta, C_{\mathrm{pair}}\zeta\rangle = 0$, while the written state $\sum_x \zeta_x |x\rangle \otimes |x\rangle$ has a

different record label on each coordinate. The extended clock does not change the record, so it connects no two terms of the written state, which therefore reads C_{pair} only on the diagonal, as in (i); the diagonal potential is the same for both states. □

So the native ready state is not a zero-resource origin. The blank, the seed, the apparatus and the resource that pays the price are supplied in these statements, not derived.

2.8 What stays unselected

The start and the clock that counts from it are fixed; which first post occurs and which state the universe was prepared in are not. Table 2.2 lists what stays open, with the proved part of each item.

Table 2.2. What the results of this chapter leave unselected.

Item	What is proved	What would settle it
The first post	$2d = 6$ first posts at $d = 3$; distinct posts give distinct first records (Proposition 2.19(iv)); no result here distinguishes them	a selection principle, or the premise that every lawful history is carried at once (Chapter 4)
The genesis atom	no translation-closed rule selects one atom (Theorem 2.32); proved underdetermination by structure	genesis data, or a physical superselection law
First event or backward extension	the reversible predecessor exists and lies outside the recorded histories (Theorem 2.24(iii))	a proof from accepted premises, or two proved models that differ on it
Preparation and funding	readout and recording cost exactly $7E_{\mathrm{nat}}$ (Proposition 2.31); no closed unitary resets the carrier to blanks (Theorem 2.28(iv))	an enacted preparation with its resource and controller derived
Blank readiness	equal energy leaves readiness open at every time (Theorem 2.28(ii))	a selection rule from accepted premises
The state beyond the empty record	energy selects neither a blank (Theorem 2.26) nor the seed (Proposition 2.27); no homogeneous normalized source (Proposition 2.29)	a law selecting the initial source, attachment and population

2.9 The genesis state in boundary theory

The Recognition Boundary Theory manuscript [93] works on its own carrier. It is built from the same stages and posting maps as Section 2.6, but it completes them differently and fixes a different set of admitted operations. Its results on the beginning answer a question the record cannot: once a history has started, how much of its start does it keep? We state them on that carrier; they are proved in [93], and none is combined here with a result on another carrier.

The carrier. The stages are the sectors $\mathscr{H}_k$ of Section 2.6, with the posting isometries P_k. The complete history space $\mathscr{H}_\infty$ is their inductive limit, not their orthogonal sum: it comes

with isometries $\jmath_k : \mathscr{H}_k \to \mathscr{H}_\infty$ satisfying $\jmath_{k+1} P_k = \jmath_k$, whose ranges have dense union. A *genesis state* is a vector $v \in \mathscr{H}_0$; its posted history after any number of postings is the single vector $\jmath_0 v$. A *coordinate* $a = (k, n, z, h)$, with n the loop index, z a cell basis vector and h a list of k labels, has the vector $\chi_a = \jmath_k(\delta_n \otimes \delta_h \otimes |z\rangle)$ and the *event* $F_a = |\chi_a\rangle\langle\chi_a|$; the norm-closed algebra $\overline{\mathscr{E}}$ the events generate is the set of admitted operations, an identification made in [93]. Two coordinates are *connected* if a chain of nonzero overlaps joins them; the *atom* of a is the closed span of the coordinates connected to it; a *boundary* is a closed subspace invariant under every event, and $\mathscr{C}(\xi)$ is the least boundary containing ξ. A *lineage* is a pair (x, n) of an original occupation and an original loop flux, and $\Lambda(x, n)$ is its atom.

Theorem 2.32 (Genesis results of boundary theory)**.** *On the carrier just described:*

(i) Atoms are lineages. *A coordinate of genesis origin is connected to another exactly when they share the original occupation and the original loop [93, Thm. 4.9].*

(ii) Atoms are superselection sectors. *A bounded operator commutes with every event if and only if it is one scalar on each atom; the expectation of every element of $\overline{\mathscr{E}}$ in $\xi_1 + e^{i\theta}\xi_2$, with nonzero ξ_1, ξ_2 in two atoms, is independent of θ if and only if the atoms differ [93, Thms 4.14, 4.15].*

(iii) The boundary is fixed at genesis. *$\mathscr{C}(\jmath_0 v)$ is the closed join of the lineage atoms on which v has amplitude, at every stage of the posted history, and the weight of $\jmath_0 v$ on each atom is the same at every stage [93, Thms 4.21, 5.4, Cor. 5.5].*

(iv) The genesis whole. *A genesis state with nonzero amplitude at every genesis coordinate generates a boundary that contains every lineage atom and is not itself an atom [93, Prop. 4.13].*

(v) No symmetric rule selects an atom. *No set of atoms that is closed under translation of the loop flux consists of a single atom; under all posting symmetries a selecting rule can read only one bit of a lineage, whether its occupation is balanced [93, Thms 5.9, 5.11].*

(vi) Six numbers of genesis. *For a unit genesis state v, the probability that the postings after the nth write a given word w depends on v only through six numbers, up to an error at most $C_v C_w 2^{-\lfloor n/8 \rfloor}$ with constants depending only on v and only on w. The six are five conserved class weights and one oscillation phase, and the record holds all of them. Every other mode of the record content decays; the slowest decays by the factor $\rho_* \in (0.7792, 0.7793)$ per posting, the real root of $8x^5 + 4x^4 - 2x^2 - 2x - 1$, and eight postings at least halve every such mode [93, Thm. 11.6].*

The genesis state is an input to every one of these statements, and no result selects or prepares one. Which sector is real is initial data; structure cannot decide it, and the admitted operations cannot even ask it, since $\overline{\mathscr{E}}$ contains no projection onto a boundary other than $\{0\}$ and $\mathscr{H}_\infty$, although finite reads at a growing stage can [93, Thm. 5.16]. Reading the posted genesis state as the physical history of the ledger and the genesis lineages as its physical initial conditions are identifications, named in [93] and not used in its theorems. Part (vi) is a memory of the beginning that survives into a long record: six numbers, recorded, and everything else forgotten at a proved rate.

The empty record, balanced and costing nothing, is where every recorded history starts; that the universe's books opened there is the opening premise. Chapter 3 takes the first step away from it: the first change of the record, what it can be, and what it costs.

Chapter 3

The first event and its price

Chapter 2 ended at the empty record: no posts, cost zero, net flow zero at every site, and no law that selects a first change. This chapter asks two exact questions about the step out of that state. Which change of the empty record comes first? And what does it cost?

The answers come in three layers that differ in what they assume. The ledger and its cost alone make the cheapest change one unit post, and the first tick costs $2J(r)$ with its ratio r free. The adopted premise (R1) gives the first event one price at every strain, and no first-event price follows from J alone. The adopted premise (R2) reads a post as the unit source of a field; the golden price then depends on the graph that holds the field, and which graph that is remains open. Most of the chapter prices the first event on every finite graph [108, 43].

The account ledger and the event ledger belong to carrier (a), the weighted graphs that hold a field to carrier (b) (Appendix A); no correspondence theorem joins them. The adopted readings (R1) and (R2) are the only bridge this chapter uses between them, and every price of the first event below carries them. The statements about the field law on graphs are theorems with no premise beyond their stated hypotheses.

3.1 The cheapest change is one post

Start with the empty ledger of Chapter 2 on three accounts. Each account keeps a debit count and a credit count, all zero. A debit post on account 2 raises one count by one. It changes one balance, from 0 to 1, and one parity bit, the second. Is there a cheaper way to change the ledger? To answer we need a price for every change.

Definition 3.1 (account ledger, unit post)**.** An account ledger on $d \geq 1$ accounts is a book $L = (D, C) \in \mathbb{Z}^d \times \mathbb{Z}^d$ of Definition 2.8. Its balance is $x(L) = D - C$ (the φ_k of Chapter 2; here φ is the golden ratio) and its parity pattern is $(x_1 \bmod 2, \dots, x_d \bmod 2) \in \{0,1\}^d$. The unit post $P_{k,s}(L)$, for an account k and a side $s \in \{\text{debit}, \text{credit}\}$, is the posting step that raises D_k or C_k by one. A transition $L \to L'$ is monotone if no count decreases, and nontrivial if $L' \neq L$.

A post presents a logarithmic ratio, its strain q, and n posts on one side of one account present qn. With $J_{\log}(t) = J(e^t) = \cosh t - 1$, the transition is priced count by count:

$$c_q(L, L') = \sum_{i=1}^{d} J_{\log}\big(q\,\Delta D_i\big) + \sum_{i=1}^{d} J_{\log}\big(q\,\Delta C_i\big), \qquad \Delta D = D' - D,\ \Delta C = C' - C. \tag{3.1}$$

On one account a monotone change raises the debit count by $n \geq 0$ and the credit count by $n' \geq 0$, not both zero, at the price $J_{\log}(qn) + J_{\log}(qn')$. Since $J_{\log}(0) = 0$ and $J_{\log}(qn) > J_{\log}(q)$ for every integer $n \geq 2$, the cheapest choices are one debit post or one credit post. The same count works for any number of accounts.

Theorem 3.2 (the cheapest change is one post)**.** *Let $d \geq 1$ and $q > 0$. Every unit post costs $c_q(L, P_{k,s}(L)) = J_{\log}(q)$. Every monotone nontrivial transition costs at least $J_{\log}(q)$, with equality exactly when it is a unit post. Hence a monotone nontrivial transition has the least price among all monotone nontrivial transitions from L if and only if it is a unit post.*

Proof. The deltas of a monotone transition are nonnegative integers, not all zero. For an integer $n \geq 1$, $qn \geq q > 0$ and cosh is strictly increasing on $[0, \infty)$, so $J_{\log}(qn) \geq J_{\log}(q)$ with equality only at $n = 1$; for $n = 0$ the term vanishes. Every term of (3.1) is nonnegative, so $c_q \geq J_{\log}(q)$, and equality forces one delta to be 1 and the others 0, which is a unit post. A unit post has that delta pattern. □

Corollary 3.3 (one parity bit; six first posts)**.** *A unit post on account k, on either side, changes x_k by ± 1 and no other balance, so its parity pattern differs from the old one in exactly bit k. From the empty ledger on $d = 3$ accounts there are exactly six first posts, three accounts times two sides. They lead to six distinct ledgers, whose parity patterns are three corners of the cube $\{0, 1\}^3$, two posts to each corner.*

Proof. A debit post raises $x_k = D_k - C_k$ by one and a credit post lowers it by one; either flips the parity of x_k alone. From the empty ledger the six posts change six different counts, so the six ledgers differ, and the debit and credit posts on account k both reach the pattern with bit k set. □

The conclusion does not depend on the size of a post in cost units, and a change of two units is never a single least-cost step.

Proposition 3.4 (a double entry is two posts)**.** *Let j, k be accounts. The double entry that raises D_j and C_k by one is the composition of two unit posts, the debit post on j and then the credit post on k. It changes two counts by one each, so it is not a unit post, and by Theorem 3.2 it is not a least-price monotone nontrivial transition at any $q > 0$; it is not one under the count of changed units either, where it moves two units and a post moves one.*

Proof. Both sides raise D_j and C_k by one and change nothing else. A unit post has exactly one nonzero delta. □

3.2 The first tick and the strain of a post

The comparisons themselves are recorded as events, each with a ratio and paired with its reciprocal.

Definition 3.5 (events and the event ledger)**.** Recognition events $e = (s, t, r)$, event ledgers, their cost and their net flow σ_a at a site a are those of Section 2.2; the reciprocal of e is written $\bar{e} = (t, s, r^{-1})$ here, and e costs $J(r)$. Posting e onto a ledger adds e and $\bar{e}$. The first tick of e_0 is the ledger $T(e_0) = [e_0, \bar{e}_0]$ obtained by posting e_0 onto the empty ledger.

Proposition 3.6 (the first tick)**.** *For every event $e_0 = (s, t, r)$ the first tick $T(e_0)$ has two events, net flow 0 at every site and cost $2J(r)$, which is positive exactly when $r \neq 1$. Posting any further list of events keeps the ledger balanced and adds two events per posting.*

Proof. $J(r) + J(r^{-1}) = 2J(r)$ by the symmetry of J, and $J(r) = (r-1)^2/(2r)$ vanishes only at $r = 1$. A site touched by e_0 is touched by $\bar{e}_0$, with contributions $\log r$ and $-\log r$ to its net flow. Each posting adds an event together with its reciprocal. □

When $r \neq 1$ the first tick lifts the event ledger off zero cost and keeps the net flow zero: content without imbalance. The ratio r is an input; nothing in the event ledger fixes it.

The same event can be read as a field. Put its two sites a, b at the ends of one bond of weight one and give them the logarithmic field $\phi_a = \log r$, $\phi_b = 0$. The bond action, summed over the two orientations,

$$\sum_{i,j\in\{a,b\}} w_{ij}\big(\cosh(\phi_i - \phi_j) - 1\big) = 2(\cosh\log r - 1) = 2J(r), \tag{3.2}$$

equals the cost of the first tick. The reciprocal is the reverse orientation of the same bond.

Definition 3.7 (strain; the premise (R1))**.** A post presents strain q if the ratio it presents is e^q. We adopt the reading **(R1)**: *the first event is one post*, so that the first event's ratio is $r_0 = e^q$, with q the strain of a post.

Proposition 3.8 (one event, one price)**.** *Assume (R1). For every real q the following are all equal to $2J_{\log}(q) = 2(\cosh q - 1)$, twice the price of one post: (i) the cost of the first tick $T(e_0)$; (ii) the action* (3.2) *of the field with drop q across a single bond of weight one; (iii) the price c_q of a double entry, a debit post on any account j with a credit post on any account k.*

Proof. (i) is Proposition 3.6 at $r = e^q$. (ii) is (3.2) with $\log r = q$. (iii) The double entry has exactly two nonzero deltas, each equal to 1, also when $j = k$. □

So the first event has one price at each strain, however it is read. What fixes the strain?

Proposition 3.9 (the cost law leaves the strain free)**.** *For every $q > 0$: (a) the price $P(n) = J_{\log}(qn)$ of n posts on one side of one account satisfies the composition law $P(m+n) + P(m-n) = 2P(m)P(n) + 2P(m) + 2P(n)$ for all $m, n \in \mathbb{Z}$; (b) every unit post costs $J_{\log}(q)$ and flips exactly one parity bit; (c) the unit posts are exactly the least-price monotone nontrivial transitions. The composition law of J and its calibration do not involve q.*

Proof. (a) is the composition law $J_{\log}(s+t) + J_{\log}(s-t) = 2J_{\log}(s)J_{\log}(t) + 2J_{\log}(s) + 2J_{\log}(t)$, d'Alembert's equation for $1 + J_{\log} = \cosh$ [1], at $s = qm$, $t = qn$. (b) and (c) are Theorem 3.2 and Corollary 3.3. □

The same holds for the other forced statements of the foundation (Chapter 1) that read a post or its cost, among them the golden weighting φ^{-n} of n recognition steps: each holds at every positive strain, so none of them can tell $q = 1$ from any other positive value. The strain is a free unit. There is no first-event price from J alone.

Remark 3.10 ($q = 1$ is a calibration in the count). Measured in the count n, the price of n posts is $\cosh(qn) - 1$, whose curvature at $n = 0$ is q^2. Unit curvature in the count coordinate, the calibration of J read on counts, holds for $q > 0$ exactly when $q = 1$. Then one post presents the ratio e and costs $\cosh 1 - 1$. This is a choice of unit.

Three choices of this unit are in use, and they cannot be combined.

Proposition 3.11 (three readings of the unit exclude one another)**.** *Consider three readings of the size of a post: (i) the count reading of Remark 3.10, $q = 1$, with post price $\cosh 1 - 1 \approx 0.5431$; (ii) the unit-source reading of Section 3.3, $q = \log\varphi$, with post price $\varphi - \frac{3}{2}$; (iii) the native-action reading, in which one post costs the native action quantum $\varphi^{-5} \approx 0.0902$, the value Chapter 9 takes for $\hbar$ in native units under (N2), so that $q = \operatorname{arcosh}(1 + \varphi^{-5}) \approx 0.4215$. The three strains are pairwise different. Under (R1), the count reading gives the first event the ratio e, and the field of that ratio on a single unit bond is stationary only for a source of strength $2\sinh 1 \neq 1$, so the count reading and the unit-source premise of Theorem 3.15 cannot both hold.*

Proof. $\log\varphi = \operatorname{arsinh}\frac{1}{2} < \frac{1}{2} < 1$ (Theorem 3.15). The price $\cosh q - 1$ is strictly increasing in $q \geq 0$, and the three prices $\cosh 1 - 1 > 0.54$, $\varphi - \frac{3}{2} > 0.118$ and $\varphi^{-5} < 0.091$ are different. The last statement is Lemma 3.14 at $q = 1$. □

An account of the first event can adopt one of these units, not two.

3.3 The golden price, conditional

The third layer reads a post as the source of a field. The field obeys the stationarity condition of the cost on a weighted graph, and the graph that holds it will matter.

Definition 3.12 (carrier, field, the exact-J law)**.** A carrier is a finite set V of sites with weights $w_{ij} = w_{ji} \geq 0$; a bond is a pair with $w_{ij} > 0$. A field is a function $\phi : V \to \mathbb{R}$, read as a logarithmic potential. Its current from i to j is $I_{ij} = w_{ij}\sinh(\phi_i - \phi_j)$ and its action is

$$A(\phi) = \sum_{i,j\in V} w_{ij}\big(\cosh(\phi_i - \phi_j) - 1\big), \tag{3.3}$$

summed over ordered pairs. For sites $a \neq b$ and a strength $m \in \mathbb{R}$, the field obeys the exact-J law with source a, sink b and strength m if

$$2\sum_{j\in V} w_{ij}\sinh(\phi_i - \phi_j) = m\,(\delta_{ia} - \delta_{ib}) \qquad \text{for every } i \in V. \tag{3.4}$$

Its drop is $\phi_a - \phi_b$.

Current is conserved at every site except two: $m/2$ leaves the source and $m/2$ enters the sink. This is Gauss's law with the hyperbolic sine of the field difference as the current, and it is the stationarity condition of the cost.

Lemma 3.13 (stationarity)**.** *A field obeys* (3.4) *if and only if $\frac{d}{d\epsilon}A(\phi + \epsilon v)\big|_{\epsilon=0} = m(v_a - v_b)$ for every $v : V \to \mathbb{R}$.*

Proof. The derivative is $\sum_{i,j} w_{ij}\sinh(\phi_i - \phi_j)(v_i - v_j)$. Since w is symmetric and sinh is odd, $I_{ji} = -I_{ij}$, so the sum equals $2\sum_i v_i \sum_j I_{ij}$, which is $m(v_a - v_b)$ for every v exactly when (3.4) holds. □

We now adopt the second reading, **(R2)**: *the field of a post with bond (a, b) obeys the exact-J law on its carrier with source a, sink b and strength equal to the number of counts the post changes, and the post's strain is the drop $\phi_a - \phi_b$ across its own bond.* A unit post changes one count, so (R2) fixes $m = 1$. Theorem 3.22 below shows that the drop does not depend on which solution is taken. Neither (R1) nor (R2) is derived; which carrier holds the field is a further question, open, and stated in Section 3.10.

Lemma 3.14 (a single bond)**.** *On two sites a, b joined by one bond of weight one, a field obeys the exact-J law with strength m if and only if $2\sinh(\phi_a - \phi_b) = m$.*

Proof. At a the law reads $2\sinh(\phi_a - \phi_b) = m$, at b the same equation negated. □

Theorem 3.15 (the golden price, conditional)**.** *Assume (R1) and (R2), with the field of the first post on a single bond of weight one. Then the strain is*

$$q = \operatorname{arsinh} \tfrac{1}{2} = \log \varphi \approx 0.481212, \tag{3.5}$$

the first event has ratio $r_0 = \varphi$, which satisfies $1 + r_0^{-1} = r_0$, one post costs $J(\varphi) = \varphi - \frac{3}{2} \approx 0.118034$, and the first event with its reciprocal costs

$$2J(\varphi) = \sqrt{5} - 2 \approx 0.236068. \tag{3.6}$$

Proof. By Lemma 3.14 with $m = 1$, $2\sinh q = 1$, that is $r_0 - r_0^{-1} = 1$ with $r_0 = e^q$, or $r_0^2 - r_0 - 1 = 0$. Its positive root is φ. Since $\sinh \log \varphi = \frac{1}{2}(\varphi - \varphi^{-1}) = \frac{1}{2}$, $\log \varphi = \operatorname{arsinh} \frac{1}{2}$. With $\varphi^{-1} = \varphi - 1$, $J(\varphi) = \frac{1}{2}(2\varphi - 1) - 1 = \varphi - \frac{3}{2}$ and $2J(\varphi) = 2\varphi - 3 = \sqrt{5} - 2$. □

The theorem carries three conditions: (R1), (R2), and a carrier on which the post's bond is a single bond, or, by Theorem 3.24, a bridge of weight one. The sentence "the first event costs $\sqrt{5} - 2$" is true under these three conditions and is not established without them.

Proposition 3.16 (what a single bond excludes)**.** *On a single bond of weight one: (a) a field with drop q obeys the law with strength m exactly when $m = 2\sinh q$, so the count reading $q = 1$ needs $m = 2\sinh 1 > 1$; (b) two units of source, $m = 2$, give $q = \operatorname{arsinh} 1 = \log(1 + \sqrt{2})$, not $2\log\varphi$; (c) under the quadratic law, whose current is the field difference itself, a unit source gives the drop $\frac{1}{2}$, not $\log\varphi$.*

Proof. (a) is Lemma 3.14. (b) If $\operatorname{arsinh} 1 = 2\log\varphi$, then

$$1 = \sinh(2\log\varphi) = 2\sinh(\log\varphi)\cosh(\log\varphi) = \cosh\log\varphi > 1.$$

(c) The quadratic law reads $2(\phi_a - \phi_b) = 1$, while $2\log\varphi < 2\sinh\log\varphi = 1$. □

Under the source reading posts do not accumulate strain on one bond: two posts there give the ratio $1 + \sqrt{2}$, not φ^2, so a sequence of golden ratios would need a new bond for each post. Item (c) shows that the hyperbolic sine matters.

3.3.1 Self-grounding: a fixed point, not a selection

The golden ratio satisfies $1+\varphi^{-1}=\varphi$. Write $T(r)=1+r^{-1}$ for $r>0$, and call r self-grounding when $T(r)=r$.

Proposition 3.17 (self-grounding)**.**

(i) *For $r>0$ the following are equivalent: $T(r)=r$; $2\sinh(\log r)=1$; the field $(\log r, 0)$ on a single bond of weight one obeys the exact-J law with strength 1; $r=\varphi$.*
(ii) *For $r\geq 1$ and every n, $|T^n(r)-\varphi|\leq\varphi^{-n}|r-\varphi|$, and $T^n(r)\to\varphi$ for every $r>0$.*
(iii) *The golden field $(\log\varphi, 0)$ on a single bond of weight one obeys the exact-J law only at strength 1.*

Proof. (i) $2\sinh\log r=r-r^{-1}$, which equals 1 exactly when $1+r^{-1}=r$; Lemma 3.14 gives the third form, and the positive root of r^2-r-1 is φ. (ii) $|T(r)-\varphi|=|r^{-1}-\varphi^{-1}|=|r-\varphi|/(r\varphi)\leq|r-\varphi|/\varphi$ for $r\geq 1$, and $T(r)>1$ for $r>0$, so after one step the contraction applies. (iii) Lemma 3.14 with $2\sinh\log\varphi=1$. □

The proposition gives a fixed point and its attraction; it does not select the ratio of the first event. Under (R1) and (R2) on a single bond the first ratio is the fixed point; without those premises the ratio is free (Proposition 3.9), and attraction says only what repeated application of T would do to a ratio, not that the first event applies it.

Proposition 3.18 (the golden past of the recurrence)**.**

(i) *For every $n\geq 0$ and $r>0$, $T^n(r)=\varphi$ if and only if $r=\varphi$.*
(ii) *The seeds 2 and 3 both converge to φ under T, and $T^n(2)\neq T^n(3)$ for every n.*
(iii) *If $(p_n)_{n\geq 0}$ is a sequence of positive numbers with $p_n=T(p_{n+1})$ for every n, then $p_n=\varphi$ for every n.*

Proof. T is injective on $(0,\infty)$ and fixes φ, so each T^n is injective, which gives (i) and the second half of (ii); the first half is Proposition 3.17(ii). (iii) Every $p_n=T(p_{n+1})$ exceeds 1, so $p_{n+1}>1$ gives $p_n<2$, and $|p_n-\varphi|<1$. Since $p_0=T^n(p_n)$, Proposition 3.17(ii) gives $|p_0-\varphi|\leq\varphi^{-n}$ for every n, so $p_0=\varphi$; the same argument applies to every shifted sequence. □

Item (iii) derives the fixed point from an ancestry of unlimited depth; it does not say that the universe has such an ancestry, and the depth of a numerical recurrence is not past physical time. Items (i) and (ii) refute the picture of a cascade of seeds that "crystallizes" into φ: a finite cascade lands on φ only if it started there, and two different seeds never meet on the way to φ.

3.3.2 The first event fixes its ledger

Theorem 3.15 puts the first post on a carrier of two sites and one bond. The record of the first event singles out that carrier, as follows. On a finite set V with sites $a\neq b$, the pair graph P_{ab} has $w_{ab}=w_{ba}=1$ and every other weight 0. For a field ϕ, the field-read first event from a to b is the event with ratio $e^{\phi_a-\phi_b}$.

Theorem 3.19 (the first event fixes its ledger)**.** *Let $a\neq b$ be sites of a finite set V.*

(i) *For every field ϕ the field-read first event, recorded with its reciprocal, costs* $2(\cosh(\phi_a - \phi_b) - 1) = A_{P_{ab}}(\phi)$. *Every first event from a to b is field-read, from the field equal to* $\log r(e_0)$ *at a and* 0 *elsewhere.*

(ii) *A carrier on V is determined off the diagonal by its action: for $i \neq j$ and every $t \neq 0$,*

$$w_{ij} = \frac{A(t\delta_i) + A(t\delta_j) - A\big(t(\delta_i + \delta_j)\big)}{4(\cosh t - 1)}.$$

(iii) *P_{ab} is the only carrier on V with zero diagonal whose action equals, at every field, the cost of the field-read first event recorded with its reciprocal.*

Proof. (i) $J(e^u) = \cosh u - 1$ is even in u, and the action of P_{ab} has the two ordered terms (a, b) and (b, a). (ii) Put $d_i = \sum_{k \neq i} w_{ik}$. At $\phi = t\delta_i$ only ordered pairs with exactly one end at i contribute, so $A(t\delta_i) = 2d_i(\cosh t - 1)$; at $t(\delta_i + \delta_j)$ the pair $\{i, j\}$ has no difference and $A = 2(d_i + d_j - 2w_{ij})(\cosh t - 1)$. Subtract. (iii) follows from (ii) and (i). □

Two limits keep this theorem at its scope.

Proposition 3.20 (what the record does not identify)**.** *(i) For a fixed event ledger L of positive cost, no carrier has action equal to the cost of L at every field, since the action vanishes at the zero field. (ii) On the period-three torus of Section 3.5, the action of the whole graph that joins every pair of neighbouring sites with weight one is not, at every field, the cost of the field-read first event between two neighbours: the third site of their axis line is joined to both, and Theorem 3.19(ii) reads that bond off the action.*

Proof. (i) $A(0) = 0 < \mathrm{cost}(L)$. (ii) On the period-three torus the third site c of the axis line through a, b has $w_{ca} = 1$ while P_{ab} has weight 0 there; Theorem 3.19(ii) would make the two weights equal. □

So the record determines the pair graph through the whole family of field-read events, not through one recorded cost.

3.4 When the flux stays on its own bond

A single bond is the simplest carrier. The next simplest shows what matters in a larger one.

Example 3.21 (a path of three sites)**.** Take sites a, b, c with unit bonds ab and bc, source a, sink b, and the field $(\log \varphi, 0, 0)$. At a the law reads $2 \sinh \log \varphi = 1$; at b it reads $2(\sinh(-\log \varphi) + \sinh 0) = -1$; at c both differences vanish. The whole flux crosses the post's bond and the drop is $\log \varphi$. The extra site hangs off the sink and changes nothing. What matters is whether the source has another way out.

Theorem 3.22 (drops and currents are unique)**.** *If ϕ and ψ obey the exact-J law on the same carrier with the same source, sink and strength, then $\phi_i - \phi_j = \psi_i - \psi_j$ on every bond, and the two fields carry the same current on every pair of sites.*

Proof. Apply Lemma 3.13 to both fields with $v = \phi - \psi$ and subtract: $\sum_{i,j} w_{ij}\big(\sinh(\phi_i - \phi_j) - \sinh(\psi_i - \psi_j)\big)\big((\phi_i - \phi_j) - (\psi_i - \psi_j)\big) = 0$. Since sinh is strictly increasing, every term is nonnegative, so every term vanishes, and on a bond this forces equal differences. □

Uniqueness of currents in networks with monotone current laws is classical [39, 80]. It makes the drop across a post's bond a property of the carrier and the law alone.

Theorem 3.23 (the source holds the maximum)**.** *Let ϕ obey the exact-J law with source a, sink b and strength $m \geq 0$. Then (i) $\phi_j \leq \phi_a$ for every site j joined to a by a path of bonds; (ii) every current out of the source is nonnegative; (iii) if (a, b) is a bond of weight w, then $\phi_a - \phi_b \leq \operatorname{arsinh}(m/(2w))$.*

Proof. Spreading step: if $i \neq a$ has $\phi_i \geq \phi_j$ for every bond neighbour j, each $I_{ij} \geq 0$ while the law gives $\sum_j I_{ij} = -\frac{m}{2}\delta_{ib} \leq 0$; so every $I_{ij} = 0$ and every bond neighbour of i has the value ϕ_i. (i) If the maximum of ϕ over the sites joined to a exceeded ϕ_a, the set of joined sites where it is attained would avoid a, be closed under bond neighbours by the spreading step, and so contain a. (ii) follows from (i). (iii) The law at a reads $2w\sinh(\phi_a - \phi_b) + 2\sum_{j\neq b} I_{aj} = m$, and the sum is nonnegative by (ii). □

A bond is a bridge when it lies on no cycle, equivalently when its ends are joined by no path of bonds that avoids it [32].

Theorem 3.24 (bridge criterion)**.** *Let (a, b) be a bond of weight $w > 0$ and let $m > 0$.*

(i) *If (a, b) is a bridge, the field equal to h on the sites joined to a without using the bond and to $h - \operatorname{arsinh}(m/(2w))$ elsewhere obeys the exact-J law, and no current leaves a except along (a, b).*

(ii) *If (a, b) lies on a cycle, every solution sends a nonzero current out of a along some other bond, and $\phi_a - \phi_b < \operatorname{arsinh}(m/(2w))$.*

Hence for every solution the following are equivalent: (a, b) is a bridge; no current leaves a except along (a, b); $\phi_a - \phi_b = \operatorname{arsinh}(m/(2w))$.

Proof. (i) Let S be the set of sites joined to a by paths avoiding the bond. Then $b \notin S$, and (a, b) is the only bond between S and its complement; every other bond carries no difference. The law holds at a, at b with the opposite sign, and trivially elsewhere. (ii) Suppose every current out of a along other bonds vanishes. Then each bond neighbour $j \neq b$ of a has $\phi_j = \phi_a$, the maximum. Along a path $a = v_0, v_1, \ldots, v_k = b$ avoiding the bond, the spreading step of Theorem 3.23 carries the value ϕ_a from v_t to v_{t+1}, so $\phi_b = \phi_a$, against $2w\sinh(\phi_a - \phi_b) = m > 0$. Some other current out of a is then positive, and the law at a gives the strict inequality. The equivalences follow with Theorem 3.22. □

Corollary 3.25 (the golden price is a graph property)**.** *Assume (R1) and (R2), and let the post's bond (a, b) have weight $w > 0$ on the carrier of its field, joined by a path of bonds. The flux leaves the source only along the bond exactly when the bond is a bridge, and then the strain is $\operatorname{arsinh}(1/(2w))$; it is $\log\varphi$ exactly for a bridge of weight one. For a bond of weight one, every solution has drop $\log\varphi$ on a bridge and a drop d with $0 < d < \log\varphi$ on a cycle, so the first event costs exactly $\sqrt{5} - 2$ on a bridge and a price strictly between 0 and $\sqrt{5} - 2$ on a cycle.*

Proof. Theorem 3.24 with $m = 1$, and arsinh is injective. For $d > 0$: by Theorem 3.23(ii) the current $\sinh d$ is nonnegative; if $d = 0$, then $\phi_b = \phi_a$ is the maximum, every current out of b is nonnegative, and the law at b, $2\sum_j I_{bj} = -1$, fails. The price $2(\cosh d - 1)$ is strictly increasing in $d \geq 0$ and equals $\sqrt{5} - 2$ at $\log\varphi$. □

If the post's bond is the only connection between the source's side and the rest, all of the flux crosses it (Figure 3.1(a)); if there is another way round, part of the flux takes it and the drop falls. Under (R1) and (R2), whether the first event costs $\sqrt{5}-2$ is decided by the shape of the graph that holds its field and by the unit weight of its bond.

3.5 The two carriers the foundation defines

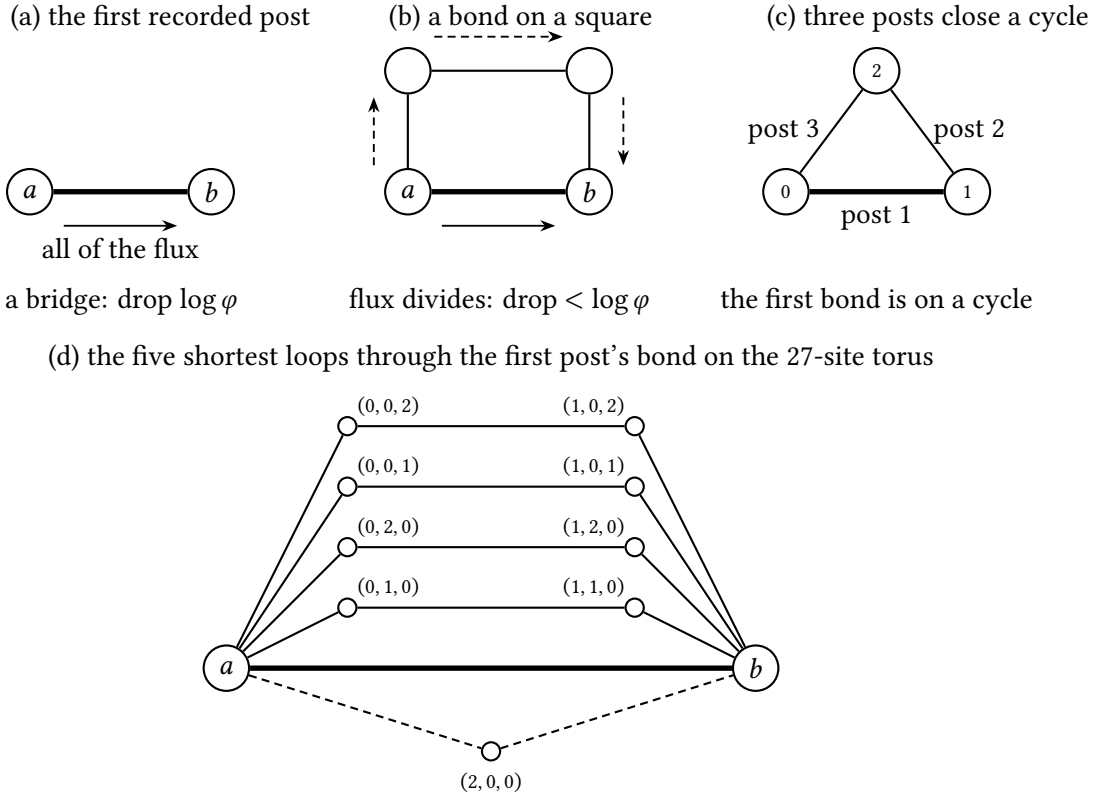


Figure 3.1. Carriers of a post's field; the thick bond is the post's own bond, from a to b. (a) The record carrier of the first post is one bond, a bridge, so the whole flux crosses it. (b) On a square part of the flux goes round (dashed arrows) and the drop falls. (c) On the 27-site torus three forward posts along one axis close a cycle through the first bond (Proposition 3.31). (d) On the possibility graph on T_3, with $a=(0,0,0)$ and $b=(1,0,0)$, the triangle through $(2,0,0)$ (dashed, it wraps round the period) and four squares; they share no bond, and together they use every bond at a and at b (Corollary 3.53).

The foundation does not assert which graph holds a post's field. Its construction of posting fields defines two kinds of carrier: one joins every pair of sites that a single post could join; the other contains only the bonds of posts actually recorded. In the spatial picture of that construction a post is a step between neighbouring sites of the discrete three-torus $T_N=(\mathbb{Z}/N\mathbb{Z})^3$. This picture is the foundation's construction of the posting field, a reading the book does not derive: imbalance is not position (Proposition 8.20).

Definition 3.26 (possibility graph)**.** Two sites of T_N are neighbours if they differ by ±1 in exactly one coordinate. A possibility graph is a carrier on T_N with positive weight on every pair of neighbours. Two versions in the foundation give each pair weight one; a third, the signed version, gives weight 1 for $N\geq 3$ and 2 for $N=2$, where the forward and backward steps along an axis reach the same site, so that every site has total weight 6.

Theorem 3.27 (no neighbour bond is a bridge)**.** *For $N \geq 2$, every neighbour bond of every possibility graph on T_N lies on a cycle of length four. Under (R2) the flux of every post divides, and the undivided-flux condition of Corollary 3.25 fails for every post at every strain.*

Proof. Let the bond be $(p, p + s e_k)$ with $s = \pm 1$, and let $k' \neq k$. The sites p, $p + e_{k'}$, $p + s e_k + e_{k'}$, $p + s e_k$ are distinct because $1 \neq 0$ and $s \neq 0$ in $\mathbb{Z}/N\mathbb{Z}$, and consecutive ones are neighbours, so the bond lies on a square. Theorem 3.24(ii) applies. □

A carrier whose bonds are exactly the neighbour pairs, which is what a selection of the full minimum-cost support of a post would give, therefore excludes the golden price on every torus.

Theorem 3.28 (the first post on the 27-site torus)**.** *On the possibility graph on T_3 with unit weights, take the exact-J law with source $(0, 0, 0)$, sink $(1, 0, 0)$ and strength 1. Let $G(x) = \sinh 2x + \sinh x$, and for $c \geq 0$ put $B(c) = c + \operatorname{arsinh} \frac{G(c)}{2}$, $A(c) = B(c) + \operatorname{arsinh}\big(G(B(c)) + 2\sinh(B(c) - c)\big)$ and $S(c) = 2\big(G(A(c)) + 4\sinh(A(c) - B(c))\big)$.*

(i) *$S(c) = 1$ has exactly one root $c_* > 0$. The three-level field with value $A = A(c_*)$ at the source, $B = B(c_*)$ at the four sites $(0, p_1, p_2)$ with exactly one of p_1, p_2 zero, c_* at the four sites with neither zero, the negatives of these on the slice $p_0 = 1$, and 0 on the slice $p_0 = 2$, obeys the law. Every solution has drop $2A$.*

(ii) *The own bond carries the share $2\sinh(2A) < 1$ of the unit flux, so $2A < \log \varphi$.*

(iii) *The drop and the share are certified:*

$$0.1601 < 2A < 0.1603, \qquad 0.32 < 2\sinh(2A) < \tfrac{1}{3}, \qquad 2A < \tfrac{1}{3}\log \varphi.$$

The proof is in Appendix B.1. With six ways out of the source the field sends about a third of its flux across the post's own bond. A numerical solution of the shell equations, which is not part of the certified statement, gives $e^{2A} \approx 1.1737$, with shares of the unit flux near 0.322 on the post's bond, 0.160 toward the third site of the axis line and 0.129 toward each of the four other neighbours of the source.

Definition 3.29 (record carrier)**.** For a history of posts, a list of ordered pairs of sites $(a_1, b_1), \dots, (a_n, b_n)$, the record carrier has weight w_{ij} equal to the number of recorded posts that join i and j, in either orientation.

The record carrier of neighbour posts is the possibility graph restricted to what has happened, counted with multiplicity.

Proposition 3.30 (the first post on its record carrier)**.** *Let the first post have bond (a, b), $a \neq b$.*

(i) *Its record carrier gives the bond weight 1, the source has no other bond, and the bond is a bridge. Under (R2) the strain is $\log \varphi$, and under (R1) as well the first event costs $\sqrt{5} - 2$.*

(ii) *For every field u the first tick of the event with ratio $e^{u_a - u_b}$ costs exactly the action of this record carrier, $2(\cosh(u_a - u_b) - 1)$: the event and its reciprocal form one bond of weight one.*

(iii) *If the reciprocal were recorded as a second post (b, a), the bond would have weight 2, the history would cost two first ticks, and under (R2) the strain would be $\operatorname{arsinh} \frac{1}{4} \approx 0.247466 \neq \log \varphi$.*

Proof. (i) A source whose only bond goes to b cannot reach b another way; apply Corollary 3.25 with $w = 1$. (ii) is Theorem 3.19(i) for the record carrier, which is P_{ab}. (iii) Corollary 3.25 with $w = 2$. □

So the weight of the first bond is settled: the first tick, an event with its reciprocal, is one record bond of weight one.

Proposition 3.31 (later posts)**.** *If no later post touches the first post's source except along the first bond, the first bond remains a bridge of its record carrier, and under (R2) its strain is* $\operatorname{arsinh}(1/(2c))$*, with c the number of posts recorded on it. On T_3, after two forward posts along one axis from the origin the first bond is still a bridge; a third forward post closes the axis cycle $(0,0,0) \to (1,0,0) \to (2,0,0) \to (0,0,0)$, and the undivided-flux condition for the first bond then fails at every strain.*

Proof. The source is a leaf of weight c; apply Corollary 3.25. The third post adds the bond between $(2,0,0)$ and $(0,0,0)$, which with the second post joins the first bond's ends without it; apply Theorem 3.24(ii). □

Corollary 3.32 (two carriers, two answers on the same 27 sites)**.** *Assume (R2). For the first post $(0,0,0) \to (1,0,0)$ on T_3, the strain read on its record carrier is* $\log\varphi$*, with undivided flux; read on the possibility graph it is the drop $2A$ of Theorem 3.28, below $\frac{1}{3}\log\varphi$, with divided flux. On the whole signed possibility graph at period three the unit source has $1 < e^{2A} < \varphi$: the golden drop of the single bond does not extend to the whole graph.*

Proof. Proposition 3.30(i), Theorem 3.27 and Theorem 3.28; at $N = 3$ the signed version has unit weights. □

Table 3.1 collects the strains.

Table 3.1. Strain of one post under (R2), by carrier. One post costs $\cosh q - 1$; under (R1) the first event costs twice that.

Carrier of the field, or reading	Strain q	Price of one post
single bond, or any bridge of weight 1 (first record, three-site path)	$\log\varphi = \operatorname{arsinh}\frac{1}{2}$	$\varphi - \frac{3}{2}$
bridge of weight w (c posts on one bond: $w = c$)	$\operatorname{arsinh}\frac{1}{2w}$	$\cosh q - 1$
reciprocal recorded as a second post ($w = 2$)	$\operatorname{arsinh}\frac{1}{4}$	$\cosh q - 1$
bond of weight 1 on a cycle	in $(0, \log\varphi)$	below $\varphi - \frac{3}{2}$
possibility graph on T_3, first post	in $(0.1601,\ 0.1603)$	below $\varphi - \frac{3}{2}$
two units of source on one bond	$\operatorname{arsinh} 1 = \log(1+\sqrt{2})$	$\sqrt{2} - 1$
quadratic law, unit source, one bond	$\frac{1}{2}$	$\cosh\frac{1}{2} - 1$
count reading $q = 1$	1	$\cosh 1 - 1$

3.6 Loops priced exactly

A cycle through the post's bond lowers its drop (Theorem 3.24). On a ring, and on every carrier made of the post's bond and paths that share only its ends, the drop and the price are exact. Throughout, the price is $2(\cosh d - 1)$ for the own-bond drop d; under (R1) and (R2) it is the price of the first event.

Example 3.33 (a triangle)**.** Take unit bonds $\{0,1\}$, $\{1,2\}$, $\{2,0\}$, source 0, sink 1, and the field $(d, 0, \frac{d}{2})$. At site 2 the currents $\sinh(-\frac{d}{2})$ and $\sinh\frac{d}{2}$ cancel; at the source the law reads $2\sinh d + 2\sinh\frac{d}{2} = 1$. The left side is strictly increasing, vanishes at 0 and exceeds 1 at $\log\varphi$, so there is one root d_3, and every solution has that drop (Theorem 3.22). The certified value is $0.328865 < d_3 < 0.328866$, and the first tick costs between 0.109130 and 0.109132, about 46% of $\sqrt{5}-2$.

Definition 3.34 (ring, theta carrier)**.** For $r \geq 1$ and lengths $l_1, \dots, l_r \geq 2$, the theta carrier $\Theta(l_1, \dots, l_r)$ consists of the post's bond ab and r paths from a to b, path i with l_i bonds, no two paths sharing a site other than a, b; every bond has weight one. The ring C_k of $k \geq 3$ unit bonds is $\Theta(k-1)$.

Theorem 3.35 (each loop returns its own share)**.** *Let $l = (l_1, \dots, l_r)$.*

(i) *For all real d and m, the field equal to d at a, 0 at b and $t\,d/l_i$ at the interior site of path i that lies t bonds from b obeys the exact-J law with strength m if and only if*

$$2\sinh d + \sum_{i=1}^{r} 2\sinh\frac{d}{l_i} = m. \tag{3.7}$$

(ii) *At $m = 1$ this has one real root $d(l)$, with $0 < d(l) < \log\varphi$; every solution has own-bond drop $d(l)$, and the price $p(l) = 2(\cosh d(l) - 1)$ lies in $(0, \sqrt{5}-2)$.*

(iii) *Adding a path of length at least 2, or shortening a path to a length of at least 2, strictly lowers the price; so $p(l) \leq p_{l_i+1}$ for every i, the price of the ring C_{l_i+1} made of the post's bond and path i alone.*

(iv) *On the ring C_k the drop d_k solves $2\sinh d + 2\sinh\frac{d}{k-1} = 1$; d_k and the price $p_k = 2(\cosh d_k - 1)$ rise strictly with k, and $d_k \to \log\varphi$, $p_k \to \sqrt{5}-2$, never reached.*

(v) *For the carriers of Table 3.2 the drops and prices lie in the certified intervals printed there.*

Proof. (i) Along path i the field falls by d/l_i across each bond, so the currents at an interior site cancel; at a the currents are $\sinh d$ and $\sinh(d/l_i)$ for each path, and at b the law is the negative. (ii) The left side g_l of (3.7) is continuous, odd and strictly increasing, and $g_l(\log\varphi) > 2\sinh\log\varphi = 1$; uniqueness is Theorem 3.22. (iii) At fixed $d > 0$ each change raises $g_l(d)$, which moves the root down, and the price follows the drop. (iv) For $k < k'$, $g_{k'}(d_k) < g_k(d_k) = 1$, so $d_{k'} > d_k$; and $\frac{1}{2} - \sinh\frac{\log\varphi}{k-1} < \sinh d_k < \frac{1}{2}$ gives the limit. (v) For a rational $x \in [0,1]$, replace each sinh on the left of (3.7) by its Taylor polynomial of degree 9 plus or minus the remainder bound $11x^{10}/(10\cdot 10!)$; if the upper bound at x is below 1 then $x < d(l)$, and if the lower bound is above 1 then $d(l) < x$. The printed endpoints pass these tests in exact rational arithmetic, and so do the price endpoints, with the same bounds for cosh. □

Table 3.2. Own-bond drop and first-tick price $2(\cosh d - 1)$ of a unit source on carriers with loops through the post's bond, with the carrier's conductance C between the post's ends (Section 3.7). The intervals are certified; the last row is Theorem 3.28 and Corollary 3.56. Under (R1) and (R2) the price column is the price of the first event.

Carrier	C	Drop d	Price $2(\cosh d - 1)$
any bridge of weight 1	1	$\log\varphi \approx 0.481212$	$\sqrt{5}-2 \approx 0.236068$
ring C_{12}	12/11	(0.444749, 0.444750)	(0.201083, 0.201085)
ring C_6	6/5	(0.407140, 0.407141)	(0.168065, 0.168067)
square C_4	4/3	(0.368472, 0.368473)	(0.137314, 0.137316)
triangle C_3	3/2	(0.328865, 0.328866)	(0.109130, 0.109132)
two triangles $\Theta(2,2)$	2	(0.248399, 0.248400)	(0.062019, 0.062021)
triangle and four squares $\Theta(2,3,3,3,3)$	17/6	(0.176061, 0.176062)	(0.031077, 0.031079)
possibility graph on T_3	81/26	(0.1601, 0.1603)	(0.025686, 0.025743)

Each path from the source to the sink is a separate return channel: a path of l bonds carries $2\sinh(d/l)$ of the unit flux, whatever the other paths do. On the 27-site torus the post's bond lies on the triangle through $(2,0,0)$ and on four squares, through $(0,\pm1,0)$ and $(0,0,\pm1)$; these five loops meet only at the bond's ends and form $\Theta(2,3,3,3,3)$, and the torus drop is lower still because the torus has further ways round, which share sites (Section 3.9).

3.6.1 Series-parallel carriers

Definition 3.36 (series-parallel networks, characteristic)**.** A series-parallel network X is a carrier with a first and a second terminal, built from single bonds of weight $w > 0$ by series composition (the second terminal of Y joined to the first of Z at a new site) and parallel composition (first terminals identified, second terminals identified, merged bonds adding their weights). Along the same rules define the characteristic $\Psi_X : \mathbb{R} \to \mathbb{R}$ and the number $C_X > 0$: a bond has $\Psi_X(d) = 2w\sinh d$ and $C_X = w$; series gives $\Psi_X^{-1} = \Psi_Y^{-1} + \Psi_Z^{-1}$ and $C_X = C_Y C_Z/(C_Y + C_Z)$; parallel gives $\Psi_X = \Psi_Y + \Psi_Z$ and $C_X = C_Y + C_Z$.

$\Psi_X(d)$ is the strength X passes when its terminals differ by d: parallel parts share the drop and add strengths, series parts pass one strength and add drops [40].

Theorem 3.37 (series-parallel carriers)**.** *Let X be a series-parallel network with terminals a, b.*

(i) *Ψ_X is a continuous, odd, strictly increasing bijection of $\mathbb{R}$.*
(ii) *For every m the exact-J law on X with source a, sink b and strength m has a solution, and every solution has drop $\Psi_X^{-1}(m)$.*
(iii) *If the post's carrier is a bond ab of weight $w_0 > 0$ in parallel with X, the drop is the unique root of $2w_0\sinh d + \Psi_X(d) = m$; at $w_0 = m = 1$ the price is below $\sqrt{5}-2$.*
(iv) *For $m > 0$, adding a branch in parallel to any part of X, raising the weight of any bond, or removing one factor of a series composition, strictly lowers that root and the price.*
(v) *For $d > 0$, $2C_X d < \Psi_X(d) \le 2C_X\sinh d$, and C_X is the conductance of Definition 3.42.*
(vi) *A chain of L unit bonds has $\Psi(d) = 2\sinh(d/L)$ and $C = 1/L$; for $\Theta(l_1,\dots,l_r)$ the root equation is (3.7) and $C = 1 + \sum_i 1/l_i$.*

The proof is in Appendix B.1. Under (R1) and (R2), on every carrier made of the post's bond

and a series-parallel rest, with any positive weights, the price of the first event is fixed by one equation in one unknown.

3.6.2 The strength that would restore the golden strain on a torus

Under (R2) the strength of a post's source is one, and on the possibility graph of a torus that falls short of the golden strain. The law itself makes sense at every strength. We use the signed possibility graph of Definition 3.26, on which every site has total weight 6; put $R_N = (1 - N^{-3})/6$.

Lemma 3.38 (the linear dipole on a torus)**.** *Let $N \geq 2$ and let a, b be neighbours. Every $u : T_N \to \mathbb{R}$ with $\sum_j w_{ij}(u_i - u_j) = \frac{1}{2}(\delta_{ia} - \delta_{ib})$ at every site has $u_a - u_b = R_N$ and $\sum_{i,j} w_{ij}(u_i - u_j)^2 = R_N$.*

Proof. The characters $\chi_\kappa(x) = e^{2\pi i \kappa \cdot x/N}$ diagonalize the left side with eigenvalues $\lambda(\kappa) = \sum_{k=1}^{3}(2 - 2\cos\theta_k)$, $\theta_k = 2\pi\kappa_k/N$, which vanish only at $\kappa = 0$. With $n = N^3$, $u_a - u_b = \frac{1}{2n}\sum_{\kappa \neq 0}(2 - 2\cos\theta_{k_0})/\lambda(\kappa)$, where k_0 is the axis of the bond. The sum is the same for every axis, since λ is symmetric in the coordinates; averaging over the three axes replaces the numerator by $\lambda(\kappa)/3$ and gives $(n-1)/(6n) = R_N$. Multiplying the equation by $2u_i$ and summing gives the last identity. □

Theorem 3.39 (the strength that gives the golden strain)**.** *Let $N \geq 2$ and let a, b be the ends of a post's bond on the signed possibility graph on T_N.*

(i) *For every real m the exact-J law with strength m has a solution, and all solutions have one drop $D_N(m)$.*
(ii) *D_N is odd and strictly increasing, and $|D_N(m) - D_N(m')| \leq R_N|m - m'|$.*
(iii) *For $m > 0$, $\operatorname{arsinh}(m/12) < D_N(m) < R_N m$; for the unit source $\operatorname{arsinh}\frac{1}{12} \approx 0.0832 < D_N(1) < (1 - N^{-3})/6$.*
(iv) *Exactly one strength m_N^* gives the golden strain $D_N(m_N^*) = \log\varphi$, and $6\log\varphi/(1 - N^{-3}) < m_N^* < 6$. In particular $m_N^* \neq 1$; on the 27-site torus $2.998 < m_3^* < 6$.*
(v) *For $m \geq 0$ any two values of a solution differ by at most $D_N(m)$.*

The proof is in Appendix B.1. The law fixes one golden strength for each torus without saying that a post has it; the upper end $6 = 12\sinh\log\varphi$ is the strength at which every bond out of the source would carry the golden drop.

3.6.3 The linear field behind the torus numbers

Replacing sinh by its argument gives the quadratic law, whose field is the lattice Coulomb field of a dipole. It is the small-strength limit of the exact field and explains the torus numbers.

Proposition 3.40 (the lattice dipole)**.**

(i) *On the signed possibility graph on T_N, every solution of the quadratic law $2\sum_j w_{ij}(u_i - u_j) = m(\delta_{ia} - \delta_{ib})$ for neighbours a, b drops by $mR_N = m(1 - N^{-3})/6$ across the post's bond, which tends to $m/6$ as $N \to \infty$. At $N = 3$ the drop is $\frac{13}{81}m$, and the post's bond carries $\frac{26}{81}$ of the flux and the five other bonds at the source $\frac{55}{81}$.*

(ii) *On the 27-site torus the exact drop per unit strength tends to the linear one,* $D_3(m)/m \to \frac{13}{81}$ *and* $2\sinh(D_3(m))/m \to \frac{26}{81}$ *as* $m \to 0$. *At* $m = 1$ *the exact drop is below* $\frac{13}{81}$ *and the exact share above* $\frac{26}{81}$.

(iii) *On a cubic box* $\{0, \dots, L-1\}^3$ *with unit bonds and no periodic wrap, the effective resistance across any neighbour bond is at least* $\frac{1}{3}$, *and across a bond at a corner at least* $\frac{2}{5}$; *on the torus it is* $(1 - N^{-3})/3 < \frac{1}{3}$. *The resistances across the bond at the centre of growing boxes decrease to a limit of at least* $\frac{1}{3}$. *Whether that limit equals* $\frac{1}{3}$ *is open here.*

Proof. (i) is Lemma 3.38 scaled by m; the share is $2 \cdot \frac{13}{81} = \frac{26}{81}$. (ii) The three-level field of Theorem 3.28 solves the law at strength $S(c)$ for every c, with $A'(0) = 13$ and $S'(0) = 162$, which gives the limit; the unit-strength comparison is the upper end of the bracket of Theorem 3.43 with $C = \frac{81}{26}$, and the share follows from $2A > 0.1601 > \operatorname{arsinh} \frac{13}{81}$ (Theorem 3.28). (iii) Adding bonds never raises a resistance (Rayleigh's monotonicity, from Dirichlet's principle [36]). An open box has at least the resistance of the torus of the same side, which adds the wrap-around bonds, and those torus values $(1-N^{-3})/3$ tend to $\frac{1}{3}$. The centred box of side $n+2$ sits inside the centred box of side $n+3$ with its central bond on the central bond, so the centred resistances decrease to a limit, which is at least $\frac{1}{3}$; and any neighbour bond of any box embeds into a large centred box, so its resistance is at least that limit. At a corner, the test potential equal to 1 at the corner, $\frac{1}{4}$ at its two neighbours off the bond and 0 elsewhere has unit difference across the bond and energy $\frac{5}{2}$, so the conductance is at most $\frac{5}{2}$ by Dirichlet's principle. □

The effective resistance is twice the linear unit-post drop, $1/C$ in the notation of the next section. For the infinite cubic lattice the classical value of the neighbour resistance is $\frac{1}{3}$, by superposition of a source and a sink [36]; whether the finite open boxes reach it is open here.

3.7 The price on every carrier

On every carrier whose bonds join the post's two ends the drop is one number, and the carrier's conductance between the ends confines it to a window with closed-form ends.

Theorem 3.41 (one drop on every carrier). *Take a carrier, sites* $a \neq b$ *and a strength* $m \in \mathbb{R}$.

(i) *If* a *and* b *are joined by a path of bonds, the exact-J law with source* a, *sink* b *and strength* m *has a solution, and all solutions have the same drop, written* $D(m)$.

(ii) *If* $m \neq 0$ *and the law has a solution, then* a *and* b *are joined by a path of bonds.*

Proof. Let K be the set of sites joined to a; no bond leaves K. (i) Minimize $\Phi(\phi) = A(\phi) - m(\phi_a - \phi_b)$ over the fields that vanish at b and off K. If $|\phi|$ reaches M at a site of K, some bond on a path of at most $n-1$ bonds from b carries a difference of at least $M/(n-1)$, so $A(\phi) \geq 2w_0(\cosh \frac{M}{n-1} - 1)$ with w_0 the least bond weight, while $|m(\phi_a - \phi_b)| \leq |m|M$. So Φ is coercive and attains a minimum, where its derivative vanishes in every admissible direction. Directions constant on K, or vanishing on K, give zero on both sides of Lemma 3.13, so the identity holds for every v and the minimizer obeys the law. Uniqueness of the drop is Theorem 3.22 summed along a path. (ii) Summing the law over K, currents inside K cancel in pairs and none leaves it, so if $b \notin K$ the sum is $0 = m$. □

Under (R1) and (R2), every carrier joining the post's two ends prices the first event once: the strain is $D(1)$ and the price $2(\cosh D(1) - 1)$. A carrier in which the two ends are not joined holds no field of a unit source at all.

Definition 3.42 (conductance)**.** For a carrier and sites $a \neq b$, the conductance between them is

$$C = \inf\Big\{\tfrac{1}{2}\sum_{i,j} w_{ij}(g_i - g_j)^2 \;:\; g : V \to \mathbb{R},\ g_a - g_b = 1\Big\},$$

the least quadratic cost of a unit difference, the effective conductance of the linear network with conductance w_{ij} on each bond (Dirichlet's principle [36]); $1/C$ is the effective resistance.

A single bond of weight w has $C = w$, and C never falls when weights rise. On the triangle, $g = (1, 0, t)$ costs $1 + t^2 + (1-t)^2$, least at $t = \frac{1}{2}$, so $C = \frac{3}{2}$; a single bond of weight $\frac{3}{2}$ would carry a unit source with drop $\operatorname{arsinh} \frac{1}{3} \approx 0.32745$ under the exact law and $\frac{1}{3}$ under the quadratic law, and the triangle's drop $d_3 \approx 0.32887$ lies between them. That is the general rule.

Theorem 3.43 (the conductance bracket)**.** *Let $a \neq b$ be joined by a path of bonds, let $m > 0$, and let C be their conductance. Then $C > 0$,*

$$\operatorname{arsinh} \frac{m}{2C} \;\leq\; D(m) \;<\; \frac{m}{2C}, \tag{3.8}$$

and every solution has $|\phi_i - \phi_j| \leq D(m)$ on every bond.

Proof. $C > 0$: a unit difference spread over a path of at most $n - 1$ bonds puts at least $1/(n-1)$ on one of them. Let ϕ solve the law with $d = D(m)$. By Theorem 3.23 applied to ϕ and to $-\phi$ (which has source b), ϕ_a is the largest and ϕ_b the smallest value on K; off K the field is constant on each component by the spreading step. So $|\phi_i - \phi_j| \leq d$ on every bond. Lemma 3.13 gives, for every v,

$$\sum_{i,j} w_{ij} \sinh(\phi_i - \phi_j)(v_i - v_j) = m(v_a - v_b). \tag{3.9}$$

Upper end: at $v = \phi$, $\sum w_{ij}(\phi_i - \phi_j)\sinh(\phi_i - \phi_j) = md$; since $x \sinh x \geq x^2$ with equality only at 0, and some bond at a carries a nonzero difference, the quadratic action $A_2(\phi) = \frac{1}{2}\sum w_{ij}(\phi_i - \phi_j)^2$ satisfies $2A_2(\phi) < md$, so $d > 0$; and $C \leq A_2(\phi/d)$ gives $2Cd^2 < md$. Lower end: for g with $g_a - g_b = 1$, Cauchy-Schwarz in (3.9) gives $m^2 \leq S \cdot 2A_2(g)$ with $S = \sum w_{ij} \sinh^2(\phi_i - \phi_j)$. Convexity of sinh on $[0, d]$ gives $d \sinh^2 x \leq \sinh d \cdot x \sinh x$ for $|x| \leq d$, so $S \leq m \sinh d$ by (3.9) at $v = \phi$. Hence $m \leq 2 \sinh d\, A_2(g)$ for every such g, and $m \leq 2C \sinh d$. □

The window has width $x - \operatorname{arsinh} x$ at $x = m/(2C)$. For a unit source it is 0.0188 on a single unit bond and 0.0007 on the 27-site torus: the carrier acts on the drop like a single bond of weight C, up to the difference between the exact and the quadratic law on that bond.

Corollary 3.44 (the golden window: a test for every carrier)**.** *Let $a \neq b$ be joined, with conductance C. If $D(1) = \log \varphi$, then*

$$1 \leq C < \frac{1}{2 \log \varphi} \approx 1.039043.$$

If $w \leq w'$ bond by bond and $m > 0$, then $D'(m) - D(m) < x - \operatorname{arsinh} x$ with $x = m/(2C)$.

Proof. At $m = 1$ the bracket reads $\operatorname{arsinh}\frac{1}{2C} \le \operatorname{arsinh}\frac{1}{2} < \frac{1}{2C}$. For the comparison, $C \le C'$, so $D'(m) < m/(2C') \le x$ while $D(m) \ge \operatorname{arsinh} x$. □

A single unit bond sits at the lower end of the window. The test is necessary, not sufficient, and it does not choose a carrier.

Proposition 3.45 (the conductance of the 27-site torus)**.** *(i) On any carrier, if ψ satisfies $\sum_j w_{ij}(\psi_i - \psi_j) = I(\delta_{ia} - \delta_{ib})$ with $\psi_a > \psi_b$, then $C = I/(\psi_a - \psi_b)$. (ii) On the possibility graph on T_3 with unit weights, the conductance between $(0,0,0)$ and $(1,0,0)$ is $C = \frac{81}{26}$. (iii) The drop of Theorem 3.28 lies in $[\operatorname{arsinh}\frac{13}{81}, \frac{13}{81}) \subset [0.15981, 0.16050)$, which contains the certified interval $(0.1601, 0.1603)$. The torus fails the golden test.*

Proof. (i) Thomson's principle [36]: write a unit-difference field as $\psi/V + h$ with $V = \psi_a - \psi_b$ and $h_a = h_b$; the cross term vanishes by the equation for ψ, so $A_2 \ge I/V$ with equality at $h = 0$. (ii) The three-level shape of Theorem 3.28 with the values $\alpha = \frac{13}{81}$, $\beta = \frac{5}{162}$, $\gamma = \frac{1}{81}$ satisfies the linear law with $I = 1$: at the source $7\alpha - 4\beta = 1$, at a β site $6\beta - \alpha - 2\gamma = 0$, at a γ site $5\gamma - 2\beta = 0$. Its drop is $\frac{26}{81}$. (iii) Theorem 3.43, with $\operatorname{arsinh}\frac{13}{81} > 0.15981$ and $\frac{13}{81} < 0.16050$. □

Theorem 3.46 (the drop as a function of the strength)**.** *Let $a \ne b$ be joined, with conductance C.*

(i) *D is odd, continuous and strictly increasing, maps $\mathbb{R}$ onto $\mathbb{R}$, and $|D(m) - D(m')| \le |m - m'|/(2C)$.*
(ii) *Its inverse Ψ, the characteristic of the carrier, satisfies $2Cd < \Psi(d) \le 2C\sinh d$ for $d > 0$, and $\Psi(d)/d \to 2C$ as $d \to 0$. On a series-parallel network $\Psi = \Psi_X$.*
(iii) *Multiplying every weight and the strength by the same $c > 0$ leaves the drop unchanged.*
(iv) *$D(1) = \log\varphi$ if and only if $\Psi(\log\varphi) = 1$.*

Proof. (i) For solutions ϕ, ψ with strengths m, m' and drops d, d', subtracting (3.9) at $v = \phi - \psi$ gives $\sum w_{ij}(\sinh\Delta\phi_{ij} - \sinh\Delta\psi_{ij})(\Delta\phi_{ij} - \Delta\psi_{ij}) = (m - m')(d - d')$, and since $\cosh \ge 1$ the left side is at least $2A_2(\phi - \psi) \ge 2C(d - d')^2$. If $d = d'$, the two fields have the same currents and $m = m'$. So $m \ne m'$ gives $2C|d - d'| \le |m - m'|$ with $(m - m')(d - d') > 0$. Oddness: $-\phi$ solves at $-m$. Ontoness follows from the lower end of (3.8). (ii) The bracket at $m = \Psi(d)$; on a series-parallel network, Theorem 3.37(ii). (iii) Both sides of (3.4) scale by c. (iv) Ψ is the inverse of D. □

Under (R1) and (R2) a carrier enters the first event only through its characteristic: the golden strain sits on it exactly when $\Psi(\log\varphi) = 1$.

3.8 A stronger carrier can raise the drop; at unit strength the rise is capped

For linear networks Rayleigh's monotonicity law says that strengthening a conductor never raises the effective resistance [36]. For the exact-J law that fails, and a precise cap replaces it. The two statements belong together: a stronger carrier can raise the first post's drop locally, by less than the width of its window, and, at unit strength with the post's bond of weight at least one, never above $\log\varphi$.

Definition 3.47 (the Wheatstone carrier)**.** For $\alpha, \beta, \kappa, w_0 \geq 0$, the carrier $B(\alpha, \beta, \kappa, w_0)$ has six sites a, b, p, q, p', q', bonds ap' and $p'p$ of weight α, pb of weight β, aq of weight β, qq' and $q'b$ of weight α, the cross bond pq of weight κ, and the post's bond ab of weight w_0; weight 0 means the bond is absent. Source a, sink b.

Theorem 3.48 (strengthening can raise the drop; the rise is capped)**.**

(i) *On* $B(4, \frac{1}{2}, \kappa, 0)$ *at strength* $\frac{26944}{1365}$*, raising the cross bond from* $\kappa = \frac{24}{23}$ *to* $\frac{128}{37}$ *raises the drop from exactly* $\log \frac{819}{5} \approx 5.0986$ *by more than* $\frac{517}{20992} > 0.0246$*. On* $B(\frac{49}{12}, \frac{1}{2}, \kappa, 0)$ *at strength* $\frac{399}{20}$*, adding the cross bond* $\kappa = \frac{23}{18}$ *raises the drop from exactly* $\log \frac{784}{5}$ *by more than* $\frac{13}{405} > 0.0320$*. For every strength* $\mu > 0$ *there are carriers* $w \leq w'$ *on six sites, with the ends joined in* w*, such that* $D_w(\mu) < D_{w'}(\mu)$*.*

(ii) *Inside the golden window: with* $\alpha_W = \frac{3391}{77520}$*,* $\beta_W = \frac{3391}{156672}$ *and a cross bond* $\kappa_W \approx 0.80472$*, the carriers* $w = B(\alpha_W, \beta_W, 0, 1)$ *and* $w' = B(\alpha_W, \beta_W, \kappa_W, 1)$ *have a post's bond of weight* 1 *and conductances in* $[1, 1/(2\log\varphi))$*;* $D_w(1) = \log \frac{513}{320} \approx 0.471955$*, and* $\frac{1}{24960016} < D_{w'}(1) - D_w(1) < 10^{-7}$*.*

(iii) *The cap: let* $w \leq w'$ *bond by bond, with the post's bond of weight* $w_{ab} \geq 1$ *in* w*, and* $x = 1/(2C)$ *for the conductance* C *of* w*. Then* $D_{w'}(1) - D_w(1) < x - \operatorname{arsinh} x \leq \frac{1}{2} - \log\varphi < 0.0191$*; both drops are at most* $\log\varphi$*; and if* $D_w(1) = \log\varphi$*, then* $D_{w'}(1) \leq D_w(1)$*.*

(iv) *On a series-parallel network, raising the weight of one bond strictly lowers the drop at every strength* $m > 0$*.*

Proof. (iii) The first inequality is Corollary 3.44. The post's bond gives $C \geq 1$, so $x \leq \frac{1}{2}$, and $x - \operatorname{arsinh} x$ increases with x, with value $\frac{1}{2} - \log\varphi \approx 0.018788$ at $\frac{1}{2}$. Theorem 3.23(iii) bounds both drops by $\operatorname{arsinh} \frac{1}{2} = \log\varphi$, so a rise from $\log\varphi$ is impossible. (iv) is Theorem 3.37(iv). Parts (i) and (ii) are exact certificates: the exact solutions and the bounds are in Appendix B.1. □

Under (R1) and (R2) the price of the first event is therefore not monotone in the carrier: a carrier at least as strong on every bond can give a larger strain and price, also at unit strength inside the golden window. At unit strength with the post's bond of weight at least one, every such rise is small, below $\frac{1}{2} - \log\varphi \approx 0.018788$, and nothing raises the strain above $\log\varphi$. Where the strain is exactly golden, as on the record carrier of the first post, adding bonds can only keep or lower the price. The reversal is an instance of Braess's paradox [9], whose electrical and mechanical analogues are known [23]; Calvert and Keady showed that for every conductor law that is not a power law some network of at least six conductors shows it, and that two-terminal series-parallel networks never do [15]. The law $2w\sinh u$ is not a power law. What is new here is the exact certificate at the first event's own operating point and the cap. Carriers must therefore be compared through their windows or their characteristics, never by inclusion alone.

3.9 Ceilings from loops

What holds on every carrier is a ceiling: loops through the post's bond cap its drop. The mechanism is a cut. On a ring, take any step of the way round from b to a; the sites at or above its upper end contain a and not b, and exactly two bonds leave them, the post's bond and the step. Summing the law over those sites, the inner currents cancel in pairs and the two

leaving currents add to $\frac{m}{2}$, which is the ring's balance. On a larger carrier holding the same loop, a cut of the same kind crosses the post's bond, one step of the loop and possibly more bonds, all with outward currents, so those two currents add to at most $\frac{m}{2}$.

Theorem 3.49 (one loop caps the drop)**.** *Let the post's bond ab have weight at least $w_0 > 0$, let $m > 0$, and let $b = p_0, p_1, \dots, p_{k-1} = a$ be distinct sites, $k \geq 3$, with $w_{p_{i-1}p_i} \geq u_i > 0$. Let P be the chain of bonds of weights $u_1, \dots, u_{k-1}$, so that $\Psi_P^{-1}(c) = \sum_i \operatorname{arsinh}(c/(2u_i))$, and let $D^*(m)$ be the root of $2w_0 \sinh d + \Psi_P(d) = m$, the drop on a bond of weight w_0 beside P alone. Then $\operatorname{arsinh}(m/(2C)) \leq D(m) \leq D^*(m)$, and the price obeys $2(\cosh D(m) - 1) \leq 2(\cosh D^*(m) - 1)$. At unit weights and $m = 1$, $D^*(1) = d_k$, the drop of the ring C_k, which the ring attains.*

Proof. Let $d = D(m) > 0$, $c = \Psi_P(d)$ and $r_i = \operatorname{arsinh}(c/(2u_i))$, so $\sum_i r_i = d$. The rises of ϕ along the path also add to d, so some step i rises by at least r_i. Let S be the set of sites with $\phi \geq \phi_{p_i}$; it contains a and not b (Theorem 3.23 for ϕ and $-\phi$). Summing (3.4) over S gives $2\sum_{x\in S, y\notin S} I_{xy} = m$ with every term nonnegative. Two of the terms are $w_{ab} \sinh d \geq w_0 \sinh d$ and the current on step i, at least $u_i \sinh r_i = c/2$; they are different bonds since $k \geq 3$. Hence $2w_0 \sinh d + \Psi_P(d) \leq m$, and $d \leq D^*(m)$. At unit weights $\Psi_P(d) = 2\sinh(d/(k-1))$ (Theorem 3.37(vi)). □

Proposition 3.50 (two loops do not cap the drop together this way)**.** *On the carrier w' of Theorem 3.48(ii) the post's bond lies on the loops b, p, p', a and b, q', q, a, which share no site other than a, b. The carrier made of the post's bond and these two paths alone is w, with drop $\log \frac{513}{320}$ at unit strength, and $D_{w'}(1)$ is larger. On both carriers $0.471668 < D(1) < 0.476543$, by the conductance and by the single-loop ceilings, and the price is below 0.231424.*

Proof. The two paths and the post's bond are exactly the bonds of w; Theorem 3.48(ii) gives the comparison. The bracket is Theorem 3.43 with $C \leq \frac{30415361}{29767680}$ and Theorem 3.49 for each loop at its own weights, certified as in Theorem 3.35(v). □

Loops that share no bond do cap the drop together, through power. Taking $v = \phi$ in (3.9),

$$\sum_{i,j} w_{ij}(\phi_i - \phi_j)\sinh(\phi_i - \phi_j) = m\,d, \tag{3.10}$$

which says that the bonds together take the power $\frac{m}{2}d$ the source delivers, an instance of Tellegen's theorem [120]; every term is nonnegative.

Lemma 3.51 (least power of a path)**.** *For a chain P of $l \geq 1$ bonds of weights $u_1, \dots, u_l > 0$ let $E_P(d) = \inf\{\sum_i u_i r_i \sinh r_i : r_i \geq 0, \sum_i r_i = d\}$. Then (i) $E_P(d) \leq \sum_i u_i \rho_i \sinh \rho_i$ for all real ρ_i with $\sum_i \rho_i = d > 0$; (ii) E_P is continuous, $0 \leq E_P(d) \leq (\sum_i u_i) d \sinh d$, and $E_P(d)/d$ is nondecreasing; (iii) if every $u_i = 1$ then $E_P(d) = d\sinh(d/l)$, and if every $u_i \leq 1$ then $E_P(d) \leq d\sinh(d/l)$; (iv) $2E_P(d)/d \leq \Psi_P(d)$.*

Proof. (i) Rescale $r_i = |\rho_i| d / \sum_j |\rho_j| \leq |\rho_i|$; $x\sinh x$ is even and increasing in $|x|$. (ii) Minimize over the simplex $r = ds$; and a split of d' scaled by $d/d' \leq 1$ is a split of d costing at most $\frac{d}{d'}$ times as much. (iii) $x\sinh x$ is convex, so the equal split is least. (iv) The bond drops of P at strength $\Psi_P(d)$ split d and give $\sum_i u_i r_i \sinh r_i = \frac{1}{2}\Psi_P(d)d$. □

Theorem 3.52 (loops cap the drop together)**.** *Let the post's bond have weight at least $w_0 > 0$ and $m > 0$. For $j = 1, \dots, R$ let $b = p_{j,0}, \dots, p_{j,l_j} = a$ be distinct sites, $l_j \geq 2$, with $w_{p_{j,i-1}p_{j,i}} \geq u_{j,i} > 0$, and let E_j be the least power of the chain $u_{j,1}, \dots, u_{j,l_j}$. Let $\lambda_j > 0$ be weights whose sum over the paths that use a given pair of sites as a bond is at most 1, for every pair; paths that share no bond meet this with every $\lambda_j = 1$. Then $F(d) = 2w_0 \sinh d + \sum_j 2\lambda_j E_j(d)/d$ is strictly increasing with one positive root $D^E(m)$ of $F = m$, and*

$$\operatorname{arsinh} \frac{m}{2C} \leq D(m) \leq D^E(m).$$

At unit weights, $\lambda_j = 1$, paths sharing no bond and $m = 1$, the ceiling is the drop $d(l_1, \dots, l_R)$ of the theta carrier those loops form. For one path the ceiling is at least that of Theorem 3.49.

Proof. F is continuous and strictly increasing by Lemma 3.51(ii), tends to 0 at 0 and to ∞. In (3.10) the post's bond contributes at least $2w_0 d \sinh d$. A bond β crossed by steps of several paths contributes $2w_\beta r_\beta \sinh r_\beta \geq \sum 2\lambda_j u_{j,i} \rho_{j,i} \sinh \rho_{j,i}$ over those steps, since their λ_j add to at most 1 and $u_{j,i} \leq w_\beta$; no step is the post's bond, since $l_j \geq 2$. Summing and applying Lemma 3.51(i) to each path, $md \geq 2w_0 d \sinh d + \sum_j 2\lambda_j E_j(d)$, that is $F(D(m)) \leq m$. At unit weights Lemma 3.51(iii) turns F into the left side of (3.7). For one path, Lemma 3.51(iv) bounds F by the left side of the single-loop equation. □

Corollary 3.53 (the 27-site torus under its five shortest loops)**.** *On the possibility graph on T_3 with unit weights, $a = (0,0,0)$, $b = (1,0,0)$, the five paths from b to a through $(2,0,0)$; through $(1,1,0), (0,1,0)$; through $(1,2,0), (0,2,0)$; through $(1,0,1), (0,0,1)$; and through $(1,0,2), (0,0,2)$ share no bond, and $2A < d(2,3,3,3,3) < 0.176062$. The price is below 0.031079.*

Proof. The nine interior sites are distinct, so no two paths share a bond. Theorem 3.52 and Table 3.2; the inequality is strict since $2A < 0.1603$. □

Theorem 3.54 (the five ports)**.** *On the possibility graph on T_3 with unit weights, at $m = 1$ and $0 < w_0 \leq 1$, no family of paths and weights as in Theorem 3.52 gives a ceiling below $d(2,3,3,3,3)$, and a family containing a path of at least four bonds gives a strictly higher one. Every such ceiling exceeds $2A$ by more than 0.015761.*

Proof. Every path leaves b through one of its five other neighbours, its ports; the paths through one port share that bond, so their λ_j add to at most 1. Only the port $(2,0,0)$ is a neighbour of a, so a path through any other port has at least three bonds. With all weights at most one, Lemma 3.51(iii) gives $2E_j(d)/d \leq 2\sinh(d/l_j)$, and grouping by port, $F(d) \leq 2\sinh d + 2\sinh \frac{d}{2} + 8\sinh \frac{d}{3}$, strictly if some path has four or more bonds. The right side is the theta balance of $\Theta(2,3,3,3,3)$, whose root exceeds 0.176061, while $2A < 0.1603$. □

The gap sits at the ports. In the exact field the port $(2,0,0)$ lies on the slice where the field vanishes, and its bond to b drops by A, half the post's drop, exactly the triangle's equal split. In the linear field of Proposition 3.45(ii) each square's port bond carries $\frac{7}{54}$ instead of the equal split $\frac{26}{243}$, and those four excesses, $\frac{22}{243}$ in all, are exactly the current the torus passes beyond its five loops: $1 - \frac{26}{81}(1 + \frac{1}{2} + \frac{4}{3}) = \frac{22}{243}$. A sharper ceiling must therefore see more than loops. Keeping the post's bond exact and treating only the rest of the carrier linearly does.

Theorem 3.55 (the split ceiling). *Let $a \neq b$, let the post's bond have weight $w_{ab} > 0$, let $m > 0$, and let C_R be the conductance between a and b of the carrier with that bond removed.*

(i) *$C_R = C - w_{ab}$; $C_R = 0$ if the bond is a bridge and $C_R > 0$ otherwise.*
(ii) *$2w_{ab} \sinh d + 2C_R d = m$ has exactly one positive root $D^S(m)$, and $D(m) \leq D^S(m)$, with equality exactly when the bond is a bridge. Also $D^S(m) < m/(2C)$ and $D^S(m) \leq \operatorname{arsinh}(m/(2w_{ab}))$.*
(iii) *If $m = 1$ and $D(1) = \log\varphi$, then $w_{ab} + 2C_R \log\varphi \leq 1$; if moreover $w_{ab} \geq 1$, the bond is a bridge of weight exactly 1.*

Proof. (i) The quadratic action splits as that of the rest plus $w_{ab}(g_a - g_b)^2$; minimize. On a bridge the indicator of the source's side of the rest has unit difference and zero rest energy. (ii) In (3.10), separate the post's bond: $md = 2w_{ab} d \sinh d + \sum_{\text{rest}} w_{ij}(\phi_i - \phi_j)\sinh(\phi_i - \phi_j)$, and the rest is at least $2C_R d^2$ since $x \sinh x \geq x^2$; strictly unless the rest carries no difference, which on a joined rest is impossible. On a bridge $D^S = \operatorname{arsinh}(m/(2w_{ab})) = D(m)$ by Theorem 3.24. At $x = m/(2C)$ the left side exceeds $2(w_{ab} + C_R)x = m$, and at $y = \operatorname{arsinh}(m/(2w_{ab}))$ it equals $m + 2C_R y \geq m$. (iii) The inequality of (ii) at $d = \log\varphi$, with $2\sinh\log\varphi = 1$. □

Corollary 3.56 (the 27-site torus under the split ceiling). *On the possibility graph on T_3, $C_R = \frac{55}{26}$, and $D^S(1)$ is the only real root of $2\sinh d + \frac{55}{13}d = 1$, with $0.160273 < D^S(1) < 0.160274$. Hence $2A < D^S(1)$, $D^S(1) - 2A < 0.000174$, and the first-tick price satisfies*

$$0.025686 < 2(\cosh 2A - 1) < 0.025743.$$

Proof. $C = \frac{81}{26}$ and $w_{ab} = 1$; the enclosure is certified as in Theorem 3.35(v); the bond lies on a cycle, so the inequality is strict; the price bounds follow from $0.1601 < 2A < 0.160274$. □

So if the field of the first post lived on the possibility graph on the 27-site torus, then under (R1) and (R2) its strain would lie between 0.1601 and 0.160274, about a third of $\log\varphi$, and the first event would cost between 0.025686 and 0.025743, about a ninth of $\sqrt{5} - 2$. On a bridge of unit weight the split ceiling is the drop itself: the golden price of the record carrier is its equality case. On the torus every ceiling built from loops lies more than 0.015761 above the drop, the split ceiling within 0.000174; how close it comes on a general carrier, and whether it has a matching floor, is open.

3.10 Which carrier holds the first post's field

Everything above prices the first event on a given carrier; which carrier is given remains open.

Remark 3.57 (open: the carrier of the first post's field). Under (R1) and (R2), decide whether the field of the first post lives on its record carrier, one bond of weight one, or on a carrier in which that bond lies on a cycle, such as the possibility graph. A theorem tying the bond weights of the field's carrier to recorded posts would settle the question for the record carrier, and the first event would then cost $\sqrt{5} - 2$ with no condition on the carrier left. A theorem selecting the full neighbour support for the field would exclude the golden price (Theorem 3.27) and, on the 27-site torus, put the price in $(0.025686, 0.025743)$. Any proposed carrier must pass the test of Corollary 3.44: conductance between the post's ends in $[1, 1/(2\log\varphi))$.

Further open statements bear on this one, and Chapter 11 states them exactly, among them a derivation of (R1) and (R2) themselves; whether the action of the physical carrier at the first event equals, at every field, the cost of the first event's record (by Theorem 3.19(iii) that equality would select the pair graph, and by Proposition 3.20 it fails for the whole period-three graph); and a construction that attaches the actual first event to a field. On the single bond the first tick has net flow zero at every site and, at the golden ratio, its field has a unit source; these are two readings of the same pair, and equal scalar costs alone do not identify them.

3.11 After the first event

The first event leaves the next one open, and the golden field does not carry through it.

Proposition 3.58 (the next event is underdetermined)**.** *Let L be any balanced event ledger. The events $e_1 = (0, 1, 2)$ and $e_2 = (2, 3, 2)$ are distinct, neither is its own reciprocal, each can be completed to a balanced ledger by posting its reciprocal next, appending either advances the record by one entry, and both cost $J(2) = \frac{1}{4}$. More generally every event with ratio $\neq 1$ can be opened onto L and completed the same way, while a rule requiring the ledger to stay balanced after every single entry forbids every such event.*

Proof. An event with ratio $r \neq 1$ differs from its reciprocal, since $r = r^{-1}$ forces $r = 1$ for $r > 0$. L followed by e and $\bar{e}$ is balanced, being the concatenation of two balanced lists. $J(2) = \frac{1}{2}(2 + \frac{1}{2}) - 1 = \frac{1}{4}$. After appending e alone, e occurs once more than $\bar{e}$. □

So the premises of the record (append-only entry, balance as a completion condition, the reciprocal structure and its cost symmetry) choose neither the next event nor the first post among the six of Corollary 3.3. Nor does a field frozen at the golden value.

Proposition 3.59 (a frozen golden field through a reciprocal second event)**.** *Let the first post $a \to b$ be followed by the reciprocal post $b \to a$, so that the record carrier is one bond of weight 2 and the two unit sources cancel. The field $(\log r, 0)$ obeys the exact-J law on this carrier with zero source if and only if $r = 1$. In particular the golden field $r = \varphi$ is not stationary through a reciprocal second event.*

Proof. At a the law reads $2 \cdot 2 \sinh(\log r) = 0$, so $\log r = 0$. □

The golden strain is a fact about the first post on its own bond, under (R1) and (R2). It is not a state that the record keeps through what follows.

Nothing says which of the six first posts occurs, and from every later ledger the posting law again offers six. Chapter 4 takes all of them at once: it counts every lawful history, finds the least complete pass of eight ticks, and asks how the branches weigh.

Chapter 4

Time: the eight-tick order, the native law and the arrow

Physics measures time with clocks, and a clock is a process that repeats while something else keeps count. Before there is any clock, what in the recognition account puts events in order, and does that order have a direction? We answer for one recognizer, a ledger of three accounts: the record orders a history, the parity pattern fixes a least complete pass of eight ticks, the native law K weighs the branches, and the direction of time sits in the record. None of these orders is a time in seconds. The counts and the eight-tick results are proved from definitions; everything said about K holds under the adopted premises (P1)–(P4) and three readings, introduced in Section 4.4.

4.1 Every lawful history, carried at once

The carrier of this section and the next is the account ledger with $d = 3$ accounts (carrier (a); Definition 3.1), with balance $x(L) = D - C \in \mathbb{Z}^3$, which we call the *imbalance* of one recognizer and write s when it serves as the state of a law. Every ledger has exactly six cheapest successors, the unit posts $P_{k,s}$, all at one price (Theorem 3.2). The posting law does not say which of them happens, and two different next events meet every requirement of the record at equal cost (Proposition 3.58). We therefore carry every lawful history at once. The law picks no branch; a weight on branches is a further law, and Section 4.4 supplies one under adopted premises.

Definition 4.1 (History, its ledger)**.** A *history* of length n is a word $h = (h_1, \dots, h_n)$ of unit posts; read as a list of posts, oldest first, it is a record of length n (Definition 2.8). Its *run* is $L_0 = \emptyset$, the empty ledger, and $L_t = h_t(L_{t-1})$; its *ledger* is L_n, the tally of that record, written $c = (D; C)$, with $|c| = \sum_k D_k + \sum_k C_k$ posts. A *tick* of a history is one of its steps.

A history is the same thing as a chain of lawful steps from the empty ledger: a unit post is identified by the count it raises, so each chain is the run of exactly one word. Counting histories counts lawful branches.

One account, worked by hand. With a single account a history is a word in two letters, debit and credit. After n posts the ledger is a pair (a, b) with $a + b = n$; there are $n + 1$ ledgers, and $\binom{n}{a}$ histories write (a, b). These are the rows of Pascal's triangle. The balance $a - b$ returns to 0 at every even n, while each ledger sits on exactly one row: a ledger fixes the tick at which it was written, and the balance does not. Three accounts give the same picture with six letters.

Proposition 4.2 (Ledgers and histories)**.** *(i) Posts commute: $P_{j,\tau}(P_{k,s}L) = P_{k,s}(P_{j,\tau}L)$ for every ledger L.*

(ii) A ledger is the ledger of a history of length n if and only if all its counts are nonnegative and they add up to n. There are exactly $\binom{n+5}{5}$ such ledgers.

(iii) The number of histories of length n with ledger $c = (D; C)$, $|c| = n$, is the multinomial coefficient

$$\binom{n}{c} = \frac{n!}{D_1!\, D_2!\, D_3!\, C_1!\, C_2!\, C_3!}, \tag{4.1}$$

and these numbers add up to 6^n over the records of length n.

Proof. (i) Both sides raise the same two counts by one. (ii) The characterization is Proposition 2.14. By (i) the ledger depends only on how often each of the six posts occurs, and the vectors in $\mathbb{N}^6$ with sum n number $\binom{n+5}{5}$. (iii) The words of length n in which post p occurs a_p times number $n!/\prod_p a_p!$; every word has one ledger, so the sum is 6^n. □

The ledger is what the books show; the history is the order in which the entries were made. Because entries commute, the ledger forgets the order, and (4.1) counts how many orders it forgets. The imbalance forgets more, since different ledgers share it.

Theorem 4.3 (Imbalance counts)**.** *Let $N_n(q)$ be the number of histories of length n whose ledger has imbalance $q \in \mathbb{Z}^3$. Then*

$$N_n(q) = \sum_{m_1+m_2+m_3=n} \frac{n!}{m_1!\, m_2!\, m_3!} \prod_{k=1}^{3} a(m_k, q_k), \tag{4.2}$$

summed over $m \in \mathbb{N}^3$, where $a(m, z) = \binom{m}{(m+z)/2}$ when $m \equiv z \pmod 2$ and $|z| \le m$, and $a(m, z) = 0$ otherwise. $N_n(q) > 0$ if and only if $|q|_1 \le n$ and $|q|_1 \equiv n \pmod 2$. In particular $N_n(0) = 0$ for odd n, and

$$N_{2j}(0) = \sum_{a+b+c=j} \frac{(2j)!}{(a!\, b!\, c!)^2}, \tag{4.3}$$

which is $1, 6, 90, 1860, 44{,}730$ for $2j = 0, 2, 4, 6, 8$: of the $6^8 = 1{,}679{,}616$ histories of eight posts, $44{,}730$ end balanced.

Proof. Group the ledgers by the account totals $m_k = D_k + C_k$; the multinomial (4.1) factors as $\frac{n!}{m_1!m_2!m_3!} \prod_k \binom{m_k}{D_k}$, and given m_k the imbalance $q_k = D_k - C_k$ fixes $D_k = (m_k + q_k)/2$, a count between 0 and m_k exactly when $m_k \equiv q_k \pmod 2$ and $|q_k| \le m_k$. For the support, such m with sum n exists exactly when $n - |q|_1$ is even and nonnegative: take $m_k = |q_k|$ and add the even remainder to one account. Balance means $D_k = C_k$ for each k, so $n = 2(D_1 + D_2 + D_3)$ is even, and (4.1) counts the histories behind each balanced ledger $(a, b, c; a, b, c)$. □

The numbers (4.3) are the closed walks of length $2j$ on the simple cubic lattice [89]: the imbalance performs exactly that walk, one unit step per post. Table 4.1 lists the branch set through eight posts. The counts depend on q only through $|q_1|, |q_2|, |q_3|$ in any order, because relabelling the accounts and exchanging debit with credit on an account maps lawful histories to lawful histories; equal counts need not come from such a relabelling, since $N_8(0, 1, 5) = N_8(2, 3, 3) = 560$.

These are numbers of branches. None of them is a probability.

Table 4.1. The branch set after n posts from the empty record: histories, distinct ledgers, distinct imbalances reached, and histories that end balanced (Proposition 4.2, Theorem 4.3).

n	histories 6^n	ledgers $\binom{n+5}{5}$	imbalances	balanced $N_n(0)$
0	1	1	1	1
1	6	6	6	0
2	36	21	19	6
3	216	56	44	0
4	1,296	126	85	90
5	7,776	252	146	0
6	46,656	462	231	1,860
7	279,936	792	344	0
8	1,679,616	1,287	489	44,730

4.2 The eight-tick pass

Each post changes one balance by one, so it flips exactly one bit of the parity pattern $x \bmod 2 \in \{0,1\}^3$ (Corollary 3.3). A history therefore walks along the edges of the cube $\{0,1\}^3$, one edge per post, and the account posted chooses the edge; the side does not matter to the walk. When has the walk gone all the way round?

Definition 4.4 (Pass)**.** Let $d \geq 1$. A *pass* of length $T \geq 1$ on the d-cube is a sequence $(q_0, \dots, q_{T-1})$ of patterns in $\{0,1\}^d$ in which q_{i+1} differs from q_i in exactly one bit for every i, indices taken modulo T, so that the step from q_{T-1} back to q_0 is one bit too. A pass is *complete* if every pattern of $\{0,1\}^d$ occurs in it. A *tour* is a complete pass of length 8 on the 3-cube with $q_0 = 000$.

Completeness alone does not fix the length: $000, 100, 000, 100, 110, 010, 011, 111, 101, 001$ steps back and forth once and then goes round, a complete pass of length ten. What completeness fixes is a floor.

Theorem 4.5 (The least complete pass)**.** *Let $d \geq 1$. Every complete pass has length $T \geq 2^d$; a complete pass has length exactly 2^d if and only if it repeats no pattern; and the least length of a complete pass is 2^d. For the three-account ledger it is 8.*

Proof. A complete pass is a map from its T positions onto the 2^d patterns, so $T \geq 2^d$; at $T = 2^d$ an onto map between finite sets of equal size is one-to-one, and a pass without repeats has $T \leq 2^d$. The reflected binary code of order d lists all patterns with consecutive ones, the last and the first included, differing in one bit [52]; for $d = 3$ it is

$$000 \to 100 \to 110 \to 010 \to 011 \to 111 \to 101 \to 001 \to 000, \tag{4.4}$$

with axis word $(1, 2, 1, 3, 1, 2, 1, 3)$. □

A pass returns to its start by definition. Is return forced? Drop the closing step and look at infinite walks.

Definition 4.6 (Process)**.** A *process* is a sequence $w = (w_0, w_1, \dots)$ in $\{0,1\}^d$. It *completes* if for every tick n some window $w_n, \dots, w_{n+T-1}$ contains every pattern. It is *state-local* if $w_i = w_j$ implies $w_{i+1} = w_{j+1}$: the next pattern is a function of the present one. A *period* is an integer $p > 0$ with $w_{n+p} = w_n$ for all n.

Example 4.7 (A completing walk that does not return)**.** On the square, $d = 2$, repeat $00, 01, 11, 10, 11, 01, 00, 10$ forever. Every step flips one bit, the step $10 \to 00$ between blocks included; every block of four ticks starting at a multiple of four contains all four patterns, so the process completes. At tick 4 it is at 11, so 4 is not a period, although 8 is. It is not state-local: after 01 comes 11 at tick 1 and 00 at tick 5.

Theorem 4.8 (Eight is a period)**.** *Let $d \geq 1$ and let w be a process on $\{0,1\}^d$ that completes.*

(i) If w is state-local, its least period is exactly 2^d; for $d = 3$ it is 8.
(ii) w is state-local if and only if it has period 2^d.

Proof. (i) Every pattern occurs in w. For each pattern q pick i with $w_i = q$ and put $f(q) = w_{i+1}$; state-locality makes f well defined, so $w_n = f^n(w_0)$. Completion from tick 1 makes every pattern equal to some $f(w_{i-1})$, so f is onto, hence a bijection of the finite set $\{0,1\}^d$. Let p be least with $f^p(w_0) = w_0$. Then p is a period, $w_0, \dots, w_{p-1}$ are distinct, so $p \leq 2^d$, and every tick reduces modulo p into $\{0, \dots, p-1\}$, so these p positions carry every pattern and $p \geq 2^d$. Any period has the same covering property, so 2^d is the least. (ii) One direction is (i). If w has period 2^d, the positions $0, \dots, 2^d - 1$ carry all 2^d patterns, each exactly once; so $w_i = w_j$ forces $i \equiv j \pmod{2^d}$ and then $w_{i+1} = w_{j+1}$. □

The theorem uses neither the one-bit step nor a return clause: return follows from reading only the present pattern, and Example 4.7 shows that this condition does the work. Both premises are conditions on a deterministic pattern rule, not consequences of the posting law, under which any of the three accounts may be posted next from any pattern; the rest of this section counts the lawful histories that actually complete a pass. Chapter 1 states the eight-tick floor of the foundation and its scope (Theorem 1.12).

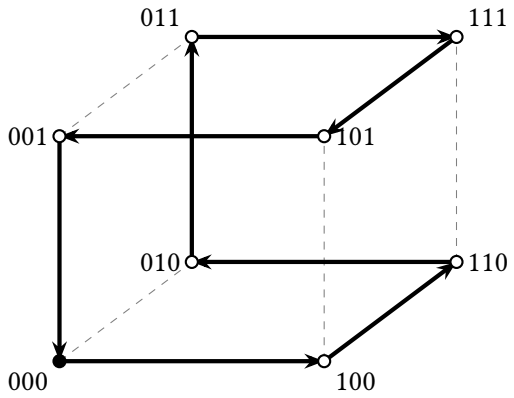


Figure 4.1. The parity patterns of three accounts are the corners of a cube; each post moves the pattern along one edge, the first bit for account 1. Thick arrows: the reflected binary tour (4.4), which occupies every corner and closes in eight ticks. Dashed: the four edges it does not walk, one at each corner. The empty pattern 000 is filled. Twelve tours leave 000 (Theorem 4.9).

Theorem 4.9 (Tours and eight-tick histories). *(i) Exactly* 12 *tours start at* 000. *Their axis words are* (a, b, a, c, a, b, a, c) *and* (a, b, c, b, a, b, c, b), *where* (a, b, c) *runs over the six orderings of the accounts.*

(ii) Exactly $3{,}072 = 12 \cdot 2^8$ *of the* 6^8 *eight-post histories complete a pass at tick* 8, *that is, have a tour as their first eight patterns and return to* 000.

(iii) Exactly 288 *of them end balanced, and exactly* 96 *keep every balance inside one unit cell* $\sigma + \{0, 1\}^3$ *of* $\mathbb{Z}^3$; *each of these* 96 *ends balanced.*

Proof. (i) Let a be the first account posted and $b \neq a$ the second (posting a again would return to 000), and c the third account. In a tour every corner has exactly two tour neighbours. If the third post is on a, the face spanned by e_a, e_b is used up, the fourth post is on c, the last corner must be e_c, and the rest is forced: (a, b, a, c, a, b, a, c). If the third post is on c, the corner $e_a \oplus e_c$ cannot use e_a as a tour neighbour, so its neighbours are e_c and $e_a \oplus e_b \oplus e_c$, and the walk is forced to (a, b, c, b, a, b, c, b). That gives $3 \cdot 2 \cdot 2 = 12$ tours, distinct because one shape posts its first account four times and the other twice. (ii) Completing at tick 8 depends only on the axis word, and the eight sides are free. (iii) In both shapes one account is posted four times and two accounts twice; ending balanced needs two debits among the four posts of the first, $\binom{4}{2} = 6$ ways, and one of each on the others, so 24 side words per tour and $12 \cdot 24 = 288$. A balance stays in an interval of length one exactly when the posts on its account alternate in side, two choices per account: $2^3 = 8$ per tour, 96 in all, and an even number of alternating posts ends at zero. □

Completing a pass in eight ticks is a narrow class of histories: 3,072 of 1,679,616. Ending balanced is a different condition: 44,730 histories end balanced and only 288 of them complete a pass on the way. A unit cell here is a cell of the imbalance lattice $\mathbb{Z}^3$; nothing in this section places the imbalance in space.

4.2.1 First completion at every tick

A history of n posts *closes a pass at tick* $t \leq n$ if $p_t = p_0$ and every pattern occurs among $p_0, \dots, p_t$; it *first completes* at n if it closes a pass at n and at no earlier tick. Let N_n count such histories and A_n the axis words that do.

Theorem 4.10 (First-completion counts). *(i)* $N_n = 2^n A_n$ *for every* n.

(ii) Follow the walk together with its set V *of visited corners: the* 2048 *states* (V, q) *and the step* $(V, q) \mapsto (V \cup \{q \oplus e_k\}, q \oplus e_k)$ *define a transfer operator whose* n*-th iterate, with the completed state* $(\{0, 1\}^3, 000)$ *absorbing, gives* A_n *exactly.*

(iii) $N_n = 0$ *for* $n < 8$ *and for every odd* n, *and* $N_n \geq 2^n$ *for every even* $n \geq 8$: *first completions occur exactly at the even ticks from eight on.*

(iv) A history first completes at tick 8 *if and only if it completes a pass at tick* 8; *so* $N_8 = 3{,}072$.

Proof. (i) The event depends only on the axis word. (ii) The state at tick $t+1$ is the step applied to the state at tick t and the next account; a word first completes at $t + 1$ exactly when its prefix has not closed a pass and its last step enters the completed state. (iii) A first completion at n makes $(p_0, \dots, p_{n-1})$ a complete pass, so $n \geq 8$ by Theorem 4.5. The number of ones in p_t has the parity of t, and $p_n = 000$, so n is even. For $n = 8 + 2j$, the axis word of j copies of $(1, 1)$ followed by $(1, 2, 1, 3, 1, 2, 1, 3)$ first completes at n. (iv) If the walk closes at tick 8, the

positions $0, \dots, 7$ already cover all corners, so they form a tour; a tour is not at 000 at ticks 1 to 7. □

The recursion of (ii) was evaluated in exact integer arithmetic through tick 24, a direct exact computation; Table 4.2 gives the values.

Table 4.2. Axis words A_n and histories $N_n = 2^n A_n$ whose parity walk first completes a pass at tick n, at every even tick from 8 to 24 (Theorem 4.10). Both vanish at every odd tick and below tick 8.

tick n	axis words A_n	histories N_n
8	12	3,072
10	654	669,696
12	13,572	55,590,912
14	198,168	3,246,784,512
16	2,393,178	156,839,313,408
18	25,658,928	6,726,334,021,632
20	253,694,988	266,018,475,737,088
22	2,366,347,374	9,925,180,256,157,696
24	21,130,701,204	354,514,338,330,968,064

The first complete pass is therefore not one event at one time but a family of branches spread over the even ticks, every one of them carried.

One tour has often been read as the event history itself: the reflected binary tour (4.4), run deterministically, as the order in which the first events occur. The invariants of the event ledger do not select it.

Proposition 4.11 (The ledger's invariants do not select the Gray tour)**.** *Label the corners of the cube* $0, \dots, 7$ *in binary. Post, as an event of ratio* φ *followed by its reciprocal, each edge of the Gray tour* $0 \to 1 \to 3 \to 2 \to 6 \to 7 \to 5 \to 4 \to 0$*, or each edge of the two disjoint square faces* $0 \to 1 \to 3 \to 2 \to 0$ *and* $4 \to 5 \to 7 \to 6 \to 4$*. Both event ledgers have* 16 *events, use all eight labels and only cube edges, have the same cost and net flow* 0 *at every label. The first crosses between the faces* $\{0, 1, 2, 3\}$ *and* $\{4, 5, 6, 7\}$*; the second never does.*

Proof. Each schedule posts eight events of ratio φ and their eight reciprocals, so the counts and costs agree, and an event with its reciprocal contributes zero net flow. The Gray tour uses the edges $2 \to 6$ and $4 \to 0$; every edge of the second schedule stays within one face. □

Count, cost and balance therefore do not require one covering tour; a requirement of a single tour is a further premise, and the deterministic Gray tour as the history of the first events is a refuted route.

4.3 What carries the eight: the register, and why writing is not a tick

Beside its ledger, a recognizer can carry an amplitude register: eight complex numbers, one per tick of the eight-tick cycle [111]. Its carrier is $\mathbb{C}^{\mathbb{Z}/8}$, functions f on the eight slots, with the

shift $(Sf)(t) = f(t+1)$, $S^8 = 1$. With $\omega = e^{-i\pi/4}$ the modes $e_k(t) = \omega^{kt}/\sqrt{8}$ are orthonormal and $Se_k = \omega^k e_k$, so four shifts act on mode k as $(-1)^k$. The *core* is the span of the odd modes; a register lies in it exactly when $f(t+4) = -f(t)$ for every slot. One tick of the register shifts it by one slot and then keeps a set A of Fourier modes: A contains the odd modes and not the constant mode 0, and the tick is $R_A = P_A S$ with P_A the orthogonal projector onto the span of the kept modes. A set A is fixed by which of the even modes $2, 4, 6$ it keeps, so there are exactly eight versions of the tick.

Proposition 4.12 (The quarter-turn shift and the open choice)**.** *(i) On the core every version of the tick is the shift: $R_A f = Sf$. Four ticks negate every core register and eight return it.*

(ii) The minimal version $A = \{1, 3, 5, 7\}$ has $P_A = \frac{1}{2}(1 - S^4)$, so $(R_A f)(t) = \frac{1}{2}\big(f(t+1) - f(t+5)\big)$ and $R_A^n = \frac{1}{2}(S^n - S^{n+4})$ for $n \geq 1$: each new amplitude is half the difference of two old ones four slots apart.

(iii) Distinct sets A give distinct ticks.

(iv) On the registers with $\sum_t f(t) = 0$, R_A preserves inner products exactly when $A = \{1, \dots, 7\}$; no version preserves inner products on all registers.

Proof. (i) P_A fixes the odd modes and S maps the core into itself; $S^4 = -1$ on odd modes. (ii) $\frac{1}{2}(1 - S^4)e_k = \frac{1}{2}(1 - (-1)^k)e_k$ is e_k for odd k and 0 for even k; P_A is a polynomial in S, so it commutes with S, and $P_A^2 = P_A$ gives $(P_A S)^n = P_A S^n$. (iii) If A keeps an even mode k that A' drops, $R_A e_k = \omega^k e_k \neq 0 = R_{A'} e_k$. (iv) Every R_A sends the constant register to 0. On the span of the nonconstant modes, R_A is S followed by the projector onto the kept modes, an isometry exactly when every nonconstant mode is kept. □

The recognition operator of the eight-tick foundation acts on its quarter-turn core as this same cyclic shift, with four shifts negating, which is part (i) in the foundation's notation. Which version nature uses is open. Every law stated for the update holds in all eight versions; six equivalent conditions single out the minimal version, for instance that every nonzero output needs a full eight-tick pass to return, but none of them follows from the stated laws, while the unitarity reading of part (iv) selects the full version instead [111, Theorems 6–8]. Replayed from the standard start, the posts of a history leave one of eight registers and keep at most two bits of the order of the posts beyond the record, and nothing of their sides [111, Theorem 16]. We state these two results with their source; the proposition above is the part proved on the page.

Writing a record is not a tick. For every reading of source and recorder states as recorded histories, a reversible writer used twice is not two successive postings, a return to blank is not one posting, and an early signal is not a completed posting (Proposition 2.30, a negative statement about correspondences between two carriers). The writer therefore cannot serve as the clock of the record.

4.4 How branches weigh: the native law

The counts of Section 4.1 weigh every branch alike. A law that says how likely each tick is has to choose a state on which it acts and conditions that it meets. The obvious state is the ledger itself, and it fails at once.

Proposition 4.13 (The ledger cannot carry a balanced law)**.** *Let P be a function on pairs of ledgers with $P(L, L') \neq 0$ only when $L' = L$ or L' is one unit post from L, and let $w > 0$ be any weight with $w(L)P(L, L') = w(L')P(L', L)$ for all L, L'. Then $P(L, L') = 0$ whenever $L' \neq L$.*

Proof. If $P(L, L') \neq 0$, the balance gives $P(L', L) \neq 0$, so each of L, L' is one post from the other. Then every count of L' is at least that of L and conversely, so $L = L'$. □

Counts only rise, so no post on the ledger can be undone, and a law in detailed balance on the ledger never posts. The proposition excludes the ledger; it does not single out the imbalance among other states on which a post can be undone. We therefore adopt three readings, named here once:

(a) the state of one recognizer is its imbalance $s \in \mathbb{Z}^3$, whose six neighbours $s \pm e_k$ are exactly the images of the ledger's six unit posts;
(b) a tick may hold, leaving s unchanged;
(c) the weight of s is the weight φ^{-k} of the recognition level $k = |s|_1$,

$$w(s) = \varphi^{-|s|_1}, \qquad |s|_1 = |s_1| + |s_2| + |s_3|. \tag{4.5}$$

What the foundation supplies is the values φ^{-k} of the level weights, through the golden scaling of the recognizing join (Theorem 1.7). Placing an imbalance at level $|s|_1$ is reading (c). We write $t \sim s$ when $t = s \pm e_k$.

Lemma 4.14 (The long-run weights)**.** $\sum_{s \in \mathbb{Z}^3} w(s) = \varphi^9$*, so*

$$\pi(s) = \varphi^{-9}\varphi^{-|s|_1} \tag{4.6}$$

is a probability law, with $\pi(0) = \varphi^{-9} \approx 0.0131556$. The shell $|s|_1 = k \geq 1$ has $4k^2 + 2$ states. Under π the mean of $|s|_1$ is 6 and the entropy is $15 \log \varphi$.

Proof. With $\rho = \varphi^{-1}$, $\sum_{k \in \mathbb{Z}} \rho^{|k|} = (1+\rho)/(1-\rho) = \varphi/\varphi^{-2} = \varphi^3$, since $1+\rho = \varphi$ and $1-\rho = \rho^2$; cube it. A state on the shell $k \geq 1$ is a composition of k into three parts with signs on the nonzero parts, which gives $4k^2 + 2$. The mean of $|k|$ under $\rho^{|k|}/\varphi^3$ is $2\rho/((1+\rho)(1-\rho)) = 2$, and $-\log \pi(s) = (9 + |s|_1) \log \varphi$ averages to $15 \log \varphi$. □

Definition 4.15 (The four conditions)**.** A *transition law* is a real function P on pairs of states, read as the chance that one tick takes the recognizer from s to t. The four conditions are:

(P1) *Markov*: $P \geq 0$, and the hold and the six posts out of each state have total chance one, $P(s, s) + \sum_{t \sim s} P(s, t) = 1$;
(P2) *hold or one post*: $P(s, t) \neq 0$ only if $t = s$ or $t \sim s$;
(P3) *detailed balance* against the weight: $w(s)P(s, t) = w(t)P(t, s)$ for all s, t;
(P4) *hold-return balance*: $P(s, s)P(t, t) = P(s, t)P(t, s)$ whenever $t \sim s$.

We write $h(s) = P(s, s)$ for the hold. A law with (P1)–(P3) whose six posts out of 0 are equally likely is a *voxel law*.

Condition (P3) says that in the long run the flow across each link is the same both ways; (P4) says that holding at s and then at t is exactly as likely as stepping from s to t and back. The three readings and (P1)–(P4) are adopted premises. None of them is derived; deriving them from the foundation is open, and every statement below about K carries them. Every voxel law leaves π unchanged: by (P3) and (P1), $\sum_s \pi(s)P(s, t) = \pi(t) \sum_s P(t, s) = \pi(t)$.

Example 4.16 (The first link)**.** On the link from 0 to e_1, (P3) gives $P(e_1,0) = \varphi P(0,e_1)$, since $w(e_1)/w(0) = \varphi^{-1}$, and (P4) gives $h(0)h(e_1) = P(0,e_1)P(e_1,0) = \varphi P(0,e_1)^2$. So $P(0,e_1) = \varphi^{-1/2}\sqrt{h(0)h(e_1)}$ and $P(e_1,0) = \varphi^{1/2}\sqrt{h(0)h(e_1)}$: the two holds fix both crossings of the link.

The same computation on every link writes every post through the holds. With the *hold amplitude* $u(s) = \sqrt{h(s)}$ and $c(s,t) = \sqrt{w(t)/w(s)}$, which is $\varphi^{-1/2}$ on an outward step ($|t|_1 = |s|_1 + 1$) and $\varphi^{1/2}$ on an inward one, (P3) and (P4) give $P(s,t) = c(s,t)u(s)u(t)$, and (P1) becomes

$$u(s)\Big(u(s) + \varphi^{1/2}\sum_{t\ \text{inward}} u(t) + \varphi^{-1/2}\sum_{t\ \text{outward}} u(t)\Big) = 1 \qquad (s \in \mathbb{Z}^3). \tag{4.7}$$

Theorem 4.17 (The native law)**.** *Equation* (4.7) *has exactly one positive solution u on $\mathbb{Z}^3$, and it satisfies $1/13 \le u \le 1$. A transition law satisfies (P1)–(P4) if and only if it is*

$$K(s,s) = u(s)^2, \qquad K(s,t) = c(s,t)\,u(s)\,u(t)\ \ (t \sim s), \qquad K(s,t) = 0 \text{ otherwise.} \tag{4.8}$$

Every hold and every post of K has positive chance; $K(gs,gt) = K(s,t)$ for each of the 48 signed permutations g of the three accounts; and the six posts out of 0 are equally likely, so K is a voxel law.

Proof of the reduction. Let P satisfy (P1)–(P4). If $h(s) = 0$, (P4) gives $P(s,t)P(t,s) = 0$ for every neighbour, and (P3) with $w > 0$ makes both factors vanish, so every post out of s vanishes and the row sum (P1) fails. So every hold is positive, $u = \sqrt{h} > 0$ solves (4.7), and (4.8) holds once (4.7) has only one positive solution. Conversely (4.8) satisfies (P1) by (4.7), (P2) by construction, (P3) because $w(s)c(s,t)u(s)u(t) = \sqrt{w(s)w(t)}\,u(s)u(t)$ is symmetric, and (P4) because $c(s,t)c(t,s) = 1$. A signed permutation g maps neighbours to neighbours and preserves $|\cdot|_1$, so $u \circ g$ solves (4.7) and equals u. Existence and uniqueness of u, by a lattice fixed-point argument and a ratio bound, are proved in Appendix B.2. □

We call K the *native law* of one recognizer and $h_0 = K(0,0)$ its hold at the empty record (the state $s = 0$, the imbalance of the empty record and of every balanced ledger). Under the adopted premises the four conditions leave one free number per state, the hold, and (4.7) fixes all of them at once. A tick toward balance is favoured over its reverse by exactly φ, the weight of one unit of imbalance, and the size of each move is the geometric mean of the two holds it joins.

Proposition 4.18 (Each premise is load-bearing; φ enters only through (P3))**.** *Each of the following obeys three of the four conditions and fails the fourth.*

(a) Post from 0 to each neighbour with chance $1/6$, from $\pm e_k$ to 0 with chance $\varphi/6$, along every other link from s to t with chance $\frac{1}{9}\min\{1, w(t)/w(s)\}$, and hold otherwise: it meets (P1)–(P3), has equal posts out of 0, and fails (P4).

(b) The uniform law, holding and making each post with chance $1/7$: it meets (P1), (P2), (P4) and fails (P3).

(c) The doubled law $2K$: it meets (P2)–(P4) and fails (P1).

(d) K on holds and posts and $P(s,t) = w(t)$ on every pair that is neither equal nor neighbouring: it meets (P1), (P3), (P4) and fails (P2).

Proof. (a) The exits from 0 total 1, so $h(0) = 0$ while $P(0, e_1)P(e_1, 0) = \varphi/36 > 0$. Detailed balance holds at 0 because $w(0) \cdot \frac{1}{6} = \varphi^{-1} \cdot \frac{\varphi}{6}$, and elsewhere because $w(s) \min\{1, w(t)/w(s)\} = \min\{w(s), w(t)\}$. (b) $(1/7)^2 = (1/7)^2$, while $w(0)/7 \neq w(e_1)/7$. (c) Rows of $2K$ sum to 2; the other conditions are homogeneous. (d) (P1) and (P4) concern only holds and posts, and $w(s)w(t)$ is symmetric; the value at $(0, 2e_1)$ is nonzero. □

The uniform law meets every condition except detailed balance against w, and it knows nothing of φ. All of the golden structure of K comes in through (P3). Condition (P4) has two equivalent readings among voxel laws: the long-run flow across every link is the geometric mean of the long-run holding masses at its ends, $\pi(s)P(s,t) = \sqrt{\pi(s)h(s)\pi(t)h(t)}$, and the law has the largest entropy rate $\sum_s \pi(s) \sum_o -P(s,o) \log P(s,o)$ among voxel laws; each holds exactly for K [110, Theorem 9]. These readings say what a derivation of (P4) would have to establish. They do not derive it.

4.4.1 Certified values

K has no known closed form. Its values are bounded from both sides by a finite exact computation over the rationals: enclosures of the hold amplitudes, each checked by an exact inequality and proved to contain u at every site of $\mathbb{Z}^3$, are swept over a finite ball (proof of Theorem 4.19). A bound obtained this way is *certified*. Table 4.3 collects the values used in this book.

Table 4.3. Certified values of the native law K, under the adopted premises. The computed column evaluates enclosures proved to contain the true values.

quantity	certified bound	computed
hold at the empty record, $h_0 = K(0,0)$	(0.1839, 0.1840)	0.1839463127
exit from the empty record, $1 - h_0$	(0.8160, 0.8161)	0.8160536873
mean posts in the first two ticks from 0	(1.6494, 1.6495)	1.6494320655
mean posts in the first three ticks from 0	(2.4876, 2.4877)	2.4876973999
hold with one account unbalanced	(0.162, 0.167)	
hold with two accounts unbalanced	(0.147, 0.153)	
hold with three accounts unbalanced	(0.135, 0.141)	
least hold, at $(1,1,1)$	(0.136067, 0.1360695)	
long-run post chance $\nu = \sum_s \pi(s)(1 - h(s))$	(0.85, 0.856)	

Theorem 4.19 (K holds longest at the empty record)**.** *Let $m(s)$ be the number of nonzero coordinates of s. Under the adopted premises the following are certified.*

(i) $0.1839 < h_0 < 0.1840$, *and* $h(s) < 1/6$ *for every* $s \neq 0$, *so* $h(s) + 1/75 < h_0$*: the empty record is the only state with the largest hold.*
(ii) If $m(s) < m(t)$ *then* $h(t) < h(s)$*, with the bands of Table 4.3.*
(iii) Every hold exceeds 0.136067*; in particular every hold exceeds* $1/169$.

Proof. The bounds rest on enclosures of u valid at every site of $\mathbb{Z}^3$. With $r(S) = \frac{1}{2}(\sqrt{S^2 + 4} - S)$, the positive root of $y(y + S) = 1$, write (4.7) as $u(s) = r(S_s)$ with $S_s = \sum_t c(s,t)u(t)$ over the six neighbours t of s. Suppose a table gives intervals $[\ell_t, v_t] \ni u(t)$ at every site. Let S_s^+ be the sum

formed from the upper bounds v_t with each coefficient rounded up to a rational, and S_s^- the sum formed from the lower bounds with each coefficient rounded down; then $S_s^- \le S_s \le S_s^+$. A candidate lower bound $\ell \ge 0$ at s is accepted only after the exact rational check $\ell(\ell + S_s^+) \le 1$, and a candidate upper bound $v \ge 0$ only after $1 \le v(v + S_s^-)$.

Soundness. The function $y \mapsto y(y + S)$ increases in $y \ge 0$ and in S. If $\ell(\ell + S_s^+) \le 1$, then $\ell(\ell + S_s) \le 1 = u(s)(u(s) + S_s)$, so $\ell \le u(s)$; if $1 \le v(v + S_s^-)$, then $u(s)(u(s) + S_s) = 1 \le v(v + S_s)$, so $u(s) \le v$. Starting from $[1/13, 1]$ at every site, which is valid by Theorem 4.17, and sweeping accepted updates over a finite ball while every site outside it keeps the starting interval, every table of this family encloses u at every site, for every choice of precision, radius and number of sweeps. This is a standard interval construction [81].

The certificate. Statements (i) and (iii) compare finitely many rational endpoints of one such table, together with the starting interval outside the ball; each comparison is an exact rational inequality. Statement (ii) uses a table indexed by classes: each site is reduced, with the absolute values of its coordinates sorted and clamped to a fixed box, to one of finitely many classes, and the same update and checks, applied to classes, give one interval per class that encloses u at every site of the class. Each inequality is then a comparison of finitely many rational intervals. The holds are $h = u^2$, and every post of K is $c(s,t)u(s)u(t)$, so the same tables bound every entry of K and every path weight (4.9), from below and above, with rational bounds on $\varphi^{\pm 1/2}$. The pull bounds of Theorem 4.21 and the long-run post chance follow from the same class tables. □

4.4.2 Histories weigh by where they linger

Under a transition law P a history of n ticks from s_0, each tick a hold or one post, traces a path $s_0, \dots, s_n$ and weighs $W_P = \prod_{i=1}^{n} P(s_{i-1}, s_i)$. By (P1) the 7^n histories of n ticks from any start have total weight one.

Theorem 4.20 (Weight from holds)**.** *For $n \ge 1$ and every path whose steps are holds or posts,*

$$W_K(s_0, \dots, s_n) = \sqrt{h(s_0)h(s_n)}\,\sqrt{\frac{w(s_n)}{w(s_0)}}\,\prod_{0<i<n} h(s_i). \tag{4.9}$$

Consequently two histories with the same ends whose interior visits differ in one visit, to s against t, have weights in the ratio $h(s) : h(t)$.

Proof. Each factor is $c(s_{i-1}, s_i)u(s_{i-1})u(s_i)$, with $c = 1$ on a hold; the coefficients telescope to $\sqrt{w(s_n)/w(s_0)}$, each interior amplitude appears twice and each end once. □

K weighs orderings, not ledgers. From the empty record the histories $(+e_1, +e_1, -e_1)$ and $(+e_1, -e_1, +e_1)$ write one ledger and both end at e_1; their interiors are $\{e_1, 2e_1\}$ and $\{e_1, 0\}$, so the second, which passes back through balance, is heavier by the factor $h_0/h(2e_1)$, and by Theorem 4.19 any history whose single differing visit is to balance rather than elsewhere is heavier by more than $27/25$. A weighted history must keep the order of its posts.

4.4.3 Drift toward balance and relaxation

Let $D_i(s) = K(s, s + e_i) - K(s, s - e_i)$ be the mean change of the coordinate s_i in one tick.

Theorem 4.21 (The pull toward balance)**.** *Under the adopted premises, for every state s and account i: $D_i(s) = 0$ if and only if $s_i = 0$; $D_i(s)$ has the sign opposite to s_i; and if $s_i \neq 0$ then $0.06325 < |D_i(s)| < 0.09368$ (certified).*

Proof of the sign. If $s_i = 0$, the reflection of account i fixes s and exchanges $s \pm e_i$, so $D_i(s) = 0$ by the symmetry of Theorem 4.17. If $s_i > 0$, $D_i(s) = -u(s)\big(\varphi^{1/2}u(s - e_i) - \varphi^{-1/2}u(s + e_i)\big)$, and certified enclosures bound the bracket away from zero on both sides; $s_i < 0$ follows by symmetry. □

Every unbalanced account is drawn back toward zero and a balanced account is pushed out: the law pulls back rather than spreads. These are properties of K as a law of chances; they do not identify the pull with a physical force. Write $P^n(a, s)$ for the chance of being at s after n ticks from a, and $d_n(a) = \sum_s |P^n(a, s) - \pi(s)|$, twice the total-variation distance.

Theorem 4.22 (Relaxation to π)**.** *(i) Every voxel law holds at some state. Every voxel law that moves along every link, $P(s, t) > 0$ for all $t \sim s$, has $d_n(a) \to 0$ from every start, and π is its only stationary probability law. K is such a law.*

(ii) For every voxel law the chance of balance at even ticks never rises, $P^{2n+2}(0, 0) \leq P^{2n}(0, 0)$, and never falls below φ^{-9}.

(iii) Under the adopted premises, 0.00836 is a Poincaré gap of K and no gap of K exceeds 0.008398 (certified); for every n and every start a,

$$d_n(0) \leq 9\,(1 - 0.00836)^n, \qquad d_n(a) \leq \sqrt{\varphi^{9+|a|_1} - 1}\,(1 - 0.00836)^n. \tag{4.10}$$

(iv) Under K the chance of a post tends to the long-run post chance $\nu \in (0.85, 0.856)$, and the mean of $|s|_1$ tends to 6.

Proof. (i) Every post changes the parity of $|s|_1$, and $\sum_s (-1)^{|s|_1}\pi(s) = \varphi^{-9}(\sqrt{5} - 2)^3 > 0$, since the even terms of $\sum_k \varphi^{-|k|}$ add to $\sqrt{5}$ and the odd ones to 2; a law that never held would carry the even mass of π onto the odd states and back, and leaving π unchanged it would make the two masses equal. Moving along every link makes the chain irreducible on $\mathbb{Z}^3$, a hold makes it aperiodic, and it has the stationary probability law π; the convergence theorem for such chains [83] gives $P^n(a, s) \to \pi(s)$, Scheffé's lemma upgrades this to $d_n(a) \to 0$, and an irreducible positive recurrent chain has one stationary law. For K every post has positive chance by Theorem 4.17. (ii) Chaining (P3) gives $\pi(s)P^n(s, 0) = \pi(0)P^n(0, s)$, so $P^{2n}(0, 0) = \pi(0)\sum_s P^n(0, s)^2/\pi(s) \geq \pi(0)$ by Cauchy-Schwarz; one tick is self-adjoint for the inner product $\sum_s X(s)Y(s)/\pi(s)$ by (P3) and is a contraction for it, so the sum does not increase. (iii) The gap is certified as described in Appendix B.2; the rate follows from it and the hold floor of Theorem 4.19(iii), with $\sqrt{\varphi^9 - 1} < 9$. (iv) The post chance and $|s|_1$ are functions of the state with $P^n(0, s) \leq w(s)$ at every s, so their means converge with the law to their values under π. □

The proved bound is a guarantee; the actual distance is much smaller. Evaluating proved enclosures of $d_n(0)$, the distance falls below 1 at tick 9, below $1/2$ at tick 26, below $1/10$ at tick 99 and below $1/1000$ at tick 456, and it never increases afterwards [110]. The long-run contraction per tick, $\lambda^{1/2} = \lim d_N^{1/N}$, lies between 0.879 and 0.99164 on proofs alone; an

evaluation of the enclosures at ticks 800 and 802 that is not itself proved would, if exact, raise the lower end to 0.9898. No settling bound can be uniform in the start: from a at least $|a|_1$ ticks pass before any mass reaches balance. For large starting imbalances the time to settle grows linearly, at between about 12.03 and 15.81 ticks per unit of the largest account [110, Theorem 33].

Corollary 4.23 (Return to the empty record)**.** *Under the adopted premises the recognizer returns to every state a with certainty, after $\varphi^{9+|a|_1}$ ticks on average; to the empty record after*

$$\varphi^9 = 38 + 17\sqrt{5} \approx 76.013 \text{ ticks.} \tag{4.11}$$

Proof. Kac's formula [63]: in an irreducible positive recurrent chain the mean return time to a is $1/\pi(a)$. Expanding φ^9 with $\varphi^2 = \varphi + 1$ gives $34\varphi + 21 = 38 + 17\sqrt{5}$. □

The return is certain and its mean is long, but the distribution is skewed: evaluated, not proved, more than half of the returns happen by tick 8 and 99% by tick 717.

4.4.4 Completion is late under the native law

Under K a tick may hold, and a hold changes nothing about a pass: the pattern stays, the set of occupied corners does not grow, and a word closes a pass at tick $t + 1$ exactly when it closes one at t. So a first completion under K ends with a post, makes at least eight posts, and at tick 8 is one of the 3,072 hold-free histories of Theorem 4.9. Let F_n be the probability that the first pass completes at tick n and L_n the probability that no pass has completed by tick n.

Proposition 4.24 (Completion balance)**.** *For every voxel law and every $n \geq 0$, $\sum_{t \leq n} F_t + L_n = 1$; for $N \leq M$, $\sum_{t=N+1}^{M} F_t + L_M = L_N$, so a first pass after tick N weighs at most L_N.*

Proof. At $n = 0$ the empty word is live and $F_0 = 0$. Each live word of length n has seven one-tick extensions, each either first completing at $n + 1$ or live at $n + 1$, and by (P1) their weights add up to its weight; a word that has completed has no extension of either kind. So $F_{n+1} + L_{n+1} = L_n$; add. □

Under K the weight of a first pass is a sum of path weights (4.9), each a product of holds. Its enclosures are proved sound: for every table of the enclosure method of Theorem 4.19, rational lower and upper bounds computed from the table contain F_n and L_n at every tick, and a computed upper bound on L_n below $\frac{1}{2}$ proves that more than half the weight has completed by tick n. The endpoints themselves, and the comparisons between them, are direct exact computations in integer arithmetic on a ball of radius 56 (through tick 33) and of radius 82 (through tick 83) [109]. Table 4.4 gives the result. One weighted statement is proved outright, its finite certificate included: the 96 quickest passes that stay in one unit cell, each weighing h_0 times the holds at the seven other corners of its cell, carry more than 0.3506 and less than 0.3507 of the weight of the 288 balanced quickest passes, slightly more than their third by count.

The values of Table 4.4 and of this paragraph are computed endpoints of proved enclosures. The quickest pass has probability 0.000409204. Odd ticks carry weight, $F_9 > 0$, because a hold delays the walk by one tick without moving it, so the parity argument of Theorem 4.10 no

Table 4.4. The first pass under the native law, under the adopted premises: the probability F_n that it completes at tick n and the probability $C_n = 1 - L_n$ that it has completed by tick n, rounded to six significant figures from certified enclosures of relative width below 5×10^{-17}. $F_n = C_n = 0$ for $n < 8$.

n	F_n	C_n	n	F_n	C_n
8	0.000409204	0.000409204	24	0.0296349	0.253064
9	0.000502596	0.000911800	27	0.0308483	0.344870
10	0.00218757	0.00309937	32	0.0289589	0.495021
12	0.00558383	0.0116827	33	0.0281821	0.523203
16	0.0152785	0.0567528			

longer applies. F_n rises at every tick from 8 to 27 and falls at every tick from 27 to 83; it is largest at tick 27 among all ticks. Half the weight has completed its first pass by tick 33 and not by tick 32: the median first completion is tick 33, about four times the least period, and only 5.7% of the weight has completed by tick 16.

So eight is the least period of a complete pass, not the period of a recognizer under its law. The sentence "every recognizer runs an eight-tick cycle" is false as written: under K about 0.04% of the weight completes a pass at tick 8.

4.5 The arrow of time sits in the record

A reversible law has no preferred direction, yet a recognizer's history does. Where does the direction sit? The answer has two parts on the account ledger and the imbalance carrier, and a third on another carrier.

4.5.1 The record grows; the imbalance returns

Proposition 4.25 (Ledger and imbalance)**.** *On the account ledger with three accounts:*

(i) a ledger with n posts is written only after exactly n posts, and along any history no ledger recurs;

(ii) some history of n posts ends at the imbalance q if and only if $|q|_1 \leq n$ *and* $|q|_1 \equiv n \pmod 2$*; balance is reached at every even n, and of the* 6^8 *histories of eight posts* 44,730 *end at balance while none ends on a ledger it held before;*

(iii) among ledgers with nonnegative counts, the empty ledger is the only one that no post leads to;

(iv) if a tick may hold, the ledger counts posts, not ticks: it bounds the tick from below and still never returns to a value it has left.

Proof. (i) Each post raises the total of the counts by one. (ii) is Theorem 4.3 with (i). (iii) is Theorem 2.15(iii). (iv) A hold leaves the ledger unchanged and a post raises its total by one. □

The record is a clock in the weakest sense: it fixes how many posts have been made, and it never returns. The imbalance is not: whatever it reaches at one tick it reaches again two ticks

later, and a state that recurs cannot by itself order the moments of a history. One relation shows the split: each post raises the ledger total by one, so the posting relation has no cycle of any length, while a debit post followed by a credit post on the same account returns the balances exactly and the record grows by two. One relation is cyclic on the balances and acyclic on the record; Chapter 2 proves the record's side as the origin of time (Theorem 2.12).

4.5.2 Arrows that fail

Two older candidates for the direction of time are refuted.

Proposition 4.26 (Cost cannot orient time)**.** *$J(x) = J(1/x)$ for every $x > 0$, and for every ledger L and unit post $P_{k,s}$ the count-by-count cost of the change $L \to P_{k,s}L$ equals the cost of the reverse change $P_{k,s}L \to L$. Hence cost assigns the same value to a change and its reverse, and cost alone orders no pair of moments.*

Proof. $J(1/x) = \frac{1}{2}(x^{-1} + x) - 1 = J(x)$. The reverse change negates every count difference, and $J_{\log}(t) = \cosh t - 1$ is even. □

The direction of a post sits in its legality, that counts never decrease, not in its cost; the cost $2J(r)$ of the first tick (Chapter 3) is the same read either way. On the configuration carrier of Section 2.5 the descent of the cost runs toward its minimum, the attractor of the dynamics and not its origin, and the defect is not the currency of time (Propositions 2.22 and 2.23).

A horizon does not supply the direction either, and the reason locates the arrow precisely.

Proposition 4.27 (A horizon supplies no orientation; a record does)**.** *(i) A seam between two sides A and B whose recorded count is fixed by the sum $\log_2 |A| + \log_2 |B|$ of the two sides' counts is unchanged by exchanging the sides: it names no side.*

(ii) On a cell carrying one bit at each of its eight vertices, whose record is the closure bit posted on each of its six faces, the record potential w, the number of posted face bits, satisfies $0 \le w \le 6$. Every achievable record has an achievable complement, every bit negated, with potential $6 - w$; so every direction read from differences of w is reversed by a symmetry of the achievable records.

(iii) A count bounded below and unbounded above, such as the length of a record, admits no order-reversing relabelling.

Proof. (i) Addition commutes. (ii) Negating every bit turns w posted bits into $6 - w$. The two faces of one axis constrain the record only through their exclusive or, which complementing preserves, so the achievable records are closed under complement; a finite check over the $2^8 = 256$ vertex configurations of the cell confirms it. (iii) An order-reversing bijection would send the least element to a greatest one, and an unbounded count has none. □

A finite aperture gives one-sidedness and a capacity bound, but a bounded potential is reversible, so it cannot give orientation. What orients is the missing ceiling of the record. We state (ii) at the scope of its cell; a horizon that distinguishes its sides by a further rule is not covered.

4.5.3 The information arrow

The law of the imbalance carries a second direction. Take a voxel law P (Definition 4.15), which we call *reversible* because of (P3); K is one. Write $P^n(a,\cdot)$ for the law after n ticks from a, supported in $B_{|a|_1+n}$, where $B_N = \{s : |s|_1 \le N\}$. The information the law still carries about its start, against π, is the relative entropy in nats

$$D_n(a) = \sum_s P^n(a,s) \log \frac{P^n(a,s)}{\pi(s)}, \qquad D_0(a) = (9 + |a|_1) \log \varphi, \tag{4.12}$$

the value at $n = 0$ being $\log(1/\pi(a))$ by (4.6). From the empty record $D_0(0) = 9 \log \varphi \approx 4.3309$ nats, about 6.248 bits.

To say what a record of one moment loses, use a resample. Let X be a finite set, ν a law on X positive at every point, and $r : X \to R$ any map, the *record* of a moment (here a summary of a path, not a list of posts), with fibre $F_x = \{y : r(y) = r(x)\}$. The *resample* through r keeps the law of the record and redraws within each fibre from the reference, $(R_r\mu)(x) = \mu(F_x)\nu(x)/\nu(F_x)$, and the *loss* of the record is $L_r(\mu) = D(\mu \,\|\, R_r\mu)$.

Lemma 4.28 (What a record loses)**.** *$L_r(\mu) = D(\mu\|\nu) - D(r_*\mu\|r_*\nu)$, with r_* the image law. $L_r(\mu) \ge 0$, with equality exactly when μ/ν is constant on every fibre. If $r' = g \circ r$ then $L_{r'}(\mu) \ge L_r(\mu)$, and a one-to-one record loses nothing. $R_r(R_r\mu) = R_r\mu$.*

Proof. $R_r\mu(x)/\nu(x) = (r_*\mu/r_*\nu)(r(x))$; expand $D(\mu\|R_r\mu)$. The rest is Gibbs' inequality, the data-processing inequality for relative entropy [27], and the fact that resampling again redraws the same mass from the same reference. □

Some law loses information through a record exactly when the record is not one-to-one, and coarsening a record can only raise the loss: forgetting is the source of the loss. Now fix a, a number $n \ge 0$ of ticks already made, and a stretch of $m \ge 1$ further ticks. On the finite set X of paths $(s_0, \dots, s_m)$ with positive steps and present in B_N, $N \ge |a|_1 + n + m$, put the forward path law $\mu(x) = P^n(a, s_0) \prod_i P(s_i, s_{i+1})$ and the stationary path law $\nu \propto \pi(s_0) \prod_i P(s_i, s_{i+1})$, and let $r_j(x) = s_j$ record the j-th moment.

Theorem 4.29 (The window identity)**.** *For every voxel law, start a, $n \ge 0$, $m \ge 1$, $N \ge |a|_1 + n + m$ and $0 \le j \le m$,*

$$L_{r_j}(\mu) = D_n(a) - D_{n+j}(a).$$

The start of a stretch loses nothing; the present loses what the m ticks shed; and the loss of the j-th moment does not depend on what follows it.

The proof is in Appendix B.2. The two path laws share every step chance and differ only in the law of the start, so keeping the start and redrawing the rest loses nothing, while keeping a later moment loses what the ticks before it shed. A single moment is a worse summary of the path the later it lies.

Theorem 4.30 (Every tick loses)**.** *Let P be a voxel law, a a start and $n \ge 0$.*

(i) $D_{n+1}(a) \le D_n(a)$. If P moves along every link, then $D_{n+1}(a) < D_n(a)$ for every a and n, $D_n(a) \to 0$, and the losses of all ticks add up to $D_0(a) = (9 + |a|_1) \log \varphi$; from the empty record, $9 \log \varphi$.

(ii) For every map g on states, the record $g(s_1)$ of a one-tick stretch loses at least $D_n(a) - D_{n+1}(a)$. This holds for K.

Proof. (i) Monotonicity is Theorem 4.29 with $j = m = 1$ and Lemma 4.28. By that lemma a tick loses nothing only if $q = P^n(a, \cdot)$ has q/π equal at any two states with a common successor. Let s be a state of largest $|s|_1 = R$ with $q(s) > 0$, pick k and a sign σ with $|s_k + \sigma| = |s_k| + 1$, and put $t = s + \sigma e_k$, $s' = s + 2\sigma e_k$. Both s and s' step to t with positive chance, while $q(s)/\pi(s) > 0 = q(s')/\pi(s')$. For the limit, $\log x \le x - 1$ and chained detailed balance give $D_n(a) \le P^{2n}(a, a)/\pi(a) - 1$, which tends to 0 by Theorem 4.22(i); the losses telescope. (ii) Lemma 4.28 with $r' = g \circ r_1$. □

At the outer edge of the law's support one predecessor of an outer state is occupied and the other is not; so every tick sheds information, from every start and at every age, for every law that can make every post. Reading the present more coarsely, for instance by its size $|s|_1$ alone, only loses more. Monotonicity is the classical second law for Markov chains [27], and the window identity at the present is the known identification of entropy production with the relative entropy between a path law and its time reverse [48, 69]. What is particular here is the identity at every moment of a stretch, the strictness without any condition on the holds, and the values below.

Theorem 4.31 (The first tick and its ceiling)**.** *For every voxel law with hold h at balance, the first tick from the empty record loses*

$$D_0(0) - D_1(0) = F(h) := -h \log h - (1 - h) \log(1 - h) + (1 - h) \log \frac{6}{\varphi}.$$

With $h_* = \varphi/(\varphi + 6) \approx 0.21240$, $F(h) = \log(6\varphi - 5) - d(h\|h_*)$, where d is the relative entropy of the two-point laws $(h, 1 - h)$ and $(h_*, 1 - h_*)$. So $F(h) \le \log(6\varphi - 5) \approx 1.549307$, with equality exactly at $h = h_*$, and F increases on $[0, h_*]$. Over n ticks from balance no voxel law sheds more than

$$\Lambda_n = \log \sum_{s \in B_n} w(s) = \log\Big(1 + \sum_{k=1}^{n} (4k^2 + 2)\varphi^{-k}\Big),$$

with $\Lambda_1 = \log(6\varphi - 5)$, $\Lambda_2 = \log(31 - 12\varphi) \approx 2.44959$, $\Lambda_3 = \log(64\varphi - 83) \approx 3.02306$, and Λ_n rising to $9 \log \varphi$; at $n = 2$ no voxel law meets the ceiling.

Proof. After one tick from balance the law sits at 0 with chance h and at each unit imbalance with chance $(1 - h)/6$; with $\pi(0) = \varphi^{-9}$ and $\pi(e) = \varphi^{-10}$ this gives $D_1(0) = h \log(h\varphi^9) + (1 - h) \log\big((1 - h)\varphi^{10}/6\big)$ and the formula for F. Since $1 - h_* = 6/(\varphi + 6)$ and $(\varphi + 6)/\varphi = 6\varphi - 5$, expanding $d(h\|h_*)$ gives the second form. The ceiling and the case $n = 2$ are proved in Appendix B.2. □

Corollary 4.32 (The native law's first ticks)**.** *Under the adopted premises, from the empty record K loses between 1.54679 and 1.54682 nats on its first tick, between 35.715% and 35.716% of the total $9 \log \varphi$, and falls short of the ceiling $\log(6\varphi - 5)$ by between 0.00248 and 0.00252 nats; $2.78408 < D_1(0) < 2.78412$. Its second tick loses less than 0.90281 nats, less than its first.*

Proof. $0.1839 < h_0 < 0.1840 < h_*$ and F increases on $[0, h_*]$, so $F(0.1839) < D_0 - D_1 < F(0.1840)$, with $F(0.1839) = 1.546793\ldots$ and $F(0.1840) = 1.546811\ldots$; divide by $9 \log \varphi = 4.330906\ldots$ and subtract from $\log(6\varphi - 5) = 1.549306\ldots$. For the second tick, $D_0 - D_2 \leq \Lambda_2 < 2.44960$. □

The native law, fixed by its four conditions without reference to this optimum, holds at balance with chance just below h_* and sheds more than a third of its total budget on its first tick. The information is measured in nats against the stationary weights. It is information, not heat; it becomes entropy production only on a thermal model of the tick, which Chapter 9 states with its own premises.

Theorem 4.33 (A tick is not a forget-and-redraw step)**.** *(i) For every record and positive reference on a finite set, R_r changes a law at most once: if $R_r\mu_0 = \mu_1$ and $R_r\mu_1 = \mu_2$, then $\mu_2 = \mu_1$.*

(ii) Let P be a voxel law that moves along every link, a a start, $n \geq 0$, $N \geq |a|_1 + n + 2$ and $p_k = P^k(a, \cdot)$ on B_N. No record on B_N and no positive reference give $R_r p_n = p_{n+1}$ and $R_r p_{n+1} = p_{n+2}$.

(iii) For such a law, one tick from a sure start a is never a resample through a record that tells a apart from every other state.

All three hold for K.

Proof. (i) is the last statement of Lemma 4.28. (ii) By (i), $p_{n+2} = p_{n+1}$, so $D_{n+2}(a) = D_{n+1}(a)$, which Theorem 4.30 excludes. (iii) The fibre of a is $\{a\}$, so the resample of the point mass at a is that point mass, which equals $P(a, \cdot)$ only if $P(a, a) = 1$, impossible when every post has positive chance. □

A fixed coarse-graining followed by redistribution within cells reaches its result in one step and then stops. The recognizer's law keeps shedding information at every tick, so no single coarse-graining with a fixed reference produces two of its ticks. The tick itself is a reversible move of the state; the loss belongs to what a record of one moment retains about the path. This is the sense in which the direction of time sits in the record: the record never returns, and what a record of the present retains about the path falls strictly at every tick.

4.5.4 The arrow read by a record, on the boundary-theory carrier

The Recognition Boundary Theory manuscript [93] works on its own carrier (carrier (j); Section 2.9), and no correspondence theorem joins it to the imbalance carrier, so we combine nothing across the two. On that carrier a long record carries the present onto pointer lines, where the present moves as a classical Markov chain with five closed classes. The chain is reversible exactly on the three classes that hold its sure subjects; on the other two it has one-way steps, its stationary current circulates, and its irreversibility, the total-variation distance between the stationary two-step law and its reverse, is $\frac{23}{24}$ and $\frac{20}{21}$ [93, Theorem 11.4]. The record reads this direction: the pair of lines the present occupies at two consecutive stages is named by the written labels up to an error $30 \cdot 4096\,(41/48)^{\lfloor s/2 \rfloor}$ after s further postings, and a fixed read of that pair tells forward from backward with a bias tending to the class weight times the irreversibility, while an oscillating mode gives a read that tells the stage parity but

no direction: a clock, not an arrow [93, Theorem 11.7]. That manuscript reads the circulation as an arrow of time inside its ledger and names that reading as an identification; we report it at that scope.

4.6 Clocks

The orders of this chapter are orders, not durations. Which clocks does the account have? Four results answer, each on its own carrier, and one closes off the hope that the native law sets a tick length by itself.

Rank is a chart. On the record cell of Chapter 5, a linear chart of the sixteen records takes the rank of a record, its number of postings, as its time coordinate; the cell's tick clock equals rank given the premises stated there. A chart is a choice of coordinates: it orders records and counts postings, it does not supply seconds, and a duration needs the unit premises of Chapter 9. Reading the tick count as time is an adopted identification wherever it is used. The postings themselves do not fill the cell's causal cone (Proposition 5.9). On the clock-photon lattice, by contrast, the group velocities of the photon modes generate the whole future cone in units with $C_E = 1$; that is a cone of modes, not of signals, and no theorem relates it to the record cone.

No global serial clock. A single clock for many recognizers would let at most one of them post per tick. The native law forbids it.

Proposition 4.34 (No global serial clock)**.** *Let V be a finite set of recognizers, and let a joint law give, at every reached state, each recognizer a hold or one of its six posts, with chances that are nonnegative and add to one.*

(i) If $|V| \geq 2$ and every recognizer's own moves have the chances of K at its own imbalance (the native target of Chapter 8), then at every state some move in which two recognizers post at once has positive chance: no law with at most one post per tick gives two recognizers their native laws.

(ii) If instead each recognizer's native chances of posting are scaled by a rate $a \geq \alpha_0 > 0$ and at most one recognizer posts per tick, then $\frac{51}{100}\,|V|\,\alpha_0 \leq 1$.

Proof. Under K every state is left with chance at least 0.51 (Theorem 4.19: $1 - h(s) > 0.8160$ at 0 and $> 5/6$ elsewhere). (i) Two recognizers each post with chance above $\frac{1}{2}$, so the events "v posts" and "v' posts" cannot be disjoint. (ii) With at most one post per tick, the events "v posts" are disjoint across v, and each has chance at least $0.51\,\alpha_0$. □

A serial schedule needs global data: keeping one slot as recognizers are added forces every rate down with $|V|$, a number no single recognizer holds. There is no shared global tick.

Refinement clocks. A record can be refined: each edge of a relation at depth n splits into two edges at depth $n+1$. Let R_n be a relation on a set A_n at each depth, $\iota_n : A_n \to A_{n+1}$ refinement maps such that every edge $v \to w$ of R_n has a two-edge realization $\iota_n v \to m \to \iota_n w$ in R_{n+1}, and $c_n : A_n \to \mathbb{R}$ clock readings that agree on refined endpoints, $c_{n+1}(\iota_n v) = c_n(v)$, and advance by the same increment τ_n along every edge of depth n.

Proposition 4.35 (Refinement clocks)**.** *Under these conditions, if every depth has an edge then $\tau_n = \tau_0/2^n$; in particular no nonzero increment is the same at every depth. And no map of every edge of every depth into one primitive relation, carrying a counter that rises by a fixed $\delta \neq 0$ along each primitive edge, can keep the counter on refined endpoints.*

Proof. An edge at depth n advances the clock by τ_n; its realization advances it by $2\tau_{n+1}$ between the same readings. For the counter, the coarse edge advances it by δ and its two-edge realization by 2δ, between endpoints where the counter agrees. □

Refinement forces dyadic increments, up to a base unit that it does not fix, and no single native relation with a fixed nonzero integer step realizes every depth.

Clocks read from the state. A clock need not be an independent oscillator; a state can carry its own. On the self-consistent photon carrier of Chapter 6, near the vacuum the state is a function of its own two-photon reading at any one site, with a local inverse (Theorem 6.11). On the selected vacuum of the same carrier, the real cross coherence between the initial source-vacuum amplitude and the photon amplitude along the initial velocity is a phase-invariant bounded reading of the state that starts at 0 with derivative exactly 1; so near the start it increases strictly and has a local inverse, and it orders the events of the trajectory without an independent oscillator. On the source and recorder carrier of Chapter 2, started from the coherent seed with supplied profiles $z_\pm$, the Born weight $p(t)$ of the pair-coordinate one-photon component satisfies $p(t)/t^2 \to \|\omega(z_+ - z_-)\|^2/32$ as $t \to 0$, and for $C_E > 0$ and $z_+ \neq z_-$ it increases strictly on an initial interval: a local probability clock. Each of these is local and defined near one instant. None is a global time, and none fixes a unit.

No clock built from the native law fixes a tick length. One could try to read a physical tick off K by letting a clock decide when the recognizer acts.

Proposition 4.36 (Clocks built from K)**.** *Under the adopted premises:*

(i) if at each tick the recognizer acts by K with a fixed chance $r \in (0, 1]$ and holds otherwise, its law after n ticks is $\sum_{k\le n} \binom{n}{k} r^k (1-r)^{n-k} P_K^k$, the law of K at a binomial number of its own ticks, and its long-run post chance lies strictly between $0.85\,r$ and $0.856\,r$;

(ii) if the chance $\rho(s) \in [0, 1]$ of acting depends on the recognizer's own imbalance, π stays stationary if and only if ρ is constant.

Proof. (i) The slowed law is $rK + (1-r)I$; expand its n-th power, since K and I commute. Its stationary law is π, and its post chance is rv. (ii) The slowed law $\rho K + (1-\rho)I$, with ρ acting on the row, leaves π unchanged exactly when $\sum_s \pi(s)\rho(s)K(s,t) = \pi(t)\rho(t)$ for all t; by detailed balance the left side is $\pi(t)\sum_s K(t,s)\rho(s)$, so $K\rho = \rho$ and hence $K^n\rho = \rho$ for every n. Since ρ is bounded and $d_n(t) \to 0$ by Theorem 4.22, $(K^n\rho)(t) \to \sum_s \pi(s)\rho(s)$ at every t, so ρ is constant. The converse is immediate. □

A steady clock only stretches the native tick, and a clock that reads the recognizer's own imbalance either changes its long-run law or is steady; neither fixes how long a tick lasts. A duration in seconds needs a dimensional input (Chapter 9).

Chapter 5 asks what geometry the smallest complete record carries: whether the record cell of four accounts, with rank as its time, is a spacetime cell.

Chapter 5

The first spacetime cell

An expanding universe is a statement about geometry. Space has to stretch, light has to travel along a cone, and something has to say which separations are timelike, which are null and which are spacelike. The records of Chapters 2–4 have an order and a count, and no geometry. This chapter asks what the smallest four-dimensional cell of records fixes about time and signature, whether light can be placed on it, and how the standard gravitational action of the lattice behaves near flatness. The record cell, the electromagnetic lattice field and the Euclidean Regge lattice are separate constructions; we set their results side by side and do not chain them. The results on the record cell and on Regge's action appeared first in [94].

5.1 Sixteen records and their twenty-four histories

Start with two accounts. The records are the four subsets of $\{1, 2\}$: nothing posted, account 1 posted, account 2 posted, both posted. Put them at the corners of a unit square. One record extends another when it holds every posting the other holds. The empty record is extended by every record and the full one extends every record, while $\{1\}$ and $\{2\}$ are incomparable. A history posts one account and then the other, so there are two histories, and their triangles $\emptyset, \{1\}, \{1, 2\}$ and $\emptyset, \{2\}, \{1, 2\}$ cut the square along the diagonal from the empty record to the full one. The other diagonal joins two records neither of which extends the other. Of the two ways to cut a square into triangles, the record order selects one. The same happens in four dimensions.

Definition 5.1 (The record cell)**.** Let $e_1, \dots, e_4$ be the unit vectors of $\mathbb{R}^4$, write $e_B = \sum_{a \in B} e_a$ for $B \subseteq \{1, 2, 3, 4\}$, and fix a vertex $x \in \mathbb{Z}^4$. The *record* $x + e_B$ is the state in which exactly the accounts of B hold a posting, each account being posted at most once; $x + e_{B'}$ *extends* $x + e_B$ when $B \subseteq B'$, and the *rank* of $x + e_B$ is $|B|$. The sixteen records with this order form the *record cell.* A *history* of the cell is a sequence of five records, each strictly extending the one before. A *displacement* is a vector $y \in \mathbb{R}^4$; the step from $x + e_B$ to $x + e_{B'}$ has displacement $e_{B' \setminus B}$.

The record cell is carrier (e) (Appendix A). Its four accounts hold zero or one posting each, while the ledger of Chapters 2–4 has three accounts, two sides and integer counts; no correspondence between the two carriers is proved, and nothing proved on the ledger is used here.

For an ordering σ of $\{1, 2, 3, 4\}$ the *chain simplex* has the five vertices $x,\ x + e_{\sigma_1},\ x + e_{\sigma_1} + e_{\sigma_2},\ x + e_{\sigma_1} + e_{\sigma_2} + e_{\sigma_3},\ x + e_{\{1,2,3,4\}}$. The 24 chain simplices with base x fill the unit four-cube; this is the subdivision of Freudenthal and Kuhn [47, 70], the one lattice gravity uses on the cubic lattice [99]. The edge from x to $x + e_{\{1,2,3,4\}}$ is the *main diagonal.*

Theorem 5.2 (Histories are the chain simplices)**.** *On the record cell:*

(i) every history starts at the empty record x, ends at the full record $x + e_{\{1,2,3,4\}}$ *and adds exactly*

one posting per step; the histories are exactly the vertex sequences of the 24 *chain simplices, one history for each order of the four postings;*

(ii) no six records form a sequence in which each strictly extends the one before;

(iii) any two records one of which extends the other lie on a common history;

(iv) if a function q on records vanishes on the empty record, takes the value 1 *on each record of rank one, and adds over disjoint postings, $q(x + e_{B\cup C}) = q(x + e_B) + q(x + e_C)$ for disjoint B, C, then $q(x + e_B) = |B|$; the edge of a chain simplex from its i-th to its j-th vertex then has q-length $j - i$;*

(v) reflecting the cube in one axis gives a triangulation isometric to the chain subdivision, and none of its simplices has the records of a history as vertices.

Proof. (i) A strict extension adds at least one posting, so the ranks of the five records of a history strictly increase inside $\{0, 1, 2, 3, 4\}$. They are therefore $0, 1, 2, 3, 4$: the history starts empty, ends full, and adds one posting per step. It is fixed by the order in which the four postings are added, and that order lists the vertices of a chain simplex; conversely the vertices of every chain simplex form a history. (ii) Six strictly increasing ranks do not fit in $\{0, \dots, 4\}$. (iii) If $B \subseteq B'$, post the accounts of B, then those of $B' \setminus B$, then the rest. (iv) Splitting B into single postings gives $q(x+e_B) = |B|$. The vertices $i < j$ of a chain simplex differ by $j-i$ postings. (v) The reflection in the first axis sends $x + e_B$ to $x + e_{B\triangle\{1\}}$ and preserves Euclidean distances. It sends the first vertex x of a chain simplex to $x + e_1$ and the last vertex to $x + e_{\{2,3,4\}}$, two records neither of which extends the other; the records of a history are totally ordered by extension. □

So the record order selects the subdivision among the triangulations of the cube, up to the permutations of the accounts and the inversion $B \mapsto \{1, 2, 3, 4\}\setminus B$ read along reversed histories; a single reflection is not among these symmetries. An edge of the subdivision joins $x + e_B$ to $x + e_{B'}$ with $B \subsetneq B'$, and its displacement e_C, $C = B' \setminus B$, falls in one of 15 classes, one for each nonempty set C of postings. Part (iv) is the Euclidean reading of lengths: one unit per posting, so an edge spanning m postings has squared length m.

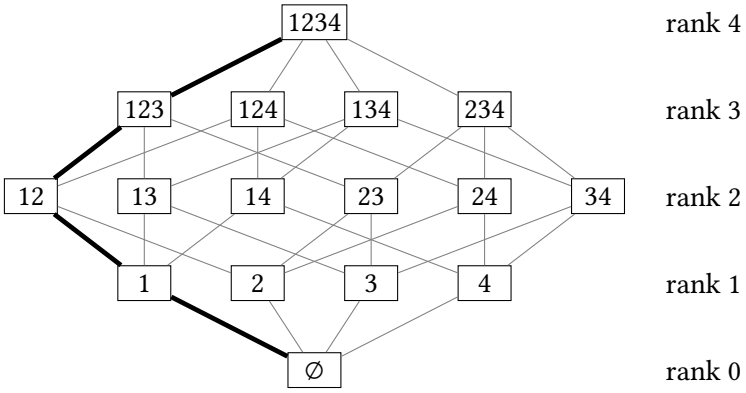


Figure 5.1. The record cell. Each box is a record, labelled by the set of accounts that hold a posting; lines join records that differ by one posting. The thick chain is the history that posts accounts $1, 2, 3, 4$ in that order. The 24 histories are the 24 chains from the empty record at the bottom to the full record at the top, and each is a simplex of the chain subdivision of the four-cube (Theorem 5.2).

5.2 The record cell is not yet a spacetime cell

An *interval* is a quadratic form $Q(y) = y^\top G y$ on displacements, with G a real symmetric 4×4 matrix. A history is *causal* for Q when each of its steps has $Q \le 0$. Lattice field theory usually reads one lattice axis as time and the other three as space. On the record cell that reading fails at once.

Proposition 5.3 (The order singles out no axis)**.** *On the record cell:*

(i) along every history each of the four coordinates is nondecreasing;
(ii) each account is posted at each of the four steps by exactly six of the 24 histories, and every permutation of the accounts maps histories to histories;
(iii) every history is causal for Q exactly when every single posting is, $G_{aa} \le 0$ for each account a; so if some posting is spacelike, no history is causal; this excludes the Euclidean count $\sum_a y_a^2$ and the Minkowski interval $y_1^2 + y_2^2 + y_3^2 - y_4^2$ with the fourth account as time, and every reading that gives three postings unit positive length;
(iv) if G has no cross terms between accounts and every history is causal, then $Q \le 0$ everywhere, so no linear time function τ makes Q positive on the nonzero displacements τ does not advance.

Proof. (i) follows from Theorem 5.2(i). (ii) The histories that post account a at step k are fixed by the order of the other three accounts, $3! = 6$ of them; a permutation of the accounts maps orders to orders. (iii) Every step of a history is a single posting e_a, with $Q(e_a) = G_{aa}$, and every posting is a step of some history. (iv) Without cross terms $Q(y) = \sum_a G_{aa} y_a^2 \le 0$. Every linear τ on $\mathbb{R}^4$ vanishes on a nonzero displacement. □

In the axis reading three of the four steps of every history are spacelike moves with no advance of time, and the order, symmetric in the accounts, chooses no account as time. The cell needs a time direction that every posting advances.

5.3 The cell's time is its rank

Definition 5.4 (Tick clock and spacetime interval)**.** A *tick clock* is a function T on records with $T(x) = 0$ that rises by one at every step of every history. An interval Q together with a linear time function τ on displacements is a *spacetime interval of the cell* when

(a) every history step advances τ and is causal, $Q \le 0$;
(b) Q is positive on every nonzero displacement that τ does not advance, so that these displacements form a space;
(c) Q is negative on some displacement.

The interval is *name-blind* when $Q(y_{\sigma(1)}, \dots, y_{\sigma(4)}) = Q(y)$ for every permutation σ of the accounts.

The first record-cell premise of Chapter 1 enters here. *We adopt that the cell's interval is name-blind.* Proposition 5.3(ii) makes it natural, since the histories are symmetric under every exchange of accounts, but nothing derives it, and Section 5.5 shows a spacetime interval that

is not name-blind. We write

$$Q_\kappa(y) = \sum_a y_a^2 - \kappa\Big(\sum_a y_a\Big)^2.$$

Theorem 5.5 (Time and signature of the record cell)**.** *On the record cell:*

(i) every tick clock is the rank, $T(x + e_B) = |B|$; the fourth coordinate y_4 is not a tick clock; and the only linear time function with $\tau(e_B) = |B|$ for every B is $\tau(y) = y_1 + y_2 + y_3 + y_4$;

(ii) with this τ, a name-blind interval Q is a spacetime interval of the cell exactly when $Q = \lambda Q_\kappa$ with $\lambda > 0$ and $\kappa \geq 1$; then the main diagonal $(1, 1, 1, 1)$ is timelike, the displacements Q-orthogonal to it are exactly those with $\sum_a y_a = 0$, the differences of records of equal rank, $Q = \lambda \sum_a y_a^2$ on them, and $Q(e_a - e_b) = 2\lambda$ for $a \neq b$, so that in space the four postings point to the corners of a regular tetrahedron;

(iii) the displacement e_C with $|C| = m$ has $\lambda Q_\kappa(e_C) = \lambda(m - \kappa m^2)$; both $\kappa = 1$ and $\kappa = 2$ satisfy (ii), and their class lengths are not proportional, so the order and the count do not fix the lengths.

Proof. (i) The record $x + e_B$ is the $|B|$-th record of every history that posts the accounts of B first; since T starts at zero and rises by one per step, $T(x + e_B) = |B|$. The coordinate y_4 does not rise on the first step of the history that posts account 1 first. A linear τ with $\tau(e_B) = |B|$ has $\tau(e_a) = 1$ for each a.

(ii) Name-blindness makes the diagonal entries of G equal, say d, and the off-diagonal entries equal, say o, so $Q(y) = (d - o)\sum_a y_a^2 + o(\sum_a y_a)^2$. The displacement $e_1 - e_2$ is not advanced by τ, and $Q(e_1 - e_2) = 2(d - o)$, so (b) gives $\lambda := d - o > 0$. Every posting is a history step, so (a) gives $Q(e_a) = d \leq 0$, that is $\kappa := -o/\lambda \geq 1$. Conversely let $\lambda > 0$, $\kappa \geq 1$. Each history step e_a has $\tau(e_a) = 1$ and $Q(e_a) = \lambda(1 - \kappa) \leq 0$; if $\sum_a y_a = 0$ and $y \neq 0$ then $Q(y) = \lambda \sum_a y_a^2 > 0$; and $Q(1, 1, 1, 1) = 4\lambda(1 - 4\kappa) < 0$. The bilinear form of Q pairs $(1, 1, 1, 1)$ with y as $\lambda(1 - 4\kappa)\sum_a y_a$, which vanishes exactly when $\sum_a y_a = 0$. Finally $Q(e_a - e_b) = 2\lambda$.

(iii) For a displacement with m entries equal to one, $\sum_a y_a^2 = \sum_a y_a = m$. At $\kappa = 1$ the classes read $0, -2, -6, -12$ for $m = 1, 2, 3, 4$; at $\kappa = 2$ they read $-1, -6, -15, -28$. A common factor would have to send -1 to 0. □

The signature does not need name-blindness. By Sylvester's law of inertia, a form that is positive on the three-dimensional space $\tau = 0$ and nonpositive on every posting has three positive directions and one negative one: a posting that were null and orthogonal to that space would give $e_a - e_b$ the value $Q(e_b) \leq 0$ for every other posting e_b, against (b). Name-blindness then puts the time axis on the main diagonal and leaves the single number κ.

In plain terms: the time function counts postings, the time axis is one posting on every account, and space is the set of differences between records of equal rank. That the tick clock equals the rank is a theorem about clocks that advance once per posting; reading the rank as elapsed physical time is an identification, kept apart in Chapter 4, and seconds need the unit premises of Chapter 9. Table 5.1 lists the edge classes and their lengths.

Table 5.1. Edge classes of the chain subdivision of one four-cube. An edge spanning m postings has displacement e_C with $|C| = m$. Squared lengths: Euclidean (Theorem 5.2(iv)), general causal (Theorem 5.5(iii), $\kappa \geq 1$), and null (Theorem 5.6, $\kappa = 1$).

postings m	classes $\binom{4}{m}$	edges per cell $\binom{4}{m}2^{4-m}$	Euclidean	λQ_κ	λQ_1
1	4	32	1	$\lambda(1-\kappa)$	0
2	6	24	2	$\lambda(2-4\kappa)$	-2λ
3	4	8	3	$\lambda(3-9\kappa)$	-6λ
4	1	1	4	$\lambda(4-16\kappa)$	-12λ

5.4 Light and the null interval

One number, $\kappa \geq 1$, is left. A further premise fixes it: that light moves along histories. Call a *history direction* a nonzero displacement with no negative entry, and say that *light is a history* for Q when some history direction is null.

Theorem 5.6 (Light moving by postings fixes the interval)**.** *Let $\lambda > 0$ and $\kappa \geq 1$.*

(i) Light is a history for λQ_κ exactly when $\kappa = 1$. Consequently a name-blind spacetime interval of the cell for which light is a history is a positive multiple of

$$Q_1(y) = \sum_a y_a^2 - \Big(\sum_a y_a\Big)^2 = -2\sum_{a<b} y_a y_b.$$

(ii) Under λQ_1, two records of a history that differ by m postings are at squared distance $-\lambda m(m-1)$, that is $0, -2\lambda, -6\lambda, -12\lambda$; the null history directions are the positive multiples of single postings.

(iii) Every triangle whose vertices are three records of one history is timelike under λQ_1: its two edges from the first vertex, spanning m and $m+r$ postings, have Gram determinant $-\lambda^2 mr(m+r-1) < 0$.

Proof. (i) For a history direction y, $\sum_a y_a^2 \leq (\sum_a y_a)^2$ because the cross terms are nonnegative, and $\sum_a y_a > 0$; so $Q_\kappa(y) \leq (1-\kappa)(\sum_a y_a)^2 < 0$ when $\kappa > 1$, while $Q_1(e_a) = 0$. The second sentence follows with Theorem 5.5(ii). (ii) The value $m - m^2$ is Theorem 5.5(iii) at $\kappa = 1$. With no negative entry, $-2\sum_{a<b} y_a y_b = 0$ forces every product $y_a y_b$ with $a \neq b$ to vanish, so at most one entry is nonzero. (iii) Take records $i < j < k$ of a history and the edge displacements u (with $m = j - i$ ones) and w (with $m + r$ ones, $r = k - j$); every one of u is a one of w. The bilinear form of Q_1 gives $\langle u, w\rangle = m - m(m+r) = -m(m+r-1)$, so

$$Q_1(u)Q_1(w) - \langle u, w\rangle^2 = m(m-1)(m+r)(m+r-1) - m^2(m+r-1)^2 = -mr(m+r-1).$$

A plane on which the Gram determinant is negative contains timelike displacements. □

The qualifier $\kappa \geq 1$ matters: at $\kappa = 1/2$ the history direction $(1,1,0,0)$ is null, $Q_{1/2}(1,1,0,0) = 2 - 2 = 0$, while every posting is spacelike. Within the causal family, light along a history forces $\kappa = 1$; for $\kappa > 1$ no history direction is null and light would follow no history at all. *That physical light propagates by single postings is an identification.* We name it here once; nothing in this book derives it (Section 5.10).

Proposition 5.7 (Euclidean lengths and Wick rotations)**.** *The Euclidean count of Theorem 5.2(iv) is $Q_0(y) = \sum_a y_a^2$, the member $\kappa = 0$ of the family; it makes every posting spacelike, so no history is causal. More generally, for a unit vector $w \in \mathbb{R}^4$ and $c \in \mathbb{R}$, the interval $\sum_a y_a^2 - c(w \cdot y)^2$ makes every history causal only if $c \geq 4$. A Wick rotation of the Euclidean lengths along any direction ($c = 2$) therefore makes no history causal, while $c = 4$ with $w = \frac{1}{2}(1,1,1,1)$ gives Q_1.*

Proof. The posting e_a has value $1 - c\,w_a^2$. If $c \leq 0$ this is positive. If $c > 0$, causality of every posting needs $w_a^2 \geq 1/c$ for all four accounts, so $1 = \sum_a w_a^2 \geq 4/c$. With $c = 4$ and $w_a = \frac{1}{2}$ the interval is $\sum_a y_a^2 - (\sum_a y_a)^2$. □

The null interval has a familiar frame. Let $u_1, \dots, u_4$ be unit vectors of $\mathbb{R}^3$ pointing to the corners of a regular tetrahedron, $u_a \cdot u_b = -\frac{1}{3}$ for $a \neq b$, so that $|\sum_a y_a u_a|^2 = \frac{4}{3}\sum_a y_a^2 - \frac{1}{3}(\sum_a y_a)^2$.

Proposition 5.8 (The null frame and the step-speed family)**.** *(i) With $t = \frac{\sqrt{3\lambda}}{2}\sum_a y_a$ and $v = \frac{\sqrt{3\lambda}}{2}\sum_a y_a u_a$, $\lambda Q_1(y) = -t^2 + |v|^2$.*

(ii) For $\alpha \neq 0$, $-(\sum_a y_a)^2 + \alpha^2|\sum_a y_a u_a|^2 = \frac{4\alpha^2}{3}Q_\kappa(y)$ with $\kappa = (3+\alpha^2)/(4\alpha^2)$, and $\kappa \geq 1$ exactly when $\alpha^2 \leq 1$.

Proof. Both are the expansion of $|\sum_a y_a u_a|^2$ above; for (ii), $\frac{4\alpha^2}{3}\sum_a y_a^2 - (1 + \frac{\alpha^2}{3})(\sum_a y_a)^2 = \frac{4\alpha^2}{3}Q_\kappa(y)$, and $3 + \alpha^2 \geq 4\alpha^2$ exactly when $\alpha^2 \leq 1$. □

In the frame of (i), time counts postings, one posting on every account is rest, and each single posting is a light ray toward one corner of a regular tetrahedron: four null vectors with equal mutual products, a symmetric null frame in the sense of Coll and Morales [25]. Part (ii) is the hyperdiamond lattice of Foster and Jacobson [46]: its links $t + \alpha u_a$ have step speed α, and their choice $\alpha = 3$, which makes the faces of the cone of future links null, is $\kappa = 1/3$, with spacelike links. The record cell's histories are causal exactly when the step speed is at most that of light, and light moves by postings exactly at $\alpha = 1$.

Proposition 5.9 (The cone of record displacements)**.** *The displacements $\sum_k w_k e_{a_k}$ of finite sequences of postings with weights $w_k \geq 0$ form the closed orthant $\{y : y_a \geq 0 \text{ for every } a\}$, which contains the displacement from any record of a history to any later record of it. The orthant lies in the future cone $\{y : \sum_a y_a \geq 0,\ Q_1(y) \leq 0\}$ and contains neither the future null displacement $(-1,1,1,1)$ nor the future timelike displacement $(-1,2,2,2)$, for which $Q_1 = -12$. In the frame of Proposition 5.8(i), a displacement in the orthant has velocity $v/t = \sum_a\big(y_a/\sum_b y_b\big)u_a$, a point of the regular tetrahedron inscribed in the unit sphere of light speeds, whose faces lie at speed $\frac{1}{3}$ from rest; $(-1,2,2,2)$ has velocity $-\frac{3}{5}u_1$.*

Proof. A nonnegative combination of postings has nonnegative entries, and conversely $y = \sum_a y_a e_a$. For y in the orthant, $\sum_a y_a \geq 0$ and $Q_1(y) = -2\sum_{a<b} y_a y_b \leq 0$. The displacement $(-1,1,1,1)$ has $\sum_a y_a = 2$ and $Q_1 = 4 - 4 = 0$; $(-1,2,2,2)$ has $\sum_a y_a = 5$ and $Q_1 = 13 - 25 = -12$. The velocity formula is the frame of Proposition 5.8(i); the face opposite u_1 has centroid $\frac{1}{3}(u_2 + u_3 + u_4) = -\frac{1}{3}u_1$, and $(-u_1 + 2u_2 + 2u_3 + 2u_4)/5 = -\frac{3}{5}u_1$ because $\sum_a u_a = 0$. □

From one cell, records reach the speed of light only toward the four corners and no more than a third of it opposite a corner. Filling the light cone needs displacements of both signs or a law that carries records into moving frames; the order of one cell supplies neither. The same proposition shows that the order of the cell is not the causal order of Q_1: from the record $x + e_4$ to the record $x + e_{\{1,2,3\}}$ the displacement $(1, 1, 1, -1)$ is null and future-directed, yet neither record extends the other.

5.5 The two premises reread

Name-blindness and light along a history are the two premises of Theorems 5.5 and 5.6. They can be traded for two other readings, which shows what each one carries. Recall the recognition cost J of Chapter 1, with $J(e^t) = J_{\log}(t) = \cosh t - 1$. The cost of the move $s\,y$ of the four accounts away from balance is $\sum_a J_{\log}(s y_a)$; it names no account, and its second derivative at $s = 0$ is $\sum_a y_a^2$, the *cost Hessian* along y. We name two readings.

Space from the cost.
On displacements with $\sum_a y_a = 0$ the interval is λ times the cost Hessian, $Q(y) = \lambda \sum_a y_a^2$.

One voxel per tick.
There is a time vector T with $\sum_a T_a = 1$, Q-orthogonal to every displacement with $\sum_a y_a = 0$, with $Q(T) = -1$, and each posting's spatial part has $Q(e_a - T) = 1$: in units of one voxel and one tick, every posting moves one voxel in one tick. This is the convention that the unit premise (N1) of Chapter 9 states, applied to the posting.

Theorem 5.10 (The premises reread)**.** *Let $Q(y) = y^{\top} G y$ be an interval on the record cell.*

(i) Every name-blind interval reads space from the cost with $\lambda = G_{11} - G_{12}$, and every Q_κ is the cost Hessian minus $\kappa(\sum_a y_a)^2$.

(ii) If Q reads space from the cost, then $Q(y) = \lambda \sum_a y_a^2 + (G_{11} - \lambda)(\sum_a y_a)^2$ for all y exactly when all four postings read the same, $G_{11} = G_{22} = G_{33} = G_{44}$.

(iii) The interval $Q_{\mathrm{tilt}}(y) = Q_2(y) + 2(\sum_a y_a)(w \cdot y)$ with $w = (\frac{1}{2}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6})$ reads space from the cost with $\lambda = 1$, is a spacetime interval of the cell with the rank as time, has a null posting, $Q_{\mathrm{tilt}}(e_1) = 0$, and is neither name-blind nor a multiple of Q_1.

(iv) If Q reads space from the cost and every posting is null, then $Q = \lambda Q_1$, and Q is name-blind.

(v) If Q reads space from the cost and one voxel per tick holds, then $\lambda = \frac{4}{3}$, $T = \frac{1}{4}(1, 1, 1, 1)$, $Q = \frac{4}{3} Q_1$, Q is name-blind, it is a spacetime interval of the cell with the rank as time, and the displacements orthogonal to the time axis are exactly those that add no rank. Both readings hold for $\frac{4}{3} Q_1$.

Proof. (i) is the formula $Q = (d - o)\sum_a y_a^2 + o(\sum_a y_a)^2$ of the proof of Theorem 5.5. (ii) If the diagonal entries are equal, polarizing the reading on $e_a - e_b$ fixes every symmetric cross sum $G_{ab} + G_{ba} = 2G_{11} - 2\lambda$, which gives the formula; conversely the formula gives $G_{aa} = \lambda + (G_{11} - \lambda)$. (iii) On $\sum_a y_a = 0$ the tilt term vanishes and $Q_2 = \sum_a y_a^2$. The postings read $Q_{\mathrm{tilt}}(e_1) = -1 + 1 = 0$ and $Q_{\mathrm{tilt}}(e_a) = -1 - \frac{1}{3} = -\frac{4}{3}$ for $a \neq 1$, so every history is causal; $Q_{\mathrm{tilt}}(1, 1, 1, 1) = 4 - 32 + 0 < 0$. Unequal diagonal entries rule out name-blindness, and $Q_1(e_2) = 0 \neq Q_{\mathrm{tilt}}(e_2)$ rules out a multiple of Q_1. (iv) By (ii) with $G_{aa} = 0$, $Q = \lambda \sum_a y_a^2 - \lambda(\sum_a y_a)^2$; a symmetric matrix with this form has equal diagonal and equal off-diagonal entries. (v) Write $e_a = T + (e_a - T)$, with $\sum_b (e_a - T)_b = 0$; orthogonality gives $Q(e_a) = Q(T) + Q(e_a - T) = -1 + 1 = 0$, so every posting

is null and (iv) gives $Q = \lambda Q_1$. Orthogonality of T to $e_1 - e_b$ under λQ_1 gives $T_1 = T_b$, so $T = \frac{1}{4}(1,1,1,1)$, and $Q(T) = \lambda(\frac{1}{4} - 1) = -1$ gives $\lambda = \frac{4}{3}$. The rest is Theorem 5.5(ii) at $\kappa = 1$; for the last sentence, $\frac{4}{3}Q_1(e_a - T) = \frac{4}{3}\big(\frac{9}{16} + \frac{3}{16}\big) = 1$. □

This is a reduction, not a closure: the two readings replace the two premises and are themselves adopted. One remark keeps the second reading at its scope. In Chapter 8 the only interacting cone proved, under a supplied pair law, is octahedral, with speed one along the axes and $1/\sqrt{3}$ along the body diagonal; (N1) fixes a speed, and the isotropy of a light cone is a further step.

5.6 The Lorentzian sign with the time axis built in

A different argument obtains the Lorentzian sign from the records by building the time axis into the interval. We state it at its scope.

Proposition 5.11 (The sign of an axis-time reading)**.** *Let q be a function on the sixteen records that vanishes on the empty record, adds over disjoint postings, gives the single postings of accounts $1, 2, 3$ the value 1, and vanishes on the record $x + e_1 + e_4$ of one spatial posting together with a posting on account 4. Then $q(x + e_4) = -1$, q vanishes on $x + e_2 + e_4$ and $x + e_3 + e_4$, and $q(x + e_B) = |B \cap \{1,2,3\}| - [4 \in B]$, the Minkowski interval $y_1^2 + y_2^2 + y_3^2 - y_4^2$ on the records.*

Proof. Additivity gives $0 = q(x + e_1 + e_4) = 1 + q(x + e_4)$, and then $q(x + e_a + e_4) = 1 - 1 = 0$ for $a = 2, 3$; splitting any B into single postings gives the formula. On a displacement with entries 0 and 1, $y_a^2 = y_a$. □

The hypotheses make account 4 the time axis and the other three spatial with unit length: the axis-time reading that Proposition 5.3(iii) excludes, since three of each history's four steps have $q = +1$. The same construction also gives an exact static operator: on the second variation of the Euclidean Regge action on the chain-subdivided lattice, a static field of the form $\mathrm{diag}(b, b, b, -a)$ has the nearest-neighbour lattice Laplacian symbol times $-\frac{1}{2}ab + \frac{1}{4}b^2$ at every wavelength. Both statements concern a background the record cell's causal readings exclude. With time on the main diagonal the counterpart of the sign is Theorem 5.6(ii); the counterpart of the static operator has not been computed.

5.7 The cell's cone and the lattice photon

The interval of the cell says which directions are null. A field on the cell has its own cone, set by its field equation. Two questions follow: is the cell's cone the cone of a wave equation, and does the electromagnetic field of the lattice travel on it?

Proposition 5.12 (The cell's cone as a wave cone)**.** *Let $g_\kappa = I - \kappa\mathbf{J}$, with $\mathbf{J}$ the 4×4 matrix of ones, the matrix of Q_κ. Then $\det g_\kappa = 1 - 4\kappa$, and for $\kappa \neq \frac{1}{4}$, $g_\kappa^{-1} = I + \frac{\kappa}{1-4\kappa}\mathbf{J}$. Lowering, $y \mapsto g_\kappa y$, is a bijection from displacements to covectors that carries the null cone of Q_κ onto the characteristic cone $\{k : k^\top g_\kappa^{-1} k = 0\}$ of the wave operator with principal symbol $k^\top g_\kappa^{-1} k$. The covector of a single posting has symbol $1 - \kappa$. For $\kappa \geq 1$, some history direction is a characteristic ray exactly when $\kappa = 1$.*

Proof. $\mathbf{J}^2 = 4\mathbf{J}$ gives the inverse and the determinant. $(g_\kappa y)^\top g_\kappa^{-1}(g_\kappa y) = y^\top g_\kappa y = Q_\kappa(y)$, so the posting covector has symbol $Q_\kappa(e_a) = 1 - \kappa$; the last sentence is Theorem 5.6(i). □

The same matrix is the flat background of the long-wavelength symbol of the lattice's Lorentzian gravitational action; reading the zero set of that symbol as the cone of a graviton with two polarizations requires $\kappa > \frac{1}{2}$.

The lattice electromagnetic field. The electromagnetic field of the lattice is a separate carrier: real potentials $A_\mu(u)$ on the links of $\mathbb{Z}^4$, field strength $F_{\mu\nu}(u) = A_\nu(u + e_\mu) - A_\nu(u) - A_\mu(u + e_\nu) + A_\mu(u)$, and energy density $D = \frac{1}{2}\sum_{\mu<\nu} F_{\mu\nu}^2$, which is $\frac{1}{2}(E^2 + B^2)$ when the fourth direction is time. D is the quadratic part of the plaquette cost $\sum_{\mu<\nu} J_{\log}(F_{\mu\nu})$, since $J_{\log}(F) = F^2/2 + O(F^4)$. This field is not the transverse photon field of the clock-photon lattice of Chapters 6 and 7, which lives in continuous time on $\mathbb{Z}^3$. To read it on the record lattice we place the four lattice directions on the four accounts (D is unchanged by every relabelling) and split the field strength along a time direction w: the *electric part* D_{el} collects the components along w, the *magnetic part* D_{mag} those that annihilate w, and $D = D_{\mathrm{el}} + D_{\mathrm{mag}}$ for every $w \neq 0$. The reading with *electric weight* λ_e has Hamiltonian $\lambda_e D_{\mathrm{el}} + D_{\mathrm{mag}}$ and Lagrangian $\lambda_e D_{\mathrm{el}} - D_{\mathrm{mag}}$. *The energy reading*, the identification of D as the field's Hamiltonian, and the placement are identifications.

Theorem 5.13 (The lattice photon on the record cell)**.** *(i) For every time direction $w \neq 0$, the energy reading holds exactly when $\lambda_e = 1$.*

(ii) With the rank as time, $w = (1, 1, 1, 1)$, and $\lambda_e > 0$, the Lagrangian's characteristic rays on plane waves are exactly the null displacements of Q_{κ_γ} with $\kappa_\gamma = (1 + \lambda_e)/(4\lambda_e)$. At $\lambda_e = 1$, $\kappa_\gamma = \frac{1}{2}$.

(iii) At $\kappa_\gamma = \frac{1}{2}$ no posting is a ray of the photon: a posting has $Q_{1/2}(e_a) = \frac{1}{2} > 0$, and every photon ray y that advances the rank has $Q_1(y) = -\frac{1}{2}(\sum_a y_a)^2 < 0$. A single posting outruns this light.

(iv) For the energy reading $\lambda_e = 1$, among cell parameters $\kappa > \frac{1}{4}$, the photon's characteristic cone coincides with the cell's wave cone of Proposition 5.12 only at $\kappa = \frac{1}{2}$, which is not a spacetime interval of the cell. The cell's $\kappa = 1$ would need $\lambda_e = \frac{1}{3}$, which is neither the energy reading's 1 nor the action reading's -1.

(v) With the fourth lattice direction as time, the energy reading's Lagrangian is proportional to $E^2 - B^2$, and its characteristic covectors are the null covectors of $\mathrm{diag}(1, 1, 1, -1)$.

(vi) The action reading, all six plaquettes with one sign ($\lambda_e = -1$ in every frame), has no characteristic covector.

The proof is in Appendix B (Section B.3); its mechanism is that the plane-wave symbol of a relabelling-invariant quadratic form of the field strength is a Gram determinant, whose zero set is the null cone of one member of the cell family. The lattice's own field equation shows the same gap from the other side.

Theorem 5.14 (The lattice Maxwell equation of the action is Euclidean)**.** *Let the field on $\mathbb{Z}^4$ solve the Euler-Lagrange equation of the quadratic plaquette action with all six plaquette orientations of one sign, $\sum_\nu \big(F_{\rho\nu}(u) - F_{\rho\nu}(u - e_\nu)\big) = 0$ at every site u and direction ρ.*

(i) *Every component $f = F_{\mu\rho}$ solves the four-dimensional lattice Laplace equation $\sum_\nu \big(f(u+e_\nu) - 2f(u) + f(u-e_\nu)\big) = 0$, with the time direction weighted like space.*
(ii) *A real plane wave $A_\mu(u) = a_\mu \cos(k\cdot u) + b_\mu \sin(k\cdot u)$, $k \in \mathbb{R}^4$, solves the equation exactly when its field strength vanishes.*
(iii) *The pulse $A_x(u) = g(t-z)$ solves the equation exactly when g has zero second difference, so a bounded profile only with zero field; with the time terms of the divergence weighted -1 it solves the resulting equation for every profile g.*
(iv) *The transverse mode $A_x(u) = e^{-\omega t}\cos(qz)$ solves the equation exactly when $\cosh\omega - 1 = 1 - \cos q$, and every q has such a rate $\omega \geq 0$.*

The proof is in Section B.3. Along the fourth direction the action's field equation fixes real decay rates, the lattice form of $\omega = |q|$ in imaginary time; it treats that direction as a fourth direction of space. A real-time photon on $\mathbb{Z}^4$ needs a further reading, such as the signed divergence of part (iii), which no action of the lattice is known to produce.

The last question is whether the photon's light rays can be identified with the cell's by a linear map. Write $\eta = \mathrm{diag}(1,1,1,-1)$, the photon's interval with the fourth lattice direction as time (Theorem 5.13(v)).

Theorem 5.15 (The photon cone maps onto the record cone only with $\sqrt{3}$)**.** *Let K be $\mathbb{R}$ or $\mathbb{Q}$, $\kappa \in K$, and let L be an invertible 4×4 matrix over K that maps every null vector of η with entries in K to a null vector of Q_κ.*

(i) *$L^\top g_\kappa L = c\,\eta$ for some $c \neq 0$ in K: the map is conformal.*
(ii) *$4\kappa - 1 = (c^2/\det L)^2$. Hence $\kappa > \frac{1}{4}$, and $4\kappa - 1$ is the square of a nonzero element of K.*
(iii) *Conversely, if $s \in K$, $s \neq 0$, $s^2 = 4\kappa - 1$, the map sending the photon's time axis to $\frac{1}{2s}(1,1,1,1)$ and its three space axes to $\frac{1}{2}(1,1,-1,-1)$, $\frac{1}{2}(1,-1,1,-1)$, $\frac{1}{2}(1,-1,-1,1)$ satisfies $L^\top g_\kappa L = \eta$. It pulls each posting back to a photon displacement of half a cube diagonal in space and $s/2$ in time, of photon interval $1-\kappa$. Over $\mathbb{R}$, every map of (i) is this one up to scale and a Lorentz transformation of η.*
(iv) *At $\kappa = 1$, $4\kappa - 1 = 3$: no rational map, hence no map of integer lattices, carries the photon's cone into the cell's, and every identification of the two cones needs $\sqrt{3}$. At $\kappa = \frac{5}{4}$ a rational map exists. At every rational κ that admits a rational map, no posting is null.*

Proof. (i) Let $M = L^\top g_\kappa L$, so $y^\top M y = 0$ for every rational null vector y of η. The six vectors $e_i \pm e_4$ ($i = 1,2,3$) give $M_{ii} \pm 2M_{i4} + M_{44} = 0$, so $M_{i4} = 0$ and $M_{ii} = -M_{44}$. The three vectors $3e_i + 4e_j + 5e_4$ ($i < j \leq 3$) then give $(9 + 16 - 25)M_{11} + 24M_{ij} = 0$, so $M_{ij} = 0$. Thus $M = c\,\eta$ with $c = M_{11}$. If $c = 0$ then $g_\kappa = 0$, which is false. (ii) Taking determinants, $(\det L)^2(1-4\kappa) = \det(c\eta) = -c^4$. (iii) The three space images are orthogonal unit vectors of Q_κ with entries summing to zero, $Q_\kappa\big(\frac{1}{2s}(1,1,1,1)\big) = (1-4\kappa)/s^2 = -1$, and the cross terms vanish. Since $e_1 = \frac{s}{2}\cdot\frac{1}{2s}(1,1,1,1) + \frac{1}{2}(h_1+h_2+h_3)$ with h_i the space images, and similarly for the other accounts, a posting pulls back to the photon displacement with time component $s/2$ and spatial part one half of a cube diagonal, of photon interval $\frac{3}{4} - \frac{s^2}{4} = 1-\kappa$. Two maps of (i) differ by a matrix P with $P^\top \eta P = r\eta$; over $\mathbb{R}$, $r > 0$ because η and $-\eta$ have different signatures, so P is $\sqrt{r}$ times a Lorentz transformation. (iv) 3 is not the square of a rational number; $4\cdot\frac{5}{4} - 1 = 4 = 2^2$. A null posting would need $\kappa = 1$ by (iii), which (ii) excludes over $\mathbb{Q}$. □

The map is fixed only up to scale, so the ratio between the photon's lattice tick and the cell's posting count is a unit identification, and any match of the two cones is an identification, not a derivation.

5.8 Regge's action near flatness

The record cell fixes a time direction and a signature. How the geometry of the lattice changes is a further law, and here we take the standard one: Regge's discrete form of the Einstein-Hilbert action [96], in its Euclidean form, on the periodic lattice $\mathbb{Z}_N^4$ with each four-cube cut into its 24 chain simplices. *The Euclidean signature, the subdivision and Regge's action are adopted starting points of this section*; deriving the action from recognition is a step not yet made. The Euclidean form is the member Q_0 of the family of Section 5.4, which Proposition 5.7 excludes as an interval of the records.

A *geometry* assigns to every edge e of the subdivision a real number ℓ_e, its squared length; the flat geometry gives the edge from $x + e_B$ to $x + e_{B'}$ the value $|B' \setminus B|$. A geometry is *nondegenerate* when every simplex with vertices $p_0, \dots, p_4$ has positive definite Gram matrix $g_{ab} = \frac{1}{2}(\ell_{0a} + \ell_{0b} - \ell_{ab})$, $a, b = 1, \dots, 4$; it can then be drawn in $\mathbb{R}^4$ with those squared lengths. Each triangle h with squared side lengths a, b, c has area A_h with $16A_h^2 = 2ab + 2bc + 2ca - a^2 - b^2 - c^2$, and in each simplex σ containing it an interior dihedral angle $\theta_{\sigma,h} \in (0, \pi)$. The *deficit* and the action are

$$\varepsilon_h = 2\pi - \sum_{\sigma \supset h} \theta_{\sigma,h}, \qquad S(\ell) = \sum_h A_h \varepsilon_h.$$

The flat geometry is the flat torus cut into simplices, with all deficits zero and $S = 0$. By Schläfli's identity [96, 17], under a smooth change of a nondegenerate geometry $\mathrm{d}S = \sum_h \varepsilon_h \, \mathrm{d}A_h$.

The *body path* $\ell_q = \ell_{\mathrm{flat}} + q\,\delta$ changes the squared length of every main diagonal by the same amount q and leaves every other edge flat. One simplex shows the mechanism. Take $p_0 = x$ and $p_i = x + e_1 + \cdots + e_i$. The triangle $p_0 p_k p_4$ has squared sides k, $4 - k$ and $4 + q$, so its area is $\sqrt{4k(4-k) - q^2}/4$; at $q = 0$ its angle at p_k is a right angle, since $k + (4 - k) = 4$, and that is why its area does not change to first order. Each four-cube has $\binom{4}{k} = 4, 6, 4$ such triangles for $k = 1, 2, 3$, each shared by $k!(4-k)! = 6, 4, 6$ chain simplices. Only triangles containing a main diagonal change their area along the path.

Theorem 5.16 (Response to a uniform stretch)**.** *Let $N \geq 3$ and $|q| < 2$. Then ℓ_q is nondegenerate, the Gram determinant of every simplex being $(4 - q^2)/4$, and*

$$\frac{\mathrm{d}}{\mathrm{d}q} S(\ell_q) = N^4 F(q), \qquad F(q) = -q\Big(\frac{2\,\varepsilon_1(q)}{\sqrt{12 - q^2}} + \frac{3\,\varepsilon_2(q)}{2\sqrt{16 - q^2}}\Big),$$

where $\varepsilon_1 = 2\pi - 6\theta_1$ and $\varepsilon_2 = 2\pi - 4\theta_2$ are the deficits of the triangles $p_0 p_k p_4$ for $k = 1$ (equal to that for $k = 3$) and $k = 2$, with

$$\cos\theta_1(q) = \sqrt{\frac{4 - q^2}{2(8 - q^2)}}, \qquad \cos\theta_2(q) = -\frac{2q}{8 - q^2}.$$

Moreover $F(0) = F'(0) = 0$, the middle deficit falls as $\varepsilon_2(q) = -q + O(q^2)$, and

$$\lim_{q \to 0} \frac{F(q)}{q^2} = \frac{3}{8}.$$

The proof is in Section B.3.

Corollary 5.17 (Flatness is a stationary saddle)**.** *Let $N \geq 3$.*

(i) $S(\ell_q) - S(\ell_0) = N^4 \int_0^q F$ for $|q| < 2$; the flat geometry is stationary along the body path with vanishing second derivative, and $S(\ell_q) - S(\ell_0) = N^4\big(q^3/8 + o(q^3)\big)$.
(ii) There is $\delta > 0$ with $S(\ell_q) < S(\ell_0)$ for $-\delta < q < 0$ and $S(\ell_q) > S(\ell_0)$ for $0 < q < \delta$.
(iii) The flat geometry is neither a local minimum nor a local maximum of S on the nondegenerate geometries.

Proof. (i) is the fundamental theorem of calculus with Theorem 5.16: $F(0) = F'(0) = 0$ and $\int_0^q F = \int_0^q \frac{3}{8} t^2 (1 + o(1))\, \mathrm{d}t = q^3/8 + o(q^3)$. By the limit, $F(q) > 0$ for $0 < |q| < \delta$, so $q \mapsto S(\ell_q)$ is strictly increasing on $(-\delta, \delta)$, which is (ii). The path is continuous and passes through the flat geometry, so every neighbourhood of it contains ℓ_q for small q of both signs, and (ii) gives (iii). □

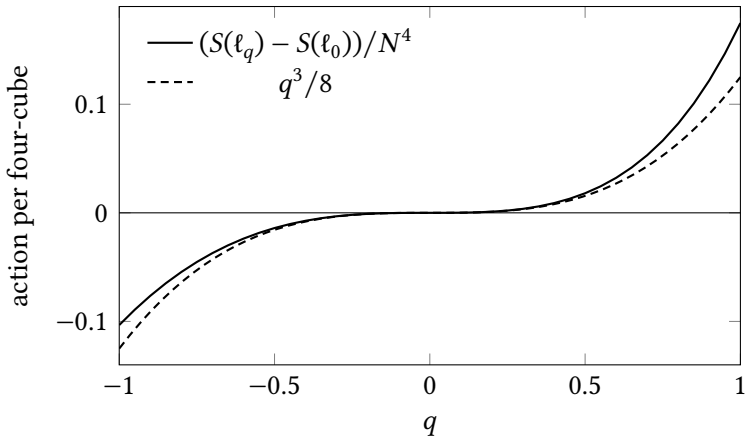


Figure 5.2. The change of Regge's action per four-cube along the body path, computed by numerical integration of the closed form of F in Theorem 5.16 (solid), and its cubic germ $q^3/8$ (dashed). The curve crosses its flat value with zero slope and zero curvature: flatness is stationary and is neither a minimum nor a maximum (Corollary 5.17).

The stretch is a genuine curvature: lengthening the main diagonals opens the dihedral angles at the middle triangles beyond a right angle, so each middle triangle has deficit $-q$ to first order. Yet, as the simplex above shows, the triangles that carry the curvature keep their area to first order, so the derivative of the action vanishes at first order. On the chain-subdivided cubic lattice every triangle area has zero derivative with respect to a main diagonal at flatness [33]; the expansion of Regge's action about a flat lattice goes back to Roček and Williams [99, 100]. Flatness is selected by stationarity, not by the least value of the action. In the continuum the Euclidean Einstein-Hilbert action is unbounded below under conformal rescalings [50]; the lattice shows the absence of a minimum at third order along a direction on which the second variation vanishes.

Theorem 5.18 (No balanced nonnegative cost gives the action)**.** *Let $N \geq 3$, and let C be a real function on a neighbourhood of 0 with $C \geq 0$ and $C(0) = 0$. There are no constants $\alpha, \beta \in \mathbb{R}$ with $S(\ell_q)/N^4 = \alpha + \beta C(q)$ for all q near 0. In particular this holds for the recognition cost of postings*

$$C(q) = \sum_{(a,b)\in P(q)} J\big(e^{\phi_a(q)-\phi_b(q)}\big) = \sum_{(a,b)\in P(q)} \big(\cosh(\phi_a(q) - \phi_b(q)) - 1\big)$$

for any finite sets $P(q)$ of posted pairs of sites of a periodic three-dimensional lattice and any log-potentials $\phi(q)$ on its sites that are balanced on the posted pairs at $q = 0$, $\phi_a(0) = \phi_b(0)$ for $(a, b) \in P(0)$.

Proof. At $q = 0$ the identity gives $\alpha = S(\ell_0)/N^4$. If $\beta \geq 0$ then $S(\ell_q) \geq S(\ell_0)$ near 0, against Corollary 5.17(ii) for small negative q; if $\beta < 0$ then $S(\ell_q) \leq S(\ell_0)$, against it for small positive q. The posting cost is nonnegative because $\cosh t \geq 1$ and vanishes at $q = 0$ by balance. □

No continuity of C is needed. The route "the gravitational action of the lattice is a balanced recognition cost" is therefore refuted along the body path (Appendix D). What the theorem leaves open is as definite: signed differences of cost between configurations, the phases of a Lorentzian action, backgrounds away from balance, and an action induced by the dynamics of fields are not excluded.

The two halves of the chapter meet on the main diagonal. Under λQ_1 the main diagonal is the time axis of each cell, of squared length -12λ, and every hinge of the subdivision, a triangle of three records of one history, is timelike (Theorem 5.6(iii)). The body path changes that squared length by the same amount in every cell, which reads as a uniform change of the duration of one complete round of postings. The three results above are proved on Euclidean lengths; whether flatness is a saddle of the Lorentzian Regge action [105] on the record interval, along the same path, is open.

5.9 The continuum light cone

The record cell has its cone within one cell. The photon field of Chapters 6 and 7 has a cone only in a limit, which we prove here; nothing below uses it. The carrier is the photon field of carrier (i) without registers, on $\mathbb{Z}^3$ read as the lattice of spacing $\ell > 0$, with the dispersion (6.1) at $C_E = 1/\ell$: momentum p has energy $E_\ell(p) = \frac{2}{\ell}\operatorname{arsinh}\big(\sqrt{\lambda(\ell p)}/2\big)$, and long waves move with speed one (Proposition 9.19). A smooth current $J : \mathbb{R}^3 \to \mathbb{R}^3$ of compact support gives the lattice current J_ℓ in which the edge of direction d with midpoint m carries $\ell^{3/2} J_d(m)$. Let $C_\ell(t) = c(J_\ell, H_\ell, t)/a$ be the source commutator defined before Proposition 6.36, per unit current scale $a > 0$, and let

$$C_0(t) = -\frac{i}{(2\pi)^3}\int_{\mathbb{R}^3} \big\langle \hat{J}(p), |p| \sin(t|p|) P_\perp(p) \hat{H}(p) \big\rangle \, \mathrm{d}p, \qquad \hat{J}(p) = \int_{\mathbb{R}^3} J(x) e^{ix\cdot p} \, \mathrm{d}x,$$

with $P_\perp(p)$ the projection onto $p^\perp$, be the same commutator for the free transverse field of speed one in the continuum.

Proposition 5.19 (The continuum light cone)**.** *Let J, H be smooth currents of compact support and $T \geq 0$. There are $C(T), D \geq 0$, independent of ℓ, with:*

(i) $|C_\ell(t) - C_0(t)| \le \ell^2 C(T) + \ell^4 D$ *for every* $\ell > 0$ *and* $|t| \le T$*;*
(ii) if $|t| \le T$ *and* $|t| < |y - x|$ *for all* x *in the support of* J *and* y *in that of* H*, then* $C_0(t) = 0$*, and* $|C_\ell(t)| \le \ell^2 (C(T) + D)$ *for* $0 < \ell \le 1$*;*
(iii) if $\lambda(\ell p) > 0$*, then* $0 < E_\ell(p) < |p|$*, and the boost of speed* $\beta = \frac{1}{2}(1 + E_\ell(p)/|p|) < 1$ *along* p *takes* $(E_\ell(p), p)$ *to a four-momentum of the same Minkowski square and negative energy, which no mode of the lattice has.*

The proof is in Appendix B.3. The limit is causal by (ii), and Poincaré covariant: its plane waves have the phases $x \cdot p - t|p|$, which translations shift and boosts preserve (Section 9.6). By (iii) no finite spacing is boost covariant. The light cone appears in the continuum limit, not in the first cell.

The closed torus. One result of the institute's related work, outside the chain of this book, also bears on the cell; we report it at its scope and do not use it. On the periodic Lorentzian Regge lattice of side $N \ge 3$ with cell parameter $\kappa > \frac{1}{2}$, assuming that every zero mode of the flat second variation is a constant strain plus a vertex move (a statement not yet proved), positive matter at rest must be balanced at first order by a uniform stress, and the Gauss law of the closed torus fixes that stress: among uniform backgrounds it selects exactly the body's mean density, and as a stress it is $(\sum_x \rho(x)/(2\tau))\mathbf{1}\mathbf{1}^\top$, with $\tau = 2\sqrt{4\kappa - 1}$ the proper length of a main diagonal, a uniform dust at rest. A stress of this form equals a cosmological term Λg^{-1} only when both vanish. Reading this as the closed-universe constraint of general relativity is an inference, not a result.

5.10 What the cell does not yet give

Three things are missing. The premises of the cell, or the two readings that replace them (Theorem 5.10), are adopted; deriving either pair from the ledger would turn Theorems 5.5 and 5.6 into consequences of the recognition account, and until then reading single postings as physical light remains an identification. Whether some Lorentzian interval that does not treat the accounts alike makes the extension order of the sixteen records exactly its causal order, among pairs with nondecreasing rank, is open; the name-blind null interval does not (Proposition 5.9). And no theorem connects the record cell's cone with the dispersion of the clock-photon lattice of Chapters 6 and 7, nor with the lattice electromagnetic field beyond the $\sqrt{3}$ map of Section 5.7; a correspondence theorem between those carriers, or a real-time reading of the lattice photon derived from an action, would settle it.

Nothing yet takes the empty record of Chapter 2 through the first post of Chapter 3 into a spacetime of many cells with a causal reading. Chapter 6 takes up the clock-photon lattice, with its own cells and its own photon, the first carrier on which an interacting dynamics is proved.

Chapter 6

The first interacting dynamics I: finite carriers, records and regional stability

Once a record exists, something has to happen to it. A posting has to move, a field has to respond and a later record has to be written. Each of these needs a dynamics: a self-adjoint energy operator on a stated carrier, an evolution that exists for all times, and an energy account that says what pays for what. This chapter builds such dynamics on finite carriers made from the native objects and proves what they do. It also proves what they do not do, because several natural identifications fail.

Here is the effect in its smallest form. An excitation sits at one site of a lattice and the photon field is empty. A hop to a neighbouring site displaces the field by the profile z attached to that bond, so the part of the state that has hopped carries the displaced vacuum

$$W(-z)\Omega_{\mathrm{ph}} = e^{-\|z\|^2/2} \sum_{m\geq 0} \frac{(-z)^{\otimes m}}{\sqrt{m!}},$$

which holds m photons with probability $e^{-\|z\|^2}\|z\|^{2m}/m!$. A hop makes photons, with mean number $\|z\|^2$. Three questions follow: where the photons' energy comes from, what writing a record of such a process costs, and, when many registers drive one field, whether the energy of a finite region is bounded below and the dynamics has a limit as the region grows. The infinite lattice is the subject of Chapter 7.

6.1 The native field, the register and the carriers

Two objects appear in every dynamical result of this chapter and the next: the transverse photon field of the cubic lattice and the eight-state clock register. Their form is taken from the recognition foundation [125]; deriving the dispersion and the occupation projection is not part of this book. Every theorem below uses only the mathematical form stated here. We call these fixed objects native.

Definition 6.1 (The native photon field)**.** Let $\Lambda = \mathbb{Z}^3$ with oriented edges $\mathscr{E} = \Lambda \times \{0,1,2\}$, and let $\mathbb{T}^3 = (-\pi,\pi]^3$ carry the normalized measure $\mathrm{d}\mu(k) = \mathrm{d}^3k/(2\pi)^3$. Put

$$\lambda(k) = 2\sum_{d=0}^{2}(1-\cos k_d), \qquad r_d(k) = 1 - e^{ik_d}, \qquad \rho(k) = \overline{r(k)},$$

and let $P_T(k)$ be the orthogonal projection of $\mathbb{C}^3$ onto $\rho(k)^\perp$, the transverse directions of the lattice divergence. The one-photon space is $\mathscr{H}_1 = \{z \in L^2(\mathbb{T}^3,\mu;\mathbb{C}^3) : P_T z = z\}$ and the photon space is the symmetric Fock space $\mathscr{F} = \bigoplus_{m\geq 0} \mathscr{H}_1^{\otimes_s m}$ with vacuum Ω_{ph}; inner

products are conjugate-linear in the first argument. For an energy scale $C_E > 0$,

$$\omega(k) = 2C_E \operatorname{arsinh}\Big(\frac{\sqrt{\lambda(k)}}{2}\Big), \qquad hz = \omega z, \qquad H_0 = \mathrm{d}\Gamma(h), \tag{6.1}$$

on the domain of vectors with $\sum_m \|E_m \psi_m\|^2 < \infty$, where $E_m(k_1, \dots, k_m) = \sum_\ell \omega(k_\ell)$ on the m-photon sector. A source with displacement profile $g \in \mathscr{H}_1$ is $S(g) = a^*(hg) + a(hg)$. The Weyl displacement $W(g) = \exp(a^*(g) - a(g))$ is the unitary with $W(g)^* a(f) W(g) = a(f) + \langle f, g\rangle$, $W(g)^* = W(-g)$ and

$$W(g)\Omega_{\mathrm{ph}} = e^{-\|g\|^2/2}\, \varepsilon(g), \qquad \varepsilon(g) = \sum_{m \geq 0} \frac{g^{\otimes m}}{\sqrt{m!}}, \tag{6.2}$$

where the exponential vectors satisfy $\langle \varepsilon(f), \varepsilon(g)\rangle = e^{\langle f, g\rangle}$.

Since $\cosh t - 1 = 2\sinh^2(t/2)$, the dispersion says $J(e^{\omega(k)/C_E}) = \lambda(k)/2$: in units of C_E the frequency is the logarithm of the ratio whose cost is half the lattice symbol. Since $0 \leq \lambda \leq 12$, the one-photon energy is bounded, $0 \leq \omega \leq \omega_{\max} = 2C_E \operatorname{arsinh}\sqrt{3} \approx 2.6339158\, C_E$, and positive away from $k = 0$. The field energy H_0 is unbounded because photon numbers are.

Definition 6.2 (The register and its occupation)**.** A register has carrier $\mathbb{C}^8$ with basis $x = 0, \dots, 7$, read in binary as a vertex of the cube $\{0,1\}^3$. With $F_{xk} = 8^{-1/2} e^{-2\pi i x k/8}$ and $D = \mathrm{diag}(0, 1, \dots, 7)$, the native clock is $C = FDF^*$, a positive semidefinite circulant matrix with ground state $\chi = Fe_0 = 8^{-1/2}(1, \dots, 1)$. Let $\sigma_x = (-1)^{\#\{\text{ones in } x\}}$, let T swap each label with the label that differs in its lowest bit, and $\Sigma = \mathrm{diag}(\sigma_x)$. The occupation is

$$Q = \tfrac{1}{2}\big(I - \mathrm{charge}\big), \qquad \mathrm{charge} = i\Sigma T. \tag{6.3}$$

Since $T^2 = I$ and σ changes sign under the swap, the charge is a self-adjoint involution, so $Q = Q^* = Q^2$ has rank four and picks one direction in each of the four pairs $\{x, Tx\}$. An occupation basis labels a register by $(q, \alpha) \in \{0,1\} \times \{1,2,3,4\}$; all eight coordinates remain. In the native basis $C_{xy} = \frac{8\delta_{xy} - 1}{2} + \frac{i}{2}(c^{(1)}_{x-y} + \sqrt{2}\, c^{(2)}_{x-y})$ with differences modulo eight, $c^{(1)} = (0, 1, 1, -1, 0, 1, -1, -1)$ and $c^{(2)} = (0, 1, 0, 1, 0, -1, 0, -1)$.

Definition 6.3 (Current profiles)**.** Let $a \geq 0$ be a current scale. A finitely supported real current $f : \mathscr{E} \to \mathbb{R}$ has the profile

$$z_f(k) = i\, w(k) P_T(k) F_f(k), \qquad F_f(k) = \sum_{(x,d) \in \mathscr{E}} f(x,d)\, e^{ix\cdot k} e_d, \tag{6.4}$$

with the weight $w(k) = \big(a/(2C_E\sqrt{\lambda(k)}\sqrt{1 + \lambda(k)/4})\big)^{1/2}$, and the electric profile $e_f(k) = i\,(2w(k))^{-1} P_T(k) F_f(k)$. The unit current of an edge e has the profile $z_e = z_{\delta_e}$.

Near $k = 0$, $w(k)^2 = O(|k|^{-1})$, so $z_f \in \mathscr{H}_1$; no divergence-free condition is imposed, since P_T keeps the transverse part. Translating a current by $y \in \Lambda$ multiplies its profile by $e^{iy\cdot k}$.

Lemma 6.4 (A unit current pairs to one third)**.** *Let $a > 0$ and $C_E > 0$. For every edge $e = (x, d)$, $\langle e_{\delta_e}, z_e\rangle = \int_{\mathbb{T}^3} \frac{1}{2}\|P_T(k) e_d\|^2 \,\mathrm{d}\mu = \frac{1}{3}$. In particular $z_e \neq 0$.*

Proof. The phases $e^{ix\cdot k}$ and the factors i cancel in the inner product, and $(2w)^{-1} w = \frac{1}{2}$. Where $\lambda(k) > 0$, $\|P_T(k) e_d\|^2 = 1 - |r_d(k)|^2/\lambda(k)$; since $\sum_d |r_d|^2 = \lambda$, symmetry under permuting the coordinates gives $\int \|P_T e_d\|^2 \,\mathrm{d}\mu = 1 - \frac{1}{3} = \frac{2}{3}$. □

6.1.1 The carriers of this chapter

Four carriers appear; the letters are those of Appendix A. They share the photon field $\mathscr{F}$ and, where registers occur, the register of Definition 6.2. They are separate constructions, and no theorem maps the dynamics of one onto another.

- *The self-consistent lattice*, carrier (h) (Sections 6.2–6.4): one excitation on the period-N lattice $\Lambda_N = (\mathbb{Z}/N\mathbb{Z})^3$, with the full photon space at each site and a signed scalar field that the excitation sources itself.
- *The source and recorder*, carrier (g) of Section 2.7 (Sections 6.5 and 6.6): one source on $\mathbb{Z}^3$ carrying the four registers of one cell ($8^4 = 4096$ internal coordinates) and the photon field, with record labels in $\mathbb{Z}^3$.
- *The atomic carrier*: the source of carrier (g) alone, $\ell^2(\mathbb{Z}^3; \mathbb{C}^{4096} \otimes \mathscr{F})$.
- *The cell carrier* of the clock-photon lattice, carrier (i) (Sections 6.7 and 6.8): every cell of $\mathbb{Z}^3$ carries four registers; a configuration assigns them labels and differs from the all-zero assignment at finitely many cells; the carrier is $\mathscr{H} = \ell^2(\mathscr{X}; \mathscr{F})$ over the countable set $\mathscr{X}$ of such configurations, with one shared photon field.

The supplied data are explicit in every theorem: the scales a and C_E, the clock strength ϵ, the profiles and their placement on edges, the background, the reference density, the region and the initial state. The all-zero configuration of the cell carrier is part of the representation; it is not a selected first state of the universe.

The only correspondences proved between these carriers are negative. The plainest concerns the atomic and cell carriers. Fix a cell r. Sending a source at r with internal state ψ to $\sum_n \delta_{\{r \mapsto n\}} \otimes \psi(n)$, where $\{r \mapsto n\}$ is the configuration with native value n at r and zeros elsewhere, is an isometric embedding of the fibre over r into the cell carrier.

Proposition 6.5 (The atomic and cell carriers do not match)**.** *(i) For two cells $r \neq s$ and a nonzero photon vector v, no linear isometry from the atomic carrier into the cell carrier extends the fixed-site embeddings of all cells.*

(ii) Write the internal value at r and the marker $(1,1,1,1)$ at $r + e_0$. This marked encoding is a linear isometry $\mathscr{M}$ of the atomic carrier into the cell carrier, and it intertwines the lattice translations of the two carriers. Under it the source at r and the source at $r + e_1$ differ in at least two cells, $r + e_0$ and $r + e_1 + e_0$, whatever their internal values.

(iii) Let the regional Hamiltonian of Section 6.8 carry the clock energy, with any background and profiles. Its matrix elements between configurations that differ at two cells vanish, so it has zero amplitude from a marked configuration at r to every marked configuration at $r + e_1$. The atomic source with coupling $g \neq 0$, applied to a nonzero single-coordinate input $e_n \otimes v$ at r (in its domain, with marked image in the regional domain), has overlap $-g\|e_n \otimes v\|^2$ with the posted state at $r + e_1$. Hence the marked image of the atomic source action differs from the regional action on the marked input. For the source whose site terms are the native clocks and the clamped Gauss-law energy, started with the photon vacuum, no domain condition is needed, and there is a unit vector at $r + e_1$ whose marked image has strictly smaller Born weight under the regional evolution than under the marked image of the atomic evolution, at every small nonzero time.

Proof. (i) The atomic states $\delta_r \otimes (e_0 \otimes v)$ and $\delta_s \otimes (e_0 \otimes v)$ are orthogonal, while both fixed-site embeddings send them to the all-zero configuration with photon vector v; an isometry cannot

identify orthogonal vectors. (ii) The map $(r, n) \mapsto$ marked configuration is injective, so the marked embeddings of different sites have orthogonal ranges and assemble into an isometry; translating the source translates value and marker together. The marker leaves $r + e_0$ and appears at $r + e_1 + e_0$. (iii) Each term of the regional fibre operator, the free field, the clock, the background and each source times a single-register occupation, changes at most one register label, and configurations that differ at two cells differ in two registers. The hopping of the atomic source sends $\delta_r \otimes u$ to a vector whose component at $r + e_1$ is $-g$ times the posted image of u, while the regional action is orthogonal to the whole marked fibre at $r + e_1$; a marked image with a nonzero component there cannot equal it. For the probability statement, take the posted image of u as the probe: both evolved Born weights vanish at $t = 0$, and their second-order terms are the squared overlaps of the two actions with the probe, $g^2\|u\|^4 > 0$ and 0. □

This is a negative correspondence for the fixed-site charts and for the marked encoding only. It says nothing about other encodings, about the Heisenberg evolution of growing regions, or about a physical occupancy marker. An embedding of the source carrier that intertwines the actual source and regional dynamics, or a proof that none exists, is an open problem.

6.2 A self-consistent field on a finite periodic lattice

On this carrier the cost J enters the motion itself, not only the bookkeeping. One excitation moves on a periodic lattice; each hop across a bond displaces the photon field by that bond's profile, and the hop is weighted by a scalar field that the excitation sources itself through a Gauss law whose bond energy is exactly the cost J. We ask whether this nonlinear dynamics exists for all times, whether it conserves an energy, whether it makes photons out of the vacuum, and where their energy comes from.

Let $\Lambda_N = (\mathbb{Z}/N\mathbb{Z})^3$, $N \geq 1$, with bonds (p, k) joining p to $p + e_k$, and $\mathscr{H}_{\Lambda_N} = \bigoplus_{p\in\Lambda_N} \mathscr{F}$ with $\|x\|^2 = \sum_p \|x_p\|^2$: x_p is the photon state that accompanies the excitation at p, and $\|x_p\|^2$ is the probability of finding it there. Each bond carries a profile $z_{p,k} \in \mathscr{H}_1$. For a real field $\phi : \Lambda_N \to \mathbb{R}$ put $c_{p,k} = \cosh(\phi_p - \phi_{p+e_k})$ and $d_\phi(p) = \sum_k (c_{p,k} + c_{p-e_k,k}) - \phi_p$.

Definition 6.6 (The self-consistent field)**.** The Hamiltonian at field ϕ is

$$(H_\phi x)_p = H_0 x_p + d_\phi(p)\, x_p - \sum_{k=0}^{2} \Big(c_{p,k} W(z_{p,k}) x_{p+e_k} + c_{p-e_k,k} W(z_{p-e_k,k})^* x_{p-e_k} \Big) \tag{6.5}$$

on the domain $\mathscr{D}$ of vectors whose every component lies in $\mathscr{D}(H_0)$. The photon energy, the bond energy and the field cost are

$$E_{\mathrm{ph}}(x) = \sum_p \langle x_p, H_0 x_p \rangle, \qquad \mathscr{B}_\phi(x) = \sum_{p,k} c_{p,k} \|x_p - W(z_{p,k}) x_{p+e_k}\|^2,$$

$$\mathscr{A}(\phi) = 2 \sum_{p,k} J\big(e^{\phi_p - \phi_{p+e_k}}\big).$$

Fix a reference distribution $\varrho : \Lambda_N \to \mathbb{R}$ with $\sum_p \varrho_p = 1$ (no sign condition) and an anchor site p_*. The field functional is

$$\mathscr{J}(\phi; x) = \mathscr{B}_\phi(x) + \sum_p \big(\varrho_p - \|x_p\|^2\big)\phi_p - \mathscr{A}(\phi).$$

For a normalized x, the selected field $\phi[x]$ is the field with $\phi_{p_*} = 0$ solving at every site

$$2\sum_{k=0}^{2}\Big(\beta_{p,k}(x)\sinh(\phi_p-\phi_{p+e_k})+\beta_{p-e_k,k}(x)\sinh(\phi_p-\phi_{p-e_k})\Big)=\varrho_p-\|x_p\|^2, \tag{6.6}$$

with screened bond weights $\beta_{p,k}(x) = 1 - \frac{1}{2}\|x_p - W(z_{p,k})x_{p+e_k}\|^2$ (and $\beta_{p,k} = 0$ on a bond that closes on itself). The native and field energy of x is $E_{\mathrm{nf}}(x) = \mathscr{J}(\phi[x]; x)$, and its total energy is $E(x) = E_{\mathrm{ph}}(x) + E_{\mathrm{nf}}(x)$. The evolution is

$$i\,\psi'(t) = H_{\phi[\psi(t)]}\,\psi(t). \tag{6.7}$$

Since $\cosh t - 1 = J(e^t)$, each bond weight is $c_{p,k} = 1 + J(e^{\phi_p-\phi_{p+e_k}})$, and the field cost is the cost of the ratios $e^{\phi_p-\phi_{p+e_k}}$ summed over bonds: in every axis frame it equals the exact cost $\sum_{i,j} w_{ij}(\cosh(\phi_i-\phi_j)-1)$ on the canonical signed posting graph of Λ_N, with coinciding bonds counted with multiplicity. Expanding the squares gives

$$\langle x, H_\phi x\rangle = E_{\mathrm{ph}}(x)+\mathscr{B}_\phi(x)-\sum_p \phi_p\|x_p\|^2, \qquad E(x)=\langle x,H_\phi x\rangle+\sum_p \varrho_p\phi_p-\mathscr{A}(\phi)\quad(\phi=\phi[x]). \tag{6.8}$$

Equation (6.6) is the nonlinear Gauss law of Chapter 3, with source the excess of the reference over the probability of finding the excitation at a site, on bonds screened by the photon field: a bond conducts fully when the photon states at its ends agree after the displacement, and not at all when they are as different as normalization allows. The field is recomputed from the state at every instant, so (6.7) is a nonlinear Schrödinger equation of Hartree type [16], and the dressing of each hop by a displacement has the form of the Lang-Firsov transformation [72]. Write $V_\phi = H_\phi - \bigoplus_p H_0$ for the bounded interaction.

Two currents describe how probability moves. The *constitutive current* of a field is $\beta_{p,k}(x)\sinh(\phi_p-\phi_{p+e_k})$, the bond term of (6.6); the *Born transport current* of a state is

$$j_{p,k}(\psi) = 2\cosh(\phi_p-\phi_{p+e_k})\ \mathrm{Im}\langle\psi_p, W(z_{p,k})\psi_{p+e_k}\rangle, \qquad \phi=\phi[\psi]. \tag{6.9}$$

Theorem 6.7 (The self-consistent dynamics)**.** *Let $N \geq 1$, $C_E \geq 0$, and fix the profiles, a reference ϱ with $\sum_p \varrho_p = 1$ and an anchor.*

(i) For every field ϕ, H_ϕ is self-adjoint on $\mathscr{D}$.

(ii) For every normalized x, (6.6) has exactly one solution with $\phi_{p_*} = 0$; it is the unique maximizer of $\phi \mapsto \mathscr{J}(\phi; x)$ among anchored fields. There is $\kappa_N > 0$, depending only on N, such that for every profile family, every normalized x, every anchor and every v with $v_{p_*} = 0$,

$$\kappa_N\sum_p v_p^2 \leq \sum_{p,k}\beta_{p,k}(x)\,(v_p - v_{p+e_k})^2, \tag{6.10}$$

and consequently $\max_p|\phi[x]_p| \leq C_N(\sum_p|\varrho_p|+1)$ with C_N depending only on N. Moreover $\mathscr{B}_0(x) \leq E_{\mathrm{nf}}(x)$, so $0 \leq E_{\mathrm{nf}}(x)$ and $0 \leq E_{\mathrm{ph}}(x) \leq E(x)$ for every normalized $x \in \mathscr{D}$.

(iii) For every normalized x_0, with no domain condition, there is exactly one curve $u : \mathbb{R} \to \mathscr{H}_{\Lambda_N}$, differentiable in norm, with $u(0) = x_0$, $\|\psi(t)\| = 1$ and $u'(t) = -i\,e^{itH_0}V_{\phi[\psi(t)]}\psi(t)$ for every t, where $\psi(t) = e^{-itH_0}u(t)$. Two such curves that agree at one time agree everywhere. Along it the site probabilities obey

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi_p(t)\|^2 = -\sum_{k=0}^{2}\big(j_{p,k}(\psi(t)) - j_{p-e_k,k}(\psi(t))\big). \tag{6.11}$$

(iv) If every component of a normalized x_0 has finite second moment of photon number, $\sum_m m^2 \|(x_{0,p})_m\|^2 < \infty$, there is a curve $\psi : \mathbb{R} \to \mathscr{D}$ with $\psi(0) = x_0$, $\|\psi(t)\| = 1$, $t \mapsto H_0\psi(t)$ continuous, $\psi'(t) = -iH_{\phi[\psi(t)]}\psi(t)$ and $E(\psi(t)) = E(x_0)$ for every t. This covers the coherent start $x_{0,p_0} = \varepsilon(f)/\|\varepsilon(f)\|$, $x_{0,p} = 0$ for $p \neq p_0$, with any $f \in \mathscr{H}_1$, whose photon number has moments of every order.

The proof is in Appendix B.4. By part (ii) the photons never hold more energy than the state has in total. The solutions select no profiles, reference, initial data, resources or time unit: all are supplied.

Proposition 6.8 (Strong solutions are unique)**.** *Every strong solution as in Theorem 6.7(iv) is the solution of part (iii) for its initial state. Hence the strong solution from a given initial state is unique.*

Proof. If ψ is as in (iv), then $u(t) = e^{itH_0}\psi(t)$ is differentiable by the product rule, because e^{itH_0} is a strongly continuous unitary group and $\psi(t) \in \mathscr{D}$; its derivative is $e^{itH_0}(iH_0\psi(t) + \psi'(t)) = -i\,e^{itH_0}V_{\phi[\psi(t)]}\psi(t)$. So u solves (iii), which has one solution. □

6.3 Photons from the vacuum

Except where stated, the results of this section are on the period-three lattice Λ_3, with 27 sites. Start the excitation at the origin o with no photons, $\xi_o = \Omega_{\mathrm{ph}}$ and $\xi_p = 0$ otherwise, and let $q = o + e_0$.

Theorem 6.9 (Photon production with exact energy exchange)**.** *Let $N = 3$, $C_E > 0$, any reference with $\sum \varrho_p = 1$, any anchor, and profiles with $z_{o,0} \neq 0$. Put $\eta = -i\big((H_{\phi[\xi]}\xi)_q\big)_1$, the one-photon part of the initial velocity at q, and*

$$\gamma(x) = \mathrm{Re}\big(\overline{\langle \Omega_{\mathrm{ph}}, x_o\rangle}\,\langle \eta, (x_q)_1\rangle\big)/\|\eta\|^2.$$

There is a global strong solution ψ of (6.7) with $\psi(0) = \xi$ and a $\delta > 0$ such that:

(i) $E(\psi(t)) = E(\xi)$ and $E_{\mathrm{ph}}(\psi(t)) = E_{\mathrm{nf}}(\xi) - E_{\mathrm{nf}}(\psi(t))$ for every real t;
(ii) for $0 < |t| < \delta$, the one-photon component of $\psi(t)_q$ is nonzero, $E_{\mathrm{ph}}(\psi(t)) > 0$ and $E_{\mathrm{nf}}(\psi(t)) < E_{\mathrm{nf}}(\xi)$;
(iii) γ is unchanged by a global phase, $t \mapsto \gamma(\psi(t))$ is continuously differentiable, $\gamma(\psi(0)) = 0$, its derivative at 0 is 1, and it is strictly increasing on $(-\delta, \delta)$.

The hypothesis $z_{o,0} \neq 0$ holds whenever the bonds carry the profiles of unit currents on lattice edges with $a > 0$, placed in any way.

Proof. ξ has finite photon moments, so Theorem 6.7(iv) gives a conserved history; $E_{\mathrm{ph}}(\xi) = 0$ because $H_0\Omega_{\mathrm{ph}} = 0$, and $E = E_{\mathrm{ph}} + E_{\mathrm{nf}}$ is constant, which is (i). For (ii), only the origin carries amplitude, so in (6.5) at $p = q$ the only surviving term is the hop from $q - e_0 = o$ (no other neighbour of q is o when $N = 3$): $(H_\phi\xi)_q = -c_{o,0}W(-z_{o,0})\Omega_{\mathrm{ph}}$, whose one-photon part is $c_{o,0}e^{-\|z_{o,0}\|^2/2}z_{o,0} \neq 0$ by (6.2). The one-photon component of $\psi(t)_q$ vanishes at $t = 0$ with this nonzero derivative, so it is nonzero for small $t \neq 0$; a nonzero one-photon vector has positive energy because $\omega > 0$ almost everywhere, and (i) gives the decrease of E_{nf}. For (iii), a phase

$e^{i\theta}$ multiplies the two factors of γ by $e^{-i\theta}$ and $e^{i\theta}$; at $t = 0$ the second factor vanishes and the derivative is $\langle \eta, \eta \rangle / \|\eta\|^2 = 1$; the derivative of ψ is continuous, so $\gamma \circ \psi$ is C^1 and increasing near 0. The last sentence is Lemma 6.4. □

The photons' energy is not supplied from outside: at every moment it equals exactly what the native and field energy has lost. The photon vacuum is not a beginning of this history: the history runs through it from both sides, and the state carries its own reading γ, which passes through zero at the vacuum with unit rate.

A localized start is a choice. Spread the excitation evenly instead: take the uniform reference $\varrho_p = 1/27$, the homogeneous vacuum X_0 with $X_{0,p} = 27^{-1/2}\Omega_{\mathrm{ph}}$ at every site, and profiles that do not depend on the site, $z_{p,k} = \zeta_k$; the axial case is $\zeta = (f, 0, 0)$. Let $\widehat{H}$ be H_ϕ at $\phi = 0$, $H_{\mathrm{red}} = H_0 + \sum_k (2 - W(\zeta_k) - W(\zeta_k)^*)$ on $\mathscr{F}$, and $X(t) = e^{-it\widehat{H}} X_0$.

Theorem 6.10 (The homogeneous vacuum)**.** *Let $N = 3$, $C_E \geq 0$, the uniform reference and site-independent profiles.*

(i) Every component of $X(t)$ is $27^{-1/2} e^{-itH_{\mathrm{red}}}\Omega_{\mathrm{ph}}$; for every t and every anchor, $\phi[X(t)] = 0$, $X(t) \in \mathscr{D}$, and X is a strong solution of (6.7).

(ii) $E(X(t)) = E(X_0)$ for every t. In the axial case

$$E(X(t)) = 2 - 2e^{-\|f\|^2/2}, \qquad 0 \leq E_{\mathrm{ph}}(X(t)) \leq 2 - 2e^{-\|f\|^2/2} < 2. \tag{6.12}$$

(iii) In the axial case with $C_E > 0$ and $f \neq 0$, the active homogeneous vacuum has strictly positive total energy although its photon energy is zero; the initial velocity has zero one-photon part and nonzero two-photon part at every site; and for every site q there is $\delta > 0$ such that for $0 < |t| < \delta$ the two-photon component of $X(t)_q$ is nonzero and $E_{\mathrm{ph}}(X(t)) > 0$.

(iv) If $C_E > 0$, a normalized $x \in \mathscr{D}$ with equal components at all sites and zero photon energy is a unit phase times X_0.

(v) For any N, any reference and any profiles, the selected field of a normalized x vanishes exactly when $\|x_p\|^2 = \varrho_p$ at every site.

Proof. (v) If the densities equal the reference, the right side of (6.6) vanishes and $\phi = 0$ solves it; by uniqueness $\phi[x] = 0$. Conversely, $\phi = 0$ makes the left side vanish. (i) At zero field every $c_{p,k} = 1$ and $d_0(p) = 6$; for x with all components equal to y, $(\widehat{H}x)_p = H_{\mathrm{red}} y$. $\widehat{H}$ commutes with lattice translations, so the equal-component subspace reduces it and $e^{-it\widehat{H}}X_0$ has the stated components. Each site then holds probability $1/27$, so $\phi[X(t)] = 0$ by (v), $H_{\phi[X(t)]} = \widehat{H}$, and Stone's theorem [117] gives $X(t) \in \mathscr{D}$ and $X' = -i\widehat{H}X$. (ii) At zero field $E(X) = \langle X, \widehat{H}X \rangle$, which the unitary group preserves; in the axial case $E(X_0) = 2 - 2\,\mathrm{Re}\langle \Omega_{\mathrm{ph}}, W(f)\Omega_{\mathrm{ph}} \rangle = 2 - 2e^{-\|f\|^2/2}$ by (6.2). Since $2 - W - W^* = (I - W)^*(I - W) \geq 0$, $H_{\mathrm{red}} \geq H_0 \geq 0$, which bounds the photon energy. (iii) Positivity is (ii) with $f \neq 0$. The initial velocity has components $-i\,27^{-1/2}(2\Omega_{\mathrm{ph}} - W(f)\Omega_{\mathrm{ph}} - W(-f)\Omega_{\mathrm{ph}})$; by (6.2) its one-photon part is proportional to $f + (-f) = 0$ and its two-photon part is $i\sqrt{2}\,27^{-1/2} e^{-\|f\|^2/2} f \otimes f \neq 0$. The rest is as in Theorem 6.9. (iv) For $C_E > 0$ the only vectors H_0 sends to zero are multiples of the vacuum, so zero photon energy forces every component to be a multiple of Ω_{ph}; equal components and normalization leave a phase times X_0. □

Photons are still produced, but in pairs: the single photons emitted across the two opposite bonds of each site cancel. Since $W(-f) = (-1)^{N_{\mathrm{ph}}} W(f) (-1)^{N_{\mathrm{ph}}}$ for the photon number N_{ph}, H_{red} commutes with $(-1)^{N_{\mathrm{ph}}}$, and $X(t)$ holds only even photon numbers at every time. Zero photon energy is not zero resource: the total $2 - 2e^{-\|f\|^2/2}$ sits in the bond energy of the initial state, and the dynamics only redistributes it. The zero-field condition (v) is preserved by any unitary acting site by site, but such unitaries are not the coupled dynamics; the orbit above stays at zero field because it is translation invariant.

The pairs give the homogeneous history its own clock. For a site q let w_q be the two-photon part of the initial velocity at q and $\gamma_q(x) = \mathrm{Re}\big(\overline{\sqrt{27}\langle \Omega_{\mathrm{ph}}, x_q\rangle}\langle w_q, (x_q)_2\rangle\big)/\|w_q\|^2$.

Theorem 6.11 (A clock read from the state)**.** *Let $N = 3$, $C_E > 0$, the uniform reference and an axial profile $f \neq 0$. There is a global strong solution u of (6.7) with $u(0) = X_0$, conserving E and with $E_{\mathrm{ph}}(u(t)) = E_{\mathrm{nf}}(X_0) - E_{\mathrm{nf}}(u(t))$ for all t, such that for every site q there is a function g with $g(0) = 0$, $g'(0) = 1$, and, near 0, $u(g(\gamma_q(u(t)))) = u(t)$ and $\gamma_q(u(g(\tau))) = \tau$; at every small $\tau \neq 0$, $g(\tau) \neq 0$, the two-photon component of $u(g(\tau))_q$ is nonzero, $E_{\mathrm{ph}}(u(g(\tau))) > 0$ and $E_{\mathrm{nf}}(u(g(\tau))) < E_{\mathrm{nf}}(X_0)$.*

Proof. $t \mapsto \gamma_q(u(t))$ is C^1, vanishes at 0 and has derivative $\langle w_q, w_q\rangle/\|w_q\|^2 = 1$ there, by Theorem 6.10(iii); the inverse function theorem gives the local inverse g. The remaining statements are Theorem 6.10(ii)–(iii) and the exchange identity at zero field. □

Near the vacuum, the state is a function of its own two-photon reading at any one site: the history carries a local clock. This clock is a reading of the state, defined near one instant; it is not a global time and it fixes no unit (Chapter 4).

The bounded budget (6.12) rules out a tempting identification. The paired cost of a first tick, $2J(r)$ (Chapter 3), is unbounded in r, while the homogeneous resource is below two.

Proposition 6.12 (No fixed conversion to paired cost)**.** *Let $N = 3$, $C_E \geq 0$, the uniform reference and axial profiles.*

(i) If the total energy of the homogeneous axial orbit equals $k \cdot 2J(r)$ at some time, then $k \cdot 2J(r) < 2$.

(ii) For any fixed $k > 0$ there is no family of profiles $r \mapsto f(r)$ with $E(X_0) = k \cdot 2J(r)$ for all $r > 0$, and no family of profiles and reading times with $E_{\mathrm{ph}} = k \cdot 2J(r)$ for all $r > 0$.

Proof. (i) is (6.12). (ii) Given $k > 0$ and any bound $c \geq 0$, the ratio $r = 2 + (c+1)/k$ has $k \cdot 2J(r) > c$, because $2J(r) = r + 1/r - 2 \geq r - 2$. Taking $c = 2$ contradicts (i) for both readings. □

6.4 Currents, bond response and what the vacuum fixes

The self-consistent carrier has three natural currents: the constitutive current of the Gauss law, the Born transport current (6.9), and the transverse profile of a lattice current cut out of the torus. To compare the last with the others, read the sites of Λ_N as the integer points $\{0, \dots, N-1\}^3$ and each bond, wrap bonds included, as a literal edge of $\mathbb{Z}^3$; the *cut profile* of a

bond current is the profile (6.4) of the resulting finite current. The localized vacuum at q is the state with Ω_{ph} at q and zero elsewhere.

The localized vacuum on Λ_3 with the uniform reference is an exact finite computation. Its Born source is $\frac{1}{27} - \delta_{p,q}$; every bond touching q has screened weight $\frac{1}{2}$ (one end is empty) and every other bond weight 1. Group the sites by the number n of nonzero coordinates of $p - q$ (shells of sizes $1, 6, 12, 8$); a lattice step stays in its shell or moves to a neighbouring one. The potential V of Table 6.1 satisfies $2\sum_{j\sim p}(V_p - V_j) = \frac{1}{27} - \delta_{p,q}$ at every site, which in units of $1/324$ is the integer identity $2\sum_{j\sim p}(324V_p - 324V_j) = 12 - 324\,\delta_{p,q}$, checked shell by shell. The constitutive identity $\beta\sinh(\Delta\phi) = \Delta V$ then gives the field drops.

Table 6.1. The selected field of the localized vacuum on Λ_3 (uniform reference), by shell n around the occupied site. Direct exact computation.

shell n	sites	$324\,V$	bond weight to shell $n+1$	field drop to shell $n+1$
0	1	-33	$1/2$	$\operatorname{arsinh}(-13/81)$
1	6	-7	1	$\operatorname{arsinh}(-5/324)$
2	12	-2	1	$\operatorname{arsinh}(-1/162)$
3	8	0	none	none

Proposition 6.13 (The currents are not interchangeable)**.** *Let $a > 0$ and $C_E > 0$.*

(i) (Localized vacuum, $N = 3$, uniform reference.) *For every profile family, anchor and site q, the constitutive current of the selected field of the localized vacuum at q is the exact lattice gradient of the potential V of Table 6.1, so it pairs to zero with every divergence-free periodic current; and its Born transport current vanishes identically. For $q \neq o$ the two currents differ, the cut profile of the constitutive current is nonzero, and there is exactly one nonzero axial profile f equal to the cut profile of the screened current that f itself selects.*

(ii) (Any N.) *A zero cut profile forces zero divergence at the corner o; so a nonzero Born source at the corner forces a nonzero cut profile of the screened constitutive current.*

(iii) (A longitudinal source, $N = 2$.) *The source equal to twice the divergence of the current of a field spike of height $h > 0$ is nonzero, the spike's current is nonzero, and every solution of the Gauss law for this source has zero cut profile.*

(iv) (A dipole, $N = 2$.) *For a dipole with one end at the corner and nonzero strength, every Gauss solution has nonzero cut profile.*

(v) (Range.) *For real currents f, g with $z_f \neq 0$, $z_g = c\,z_f$ forces $c \in \mathbb{R}$; in particular iz_f is the profile of a real current only if $z_f = 0$.*

Proof. (i) The field constant on each shell, with the drops of the table, solves the screened Gauss law, as checked (the constitutive identity turns the law into the integer identity for V); by Theorem 6.7(ii) the anchored solution is unique, so the selected field has the drops of the table and its constitutive current is $V_p - V_{p+e_k}$, a gradient. Summation by parts on the torus pairs a gradient to zero with any divergence-free current. The Born current (6.9) of the localized vacuum is zero because every bond has an empty end. Nonzero cut profile: by (ii), since the Born source at the corner is $1/27 \neq 0$. For the localized vacuum the screened weights do not depend on the profiles, so neither do the selected field and its cut profile, which gives the unique self-consistent f. (ii) If a profile vanishes, every plaquette circulation of its

current vanishes; summing them around the corner of the cut cube expresses the corner edges through edges outside the cube, which are zero, so the divergence at the corner vanishes. (iii) The cut current of the spike is $\sinh h$ times the gradient of a unit charge, and P_T annihilates gradients. (iv) At $N = 2$ the plaquette relations make the incoming corner edges the negatives of the outgoing ones, so a zero profile would give zero divergence at the corner, contradicting the dipole. (v) Inner products of real-current profiles are real, because $k \mapsto -k$ conjugates the integrand; so $c\|z_f\|^2 = \langle z_f, z_g\rangle \in \mathbb{R}$. □

Proposition 6.14 (Two boundaries)**.** *Let $N = 3$.*

(i) Let $a \geq 0$, $C_E > 0$, the uniform reference and an axial profile f. The profile f equals the cut profile of the field selected along the homogeneous orbit at some time if and only if $f = 0$; and then the photon energy of the orbit is zero at all times.

(ii) Let $C_E \geq 0$. For every profile family, reference and anchor, the localized vacuum has bond energy $\mathscr{B}_0 = 6$ at zero field and total energy at least 6. With the uniform reference and an axial profile, its total energy exceeds that of the homogeneous vacuum by more than 4, and the homogeneous orbit never reaches a localized vacuum.

Proof. (i) The selected field is zero along the homogeneous orbit (Theorem 6.10), so its constitutive current and cut profile vanish; $f = 0$ makes the coupling trivial and the vacuum stationary. (ii) At zero field each of the six bonds at q contributes $\|\Omega_{\mathrm{ph}}\|^2 = 1$ and all others zero, and Theorem 6.7(ii) gives $E \geq \mathscr{B}_0 = 6$. The homogeneous energy is $2 - 2e^{-\|f\|^2/2} < 2$ by (6.12). Energy is conserved, so the orbit cannot reach a state of different energy. □

Identifying the endpoint of a homogeneous history with a localized one would need more than four units of bond energy from outside.

6.4.1 The vacuum response fixes the profile, not the operator

A vacuum state is a vector whose every component is a multiple of Ω_{ph}, and $\xi^{(p)}$ is the localized vacuum at p. Write Π_n for the projection onto the n-photon sector and W_u for $W(u)$ acting on every component. For a bounded operator B on $\mathscr{H}_{\Lambda_N}$, which may mix sites and photon numbers, put $H_B = H_\phi + B$ on $\mathscr{D}$; the *work vector* of the displacement u from the vacuum at p is $w_B(u, p) = H_B W_u \xi^{(p)} - W_u H_B \xi^{(p)}$.

Theorem 6.15 (What the vacuum response fixes)**.** *Let $C_E \geq 0$, any field ϕ and any profiles.*

(i) ($N = 3$.) The vacuum amplitude $\langle \Omega_{\mathrm{ph}}, (H_\phi \xi^{(o)})_q\rangle$ is nonzero and

$$z_{o,0} = -\frac{\big((H_\phi \xi^{(o)})_q\big)_1}{\langle \Omega_{\mathrm{ph}}, (H_\phi \xi^{(o)})_q\rangle} \ :$$

the first response of the localized vacuum reads the bond profile.

(ii) Let $n \geq 1$, $\kappa \in \mathbb{R}$ and $H' = H_\phi + \kappa \Pi_n$. Then H' is self-adjoint on $\mathscr{D}$, $H'x = H_\phi x$ for every vacuum state x, so (i) returns the same profile; $\langle x, H'x\rangle = \langle x, H_\phi x\rangle + \kappa \sum_p \|\Pi_n x_p\|^2$; and on a nonzero n-photon state at one site H' and H_ϕ differ exactly when $\kappa \neq 0$.

(iii) Let $A \subset \mathscr{H}_1$ be dense. If $H_B \xi^{(p)} = H_\phi \xi^{(p)}$ and $w_B(u, p) = w_0(u, p)$ for every site p and every $u \in A$, then $B = 0$.

(iv) Let $a > 0$ and $C_E > 0$. The finite sums of freely evolved profiles $e^{-it\omega}g$, with $t \in \mathbb{R}$ and g the profile of an integer current on the edges of $\mathbb{Z}^3$, form a dense subset of $\mathscr{H}_1$; so (iii) holds with these displacements.

Proof. (i) Only the hop from o reaches q, so $(H_\phi \xi^{(o)})_q = -cW(-z_{o,0})\Omega_{\mathrm{ph}}$ with $c = \cosh(\phi_o - \phi_q) > 0$; by (6.2) its vacuum amplitude is $-ce^{-\|z\|^2/2} \neq 0$ and its one-photon part is $ce^{-\|z\|^2/2} z_{o,0}$. Divide. (ii) Π_n is a bounded symmetric projection that annihilates the vacuum, so H' is self-adjoint on the same domain [95] and agrees with H_ϕ on vacuum states; on a state x with $\Pi_n x = x \neq 0$, $H'x - H_\phi x = \kappa x$. (iii) Since B is bounded and W_u preserves $\mathscr{D}$, $w_B - w_0 = BW_u\xi^{(p)} - W_u B\xi^{(p)}$; the first hypothesis gives $B\xi^{(p)} = 0$, so $BW_u\xi^{(p)} = 0$ for $u \in A$. By (6.2), $\|W(u)\Omega_{\mathrm{ph}} - W(v)\Omega_{\mathrm{ph}}\|^2 = 2 - 2\,\mathrm{Re}\, e^{-(\|u\|^2+\|v\|^2)/2+\langle u,v\rangle}$, so $u \mapsto W(u)\Omega_{\mathrm{ph}}$ is continuous and B vanishes on every exponential vector at every site; exponential vectors span a dense subspace of $\mathscr{F}$ [51, 84]. (iv) is proved in Appendix B.4. □

One hop out of a localized vacuum reads its bond's profile exactly, but not the operator; the work of displacing a localized vacuum along a dense set of displacements excludes every bounded correction. Unbounded corrections and corrections that change the domain lie outside this statement. Matching the vacuum action and the work are hypotheses of (iii); the identification of the work vector with the work exchanged in an actual displacement of the field is a physical premise.

6.5 The source and its recorder

A record is written by a physical interaction. The source and recorder, carrier (g), is defined in Section 2.7, where it shows that energy does not select a blank record. We use its notation: the source space $\mathcal{S} = \ell^2(\mathbb{Z}^3; \mathbb{C}^{4096} \otimes \mathscr{F})$ of one negative unit charge carrying the four registers of one cell and the photon field, with a positive unit charge fixed at the origin; the source Hamiltonian H_{src}, whose hop from $r - e_d$ to r applies the unitary $U_d(r)$ that displaces the field by the unit-current profile of the edge crossed and advances the registers, with coupling g; the joint carrier $\mathscr{H}_{\mathrm{SR}}$ of source and record label $q \in \mathbb{Z}^3$, with blank label 0; the blank embedding B; the writer W, a unitary involution that swaps the labels 0 and r; the contact $C_\kappa = \frac{\kappa}{2}(I - W)$ at rate $\kappa \in \mathbb{R}$; the joint Hamiltonian $H_\kappa = H_{\mathrm{src}} \otimes I + C_\kappa$; and its energy $E_\kappa(\Psi) = \mathrm{Re}\langle \Psi, H_\kappa \Psi\rangle$. Here $\Psi(r)$ denotes the whole fibre over the source site r (all labels, codes and photon states), the copy is $(\mathrm{copy}\,\psi)(r, q) = \delta_{qr}\psi(r)$, so that $WB\psi = \mathrm{copy}\,\psi$, and during a hop the record label is held fixed. We prove what the dynamics does: it conserves probability locally, it splits its energy exactly, and writing a record has a definite price.

Theorem 6.16 (Joint generator, continuity and energy). *Let $a > 0$, $C_E > 0$, and g, ϵ, κ real.*

(i) H_κ is self-adjoint on the domain of its unbounded diagonal part, which does not depend on κ; e^{-itH_κ} preserves it and solves the Schrödinger equation there. No spatial or photon-number cutoff is used.

(ii) Continuity. *For every Ψ in the domain, every t and site r, with $\Psi_t = e^{-itH_\kappa}\Psi$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Psi_t(r)\|^2 = \sum_{d=0}^{2}\big[J_d(r - e_d) - J_d(r)\big], \qquad J_d(r) = 2g\ \mathrm{Im}\langle \Psi_t(r), U_d(r + e_d)^{-1}\Psi_t(r + e_d)\rangle.$$

The contact and the site terms contribute nothing: they act inside each fibre.

(iii) Energy. $E_\kappa(\Psi) = \mathrm{Re}\langle\Psi, (H_{\mathrm{src}} \otimes I)\Psi\rangle + \mathrm{Re}\langle\Psi, C_\kappa\Psi\rangle$ *on the domain,* E_κ *is constant along the flow, any gain of source energy equals the loss of contact energy, and*

$$\mathrm{Re}\langle\Psi, C_\kappa\Psi\rangle = \frac{\kappa}{4}\|\Psi - W\Psi\|^2.$$

Consequently, for $\kappa \geq 0$ *the contact energy is nonnegative and the source energy at every later time is at most the initial total energy, with no positivity assumed for the bare source; and for two rates,* $E_{\kappa_1}(\Psi) - E_{\kappa_0}(\Psi) = \frac{\kappa_1 - \kappa_0}{4}\|\Psi - W\Psi\|^2$.

(iv) Writing. *For a source state* ψ *in the domain of* H_{src} *and a site* $r \neq 0$*, the matching amplitude* $\Psi_t(r, r)$ *of* $\Psi = B\psi$ *vanishes at* $t = 0$ *with derivative* $i\frac{\kappa}{2}\psi(r)$*. If* $\kappa \neq 0$ *and* $\psi(r) \neq 0$*, it has strictly positive squared norm for all small* $t > 0$*. If* $\kappa > 0$ *and* $\psi(r) \neq 0$*, the blank record* $B\psi$ *has strictly positive contact energy.*

Proof. (i) The hopping and the contact are bounded and symmetric perturbations of a self-adjoint direct sum [95]. (ii) Differentiate $\|\Psi_t(r)\|^2$ along the flow: $\frac{\mathrm{d}}{\mathrm{d}t}\|\Psi_t(r)\|^2 = 2\,\mathrm{Im}\langle\Psi_t(r), (H_\kappa\Psi_t)(r)\rangle$. The site operator and the contact act within the fibre over r and are symmetric, so they contribute zero imaginary part; the hopping contributes the currents across the six edges at r. (iii) The split is linearity of the expectation; conservation holds because the flow commutes with H_κ on its domain. Since W is a self-adjoint unitary, $\|\Psi - W\Psi\|^2 = 2\|\Psi\|^2 - 2\,\mathrm{Re}\langle\Psi, W\Psi\rangle = 2\,\mathrm{Re}\langle\Psi, (I - W)\Psi\rangle$. The ceiling follows from conservation and $\mathrm{Re}\langle\Psi, C_\kappa\Psi\rangle \geq 0$. The source part and the domain do not depend on the rate, which gives the rate difference. (iv) The source part does not change the label, and the contact maps the label-0 amplitude at r to label r: $(H_\kappa B\psi)(r, r) = -\frac{\kappa}{2}\psi(r)$. For $\kappa > 0$, $WB\psi = \mathrm{copy}\,\psi \neq B\psi$ at r. □

The isolated writer pulse, e^{-itC_κ} with $\kappa t = \pi$, is exactly W; the results above concern the full joint evolution, with the source acting throughout.

How much does the source move? On $\mathcal{S}$ the edge current of a state is $j_{r,d}(\psi) = 2g\ \mathrm{Im}\langle\psi(r), U_d(r + e_d)^{-1}\psi(r + e_d)\rangle$, and its activity is $\mathcal{T}(\psi) = \sum_d \sum_r |j_{r,d}(\psi)|$, the least traffic that realizes the currents.

Proposition 6.17 (Activity, and records that move)**.** *Let* $a > 0$*,* $C_E > 0$*,* g, ϵ *real.*

(i) $\mathcal{T}(\psi) \leq 6|g|\|\psi\|^2$ *and* $|\mathcal{T}(\psi) - \mathcal{T}(\chi)| \leq 6|g|(\|\psi\| + \|\chi\|)\|\psi - \chi\|$*. Along the source evolution the activity stays below* $6|g|\|\psi_0\|^2$*, and the traffic over a time interval* $[s, t]$*, the integral of the activity, is at most* $6|g|\|\psi_0\|^2(t - s)$*.*

(ii) With record labels held fixed during hops, the copied state $\mathrm{copy}\,\psi$ *carries zero current on every edge. If instead the label moves with the source (*$q \mapsto q + e_d$ *on a hop along* d*), the current of* $\mathrm{copy}\,\psi$ *is the source's own edge current* $j_{r,d}(\psi)$*.*

Proof. (i) Each axis contributes at most $2|g|\|\psi\|^2$ by the Cauchy-Schwarz inequality and unitarity of U_d; the stability bound is the same estimate applied to the difference. The evolution is unitary. (ii) In $\mathrm{copy}\,\psi$ the fibre at r sits on label r and the fibre at $r + e_d$ on label $r + e_d$; a label-preserving hop pairs orthogonal labels. A moving label maps label r to $r + e_d$ and restores the overlap. □

The activity is an intensity, not a law of event paths: it bounds how much probability moves without saying which path a history takes. A sharp copied position label stops the current, and a tag that moves with the source restores the current only by replacing its old value at every step: a moving tag is not a retained history.

6.6 Records and controllers

A reversible writer reused on one label erases what it wrote (Chapter 2). A history that keeps its past needs a new place for each new record.

Give the source two record labels, an earlier q_1 and a new q_2; the first writer swaps 0 and r in q_1, the second in q_2.

Proposition 6.18 (Two writers)**.** *The two writers commute on every state. The second preserves the squared norm of the amplitude at every fixed (r, q_1), and if that amplitude is blank in q_2, the second writer moves it to $q_2 = r$ and leaves q_1 untouched.*

The proof is a direct check: the writers act on different labels. For the full evolution, attach a blank new record to each branch of the earlier record and evolve by e^{-itH_κ}. More generally, let $\mathscr{Z}$ be any Hilbert space, a retained reference of arbitrary dimension, and let $L_t = (e^{-itH_\kappa}B) \otimes I_{\mathscr{Z}}$ on the completed tensor product $\mathscr{S} \otimes \mathscr{Z}$.

Theorem 6.19 (Fresh records retain the past)**.** *Let $a > 0$ and $C_E > 0$.*

(i) L_t is an isometry, independent of the basis used to build it, and preserves every Gram entry between reference coordinates, entangled inputs included. In particular, for the earlier-record branches u_q, $\langle L_t u_q, L_t v_p\rangle = \langle u_q, v_p\rangle$ for all q, p and t, and L_0 is the blank embedding.

(ii) If $\kappa \neq 0$ and a branch lies in the domain of H_{src} with nonzero amplitude at a site $r \neq 0$, its new matching record has strictly positive weight for all small $t > 0$.

(iii) Iterate: at each step attach a blank to every branch and evolve. For any schedule of times the map is an isometry and every Gram entry of earlier branches survives. Every record slice of a domain state lies in the source domain, so a newly written matching branch is ready again.

(iv) Hence, if $\kappa \neq 0$, $r \neq 0$ and a branch is ready at r, then for every length m and every bound $T > 0$ there are times $0 < t_1, \dots, t_m < T$ for which the chain of matching records has positive weight at each step, with readiness, norm and all earlier Gram entries preserved.

Proof. (i) Attaching a blank is an isometry and e^{-itH_κ} is unitary; tensoring with $I_{\mathscr{Z}}$ keeps both, and isometries preserve inner products. (ii) is Theorem 6.16(iv) applied to the branch. (iii) The evolution preserves the domain of H_κ, and a component of a domain vector at one label lies in the domain of H_{src}; the positive matching weight is a nonzero amplitude at r. (iv) follows by induction on m from (ii) and (iii). □

A history that keeps its past is therefore made of a supply of fresh blanks, one per record, each written by the same contact. The blanks and the schedule are supplied.

To switch the contact on without changing the Hamiltonian by hand, hold both records on one carrier and add a two-level controller. For each earlier label the source and new record evolve under H_κ; the direct sum over earlier labels is a self-adjoint operator $H_\kappa^{(2)}$ with rate-independent domain, whose second writer W_2 commutes with the first and whose contact

obeys $\mathrm{Re}\langle\Psi, C_\kappa^{(2)}\Psi\rangle = \frac{\kappa}{4}\|\Psi - W_2\Psi\|^2$. In controller level $i \in \{0,1\}$ the records evolve under $H_{\kappa_i}^{(2)}$, and a Hermitian 2×2 matrix K acts on the levels:

$$G = \begin{pmatrix} H_{\kappa_0}^{(2)} & 0 \\ 0 & H_{\kappa_1}^{(2)} \end{pmatrix} + K \otimes I.$$

Theorem 6.20 (Controlled contact). *Let $a > 0$, $C_E > 0$, K Hermitian. G is self-adjoint on $\mathscr{D}(H_{\kappa_0}^{(2)}) \times \mathscr{D}(H_{\kappa_1}^{(2)})$; along $\Psi(t) = e^{itG}\Psi$ the energy $E_G = \mathrm{Re}\langle\Psi, G\Psi\rangle$ is constant and splits into the energies of the two conditional record states plus the controller energy $\mathrm{Re}\langle\Psi, (K\otimes I)\Psi\rangle$. Suppose $\kappa_1 > 0$, level 1 is initially empty, $K_{10} \neq 0$ and $W_2\Psi_0 \neq \Psi_0$. Let $\mathscr{C}(t)$ be the contact energy in level 1 and $\mathscr{R}(t) = E_G - \mathscr{C}(t)$ the rest. Then $\mathscr{C}(0) = 0$, and for every small $t > 0$, $\mathscr{C}(t) > 0$, $\mathscr{R}(t) < \mathscr{R}(0)$ and $\mathscr{R}(t) + \mathscr{C}(t) = \mathscr{R}(0)$.*

Proof. G is a direct sum of self-adjoint operators plus a bounded Hermitian matrix; its flow preserves the domain and commutes with G. For small t, $\Psi_1(t) = itK_{10}\Psi_0 + o(t)$, so $\mathscr{C}(t) = t^2|K_{10}|^2\frac{\kappa_1}{4}\|\Psi_0 - W_2\Psi_0\|^2 + o(t^2) > 0$, and conservation gives the rest. □

The hypotheses hold for an explicit normalized state, a source at a site $r \neq 0$ in a native basis code with no photons, a sharp earlier label and a blank new record in level 0, and for any source state in the domain with nonzero amplitude away from the blank label. They also hold when a supplied pair amplitude v on registers 0 and 1 of the source, with the two other registers held at fixed values and the photon vacuum, has some $v(i,j) \neq 0$: a pair state drives the contact. The budget does not say which part pays: the rest contains the controller energy and both conditional record energies. The controller's energy remainder is an unattributed accounting term, and preparing the controller, the rate, the blanks and the initial state is not derived.

A held register does not stay held. In the native code (i,j,s,t) of the four registers, with (s,t) the spectators beside a pair (i,j):

Proposition 6.21 (Spectators feel the source). *(i) The source clock energy has the matrix element ϵC_{10} from the code $(0,0,0,0)$ to $(0,0,1,0)$, and $\mathrm{Re}\, C_{10} = -\frac{1}{2}$; so for $\epsilon \neq 0$ the clock term moves amplitude out of the sector with spectators fixed at $(0,0)$.*

(ii) Let $a > 0$, $C_E > 0$, $g \neq 0$. With every input of the fixed-charge source constructed, the source sends the normalized pair vacuum carrying two spectators in the clock ground state χ at r to an output at $r + e_2$ outside the subspace in which both spectators are in χ.

Proof. (i) The clock term acts on register two as ϵC, and the formula for C after Definition 6.2 gives $C_{10} = -\frac{1}{2} + \frac{i}{2}(1+\sqrt{2})$. (ii) The component of the output at $r + e_2$ is $-g$ times the posting unitary of axis two applied to the input. That unitary displaces the field and applies the parity signs σ to register two and the low-bit flip to register three. The flip fixes χ; the signs send χ to $\Sigma\chi$, which is orthogonal to χ because $\sum_x \sigma_x = 0$, so register two leaves its clock ground state. □

This concerns the clock term and one hop at one instant; whether the full coupled evolution leaves the spectator sector at all small times is not established here.

6.7 Many cells: the carrier and the clock chart

The cell carrier $\mathscr{H} = \ell^2(\mathscr{X}; \mathscr{F})$ gives every cell four registers and keeps one photon field for the whole lattice. Its reference vector $\Omega = \delta_0 \otimes \Omega_{\mathrm{ph}}$, the all-zero configuration with no photons, has norm one and is invariant under relabelling the cells, and the free field energy $\widehat{H}_0 = \bigoplus_{\xi \in \mathscr{X}} H_0$ appears once and is self-adjoint with $\widehat{H}_0 \Omega = 0$. Finite patches of cells embed isometrically, and an operator on one cell's internal space acts on that cell's embedding as it acts on the cell. The carrier is a finite-deviation sector around the zero configuration, a chart origin (Section 6.1), not an infinite tensor product.

In the clock's spectral chart the clocks alone are diagonal. Let $\mathscr{E}_\epsilon(\xi) = \epsilon \sum_r \sum_k m_{r,k}(\xi)$, where $m_{r,k} \in \{0, \dots, 7\}$ is the mode number of register k of cell r.

Proposition 6.22 (The clock spectral chart)**.** *Let $C_E \geq 0$ and ϵ real. $H_\epsilon^{\mathrm{sp}} = \bigoplus_\xi (H_0 + \mathscr{E}_\epsilon(\xi))$ is self-adjoint; ψ lies in its domain exactly when every component lies in $\mathscr{D}(H_0)$ and $\xi \mapsto (H_0 + \mathscr{E}_\epsilon(\xi))\psi(\xi)$ is square summable; and on a vector supported on one configuration ξ, the finite-region clock Hamiltonian of any region containing the support of ξ agrees with H_ϵ^{sp}.*

The proof is that $C = FDF^*$ is diagonal after the Fourier change of each register. This chart carries the clocks and the free field, not the occupation coupling, and no global unitary identifying it with the native representation is proved. The interacting operator on finitely many registers, the physical join, is the fibre operator of the next section (Theorem 6.25), with all 4096 coordinates of each cell kept. Applied to any register state times the photon vacuum, it produces no two-photon component at that instant, since each source changes the photon number by one. Its state and parameters are not selected.

6.8 Finite regions of the clock-photon lattice

We now let many registers drive the one photon field. The question is whether a finite region of complete registers, with no truncation of the registers or the field, has a well-defined energy operator that is bounded below, and what happens to it as the region grows.

6.8.1 The regional operator

For a finite region $R \subset \Lambda$ write $n = 4|R|$ and $\mathscr{N}_R = (\mathbb{C}^8)^{\otimes n}$; Q_j and C_j act on register j. The clock energy of the region is

$$K_R = \epsilon \sum_{j=1}^{n} C_j, \qquad \epsilon \geq 0, \tag{6.13}$$

and the construction allows any Hermitian native matrix K on $\mathscr{N}_R$, not necessarily a sum of local terms. Splitting a configuration into its labels inside R and its exterior labels η is a unitary $J_R : \mathscr{H} \to \ell^2(\mathscr{X}_{R^c}; \mathscr{N}_R \otimes \mathscr{F})$ that only reindexes the carrier: registers outside R are kept as passive labels, and the field is never truncated.

Definition 6.23 (Regional Hamiltonian)**.** Given a Hermitian K, a background $b \in \mathscr{H}_1$ and register profiles $z_1, \dots, z_n \in \mathscr{H}_1$, the fibre operator is

$$H_R^{\mathrm{fib}} = I \otimes H_0 + K \otimes I + I \otimes S(b) + \sum_{j=1}^{n} Q_j \otimes S(z_j) \tag{6.14}$$

on $\mathscr{D}_R = \mathscr{N}_R \otimes \mathscr{D}(H_0)$, and the regional Hamiltonian is $H_R = J_R^*\big(\bigoplus_\eta H_R^{\mathrm{fib}}\big)J_R$, on the vectors ψ with $u = J_R\psi$ such that every $u_\eta \in \mathscr{D}_R$ and $\sum_\eta \|H_R^{\mathrm{fib}} u_\eta\|^2 < \infty$.

The second domain condition matters: membership of every fibre does not by itself put ψ in the domain. The operator is uncompensated: it contains the sources as they are, with no added counterterm for the energy each occupation pattern gains by displacing the field. That is its physical form, and the stability theorem below is about this operator; Chapter 7 shows why the uncompensated form matters for the infinite lattice.

A linear source on a boson field is removed by shifting the field, at a definite energy cost; this is van Hove's identity [121, 30].

Lemma 6.24 (Compensated source)**.** *For every $g \in \mathscr{H}_1$, $W(g)$ maps $\mathscr{D}(H_0)$ onto itself, and on that domain*

$$A(g) := H_0 + S(g) + \|\sqrt{h}\, g\|^2 I = W(g)^* H_0 W(g).$$

Hence $A(g)$ is self-adjoint and nonnegative on $\mathscr{D}(H_0)$.

Proof. For $\psi \in \mathscr{D}(H_0)$, $\|a(hg)\psi\| \le \|\sqrt{h}\, g\| \|H_0^{1/2}\psi\|$ and $\|a^*(hg)\psi\|^2 = \|a(hg)\psi\|^2 + \|hg\|^2\|\psi\|^2$. With $\|H_0^{1/2}\psi\|^2 \le \|\psi\|\|H_0\psi\|$ this gives $\|S(g)\psi\| \le \delta\|H_0\psi\| + (\|hg\| + \|\sqrt{h}\, g\|^2/\delta)\|\psi\|$ for every $\delta > 0$, so $A(g)$ is self-adjoint on $\mathscr{D}(H_0)$ by the Kato-Rellich theorem [95]. On exponential vectors $W(g)\varepsilon(f) = e^{-\|g\|^2/2 - \langle g, f\rangle}\varepsilon(f+g)$, which lie in $\mathscr{D}(H_0)$ because h is bounded; differentiating in f gives $H_0 W(g)\psi = W(g)A(g)\psi$ on their span. That span is a core for H_0 and, by the relative bound, for $A(g)$; closedness extends the identity to $\mathscr{D}(H_0)$, and the same argument for $-g$ gives the reverse inclusion. □

Choose an orthonormal occupation basis for each register and let U carry it to the native basis. A full occupation index is $s = ((q_j(s), \alpha_j(s)))_{j \le n}$. Put

$$g_s = b + \sum_j q_j(s) z_j, \qquad d_s = \|\sqrt{h}\, g_s\|^2, \qquad M = U^* K U - \mathrm{diag}(d_s). \tag{6.15}$$

Real linearity of S and the diagonal form of each Q_j give

$$U^* H_R^{\mathrm{fib}} U = \bigoplus_s A(g_s) + M \otimes I. \tag{6.16}$$

Theorem 6.25 (Regional construction)**.** *Let $C_E > 0$. For every finite region, Hermitian K, background b and profiles z_j, the operators H_R^{fib} and H_R are self-adjoint on the domains of Definition 6.23. With the unitary $(Vu)_s = W(g_s)u_s$,*

$$V U^* H_R^{\mathrm{fib}} U V^* = I \otimes H_0 + B, \qquad (Bu)_s = \sum_t M_{st} W(g_s) W(g_t)^* u_t,$$

and $B = V(M \otimes I)V^*$ is bounded and self-adjoint with $\|B\| \le \|M \otimes I\|$. The native clock, the background and all occupation couplings are retained, and every exterior configuration is kept.

Proof. By (6.16), $U^* H_R^{\mathrm{fib}} U$ is a finite direct sum of the self-adjoint operators of Lemma 6.24 plus a bounded Hermitian matrix, hence self-adjoint on the componentwise domain, which U preserves. The maximal orthogonal direct sum over exterior configurations is self-adjoint on the stated domain, and J_R carries it to H_R. Lemma 6.24 shows that V preserves the componentwise domain; conjugating (6.16) by V gives the dressed form. □

Each occupation pattern drags the field into its own displaced vacuum, and the native matrix moves the registers between patterns; the Weyl products keep their phases, so the mixing remembers how far apart two patterns displace the field. The bound $\|B\| \le \|M \otimes I\|$ holds at fixed M. It is not a stability bound uniform in the region at fixed physical K, because M contains the compensations d_s, and those decide how low the energy can go.

6.8.2 Currents and regional stability

Lemma 6.26 (Current energy)**.** *Let $a \ge 0$ and $C_E > 0$. For every finitely supported real current f,*

$$\|\sqrt{h}\, z_f\|^2 \le \frac{a}{2} \sum_{e \in \mathscr{E}} f(e)^2.$$

Proof. Where $\lambda(k) > 0$, $\omega w^2 = a \operatorname{arsinh}(\sqrt{\lambda}/2)/(\sqrt{\lambda}\sqrt{1+\lambda/4}) \le a/2$, since $\operatorname{arsinh} u \le u$. The projection P_T contracts, and Parseval's identity for μ gives $\int \|F_f\|^2 \, \mathrm{d}\mu = \sum_e f(e)^2$. □

No divergence-free condition is used; the weight $\omega w^2/a$ approaches its bound $\frac{1}{2}$ only at long wavelength.

Theorem 6.27 (Regional stability)**.** *Let $a \ge 0$, $C_E > 0$, $\epsilon \ge 0$, let $R \subset \mathbb{Z}^3$ be finite, and let $e : \{1, \dots, 4|R|\} \to \mathscr{E}$ be injective. Take no background and the unit profiles $z_j = z_{e(j)}$ on all $4|R|$ registers. For every positive semidefinite native matrix K, in particular the clock energy K_R, the regional Hamiltonian is self-adjoint and*

$$\operatorname{Re}\langle \psi, H_R \psi \rangle \ge -2a|R| \, \|\psi\|^2 \qquad (\psi \in \mathscr{D}(H_R)). \tag{6.17}$$

In the occupation basis the physical operator is the nonnegative compensated part $\bigoplus_s A(g_s) + U^*KU \otimes I$ minus the compensation d_s of each occupation pattern, and $0 \le d_s \le 2a|R|$.

Proof. Real linearity of $f \mapsto z_f$ gives $g_s = z_{f_s}$ with $f_s = \sum_j q_j(s) \delta_{e(j)}$. Because e is injective the coefficients do not pile up on one edge: $\sum_e f_s(e)^2 = \sum_j q_j(s) \le 4|R|$, so $0 \le d_s \le 2a|R|$ by Lemma 6.26. For u in the componentwise domain, (6.16) gives

$$\langle u, U^* H_R^{\mathrm{fib}} U u \rangle = \sum_s \langle u_s, A(g_s) u_s \rangle + \langle u, (U^*KU \otimes I) u \rangle - \sum_s d_s \|u_s\|^2 \ge -2a|R| \|u\|^2.$$

For $u = J_R \psi$ with ψ in the domain the fibre sums converge absolutely by the Cauchy-Schwarz inequality; summing over η and using the unitarity of J_R gives (6.17). The clock energy is positive semidefinite because $C \ge 0$ and $\epsilon \ge 0$. □

The floor is $-2a$ per active cell because each of the four registers pays at most $a/2$, however many photons are present. The constant bounds the energy from below; the ground-state energy density is not computed. Zero background and distinct edges are hypotheses.

6.8.3 The clock ground state fluctuates

A floor says nothing about how far the Hamiltonian moves a given state, and a limit of regional Hamiltonians on one fixed vector needs exactly that motion to stay bounded. The natural vector is every active register in its clock ground state χ with no photons.

Take one register, no background, and $v = \chi \otimes \Omega_{\mathrm{ph}}$. The free energy annihilates the vacuum and the native matrix keeps the photon number, so the one-photon output is $(Q \otimes S(z))v = Q\chi \otimes hz$. Write $Q = \frac{1}{2}I + (Q - \frac{1}{2}I)$. The second part is $-\frac{1}{2}$ charge, half a self-adjoint involution, so it halves every norm; and $(Q - \frac{1}{2})\chi \perp \chi$, because $T\chi = \chi$ and $\sum_x \sigma_x = 0$. The response splits into perpendicular pieces, $\|Q\chi \otimes hz\|^2 = \|\frac{1}{2}hz\|^2 + \frac{1}{4}\|hz\|^2$. The first is the mean occupation $\frac{1}{2}$, which a background $b = -z/2$ removes. The second is the fluctuation: since $Q^2 = Q$ the occupation has variance $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ in χ, the variance of a fair coin, and no background removes it because a background acts identically on both outcomes.

Proposition 6.28 (Occupation statistics)**.** *Q is Hermitian, $Q - \frac{1}{2} = -\frac{1}{2}$ charge, $(Q - \frac{1}{2})^2 = \frac{1}{4}I$ and $\|(Q - \frac{1}{2})u\|^2 = \frac{1}{4}\|u\|^2$ for every $u \in \mathbb{C}^8$. In the clock zero mode χ the occupation has mean $\frac{1}{2}$, centred mean 0 and variance $\frac{1}{4}$.*

The proof is the computation just given. For n registers the coins are independent and their contributions add. Let $v_R = \chi^{\otimes n} \otimes \Omega_{\mathrm{ph}}$ and let P_1 project onto the one-photon sector.

Theorem 6.29 (Exact reference response)**.** *Let $C_E > 0$. For any native matrix K, Hermitian or not, any $b, z_j \in \mathscr{H}_1$ and any $c \in \mathbb{C}$,*

$$\|P_1(H_R^{\mathrm{fib}} - c)v_R\|^2 = \Big\|hb + \frac{1}{2}\sum_j hz_j\Big\|^2 + \frac{1}{4}\sum_j \|hz_j\|^2, \qquad \text{so} \qquad \|(H_R^{\mathrm{fib}} - c)v_R\|^2 \geq \frac{1}{4}\sum_j \|hz_j\|^2.$$

In particular v_R is an eigenvector of H_R^{fib} only if $b = 0$ and every $z_j = 0$, and its energy variance is at least $\frac{1}{4}\sum_j \|hz_j\|^2$.

Proof. H_0 annihilates the vacuum, and K and cI keep it in the vacuum sector, so after P_1 only the sources survive: $P_1(H_R^{\mathrm{fib}} - c)v_R = \chi^{\otimes n} \otimes (hb + \frac{1}{2}\sum_j hz_j) + \sum_j u_j \otimes hz_j$ with $u_j = (Q_j - \frac{1}{2})\chi^{\otimes n}$. By Proposition 6.28, $\|u_j\| = \frac{1}{2}$; $u_j \perp \chi^{\otimes n}$, and $u_j \perp u_k$ for $j \neq k$ because their inner product contains $\langle \chi, (Q - \frac{1}{2})\chi \rangle = 0$ twice. The $n + 1$ terms are orthogonal, which gives the identity; projection contracts, which gives the inequality. An eigenvalue c makes the left side vanish, so $hz_j = 0$ for every j and then $hb = 0$; h is injective on $\mathscr{H}_1$ because $\omega > 0$ away from $k = 0$. The variance statement is the inequality at $c = \langle v_R, H_R^{\mathrm{fib}} v_R \rangle$. □

To compare regions on one carrier, rotate each active register to its clock basis. Let $T_R = J_R^*\big(\bigoplus_\eta F^{\otimes n} \otimes I\big)J_R$, a unitary acting on the finitely many registers of R, and $\widetilde{H}_R = T_R^* H_R T_R$.

Proposition 6.30 (Clock-chart conjugation)**.** *For every finite region, native matrix, background and profiles, $\widetilde{H}_R$ is the fibrewise conjugate $J_R^*\big(\bigoplus_\eta (F^{\otimes n} \otimes I)^* H_R^{\mathrm{fib}} (F^{\otimes n} \otimes I)\big)J_R$, domains included, and T_R carries the reference to the uniform clock state of the registers of R: $T_R\Omega$ has fibre v_R at the zero exterior configuration and vanishes elsewhere. Consequently, for every $c_R \in \mathbb{C}$,*

$$\|(\widetilde{H}_R - c_R)\Omega\|^2 \geq V_R := \frac{1}{4}\sum_{j=1}^{4|R|} \|hz_j\|^2. \tag{6.18}$$

Proof. $J_R J_R^* = I$, and conjugation by a direct sum of unitaries acts fibre by fibre, domains included. $F^{\otimes n} e_0^{\otimes n} = \chi^{\otimes n}$, so $T_R\Omega$ has the stated fibre, and Theorem 6.29 gives (6.18). □

No limit of the unitaries T_R is used; on n registers the overlap of v_R with $e_0^{\otimes n} \otimes \Omega_{\mathrm{ph}}$ is $8^{-n/2}$.

Theorem 6.31 (Reference-vector obstruction)**.** *Let $C_E > 0$. Take any family of finite regions indexed along a non-degenerate filter (a sequence, or the finite subsets of $\mathbb{Z}^3$ ordered by inclusion), with native matrices K_R, backgrounds b_R and profiles for which $V_R \to +\infty$. Then for any scalars $c_R \in \mathbb{C}$ the vectors $(\widetilde{H}_R - c_R)\Omega$ have no norm limit in $\mathscr{H}$.*

Proof. Their squared norms tend to $+\infty$ by (6.18); a norm limit would make the norms converge. □

The hypothesis is already met when one register per cell drives a unit current. Translation multiplies a unit profile by a phase, so $v_d := \frac{1}{4}\|h z_{(x,d)}\|^2$ does not depend on $x \in \Lambda$, and $v_d > 0$ by Lemma 6.4 and the injectivity of h.

Proposition 6.32 (One unit current per cell)**.** *Let $a > 0, C_E > 0$, a direction d and any placement $p : \Lambda \to \Lambda$. In each cell $r \in R$ give register zero the profile $z_{(p(r),d)}$ and the other three registers the zero profile, all four registers being kept. Then $V_R = |R|\, v_d$, and along the finite subsets ordered by inclusion $V_R \to +\infty$; Theorem 6.31 applies for every choice of native matrices, backgrounds and scalars. The placement need not be injective.*

This family leaves three registers per cell undriven and need not use distinct edges, so Theorem 6.27 does not cover it. Both results hold for one family once every register keeps its own edge in every region.

Definition 6.33 (Edge assignment)**.** An edge assignment is an injective map $\pi : \Lambda \times \{0,1,2,3\} \to \mathscr{E}$; register k of cell r carries the unit current of the edge $\pi(r,k)$ in every region that contains r. The sample assignment is $\pi(r,k) = ((4r_0 + k, r_1, r_2), 0)$: the four registers of a cell sit on four consecutive edges along the first axis.

The sample assignment is injective: equal images have equal r_1, r_2 and equal $4r_0 + k$ with $0 \le k \le 3$, hence equal r_0 and k. In a region R list its registers as $j = (r_j, k_j)$ and put $e_R(j) = \pi(r_j, k_j)$, injective because π is; let $d(j)$ be the direction of $e_R(j)$ and $v_{\min} = \min_d v_d$.

Theorem 6.34 (One family carries both regional results)**.** *Let $a > 0$, $C_E > 0$, $\epsilon \ge 0$ and an edge assignment π. In every finite region use no background, the clock energy K_R and the profiles $z_j = z_{e_R(j)}$. Then:*

(i) for every R, both H_R and $\widetilde{H}_R$ are self-adjoint and both satisfy the floor (6.17)*;*
(ii) $V_R = \sum_j v_{d(j)} \ge 4|R|\, v_{\min}$ with $v_{\min} > 0$;
(iii) along the finite subsets of $\mathbb{Z}^3$ ordered by inclusion, for any scalars c_R, the vectors $(\widetilde{H}_R - c_R)\Omega$ have no norm limit.

Parts (ii) and (iii) hold for every choice of native matrices K_R and backgrounds b_R.

Proof. (i) Theorem 6.27 applies to H_R, since e_R is injective and $K_R \geq 0$; T_R maps $\mathscr{D}(\widetilde{H}_R)$ onto $\mathscr{D}(H_R)$, so $\widetilde{H}_R$ is self-adjoint and $\mathrm{Re}\langle\psi, \widetilde{H}_R\psi\rangle = \mathrm{Re}\langle T_R\psi, H_R T_R\psi\rangle \geq -2a|R|\|\psi\|^2$. (ii) Register j contributes $v_{d(j)} \geq v_{\min}$ wherever its edge is; no injectivity is used. (iii) $|R| \to \infty$, so $V_R \to \infty$, and Theorem 6.31 applies; neither V_R nor that theorem depends on K_R or b_R. □

The floor and the obstruction measure different quantities: the floor uses $\|\sqrt{h}\, z\|^2$, the response uses $\|hz\|^2$. For the sample assignment every edge points along the first axis and $V_R = 4|R|\, v_0$ exactly. The clock ground state of the infinite lattice is a product of infinitely many independent fair coins, each sending a fixed fluctuation into the shared field, and no native matrix, background or zero of energy cancels the sum. The obstruction is about one vector: it excludes the passage to infinite volume through the action of the Hamiltonians on the reference vector. It does not decide strong-resolvent or unitary-group convergence, the evolution of local observables, other reference states or a physical finite activation. Whether the divergence holds for physical profiles beyond the unit-current families is open.

6.8.4 Tools for the motion of regional observables

The obstruction leaves the standard route open: evolve local observables and ask whether their evolution settles as the region grows [10, Sec. 6.2]. The finite-region facts that route needs are proved here; Chapter 7 carries it out. A register observable $B \in M_8(\mathbb{C})$ on register k of cell r is the operator $O_{r,k}(B)$ on $\mathscr{H}$ that changes that register's label and leaves every other label and the field alone:

$$(O_{r,k}(B)\psi)(\xi) = \sum_{v=0}^{7} B_{\xi(r)_k,\, v}\, \psi\big(\xi[(r,k) \to v]\big).$$

Write $U_R(t) = e^{itH_R}$.

Theorem 6.35 (Regional observables)**.** *Let $C_E > 0$, the native matrices Hermitian, and the backgrounds and profiles arbitrary.*

(i) One observable, one domain. *$O_{r,k}(B)$ is the same operator in every region containing r, with $\|O_{r,k}(B)\| \leq \|B\|$; observables on different registers commute. Every regional Hamiltonian has the domain $\mathscr{D}(\widehat{H}_0)$, which every $U_R(t)$ and every $O_{r,k}(B)$ preserve.*

(ii) Local commutator. *With the clock energy K_R, fibre by fibre on that domain, $[H_R, B_j] = \epsilon\,[C, B]_j + [Q, B]_j\, S(z_j)$: only register j's own clock and source survive.*

(iii) Heisenberg equation. *For $\psi \in \mathscr{D}(H_R)$ and every register observable O, $\frac{\mathrm{d}}{\mathrm{d}t}\langle U_R(t)\psi, O U_R(t)\psi\rangle = -i\langle U_R(t)\psi, [H_R, O]\, U_R(t)\psi\rangle$.*

(iv) Added cells. *For nested regions $R \subseteq S$ with the clock energy and the same background and lattice-wide profiles, $H_S - H_R = \sum_{r\in S\setminus R} I_r$ on the common domain, where I_r is the clock and source action of the four registers of cell r. If $\|\sum_{r\in S\setminus R} I_r U_S(s)\psi\| \leq c$ for s between 0 and t, then $\|U_S(t)\psi - U_R(t)\psi\| \leq |t|\, c$; for any two regions and every ψ in the common domain such a c exists.*

(v) Locality. *With the clock energy, the commutators of a register observable with the regional Hamiltonians agree in any two regions containing its cell, and the complete interaction I_s of a cell $s \neq r$ commutes with every register observable of r.*

Proof. (ii) B_j commutes with H_0, $S(b)$, the other clocks and occupations, and their sources, which act on the field and multiply by an occupation that B_j does not touch. (i) Each H_R differs from $\widehat{H}_0$ by finitely many sources, each infinitesimally small relative to H_0 (proof of Lemma 6.24), and a bounded matrix, so the graph norms are equivalent; a register observable acts on labels only. (iii) is the derivative of an expectation along a unitary group whose generator's domain the observable preserves. (iv) is the Duhamel identity $U_S(t)\psi - U_R(t)\psi = i\int_0^t U_R(t-s)(H_S - H_R)U_S(s)\psi \, \mathrm{d}s$ with U_R unitary; the graph norm of H_S on the orbit gives a constant. (v) follows from (ii) and from the fact that the terms of I_s act on the registers of s and the field only. □

Everything that couples distant cells passes through the photon field, so the decisive quantity is the commutator of two sources at different times. For finite real currents f, g let $c(f,g,t)I = [S(e^{ith}z_f), S(z_g)]$, so that $c(f,g,t) = \langle he^{ith}z_f, hz_g\rangle - \langle hz_g, he^{ith}z_f\rangle$.

Proposition 6.36 (The source commutator)**.** *Let $a \geq 0$ and $C_E > 0$.*

(i) For every t, $\sum_{e\in\mathcal{E}} |c(f,\delta_e,t)|^2 \leq 2a\,\omega_{\max}\|hz_f\|^2$; the tails outside finite edge sets tend to zero.
(ii) With $\mathrm{D}(u) = \operatorname{arsinh}(u)/u$, $\mathrm{D}(0) = 1$, $\operatorname{sinc} y = \sin y/y$ and

$$\varrho_t(x) = \frac{atC_E^2}{2}\,\mathrm{D}\Big(\tfrac{\sqrt{x}}{2}\Big)^3\Big(1+\frac{x}{4}\Big)^{-1/2}\operatorname{sinc}\Big(2tC_E \operatorname{arsinh}\frac{\sqrt{x}}{2}\Big),$$

one has $c(f,g,t) = -2i\int_{\mathbb{T}^3}\varrho_t(\lambda(k))\langle F_f(k), F_{Lg}(k)\rangle\,\mathrm{d}\mu(k)$, where L is the lattice curl curl on currents, with Fourier multiplier $\lambda(k)P_T(k)$. The integrand is continuous on the whole zone: $\omega^2 w^2\sin(t\omega) = \lambda\,\varrho_t(\lambda)$ cancels the $1/\lambda$ of the transverse projection, and $|\varrho_t| \leq a|t|C_E^2/2$.
(iii) Replacing ϱ_t by its Bernstein polynomial of degree N gives a commutator $A_N(f,g,t)$ of finite range $N+2$, and

$$|c(f,g,t) - A_N(f,g,t)| \leq \frac{12K_t}{\sqrt{N}}\,M(f)\,M(Lg), \qquad K_t = \frac{a|t|C_E^2}{2}\Big(\frac{1}{2} + \frac{t^2C_E^2}{2}\Big),$$

with $M(f) = \sum_e |f(e)|$.

The proof is in Appendix B.4. None of these facts is an infinite-region dynamics; what they do not give is a bound on the spread of an interacting observable that is uniform as the regions grow, of the kind Lieb and Robinson introduced [75, 82].

That is where the finite theory hands over. On the clock-photon lattice the same family of sources gives a floor in every region and no limit on the clock ground state; Chapter 7 takes the other road to infinite volume, through the evolution of local observables and a state on the algebra they generate.

Chapter 7

The first interacting dynamics II: the infinite lattice, the reference state and dephasing

Chapter 6 built the clock-photon lattice one finite region at a time: in every finite region the registers and the shared photon field have a self-adjoint Hamiltonian with energy at least $-2a$ per cell when every register drives a unit current on its own edge. One object did not survive the growth of the region: the clock ground state, every register in the zero mode of its clock and no photons. The regional Hamiltonians move it by an amount that grows with the region, and no energy shift gives the moved vectors a limit. The lattice itself has no boundary at which a region stops, so the first interacting dynamics has to be found on the whole of it. This chapter asks three questions about it. Does the whole lattice have a dynamics? What becomes of the reference on the whole lattice? And what does the shared field do to the registers it couples?

In short: the whole lattice has a dynamics, carried by observables rather than by vectors, at every clock strength; the reference survives as a state on the algebra of that dynamics, although no vector of the carrier gives it. With the clock off its evolution is exact; on edge assignments that fill the directions they use, each register dephases into its occupation basis, and jointly the field reads only the combined transverse current; at every nonzero time the evolved state leaves the reference sector; and the limit points of the long-time averages of the reference are states the dynamics preserves.

Every result here concerns one carrier, the clock-photon lattice (carrier (i), Definition A.9), with its supplied data a, C_E, ϵ, the edge assignment, the background, the region and the initial state. The reference configuration is a choice of representation, not a selected first state.

7.1 The lattice, its registers and its data

The objects are those of Chapter 6: the native photon field of Definition 6.1 (one-photon space $\mathscr{H}_1$, Fock space $\mathscr{F}$ with vacuum Ω_{ph}, one-photon energy h with $0 \le \omega \le \omega_{\max} \approx 2.6339158\, C_E$, free energy H_0, sources $S(g)$, Weyl displacements $W(u)$); the register of Definition 6.2 (clock C, zero mode χ, occupation Q); the cell carrier $\mathscr{H} = \ell^2(\mathscr{X}; \mathscr{F})$ of Section 6.7; the unitary J_R and the regional Hamiltonian H_R of Definition 6.23; and the current profiles of Definition 6.3. Two symbols differ from Chapter 6, because λ and w name other objects below: we write $s(k)$ for the lattice symbol $\lambda(k)$ and $\varrho(k)$ for the weight $w(k)$, so that

$$z_f(k) = i\,\varrho(k) P_T(k) F_f(k), \qquad \varrho(k)^2 = \frac{a}{2C_E\sqrt{s(k)}\sqrt{1 + s(k)/4}}, \tag{7.1}$$

with $P_T(k)v = v - s(k)^{-1}\big(\sum_d r_d(k)v_d\big)\rho(k)$ for $k \neq 0$. We write z_e for the profile of the unit current on the edge e. The proofs use three further facts. The Weyl displacements satisfy

$$W(u)W(v) = e^{-i\,\mathrm{Im}\langle u,v\rangle}W(u+v), \qquad e^{itH_0}W(u)e^{-itH_0} = W(e^{ith}u),$$
$$\langle \Omega_{\mathrm{ph}}, W(u)\Omega_{\mathrm{ph}}\rangle = e^{-\|u\|^2/2}; \tag{7.2}$$

the compensated-source identity of Lemma 6.24 reads

$$H_0 + S(g) = W(g)^* H_0 W(g) - \|\sqrt{h}\,g\|^2 \tag{7.3}$$

on the domain of H_0; and since $T\chi = \chi$ and $\sum_x \sigma_x = 0$ (Proposition 6.28),

$$\langle \chi, Q\chi\rangle = \tfrac{1}{2}, \qquad \langle \chi, B\chi\rangle = \tfrac{1}{8}\sum_{x,y=0}^{7} B_{xy} \quad \big(B \in M_8(\mathbb{C})\big)\ : \tag{7.4}$$

in the zero mode the occupation is a fair coin.

Blocks. Write $Q^1 = Q$ and $Q^0 = I - Q$. A *block* of $A \in M_8(\mathbb{C})$ is $Q^{\beta_1}AQ^{\beta_2}$ with $\beta_1, \beta_2 \in \{0,1\}$; its *flip* is $\delta = \beta_1 - \beta_2 \in \{-1, 0, 1\}$. A block keeps the occupation when $\delta = 0$ and changes it otherwise, and every matrix is the sum of its four blocks.

Registers and operators. A register is $p = (r, k)$ with cell $r \in \mathbb{Z}^3$ and $k \in \{0, 1, 2, 3\}$; for a set R of cells, $R_4 = R\times\{0, 1, 2, 3\}$. The register observable $O_p(B)$ is the operator $O_{r,k}(B)$ of Theorem 6.35; it has norm $\|B\|$, $B \mapsto O_p(B)$ preserves sums, products and adjoints, and observables of different registers commute. The operator $\widehat{W}(u)$ acts by $W(u)$ on the field of every configuration. An operator X is *fiberwise over* R if $J_R X J_R^* = \bigoplus_\eta F$ for one operator F on $\mathscr{N}_R \otimes \mathscr{F}$: it acts in the same way in every exterior configuration η.

The regional dynamics. With a clock strength $\epsilon \geq 0$, a background $b \in \mathscr{H}_1$ and a profile z_p for every register, H_R is the regional Hamiltonian with the clock energy (6.13), whose fiber is

$$H_R^{\mathrm{fib}} = I \otimes H_0 + \epsilon \sum_{p\in R_4} C_p \otimes I + I \otimes S(b) + \sum_{p\in R_4} Q_p \otimes S(z_p), \tag{7.5}$$

self-adjoint by Theorem 6.25. For bounded X write $\tau_t^R(X) = e^{itH_R}Xe^{-itH_R}$. Each τ_t^R is linear, multiplicative, isometric and compatible with adjoints, and maps operators fiberwise over R to operators fiberwise over R.

Edge assignments and the unit-edge family. An edge assignment π (Definition 6.33) gives register p the profile $z_p = z_{\pi(p)}$. Besides the sample assignment $\pi_{\mathrm{s}}(r,k) = \big((4r_0 + k, r_1, r_2), 0\big)$, which uses every edge of direction 0 exactly once, we use the *full assignment* $\pi_{\mathrm{full}}(r,k) = \big((\lfloor N/3\rfloor, r_1, r_2), N \bmod 3\big)$, $N = 4r_0 + k$, a bijection of the registers onto all edges. An assignment *fills the directions it uses* if, whenever some register sits on an edge of direction d, every edge of direction d carries a register; both examples do, and so does every assignment onto all edges. The *unit-edge family* means, throughout: $a > 0$, $C_E > 0$, an edge assignment π, the profiles $z_p = z_{\pi(p)}$, no background and a stated clock strength $\epsilon \geq 0$. For it Theorem 6.34 gives the floor $-2a|R|$ in every region and the obstruction on the clock ground state; we start there.

Definition 7.1 (Convergence on the whole lattice)**.** A bounded operator X on $\mathscr{H}$ *converges* under the regional dynamics if for every $T \geq 0$ and $\eta > 0$ there is a finite region R_0 with

$\|\tau_t^R(X) - \tau_t^S(X)\| \le \eta$ for all finite regions $R, S \supseteq R_0$ and all $|t| \le T$. Its evolution on the whole lattice is then $\tau_t(X) = \lim_R \tau_t^R(X)$, the limit in operator norm along the finite regions ordered by inclusion, uniform for t in bounded intervals; it exists because the bounded operators form a complete space.

The definition concerns operators, not vectors, so Chapter 6's obstruction does not decide it.

7.2 One flip and its photon cloud

With the clock off, $\epsilon = 0$, every occupation Q_p commutes with (7.5), so no register changes its occupation; the registers still act on one another through the field. Choose an occupation basis in each register of a region R. A pattern s gives every register $p \in R_4$ an occupation $q_p(s) \in \{0, 1\}$, and on the vectors with these occupations the fiber operator acts as $H_0 + S(g_s)$, with the combined profile $g_s = b + \sum_{p \in R_4} q_p(s) z_p$. By (7.3) the pattern evolves by

$$\Phi_{g_s}(t) = e^{-itd(g_s)}\, W(g_s)^* e^{itH_0} W(g_s), \qquad d(g) = \|\sqrt{h}\, g\|^2. \tag{7.6}$$

An observable that flips registers moves a pattern from one combined profile to another, and what the field does to it is contained in one identity. For $u, v \in \mathscr{H}_1$ and real t put

$$\theta(u, v, t) = \operatorname{Im}\langle u, e^{ith} v - v\rangle - t \operatorname{Re}\langle \sqrt{h}\, u, \sqrt{h}\, v\rangle, \tag{7.7}$$

a real number, real-linear in u and in v. Its first term comes from the photons that two profiles exchange, its second from their static interaction energy.

Lemma 7.2 (Passing between two patterns)**.** *For all $g, w \in \mathscr{H}_1$ and real t,*

$$\Phi_{g+w}(t)\, \Phi_g(-t) = e^{i(2\theta(g,w,t) + \theta(w,w,t))}\, W\big(e^{ith} w - w\big).$$

Proof. Write $w_t = e^{ith} w$. By (7.2), $W(g+w)W(-g) = e^{i \operatorname{Im}\langle w, g\rangle} W(w)$, whose conjugate by e^{itH_0} is $e^{i \operatorname{Im}\langle w, g\rangle} W(w_t)$; further $W(-g-w)W(w_t) = e^{i \operatorname{Im}\langle g+w, w_t\rangle} W(w_t - g - w)$ and $W(w_t - g - w)W(g) = e^{-i \operatorname{Im}\langle w_t - w, g\rangle} W(w_t - w)$. With $d(g+w) - d(g) = 2 \operatorname{Re}\langle \sqrt{h}\, g, \sqrt{h}\, w\rangle + d(w)$, $\operatorname{Im}\langle u, v\rangle = -\operatorname{Im}\langle v, u\rangle$ and $\operatorname{Im}\langle g, g\rangle = 0$, the phases collect to

$$2\big(\operatorname{Im}\langle g, w_t - w\rangle - t \operatorname{Re}\langle \sqrt{h}\, g, \sqrt{h}\, w\rangle\big) + \big(\operatorname{Im}\langle w, w_t - w\rangle - t\, d(w)\big) = 2\theta(g, w, t) + \theta(w, w, t). \qquad \square$$

Example 7.3 (Two registers in one cell)**.** Take the region of one cell, no background and the clock off. Give two of its registers, p and q, the profiles z_p and z_q, and the other two the profile zero. Put the four registers in their zero modes and the field in its vacuum, $u = \chi^{\otimes 4} \otimes \Omega_{\mathrm{ph}}$, and let $B = QA(I - Q)$, a block of p that moves it from its empty to its occupied half. On a pattern with p empty and q at occupation q_q, the block adds z_p to the combined profile $g = q_q z_q$ and does not act on the field, so by Lemma 7.2 its evolution acts there as B_p times

$$e^{i(2q_q\theta(z_q,z_p,t) + \theta(z_p,z_p,t))}\, W\big(e^{ith} z_p - z_p\big);$$

on patterns with p occupied it vanishes. The phase $e^{2iq_q\theta(z_q,z_p,t)}$ is the eigenvalue of $I + (e^{i\varphi_q} - 1)Q$ on register q, with $\varphi_q = 2\theta(z_q, z_p, t)$. Since u is a product vector, the expectation is a product of

three numbers: $\langle \chi, B\chi\rangle$ on register p; $e^{-\|g\|^2/2}$ on the field, with $g = e^{ith}z_p - z_p$, by (7.2); and on register q, by (7.4),

$$\langle \chi, \big(I + (e^{i\varphi_q} - 1)Q\big)\chi\rangle = 1 + \frac{e^{i\varphi_q} - 1}{2} = \frac{1 + e^{i\varphi_q}}{2}.$$

Altogether

$$\langle u, e^{itH_R^{\mathrm{fib}}} B_p\, e^{-itH_R^{\mathrm{fib}}} u\rangle = e^{i\theta(z_p, z_p, t)}\, e^{-\|g\|^2/2}\, \langle \chi, B\chi\rangle\, \frac{1 + e^{i\varphi_q}}{2}.$$

The example already shows the whole mechanism (Figure 7.1). Register q sits in an equal superposition of its two occupations. When p changes its occupation, the occupied half of q acquires the phase $e^{i\varphi_q}$ and the empty half does not, so q now carries a trace of what p did. The flip also carries the displacement $W(e^{ith}z_p - z_p)$: read physically, it emits its own coherent photon cloud and reabsorbs the static one, and the vacuum overlap $e^{-\|g\|^2/2}$ is what the cloud costs the coherence. Neither factor removes the coherence. Since $\|g\| \le 2\|z_p\|$, the cloud factor stays above $e^{-2\|z_p\|^2}$, and $|\cos(\varphi_q/2)|$ returns to one whenever φ_q passes a multiple of 2π. Removing the coherence takes many registers whose phases do not return together; the occupations of p are untouched throughout.

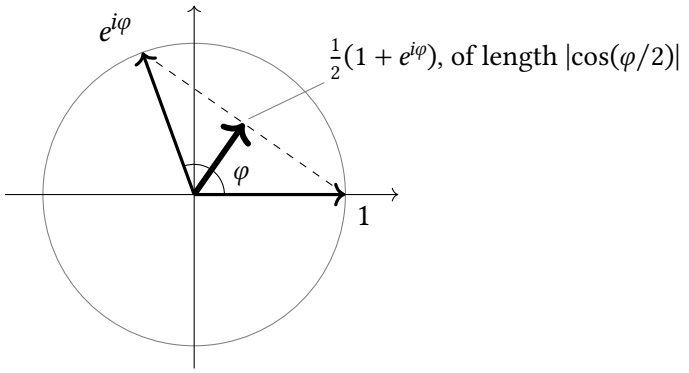


Figure 7.1. What one other register does to a coherence. When register p changes its occupation, the occupied half of q acquires the interaction phase $e^{i\varphi}$ and the empty half does not, so the coherence of p is multiplied by the average of the two unit vectors (thick arrow), of length $|\cos(\varphi/2)|$.

7.3 The clock-free dynamics of the whole lattice

For a register q and a real φ let $\Pi_q(\varphi) = O_q\big(I + (e^{i\varphi} - 1)Q\big)$, the unitary that multiplies the occupied half of q by $e^{i\varphi}$, and for a finite set J of registers and phases ψ_q write $\Pi_J(\psi) = \prod_{q\in J}\Pi_q(\psi_q)$. These are commuting unitaries, and since $\Pi_q(\psi_q) - I = (e^{i\psi_q} - 1)O_q(Q)$ with $\|O_q(Q)\| = 1$,

$$\|\Pi_J(\psi) - I\| \le \sum_{q\in J} \big|e^{i\psi_q} - 1\big|. \tag{7.8}$$

Theorem 7.4 (Clock-free evolution in closed form)**.** *On the clock-photon lattice let $C_E > 0$ and $\epsilon = 0$, and let the background b and the profiles z_q be arbitrary. Let the finite region R contain the cell of the register p, and let $B = Q^{\beta_1} A Q^{\beta_2}$ be a block with flip δ. With $z = z_p$, $\varphi_q = 2\delta\theta(z_q, z, t)$ and $\Theta = 2\delta\theta(b, z, t) + \delta^2\theta(z, z, t)$,*

$$\tau_t^R\big(O_p(B)\big) = e^{i\Theta}\, O_p(B)\, \widehat{W}\big(\delta(e^{ith}z - z)\big)\, \Pi_{R_4}(\varphi).$$

Only the last factor depends on the region. For finite regions $R \subseteq S$ containing the cell of p and every $A \in M_8(\mathbb{C})$,

$$\big\|\tau_t^S\big(O_p(A)\big) - \tau_t^R\big(O_p(A)\big)\big\| \le \sum_{\beta_1,\beta_2} \|Q^{\beta_1} A Q^{\beta_2}\| \sum_{q \in (S\setminus R)_4} \big|e^{2i(\beta_1-\beta_2)\theta(z_q,z_p,t)} - 1\big|.$$

Proof. The fiber operator is the same in every exterior configuration, so it suffices to compute in one fiber. There $O_p(B)$ annihilates the patterns with $q_p(s) \neq \beta_2$ and takes a pattern with $q_p(s) = \beta_2$ to patterns that differ from it only at p, where the occupation is β_1; their combined profile is $g_s + \delta z$. The block does not act on the field, so its evolution acts on the pattern s as $O_p(B)$ times $\Phi_{g_s+\delta z}(t)\,\Phi_{g_s}(-t)$, which Lemma 7.2 with $w = \delta z$ evaluates. Since θ is real-linear in its first argument, $2\delta\theta(g_s, z, t) = 2\delta\theta(b, z, t) + \sum_q q_q(s)\varphi_q$, and $e^{iq_q(s)\varphi_q}$ is the eigenvalue of $\Pi_q(\varphi_q)$ on the pattern s. For the comparison, write A as the sum of its four blocks. By the closed form the two evolutions of a block differ by a product of unitaries times $\Pi_{(S\setminus R)_4}(\varphi) - I$, and (7.8) bounds that factor. □

The phase that register q reads is a kernel of the two currents. For finite real currents f, g define the *phase kernel* $P_{f,g}(t) = \langle z_f, (th - \sin(th))\, z_g\rangle$. Since $z_f = i\varrho P_T F_f$ and P_T is an orthogonal projection, every kernel of this kind is a Brillouin-zone integral,

$$\langle z_f, \phi(h)\, z_g\rangle = \int \phi(\omega)\, \varrho^2\, \langle F_f, P_T F_g\rangle \, \mathrm{d}\mu, \tag{7.9}$$

for every Borel function ϕ bounded on $[0, \omega_{\max}]$.

Lemma 7.5 (The phase kernel)**.** *Let $a \ge 0$ and $C_E > 0$, and let f, g be finite real currents.*

(i) For real ϕ the kernel (7.9) *is real. In particular $P_{f,g}(t)$ is real and odd in t, and $\theta(z_f, z_g, t) = -P_{f,g}(t)$.*

(ii) Let $T \ge 0$. For every t, $\sum_{e\in\mathscr{E}} |P_{1_e,g}(t)| < \infty$, and for every $\eta > 0$ there is a finite set of edges E_η with $\sum_{e\notin E_\eta} |P_{1_e,g}(t)| \le \eta$ for all $|t| \le T$.

Proof. (i) For a real current $F_f(-k) = \overline{F_f(k)}$; ω and ϱ are even and $P_T(-k) = \overline{P_T(k)}$. The substitution $k \mapsto -k$ preserves μ and conjugates the integrand, so the integral is real. Oddness holds because $th - \sin(th)$ is odd in t. Realness gives $\mathrm{Im}\langle z_f, e^{ith} z_g - z_g\rangle = \langle z_f, \sin(th) z_g\rangle$ and $\mathrm{Re}\langle \sqrt{h}\, z_f, \sqrt{h}\, z_g\rangle = \langle z_f, h z_g\rangle$, hence $\theta(z_f, z_g, t) = \langle z_f, (\sin(th) - th) z_g\rangle = -P_{f,g}(t)$. Part (ii) is proved in Appendix B.5: the time factor cancels the singular weight ϱ^2 at long wavelength, so $P_{1_e,g}$ is a Fourier coefficient of a smooth function. □

In the time factor $th - \sin(th)$ (Figure 7.2), the term th comes from the static interaction energy of two occupied registers and $\sin(th)$ from the retarded exchange of photons between them. They enter with opposite signs, and $0 \le \omega t - \sin \omega t \le (\omega t)^3/6$ for $t \ge 0$, so the phase one register reads of another starts at third order in time. This is the combination of the exactly solvable independent boson model [79]; for finitely many two-level systems sharing one bath it appears as the phase induced between every pair [97]. Part (ii) is what the lattice adds: the phases of all registers add up to a finite total.

Theorem 7.6 (The whole lattice with the clock off)**.** *For the unit-edge family with $\epsilon = 0$, every register observable converges in the sense of Definition 7.1.*

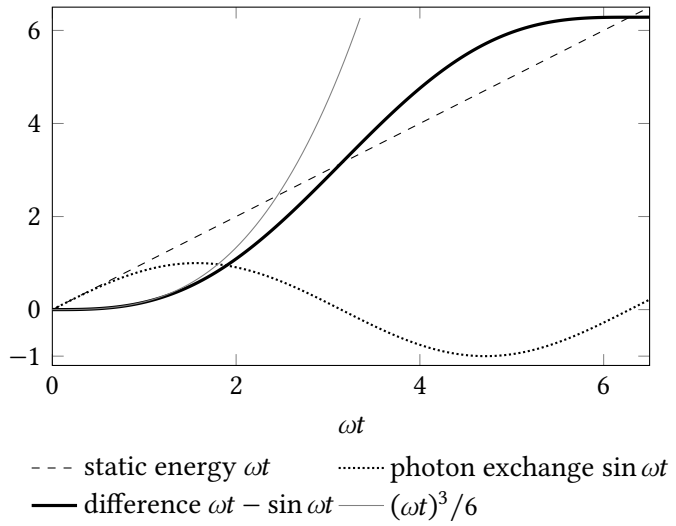


Figure 7.2. The time factor of the phase kernel at one photon frequency ω: static energy ωt (dashed) less photon exchange $\sin \omega t$ (dotted). The difference (thick) starts as $(\omega t)^3/6$ (thin gray) and stays between zero and that cubic for $t \geq 0$.

Proof. Fix $O_p(A)$, $T \geq 0$ and $\eta > 0$; put $g = 1_{\pi(p)}$ and $C_A = \sum_{\beta_1,\beta_2} \|Q^{\beta_1} A Q^{\beta_2}\|$. Lemma 7.5(ii) gives a finite set of edges E' with $\sum_{e \notin E'} |P_{1_e,g}(t)| \leq \eta/(4C_A+1)$ for all $|t| \leq T$. Since π is injective, only finitely many registers sit on edges of E'; let R_0 consist of their cells and the cell of p. For $R_0 \subseteq R \subseteq S$, the registers of $(S \setminus R)_4$ sit on distinct edges outside E'. Theorem 7.4, $|e^{ix} - 1| \leq |x|$, $|\delta| \leq 1$ and $\theta(z_q, z_p, t) = -P_{1_{\pi(q)},g}(t)$ give $\|\tau_t^S(O_p(A)) - \tau_t^R(O_p(A))\| \leq 2C_A\eta/(4C_A + 1)$ for $|t| \leq T$. Two regions containing R_0 are compared through their union, which bounds their distance by $4C_A\eta/(4C_A + 1) < \eta$. □

Convergence passes to every polynomial in register observables and to the norm closure of these polynomials (Theorem 7.16). With the clock off the dynamics of the whole lattice comes from an exact solution, not from a locality estimate.

7.4 The clock on, at every clock strength

With the clock on, registers change their occupations and the closed form no longer holds. The clock energy of a finite region is bounded, however, so the evolution with the clock is a bounded perturbation of the clock-free evolution, which is known exactly. Comparing two regions then becomes a linear integral inequality over the registers. Throughout this section the unit-edge family is fixed with a clock strength $\epsilon \geq 0$. We write H_R^0 for H_R at $\epsilon = 0$, $\tau_t^{R,0}$ for its evolution, and $C_R = \sum_{q \in R_4} O_q(C)$, so that $H_R = H_R^0 + \epsilon C_R$ on the common domain, with C_R bounded and self-adjoint.

A flip of register q reaches an observable at register p through the phase $\theta(z_q, z_p, t)$. With the clock on, flips happen at every time, so one bound is needed for a whole interval. For finite real currents put $S_{f,g}(u) = \langle z_f, h^2 \sin(uh)\, z_g \rangle$, the photon sine kernel, and for $T \geq 0$ define the *register kernel*

$$\mathscr{K}_T(p, q) = T \int_0^T \left| S_{1_{\pi(q)},1_{\pi(p)}}(u) \right| \mathrm{d}u. \tag{7.10}$$

Lemma 7.7 (The register kernel). *For the unit-edge family and $T \geq 0$:*

(i) $|\theta(z_q, z_p, t)| \leq \mathscr{K}_T(p, q)$ *for all registers* p, q *and all* $|t| \leq T$*;*
(ii) there is $K_T \geq 0$ *with* $\sum_q \mathscr{K}_T(p, q) \leq K_T$ *for every register* p*, the sum running over all registers of the lattice.*

Proof. (i) For real ω and $t \geq 0$, $\omega t - \sin \omega t = \int_0^t (t-u)\,\omega^2 \sin(u\omega)\,\mathrm{d}u$: both sides vanish at $t = 0$ and have derivative $\omega - \omega\cos\omega t$. The integrand of (7.9) with this factor is bounded by $t\,\omega_{\max}\,\omega\varrho^2|F_f||F_g| \leq t\,\omega_{\max}\frac{a}{2}|F_f||F_g|$, so Fubini's theorem gives $P_{f,g}(t) = \int_0^t (t-u) S_{f,g}(u)\,\mathrm{d}u$ for $t \geq 0$. By Lemma 7.5(i), $\theta(z_q, z_p, t) = -P_{1_{\pi(q)}, 1_{\pi(p)}}(t)$ and P is odd, so $|\theta(z_q, z_p, t)| \leq \int_0^{|t|} (|t| - u)|S(u)|\,\mathrm{d}u \leq \mathscr{K}_T(p, q)$ for $|t| \leq T$. Part (ii) is proved in Appendix B.5. □

The sine kernel is, up to sign, the second time derivative of the phase kernel. So $\mathscr{K}_T(p, q)$ bounds how far a flip of q can turn the phase seen at p at any time of the interval, and part (ii) says that the total influence of all registers on one register is finite, with one bound for all.

Lemma 7.8 (Duhamel's formula for a bounded perturbation). *Let A be self-adjoint, V bounded and self-adjoint, and $B = A + V$ on the domain of A. For every bounded X and real t,*

$$e^{itB} X e^{-itB} - e^{itA} X e^{-itA} = \int_0^t e^{isB}\, i\big[V, e^{i(t-s)A} X e^{-i(t-s)A}\big]\, e^{-isB}\,\mathrm{d}s,$$

with a strongly continuous integrand and the integral taken strongly.

The proof is in Appendix B.5. The second ingredient is the algebra of the clock-free evolution: seen from an evolved block at another register, a clock block costs only a phase.

Lemma 7.9 (Clock blocks pass at phases). *Let $R \subseteq S$ be finite regions, $p \in R_4$, X a block at p with flip δ, and $\varphi_q(t) = 2\delta\,\theta(z_q, z_p, t)$.*

(i) A block $c = O_q(Q^{g_1} M Q^{g_2})$ satisfies $c\,\Pi_J(\psi) = e^{i(g_2 - g_1)\psi_q}\,\Pi_J(\psi)\,c$ if $q \in J$, and commutes with $\Pi_J(\psi)$ otherwise.
(ii) For a block c at a register $q \neq p$, $c\,\tau_t^{R,0}(X) = \zeta\,\tau_t^{R,0}(X)\,c$, with $\zeta = e^{i(g_2 - g_1)\varphi_q(t)}$ if $q \in R_4$ and $\zeta = 1$ otherwise.
(iii) For all real t_1, t_2, $\tau_{t_1}^{S,0}\big(\tau_{t_2}^{R,0}(X)\big) = \tau_{t_1+t_2}^{R,0}(X)\,\Pi_{(S\setminus R)_4}\big(\varphi(t_1 + t_2) - \varphi(t_2)\big)$.

Proof. (i) Since $Q^2 = Q$, $Q^g\big(I + (e^{i\psi} - 1)Q\big) = \big(I + (e^{i\psi} - 1)Q\big)Q^g = e^{ig\psi}Q^g$ for $g \in \{0, 1\}$, so $c\,\Pi_q(\psi_q) = e^{ig_2\psi_q}c$ and $\Pi_q(\psi_q)\,c = e^{ig_1\psi_q}c$; observables of different registers commute. (ii) By Theorem 7.4, $\tau_t^{R,0}(X) = e^{i\Theta} X \widehat{W}\,\Pi_{R_4}(\varphi(t))$. The block c commutes with X, which acts on another register, and with $\widehat{W}$, which acts on the field only, and it passes Π_{R_4} at the stated scalar by (i). (iii) The closed form for S and for R gives $\tau_t^{S,0}(X) = \tau_t^{R,0}(X)\,\Pi_{(S\setminus R)_4}(\varphi(t))$. With the clock off every occupation of a register of S commutes with H_S^0, so $\tau_{t_1}^{S,0}$ fixes every $\Pi_J(\psi)$ with $J \subseteq S_4$, and

$$\tau_{t_1}^{S,0}\big(\tau_{t_2}^{R,0}(X)\big) = \tau_{t_1}^{S,0}\big(\tau_{t_2}^{S,0}(X)\big)\,\Pi_{(S\setminus R)_4}(-\varphi(t_2)) = \tau_{t_1+t_2}^{R,0}(X)\,\Pi_{(S\setminus R)_4}\big(\varphi(t_1 + t_2) - \varphi(t_2)\big). \qquad \square$$

Fix a register p_0 and $T \geq 0$, let K_T be as in Lemma 7.7, and let $c_1 = \sum_{g_1,g_2} \|Q^{g_1} C Q^{g_2}\|$, the sum of the norms of the four blocks of the clock. Put $\lambda = 2 + K_T$, $\mathscr{K}'(p,q) = [p = q] + \mathscr{K}_T(p,q)$ and

$$\mu_q = \sum_{n \geq 0} \lambda^{-n} \big(\mathscr{K}'^{\,n}\big)_{p_0 q}, \qquad \sigma_R(p) = \sum_{q \notin R_4} \mathscr{K}_T(p,q), \qquad D_R = \sum_{p \in R_4} \mu_p \, \sigma_R(p), \tag{7.11}$$

where $\mathscr{K}'^{\,n}$ is the n-th power of $\mathscr{K}'$ as a matrix over all registers. The number $\sigma_R(p)$ is the influence on p of the registers outside R; the weight μ sums the chains of influence that start at p_0, each step damped by $1/\lambda$; and D_R is the weighted influence that leaves the region.

Theorem 7.10 (The whole lattice with the clock on)**.** *For the unit-edge family at every clock strength $\epsilon \geq 0$:*

(i) for every register p_0 of the cell r, every $T \geq 0$, all finite regions $R \subseteq S$ with $r \in R$, every block X at p_0 and every $|t| \leq T$,

$$\|\tau_t^S(X) - \tau_t^R(X)\| \leq \|X\| \, D_R \, (4 + 2\gamma T) \, e^{\gamma T}, \qquad \gamma = 2\epsilon c_1 (3 + 2K_T),$$

and $D_R \to 0$ as R runs through the finite regions ordered by inclusion;

(ii) every register observable converges in the sense of Definition 7.1.

The proof, in Appendix B.5, applies Lemma 7.8 over the clock-free evolutions, bounds every term of the integrand with Lemmas 7.7 and 7.9, and iterates the resulting Volterra inequality against the weight μ, which is subinvariant for the kernel. Physically, the clock moves each register at a bounded rate, and a flip of q reaches an observable at p only through the phase $\theta(z_q, z_p, t)$: retarded exchange less static interaction energy. The estimate is of Lieb-Robinson type [75]. It follows the scheme used for anharmonic lattices, a Duhamel formula over an exactly solvable dynamics and an iterated integral inequality [82], and it uses only the uniform summability of the register kernel.

7.5 Photon displacements and the algebra of the dynamics

An evolved register observable carries the displacement $\widehat{W}\big(\delta(e^{ith} z_p - z_p)\big)$, the photon cloud of its flip, so the field's own observables belong in the algebra on which the dynamics acts. For $u, y \in \mathscr{H}_1$ and real t put

$$\xi(u, y, t) = \operatorname{Im} \langle u, e^{ith} y - y \rangle,$$

the turn of the commutation phase between a source of profile u and the freely evolving displacement y.

Lemma 7.11 (A displacement passing one pattern)**.** *For all $g, y \in \mathscr{H}_1$ and real s, t:*

(i) $\Phi_g(t) \, W(y) \, \Phi_g(-t) = e^{2i\xi(g,y,t)} \, W(e^{ith} y)$;
(ii) ξ *is real-linear in each argument;*
(iii) $\xi(g, e^{ish} y, t) = \xi(g, y, t + s) - \xi(g, y, s)$;
(iv) $\xi(g, e^{ish} y - y, t) = \theta(g, y, t + s) - \theta(g, y, t) - \theta(g, y, s)$.

Proof. (i) The scalar $e^{-itd(g)}$ cancels against its conjugate. With $y_t = e^{ith} y$, (7.2) gives

$$W(g) W(y) W(-g) = e^{-2i \operatorname{Im}\langle g, y \rangle} W(y), \qquad W(-g) W(y_t) W(g) = e^{2i \operatorname{Im}\langle g, y_t \rangle} W(y_t),$$

so $\Phi_g(t)W(y)\Phi_g(-t) = e^{2i\,\mathrm{Im}\langle g,y_t\rangle - 2i\,\mathrm{Im}\langle g,y\rangle}W(y_t)$. (ii) The imaginary part of an inner product is real-linear in each argument. (iii) $e^{ith}e^{ish}y - e^{ish}y = (e^{i(t+s)h}y - y) - (e^{ish}y - y)$. (iv) $e^{ith}(e^{ish}y - y) - (e^{ish}y - y) = (e^{i(t+s)h}y - y) - (e^{ith}y - y) - (e^{ish}y - y)$, and the static terms of θ cancel because $(t+s) - t - s = 0$. □

A source does not change its own configuration, so its self-energy drops out: a displacement of the field comes back freely evolved, multiplied by one phase.

Theorem 7.12 (Clock-free evolution of a displacement)**.** *Let $C_E > 0$ and $\epsilon = 0$, let the background and the profiles be arbitrary, and let $y \in \mathscr{H}_1$. For every finite region R, with $\varphi_q(t) = 2\xi(z_q, y, t)$,*

$$\tau_t^R\big(\widehat{W}(y)\big) = e^{2i\xi(b,y,t)}\,\widehat{W}(e^{ith}y)\,\Pi_{R_4}\big(\varphi(t)\big),$$

and for finite regions $R \subseteq S$, $\|\tau_t^S(\widehat{W}(y)) - \tau_t^R(\widehat{W}(y))\| \le \sum_{q\in(S\setminus R)_4} |e^{2i\xi(z_q,y,t)} - 1|$.

Proof. The displacement maps the vectors of each pattern into themselves, so its evolution acts on the pattern s as $\Phi_{g_s}(t)W(y)\Phi_{g_s}(-t)$, which Lemma 7.11(i) evaluates. By linearity in the first argument, $2\xi(g_s, y, t) = 2\xi(b, y, t) + \sum_q q_q(s)\varphi_q(t)$. The comparison follows from (7.8). □

Whether a displacement has an evolution on the whole lattice is therefore a question about how its phases add up.

Definition 7.13 (Test profile)**.** Given profiles z_q for all registers, $y \in \mathscr{H}_1$ is a *test profile* if for every $T \ge 0$ there is a function $u_T \ge 0$ on the registers with $\sum_q u_T(q) < \infty$ and $|\xi(z_q, y, t)| \le u_T(q)$ for every register q and every $|t| \le T$.

Proposition 7.14 (Which displacements pass the test)**.** *(i) The test profiles form a real vector space, and with y every freely evolved $e^{ish}y$, $s \in \mathbb{R}$, is a test profile.*

(ii) For the unit-edge family, the photon cloud $\delta(e^{ish}z_p - z_p)$ of every register p is a test profile for all real δ and s: for $T \ge 0$, $|\xi(z_q, e^{ish}z_p - z_p, t)| \le 3\,\mathscr{K}_{T+|s|}(p, q)$ for every register q and every $|t| \le T$.

Proof. (i) By Lemma 7.11(ii) sums and real multiples are dominated by the sums and multiples of the functions u_T, and by (iii) $|\xi(z_q, e^{ish}y, t)| \le 2u_{T+|s|}(q)$ for $|t| \le T$. (ii) By Lemma 7.11(iv) the coupling of the cloud is a second difference of $\theta(z_q, z_p, \cdot)$ at three times of absolute value at most $T + |s|$; Lemma 7.7 bounds each term by $\mathscr{K}_{T+|s|}(p, q)$ and makes the bound summable. Multiplying by δ keeps the test, by (i). □

The clouds are exactly the photon factors that the dynamics itself produces. A bare current profile z_p is a different matter: the weight of its coupling $\xi(z_q, z_p, t) = \langle z_q, \sin(th)z_p\rangle$ tends to $\frac{at}{2}P_T(k/|k|)$ as $k \to 0$, a function of the direction alone, whose Fourier coefficients fall off only like $|x|^{-3}$, the tail of a dipole field. Whether z_p fails the test is open (Problem 31).

Theorem 7.15 (Displacements with the clock on)**.** *For the unit-edge family at every clock strength $\epsilon \ge 0$, let y be a test profile. Fix $T \ge 0$, let $u = u_T$, $U = \sum_q u(q)$, and for finite regions*

$R \subseteq S$ let $A_{R,S} = \sum_{q\in(S\setminus R)_4} 4u(q)$ and let $\Delta_{R,S} \geq 0$ bound $\sum_{q\in R_4} 2u(q) \sum_g \|\tau_x^S(c_{q,g}) - \tau_x^R(c_{q,g})\|$ for all $|x| \leq T$, where $c_{q,g} = O_q(Q^{g_1}CQ^{g_2})$ are the four blocks of the clock of q. Then for every $|t| \leq T$

$$\|\tau_t^S(\widehat{W}(y)) - \tau_t^R(\widehat{W}(y))\| \leq \Big(A_{R,S} + \epsilon\big((2c_1U + c_1)A_{R,S} + \Delta_{R,S}\big)T\Big)e^{2\epsilon c_1 UT},$$

and $\widehat{W}(y)$ converges in the sense of Definition 7.1. In particular every photon cloud of a flip converges.

The proof (Appendix B.5) is that of Theorem 7.10 with a single node: a displacement is passed by every clock block at a phase, and the clock blocks are register observables whose convergence is already known.

Theorem 7.16 (The algebra of the dynamics)**.** *For the unit-edge family at every clock strength $\epsilon \geq 0$, let $\mathfrak{B}$ be the set of bounded operators on $\mathscr{H}$ that converge in the sense of Definition 7.1, and let $\mathfrak{A}_\pi$ be the norm closure of the algebra generated by the register observables and the displacements $\widehat{W}(y)$ of the test profiles y; the same definition makes sense for every assignment of unit edge currents.*

(i) $\mathfrak{B}$ is a norm-closed subalgebra of the bounded operators that contains the identity, is closed under adjoints and contains $\mathfrak{A}_\pi$.

(ii) For $X \in \mathfrak{B}$ and real t, $\tau_t(X)$ lies in $\mathfrak{B}$; each τ_t is linear, multiplicative, isometric and compatible with adjoints, $\tau_t \circ \tau_s = \tau_{t+s}$ and τ_0 is the identity. The τ_t form a one-parameter group of *-automorphisms of $\mathfrak{B}$.

(iii) For every assignment of unit edge currents to the registers, injective or not, $\mathfrak{A}_\pi$ is closed under adjoints, so it is a C^*-algebra.

Proof. (i) Sums, multiples and the identity converge at once. For products, $\tau_t^R(XY) - \tau_t^S(XY) = (\tau_t^R X - \tau_t^S X)\,\tau_t^R Y + \tau_t^S X(\tau_t^R Y - \tau_t^S Y)$, and both evolutions are isometries; for adjoints the distances are equal. If $\|X - X'\| \leq \eta/3$ with $X' \in \mathfrak{B}$, then $\|\tau_t^R X - \tau_t^S X\| \leq 2\eta/3 + \|\tau_t^R X' - \tau_t^S X'\|$, so $\mathfrak{B}$ is closed. The register observables lie in $\mathfrak{B}$ by Theorem 7.10, the test displacements by Theorem 7.15, hence $\mathfrak{A}_\pi \subseteq \mathfrak{B}$. (ii) The algebraic properties pass to the norm limit. For real s, $\|\tau_t^R(\tau_s X) - \tau_{t+s}^R(X)\| \leq \|\tau_s X - \tau_s^R X\|$, which tends to zero uniformly in t; since $\tau_{t+s}^R(X)$ is Cauchy uniformly for $|t| \leq T$, $\tau_s X \in \mathfrak{B}$ and $\tau_t\tau_s X = \tau_{t+s}X$. Finally $\|X\| = \|\tau_{-t}\tau_t X\| \leq \|\tau_t X\| \leq \|X\|$. (iii) $O_p(B)^* = O_p(B^*)$ and $\widehat{W}(y)^* = \widehat{W}(-y)$, and $-y$ is a test profile whenever y is, since its couplings change sign. So the generators are closed under adjoints, and so are the algebra they generate and, the adjoint being isometric, its norm closure. □

The passage from converging generators to a group of automorphisms is the standard construction for infinite systems [10, Section 6.2]. Three things are not claimed. The theorem maps $\mathfrak{B}$ into itself; it does not show that $\mathfrak{A}_\pi$ is mapped into itself. For the clock off it is (Theorem 7.49); for $\epsilon > 0$ this is open. It gives no continuity of $t \mapsto \tau_t(X)$ in norm, and none can hold for Weyl operators even for the free field: for $u \neq 0$, $W(u)$ has the whole unit circle as spectrum, and two different Weyl operators lie at distance 2; the resolvent algebra was introduced for this reason [12]. And it selects no state, no representation and no generator. The next section supplies a state.

7.6 The reference state

The obstruction of Chapter 6 concerns vectors. We now show that the clock ground state survives as a state on the algebra of the dynamics.

Definition 7.17 (Reference vectors)**.** For a finite region R, let $u_R = \chi^{\otimes 4|R|} \otimes \Omega_{\mathrm{ph}} \in \mathscr{N}_R \otimes \mathscr{F}$, and let $\psi_R \in \mathscr{H}$ be the vector with $J_R\psi_R$ equal to u_R at the exterior configuration $\eta = 0$ and zero at every other one: every register of R in its zero mode, every register outside R at the label 0, and no photons.

The unit vector ψ_R is the clock ground state $T_R\Omega$ of Proposition 6.30, on which the regional Hamiltonians, centred by any scalars, have no limit.

Proposition 7.18 (One fiber, one value)**.** *If X is fiberwise over R with fiber F, then $\langle \psi_S, X\psi_S\rangle = \langle u_R, Fu_R\rangle$ for every finite region $S \supseteq R$.*

Proof. Read ψ_S in the chart of R. Its fiber over an exterior configuration η of R vanishes unless η is zero outside S. If it is, the labels of η on the $m = 4|S \setminus R|$ registers of $S \setminus R$ form one of 8^m patterns, and the fiber is $8^{-m/2}u_R$, because $\chi^{\otimes 4|S|}$ has amplitude $8^{-2|S|}$ on every pattern. So $J_R\psi_S = \bigoplus_\eta a_\eta u_R$ with $\sum_\eta |a_\eta|^2 = 1$, and $\langle \psi_S, X\psi_S\rangle = \sum_\eta |a_\eta|^2 \langle u_R, Fu_R\rangle = \langle u_R, Fu_R\rangle$. □

Seen from a fixed region, the reference vector of every larger region is the same zero-mode fiber spread evenly over the exterior labels, and an operator that does not look at those labels cannot tell the regions apart.

Theorem 7.19 (The reference state)**.** *On the clock-photon lattice let $a > 0$ and $C_E > 0$, and let unit edge currents be assigned to the registers, injectively or not. For every $X \in \mathfrak{A}_\pi$ the limit*

$$\omega_{\mathrm{ref}}(X) = \lim_R \langle \psi_R, X\psi_R\rangle$$

exists along the finite regions ordered by inclusion. It is linear, $\omega_{\mathrm{ref}}(I) = 1$, $|\omega_{\mathrm{ref}}(X)| \le \|X\|$, $\omega_{\mathrm{ref}}(X^*) = \overline{\omega_{\mathrm{ref}}(X)}$ and $\omega_{\mathrm{ref}}(X^*X) \ge 0$: ω_{ref} is a state on the C-algebra $\mathfrak{A}_\pi$.*

Proof. Call X local over R_0 if it is fiberwise over every finite $S \supseteq R_0$. A register observable of a register in the cell r is local over $\{r\}$, every $\widehat{W}(y)$ is local over the empty region, and sums and products of local operators are local over the union of their regions. So every element of the algebra generated by these operators is local over some finite region R_0, and by Proposition 7.18 its net of expectations is constant from R_0 on. For $X \in \mathfrak{A}_\pi$ and $\eta > 0$ choose such a Y with $\|X - Y\| \le \eta/3$; since $|\langle \psi, X\psi\rangle - \langle \psi, Y\psi\rangle| \le \|X - Y\|$ for unit vectors, the expectations of X in any two regions containing R_0 differ by at most $2\eta/3$. The net is Cauchy and converges. Linearity, the value at I, the bound and the relation for adjoints pass to the limit, and $\langle \psi_R, X^*X\psi_R\rangle = \|X\psi_R\|^2 \ge 0$. The C*-algebra property is Theorem 7.16(iii). □

Theorem 7.20 (The reference state under the dynamics)**.** *For the unit-edge family at every clock strength $\epsilon \ge 0$, every $X \in \mathfrak{A}_\pi$ and every real t, the regional expectations $\langle \psi_R, \tau_t^R(X)\psi_R\rangle$ and the vector states $\langle \psi_R, \tau_t(X)\psi_R\rangle$ converge along the finite regions ordered by inclusion, to the same number $\omega_{\mathrm{ref}}(\tau_t(X))$. The function $t \mapsto \omega_{\mathrm{ref}}(\tau_t(X))$ is continuous.*

Proof. Let X be local over R_0. For $S \supseteq R_0$, X is fiberwise over S, hence so is $\tau_t^S(X)$, and Proposition 7.18 gives $\langle \psi_{S'}, \tau_t^S(X)\psi_{S'}\rangle = e_S(t)$ for every $S' \supseteq S$, with $e_S(t) = \langle \psi_S, \tau_t^S(X)\psi_S\rangle$. Put $d_S = \sup_{|t|\le T}\|\tau_t(X) - \tau_t^S(X)\|$, which tends to zero. For $S' \supseteq S \supseteq R_0$,

$$\big|\langle \psi_{S'}, \tau_t(X)\psi_{S'}\rangle - e_S(t)\big| \le d_S, \qquad |e_{S'}(t) - e_S(t)| \le d_{S'} + d_S.$$

So $e_S(t)$ is Cauchy, uniformly for $|t| \le T$, and the vector states of $\tau_t(X)$ converge to its limit. Each e_S is continuous, because $e_S(t) = \langle e^{-itH_S}\psi_S, Xe^{-itH_S}\psi_S\rangle$ and the unitary group of a self-adjoint operator is strongly continuous. A uniform limit of continuous functions on $[-T,T]$ is continuous. For general $X \in \mathfrak{A}_\pi$, approximate by a local Y within $\eta/3$; every τ_t^R and τ_t is an isometry, so all the numbers for X and Y differ by at most $\eta/3$, uniformly in t, and the conclusions pass to X. □

The continuity concerns the numbers $\omega_{\mathrm{ref}}(\tau_t(X))$, not the operators $\tau_t(X)$. The state itself is not given by any vector of the carrier.

Proposition 7.21 (No vector of the carrier gives the reference state)**.** *Under the hypotheses of Theorem 7.19, let E_0 be the projection of $\mathbb{C}^8$ onto the basis vector of label 0, and fix $k \in \{0,1,2,3\}$. Then $\omega_{\mathrm{ref}}\big(O_{(r,k)}(E_0)\big) = \frac{1}{8}$ for every cell r, while for every unit vector $\psi \in \mathscr{H}$, $\langle \psi, O_{(r,k)}(E_0)\psi\rangle \to 1$ as r leaves every finite set. So no unit vector ψ has $\langle \psi, X\psi\rangle = \omega_{\mathrm{ref}}(X)$ for all $X \in \mathfrak{A}_\pi$.*

Proof. The first value is $\langle \chi, E_0\chi\rangle = \frac{1}{8}$ by Proposition 7.18. For the second, $\langle \psi, O_{(r,k)}(E_0)\psi\rangle = \sum_\eta \|\psi(\eta)\|^2 [\eta_{(r,k)} = 0]$, the sum over configurations. Each configuration has finitely many nonzero labels, so for each η the indicator equals one for all but finitely many cells r, and dominated convergence with the summable weights $\|\psi(\eta)\|^2$ gives the limit $\|\psi\|^2 = 1$. □

The proposition is the obstruction of Chapter 6 at the level of the state: the reference is an infinite product of zero modes with the photon vacuum, while every vector of the carrier differs from the label-0 configuration on finitely many registers only. In von Neumann's language the two belong to different incomplete tensor products [124]. Neither theorem says that ω_{ref} is the physical state; Section 7.7 shows that it is not invariant.

7.7 The clock-free evolution of the reference state

Theorem 7.22 (Exact reference values)**.** *On the clock-photon lattice let $C_E > 0$ and $\epsilon = 0$, let the background and the profiles be arbitrary, let R contain the cell of p, and let $B = Q^{\beta_1} A Q^{\beta_2}$ with δ, z, φ_q and Θ as in Theorem 7.4. With $g = \delta(e^{ith}z - z)$,*

$$\langle \psi_R, \tau_t^R\big(O_p(B)\big)\psi_R\rangle = e^{i\Theta}\, e^{-\|g\|^2/2}\, e^{i\beta_2\varphi_p}\, \langle \chi, B\chi\rangle \prod_{q \in R_4 \setminus \{p\}} \frac{1 + e^{i\varphi_q}}{2}, \tag{7.12}$$

and each factor of the product has modulus $|\cos(\varphi_q/2)| \le 1$.

Proof. By Theorem 7.4, $\tau_t^R(O_p(B))$ is fiberwise over R, with fiber $e^{i\Theta}$ times the tensor product of $B(I + (e^{i\varphi_p} - 1)Q) = e^{i\beta_2\varphi_p}B$ on register p (since $Q^{\beta_2}Q = \beta_2 Q^{\beta_2}$), $I + (e^{i\varphi_q} - 1)Q$ on every other register of R, and $W(g)$ on the field. Proposition 7.18 with $S = R$ reduces the expectation to u_R, a product vector, and the three kinds of factor are evaluated as in Example 7.3. Finally $|1 + e^{i\varphi}|^2 = 4\cos^2(\varphi/2)$. □

Corollary 7.23 (The whole lattice with the clock off)**.** *For the unit-edge family with $\epsilon = 0$, let $B = Q^{\beta_1} A Q^{\beta_2}$ be a block of register p with flip δ, and $c(t) = \omega_{\mathrm{ref}}\big(\tau_t(O_p(B))\big)$.*

(i) If $\delta = 0$, then $\tau_t(O_p(B)) = O_p(B)$ and $c(t) = \langle \chi, B\chi \rangle$ for every t.
(ii) For every finite set D of registers other than p, $|c(t)| \le |\langle \chi, B\chi \rangle| \prod_{q \in D} \big|\cos\big(\delta\theta(z_q, z_p, t)\big)\big|$.
(iii) With $g_t = \delta(e^{ith} z_p - z_p)$, $|c(t)| \le e^{-\|g_t\|^2/2} |\langle \chi, B\chi \rangle|$.

Proof. (i) With $\delta = 0$ every factor of the closed form other than $O_p(B)$ is the identity, in every region. (ii) By Theorem 7.20, $c(t)$ is the limit of the left side of (7.12). In every region containing the cells of p and of D, all factors outside D have modulus at most one, and $|\cos(\varphi_q/2)| = |\cos(\delta\theta(z_q, z_p, t))|$. (iii) As in (ii), keeping the cloud factor, which does not depend on the region. □

So the reference state keeps the occupation statistics of every register, a fair coin, at all times. On blocks that change the occupation every other register contributes one factor of Figure 7.1, and the register's own cloud contributes $e^{-\|g_t\|^2/2}$, which is below one at every nonzero time.

Lemma 7.24 (Level sets of the photon frequency)**.** *Let $C_E > 0$. Every level set $\{k : \omega(k) = c\}$ is μ-null. Consequently, if $t \neq 0$ and $e^{ith} g = g$ for some $g \in \mathscr{H}_1$, then $g = 0$.*

Proof. ω is a strictly increasing function of s, so a level set of ω is a level set of s. For fixed k_2, k_3, $s(k) = c'$ fixes $\cos k_1$, which leaves at most two values of k_1; by Fubini's theorem the level set is null. If $e^{ith} g = g$, then $g(k) = 0$ except where $t\omega(k) \in 2\pi\mathbb{Z}$, a countable union of level sets. □

Proposition 7.25 (No invariance at any nonzero time)**.** *Under the hypotheses of Corollary 7.23, let $\delta \neq 0$ and $\langle \chi, B\chi \rangle \neq 0$. Then $|c(t)| < |c(0)| = |\langle \chi, B\chi \rangle|$ for every $t \neq 0$. For the flip coherence $B = Q\chi\chi^*(I - Q)$, $c(0) = \frac{1}{4}$ and $|c(t)| < \frac{1}{4}$ for every $t \neq 0$: the reference state is not invariant at any nonzero time.*

Proof. $z_p \neq 0$ for $a > 0$, so by Lemma 7.24 $g_t \neq 0$ for $t \neq 0$, and Corollary 7.23(iii) gives the strict inequality; $c(0) = \langle \chi, B\chi \rangle$ by Proposition 7.18. For the flip coherence, $\langle \chi, Q\chi\chi^*(I - Q)\chi \rangle = \langle \chi, Q\chi \rangle \langle \chi, (I - Q)\chi \rangle = \frac{1}{4}$ by (7.4). □

The proposition holds at each time, and it does not make the coherences small: $\|g_t\| \le 2\|z_p\|$, so the cloud factor stays above $e^{-2\|z_p\|^2}$. Making them small takes the other registers.

7.8 Dephasing of one register, with no open premise

By (7.7) the phase $\theta(z_q, z_p, t)$ is a constant, plus the bounded term $\mathrm{Im}\langle z_q, e^{ith} z_p \rangle$, minus t times a rate. The question is therefore about products of squared cosines of phases that grow linearly at nonzero rates, perturbed by terms that are small on average. One factor returns to one periodically; many factors return together only when their phases align, and a count of sign patterns measures how rarely that happens.

Lemma 7.26 (Erdős's count)**.** *Let $W_1, \dots, W_m$ be nonzero reals. At most $\binom{m}{\lfloor m/2 \rfloor}$ sign vectors $\eta \in \{-1,1\}^m$ have $\sum_i \eta_i W_i = 0$. Consequently, for nonzero reals $V_1, \dots, V_N$, at most $\binom{2N}{N}$ pairs $(\sigma, \sigma') \in \{-1,1\}^N \times \{-1,1\}^N$ have $\sum_j (\sigma_j + \sigma'_j) V_j = 0$.*

Proof. Changing the sign of η_i wherever $W_i < 0$ reduces to positive weights. For positive weights let $A_\eta = \{i : \eta_i = 1\}$. If $A_\eta \subsetneq A_{\eta'}$, the two sums differ by $2 \sum_{i \in A_{\eta'} \setminus A_\eta} W_i > 0$, so the balanced sign vectors give an antichain of subsets, which by Sperner's theorem [106] has at most $\binom{m}{\lfloor m/2 \rfloor}$ members [42]. For the pairs apply this to the $2N$ weights $V_1, V_1, \dots, V_N, V_N$. □

Lemma 7.27. *For every $n \geq 0$, $\binom{2n}{n}^2 (2n+1) \leq 16^n$; hence $\binom{2N}{N} 4^{-N} \leq (2N+1)^{-1/2} \to 0$.*

Proof. Both sides equal one at $n = 0$. Since $\binom{2n+2}{n+1} = \binom{2n}{n} \frac{2(2n+1)}{n+1}$, induction gives $\binom{2n+2}{n+1}^2 (2n+3) \leq 16^n \frac{4(2n+1)(2n+3)}{(n+1)^2} \leq 16^{n+1}$, because $(2n+1)(2n+3) < 4(n+1)^2$. □

Theorem 7.28 (Time averages of products of cosines)**.** *Let $\vartheta_j(t) = \alpha_j + S_j(t) - V_j t$, $j = 1, \dots, N$, with real constants α_j, nonzero rates V_j, and continuous real functions S_j with $\frac{1}{T}\int_0^T |S_j| \, \mathrm{d}t \to 0$. For every $\eta > 0$ and all sufficiently large T,*

$$\frac{1}{T} \int_0^T \prod_{j=1}^{N} \cos^2 \vartheta_j(t) \, \mathrm{d}t \leq \binom{2N}{N} 4^{-N} + \eta.$$

Proof. Since $\cos^2 x = \frac{1}{4} \sum_{\sigma, \sigma' = \pm 1} e^{i(\sigma + \sigma')x}$, the product is $4^{-N} \sum_{(\sigma, \sigma')} e^{i(A + S(t) - \Omega t)}$ with $A = \sum_j (\sigma_j + \sigma'_j) \alpha_j$, $S = \sum_j (\sigma_j + \sigma'_j) S_j$ and $\Omega = \sum_j (\sigma_j + \sigma'_j) V_j$. Since $|e^{ix} - 1| \leq |x|$ and $|S| \leq 2 \sum_j |S_j|$, each term has $\left| \int_0^T e^{i(A + S - \Omega t)} \, \mathrm{d}t \right| \leq \left| \int_0^T e^{i(A - \Omega t)} \, \mathrm{d}t \right| + 2 \sum_j \int_0^T |S_j| \, \mathrm{d}t$, and the first integral is T when $\Omega = 0$ and at most $2/|\Omega|$ otherwise. The average of the real product is at most the modulus of the sum, so it is at most $4^{-N} \#\{\Omega = 0\} + \frac{4^{-N}}{T} \sum_{\Omega \neq 0} \frac{2}{|\Omega|} + 2 \sum_j \frac{1}{T} \int_0^T |S_j| \, \mathrm{d}t$. Lemma 7.26 bounds the first term; the others tend to zero. □

The bound is uniform in the rates, so no independence or genericity of the couplings is needed. The worst case is equal rates, where all phases return together, and even then the average of $\cos^{2N}(Vt)$ is exactly $\binom{2N}{N} 4^{-N}$ by Wallis's integral, about $(\pi N)^{-1/2}$; for six registers it is $\binom{12}{6} 4^{-6} = 924/4096 \approx 0.2256$. Unequal rates only make simultaneous returns rarer.

Two properties of the native field enter. The first concerns the photon spectrum.

Definition 7.29 (No sharp photon line)**.** The photon spectrum has *no sharp line* if $\lim_{T \to \infty} \frac{1}{T} \int_0^T |\langle f, e^{itH} g \rangle| \, \mathrm{d}t = 0$ for all $f, g \in \mathscr{H}_1$.

Proposition 7.30 (Averages without atoms)**.** *Let ν be a finite measure on a set X, $\phi : X \to \mathbb{R}$ measurable with every level set ν-null, and $h \in L^1(\nu)$. Then $F(t) = \int_X e^{it\phi} h \, \mathrm{d}\nu$ satisfies $\frac{1}{T} \int_0^T |F(t)| \, \mathrm{d}t \to 0$.*

Proof. Put $K_T(u) = \int_0^T e^{itu} \, \mathrm{d}t$, so $|K_T(u)| \leq T$ and $|K_T(u)| \leq 2/|u|$ for $u \neq 0$. Writing $|F|^2$ as a double integral and exchanging integrals, $\frac{1}{T} \int_0^T |F|^2 \, \mathrm{d}t = \iint T^{-1} K_T(\phi(x) - \phi(y)) \, h(x) \overline{h(y)} \, \mathrm{d}\nu(x) \, \mathrm{d}\nu(y)$.

The integrand is dominated by $|h(x)||h(y)|$ and tends to zero off the diagonal set $\{\phi(x)=\phi(y)\}$, which is null for $\nu\otimes\nu$ because its slices are level sets. By dominated convergence the average of $|F|^2$ tends to zero, and $|F|\le \eta/2 + |F|^2/(2\eta)$ for every $\eta>0$. □

This is the atomless case of Wiener's lemma [67], the mechanism behind the RAGE theorem.

Theorem 7.31 (No sharp photon line)**.** *For every $C_E>0$ the photon spectrum has no sharp line.*

Proof. $\langle f, e^{ith} g\rangle = \int e^{it\omega(k)}\langle f(k), g(k)\rangle\, \mathrm{d}\mu(k)$, the density is integrable by the Cauchy-Schwarz inequality, and every level set of ω is null (Lemma 7.24). □

The second property concerns the placement. The *static coupling* of two profiles is $V(u,v) = \mathrm{Re}\langle\sqrt{h}\,u, \sqrt{h}\,v\rangle$, and $V_{qp} = V(z_q, z_p)$. By (7.3) the field energy of an occupation pattern is lowered by $\|\sqrt{h}\,g\|^2$, so $-2V_{qp}$ is the photon-mediated interaction energy of q and p when both are occupied. By (7.7),

$$\theta(z_q, z_p, t) = -\,\mathrm{Im}\langle z_q, z_p\rangle + \mathrm{Im}\langle z_q, e^{ith} z_p\rangle - V_{qp}\,t. \tag{7.13}$$

For two unit currents of one direction d on the edges (y,d) and (x,d), the profiles differ only by the phase of the translation, and (7.1) gives

$$V\big(z_{(y,d)}, z_{(x,d)}\big) = \mathrm{Re}\int n_d(k)\, e^{i(x-y)\cdot k}\,\mathrm{d}\mu(k), \qquad n_d = \omega\varrho^2\,\|P_T e_d\|^2, \qquad \omega\varrho^2 = a\,\varpi(s), \tag{7.14}$$

with the *photon weight* $\varpi(L) = \operatorname{arsinh}(\sqrt{L}/2)\big/\big(\sqrt{L}\sqrt{1+L/4}\big)$. With $\xi = \sqrt{L}/2$, $\varpi = \operatorname{arsinh}\xi/(2\xi\sqrt{1+\xi^2})$, so $\operatorname{arsinh}\xi \le \xi$ gives $0 \le n_d \le a/2$. The static coupling is a Fourier coefficient of one density, and two of its values decide the matter (Figure 7.3).

- Along the current's own axis, $k = te_d$ with $0<|t|<\pi$, the divergence row is a nonzero multiple of e_d, so $P_T e_d = 0$ and $n_d = 0$.
- Along another axis, $k = te_b$ with $b\neq d$, $P_T e_d = e_d$ and $n_d = a\varpi(2-2\cos t)$, with $\xi = |\sin(t/2)| \in (0,1]$. Since $\operatorname{arsinh}\xi \ge \xi/2$ and $\sqrt{1+\xi^2}\le 2$ there, $n_d \ge a/8$.

A current does not couple to the photons that travel along it at long wavelength, and couples fully to those that travel across it.

Theorem 7.32 (Infinitely many static couplings)**.** *Let $a>0$ and $C_E>0$. For every edge (x,d), infinitely many edges (y,d) of the same direction have $V\big(z_{(y,d)}, z_{(x,d)}\big)\neq 0$.*

Proof. By (7.14) it suffices that $\mathrm{Re}\,c(m)\neq 0$ for infinitely many $m\in\mathbb{Z}^3$, where $c(m) = \int n_d e^{im\cdot k}\,\mathrm{d}\mu$. Put $N(k) = n_d(k) + n_d(-k)$; since μ is invariant under $k\mapsto -k$ and n_d is real, the m-th Fourier coefficient of N is $2\,\mathrm{Re}\,c(m)$. If only finitely many were nonzero, N would equal a trigonometric polynomial P almost everywhere, by completeness of the exponentials in $L^2(\mu)$. Off $k=0$, N is continuous, and two functions continuous on an open set and equal almost everywhere on it agree there, so $N = P$ on $\mathbb{T}^3\setminus\{0\}$. Letting $t\to 0$ along the two axes, continuity of P at 0 gives $P(0)=0$ and $\mathrm{Re}\,P(0)\ge a/4$, a contradiction. □

Theorem 7.33 (The coherences dephase)**.** *For the unit-edge family with $\epsilon = 0$, let p be a register with $V_{qp}\neq 0$ for infinitely many registers $q\neq p$, let B be a block of p with $\delta\neq 0$, and $c(t) = \omega_{\mathrm{ref}}\big(\tau_t(O_p(B))\big)$. For every $\eta>0$ there is a continuous function $G\ge|c|^2$ with $\frac{1}{T}\int_0^T G\,\mathrm{d}t\le\eta$ for all large T.*

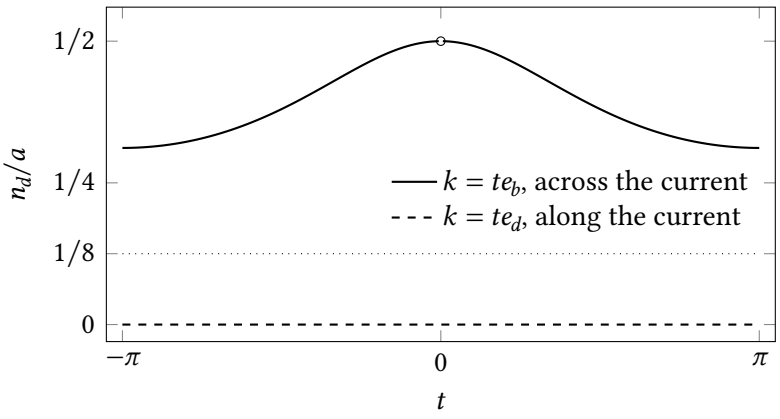


Figure 7.3. The coupling density n_d of unit currents of direction d, in units of a, along the current's own axis (dashed, zero for $t \neq 0$) and along another axis (solid, $\varpi(2 - 2\cos t)$, between $\operatorname{arsinh}(1)/(2\sqrt{2}) \approx 0.3116$ at $t = \pm\pi$ and $1/2$ as $t \to 0$). The two limits at $k = 0$ differ, so no trigonometric polynomial agrees with the density near $k = 0$, and infinitely many of its Fourier coefficients, the static couplings, are nonzero. The dotted line marks the bound $a/8$ used in the proof.

Proof. Put $M = |\langle \chi, B\chi\rangle|^2$. Choose N with $\binom{2N}{N}4^{-N} < \eta/(2(M+1))$ (Lemma 7.27) and a set D of N registers $q \neq p$ with $V_{qp} \neq 0$. Corollary 7.23(ii) gives $|c|^2 \leq G = M\prod_{q\in D}\cos^2\theta(z_q, z_p, t)$, continuous by (7.13). Write $\theta(z_q, z_p, t) = \alpha_q + S_q(t) - V_{qp}t$ as in (7.13); then $|S_q(t)| \leq |\langle z_q, e^{ith} z_p\rangle|$ has vanishing average by Theorem 7.31, and Theorem 7.28 with $\eta/(2(M+1))$ gives $\frac{1}{T}\int_0^T G \leq \eta$ for all large T. □

Theorem 7.34 (The long-time state of one register)**.** *For the unit-edge family with $\epsilon = 0$, let p be a register such that every edge of the direction of $\pi(p)$ carries a register; on the sample and the full assignment every register qualifies.*

(i) For every block B of p with $\delta \neq 0$ and $c(t) = \omega_{\mathrm{ref}}(\tau_t(O_p(B)))$, $\frac{1}{T}\int_0^T |c|^2\,\mathrm{d}t \to 0$ and $\frac{1}{T}\int_0^T c\,\mathrm{d}t \to 0$.

(ii) For every $A \in M_8(\mathbb{C})$, with $E(A) = QAQ + (I-Q)A(I-Q)$,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \omega_{\mathrm{ref}}\big(\tau_t(O_p(A))\big)\,\mathrm{d}t = \langle \chi, E(A)\chi\rangle = \operatorname{tr}(\bar\rho A), \qquad \bar\rho = \tfrac{1}{2}\chi_1\chi_1^* + \tfrac{1}{2}\chi_0\chi_0^*,$$

where $\chi_1 = \sqrt{2}\,Q\chi$ and $\chi_0 = \sqrt{2}\,(I-Q)\chi$ are the occupied and the empty half of the zero mode.

Proof. By Theorem 7.32, infinitely many edges of the direction of $\pi(p)$ couple statically to $\pi(p)$; each carries a register, distinct edges distinct registers, and all but one differ from p. Theorem 7.33 applies, and c is continuous (Theorem 7.20), so the averages of $|c|^2$ are eventually below every $\eta > 0$; since $|c| \leq \eta/2 + |c|^2/(2\eta)$, the averages of $|c|$ and hence of c tend to zero. For (ii) write A as the sum of its four blocks: the two that keep the occupation contribute the constants of Corollary 7.23(i), the two that change it average to zero. Finally $\langle \chi, QAQ\chi\rangle = \frac{1}{2}\langle \chi_1, A\chi_1\rangle$ and likewise for the empty half, and χ_1, χ_0 are unit vectors by (7.4). □

On long-time average every such register is in an equal mixture of its occupied and its empty half. Each occupation change of p is read by every other register through its interaction phase,

the photon spectrum has no line at which the retarded parts could stay coherent, and with infinitely many static couplings the readings cannot all return together. For each register, the basis the dynamics selects is its occupation basis, the one in which it drives the field or does not; this is environment-induced selection in the sense of [135, 136], with the couplings computed from the native field rather than chosen. It selects no outcome: the occupation statistics stay those of fair coins. The products of cosines have the form of the Ising-type models whose approach to equilibrium Emch and Radin proved on infinite lattices [41, 92]. The statement concerns one register at a time; the next section reads several together.

7.9 Several registers: the field reads the combined transverse current

Several registers do not dephase independently, because the field does not see them separately. It sees the current they drive together, and only the transverse part of that current. Take a finite set J of distinct registers and for each $p \in J$ a block $B_p = Q^{\beta_1(p)} A_p Q^{\beta_2(p)}$ with flip δ_p, and put

$$X_J = \prod_{p \in J} O_p(B_p), \qquad w_J = \sum_{p \in J} \delta_p \, z_p .$$

The factors act on different registers and commute. In the occupation basis of a region, X_J takes each pattern only to patterns whose combined profile is larger by w_J; we call w_J the *combined profile* of the flips, and the flips a *flip pattern*.

Theorem 7.35 (Several registers in closed form)**.** *Let $C_E > 0$ and $\epsilon = 0$, let the background b and the profiles be arbitrary, and let R contain the cells of the registers of J. With $\Theta_J = 2\theta(b, w_J, t) + \theta(w_J, w_J, t)$,*

$$\tau_t^R(X_J) = e^{i\Theta_J} \, X_J \, \widehat{W}\big(e^{ith} w_J - w_J\big) \, \Pi_{R_4}\big(2\theta(z., w_J, t)\big),$$

where $\Pi_{R_4}(2\theta(z., w_J, t))$ multiplies the occupied half of each register q of R by $e^{2i\theta(z_q, w_J, t)}$.

Proof. X_J takes the pattern s to patterns with combined profile $g_s + w_J$ and does not act on the field, so its evolution acts on the pattern s as X_J times $\Phi_{g_s + w_J}(t)\Phi_{g_s}(-t)$. Lemma 7.2 with $w = w_J$ and the real-linearity of θ in its first argument finish the proof, as for Theorem 7.4. □

For one register, $w_J = \delta z_p$ and this is Theorem 7.4. In general only the combined profile enters: two sets of flips with the same combined profile are dressed by the same photon displacement and read by every other register through the same phase.

Corollary 7.36 (Zero combined profile)**.** *Let $\epsilon = 0$ and $w_J = 0$. (i) For every background and family of profiles and every region R containing the cells of J, $\tau_t^R(X_J) = X_J$ for all t, and $\langle \psi_S, X_J \psi_S \rangle = \prod_{p \in J} \langle \chi, B_p \chi \rangle$ for every $S \supseteq R$. (ii) For every assignment of unit edge currents with no background, injective or not, $\tau_t(X_J) = X_J$ and $\omega_{\mathrm{ref}}(\tau_t(X_J)) = \prod_{p \in J} \langle \chi, B_p \chi \rangle$ for every t.*

Proof. $\theta(u, 0, t) = 0$ and $W(0) = I$, so every factor of Theorem 7.35 other than X_J is the identity. X_J is fiberwise over R with fiber $\bigotimes_{p \in J} B_p$, and Proposition 7.18 evaluates it on the product vector u_R. Part (ii) follows because every regional evolution fixes X_J. □

For unit currents the combined profile has a direct meaning. Since $f \mapsto z_f$ is linear, $w_J = z_{f_J}$ for the *combined current* $f_J = \sum_{p\in J} \delta_p \, 1_{\pi(p)}$: $+1$ on the edge of each register that the pattern fills, -1 on the edge of each register that it empties. For a finitely supported real charge q on the sites, the gradient $(\nabla q)(x,d) = q(x) - q(x+e_d)$ is the adjoint of the lattice divergence $(\operatorname{div} f)(x) = \sum_d \big(f(x,d) - f(x-e_d,d)\big)$.

Lemma 7.37 (Gradients carry no photon profile)**.** *Let $a > 0$ and $C_E > 0$. For every finitely supported real charge q, $z_{\nabla q} = 0$.*

Proof. With $\hat{q}(k) = \sum_x q(x) e^{ix\cdot k}$, shifting the summation variable gives $F_{\nabla q}(k) = \hat{q}(k) \sum_d (1 - e^{-ik_d}) e_d = \hat{q}(k)\rho(k)$, and $P_T(k)$ annihilates $\rho(k)$ for every $k \neq 0$. □

This is the lattice form of the Coulomb-gauge fact that a transverse field couples only to the transverse part of a current [24].

Proposition 7.38 (Gradient patterns of flips are conserved)**.** *Let $a > 0$, $C_E > 0$ and $\epsilon = 0$, with unit edge currents assigned to the registers (injectively or not) and no background. If the combined current of a flip pattern is a gradient, $f_J = \nabla q$, then $\tau_t(X_J) = X_J$ and $\omega_{\mathrm{ref}}(\tau_t(X_J)) = \prod_{p\in J} \langle \chi, B_p \chi \rangle$ for every t.*

Proof. $w_J = z_{\nabla q} = 0$ by Lemma 7.37, and Corollary 7.36(ii) applies. □

Example 7.39 (The star of a site)**.** The star of a site y consists of the three edges (y,d) that leave y and the three edges $(y - e_d, d)$ that enter it. The gradient of the unit charge at y is $+1$ on the leaving edges and -1 on the entering ones: a current that flows out of y along all six edges. Suppose registers p_d^+ sit on (y,d) and p_d^- on $(y - e_d, d)$, and put

$$X_{\mathrm{star}} = \prod_{d=0}^{2} O_{p_d^+}\big(Q\chi\chi^*(I-Q)\big)\, O_{p_d^-}\big((I-Q)\chi\chi^*Q\big).$$

Each factor fills a register on a leaving edge or empties one on an entering edge, so $f_J = \nabla 1_y$. Each block has value $\langle \chi, Q\chi\rangle\langle\chi,(I-Q)\chi\rangle = \frac{1}{4}$ by (7.4), and Proposition 7.38 gives

$$\omega_{\mathrm{ref}}\big(\tau_t(X_{\mathrm{star}})\big) = 4^{-6} = \frac{1}{4096} \qquad \text{for every } t,$$

although each factor alone changes the occupation and, when its direction is filled, averages to zero by Theorem 7.34 (Figure 7.4).

Theorem 7.40 (The joint long-time state is not the product of the one-register states)**.** *For the unit-edge family with $\epsilon = 0$ and an edge assignment onto all edges (for example π_{full}), there are a finite set J of registers, every one with its direction filled, and matrices A_p with*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \omega_{\mathrm{ref}}\Big(\tau_t\Big(\prod_{p\in J} O_p(A_p)\Big)\Big)\, \mathrm{d}t = 4^{-6} \neq 0 = \prod_{p\in J} \langle \chi, E(A_p)\chi\rangle.$$

So the statement that the joint long-time state is the occupation-dephased product of the one-register states is false for every such assignment.

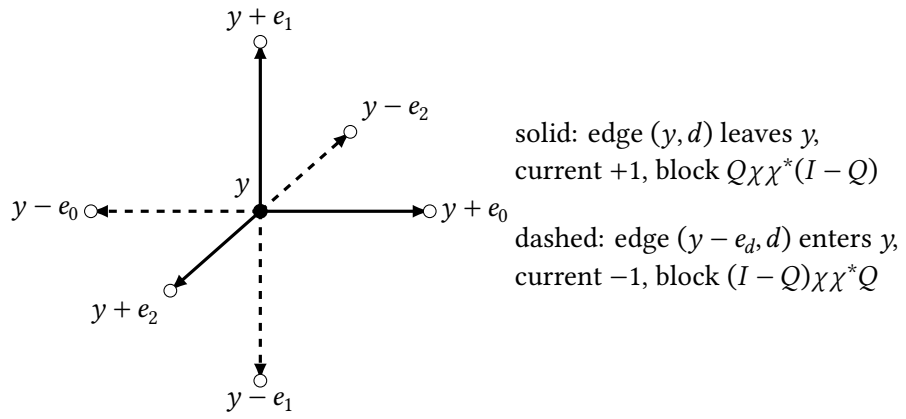


Figure 7.4. The star of a site y. The gradient $\nabla 1_y$ is $+1$ on the three edges leaving y (solid) and -1 on the three entering it (dashed), so it flows outward along all six (arrows) and has no transverse part. Registers on these edges, flipped in the pattern of this current, form a joint coherence that the field cannot see: its reference value stays 4^{-6} at all times, although each flip alone averages to zero.

Proof. The assignment is onto the edges, so the six edges of every star carry registers and every direction is filled. Take J and the A_p from X_{star}: its reference value is the constant 4^{-6}, and every $E(A_p)$ vanishes, since each A_p is a single occupation-changing block. The map π_{full} is a bijection onto the edges, because $(r_0, k) \mapsto 4r_0 + k$ is a bijection of $\mathbb{Z} \times \{0, 1, 2, 3\}$ onto $\mathbb{Z}$ and $N \mapsto (\lfloor N/3 \rfloor, N \bmod 3)$ one of $\mathbb{Z}$ onto $\mathbb{Z} \times \{0, 1, 2\}$. □

The flips the field does see are treated like one flip, with the combined profile in place of δz.

Theorem 7.41 (Exact values of products)**.** *Let $C_E > 0$ and $\epsilon = 0$, let the background and the profiles be arbitrary, and let R contain the cells of J. With Θ_J as in Theorem 7.35, $g = e^{ith}w_J - w_J$ and $\varphi_q = 2\theta(z_q, w_J, t)$,*

$$\langle \psi_R, \tau_t^R(X_J)\psi_R \rangle = e^{i\Theta_J}\, e^{-\|g\|^2/2} \prod_{p \in J} e^{i\beta_2(p)\varphi_p} \langle \chi, B_p\chi \rangle \prod_{q \in R_4 \setminus J} \frac{1 + e^{i\varphi_q}}{2}.$$

For the unit-edge family, $c_J(t) = \omega_{\mathrm{ref}}(\tau_t(X_J))$ therefore satisfies, for every finite set D of registers outside J,

$$|c_J(t)| \le \prod_{p \in J} |\langle \chi, B_p\chi \rangle| \prod_{q \in D} |\cos\theta(z_q, w_J, t)|.$$

Proof. As for Theorem 7.22, with Theorem 7.35 in place of Theorem 7.4; the bound passes to the limit by Theorem 7.20. □

The rate at which register q reads a pattern is the *combined static coupling* $V(z_q, w_J) = \sum_{p \in J} \delta_p V_{qp}$.

Theorem 7.42 (Products coupled to infinitely many registers dephase)**.** *For the unit-edge family with $\epsilon = 0$, if $V(z_q, w_J) \neq 0$ for infinitely many registers $q \notin J$, then $\frac{1}{T}\int_0^T |c_J(t)|^2 \, \mathrm{d}t \to 0$.*

Proof. The proof of Theorem 7.33 with z_p replaced by w_J and Theorem 7.41 in place of Corollary 7.23: identity (7.13) holds with any $\nu \in \mathscr{H}_1$ in place of z_p, the retarded terms average

away by Theorem 7.31, and $|c_J|^2$ is continuous by Theorem 7.20, so the envelope bound passes to $|c_J|^2$ itself. □

Whether a pattern keeps its value is thereby a question about the static couplings alone.

7.9.1 Every transverse pattern couples to infinitely many registers

Lemma 7.43 (The couplings of a pattern)**.** *Let $a > 0$ and $C_E > 0$, let δ_p be real numbers and (x_p, d_p) edges for p in a finite set J, and $w = \sum_{p\in J} \delta_p\, z_{(x_p,d_p)}$. For every edge (y, d),*

$$V\big(z_{(y,d)}, w\big) = \operatorname{Re} \int n_{J,d}(k)\, e^{-iy\cdot k}\, \mathrm{d}\mu(k), \qquad n_{J,d}(k) = \frac{a\,\varpi(s(k))}{s(k)}\, T_d(k) \quad (k \neq 0),$$

with the trigonometric polynomial

$$T_d(k) = \sum_{p\in J} \delta_p \big(s(k)[d_p = d] - \overline{r_d(k)}\, r_{d_p}(k)\big) e^{ix_p\cdot k},$$

the d-th component of $s(k)P_T(k)\hat F(k)$ for $\hat F(k) = \sum_p \delta_p e^{ix_p\cdot k} e_{d_p}$. Moreover $\overline{T_d(-k)} = T_d(k)$, so $N_{J,d}(k) = n_{J,d}(k) + \overline{n_{J,d}(-k)} = 2a\varpi(s(k))T_d(k)/s(k)$ has y-th Fourier coefficient $2V(z_{(y,d)}, w)$.

Proof. By (7.1), $V(z_{(y,d)}, z_{(x,d')}) = \operatorname{Re}\int \omega\varrho^2 \langle P_T e_d, P_T e_{d'}\rangle e^{i(x-y)\cdot k}\,\mathrm{d}\mu$, and for $k \neq 0$ the projection formula gives $\langle P_T e_d, P_T e_{d'}\rangle = [d = d'] - \overline{r_d}\, r_{d'}/s$. Summing with the real weights δ_p and using $\omega\varrho^2 = a\varpi(s)$ gives the formula; the d-th component of $sP_T\hat F = s\hat F - (\sum_{d'} r_{d'}\hat F_{d'})\rho$ is T_d. Since $r_d(-k) = \overline{r_d(k)}$, s is even and the δ_p are real, $\overline{T_d(-k)} = T_d(k)$. Finally μ is invariant under $k \mapsto -k$, so the y-th coefficient of $\overline{n(-k)}$ is the conjugate of that of n. □

Theorem 7.32 was the case of one current, where the density has two different limits at $k = 0$. For a general pattern the limits may agree, and a sharper property of the weight is needed: ϖ is not rational.

Lemma 7.44 (The photon weight is not rational)**.** *The photon weight ϖ is differentiable on $(0, \infty)$ and satisfies*

$$\varpi'(L)\,(L^2 + 4L) + (L + 2)\,\varpi(L) = 1 \qquad (L > 0). \tag{7.15}$$

There are no complex polynomials N and $D \neq 0$ with $\varpi(L)D(L) = N(L)$ for all L in an open interval (α, β), $0 \le \alpha < \beta$.

The proof (Appendix B.5) differentiates $\operatorname{arsinh}(\sqrt{L}/2)$ for (7.15), turns a rational solution into the vanishing of the polynomial $(N'D - ND')X(X + 4) + (X + 2)ND - D^2$, and excludes that by looking at the roots of D.

Theorem 7.45 (A nonvanishing numerator couples to infinitely many edges)**.** *Let $a > 0$ and $C_E > 0$, and let J, δ_p, (x_p, d_p) and w be as in Lemma 7.43. If $T_d(k_0) \neq 0$ for some k_0, then infinitely many edges (y, d) have $V\big(z_{(y,d)}, w\big) \neq 0$.*

The proof is in Appendix B.5: if only finitely many couplings were nonzero, the symmetrized density would agree off $k = 0$ with a trigonometric polynomial, and comparing the two along a line through k_0 would make ϖ rational on an interval, which Lemma 7.44 excludes.

Theorem 7.46 (Every nonzero combined profile couples to infinitely many registers)**.** *Let $a > 0$ and $C_E > 0$, and let unit edge currents be assigned to the registers (injectively or not) by an assignment that fills the directions it uses. For every flip pattern with $w_J \neq 0$, infinitely many registers q have $V(z_q, w_J) \neq 0$; in particular infinitely many of them lie outside J.*

Proof. Write $\pi(p) = (x_p, d_p)$. Some register $p_0 \in J$ has $T_{d_{p_0}}$ not identically zero. Otherwise, for every $p \in J$ the density n_{J,d_p} vanishes, so $V(z_{(x,d_p)}, w_J) = 0$ for every site x, in particular $V(z_p, w_J) = 0$; then $\|\sqrt{h}\, w_J\|^2 = V(w_J, w_J) = \sum_{p\in J} \delta_p V(z_p, w_J) = 0$, and since $\omega > 0$ almost everywhere, $w_J = 0$. Theorem 7.45 gives infinitely many edges of direction d_{p_0} that couple to w_J. The assignment fills that direction, so each of these edges carries a register, and distinct edges carry distinct registers. Since J is finite, infinitely many of these registers lie outside J. □

Theorem 7.47 (The clock-off joint dichotomy)**.** *For the unit-edge family with $\epsilon = 0$ and an edge assignment that fills the directions it uses, every product X_J of register blocks satisfies exactly one of the following:*

(i) $w_J = 0$, $\tau_t(X_J) = X_J$, and $\omega_{\mathrm{ref}}(\tau_t(X_J)) = \prod_{p\in J}\langle \chi, B_p \chi\rangle$ for every t;
(ii) $w_J \neq 0$ and $\frac{1}{T}\int_0^T \left|\omega_{\mathrm{ref}}(\tau_t(X_J))\right|^2 \mathrm{d}t \to 0$.

Proof. Corollary 7.36(ii) for $w_J = 0$; Theorems 7.46 and 7.42 for $w_J \neq 0$. □

It holds on the sample and the full assignment and on every assignment onto all edges.

Summed over the blocks of each register, the dichotomy gives the long-time state of the reference on every product of register observables.

Theorem 7.48 (The long-time state on the registers)**.** *For the unit-edge family with $\epsilon = 0$ and an edge assignment that fills the directions it uses, let J be a finite set of registers and $A_p \in M_8(\mathbb{C})$ for $p \in J$. Call a choice of one block $B_p = Q^{\beta_1(p)} A_p Q^{\beta_2(p)}$ for every $p \in J$ a block pattern, with combined profile w_J. Then*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \omega_{\mathrm{ref}}\Big(\tau_t\Big(\prod_{p\in J} O_p(A_p)\Big)\Big)\,\mathrm{d}t = \sum_{w_J=0} \prod_{p\in J}\langle \chi, B_p\chi\rangle,$$

the sum running over the block patterns with $w_J = 0$. The right side is the value of ω_{ref} on the zero-profile part of the product, the sum of $\prod_{p\in J} O_p(B_p)$ over those patterns, and that part is invariant under τ_t.

Proof. Every matrix is the sum of its four blocks, so $\prod_p O_p(A_p)$ is the sum of the products $X_J = \prod_p O_p(B_p)$ over the $4^{|J|}$ block patterns. A product with $w_J = 0$ is invariant and has the value $\prod_p \langle \chi, B_p\chi\rangle$ at every time (Corollary 7.36(ii)). A product with $w_J \neq 0$ has $\frac{1}{T}\int_0^T |c|^2\,\mathrm{d}t \to 0$ for $c(t) = \omega_{\mathrm{ref}}(\tau_t(X_J))$ by Theorem 7.47, hence $\left|\frac{1}{T}\int_0^T c\,\mathrm{d}t\right| \le \left(\frac{1}{T}\int_0^T |c|^2\,\mathrm{d}t\right)^{1/2} \to 0$. □

With the clock off the reference relaxes on the registers to the classical joint statistics of the occupations, except for the coherences of patterns with zero combined profile, which it keeps.

Theorem 7.49 (States the clock-off dynamics preserves)**.** *For the unit-edge family with $\epsilon = 0$ (so $a > 0$, $C_E > 0$, an injective edge assignment π and no background):*

(i) for every real t, τ_t maps $\mathfrak{A}_\pi$ onto itself, and restricted to $\mathfrak{A}_\pi$ the τ_t form a one-parameter group of *-automorphisms of $\mathfrak{A}_\pi$;

(ii) the long-time means $\omega_T(X) = \frac{1}{T}\int_0^T \omega_{\mathrm{ref}}(\tau_t(X))\,\mathrm{d}t$ ($X \in \mathfrak{A}_\pi$, $T > 0$) have cluster points as $T \to \infty$ for pointwise convergence on $\mathfrak{A}_\pi$; every cluster point ϕ is a state on $\mathfrak{A}_\pi$ with $\phi \circ \tau_t = \phi$ for every t, and ω_{ref} is not such a state;

(iii) every state ϕ on $\mathfrak{A}_\pi$ with $\phi \circ \tau_t = \phi$ for all t has $\phi(\tau_t(X)^*\tau_t(Y)) = \phi(X^*Y)$ for $X, Y \in \mathfrak{A}_\pi$, and in its representation $(\mathscr{H}_\phi, \Gamma_\phi, \Omega_\phi)$ [8] the map $\Gamma_\phi(X)\Omega_\phi \mapsto \Gamma_\phi(\tau_t(X))\Omega_\phi$ extends to a unitary U_t with $U_t\Omega_\phi = \Omega_\phi$, $U_tU_s = U_{t+s}$ and $\Gamma_\phi(\tau_t(X)) = U_t\Gamma_\phi(X)U_t^*$;

(iv) if π fills the directions it uses, every cluster point ϕ of (ii) gives each product $\prod_{p\in J} O_p(A_p)$ the long-time value of Theorem 7.48; if all edges of π have one direction and every edge of that direction carries a register, as on the sample assignment, $\phi\big(\prod_{p\in J} O_p(A_p)\big) = \prod_{p\in J}\langle\chi, E(A_p)\chi\rangle$, and ϕ gives every flip coherence $O_p\big(Q\chi\chi^*(I-Q)\big)$ the value 0, where ω_{ref} gives $\frac{1}{4}$.

The proof is in Appendix B.5; uniqueness, the field values and the continuity of U_t are open (Problem 29).

Proposition 7.50 (States the dynamics preserves, at every clock strength)**.** *For the unit-edge family at every clock strength $\epsilon \geq 0$, with $a > 0$, $C_E > 0$, an injective edge assignment π and no background, let $\mathfrak{D}_\epsilon$ be the norm closure of the algebra generated by the $\tau_t(X)$, $X \in \mathfrak{A}_\pi$, $t \in \mathbb{R}$; so $\mathfrak{A}_\pi \subseteq \mathfrak{D}_\epsilon$. Then $\mathfrak{D}_0 = \mathfrak{A}_\pi$; the τ_t form a one-parameter group of *-automorphisms of $\mathfrak{D}_\epsilon$; $\omega_{\mathrm{ref}}(X) = \lim_R \langle\psi_R, X\psi_R\rangle$ defines a state on $\mathfrak{D}_\epsilon$, with $t \mapsto \omega_{\mathrm{ref}}(\tau_t(X))$ continuous; and Theorem 7.49(ii), without its last clause, and (iii) hold with $\mathfrak{D}_\epsilon$ in place of $\mathfrak{A}_\pi$.*

Proof. By Theorem 7.16, $\mathfrak{D}_\epsilon$ is a norm-closed subalgebra of $\mathfrak{B}$, closed under adjoints since $\tau_t(X)^* = \tau_t(X^*)$, and each τ_s, an isometric *-homomorphism with $\tau_s\tau_t = \tau_{s+t}$, maps it onto itself; $\mathfrak{D}_0 = \mathfrak{A}_\pi$ by Theorem 7.49(i). For X local over R_0 and $S \supseteq R_0$, $\tau_t^S(X)$ is fiberwise over S and tends to $\tau_t(X)$ uniformly for bounded t, and products of operators fiberwise over S are fiberwise over S. So every element of $\mathfrak{D}_\epsilon$ is a norm limit of operators fiberwise over finite regions, and the proofs of Theorems 7.19, 7.20 and 7.49(ii), (iii) apply on $\mathfrak{D}_\epsilon$. □

Which patterns have zero profile? Gradient patterns do (Lemma 7.37). Whether every pattern with zero profile has a gradient current is open, and the dichotomy does not need it. Two simple classes are transverse.

Proposition 7.51 (Patterns that cannot hide)**.** *Let $a > 0$ and $C_E > 0$, and let $f \neq 0$ be a finitely supported real current.*

(i) If f is supported on edges of at most two directions, then $z_f \neq 0$.

(ii) If $\operatorname{div} f = 0$, then $z_f \neq 0$.

Consequently, for the unit-edge family with $\epsilon = 0$ and an edge assignment that fills the directions it uses, every flip pattern with at least one flip whose flipped registers sit on edges of at most two directions, or whose combined current has no divergence, loses its value on time average. On the

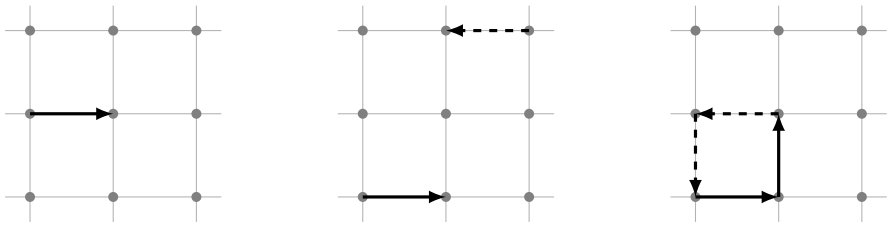
sample assignment this holds for every flip pattern with at least one flip, and for every finite J and all $A_p \in M_8(\mathbb{C})$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \omega_{\mathrm{ref}}\Big(\tau_t\Big(\prod_{p\in J} O_p(A_p)\Big)\Big)\,\mathrm{d}t = \prod_{p\in J}\langle \chi, E(A_p)\chi\rangle \ : \tag{7.16}$$

there the joint long-time state on register products is the product of the one-register states.

Proof. (i) Since $\varrho > 0$ off $k = 0$, $z_f = 0$ would give $P_T F_f = 0$ almost everywhere, that is $F_f(k) = c(k)\rho(k)$ for almost every k. Let d'' be a direction that f does not use. Then $0 = F_{f,d''} = c\,\rho_{d''}$, and $\rho_{d''}(k) = 1 - e^{-ik_{d''}}$ vanishes only where $k_{d''} = 0$, a null set; so $c = 0$ and $F_f = 0$ almost everywhere. A trigonometric polynomial that vanishes almost everywhere has zero coefficients, and the coefficients of F_f are the values of f; so $f = 0$. (ii) The Fourier transform of div f is $\sum_d r_d F_{f,d}$, so div $f = 0$ gives $P_T F_f = F_f \neq 0$ and $z_f = i\varrho F_f \neq 0$. For a flip pattern on an edge assignment, f_J takes the value δ_p on the edge of p, so it is nonzero as soon as one block changes the occupation, and $w_J = z_{f_J} \neq 0$; Theorem 7.47(ii) applies. On the sample assignment every edge has direction 0. For (7.16), write each A_p as the sum of its four blocks and expand the product. The terms in which every block keeps the occupation are constant with value $\prod_p \langle \chi, B_p\chi\rangle$ (Corollary 7.36), and they add up to $\prod_p \langle \chi, E(A_p)\chi\rangle$. Every other term $c(t)$ has $\frac{1}{T}\int_0^T |c|^2 \to 0$, hence $\big|\frac{1}{T}\int_0^T c\big| \le \big(\frac{1}{T}\int_0^T |c|^2\big)^{1/2} \to 0$. □

Every product over two registers in which at least one block changes the occupation is covered by part (i), since its flipped registers use at most two directions; the plaquette loop of Figure 7.5(c) is covered by both parts. On the sample assignment, where all currents are parallel, no flip pattern hides from the field and the joint long-time state is the product of the dephased registers; on assignments onto all edges the star hides, and it is not (Theorem 7.40). The geometry of the currents decides which correlations survive.



(a) one flip: $N_0 = 1$ (b) a hop: $N_0 = 0$ (c) a loop: $N_0 = N_1 = 0$

Figure 7.5. Flip patterns in one plane, with e_0 to the right and e_1 upward. A solid arrow marks a block that fills the register on its edge (current +1 along the edge), a dashed arrow a block that empties it (current −1, drawn in the direction the current flows). $N_d = \sum_{d_p=d} \delta_p$ is the net current along e_d. (a) One flip carries a net current. (b) A hop carries none. (c) The four flips around a plaquette carry no net current along either direction and circulate without divergence. All three have a nonzero transverse part and lose their value on time average on an assignment that fills the directions they use (Theorem 7.47). The star of Figure 7.4 has no net current either, but it is a gradient and is conserved.

What does the work in Theorem 7.45 is the lattice dispersion. If $\omega\varrho^2$ were a polynomial in s, for instance its long-wavelength value $a/2$, a divergence-free pattern, for which $T_d = s\hat{F}_d$, would

have the trigonometric-polynomial density $\omega\varrho^2\hat{F}_d$ and would couple to finitely many registers only. The transcendence of ϖ links every transverse pattern to infinitely many registers.

The physical reading is this. The shared field is transverse and couples to a current only through its transverse part. Two configurations of the registers whose combined currents differ by a gradient drive the same field; the field cannot tell them apart, and their superposition keeps its coherence at all times. These superpositions form a decoherence-free sector of the registers, in the sense of Duan and Guo, Zanardi and Rasetti, and Lidar, Chuang and Whaley [37, 134, 74]. Jointly, then, the clock-off field tells register configurations apart by the transverse part of their combined current, and by nothing finer; it does not select the configuration. Each register alone is still dephased into its occupation basis (Theorem 7.34), and the star shows that correlations among registers survive that dephasing.

7.10 The evolved state leaves the reference sector

A local operation prepares a new state from ω_{ref} by acting with finitely many local operators. Mixtures of such preparations, and their norm limits, are the states that the representation built from ω_{ref} describes by density matrices. Does the dynamics keep the reference state among them? With the clock off it does not, for any assignment, at any nonzero time.

Definition 7.52 (Local excitations)**.** Let $\mathfrak{A}_\pi^{\mathrm{loc}}$ be the algebra generated by the register observables and the test displacements, so that $\mathfrak{A}_\pi$ is its norm closure. A *local excitation* of ω_{ref} is a functional on $\mathfrak{A}_\pi$ of the form $\phi(X) = \sum_{i=1}^{n} \omega_{\mathrm{ref}}(A_i^* X A_i)$ with $A_i \in \mathfrak{A}_\pi^{\mathrm{loc}}$ and $\sum_i \omega_{\mathrm{ref}}(A_i^* A_i) = 1$.

Every local excitation is a state on $\mathfrak{A}_\pi$, and ω_{ref} is one ($n = 1$, $A_1 = I$). Far from the cells its operators touch, a local excitation looks exactly like the reference state.

Lemma 7.53 (Locality and factorization)**.** *(i) For every $A \in \mathfrak{A}_\pi^{\mathrm{loc}}$ there is a finite set S of cells such that A and A^* commute with every register observable $O_{(r,k)}(B)$ with $r \notin S$. (ii) Let p be a register, $P = O_p(\chi\chi^*)$, and $Z \in \mathfrak{A}_\pi$ with $PZ = ZP$. Then $\omega_{\mathrm{ref}}(ZO_p(B)) = \langle \chi, B\chi\rangle\, \omega_{\mathrm{ref}}(Z)$ for every $B \in M_8(\mathbb{C})$.*

Proof. (i) A register observable of (r', k') commutes with those of other registers, so take $S = \{r'\}$; a displacement acts on the field alone, so take $S = \varnothing$, as for scalars. Sums and products need the union of the sets, and $O_p(B)^* = O_p(B^*)$ handles adjoints. (ii) For every region containing the cell of p, register p of ψ_R is in its zero mode, so $P\psi_R = \psi_R$; and $P^* = P$, $PO_p(B)P = \langle \chi, B\chi\rangle P$. Hence $\langle \psi_R, ZO_p(B)\psi_R\rangle = \langle \psi_R, ZPO_p(B)P\psi_R\rangle = \langle \chi, B\chi\rangle\langle \psi_R, Z\psi_R\rangle$, and Theorem 7.19 takes the limit. □

Corollary 7.54 (Local excitations far away)**.** *For every local excitation ϕ there is a finite set S of cells with $\phi\big(O_{(r,k)}(B)\big) = \langle \chi, B\chi\rangle$ for every cell $r \notin S$, every k and every B.*

Proof. Take S as the union of the sets of Lemma 7.53(i) for the A_i. Let $r \notin S$ and $p = (r, k)$. Then $A_i^* O_p(B) A_i = A_i^* A_i\, O_p(B)$, and $A_i^* A_i$ commutes with $O_p(\chi\chi^*)$; Lemma 7.53(ii) gives $\phi(O_p(B)) = \langle \chi, B\chi\rangle \sum_i \omega_{\mathrm{ref}}(A_i^* A_i) = \langle \chi, B\chi\rangle$. □

The evolved state, on the other hand, differs from the reference state on every register by an amount that does not shrink with distance: every register carries the same cloud.

Lemma 7.55 (The cloud of a register)**.** *Let $a > 0$ and $C_E > 0$. For every site x and direction d, $\|(e^{ith} - 1)z_{(x,d)}\| = \|(e^{ith} - 1)z_{(0,d)}\|$, and this number is positive for every $t \neq 0$.*

Proof. By (7.1), $z_{(x,d)}(k) = e^{ix\cdot k} z_{(0,d)}(k)$, and multiplication by $e^{ix\cdot k}$ is unitary and commutes with e^{ith}. Positivity is Lemma 7.24, since $z_{(0,d)} \neq 0$. □

Theorem 7.56 (The evolved state leaves the reference sector)**.** *For the unit-edge family with $\epsilon = 0$ and every $t \neq 0$, put*

$$m_t = \min_{d\in\{0,1,2\}} \|(e^{ith} - 1)z_{(0,d)}\|, \qquad \Delta_t = \tfrac{1}{4}\big(1 - e^{-m_t^2/2}\big) > 0,$$

and $F = Q\chi\chi^*(I - Q)$. For every local excitation ϕ of ω_{ref} there is a cell r such that every register $p = (r, k)$ of that cell has $\|O_p(F)\| \leq \frac{1}{2}$ and

$$\big|\omega_{\mathrm{ref}}\big(\tau_t(O_p(F))\big) - \phi\big(O_p(F)\big)\big| \geq \Delta_t.$$

Proof. Take S from Corollary 7.54 and a cell $r \notin S$. Then $\phi(O_p(F)) = \langle \chi, F\chi\rangle = \frac{1}{4}$. The block F changes the occupation, so Corollary 7.23(iii) gives $|\omega_{\mathrm{ref}}(\tau_t(O_p(F)))| \leq \frac{1}{4}e^{-\|g_t\|^2/2}$, and $\|g_t\| \geq m_t$ by Lemma 7.55. So the difference is at least Δ_t, and $m_t > 0$. For the norm, $F^*F = \frac{1}{2}C$ with $C = (I - Q)\chi\chi^*(I - Q)$ and $C^2 = \frac{1}{2}C$, so C is half an orthogonal projection and $\|F\|^2 = \|F^*F\| = \frac{1}{4}$. □

The bound Δ_t does not depend on ϕ. A local excitation can imitate the evolved state on finitely many cells but not on all of them, because every register's coherence has been cut by its own photon cloud, by the same amount at every distance.

Remark 7.57 (The reference representation). Let $(\mathscr{H}_{\mathrm{ref}}, \Gamma_{\mathrm{ref}}, \Omega)$ be the representation of $\mathfrak{A}_\pi$ built from ω_{ref} [8]. The vectors $\Gamma_{\mathrm{ref}}(A)\Omega$, $A \in \mathfrak{A}_\pi^{\mathrm{loc}}$, are dense, so truncating the spectral sum of a density matrix Λ on $\mathscr{H}_{\mathrm{ref}}$ and approximating its eigenvectors gives, for every $\eta > 0$, a local excitation within $\eta\|X\|$ of $X \mapsto \operatorname{Tr}\Lambda\Gamma_{\mathrm{ref}}(X)$ for all X. The observable of Theorem 7.56 has norm at most $\frac{1}{2}$, so at every $t \neq 0$ the evolved state $X \mapsto \omega_{\mathrm{ref}}(\tau_t(X))$ lies at norm distance at least $2\Delta_t$ from every normal state of the reference representation. In particular it is given by no vector of $\mathscr{H}_{\mathrm{ref}}$. With the clock off, where τ_t maps $\mathfrak{A}_\pi$ onto itself (Theorem 7.49), no unitary U_t of $\mathscr{H}_{\mathrm{ref}}$ implements it: $\Gamma_{\mathrm{ref}}(\tau_t(X)) = U_t\Gamma_{\mathrm{ref}}(X)U_t^*$ for all X would make the evolved state the state of the vector $U_t^*\Omega$.

Theorem 7.56 is the obstruction of Chapter 6 for states: the reference representation does not carry the dynamics, and a representation in which unitaries carry it must start from another state, such as the long-time limit points of Theorem 7.49. The theorem is closest to two known phenomena: in van Hove's static-source model the dressed ground state belongs to another representation when the source's displacement is not square integrable [121, 30], and two product states that differ by a fixed amount at every site generate inequivalent representations [65, 91]. Here the evolved state leaves the folium at every nonzero time, not only in the infinite-time limit as in Hepp's models of measurement [59].

The open problems of this chapter are Problems 27 to 33 of Chapter 11. The lattice has no rule that selects its couplings, its placement or its first state; its registers interact only through a supplied field and a supplied assignment. Chapter 8 asks what the law of one recognizer fixes for many coupled ones.

Chapter 8

Many recognizers

Chapter 4 fixed how the branches of one recognizer weigh. Under the adopted conditions (P1)–(P4) and the three readings (the state is the imbalance $s \in \mathbb{Z}^3$, a tick may hold, the weight of s is $w(s) = \varphi^{-|s|_1}$), there is exactly one transition law, the native law K (Theorem 4.17). A universe holds many recognizers, and a law for all of them has to say how they move together: whether their posts are independent, whether one recognizer's post is matched by another's, and how far a post's influence reaches. This chapter asks what K fixes for many recognizers and what it leaves open.

The results live on two of the book's carriers (Appendix A): one recognizer's imbalance carrier $\mathbb{Z}^3$ under K, carrier (c), and finite sets of recognizers with joint laws on their joint moves, carrier (d). Taking K as the law of a recognizer rests on the adopted premises of Chapter 4; a statement that does so says *under K*. The full statements and proofs are in the companion paper [62] and the stage papers [110, 112, 113]. We write the local moves as $\mathcal{O} = \{0, \pm e_1, \pm e_2, \pm e_3\}$, with 0 the hold and $\pm e_k$ a debit or a credit on account k, and $K(s, o) = K(s, s + o)$. Besides the root equation of K, we use its sign symmetry $K(-s, -t) = K(s, t)$ and the positivity of every hold and post (Theorem 4.17), the bound that K posts with chance above 0.8160 at every imbalance ($h_0 < 0.1840$ and $h(s) < 1/6$ for $s \neq 0$, Theorem 4.19), the inequality $K(e_1, 0) \neq K(e_1, 2e_1)$, and the certified hold at balance of Table 4.3, with the six posts from balance equally likely:

$$h_0 = K(0, 0) \in (0.1839,\ 0.1840), \qquad k_0 = K(0, \pm e_j) = \tfrac{1-h_0}{6} \in (0.13600,\ 0.13602). \tag{8.1}$$

8.1 Two recognizers at their first tick

Take two recognizers, both at the empty record. Here are two ways for them to move. In the *product law* each draws its move from $K(0, \cdot)$ independently. In the *mirror law* the first draws its move o from $K(0, \cdot)$ and the second makes the reverse move $-o$, so a debit at one is matched by a credit on the same account at the other. Seen alone, each recognizer moves by K under both laws: under the mirror law the second recognizer makes $-o$ with chance $K(0, o) = K(0, -o)$. The laws differ only in what the two do together (Table 8.1).

Three things are already visible. The one-recognizer law does not decide the joint event: both laws give every single-recognizer observable the same value. The mirror law conserves the total imbalance, and the product law does not. And nothing in either law refers to where the recognizers are.

Table 8.1. The first tick of two recognizers from the empty record under the product and the mirror law. The intervals follow from (8.1) by exact arithmetic; $p = 2k_0$. Each recognizer's own chances are the same under both laws.

joint event at the first tick	product law	mirror law
both hold	$h_0^2 \in (0.03381, 0.03386)$	h_0
first posts $+e_1$, second posts $-e_1$	$k_0^2 \in (0.018496, 0.018501)$	k_0
both post $+e_1$	$k_0^2 \in (0.018496, 0.018501)$	0
exactly one of them posts on account 1	$2p(1-p) \in (0.39603, 0.39607)$	0
total imbalance changes	$1 - h_0^2 - 6k_0^2 > 0.85$	0

8.2 Laws of many recognizers

Definition 8.1 (law of recognizers, keeping the books, native target)**.** Let V be a finite set of recognizers. A *joint move* $\mu \in \mathcal{O}^V$ gives each recognizer a local move; there are $7^{|V|}$ of them, η is the move in which every recognizer holds, and the reverse move $\bar{\mu}$ reverses every post and keeps every hold. A *law of the recognizers* has a set of states x, which may carry anything (the order of past posts, fields, phases), a start, a successor $x\mu$ for each joint move and a weight $P(x, \mu) \in \mathbb{R}$. Every state shows each recognizer's account ledger, with imbalance $s_v(x)$, a register of eight complex slots, and one event journal (Chapter 3). The law *keeps the books* when it starts at the empty record (every ledger zero, the journal empty), each ledger changes by its recognizer's local move, $s_v(x\mu) = s_v(x) + \mu_v$, each post appends one reciprocal pair of events to the journal, neither at ratio 1, and every register keeps zero sum. The *reached states* are the start and every $x\mu$ with x reached and $P(x, \mu) \neq 0$. The law *meets the native target* when it keeps the books and, at every reached state, the weights are nonnegative, add to one, and give every recognizer its native law:

$$\sum_{\mu\colon \mu_v = o} P(x, \mu) = K\big(s_v(x), o\big) \qquad \text{for every } v \in V \text{ and } o \in \mathcal{O}. \tag{8.2}$$

The journal clause, a post booked as one reciprocal pair of events of ratio other than 1, is adopted: it ties the imbalance carrier to the event ledger of Chapter 3, and nothing here derives it. The target is required at reached states. Requiring it at every state, reached or not, is a different specification, and the next proposition shows that the choice matters.

Proposition 8.2 (the target at every state excludes all pairing)**.** *Under K, let a law keep the books and meet* (8.2) *with nonnegative weights adding to one at every state, reached or not, on a nonempty set V. Then some state carries a move of nonzero weight in which the posting recognizers cannot be split into pairs making opposite posts on one account.*

Proof. Suppose every move of nonzero weight at every state splits into such pairs. Let x be the state after the move in which every recognizer posts $+e_1$; every $s_v(x) = e_1$, whatever weight that move has. In a split move as many recognizers make $+e_1$ as make $-e_1$, so averaging over the moves at x and using (8.2) gives $|V|\,K(e_1, e_1) = |V|\,K(e_1, -e_1)$, that is $K(e_1, 2e_1) = K(e_1, 0)$, which is false. □

So the every-state form already rules out the conserving laws of Section 8.4 on every number

of recognizers, through a state no lawful history visits. We use the reached-state form; it is the weaker requirement, and every law below that meets it at every state says so.

Proposition 8.3 (the target is necessary, not sufficient)**.** *Under K, the product law $P_\times(x,\mu) = \prod_v K(s_v(x),\mu_v)$, with states the ledgers, the journal and zero registers and each post booked as one reciprocal pair of a fixed event of ratio other than* 1*, meets the target at every state. If $|V| \geq 2$ and a law meets the target at every state, then at every state some move with two or more posts has positive weight: no law meeting the target posts one recognizer at a time.*

Proof. Summing $\prod_u K(s_u,\mu_u)$ over the moves of the recognizers other than v leaves $K(s_v,\mu_v)$; the bookkeeping is that of the state described. For the second claim, if every move of positive weight at x had at most one post, the chances of posting would add to at most one, $\sum_v(1 - K(s_v,0)) \leq 1$; but every recognizer posts with chance above $1/2$ at every imbalance under K, so the sum exceeds 1 once $|V| \geq 2$. □

So the target does not by itself say how recognizers interact. The first consequence holds for every coupling.

Theorem 8.4 (each recognizer keeps its own law)**.** *Under K, let a law meet the native target. For every reached state x, every recognizer v, every $n \geq 0$ and all local moves $o_1,\dots,o_n$, the chance that v makes the moves $o_1,\dots,o_n$ in the next n ticks is*

$$\prod_{i=1}^{n} K(t_{i-1},o_i), \qquad t_0 = s_v(x), \quad t_i = t_{i-1} + o_i.$$

Proof. Induct on n; at $n = 0$ both sides are 1. The chance of joint moves $\mu^1,\dots,\mu^n$ from x is the product of the weights along $x, x\mu^1, x\mu^1\mu^2, \dots$. Sorting by the first move, the chance in question is $\sum_{\mu:\ \mu_v = o_1} P(x,\mu)\, c(x\mu)$, where $c(y)$ is the chance that v makes $o_2,\dots,o_n$ in the $n-1$ ticks after y. If $P(x,\mu) \neq 0$, then $x\mu$ is reached and $s_v(x\mu) = t_1$, so by induction $c(x\mu) = \prod_{i\geq 2} K(t_{i-1},o_i)$, the same for every μ in the sum; the remaining sum is $K(t_0,o_1)$ by (8.2). □

Every statement of Chapter 4 about one recognizer's history therefore holds for each recognizer of every law that meets the target, among them its holds (Theorem 4.19), its relaxation to π (Theorem 4.22) and its mean return time φ^9 (Corollary 4.23). A coupling can show only in joint observables, records read across recognizers. The next corollary says which joint readings the target still fixes.

Corollary 8.5 (single-recognizer readings fix means, not distributions of sums)**.** *Under K, let a law meet the native target on a nonempty finite set V, and read every recognizer's moves over the first n ticks from the empty record.*

(i) Every function F of one recognizer's n moves has the mean it has under K alone.

(ii) For functions F_v, one per recognizer, the sum $\sum_v F_v$ has mean $\sum_v \mathbb{E}_K F_v$. Its distribution is not fixed by the target (Proposition 8.14).

(iii) Give every recognizer v a nonempty set B_v of its accounts, and say that v's own window on B_v has passed by tick n when at least eight of v's first n moves are posts on accounts of B_v. The chance that every recognizer's own window has passed tends to 1 *as $n \to \infty$, under every such law.*

Proof. (i) is Theorem 8.4 summed against F; (ii) follows by linearity. For (iii), the chance that all windows have passed is at least $1 - \sum_v (1 - q_v(n))$, where $q_v(n)$ is the chance for v alone; by (i) $q_v(n)$ is its value under K alone, which tends to 1 because each post of K has chance bounded below uniformly in the imbalance, so eight posts on B_v occur by tick n with chance tending to one. □

These are readings of records. They fix how often each recognizer completes its own window; they select neither an account labelling nor a coupling.

8.3 The joint law is not fixed, and the balances force independence

The target fixes the marginals of the joint law and leaves the rest open. One way to see the freedom is to add to the product law a correction whose sums over each recognizer's moves vanish.

Theorem 8.6 (the joint law is not unique)**.** *Under K, let $|V| \geq 2$, and let τ be 1 on the hold, -1 on the post $+e_1$ and 0 on the other five posts. With $m(x) = \min_\mu P_\times(x, \mu) > 0$, the correlated law*

$$P_c(x, \mu) = P_\times(x, \mu) + m(x) \prod_{v \in V} \tau(\mu_v),$$

with the bookkeeping of the product law, meets the native target at every state, and at every state the chance that all recognizers hold is larger by $m(x)$ than under $P_\times$.

Proof. τ sums to zero over $\mathcal{O}$, so summing $\prod_v \tau(\mu_v)$ over the moves with $\mu_v = o$ gives $\tau(o) \prod_{u \neq v} \sum_{o'} \tau(o') = 0$ because some $u \neq v$ exists; hence P_c has the marginals of $P_\times$ and its weights add to one. Since $|\tau| \leq 1$ and $P_\times \geq m$, $P_c \geq 0$. At η the product of the τ is 1. □

Theorem 8.7 (joint balance forces the product)**.** *Let a law keep the books and have nonnegative weights adding to one at every state. Put $W(x) = \prod_v w(s_v(x))$. Suppose that at every state x and for every joint move μ*

$$W(x)P(x, \mu) = W(x\mu)P(x\mu, \bar{\mu}), \qquad P(x, \eta) > 0, \qquad P(x, \eta)P(x\mu, \eta) = P(x, \mu)P(x\mu, \bar{\mu}), \tag{8.3}$$

joint detailed balance and joint holding-and-return balance. Then $P = P_\times$, the product of the rows of K, at every state, whatever else the states carry. No such law keeps the total imbalance $\sum_v s_v$ fixed.

Proof. Write $h(x) = P(x, \eta)$. Since $W(x\eta) = W(x)$, detailed balance at η gives $h(x\eta) = h(x)$. For any μ, detailed balance gives $P(x\mu, \bar{\mu}) = W(x)P(x, \mu)/W(x\mu)$, and holding-and-return balance then gives $W(x)P(x, \mu)^2 = W(x\mu)h(x)h(x\mu)$. With $G = (Wh)^{1/2} > 0$,

$$P(x, \mu) = \frac{G(x)G(x\mu)}{W(x)}, \qquad G(x) \sum_\mu G(x\mu) = W(x). \tag{8.4}$$

The product law has this form with $G_*(x) = \prod_v g(s_v(x))$, where $g = w^{1/2}u$, with u the hold amplitude of Theorem 4.17; by (4.7), g is the positive solution of $g(s)\big(g(s) + \sum_{t \sim s} g(t)\big) = w(s)$,

and $K(s,t) = g(s)g(t)/w(s)$. It remains to show $G = G_*$, by the uniqueness argument for g run on joint moves. The hold term of (8.4) is $G(x)^2$, so $G \le W^{1/2}$; a joint move changes each $w(s_v)$ by a factor 1 or $\varphi^{\pm 1}$, so $W(x\mu) \le \varphi^{|V|} W(x)$, and with $7^{|V|}$ joint moves $G \ge W^{1/2}/c$ for $c = 7^{|V|}\varphi^{|V|/2}$. The same bounds hold for G_*. So $r = G/G_*$ lies in $[1/c, c]$, $r(x\eta) = r(x)$, both laws hold with chance at least $p_0 = c^{-2}$, and dividing (8.4) by $W(x)$ gives $r(x)\sum_\mu P_\times(x,\mu)\, r(x\mu) = 1$. Let $B \ge 1$ be the supremum of r and $1/r$. At a state with $r(x) \ge 1$, bound $r(x\eta) = r(x) \ge 1$ in the hold term and $r(x\mu) \ge 1/B$ in the others: with $p = P_\times(x,\eta) \ge p_0$,

$$1 \;\ge\; r(x)\Big(p + \frac{1-p}{B}\Big) \;\ge\; r(x)\,\frac{p_0 B + 1 - p_0}{B}.$$

Exchanging G and G_* bounds $1/r$ the same way, so $B \le B/(p_0 B + 1 - p_0)$, which forces $B = 1$; hence $P = P_\times$. Finally, every value of K on a hold or a post is positive, so under $P_\times$ the move in which one recognizer posts and the others hold has positive chance and changes the total. □

In probabilistic terms a law meeting the target is a Markovian coupling of the one-recognizer chains [78, 73], and Theorem 8.4 is the statement that such a coupling, with arbitrary extra state, has the native chains as its coordinates. Detailed balance against a product weight does not force independence in general: the symmetric exclusion process is reversible with respect to Bernoulli product measures and interacting [76]. Holding-and-return balance on joint moves is what removes that freedom here. Interacting recognizers, and every conserving law, therefore break at least one of the two joint balances, and each balance alone leaves a continuum.

Proposition 8.8 (each balance alone admits coupling)**.** *Under K:*

(i) Detailed balance alone. *Let $Q = \{0, e_1, e_2, e_1 + e_2\}$, the unit square on accounts 1 and 2, and let $\sigma(s,o)$ be $+1$ if $s, s+o \in Q$ and $o = \pm e_1$, -1 if $s, s+o \in Q$ and $o = \pm e_2$, and 0 otherwise. On $|V| \ge 2$ recognizers there is $a_* > 0$ such that for every $0 < a \le a_*$ the law*

$$P_a(x,\mu) = P_\times(x,\mu) + \frac{a}{W(x)} \prod_v \sigma\big(s_v(x), \mu_v\big)$$

reads only the imbalances, meets the target at every state, has joint detailed balance against W at every state and breaks joint holding-and-return balance; distinct a give distinct laws.

(ii) Holding-and-return balance alone. *Let each of two recognizers carry its reduced post list y: its posts, newest first, with every adjacent pair of a post and its reverse erased; $y \cdot o$ is the list after the move o. There is a function f on lists with $f = 1$ at the empty list, $|f| \le 1$ and $\sum_o K(s,o) f(y \cdot o) = 0$ at every list y of imbalance s. For $0 < \varepsilon \le \frac{1}{2}$ the rows*

$$R_\varepsilon(z,\mu) = P_\times(z,\mu)\big(1 + \varepsilon f(y_1 \cdot \mu_1) f(y_2 \cdot \mu_2)\big)$$

have K's marginals, joint holding-and-return balance and detailed balance against a positive weight on the pair of lists, and at the two empty lists the joint hold has chance $h_0^2(1+\varepsilon)$, so distinct ε give distinct laws. Carried beside the books, $R_{1/2}$ gives a law that meets the target and joint holding-and-return balance at every state, is coupled, and breaks joint detailed balance against W.

Proof of (i). Each point of Q has exactly one neighbour in Q along account 1 and one along account 2, so $\sum_o \sigma(s,o) = 0$ for every s, and $\sigma(s+o,-o) = \sigma(s,o)$. With two or more recognizers

the correction therefore has zero sum over the moves with $\mu_v = o$, so the marginals are K's. The weighted values $W(x)P_a(x,\mu) = W(x)P_\times(x,\mu) + a\prod_v \sigma$ are unchanged by $(x,\mu) \mapsto (x\mu, \bar{\mu})$, which is joint detailed balance; nonnegativity holds for $a \le a_*$, half the $|V|$-th power of the least value of $w(s)K(s,o)$ over the points s of Q and the moves o. At the empty record and the move in which every recognizer posts $+e_1$, P_a exceeds $P_\times$ by a, so distinct a give distinct laws; by Theorem 8.7 none of them has joint holding-and-return balance. □

The function f of (ii) is built along the lists: $-h_0/(1-h_0)$ at each one-post list, and at the five lists extending a nonempty list $x::L$ of imbalance s other than by $-x$ the common value $-\big(K(s,0)f(x::L) + K(s,-x)f(L)\big)/\big(1 - K(s,0) - K(s,-x)\big)$; a hold and one post of K have chance at most $\frac{1}{2}$ together, which gives $|f| \le 1$ by induction on the length [112]. The coupling of (ii) is a shared coin: for $0 \le \varepsilon \le \frac{1}{2}$ its row is the average over a fair coin $c = \pm 1$ of $\prod_v K(s_v,\mu_v)\big(1 + c\sqrt{\varepsilon} f(y_v \cdot \mu_v)\big)$, each factor reading only one recognizer's own list, and its effect on the imbalances fades while a correlation of the lists persists [112]. The record of how each recognizer reached its imbalance thus carries a continuum of interactions with native marginals.

The coupling lives entirely in correlations, and it always costs entropy.

Proposition 8.9 (entropy is maximal at independence)**.** *Under K, let a law meet the target and let x be reached. The Shannon entropy of the joint move law at x is at most $\sum_v S_K(s_v(x))$, the sum of the entropies of the recognizers' rows of K, with equality exactly when $P(x,\cdot) = P_\times(x,\cdot)$. The deficit is the relative entropy of $P(x,\cdot)$ from $P_\times(x,\cdot)$, the total correlation of the recognizers' moves.*

Proof. Since the marginals are K's rows, $-\sum_\mu P(x,\mu)\log P_\times(x,\mu) = \sum_v S_K(s_v(x))$, so the deficit is $\sum_\mu P\log(P/P_\times) \ge 0$ by Gibbs' inequality, with equality exactly when $P = P_\times$ ($P_\times > 0$). □

Among voxel laws (Chapter 4), condition (P4) is equivalent to maximizing the entropy rate [110, Theorem 9]. Read the same way on joint moves, as the requirement that the joint row have the largest entropy among rows with K's marginals, the premise selects independence at every reached state; so does the joint form of (P3) with (P4) (Theorem 8.7). An interaction needs a requirement that neither premise supplies.

8.4 Conservation is pairing

Double entry holds inside each recognizer's journal. Can it hold for the universe as a whole, with the total imbalance $\sum_v s_v$ fixed at its starting value 0? Say that a joint move *splits into pairs* when its posting recognizers can be grouped in pairs making opposite posts, $+e_k$ and $-e_k$, while every other recognizer holds.

Proposition 8.10 (conservation is pairing)**.** *Let the laws below keep the books.*

(i) A joint move splits into pairs exactly when $\sum_v \mu_v = 0$. Hence every move of nonzero weight from a reached state splits into pairs exactly when the total imbalance is 0 at every reached state.

(ii) Under K, group the recognizers in pairs and let the pairs move independently, each by the mirror law of Section 8.1. This law meets the target, and every move of nonzero weight splits into its pairs. So every even number of recognizers carries a conserving law with native marginals.

(iii) On two recognizers every law whose moves of nonzero weight from reached states split into pairs keeps $s_2 = -s_1$ at every reached state, and under K the mirror law is the only such law meeting the target.

Proof. (i) Opposite posts add to zero; conversely, if the posts add to zero, each account carries as many $+e_k$ as $-e_k$, and matching them one to one splits the posters into pairs. A move of nonzero weight from a reached state leads to a reached state, and the total starts at 0. (ii) Each pair starts at zero and every move adds o to its first member and $-o$ to its second, so the second member's imbalance is $-s$ when the first's is s; it makes $-o$ with chance $K(s,o) = K(-s,-o)$, its own native law. The pairs are independent, so summing over the others leaves these marginals. (iii) By (i), $s_1 + s_2 = 0$ on reached states; a split move on two recognizers is $(o,-o)$, so its weight is the first recognizer's marginal $K(s_1,o)$. □

Odd numbers meet parity: a conserving tick posts an even number of times, while each balanced recognizer posts with chance about 0.816.

Proposition 8.11 (no conserving law on one, three or five recognizers)**.** *Under K, on one, three or five recognizers no law meets the target with every move of nonzero weight from a reached state split into pairs. If such a law exists on n recognizers, one exists on $n + 2$.*

Proof. On an odd number n of recognizers a split move has at most $n - 1$ posts. At the empty record (8.2) makes the expected number of posting recognizers $n(1 - h_0)$, so $n(1 - h_0) \le n - 1$, that is $h_0 \ge 1/n \ge 1/5$ for $n \le 5$, against $h_0 < 0.1840$. For the last claim, run the law on n recognizers beside an independent mirror pair. □

On seven recognizers the count at the start no longer decides, since $7(1 - h_0) \in (5.712, 5.7127)$ is below 6. Pairing imposes a balance at every reached state that does decide.

Lemma 8.12 (pairing balances the native posts)**.** *Under K, let a law meet the target with every move of nonzero weight from a reached state split into pairs. At every reached state x and for every post o,*

$$\sum_{v \in V} K\big(s_v(x), o\big) = \sum_{v \in V} K\big(s_v(x), -o\big).$$

For every recognizer v and post o some move μ with $\mu_v = o$ has nonzero weight, and $x\mu$ is reached.

Proof. In a split move the recognizers making o are matched one to one with those making $-o$, so their numbers $N_o(\mu)$ and $N_{-o}(\mu)$ agree. Averaging with the weights $P(x,\mu)$ and using (8.2), $\sum_\mu P(x,\mu) N_o(\mu) = \sum_v K(s_v, o)$, and the same with $-o$. The second claim holds because $\sum_{\mu\colon \mu_v = o} P(x,\mu) = K(s_v, o) > 0$. □

Theorem 8.13 (no conserving law on seven recognizers)**.** *Under K, on seven recognizers no law meets the target with every move of nonzero weight from a reached state split into pairs. Consequently, either no odd number of recognizers carries such a law, or there is an odd $N \ge 9$ such that an odd number n carries one exactly when $n \ge N$.*

Proof outline. A certified finite search over split moves, using rational enclosures of K in the balances of Lemma 8.12, shows that some recognizer is balanced at every state reached after three ticks; by Theorem 8.4 the expected number of balanced recognizers is then $7\,P_K(s_3 = 0 \mid s_0 = 0) < 1$ (the return chance is about 0.1015), a contradiction. Appendix B.6 gives the proof in full; the consequence follows from Proposition 8.11. □

The mirror law fits Theorem 8.7: it breaks joint detailed balance against W and, over all joint moves, joint holding-and-return balance, while it keeps holding-and-return balance on its own mirrored moves. Whether nine or more odd recognizers carry a conserving law with native marginals is open.

8.5 What the target leaves to the joint reading

Corollary 8.5 fixed every reading of one recognizer and the mean of every additive reading across recognizers. A reading that needs two recognizers at once is not fixed.

Proposition 8.14 (the cross reading is not fixed)**.** *Under K, the product law and the mirror law on two recognizers both meet the target and give every reading of one recognizer's moves the same mean at every tick. The chance that at the first tick exactly one of the two recognizers posts on its account* 1 *is* 0 *under the mirror law and* $2p(1-p) \in (0.39603, 0.39607)$*,* $p = 2k_0$*, under the product law.*

Proof. Both laws meet the target (Propositions 8.3 and 8.10), so Corollary 8.5 gives the equal means. A mirrored move posts on account 1 at both recognizers or at neither. Under the product law each recognizer posts on account 1 with chance $p = 2k_0$, independently; the interval follows from (8.1). □

How far can a correlation with native marginals go? Further than quantum mechanics allows.

Proposition 8.15 (correlations beyond the quantum bound)**.** *Under K, there is a law on two recognizers whose states are histories and whose weights read only each recognizer's own list of posts, that meets the target and joint holding-and-return balance at every history, and whose correlations exceed Tsirelson's bound. Give each recognizer a setting before a tick: setting* 0 *if its own list is a debit on account* 1 *followed by its reverse, setting* 1 *if its own list is empty; and read its next move as* $+1$ *if it posts on account* 3 *and* -1 *otherwise. Then the CHSH combination* $S = E(0,0) + E(0,1) + E(1,0) - E(1,1)$ *equals*

$$S = \frac{233}{125} + \frac{8(1-h_0)}{6} \in (2.9520,\ 2.9522) \ >\ 2\sqrt{2} \approx 2.8284.$$

At all four settings both recognizers stand at imbalance 0, so each one's outcome has the native chances and no signal passes. A shared random resource gives $|S| \le 2$ [19], and correlations of a shared quantum state give $|S| \le 2\sqrt{2}$ [20]; the witness is neither, as a Popescu-Rohrlich box is neither [90]. The law is one explicit construction among many: its rows at the test histories are chosen, and at every other history they are filled so that the marginals and the holding-and-return relations hold [112]. With Proposition 8.8, interactions compatible with every marginal range from a continuum of shared-coin couplings to correlations no quantum system produces, and the native law selects none of them.

8.6 Records, reversible pictures and cost select no coupling

Could a further structure already in hand select the coupling? We test three: reversibility, records, and least cost.

Proposition 8.16 (no reversible evolution of one recognizer on the imbalance alone)**.** *Let P be a transition law on $\mathbb{Z}^3$ meeting (P1) and (P3), with $P(0,0) < 1$. No orthonormal family $(a_s)_{s\in\mathbb{Z}^3}$ in $\ell^2(\mathbb{Z}^3)$ has $|\langle e_t, a_s\rangle|^2 = P(s,t)$ for all s,t. In particular no unitary on $\ell^2(\mathbb{Z}^3)$ has the Born weights of K.*

Proof. By Bessel's inequality $\sum_s |\langle e_0, a_s\rangle|^2 \le 1$. By detailed balance $P(n,0) = \varphi P(0,n)$ for the six neighbours n of 0, so by (P1) $\sum_s P(s,0) \ge P(0,0) + \varphi\big(1 - P(0,0)\big) > 1$. For K this column sum is $h_0 + \varphi(1-h_0) \in (1.504, 1.505)$. □

The hypothesis $P(0,0) < 1$ is needed: the law that always holds has the identity. Records remove the obstruction by enlarging the carrier.

Proposition 8.17 (a record dilation realizes every joint law and selects none)**.** *Let a record cell hold one local move, or one joint move, and write $|t, o\rangle$ for the state at imbalance t with record o.*

(i) For every transition law P on $\mathbb{Z}^3$ that holds or makes one post, the vectors $A_s = \sum_{o\in\mathcal{O}} \sqrt{P(s, s+o)}\,|s+o, o\rangle$ in $\ell^2(\mathbb{Z}^3 \times \mathcal{O})$ are orthonormal, and the Born weight of landing at t, summed over the record cell, is $P(s,t)$.

(ii) Let q be any law on the joint moves of a finite set V. There is one bounded self-adjoint operator H on $\ell^2(\mathbb{Z}^{3V} \times \mathcal{O}^V)$, of norm at most $\pi/2$, coupling each state $|z, \eta\rangle$ with a blank record only to states $|z+\mu, \mu\rangle$, such that from every blank state e^{-iH} lands at $|z+\mu, \mu\rangle$ with Born weight $q(\mu)$. Such an H exists exactly when q is a probability law.

(iii) Hence the product law and the correlated law of Theorem 8.6 both have such a record picture, with different joint laws: the existence of a fresh record or of one energy operator selects no coupling.

Proof. (i) Distinct starts have disjoint supports, since o and $s+o$ determine s, and $\|A_s\|^2 = \sum_o P(s, s+o) = 1$. (ii) Write $e_z = |z, \eta\rangle$ and, when $q(\eta) < 1$, $f_z = \sum_{\mu\neq\eta} \sqrt{q(\mu)/(1-q(\eta))}\,|z+\mu, \mu\rangle$ (a fixed post replaces this when $q(\eta) = 1$). The two-dimensional spaces spanned by e_z, f_z are mutually orthogonal. With $\beta = \arccos\sqrt{q(\eta)} \in [0, \pi/2]$, the operator $H = \beta \sum_z \big(|e_z\rangle\langle f_z| + |f_z\rangle\langle e_z|\big)$ is self-adjoint of norm β and gives $e^{-iH} e_z = \cos\beta\, e_z - i \sin\beta\, f_z$, whose Born weights are $q(\eta)$ at the blank record and $q(\mu)$ at $|z+\mu, \mu\rangle$. Conversely, e^{-iH} is unitary, so its images of the blank states are orthonormal and their Born weights add to one. (iii) Apply (ii) at each state to each law. □

A record route that bears on the coupling must therefore add a physical requirement on H or on the record environment, for example a conserved energy with a thermal record cell (Chapter 9); the existence of a dilation is no constraint. Cost does no better.

Proposition 8.18 (the static cost minimum selects no motion law)**.** *Under K:*

(i) A law that, at some reached state, gives nonzero weight only to the all-hold move, the least-cost successor, does not meet the target, since K posts with chance above 0.8160 *at every imbalance.*

(ii) On a nonempty set, a law each of whose booked events has a ratio r for which the field $(\log r, 0)$ is stationary through a reciprocal second event does not meet the target: Proposition 3.59 forces $r = 1$, and the books require ratios other than 1.

(iii) Let each recognizer's walk be the list of its posts, and price a law by a potential g of the walk at inverse temperature β: at every reached state and every move of nonzero weight in which v posts o, the chance of o times $e^{\beta g(after)}$ equals the chance of the reverse post from the successor times $e^{\beta g(before)}$. For every law meeting the target, g prices it exactly when $\beta \neq 0$ and $g(walk) = g(\varnothing) + (\log \varphi/\beta)\,|net\ imbalance\ of\ the\ walk|_1$. The condition does not mention the law, so it prices every law meeting the target or none.

Proof. (i) The posting chance $1 - K(s, 0)$ is positive at every s, so the target forbids an all-hold row. (ii) Every recognizer posting at the first tick books at least one event. (iii) Under the target the chances are K's, and by detailed balance $K(s, o)/K(s + o, -o) = w(s + o)/w(s) = \varphi^{|s|_1 - |s+o|_1}$. So the pricing condition says $\beta\big(g(\text{after}) - g(\text{before})\big) = \log \varphi\,\big(|s + o|_1 - |s|_1\big)$ at every post the law makes. Under the target every finite walk is the walk of some recognizer at some reached state, because every post has positive chance, so the condition holds along every walk, which gives the formula. □

A potential that a recognizer reads from its own walk is K's own weight up to scale and offset, forgets the path, and cannot prefer a coupled law to the product. A coupling read through a field must come from a carrier the recognizers share.

8.7 Space is not yet derived

A light cone needs a notion of distance between recognizers. Two candidates come with the native law, and neither is a position.

Proposition 8.19 (eight ticks cannot occupy space)**.** *The map $x \mapsto (x \bmod 2, \lfloor x/2 \rfloor)$, componentwise, is a bijection $\mathbb{Z}^3 \to \{0, 1\}^3 \times \mathbb{Z}^3$; so the eight parity patterns of three accounts, the states of the eight-tick pass, name a place only within a block of eight places. No map from $\{0, 1\}^3$ onto $\mathbb{Z}^3$ exists.*

Proof. $(2\alpha_i + b_i) \bmod 2 = b_i$ and $\lfloor (2\alpha_i + b_i)/2 \rfloor = \alpha_i$ for $b_i \in \{0, 1\}$; and a finite set has no surjection onto an infinite one. □

Proposition 8.20 (imbalance is not position)**.** *Let P be a transition law on $\mathbb{Z}^3$ that holds or makes one post, has detailed balance against w, makes the six posts from balance equally likely, and posts from balance with chance $\epsilon_0 = 1 - P(0, 0) > 0$. Then P is not invariant under the translations of $\mathbb{Z}^3$.*

Proof. By isotropy $P(0, -e_1) = \epsilon_0/6$, and by detailed balance $P(e_1, 0) = \varphi P(0, e_1) = \varphi\epsilon_0/6$. Translation by $-e_1$ would make these equal. □

So K, which satisfies the hypotheses, is not a hopping law on a homogeneous space, and the spread of $|s|_1$ under K is not an expansion of space. Distance must come from a placement of the recognizers, and the placement is supplied. Place recognizers at points $p_v \in \mathbb{Z}^3$ and write $[x] = \lfloor x/2 \rfloor$ for the cube of x.

Definition 8.21 (placement, touching, contact graph)**.** A placement of V is a map $p : V \to \mathbb{Z}^3$. Two recognizers $v \neq w$ *touch* when $|[p_v] - [p_w]|_1 \leq 1$: their cubes coincide or share a face. The contact graph has these edges, and the contact distance is the length of a shortest path in it, infinite when there is none. The *touching premise* (L) is the adoption of this graph as the relation along which recognizers act on each other.

Premise (L) is adopted, not derived. The natural attempt to derive it shows why.

Proposition 8.22 (what a moving unit gives)**.** *Call a map $u : \mathbb{N} \to \mathbb{Z}^3$ with $|u(t+1)-u(t)|_1 \leq 1$ a unit path, and say the book of cube α records u at t when $[u(t)] = \alpha$.*

(i) Some unit path is recorded by α at t and by β at $t+1$ exactly when $|\alpha - \beta|_1 \leq 1$. If α records u at t and β at $t+d$, then $|\alpha - \beta|_1 \leq d$; one unit is recorded by both at one tick exactly when $\alpha = \beta$.

(ii) Touching recognizers sit within $|p_v - p_w|_1 \leq 5$.

(iii) Whether a cross posting between two placed recognizers can be carried is not a function of their ledgers: moving one recognizer to a far cube while keeping both ledgers changes it.

(iv) Realized cross postings do not imply that the two recognizers are lattice neighbours: a realized cross posting can join two recognizers at distance zero.

Proof. One lattice edge changes each $\lfloor x_k/2 \rfloor$ by at most one, which gives the bounds; a unit that stays at 2α, or steps by $+e_k$ from $2\alpha + e_k$ or by $-e_k$ from 2α, gives the converse. For (ii), $x - y = 2([x]-[y]) + (b-b')$ with patterns $b, b' \in \{0,1\}^3$. For (iii), place two recognizers in one cube and move one to the cube $(5,5,5)$. For (iv), a unit that stays in one cube realizes a cross posting between two recognizers placed at one point, which are not lattice neighbours. □

A hopping unit yields touching only by assuming the hop bound of one edge per tick and the one-tick delay of a credit after its debit, and by (iii) no ledger statement supplies the placement. So (L) is a premise, and the stronger lattice-neighbour relation is not a consequence of it. With (L) adopted, conservation between touching recognizers becomes a matching problem.

Proposition 8.23 (conservation along contacts is a matching problem)**.** *Under K and (L), say a law that pairs meets the own-count reading when in every move of nonzero weight from a reached state the two members of each pair touch and have made equal numbers of posts before the move. For every placement, some law meeting the target meets the own-count reading exactly when the contact graph has a perfect matching. In particular the number of recognizers is even.*

Proof. If the contact graph has a perfect matching, let each tick draw uniformly a perfect matching of touching recognizers with equal post counts and opposite imbalances and move each pair by the mirror law; every perfect matching qualifies at the empty record and keeps qualifying after a move along it, so this law meets the target. Conversely, at a reached state let Z be the recognizers that have never posted. Each $v \in Z$ posts with positive chance, and partners have equal counts, so the posting members of Z pair among themselves along contacts and Z shrinks; by induction Z, which is V at the start, splits into touching pairs. □

The same law, on the same recognizers, can meet the touching condition under one placement and fail it under another: the placement decides, and no result places the universe's recognizers.

8.8 Light cones

With a contact graph and its distance, say what a finite speed of influence means.

Definition 8.24 (light cone, distant regions factor)**.** Fix a graph on V with distance d. For $A \subseteq V$, the n-tick law of A from a state is the law of A's moves over the next n ticks. A law has a *light cone with offset* r_0 when the n-tick law of every A is the same from any two states that show the same ledgers, journals and registers on every recognizer within distance $r_0 + n$ of A. Its *distant regions factor* with offset r_0 when the n-tick law of $A \cup B$ is the product of those of A and B whenever every recognizer of A is farther than $2n + r_0$ from every recognizer of B.

Proposition 8.25 (locality is a further condition)**.** *Under K, on a placed set of recognizers with its contact graph:*

(i) The product law has a light cone with offset 0*. A law in which, at every state, the recognizers move independently, each with chances that read only the records within a fixed distance of it, and which meets the target at every state, is the product law.*

(ii) If the members of every pair touch, the law of independent mirror pairs has a light cone with offset 0*, and its distant regions factor with offset* 0*.*

(iii) On three recognizers no two of which touch, the switched law $P_\times(x,\mu) + \chi(x)\, m(x)\, \tau(\mu_1)\tau(\mu_2)$*, with* τ *and* m *as in Theorem 8.6 and* $\chi(x) = 1$ *if* $s_3(x) = 0$*,* $\chi(x) = 0$ *otherwise, meets the target at every state and has no light cone with any offset: recognizers* 1 *and* 2 *show the same records at the start and after recognizer* 3*'s first post, yet the chance that both hold next is* $h_0^2 + 7m$ *at the start and* h_0^2 *after the post. On two recognizers that do not touch, the distant regions of the correlated law do not factor with any offset.*

Proof. (i) Under $P_\times$ the n-tick law of A is the product over $v \in A$ of the chances of Theorem 8.4, which read only A's imbalances. If a law is a product of rows $k_{x,v}$ adding to one, the marginal of v is $k_{x,v}$, which (8.2) makes $K(s_v, \cdot)$. (ii) The pairs move independently and each pair's chances read only its first member's imbalance, so the n-tick law of A depends only on the pairs that meet A, within distance $1 \le n$ of A; pairs meeting A and B far apart are distinct. (iii) The correction has zero marginal sums and keeps weights nonnegative, as in Theorem 8.6. Summing over recognizer 3's seven moves gives the stated chances; recognizer 3 is at infinite distance from 1 and 2, whose records its post leaves unchanged. For the correlated law, both hold at the start with chance $h_0^2 + m$, while each holds with chance h_0. □

A finite speed of influence is therefore a condition on the joint law, not a consequence of the target. Under it, independent local updates carry no interaction (by (i)), so an interacting local law must correlate the moves of nearby recognizers within one tick, as mirror pairs do. Mirror pairs are local, conserving and native, and these three properties still leave a choice.

Proposition 8.26 (three pairings)**.** *Under K, let four recognizers touch one another. The three groupings* $\{1,2\},\{3,4\}$*;* $\{1,3\},\{2,4\}$*;* $\{1,4\},\{2,3\}$ *give three laws of independent mirror pairs. Each meets the target, conserves the total imbalance and has a light cone with offset* 0*, and no two are equal: the first tick in which recognizer* 1 *and its partner make* $+e_1$ *and* $-e_1$ *while the other two hold has chance* $k_0 h_0 > 0$ *under that grouping and* 0 *under the other two.*

Proof. Propositions 8.10(ii) and 8.25(ii) apply to each grouping. Under the grouping that pairs 1 with j, the named move is the pair $\{1, j\}$ making $(+e_1, -e_1)$, chance k_0, times the other pair

holding, chance h_0. Under either other grouping recognizer 1 posts while its partner holds, which is not a mirror move. □

8.8.1 Locking contacts and an exact cone

The interacting laws with a cone constructed so far let touching recognizers lock. For one contact with amplitude $b \geq 0$ between recognizers at imbalances s and t: the contact locks with chance $\theta = b/(w(s) + b)$ if $s = t$ and $\theta = 0$ otherwise; locked, both make one move drawn from $K(s, \cdot)$; unlocked, each moves by its own row. The joint law of their moves (o, o') is $\theta[o = o']\,K(s, o) + (1-\theta)K(s, o)K(t, o')$, so each alone moves by K. On a graph with amplitudes $b_e \geq 0$, a set A of contacts whose two ends hold equal imbalances may lock; its clusters move as one native move each, with weight the product over clusters of w at the cluster's common imbalance, times $\prod_{e \in A} b_e$, and a *lock rule* says which sets A are allowed. A *partner rule* lets the tick and the joint imbalance choose the allowed contacts $M_t(\sigma)$; it has *reach d* if recognizers joined by chosen contacts lie within d contacts of each other, and *read radius r* if whether a contact is chosen reads the imbalances only within r contacts of its ends. The *isolated-pair rule* chooses the contacts $\{u, v\}$ with $\sigma_u = \sigma_v$ and $\sigma_w \neq \sigma_u$ for every other w touching u or v; it chooses matchings, with reach 1 and read radius 1.

Theorem 8.27 (bounded-reach locking gives an exact cone)**.** *Under K, let a contact graph carry amplitudes $b_e \geq 0$.*

(i) Every lock rule and every partner rule gives every recognizer K's row and meets the target at every state, reached or not.

(ii) If the partner rule has reach d and read radius r, two starting laws that give the ledgers within $n(d + r + 1)$ contacts of S the same law give S's ledgers the same law after n ticks. If it chooses matchings, nr suffices.

(iii) Under the isolated-pair rule S's ledger law after n ticks reads the start only within n contacts, and regions more than $2n - 1$ contacts apart have independent ledgers after n ticks from every fixed start. On a cycle of $k \geq 3$ recognizers with every amplitude positive, the two-tick law from the empty record differs from the product law.

Proof outline. (i) Each allowed lock set moves each cluster by one native move, and summing a cluster's move over the other members leaves K's row. (ii) A tick of a partner rule is the lock law of the chosen set. If C is the union of the chosen blocks meeting T, every contact with an end in C is chosen by data within $d + r + 1$ of T, so two states agreeing there choose the same contacts in C, and C steps by its own law; induction on n gives the cone. For matchings a chosen path has one contact, so $C = T$. (iii) The isolated-pair rule chooses matchings with read radius 1. On the cycle the first tick is the product (nothing is isolated at the empty record), and after a first tick in which every recognizer but 0 and 1 has posted, the contact $\{0, 1\}$ is isolated and locks with positive chance, which raises the chance that 0 and 1 end equal [113]. □

On the cubic lattice the cone has a definite shape. Place one recognizer in each cube of a box, so that touching recognizers are lattice neighbours of their cube addresses and contact distance is the Manhattan distance of the addresses.

Theorem 8.28 (the cone of the isolated-pair law is octahedral)**.** *Under K and (L), on the box with one recognizer per cube and the isolated-pair rule:*

(i) *For $b_e \geq 0$, two starts whose ledgers agree within Manhattan distance n of a region S give S the same ledger law after n ticks.*
(ii) *For $b_e > 0$ on every contact, a touching pair $\{u, v\}$, every $n \geq 1$ and every recognizer z at Manhattan distance exactly n from the nearer of u, v, there are two starts differing only at z that give the pair different ledger laws after n ticks; the chances that u and v end equal differ by $(-1)^{n-1}c$ with $c > 0$. A recognizer $z \notin \{u, v\}$ can change the pair's law within n ticks exactly when it lies within Manhattan distance n of the pair.*

So the front after n ticks is the octahedron $|x| + |y| + |z| = n$ in cube addresses: influence moves one cube side per tick along an axis, $1/\sqrt{2}$ of a side per tick along a face diagonal, and $1/\sqrt{3}$ along a body diagonal.

Proof outline. (i) is Theorem 8.27(iii) with contact distance equal to Manhattan distance. (ii) is stated in [113, Section 6.3]; we outline its proof. Fix the touching pair $\{u, v\}$ and a recognizer z with $\min(d(u, z), d(v, z)) = n \geq 1$, d the Manhattan distance of cube addresses; on the box the two ends of a contact lie at distances from z that differ by exactly one, so we may take $d(v, z) = n$ and $d(u, z) = n + 1$. A shortest lattice path from v to z, with u in front, is an induced line $p_0 = u, p_1 = v, \dots, p_{n+1} = z$: its recognizers are distinct, consecutive ones touch, and no other two do, by the triangle inequality.

Far recognizers are invisible. Put every recognizer off the line at an imbalance whose first count lies more than $2n$ from every other recognizer's. A tick moves each recognizer by at most one post, so for n ticks no off-line recognizer holds the imbalance of a neighbour; it is never chosen by the isolated-pair rule and never blocks a choice. On the line the chosen contacts are therefore those of the line alone, and the lock law of a chosen matching depends only on the matching, so n ticks on the box from such a start are n ticks of the same law on the line.

The relay. On the line, take two starts that differ only at p_{n+1}. At each stage i only the contact $\{p_i, p_{i+1}\}$ can change its choice, and the gain in the chance that p_0 and p_1 end equal is a sum over lock sets of nonnegative terms, one of them positive when every amplitude is positive. Telescoping the stages from p_{n+1} back to the pair gives two chances that differ by $(-1)^{n-1}c$ with $c > 0$: each stage reverses the sign, as a lock that makes two neighbours equal removes the isolation of the next contact. So the pair's ledger laws after n ticks differ.

Exactness. By part (i), a recognizer farther than n from the pair cannot change the pair's law within n ticks; by the relay, every recognizer at distance $t \leq n$ changes it at tick t. So the set of recognizers that can change the pair's law within n ticks is the Manhattan ball of radius n around the pair. □

The Euclidean front is not a sphere. "One recognizer per tick" is therefore an axis speed, the speed in contact steps; the unit premise (N1) of Chapter 9 must be read that way. The witnesses of (ii) are particular starts with far backgrounds, and the size of c is not estimated. The rule, its amplitudes and the placement are supplied; nothing selects them, and a derived isotropic cone is open. The record cell of Chapter 5 has its own cone, and no theorem connects the two.

Three further facts mark the limits of the construction [113]. With fixed lock blocks, every union of blocks is computed exactly from its own state and two disjoint unions evolve independently at every tick: a fixed-block law is local only as a product of isolated blocks.

When every equal-ended contact may lock, the law on a cycle reads arbitrarily far, because the chance that two neighbours lock depends on the length of the cycle. And with weak locks a light cone holds approximately over every number of ticks: on a forest of contacts in which no recognizer touches more than g others, with amplitudes at most $1/\big(64g^2(\varphi^9+1)\big)$, replacing the forest farther than r contacts from S by another forest changes the law of S's moves over the first n ticks from the empty record by at most $\varphi^9\,2^n|S|\,2^{-r}$ in ℓ^1. Whether free locking on graphs with loops has a cone is open.

An exact cone makes a region computable from its past alone. Under a schedule of lock sets with rational amplitudes $b_e \geq 0$, compute the region's law along its backward cone, the least chain of sets $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_n = S$ each path-closed for that tick's locks, using rational bounds on w and rational lower bounds on K and rounding weights down. The computed list lies below the true law of S after n ticks and its missing mass is exactly its ℓ^1 error; when the bounds on w and K are tight enough on the imbalances the region can reach, a stated choice of precisions puts the error below any rational $\varepsilon > 0$. If $|S_0| \leq c$, the computation from the empty record makes at most $n\,14^c(n+1)^{3c}$ products of a weight and a row entry, since a recognizer's imbalance after t ticks lies in the ball $|s|_1 \leq t$ of $(2t+1)(2t^2+2t+3)/3 \leq 2(t+1)^3$ points [113]. The cost is polynomial in n when the backward cone stays narrow.

A cone of the same form holds, order by order, on a different carrier. On the spatial assembly of the atomic source (cells of $\mathbb{Z}^3$, each with four eight-state registers and a photon space, coupled by a hopping transport along the six unit shifts), the n-th power of the transport at a cell reads only the cells within n hops. This is finite-order locality; it is not a strict cone for the complete unbounded generator at finite times. Likewise, the finite range of the generator in Proposition 8.17 restricts each recognizer's imbalance step, not the spatial dependence among recognizers. No correspondence theorem joins that carrier to the recognizers of this chapter, so the two cones stand side by side and are not combined.

8.9 What a law selecting the interaction must respect

A physical requirement that selects how recognizers interact must give each recognizer K at every reached state, or name the relaxation it uses, such as a local clock that slows a recognizer's opportunities. If it couples recognizers or conserves the total imbalance, it breaks one of the two joint balances (Theorem 8.7); if it conserves, it pairs and must decide which recognizers pair (Proposition 8.26); and it must supply locality as a further condition, beyond which an isotropic cone needs more than touching (Theorem 8.28).

Since each recognizer's own history has the law K under every coupling that meets the target (Theorem 8.4), while distinct couplings are distinct joint laws (Theorem 8.6), a requirement or an observation that selects the coupling has to read the records of two or more recognizers together; the records of one recognizer cannot select it.

The open problems of this chapter, the premises of K, the requirement that selects a coupling, conserving laws on odd numbers of recognizers, a derived placement with its cone, and free locking on graphs with loops, are stated exactly in Chapter 11 (Problems 13 and 34 to 37); until the first is solved, K and everything here that uses it is conditional. Chapter 9 turns from interactions to heat, under an adopted energy reading of the golden weights.

Chapter 9

Toward the hot universe

The radiation that fills the universe today is a blackbody at $T_0 = 2.72548 \pm 0.00057$ K [45], with departures from a Planck spectrum bounded by $|y| < 1.5 \times 10^{-5}$ and $|\mu| < 9 \times 10^{-5}$ at 95% confidence [44]. In the standard account the early universe was a hot plasma close to thermal equilibrium, and expansion cooled it [68, 132]. There the heat is an initial condition. An account that starts from the empty record has to produce it. This chapter asks how far the proved results get.

We ask whether the native law of one recognizer defines a temperature, where heat can come from, whether the native photon field is radiation, and what turns native numbers into kelvin and seconds; the answers differ in kind, and we say which kind each time. Four adopted premises enter, each named where it enters: the energy reading, the thermal record cell with its per-tick refresh, a supplied Bose occupation, and the native constants (N1)–(N3). Every statement about how a recognizer moves rests on the native law K of Chapter 4 and its premises (P1)–(P4).

9.1 One temperature

Start with one account. Its imbalance $k \in \mathbb{Z}$ carries the golden weight $\rho^{|k|}$, $\rho = \varphi^{-1}$. Suppose these weights are Boltzmann factors $e^{-E(k)/T}$ of some energy. A weight that falls by the same factor ρ for each unit of imbalance needs an energy that rises by the same amount ε for each unit, and then $\varepsilon/T = \log\varphi$. The temperature is $T = \varepsilon/\log\varphi$, and the partition sum is $\sum_k \rho^{|k|} = (1+\rho)/(1-\rho) = \varphi^3$. Only the energy ε of one level is left to choose; its ratio to the temperature is fixed. The same holds on three accounts.

The carrier of this section and the next two is one recognizer's imbalance carrier $\mathbb{Z}^3$ (with a cell attached in Section 9.2). We write $|s|_1 = |s_1|+|s_2|+|s_3|$ for the level of $s \in \mathbb{Z}^3$, $w(s) = \varphi^{-|s|_1} = \rho^{|s|_1}$ for the golden weight and $\pi(s) = \varphi^{-9} w(s)$; recall $\varphi^2 = \varphi + 1$ and $\rho^2 + \rho = 1$. The native law K of Chapter 4 is in detailed balance against w by (P3). We use four of its properties.

(F1) The vacuum hold $h_0 = K(0,0)$ lies in $(0.1839, 0.1840)$, and the six posts out of 0 have the common chance $(1-h_0)/6$ (Theorems 4.17 and 4.19).
(F2) From the empty record, $\sum_s |K^n(0,s) - \pi(s)| \le 9\,(1-0.00836)^n$ for every $n \ge 0$ (Theorem 4.22).
(F3) The mean level under $K^n(0,\cdot)$ tends to 6 (Theorem 4.22).
(F4) Every hold is at least c_0^2, $c_0 = 6188671/16777216$, so $K(s,s) > 0.136067$ for every s; this is proved by exact rational computation from certified lower bounds on the positive root whose squares are the holds [115].

The hold and the six posts of K are positive at every state.

Lemma 9.1 (The golden weights). *(i)* $\sum_{k\in\mathbb{Z}} \rho^{|k|} = \varphi^3$ *and* $\sum_{s\in\mathbb{Z}^3} w(s) = \varphi^9$*, so* π *is a proba-*

bility law on $\mathbb{Z}^3$.

(ii) Level $k \geq 1$ *holds* $4k^2 + 2$ *states and has* π*-mass* $(4k^2 + 2)\varphi^{-(k+9)}$*; level* 0 *has mass* φ^{-9}*. Level* 4 *has larger mass than every other level.*

(iii) $\log \pi(s) = -(|s|_1 + 9) \log \varphi$*; the mean level under* π *is* 6 *and its entropy* $-\sum_s \pi(s) \log \pi(s)$ *is* $15 \log \varphi$.

Proof. (i) $\sum_k \rho^{|k|} = 1 + 2\rho/(1-\rho) = (1+\rho)/(1-\rho)$, and $1 + \rho = \varphi$, $1 - \rho = \rho^2$, so the sum is φ^3; the weight factorizes over the three accounts. (ii) A state of level $k \geq 1$ is a composition of k into three nonnegative parts with a sign on each nonzero part; counting by the number of nonzero parts gives $3 \cdot 2 + 3(k-1) \cdot 4 + \binom{k-1}{2} \cdot 8 = 4k^2 + 2$. The ratio of the masses of levels $k+1$ and $k \geq 1$ is $\rho(4(k+1)^2 + 2)/(4k^2 + 2)$, which decreases in k, exceeds 1 for $k \leq 3$ and is below 1 from $k = 4$ on; level 1 has 6ρ times the mass of level 0. (iii) The first formula is the definition. The mean of $|k|$ under $\rho^{|k|}/\varphi^3$ is $2\rho/((1+\rho)(1-\rho)) = 2$, so the mean level is 6, and the entropy is $\sum_s \pi(s)(|s|_1 + 9) \log \varphi = 15 \log \varphi$. □

Definition 9.2 (The energy reading)**.** The *energy reading* is the adopted premise that the golden weights are Boltzmann factors: there are an energy $E : \mathbb{Z}^3 \to \mathbb{R}$ of the recognizer's imbalance and an inverse temperature $\beta \in \mathbb{R}$ with $e^{-\beta(E(s)-E(0))} = w(s)$ for every s. Boltzmann's constant is set to 1 until Section 9.8.

The reading is not derived. The golden weights could carry another meaning; the results below say what this reading forces.

Theorem 9.3 (The energy is forced)**.** *On* $\mathbb{Z}^3$*, let* $E : \mathbb{Z}^3 \to \mathbb{R}$ *and* $\beta \in \mathbb{R}$ *satisfy* $e^{-\beta(E(s)-E(0))} = w(s)$ *for every* s*. Then* $\beta(E(s)-E(0)) = |s|_1 \log \varphi$ *for every* s*, and if* $\beta \neq 0$*,* $E(s) = E(0) + |s|_1 \log \varphi/\beta$*. Conversely, every* E *of this form with* $\beta \neq 0$ *satisfies the hypothesis.*

Proof. Take logarithms: $-\beta(E(s) - E(0)) = |s|_1 \log \rho = -|s|_1 \log \varphi$. Divide by β when it is not zero. Conversely $e^{-|s|_1 \log \varphi} = \rho^{|s|_1} = w(s)$. □

Corollary 9.4 (One temperature)**.** *Under the energy reading, let one unit of level carry the energy* $\varepsilon > 0$*, so that* $E(s) = E(0) + \varepsilon|s|_1$*. The golden weights are Boltzmann factors at inverse temperature* β *exactly when* $\beta\varepsilon = \log \varphi$*, that is, at*

$$T = \frac{\varepsilon}{\log \varphi}. \tag{9.1}$$

For $\varepsilon = \log \varphi$*,* $e^{-\beta \log \varphi} = \rho$ *holds if and only if* $\beta = 1$.

Proof. Theorem 9.3 at $s = e_1$ gives $\beta\varepsilon = \log \varphi$; conversely $\beta\varepsilon = \log \varphi$ gives $\beta \neq 0$ and the form of the converse. The last claim holds because $\log \varphi > 0$ and the exponential is injective. □

A temperature other than $\varepsilon/\log \varphi$ would need other weights. Measured in units of $\log \varphi$ per level the temperature is exactly 1; measured in the energy of one level it is $1/\log \varphi = 2.0781$. A temperature in kelvin, and any change of it, can therefore come only through ε, the energy of one level in physical units, which nothing in this chapter fixes (Section 9.8).

The recognition cost J prices ratios. It is natural to try it as the energy behind the weights, reading level k as the ratio φ^k. This fails at every temperature.

Proposition 9.5 (J is not the thermal energy)**.** *No $\beta \in \mathbb{R}$ satisfies both $e^{-\beta J(\varphi)} = \rho$ and $e^{-\beta J(\varphi^2)} = \rho^2$.*

Proof. $J(\varphi) = \varphi - \frac{3}{2}$ and $J(\varphi^2) = \frac{1}{2}(\varphi^2 + \rho^2) - 1 = \frac{1}{2}$, since $\varphi^2 + \rho^2 = 3$. The two equations say $\beta J(\varphi) = \log\varphi$ and $\beta J(\varphi^2) = 2\log\varphi$, so $J(\varphi^2) = 2J(\varphi)$; but $J(\varphi^2)/J(\varphi) = 1/(2\varphi - 3) = \sqrt{5} + 2 = \varphi^3 \approx 4.236$. □

A constant factor in front of J is absorbed into β, so no choice of energy unit rescues this reading. The proposition excludes J as the energy whose Boltzmann factors are the golden weights; it says nothing about J as the energy of another observable or carrier.

Proposition 9.6 (One recognizer at temperature one)**.** *Take $E(s) = |s|_1 \log\varphi$ and $T = 1$. The partition function is $Z = \sum_s e^{-E(s)} = \varphi^9$, the Gibbs law e^{-E}/Z is π, and*

$$U = \sum_s \pi(s)E(s) = 6\log\varphi, \quad S = -\sum_s \pi(s)\log\pi(s) = 15\log\varphi, \quad F = -\log Z = -9\log\varphi, \tag{9.2}$$

with $S = U - F$.

Proof. $e^{-E(s)} = w(s)$, so $Z = \varphi^9$ and $e^{-E}/Z = \pi$ by Lemma 9.1(i). U is $\log\varphi$ times the mean level 6, and S is Lemma 9.1(iii). Since $\log\pi(s) = -E(s) - \log Z$, $S = U + \log Z = U - F$. □

quantity	exact value	decimal	source
partition function Z	$\varphi^9 = 38 + 17\sqrt{5}$	76.0132	Lemma 9.1(i)
chance of the empty record $\pi(0)$	φ^{-9}	0.013156	Lemma 9.1(i)
mean level; most probable level	6; 4		Lemma 9.1(ii), (iii)
mean energy U	$6\log\varphi$	2.8873	Proposition 9.6
entropy S	$15\log\varphi$	7.2182	Proposition 9.6
free energy F	$-9\log\varphi$	−4.3309	Proposition 9.6

Table 9.1. One recognizer at temperature one, in units in which one level carries the energy $\log\varphi = 0.48121$. The values are exact properties of the golden weights. That a recognizer's long-run law is this one holds under K, which rests on (P1)–(P4).

Condition (P3) can be written $P(s,t)\,e^{|t|_1\log\varphi} = P(t,s)\,e^{|s|_1\log\varphi}$. So every law meeting (P3), K among them, makes each post and its reverse in the ratio that, under the energy reading, is the Boltzmann ratio of the level at inverse temperature $\log\varphi$ per level. Proposition 8.18(iii) gives the form for many recognizers.

9.2 Heat from a thermal record cell

A reversible evolution of the recognizer alone cannot produce K: no orthonormal family in $\ell^2(\mathbb{Z}^3)$ has the Born weights of a law in detailed balance against w that ever leaves the empty record (Proposition 8.16). The obstruction is concrete. The empty record has the largest weight, so it receives more probability than it sends out: $\sum_s P(s,0) = P(0,0) + \varphi(1 - P(0,0)) > 1$. An environment can absorb the difference. The smallest case shows how.

Example 9.7 (A two-level exchange)**.** Let the recognizer have two states $0, 1$ with energies $0, \log\varphi$, and let a cell have two states c_0, c_1 with energies $0, \log\varphi$. At temperature one the cell's Gibbs weights are $1/(1+\rho) = \rho$ and $\rho/(1+\rho) = \rho^2$. Let H have the real entry θ between $(1, c_0)$ and $(0, c_1)$, the two joint states of total energy $\log\varphi$, and vanish elsewhere. Then e^{-iH} acts on that pair as $\cos\theta$ on the diagonal and $-i\sin\theta$ off it and fixes $(0, c_0)$ and $(1, c_1)$. Start the cell in its Gibbs state, apply e^{-iH}, and ignore the cell's final state. The recognizer moves with chances

$$P(0,1) = \rho^2 \sin^2\theta, \quad P(0,0) = \rho + \rho^2\cos^2\theta, \quad P(1,0) = \rho\sin^2\theta, \quad P(1,1) = \rho^2 + \rho\cos^2\theta.$$

Each row sums to $\rho + \rho^2 = 1$, and $w(0)P(0,1) = \rho^2\sin^2\theta = w(1)P(1,0)$: detailed balance against the golden weight, for every θ. A recognizer at 0 can rise only by taking a unit of energy from the cell, which holds it with the cell's Gibbs weight ρ^2 of c_1; so it holds at 0 with chance at least ρ.

Definition 9.8 (Thermal record cell, exchange and read)**.** A *thermal record cell* is a finite nonempty set C with energies $E_C : C \to \mathbb{R}$ and Gibbs weights $\pi_C(c) = e^{-E_C(c)}/Z_C$ at temperature one, $Z_C = \sum_c e^{-E_C(c)}$. (It is unrelated to the four-dimensional record cell of Chapter 5.) The total energy of a joint state $(s, c) \in \mathbb{Z}^3 \times C$ is $|s|_1 \log\varphi + E_C(c)$. An *exchange* is a bounded self-adjoint operator H on $\ell^2(\mathbb{Z}^3 \times C)$ whose matrix entries are real and vanish between basis states of different total energy. Its *read* is the transition law

$$P(s,t) = \sum_{c,c' \in C} \pi_C(c)\,\left|\langle t, c'|e^{-iH}|s, c\rangle\right|^2, \tag{9.3}$$

the chance of landing at t when the cell starts in its Gibbs state, summed over the cell's final state. A law P on $\mathbb{Z}^3$ has a *thermal record* if it is the read of some cell and some exchange that moves the recognizer by at most one post (entries vanish between (s, c) and (t, c') unless $t = s$ or $t = s \pm e_k$).

The read is the law of one tick that starts with the cell in its Gibbs state. Using it as the law of every tick presumes that the cell is returned to its Gibbs state before each tick. We adopt this *per-tick refresh* together with the cell; neither is derived, and both are premises of every thermal statement below that concerns more than one tick.

Theorem 9.9 (Detailed balance from a thermal record cell)**.** *For every thermal record cell (C, E_C) and every exchange H, the read* (9.3) *has $P \geq 0$, $\sum_t P(s,t) = 1$ for every s, and*

$$w(s)P(s,t) = w(t)P(t,s) \qquad \text{for all } s, t \in \mathbb{Z}^3.$$

If c_0 is a state of least energy of the cell, then $P(0,0) \geq \pi_C(c_0) > 0$.

Proof. For each value e only finitely many s have $|s|_1 \log\varphi \leq e - \min E_C$, so each shell of equal total energy is finite. Energy conservation makes H map the span of each shell into itself; on that finite-dimensional span H is a real symmetric matrix, so e^{-iH} is unitary and symmetric there, and its entries vanish between different shells. If $x = (s, c)$ and $y = (t, c')$ lie on one shell of energy e, then $w(s)\pi_C(c) = e^{-e}/Z_C = w(t)\pi_C(c')$. Hence, term by term,

$$w(s)\pi_C(c)\,|\langle y|e^{-iH}|x\rangle|^2 = w(t)\pi_C(c')\,|\langle x|e^{-iH}|y\rangle|^2,$$

both sides vanishing off a shell. Summing over c, c' gives the balance. Each column of the unitary e^{-iH} has norm one, so $\sum_t P(s,t) = \sum_c \pi_C(c) = 1$. For the hold: every (t, c') with $t \neq 0$ has total energy at least $\log\varphi + E_C(c_0) > E_C(c_0)$, so the shell of $(0, c_0)$ contains only states $(0, c')$, all amplitude from $(0, c_0)$ stays at 0, and $P(0,0) \geq \pi_C(c_0)$. □

This is the standard mechanism by which an environment in a Gibbs state imposes detailed balance on an open system [11], here for a finite cell and a level-counting energy. The native law is such an exchange.

Theorem 9.10 (The native law is a thermal exchange)**.** *Let a law P on $\mathbb{Z}^3$ meet (P1), (P2) and (P3), and let $P(s,s) \geq h$ for every s and some $h > 0$. Then P has a thermal record. By (F4), K meets these conditions with $h = c_0^2$, so K has a thermal record; the cell built in the proof has sixteen layers $i = 0, \dots, 15$ at energies $i \log\varphi$, layer i holding $\lceil 103\,\varphi^i \rceil$ states, from 103 on layer 0 to 140493 on layer 15.*

The proof is in Appendix B.7: the layers grow by a factor close to φ, so every layer carries nearly the same Gibbs mass, and on each edge the exchange rotates pairs of cell states one layer apart whose Gibbs masses stand in the ratio φ, which is detailed balance. For K, $h = c_0^2$ gives $2/h = 14.699$ and $14/h = 102.890$, hence 15 as the top layer and 103 states on layer 0.

Proposition 9.11 (What the exchange requires)**.** *(i) Every law with a thermal record is a probability law in detailed balance against w with $P(0,0) > 0$.*

(ii) Let P meet (P1) and (P2), with $P(s,s) \geq h$ for all s and some $h > 0$. Then P has a thermal record if and only if it meets (P3).

(iii) The law A that makes each of the six posts out of 0 with chance $\frac{1}{6}$, the post from a neighbour of 0 back to 0 with chance $\varphi/6$, and every other post from s to a neighbour t with chance $\frac{1}{9}\min\{1, w(t)/w(s)\}$, holding with the remaining chance, meets (P1)–(P3) and has no thermal record.

(iv) The law that always holds has a thermal record (a one-state cell and $H = 0$). So K is not the only law with a thermal record.

Proof. (i) is Theorem 9.9. (ii) One direction is (i), the other Theorem 9.10. (iii) The posts out of a state other than 0 add to at most $\varphi/6 + 5/9 < 1$, so (P1) and (P2) hold. For (P3), $w(s)\frac{1}{9}\min\{1, w(t)/w(s)\} = \frac{1}{9}\min\{w(s), w(t)\}$ is symmetric, and $w(0)\cdot\frac{1}{6} = w(e_k)\cdot\varphi/6$. But $A(0,0) = 0$, so by (i) A has no thermal record. (iv) With $H = 0$ every row of the read is concentrated on its own state, and this law differs from K, whose posts are positive. □

The balance (P3) that K adopts is therefore exactly what one tick of a reversible, energy-conserving exchange with a finite cell in its Gibbs state at temperature one produces, and K is such an exchange. This gives (P3) a physical model; it does not derive it, because the construction starts from the balance, and it does not single out K, because every balanced law with a hold floor has a thermal record. Nothing identifies the cell with a physical system: the same construction with a smaller floor gives a larger cell.

9.3 From the empty record to the Gibbs law

At the start of every run each recognizer is at the empty record $s = 0$, under the energy reading its state of least energy, since $E(s) - E(0) = \varepsilon|s|_1 \geq 0$. For a law p on $\mathbb{Z}^3$ of finite support write

$D(p) = \sum_s p(s) \log(p(s)/\pi(s))$ and $S(p) = -\sum_s p(s) \log p(s)$. Under K, with $p_n = K^n(0, \cdot)$ and $L_n = \sum_s p_n(s)|s|_1$, the relative entropy $D_n = D(p_n) = (9 + L_n) \log \varphi - S(p_n)$ falls strictly at every tick and tends to 0, so its drops add up to $D_0 = 9 \log \varphi$; the first tick gives up between 1.54679 and 1.54682 nats, 35.7% of the total (Theorem 4.30, Corollary 4.32). Read under the energy reading, this is the approach to equilibrium.

Corollary 9.12 (Every recognizer heats to temperature one)**.** *Under (P1)–(P4), let a law of finitely many recognizers meet the native target of Definition 8.1. For every recognizer v and every n, the chance that $s_v = s$ after n ticks from the empty record is $K^n(0, s)$; hence $\sum_s |K^n(0, s) - \pi(s)| \le 9(1 - 0.00836)^n$, and its relative entropy from π is the D_n above, falling strictly at every tick with total fall $9 \log \varphi$. Under the energy reading with $\varepsilon = \log \varphi$: π is the Gibbs law at temperature one; $D_n = F_n - F$, where $F_n = \sum_s p_n(s) E(s) - S(p_n)$ is the free energy of the state law and $F = -9 \log \varphi$; the recognizer starts at its least energy, its mean energy tends to $6 \log \varphi$, and the free energy it gives up on the way is $9 \log \varphi = \log Z$.*

Proof. The marginal statement is Theorem 8.4 summed over the move sequences ending at s. The rate is (F2), the entropy statements are those of Theorem 4.30, and the mean energy is (F3). The reading follows from $E(s) = |s|_1 \log \varphi$, Proposition 9.6 and the formula for D_n. □

The temperature of a recognizer is set by its own law, whatever couples it to the others: coupling shows only in correlations, and the tick with the most entropy is the tick of independent recognizers (Proposition 8.9). The corollary says nothing about energy passing between recognizers; that depends on the coupling, which is not selected (Section 9.5).

Where the free energy goes. On a thermal record of K the fall of D_n can be followed into the cell. Let (C, E_C, H) be a thermal record of K (one exists by Theorem 9.10). Draw the recognizer from p_n and, independently, the cell from π_C; apply e^{-iH}; read recognizer and cell in the basis. The joint law after the tick is

$$\mu_n(t, c') = \sum_{s,c} p_n(s)\, \pi_C(c)\, \left|\langle t, c' | e^{-iH} | s, c \rangle\right|^2,$$

with cell marginal ν_n. Write $Q_n = \sum_c \nu_n(c) E_C(c) - \sum_c \pi_C(c) E_C(c)$ for the mean energy the cell gains, I_n for the mutual information of recognizer and cell under μ_n, and $D(\nu \| \pi_C)$ for the relative entropy of a law ν on C from π_C.

Theorem 9.13 (The heat of a tick)**.** *Under (P1)–(P4), let (C, E_C, H) be a thermal record of K. For every $n \ge 0$:*

(i) the recognizer marginal of μ_n is p_{n+1}, and $Q_n = (L_n - L_{n+1}) \log \varphi$, the same for every thermal record of K;

(ii) $D_n - D_{n+1} = S(p_{n+1}) - S(p_n) + Q_n$;

(iii) $D_n - D_{n+1} = \big(S(\mu_n) - S(p_n) - S(\pi_C)\big) + I_n + D(\nu_n \| \pi_C)$, each of the three terms nonnegative; hence $S(p_{n+1}) - S(p_n) \ge -Q_n$;

(iv) $Q_0 = -(1 - h_0) \log \varphi$, strictly between $-0.8161 \log \varphi$ and $-0.8160 \log \varphi$, and $S(p_1) = -h_0 \log h_0 - (1 - h_0) \log(1 - h_0) + (1 - h_0) \log 6$.

Under the energy reading Q_n is the heat the cell receives in the tick, at temperature one; (ii) says that the fall of the recognizer's free energy is the entropy production $\Delta S + Q/T$ of recognizer and cell with $T = 1$, and the inequality in (iii) is Clausius' inequality. Reading (i)–(iii) as the heat of the tick from n to $n + 1$ of a run rests on the per-tick refresh.

Proof. Write $A(x, y) = |\langle y|e^{-iH}|x\rangle|^2$ for joint states. By the proof of Theorem 9.9, A is symmetric, vanishes between different total energies, and has rows and columns summing to one; every sum below is finite because shells are finite and p_n has finite support. (i) Summing $\mu_n(t, c')$ over c' gives $\sum_s p_n(s)K(s, t) = p_{n+1}(t)$, since K is the read. Multiply the equality $|t|_1 \log\varphi + E_C(c') = |s|_1 \log\varphi + E_C(c)$, valid whenever $A((s, c), (t, c')) \neq 0$, by $p_n(s)\pi_C(c)A((s, c), (t, c'))$ and sum: the left side gives $L_{n+1}\log\varphi + \sum v_n E_C$, the right side $L_n \log\varphi + \sum \pi_C E_C$. (ii) Subtract $D_{n+1} = (9+L_{n+1})\log\varphi - S(p_{n+1})$ from $D_n = (9+L_n)\log\varphi - S(p_n)$ and use (i). (iii) Since $\log\pi_C = -E_C - \log Z_C$, $D(v_n\|\pi_C) = Q_n - S(v_n) + S(\pi_C)$. Adding the three terms, with the recognizer marginal p_{n+1}, gives $S(p_{n+1}) - S(p_n) + Q_n$, which is (ii). $I_n \geq 0$ and $D(v_n\|\pi_C) \geq 0$ by Gibbs' inequality [27]. For the first term, $\mu_n(y) = \sum_x A(x, y)q(x)$ with $q = p_n \otimes \pi_C$, the weights $A(\cdot, y)$ add to one, and $u \mapsto -u\log u$ is concave, so $S(\mu_n) \geq S(q) = S(p_n) + S(\pi_C)$. (iv) By (F1), $L_0 = 0$ and $L_1 = 1 - h_0$; (i) gives Q_0, and $S(p_1)$ is the entropy of the law h_0 at 0 and $(1 - h_0)/6$ at each neighbour. □

In the first tick the recognizer draws on average $(1 - h_0)\log\varphi$ from the cell, between 0.39266 and 0.39272 (between 0.8160 and 0.8161 of one level), and its entropy rises from 0 to a value between 1.93948 and 1.93952; the difference is the first-tick fall of D. Over the first N ticks the heat it draws adds up to $L_N \log\varphi$, its mean energy, which tends to $6\log\varphi$ by (F3). The heat and the total fall are fixed by K; how the fall divides among correlation, the cell's departure from its Gibbs state and the entropy of reading both depends on the cell, which is not unique.

9.4 Quick routes to heat fail

Five readings would give heat quickly. The ratio cost as the energy fails (Proposition 9.5), and the spread of a recognizer's imbalance as it relaxes is not motion in a space, since no law with detailed balance against w and equal posts out of 0 that ever posts is translation invariant (Proposition 8.20). The other three fail as follows.

Proposition 9.14 (Accumulated cost, a fixed price, a lost clock origin)**.** *(i) On the account ledger, posting an event of ratio $r \neq 1$ with its reciprocal n times on the empty ledger gives a ledger of cost $2nJ(r)$ with net flow zero at every site. So the net flows stay zero while the cost exceeds every bound: no bounded sequence equals the accumulated cost, and accumulated cost is not a function of the net flows.*

(ii) There is no $E : \mathbb{Z}^3 \to \mathbb{R}$ and $q > 0$ with $E(s + o) - E(s) = q$ for every s and every post $o \in \{\pm e_1, \pm e_2, \pm e_3\}$.

(iii) Let a clock act diagonally in a finite basis X with labels $\chi : X \to \mathbb{Z}/8$, and let $\omega = e^{-i\pi/4}$. Averaging a matrix R on X over the eight possible origins of the clock, $\bar{R}(a, b) = \frac{1}{8}\sum_{n=0}^{7} \overline{\omega^{n\chi(b)}}\,\omega^{n\chi(a)} R(a, b)$, keeps $R(a, b)$ when $\chi(a) = \chi(b)$ and sets it to zero otherwise. In particular $\bar{R}(a, a) = R(a, a)$, and $\sum_a E(a)\bar{R}(a, a) = \sum_a E(a)R(a, a)$ for every energy E diagonal in X.

Proof. (i) Each posting adds $J(r) + J(1/r) = 2J(r) > 0$ and two events whose logarithms cancel (Chapter 3). (ii) The post e_1 from 0 followed by $-e_1$ returns to 0, so $0 = 2q$. (iii) $\frac{1}{8}\sum_n \omega^{n(\chi(a)-\chi(b))}$ is 1 when $\chi(a) = \chi(b)$ in $\mathbb{Z}/8$ and 0 otherwise. □

None of the five routes turns posting activity into heat.

9.5 Coupling through heat

Corollary 9.12 holds for every coupling, so it says nothing about energy passing between recognizers. Chapter 8 proves that the native target leaves the joint law free and that joint detailed balance with joint holding-and-return balance forces independence (Theorems 8.6 and 8.7). Can a thermal record cell shared by a group select a coupling? The carrier is a finite set V of recognizers with imbalance profiles $\sigma : V \to \mathbb{Z}^3$, summed level $\lambda(\sigma) = \sum_v |\sigma_v|_1$, and a cell; a joint move gives each recognizer a hold or one post.

Proposition 9.15 (A shared cell selects no coupling)**.** *(i) Let P be a law on profiles that is a probability law on joint moves at every profile, satisfies $\rho^{\lambda(\sigma)}P(\sigma,\tau) = \rho^{\lambda(\tau)}P(\tau,\sigma)$, and holds the whole group with chance at least $h > 0$ at every profile. Then P is the one-tick read of a reversible exchange of the whole group with one finite cell in its Gibbs state at temperature one, conserving $\lambda(\sigma)\log\varphi + E_C(c)$ and moving each recognizer by at most one post. The product of K over V is such a law.*

(ii) Let $|V| \geq 2$, and let P_a be the square-coupled law of Proposition 8.8(i), now for any real a with $0 < |a| \leq f^{|V|}$, where $f > 0$ is the least of $w(s)K(s, s+o)$ over the four states of the unit square and all local moves o. Then P_a has K's rows as one-recognizer marginals at every profile, is in detailed balance against $\prod_v w(\sigma_v)$, differs from the product, fails joint holding-and-return balance, and is the one-tick read of one finite cell in its Gibbs state at temperature one whose exchange is a sum $\sum_v H_v$ of commuting terms, H_v moving only recognizer v.

(iii) A law read from one shared cell as in (i) that also meets joint holding-and-return balance is the product of K.

The proof is in Appendix B.7. The coupling in (ii) is a common cause. The cell carries a label, one sign $\epsilon_v = \pm 1$ per recognizer with the product of the signs fixed by the sign of a, drawn afresh at every refresh; each recognizer exchanges energy only with its own copy of a native-law cell, at angles set by its own sign. The recognizers never act on each other, every one of them follows K, and the amplitude a is free; among the stated requirements only joint holding-and-return balance removes the coupling, and it removes all of it. Whether a cell without a conserved label, or with exchange terms that do not commute, produces a coupling other than a common cause is open.

An interaction energy. In P_a the coupling sits in the chances; the weight stays $\prod_v w(\sigma_v)$, so under the energy reading the group has no energy beyond its levels. A coupling can also sit in the weight. On two recognizers, for a real function U of the pair of imbalances write $W_U(\sigma) = w(\sigma_1)w(\sigma_2)e^{-U(\sigma)}$; under the energy reading W_U is the Boltzmann factor at temperature one of $\lambda(\sigma)\log\varphi + U(\sigma)$, and U is an interaction energy. Call U additive if $U(\sigma) = u_1(\sigma_1) + u_2(\sigma_2)$. For $b > 0$ the *contact energy* is $U_b(s,s) = -\log(1 + b/w(s))$ and

$U_b(\sigma) = 0$ when $\sigma_1 \neq \sigma_2$, so that

$$W_{U_b}(\sigma) = w(\sigma_1)w(\sigma_2) + b\,[\sigma_1 = \sigma_2]\,w(\sigma_1),$$

with $[\cdot]$ equal to 1 when the condition holds and 0 otherwise. The *contact law* R_b moves the two recognizers independently by K when their imbalances differ; when both are at s it mixes independent moves, with weight $w(s)$, and locked moves, with weight b, in which both make the same move:

$$R_b\big((s,s),(s+\mu_1,s+\mu_2)\big) = \frac{w(s)K(s,s+\mu_1)K(s,s+\mu_2) + b\,[\mu_1 = \mu_2]\,K(s,s+\mu_1)}{w(s)+b}. \tag{9.4}$$

Proposition 9.16 (An interaction energy with native marginals)**.** *Under (P1)–(P4), let $b > 0$.*

(i) R_b is a probability law on joint moves at every profile with all values positive, its one-recognizer marginals are K's rows, and it is in detailed balance against W_{U_b}; U_b is not additive. R_b is not in detailed balance against $w(\sigma_1)w(\sigma_2)$, so no shared cell of Proposition 9.15 reads it, and it fails joint holding-and-return balance.

(ii) Used as the chances at every state of a law of two recognizers that keeps the books, R_b meets the native target, so Corollary 9.12 applies to it.

(iii) If U vanishes outside a finite set of profiles and some real function R of a profile and a joint move has K's rows as marginals and is in detailed balance against W_U, then $U = 0$.

(iv) If U and R are as in (iii) without the finiteness condition and W_U is summable with total Z_U, then $\sum_t W_U(s,t) = Z_U\pi(s)$ for every s, and likewise for the other recognizer.

Proof of (i) and (ii). Off the diagonal R_b is the product of K's rows. On the diagonal the numerator is positive, sums to $w(s)+b$ over μ, and sums to $(w(s)+b)K(s,s+\mu_1)$ over μ_2 with μ_1 fixed (and symmetrically). In both cases

$$W_{U_b}(\sigma)R_b(\sigma,\sigma+\mu) = \prod_{v=1}^{2} w(\sigma_v)K(\sigma_v,\sigma_v+\mu_v) + b\,[\sigma_1=\sigma_2][\mu_1=\mu_2]\,w(\sigma_1)K(\sigma_1,\sigma_1+\mu_1),$$

and by (P3) each factor $w(\sigma_v)K(\sigma_v,\sigma_v+\mu_v)$ is unchanged when σ_v is replaced by $\sigma_v+\mu_v$ and μ_v by $-\mu_v$, while $\sigma_1=\sigma_2$, $\mu_1=\mu_2$ is preserved: detailed balance. Additivity would give $U_b(0,0)+U_b(e_1,e_2) = U_b(0,e_2)+U_b(e_1,0) = 0$, but the left side is $-\log(1+b)$. With $k = K(0,e_1)$ and $K(e_1,0) = \varphi k$,

$$\frac{w(0)^2 R_b\big((0,0),(e_1,e_1)\big)}{w(e_1)^2 R_b\big((e_1,e_1),(0,0)\big)} = \frac{\rho+b}{\rho(1+b)} \neq 1,$$

so R_b is not balanced against $w\otimes w$; every read of Proposition 9.15 is. Joint holding-and-return balance would force the product (Theorem 8.7). (ii) The marginals are K's rows at every profile, which is the native target. □

Parts (iii) and (iv) are proved in Appendix B.7. A pair at the same imbalance s carries the extra weight $b\,w(s)$, the weight of one recognizer at s: locked together, two recognizers weigh like one. By (iv), every interaction energy with a summable weight that admits a law with K's marginals leaves each recognizer's equilibrium law at π. The interaction is not confined to a few states: $U_b(s,s) = -\log(1+b\varphi^{|s|_1})$ falls without bound, and by (iii) no nonzero interaction energy admitting such a law vanishes outside a finite set. The amplitude b is free, and nothing here selects it. Whether a bounded, non-additive interaction energy admits such a law is open.

No finite cell records the contact law. A cell could conserve some energy $E(\sigma) + E_C(c)$ of the pair other than the summed level. Put the pair at the common imbalance ne_1 and let both post e_1, a locked move of positive chance. Under the energy reading the pair's energy is $E_b = -\log W_{U_b}$, and the step raises it by

$$g_n = \log \frac{\rho^n(\rho^n + b)}{\rho^{n+1}(\rho^{n+1} + b)} = \log \varphi + \log \frac{\rho^n + b}{\rho^{n+1} + b}. \tag{9.5}$$

Two independent recognizers would pay $2 \log \varphi$, one recognizer $\log \varphi$ for one level; the locked pair pays an amount in between that changes with n. For $b = 1$: $g_0 = \log 2 = 0.6931$, $g_1 = 0.6389$, $g_2 = 0.5928$, against $\log \varphi = 0.4812$.

Proposition 9.17 (No finite record for the contact law)**.** *Under (P1)–(P4), let $b > 0$.*

(i) The g_n decrease strictly in n, lie strictly between $\log \varphi$ and $2 \log \varphi$, and tend to $\log \varphi$.

(ii) Let E be a real function on profiles of two recognizers, (C, E_C) a thermal record cell, and H a bounded self-adjoint operator on $\ell^2(\mathbb{Z}^3 \times \mathbb{Z}^3 \times C)$ with real entries vanishing between basis states of different total energy $E(\sigma) + E_C(c)$ and between (σ, c) and (τ, c') unless τ arises from σ by one joint move. If its read P has $P(\sigma, \tau) > 0$, then $E(\tau) - E(\sigma) = E_C(c) - E_C(c')$ for some $c, c' \in C$; and $e^{-E(\sigma)} P(\sigma, \tau) = e^{-E(\tau)} P(\tau, \sigma)$ for all σ, τ.

(iii) No E, finite cell and operator as in (ii) has a read equal to R_b on every joint move.

(iv) If C is any nonempty set with $Z_C = \sum_c e^{-E_C(c)} < \infty$ and the read (now with infinite sums) equals R_b, then every g_n is a difference of two cell energies, C is infinite, its energies are unbounded above, and $E - E_b$ is constant.

Proof. (i) With $x = \rho^n$, $g_n = \log \varphi + \log r(x)$, $r(x) = (x+b)/(\rho x + b)$, and $r'(x) = b(1-\rho)/(\rho x + b)^2 > 0$; as n grows, x decreases strictly to 0, so g_n decreases strictly to $\log \varphi + \log r(0) = \log \varphi$. Since $\rho x < x$, $r > 1$; since $x + b < \varphi(\rho x + b)$, $r < \varphi$. (ii) If f vanishes off one value of the total energy, so do Hf, every power $H^k f$ and $e^{-iH} f$, a norm-convergent series; so amplitudes vanish between different totals, and a positive $P(\sigma, \tau)$ needs $E(\sigma) + E_C(c) = E(\tau) + E_C(c')$ for some c, c'. Real entries make H, its powers and e^{-iH} symmetric, and $e^{-E(\sigma)} \pi_C(c) = e^{-E(\tau)} \pi_C(c')$ on a shell, so the balance follows term by term as in Theorem 9.9. (iii) A read equal to R_b is balanced against e^{-E} by (ii) and against e^{-E_b} by Proposition 9.16(i), with all values positive, so $E(\tau) - E(\sigma) = E_b(\tau) - E_b(\sigma)$ along every move; along the locked moves this is g_n. By (ii) every g_n is a difference of two energies of the cell. A finite cell has finitely many such differences, and by (i) the g_n are infinitely many distinct numbers. (iv) The argument of (ii) and (iii) applies with nonnegative infinite sums, so C is infinite; if every cell energy were at most M, each term of Z_C would be at least e^{-M} and Z_C would diverge. Joint moves connect any two profiles and $R_b > 0$, so $E - E_b$ is constant. □

A finite heat bath pays only finitely many different amounts of energy, and two locked recognizers that step outward together need a different amount at every step: near $2 \log \varphi$ close to the empty record when b is small, near $\log \varphi$ far out, where the locked pair weighs like one recognizer.

Remark 9.18 (Open: a countable record for the contact law). For $b > 0$, is there a countable set C with energies E_C and $\sum_c e^{-E_C(c)} < \infty$, a real function E on profiles of two recognizers, and an operator H as in Proposition 9.17(ii) on $\ell^2(\mathbb{Z}^3 \times \mathbb{Z}^3 \times C)$ whose one-tick read is R_b? A construction, or a proof that the unbounded energies of (iv) force a contradiction with the boundedness of H or with the convergence of Z_C, would settle it.

9.6 The photon field as radiation

A hot universe needs radiation, and the photons have to behave like light. This section works on a different carrier from the last four: the transverse photon field of the clock-photon lattice, with the dispersion $\omega(k) = 2C_E \operatorname{arsinh}(\sqrt{\lambda(k)}/2)$ of Definition 6.1 and its band ceiling $\omega_{\max} = 2C_E \operatorname{arsinh}\sqrt{3} \approx 2.6339158\, C_E$ (carrier (i)). No correspondence theorem joins this field to the recognizers of Sections 9.1 to 9.5, so the temperature below is a parameter of the field and is not the temperature of Corollary 9.4. The two results stand side by side.

To compare the field with light, give the lattice a spacing $\ell \geq 0$ and the dynamics an action scale $\hbar > 0$, and let $C_E \geq 0$. For a physical momentum $p \in \mathbb{R}^3$ put

$$\omega(p) = \frac{2C_E}{\hbar} \operatorname{arsinh}\Big(\frac{\sqrt{\lambda(\ell p)}}{2}\Big), \qquad c = \frac{\ell C_E}{\hbar}. \tag{9.6}$$

Proposition 9.19 (Light at long wavelength)**.** *For every $p \in \mathbb{R}^3$, $0 \leq c|p| - \omega(p) \leq c\,\ell^2|p|^3/8$.*

Proof. With $k = \ell p$ it suffices that $0 \leq |k| - 2\operatorname{arsinh}(\sqrt{\lambda(k)}/2) \leq |k|^3/8$. Since $\lambda(k) = \sum_d 4\sin^2(k_d/2) \leq |k|^2$ and $\operatorname{arsinh} u \leq u$, the lower bound holds. For the upper bound split $|k| - 2\operatorname{arsinh}(\sqrt{\lambda}/2) = (|k| - \sqrt{\lambda}) + (\sqrt{\lambda} - 2\operatorname{arsinh}(\sqrt{\lambda}/2))$. From $\sin^2 x \geq x^2 - x^4/3$, $|k|^2 - \lambda \leq \sum_d k_d^4/12 \leq |k|^4/12$, so $|k| - \sqrt{\lambda} \leq |k|^3/12$. From $\operatorname{arsinh} u \geq u - u^3/6$ for $u \geq 0$, $\sqrt{\lambda} - 2\operatorname{arsinh}(\sqrt{\lambda}/2) \leq \lambda^{3/2}/24 \leq |k|^3/24$. The pieces add to $|k|^3/8$. □

The same bound controls plane waves: in units with $c = 1$ the native phase $e^{i(x\cdot p - t\omega(p))}$ differs from the continuum phase by at most $|t|\ell^2|p|^3/8$, and the continuum phase is invariant under boosts, so a boost changes the native phase by at most $\frac{\ell^2}{8}(|t|\,|p|^3 + |t'|\,|p'|^3)$, primes marking the momentum and the event time in the boosted frame. At wavelengths long compared with ℓ, the native field is a massless field with limiting speed $\ell C_E/\hbar$. Reading this slope as the speed of light, $c = \ell C_E/\hbar$, is an identification; the values of ℓ, C_E and $\hbar$ in metres, joules and joule-seconds are not derived (Section 9.8).

Suppose the field holds, in each mode of frequency ω, the mean energy $\omega/(e^{\omega/T} - 1)$ of the Bose occupation at a temperature $T > 0$ in energy units. This occupation is a *supplied premise*: nothing below derives it from the dynamics. For one polarization the energy per lattice cell is

$$u_1(T) = \int_{\mathbb{T}^3} \frac{\omega(k)}{e^{\omega(k)/T} - 1}\, \mathrm{d}\mu(k). \tag{9.7}$$

Theorem 9.20 (The native blackbody law)**.** *Let $C_E > 0$, and supply the Bose occupation. For every $T > 0$,*

$$T e^{-\omega_{\max}/T} \leq u_1(T) \leq T, \tag{9.8}$$

and

$$\lim_{T \to 0^+} \frac{u_1(T)}{T^4} = \frac{\pi^2}{30\, C_E^3}. \tag{9.9}$$

For every $B > 0$ some temperature has $u_1(T) < BT^4$: no quartic law holds at all temperatures.

Proof of (9.8) *and the last claim.* With $f(x) = x/(e^x - 1)$ the integrand is $T f(\omega/T)$. From $e^x - 1 \geq x$, $f \leq 1$; from $1 - e^{-x} \leq x$, $f(x) \geq e^{-x}$. The frequency vanishes only at $k = 0$ and never

exceeds $\omega_{\max}$, and μ is a probability measure. The last claim follows from $u_1(T) \le T < BT^4$ once $T > B^{-1/3}$. The limit (9.9) is proved in Appendix B.7 by rescaling and dominated convergence; it holds along every sequence of positive temperatures tending to zero. □

With the scales of (9.6), the energy per cell divided by the cell volume ℓ^3 is an energy density, and the identification $C_E = \hbar c/\ell$ turns (9.9) into

$$\frac{u_1(T)}{\ell^3} = \frac{\pi^2}{30}\,\frac{(k_B T)^4}{(\hbar c)^3}\big(1 + o(1)\big) \qquad (T \to 0),$$

with T now in kelvin. The transverse field has two polarizations at every nonzero momentum, since $P_T(k)$ has rank two there, so a field with the Bose occupation in both holds $\pi^2(k_BT)^4/(15(\hbar c)^3)$ to leading order: the Stefan-Boltzmann energy density of blackbody radiation [86, 71]. What is derived is the coefficient $\pi^2/(30C_E^3)$ in native units: the fourth power comes from the linear slope of the dispersion at long wavelength, and $\pi^2/30$ from the Bose weights; the occupation and its temperature are supplied. Near the band ceiling the lattice shows: each mode holds at most T, the energy per cell grows at most linearly, and the quartic law fails. These two regimes are those of Debye's theory of solids [29].

Where would a thermal field come from? Photons are produced: in the self-consistent field a hop out of the vacuum creates photons with energy paid exactly by the native and field functional (Theorem 6.9). Produced photons are not yet thermal, and the reason is general. Let S be a finite set of source states and $x = (x_s)_{s\in S}$ a joint state in $\bigoplus_{s\in S}\mathscr{F}$. A measurement of the photon field alone along $v \in \mathscr{F}$ reads

$$R_x(v) = \sum_{s\in S} |\langle x_s, v\rangle|^2 = \langle v, \rho_x v\rangle, \qquad \rho_x = \sum_s |x_s\rangle\langle x_s|.$$

In a finite box the Gibbs state at temperature T reads v as $\langle v, e^{-H_0/T}v\rangle/Z$. On the whole lattice $e^{-H_0/T}$ has no finite trace, so we compare with the heat reading $\Theta(v) = \theta\langle v, e^{-\tau H_0}v\rangle$ for every scale $\theta > 0$ and every $\tau \ge 0$.

Theorem 9.21 (No photon state of finite trace reads as thermal)**.** *Let $C_E > 0$, $\theta > 0$ and $\tau \ge 0$.*

(i) For every finite S and every x there is a nonzero one-photon vector v with $R_x(v) = 0$, while $\Theta(v) > 0$.

(ii) For every finite S and every x, $\sup_{\|v\|=1} |\Theta(v) - R_x(v)| \ge \theta e^{-\tau\omega_{\max}}$.

(iii) No sequence of normalized states $x^{(m)}$ on finite sets S_m has $R_{x^{(m)}}(v) \to \Theta(v)$ for every unit vector $v \in \mathscr{F}$.

(iv) Parts (ii) and (iii) hold for every density operator ρ on $\mathscr{F}$ in place of ρ_x, with $R(v) = \langle v, \rho v\rangle$. In particular Θ is the reading of no density operator on $\mathscr{F}$.

Proof. (i) The one-photon components of the x_s span at most $|S|$ dimensions of the infinite-dimensional $\mathscr{H}_1$, so some unit $v \in \mathscr{H}_1$ is orthogonal to all of them. On $\mathscr{H}_1$, $e^{-\tau H_0}$ multiplies by $e^{-\tau\omega} \ge e^{-\tau\omega_{\max}}$, so $\Theta(v) \ge \theta e^{-\tau\omega_{\max}}$; this gives (ii) too. (iii) Take an integer $n > e^{\tau\omega_{\max}}/\theta$ and orthonormal $v_1, \dots, v_n \in \mathscr{H}_1$. Bessel's inequality gives $\sum_i R_x(v_i) \le \sum_s \|x_s\|^2 = 1$ for normalized x, while $\sum_i \Theta(v_i) \ge n\theta e^{-\tau\omega_{\max}} > 1$. (iv) For a density operator, $\sum_{i\le n}\langle v_i, \rho v_i\rangle \le \operatorname{tr}\rho = 1$, so some v_i has $R(v_i) \le 1/n$ and $|\Theta(v_i) - R(v_i)| \ge \theta e^{-\tau\omega_{\max}} - 1/n$ for every n; the argument of (iii) uses only the trace bound. The heat reading gives every one-photon mode a weight of at least $\theta e^{-\tau\omega_{\max}}$, so its total weight on $\mathscr{H}_1$ is infinite. □

This is the one-mode form of the fact that the thermal state of an infinite free Bose field is not given by a density operator on its Fock space [2, 10]. The Bose occupation of Theorem 9.20 is a mean occupation per mode. Deriving it from the dynamics is therefore a statement about photon measurements in finite regions or on finitely many modes, or about a state of the infinite field outside $\mathscr{F}$; the infinite-volume dynamics of Chapter 7 is where it would have to be proved. In the observed universe the radiation reached equilibrium by scattering with the electrons and ions of the early plasma, and its spectral distortions fell below the bounds quoted at the start of this chapter [60].

9.7 The standard thermal chain and its inputs

The standard thermal history rests on three things: the equilibrium statistical mechanics of free relativistic gases, the particle content of the Standard Model, and the Friedmann equations [68, 132]. The institute's statement of this chain proves the statistical mechanics and takes the particle content and the Friedmann equations as inputs. Its carrier is the continuum ideal gas, none of the recognition carriers; nothing connects it to the results above, and the occupancy $\{0, 1\}$ of a fermion mode is a choice, since Pauli exclusion is not derived here.

Proposition 9.22 (Radiation thermodynamics with stated inputs)**.** *Let $T > 0$ and $g \geq 0$.*

(i) For an occupation kernel ϑ on $(0, \infty)$, $g \int_{\mathbb{R}^3} T\vartheta(|k|/T)\, \frac{\mathrm{d}^3k}{(2\pi)^3} = \frac{g}{2\pi^2} T^4 \int_0^\infty t^2 \vartheta(t)\, \mathrm{d}t$. With $\vartheta(t) = t/(e^t \mp 1)$, the energy density of g_B bosonic and g_F fermionic states is $\frac{\pi^2}{30}\big(g_B + \frac{7}{8} g_F\big) T^4$.

(ii) $\int_0^\infty \frac{t^3\, \mathrm{d}t}{e^t+1} \Big/ \int_0^\infty \frac{t^3\, \mathrm{d}t}{e^t-1} = \frac{7}{8} = \eta(4)/\zeta(4)$, with $\int_0^\infty t^3/(e^t - 1)\, \mathrm{d}t = \pi^4/15$.

(iii) The entropy integrals are $\int \sigma_B = 4\pi^4/45$ and $\int \sigma_F = 7\pi^4/90$, each $\frac{4}{3}$ times the corresponding energy integral; with the entropy defined as $s = \mathrm{d}P/\mathrm{d}T$ and the energy by the Legendre relation $\rho = Ts - P$, the radiation gas has $P = \rho/3$ and $s = \frac{4}{3}\rho/T$.

(iv) With the Standard Model census as input, 28 bosonic states (2(8 + 3 + 1) gauge, 4 Higgs) and 90 fermionic states (72 quark, 12 charged-lepton, 6 neutrino), $g_* = 28 + \frac{7}{8} \cdot 90 = \frac{427}{4} = 106.75$.

(v) If a, ρ and p satisfy the two Friedmann equations $\dot{a}^2 = \frac{8\pi G}{3}\rho a^2$ and $\ddot{a}\, a = -\frac{4\pi G}{3}(\rho + 3p)a^2$ with $G \neq 0$ and $a \neq 0$, then $a\dot{\rho} = -3\dot{a}(\rho + p)$. Given this continuity equation, $Ts = \rho + p$ and $\dot{p} = s\dot{T}$, the comoving entropy sa^3 is conserved; with entropy conserved through electron-positron annihilation and neutrinos streaming freely (aT_ν constant), $(T_\nu/T_\gamma)^3 = 4/11$ and $g_{*s} = 43/11$ after annihilation.

(vi) $\zeta(3) \in (1.202042, 1.202065)$, and the entropy per photon $s/n_\gamma = \pi^4(43/11)/(45\zeta(3))$ lies in $(7.0393, 7.0396)$.

Proof outline. (i) is spherical coordinates and the substitution $k = Tt$. (ii) follows from $\int_0^\infty t^3/(e^t \mp 1)\, \mathrm{d}t = \Gamma(4) \sum_n (\pm 1)^{n+1} n^{-4}$ and $\eta(4) = (1 - 2^{-3})\zeta(4)$. (iii) is integration by parts and the definitions. (iv) is arithmetic. (v) Differentiate the first Friedmann equation and eliminate $\ddot{a}$ with the second; with the Euler and Gibbs-Duhem relations, $\frac{\mathrm{d}}{\mathrm{d}t}(sa^3) = a^3(\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p))/T = 0$; the photon-plus-pair entropy $\frac{2\pi^2}{45}(2 + \frac{7}{8} \cdot 4) T^3 a^3$ before annihilation equals $\frac{2\pi^2}{45} \cdot 2\, T_\gamma^3 a^3$ after it, while aT_ν keeps its value, which gives $4/11$, and $g_{*s} = 2 + \frac{7}{8} \cdot 6 \cdot \frac{4}{11} = \frac{43}{11}$. (vi) bounds the series $\sum n^{-3}$ by a partial sum and an integral tail. These are the standard computations [68]. □

The same census with particles dropped at their measured mass thresholds, a step model, gives the standard values $g_* = 86.25$, 61.75, 17.25 and 10.75 at 10 GeV, 1 GeV, 0.14 GeV and 2 MeV, and 112 at 200 GeV with Dirac neutrinos. These reproduce the standard values from measured inputs; they are not predictions. Each input of the chain is a gap: the gauge group, the Higgs doublet, three colours and three neutrino families; the occupancy of fermion modes; and the Friedmann equations, whose derivation from the recognition account of gravity is open.

9.8 Physical scale

Every number so far is pure: counts of ticks and posts, chances, temperatures in units of $\log \varphi$ per level, energies in units of C_E. This section proves how much must come from outside to give them units.

The native units are free. Each of the book's native units is left free by the conditions that govern it. The strain of a post meets every forced condition at every positive value (Proposition 3.9). No clock built from K fixes the length of a tick (Proposition 4.36). Lattice energies scale with the spacing: on a box of spacing h the lattice operator is $\mathscr{L}_h = h^{-2}\mathscr{L}_1$, and $\mathscr{G}_1/h$ is a Green function at spacing h when $\mathscr{G}_1$ is one at unit spacing [114]. None of these conditions can single out a value, for a simple reason.

Proposition 9.23 (A property that survives doubling pins no scale)**.** *Let P be a property of real numbers with $P(x) \Rightarrow P(2x)$, and let $a > 0$ satisfy $P(a)$. Then some $y \neq a$ satisfies $P(y)$.*

Proof. Take $y = 2a \neq a$. □

So a value of any unit needs a premise that is not invariant under rescaling, or an input from outside the laws.

Theorem 9.24 (c, $\hbar$ and G are dimensionally independent)**.** *Give c, $\hbar$ and G their exponents of length, time and mass, $(1,-1,0)$, $(2,-1,1)$ and $(3,-2,-1)$. The matrix with these columns has determinant -2, and for integers a, b, d the product $c^a\hbar^b G^d$ is dimensionless only if $a = b = d = 0$.*

Proof. The exponents of $c^a\hbar^b G^d$ are $(a + 2b + 3d,\ -a - b - 2d,\ b - d)$. Mass gives $b = d$, length $a = -5b$, time $5b - b - 2b = 2b = 0$. Expanding along the first row, the determinant is $1 \cdot 3 - 2 \cdot 1 + 3 \cdot (-1) = -2$. □

A theory whose outputs are pure numbers can state c, $\hbar$ and G only in its own units, and no combination of their laboratory values is a pure number it could predict in their place [13, 38]. So at least one quantity with a unit must be supplied. Because the determinant is -2 and not ± 1, the units built from the three need square roots, as in the Planck time $t_P = (\hbar G/c^5)^{1/2}$ [85].

One quantity suffices: the tick τ_0 in seconds is the single socket. Given τ_0, take $c\tau_0$ metres per voxel and $\hbar/\tau_0$ joules per unit of action per tick. One voxel per tick is then c and one unit sustained for one tick is $\hbar$, by construction: the SI fixes c and h exactly by definition [104], so this agreement is a definitional identity, not an output. The physics is in where τ_0 comes from.

Definition 9.25 (The native constants)**.** With time counted in ticks τ_0 and length in voxels, the *native constants* are the adopted premises

(N1) $c = 1$ voxel per tick, so the recognition length $\lambda_{\mathrm{rec}} = c\tau_0$ is one voxel;
(N2) $\hbar = E_{\mathrm{coh}}\tau_0$ with $E_{\mathrm{coh}} = \varphi^{-5}$, so $\hbar = \varphi^{-5}$ in native units;
(N3) $G = \lambda_{\mathrm{rec}}^2 c^3/(4\ln 2 \cdot \hbar)$, so $G = \varphi^5/(4\ln 2)$ and $\hbar G = 1/(4\ln 2)$ in native units.

None of (N1)–(N3) is produced by the laws of the earlier chapters. (N1) is an axis speed: under the only interacting law with a proved cone, the isolated-pair law, influence moves one cube side per tick along an axis and $1/\sqrt{3}$ of a side per tick along a body diagonal (Theorem 8.28). (N2) enters only masses. (N3) has a physical reading.

Proposition 9.26 ((N3) is one bit per lattice face)**.** *Read the Bekenstein-Hawking entropy $Ac^3/(4\hbar G')$ nats of an area A [5, 57] on one lattice face, $A = \lambda_{\mathrm{rec}}^2$. For every $G' > 0$ the face carries $c^3\lambda_{\mathrm{rec}}^2/(4\ln 2 \cdot \hbar G')$ bits, and this is one exactly when G' is the G of (N3).*

Proof. Divide the entropy in nats by $\ln 2$; the count is a strictly decreasing function of G' equal to 1 at the value of (N3). □

Jacobson derived the Einstein equation from the Clausius relation with an entropy proportional to area [61]; (N3) reads that proportionality as one bit per face. A *bridge* is a triple (a_T, a_L, a_M) of seconds per tick, metres per voxel and kilograms per native mass unit; it matches the three constants when $c_{\mathrm{SI}} = c_{\mathrm{RS}}a_L/a_T$, $\hbar_{\mathrm{SI}} = \hbar_{\mathrm{RS}}a_M a_L^2/a_T$ and $G_{\mathrm{SI}} = G_{\mathrm{RS}}a_L^3/(a_M a_T^2)$, with $(c_{\mathrm{RS}}, \hbar_{\mathrm{RS}}, G_{\mathrm{RS}}) = (1, \varphi^{-5}, \varphi^5/(4\ln 2))$, $c_{\mathrm{SI}} = 299\,792\,458$ m s^{-1}, $\hbar_{\mathrm{SI}} = 1.054571817 \times 10^{-34}$ J s and $G_{\mathrm{SI}} = 6.67430(15) \times 10^{-11}$ m^3 kg^{-1} s^{-2} [21].

Theorem 9.27 (One measured number fixes the bridge)**.** *Under (N1)–(N3), every matching bridge has $a_L = c_{\mathrm{SI}}a_T$, $a_M a_T = \hbar_{\mathrm{SI}}/(\hbar_{\mathrm{RS}}c_{\mathrm{SI}}^2)$, $a_T/a_M = G_{\mathrm{SI}}/(G_{\mathrm{RS}}c_{\mathrm{SI}}^3)$, and*

$$a_T^2 = \frac{4\ln 2 \cdot \hbar_{\mathrm{SI}}G_{\mathrm{SI}}}{c_{\mathrm{SI}}^5}, \qquad a_T = 2\sqrt{\ln 2}\, t_P. \tag{9.10}$$

The tick uses only (N1) and (N3); (N2) enters only a_M.

Proof. The first condition with $c_{\mathrm{RS}} = 1$ is $a_L = c_{\mathrm{SI}}a_T$. Substituting into the second and third gives the two products. Multiplying them, $a_T^2 = \hbar_{\mathrm{SI}}G_{\mathrm{SI}}/(\hbar_{\mathrm{RS}}G_{\mathrm{RS}}c_{\mathrm{SI}}^5)$, and $\hbar_{\mathrm{RS}}G_{\mathrm{RS}} = 1/(4\ln 2)$ by (N1) and (N3). Take the positive root. □

Every SI number shown through this bridge is a calibration through one measured value, G. The bridge does not derive G; it uses it.

The missing number. To turn the calibration into a derivation, one pure number must tie a laboratory process to the tick. The electron is the natural first target. For a mass m, write its golden level against the coherence quantum $E_{\mathrm{coh}} = \hbar/\tau_0$ as

$$\gamma(m, G) = \log_\varphi \frac{mc^2\tau_0(G)}{\hbar} = \log_\varphi\Big(2\sqrt{\ln 2}\,\frac{m}{m_P(G)}\Big), \qquad m_P(G) = \Big(\frac{\hbar c}{G}\Big)^{1/2}, \tag{9.11}$$

with $\tau_0(G) = 2\sqrt{\ln 2}\, t_P(G)$ from Theorem 9.27; it increases with m and with G.

quantity	closed form	value
seconds per tick a_T	$2\sqrt{\ln 2}\,(\hbar G/c^5)^{1/2}$	8.977×10^{-44} s
metres per voxel a_L	$c\,a_T$	2.691×10^{-35} m
kilograms per native mass unit a_M	$\varphi^5\hbar/(c^2 a_T)$	1.450×10^{-7} kg
$E_{\mathrm{coh}}=\hbar/\tau_0$	$E_P/(2\sqrt{\ln 2})$	1.175×10^{9} J $=7.332\times10^{18}$ GeV
Planck time t_P	$(\hbar G/c^5)^{1/2}$	5.391×10^{-44} s

Table 9.2. The bridge under the adopted native constants (N1)–(N3), at the CODATA 2022 value of G. Each value inherits from G (relative standard uncertainty 2.2×10^{-5}) a relative uncertainty of 1.1×10^{-5}. The tick is $2\sqrt{\ln 2}=1.6651$ Planck times, 1.06 golden levels above t_P.

Theorem 9.28 (The electron level locates; it does not predict)**.** *Under (N1) and (N3), take c, h and $\hbar$ as in the SI and the CODATA 2022 values $m_e = 9.1093837139(28)\times10^{-31}$ kg and $G = 6.67430(15)\times10^{-11}$ m^3 kg^{-1} s^{-2} [21].*

(i) For every m and G' within one standard uncertainty of these values, $-106.019763 < \gamma(m,G') < -106.019703$.

(ii) At the central values the electron's Compton period $h/(m_e c^2)$ lasts between φ^{109} and φ^{110} ticks, and divided by 2π it exceeds $1.009\,\varphi^{106}$ ticks.

Proof. (i) By monotonicity it suffices to bound γ below at (m_-,G_-) and above at (m_+,G_+), the corners one standard uncertainty away. Write $\gamma=-106+\log_\varphi x$ with $x=\varphi^{106}\cdot 2\sqrt{\ln 2}\cdot mc\,\ell_P(G')/\hbar$, $\ell_P=(\hbar G'/c^3)^{1/2}$. Rational enclosures of $2\sqrt{\ln 2}\in(1.6651092,1.6651093)$, of ℓ_P at the two corners, and of $\varphi^{106}=F_{106}\varphi+F_{105}$ with the Fibonacci numbers give $0.9905375<x<0.9905606$, and enclosures of $\ln\varphi$ and the exponential series give $\varphi^{-0.019763}<0.9905375$ and $\varphi^{-0.019703}>0.9905606$. (ii) The period is $(h/\hbar)\varphi^{-\gamma}$ ticks with $h/\hbar=2\pi<6.3$; $\varphi^3=2\varphi+1<2\pi$ and $6.3<0.99\varphi^4$ give the two bounds, and $\varphi^{-\gamma}>(1+0.019703\ln\varphi)\varphi^{106}$ gives the last. The full interval arithmetic is in [114]. □

The electron's rest energy is about $\varphi^{-106.0197}$ coherence quanta. This is a statement about one measured pure number, $m_e/m_P=4.1855\times10^{-23}$, raised by the $\log_\varphi(2\sqrt{\ln 2})=1.0596$ levels between the tick and the Planck time. The nearest whole level, -106, lies 0.0197 outside the interval, about 330 times its width: under these premises and with the measured values within one standard uncertainty, a law placing the electron at a whole golden level below the coherence quantum, in this form, is excluded.

The institute's mass law counts masses in coherence quanta. It is not derived: no theorem forces its sector constants, and its sector assignments were chosen from measured mass ratios. Its canonical electron number is $L_{\mathrm{can}}=2^{-22}\varphi^{56}(1+1332/\varphi)\approx 9.924\times10^7$; two further electron numbers in the same unit are $2^{-22}\varphi^{64}\approx5.657\times10^6$ and $\varphi^3\approx4.236$. A *unit reading* is a triple (E,τ,c) of positive numbers, the coherence quantum, the tick and the speed of light in some units, with $\hbar=E\tau$, $\lambda=c\tau$, $G=\lambda^2c^3/(4\ln2\cdot\hbar)$ as in (N1)–(N3) and Planck energy $E_P=(\hbar c^5/G)^{1/2}$.

Theorem 9.29 (Units cannot move the hierarchy)**.** *(i) In every unit reading $E_P=2\sqrt{\ln 2}\,E$, so a mass of L coherence quanta has the ratio $L/(2\sqrt{\ln 2})$ to the Planck mass in every reading.*

(ii) L_{can}, $2^{-22}\varphi^{64}$ *and* φ^3 *all exceed* 2, *so each, read as coherence quanta, puts the electron above the Planck mass in every unit reading.*

(iii) Let $R = L_{\mathrm{can}}/(2\sqrt{\ln 2})$ *and* $r = 4.1855 \times 10^{-23}$, *the measured central ratio of the electron's mass to the Planck mass. If* $|\eta R - r| < r/1000$, *then* $\varphi^{-145} < \eta < \varphi^{-144}$; *no* φ^{-n} *with* $n = 0, 1, 2, \ldots$ *satisfies the hypothesis.*

(iv) Keeping $\eta = 1$ *and replacing* G *by the* G' *that gives the canonical electron the ratio* r *puts* $L_{\mathrm{can}}^2/(4\ln 2 \cdot r^2) > 10^{59}$ *bits on one lattice face, in place of the one bit of (N3).*

(v) $\ln 2/\ln\varphi$ *is irrational, so the numbers* $2^k\varphi^n$ *with integers* k, n *are dense in the positive reals.*

Proof. (i) $\hbar c^5/G = E\tau \cdot c^5 \cdot 4\ln 2 \cdot E\tau/(c^2\tau^2c^3) = 4\ln 2 \cdot E^2$. (ii) $\varphi^2 > 2$ gives $L_{\mathrm{can}} > 2^{-22}2^{28} = 64$, $2^{-22}\varphi^{64} > 2^{10}$, $\varphi^3 > 4$, and $2\sqrt{\ln 2} < 2$. (iii) The hypothesis gives $0.999\,r/R < \eta < 1.001\,r/R$; the two inequalities $\varphi^{-145}R < 0.999r$ and $1.001r < \varphi^{-144}R$ become rational inequalities after writing $\varphi^n = F_n\varphi + F_{n-1}$ and enclosing φ and $\sqrt{\ln 2}$. A whole power φ^{-n} is at least φ^{-144} for $n \le 144$ and at most φ^{-145} for $n \ge 145$. (iv) The bit count is $E_P'^2/(4\ln 2 \cdot E^2)$ with $E_P' = L_{\mathrm{can}}E/r$; $L_{\mathrm{can}} > 9.9 \times 10^7$, $\ln 2 < 0.7$ and $r < 4.19 \times 10^{-23}$ give more than 10^{59} (its value is 2.03×10^{60}). (v) $\ln 2/\ln\varphi = p/q$ would give $2^q = \varphi^p = F_p\varphi + F_{p-1}$, making φ rational; for irrational θ the numbers $k\theta + n$ are dense [53]. □

With its unit equal to the coherence quantum, the canonical electron would weigh about 5.96×10^7 Planck masses against the measured 4.19×10^{-23}. One factor η common to all masses repairs this and keeps every mass ratio of the law; agreement to one part in a thousand places it inside one golden level (the exact match is about 7.02×10^{-31}, near $\varphi^{-144.28}$). By (v), some $2^k\varphi^n$ lies as close to η as one likes, so finding such a form near the measured value derives nothing. Deriving $\gamma_e = \gamma(m_e, G)$, or η with the mass law kept, is one task.

Proposition 9.30 (Two numbers of the electric scale are open)**.** *Read a unit charge as a debit and a credit one lattice step apart, with coupling* $s\lambda/(4\pi)$, *where* s *is the action carried by one unit of cost (in units of* $\hbar$*) and* $\lambda = C/g$ *is a cost* C *assigned to the pair divided by the pair's field coefficient* g.

(i) The two admissible readings $s = 1$ *and* $s = \varphi^5$ *differ, so the readings do not determine* s.

(ii) On the infinite cubic lattice $g = \mathcal{G}(0) - \mathcal{G}(e_1) = \frac{1}{6}$, *with* $\mathcal{G}$ *the lattice Green function; confined to one cube,* $g = \frac{7}{24}$. *With* $C = J(\varphi) = \varphi - \frac{3}{2}$ *the two give* $\lambda = 6\varphi - 9 \approx 0.7082$ *and* $\lambda = \frac{24}{7}(\varphi - \frac{3}{2}) \approx 0.4047$, *so* λ *is not fixed.*

(iii) Along a lattice axis $r\lambda\,\mathcal{G}(r) \to \lambda/(4\pi)$ *as* $r \to \infty$: *the* 4π *of the coupling is the far-field coefficient of the cubic lattice.*

Proof. (i) $\varphi^5 > 1$. (ii) The value $\frac{1}{6}$ is an exact Fourier computation on the Brillouin cube and $\frac{7}{24}$ the field cost of the pair's flow on the cube's twelve edges [114]; the two levels differ since $\varphi > 1.6$. (iii) is the classical $1/(4\pi r)$ decay of the cubic lattice Green function [107]. □

The fine-structure constant $\alpha = s\lambda/(4\pi)$ therefore waits on two numbers that the native structure leaves open. It is a boundary datum today: it enters as its measured value, and the recognition chain does not derive it.

Remark 9.31 (Open: the absolute scale). Derive from the recognition law one pure number that ties a laboratory process to the tick. The first target is the electron's golden

level $\gamma_e = \log_\varphi(m_e c^2 \tau_0/\hbar)$, which measured values within one standard uncertainty place in $(-106.019763, -106.019703)$ under (N1) and (N3); with the mass law kept, $\varphi^{\gamma_e} = \eta L_{\mathrm{can}}$ and the target is $\eta \in (\varphi^{-145}, \varphi^{-144})$. By Proposition 9.23 the derivation needs a premise that is not invariant under rescaling. With γ_e derived, $\tau_0 = \hbar\varphi^{\gamma_e}/(m_e c^2)$ would rest on the electron's mass (relative uncertainty 3.1×10^{-10}) and $G = c^5\tau_0^2/(4 \ln 2 \cdot \hbar)$ would become an output to compare with its measurement. Deriving (N1)–(N3) themselves, and the numbers s and λ, is part of the same problem.

9.9 What a hot, expanding phase still needs

The proved results give nothing that grows. Eight objects are missing, each stated exactly in Chapter 11: the interaction law, a requirement beyond the native target that fixes the joint law of a group of recognizers, with a record for the contact law or its exclusion (Problems 34 and 38); growth, a lawful process that adds recognizers or distance (Problem 42); space and a light cone, of which (N1) is only an axis speed of one supplied law (Problem 36); the energy scale ε in joules, which must fall for the phase to cool (Problem 43); the thermal premises (Problem 40); a thermal state of the field in the form Theorem 9.21 allows (Problem 44); stress and expansion, with the Friedmann equations that Proposition 9.22(v) takes as inputs (Problem 45); and the absolute scale (Problem 54). With the first four and the absolute scale resolved, a simulation could run a group of recognizers from the empty record, assign each region a temperature $\varepsilon/(k_B \log \varphi)$ in kelvin at each tick, and follow it as the phase grows. The first comparison with nature would be the radiation temperature today and the epochs at which matter falls out of equilibrium with it [68]. None of these numbers is computed in this book. Chapter 10 sets out the whole gap, topic by topic, from the first records to the dark sector.

Chapter 10

What is not yet derived: the gap to the hot, expanding universe

The radiation that fills the universe today is a blackbody at $T_0 = 2.72548 \pm 0.00057$ K, within the distortion bounds quoted in Chapter 9. Standard cosmology accounts for it with a hot plasma in an expanding space that cooled as it expanded. Can the results of this book compute T_0?

Take the one temperature we have. Under the energy reading of Chapter 9, every recognizer that follows the native law has the temperature $\epsilon / \log \varphi$, where ϵ is the energy of one level. To turn it into kelvin we need ϵ in joules, and no result fixes ϵ. To make it the temperature of radiation we need the photon field of Chapters 6 and 7 in a thermal state, and no result reaches one. To make it the temperature today we need the expansion between the hot phase and now, and no result on any carrier of this book describes an expansion. Each missing step is a definite mathematical object: a value in physical units, a state of an infinite field, a law that changes distances. None of them is supplied by the chapters before this one.

We state the gap topic by topic, in the order in which standard cosmology meets the topics, each with a table and with what is missing.

Earlier work at the institute proposed closed forms for a spectral tilt, a baryon asymmetry, a dark-energy fraction, an equation-of-state point and a dark-matter fraction. We list each with the premises that carry it and its comparison with data. None is a prediction in the sense of Chapter 1: a definition or an adopted premise carries each number, or the form was chosen after the comparison. Refuted routes are stated as results.

10.1 How the ledgers are read

Every entry of the tables below has one of six statuses.

Derived.
: A theorem on a named carrier of Appendix A, from the hypotheses stated with it, whose physical reading uses no adopted premise and no identification.

Conditional.
: A theorem whose hypotheses or physical reading include an adopted premise or an identification. The premise is named in the entry.

Adopted.
: An input that is chosen rather than derived: a premise, an identification, a definition that carries a number, or content imported from established physics (the particle content of the Standard Model, the field equations of general relativity). An adopted input may be correct physics. It is not derived here.

Measured.
A datum with its dataset, its year and its uncertainty. A comparison prints the deviation $(x - x_{\mathrm{obs}})/\sigma$, recomputed from the stated numbers.

Excluded.
A route refuted by a proved counterexample, by a proved no-go result or by a measurement.

Open.
A missing object, stated as a precise question. Chapter 11 collects them.

Two rules govern the entries. Results on different carriers stand side by side; one is used on another only where the second is built on the first or an adopted premise joins them (Table A.3). Agreement with data never moves an entry from adopted to derived: a closed form that matches a measurement after the form was chosen is a candidate, with the status of the premises that carry it.

When an entry says *nothing derived*, it means that no theorem on any carrier of this book, conditional or not, produces the object in question. A few entries qualify a status. *Derived (negative)* marks a proved absence; *conditional (negative)* marks a proved absence whose hypotheses include adopted premises, which the entry names. An *adopted candidate* is a closed-form value carried by adopted premises. *Excluded by inference* marks a route closed by an argument that is not yet a proved result; it is listed so that the route is not revived, and its proof is open.

10.2 From the first records to a thermal state

The proved results of Chapters 2 to 9 end on two separate carriers: with single recognizers that relax, under the native law, to the stationary law π, a Gibbs law under the energy reading, and with a photon lattice whose dynamics exists on the whole of $\mathbb{Z}^3$ and, with the clock off and on edge assignments that fill the directions they use, dephases each register. Between these two end points and a hot plasma in an expanding space lie seven steps: a law of interaction between recognizers, a law of growth, a derived space with its light cone, the energy of one level in physical units, a thermal state, a stress that sources geometry, and an expansion. Tables 10.1 and 10.2 record where each stands.

What is missing. Each open entry is a definite object.

1. *Interaction.* A requirement beyond admissibility that selects one joint law of a group of recognizers, with its amplitude; the native marginals, the record and energy pictures and a shared cell select none (Chapter 8). Since each recognizer's own history has the law K under every coupling that meets the target (Theorem 8.4), while distinct couplings are distinct joint laws (Theorem 8.6), a requirement or an observation that selects the coupling has to read the records of two or more recognizers together; the records of one recognizer cannot select it.
2. *Growth.* A lawful process, derived from the conditions of Chapters 4 and 8, that adds recognizers or increases distances between them.
3. *Space.* A placement of recognizers, with a distance and a light cone, derived from the recognition law; together with correspondence theorems that identify the contact cone of Chapter 8, the record cone of Chapter 5 and the dispersion of the photon lattice of

Table 10.1. From the first records to a thermal state, part 1: interaction, growth and space. Carriers as in Appendix A: (c) one recognizer, (d) many recognizers, (e) the record cell, (i) the clock-photon lattice.

Item	Status	Carrier, conditions, data; where
Law K of one recognizer	conditional	(c); unique under (P1)–(P4) and the three readings; relaxes to $\pi(s) = \varphi^{-9}\varphi^{-\lvert s\rvert_1}$; Chapter 4
Each recognizer keeps K under every admissible coupling	conditional	(d); (P1)–(P4); Chapter 8
Joint law of many recognizers	conditional	(d); marginals K, so (P1)–(P4); not unique; joint detailed balance with joint holding-and-return balance forces the product law; Chapter 8
Coupling through a shared record cell	conditional	(d) with record cells; energy reading; the thermal record cell and its per-tick refresh adopted (a cell returned to its Gibbs state before every tick is an external bath); a shared cell selects no coupling; a contact energy with free amplitude exists; no finite cell records it; Chapter 9
Which requirement fixes the interaction	open	
Adding recognizers or distance	derived (negative)	(d); by its definition every admissible law acts on a fixed finite set of voxels and defines no distance; Chapters 8, 9
Imbalance as position; its spread as expansion	excluded	(c); no law meeting (P1)–(P3) with equal exits from balance that leaves balance with positive chance is translation invariant; Chapter 8
Births added on a schedule outside the ledger, read as expansion	excluded	the births are supplied, not lawful events, and their sign pattern is chosen; a recorded run gives at most 0.231 e-fold and no exit to deceleration
A law of growth	open	
Light cone of many recognizers	conditional	(d); exact octahedral cone (axis speed 1, body diagonal $1/\sqrt{3}$) under a supplied contact rule, cube adjacency adopted; admissibility alone leaves locality open; Chapter 8
Light cone of the record cell	conditional	(e); a name-blind interval with rank as time is a spacetime interval of the cell exactly when it is λQ_κ with $\lambda > 0$ and $\kappa \geq 1$ (every Q_κ with $\kappa > \frac{1}{4}$ is Lorentzian); single postings as light is an identification; Chapter 5
Light cone of the photon lattice	conditional	(i); long-wavelength light; $c = \ell C_E/\hbar$ is an identification; Chapter 9
A derived placement of recognizers; correspondence of the three cones	open	no correspondence theorem exists

Table 10.2. From the first records to a thermal state, part 2: scale, thermal state, stress and expansion. Carriers as in Table 10.1.

Item	Status	Carrier, conditions, data; where
One dimensional input needed	derived	no nontrivial integer monomial in $c, \hbar, G$ is dimensionless; Chapter 9
Tick in seconds	conditional	$2\sqrt{\ln 2}\, t_P = 8.977 \times 10^{-44}$ s under (N1), (N3) and the measured G; Chapter 9
ϵ in joules and its change	open	under the energy reading a phase cools only if ϵ falls
One temperature	conditional	(c); energy reading; $T = \epsilon/\log\varphi$, $Z = \varphi^9$, $U = 6\log\varphi$, $S = 15\log\varphi$; Chapter 9
Blackbody coefficient	conditional	(i); supplied Bose occupation; $E/T^4 \to \pi^2/(30C_E^3)$ per polarization; Chapter 9
An exact T^4 law at every temperature	excluded	(i); bounded dispersion; Chapter 9
Heat reading from finite-source pure states	excluded	(i); Chapter 9
Dephasing of the registers	conditional	(i); supplied lattice data; clock off; per register, and jointly for the flip patterns with a transverse current on placements that fill each direction they use; Chapter 7
The reference state as the physical state	excluded	(i); supplied lattice data; clock off; the evolved state leaves the reference folium at every $t \neq 0$; Chapter 7
States the dynamics preserves	conditional	(i); supplied lattice data; clock off; long-time limit points of the reference state, carried by unitaries in their representations; on the sample assignment the dephased occupation statistics of the registers; uniqueness and photon values open; at every clock strength such limit points exist on the algebra the dynamics generates, not yet identified; Chapter 7
A thermal state of the photon field	open	
Response of a lattice action at flatness	conditional	Regge's action adopted; four-dimensional Euclidean lattice; stationary only at third order (coefficient 3/8, a saddle); no balanced nonnegative cost equals its germ; Chapter 5
Stress of the photon field as a source of geometry	open	
A scale factor	open	Section 10.8
Radiation temperature today	measured	$T_0 = 2.72548 \pm 0.00057$ K (COBE FIRAS recalibrated with WMAP, 2009) [45]; $\lvert y\rvert < 1.5 \times 10^{-5}$, $\lvert\mu\rvert < 9 \times 10^{-5}$ (95%, 1996) [44]

Chapter 6, or that prove them different.

4. *Scale and cooling.* The energy ϵ of one level in joules, and a law for its change as the phase grows.
5. *A thermal state.* A state of the infinite photon field (necessarily outside the reference folium, by Chapter 7) whose restrictions to finite regions are Gibbs states at one temperature, reached by the dynamics from a physical initial state, with spectral distortions below the FIRAS bounds. Chapter 9 supplies the per-tick refresh of its record cells as a premise; a closed account has no outside to return a cell to its Gibbs state, so it must produce that return from its own dynamics (Chapter 11, Problem 41).
6. *Stress and expansion.* The stress of the photon field coupled to a Lorentzian lattice action, and a law that carries records into moving frames; with it, a scale factor (Section 10.8).

With these in hand, the temperature $\epsilon/(k_B \log \varphi)$ could be followed from the first records to today and compared with T_0. That comparison is the first one with nature that the book's results can make, and it cannot be made now.

10.3 Inflation or its replacement

In the standard account a brief phase of accelerated expansion makes the observed universe flat and causally connected and seeds its fluctuations, whose spectrum has the tilt $n_s = 0.9649 \pm 0.0042$ (Planck 2018, TT,TE,EE+lowE+lensing) [87], the amplitude $\ln(10^{10} A_s) = 3.044 \pm 0.014$ (same data) and a tensor-to-scalar ratio $r_{0.05} < 0.036$ (95%, BICEP/Keck 2018 data with Planck and WMAP) [7]. A recognition account must either derive such a phase, with its duration and its spectrum, or derive a replacement that explains the same observations. Neither exists. What exists is a test of the obvious candidate potential and a set of adopted values.

The cost of a ratio $1 + q$ is a natural first candidate for a potential: it vanishes only at $q = 0$ and is positive elsewhere. Write

$$V(q) = J(1+q) = \frac{q^2}{2(1+q)}, \qquad q > -1, \tag{10.1}$$

and, in the logarithmic coordinate $t = \log(1+q)$, $V = \cosh t - 1 = J_{\log}(t)$. Slow roll needs the first two derivative ratios small: $\varepsilon = \frac{1}{2}(V'/V)^2 \ll 1$ and $|\eta| = |V''/V| \ll 1$ in the field coordinate.

Proposition 10.1 (the cost potential does not roll slowly near its vacuum)**.** *Let V be as in* (10.1).

(i) For $q > -1$, $q \neq 0$, $V'(q)/V(q) = (q+2)/(q(q+1))$. For $0 < q \leq 1$,

$$\varepsilon(q) = \frac{1}{2}\Big(\frac{q+2}{q(q+1)}\Big)^2 \geq \frac{9}{8},$$

so $\varepsilon(q)$ does not tend to 0 as $q \to 0^+$.

(ii) In the coordinate t, $\eta(t) = V''(t)/V(t) = \cosh t/(\cosh t - 1) > 1$ for every $t \neq 0$.

Proof. (i) $V'(q) = \frac{1}{2}\big(1 - (1+q)^{-2}\big) = q(q+2)/(2(1+q)^2)$, and dividing by V gives the ratio. For $0 < q \leq 1$, $(3q+4)(q-1) \leq 0$, that is $3q^2 + q - 4 \leq 0$, which is $2(q+2) \geq 3q(q+1)$; so the ratio is at least $3/2$ and $\varepsilon \geq 9/8$. (ii) $V''(t) = \cosh t$, and $0 < \cosh t - 1 < \cosh t$ for $t \neq 0$. □

Both ratios are taken in the coordinates in which the cost is defined, with the field normalized to those coordinates. No kinetic term and no normalization of a physical scalar is derived on any carrier, so the proposition excludes the cost read directly as the potential of a field in either coordinate. A different normalization rescales ε and η by a common positive factor.

Remark 10.2 (every constant normalization). The standard slow-roll formulas suggest that no constant normalization rescues the potential, though we have not proved this. Let the canonical field be $\psi = cq$ with $c > 0$. Near $q = 0$, $V \simeq q^2/2$ is quadratic, for which $r \simeq 8/N$ at N e-folds before the end; for $q \gg 1$, $V \simeq q/2$ is linear, for which $r \simeq 4/N$. A constant c moves the scale at which the observed modes leave the horizon between these regimes, and for $50 \le N \le 60$ both bounds lie above 0.036. In the coordinate t the large-field branch $V \simeq e^t/2$ is exponential, which inflates with no end and with $r = 8(1 - n_s) \simeq 0.28$ at the observed tilt. This is an inference from standard formulas, not a result of this book.

Table 10.3. Inflation or its replacement.

Item	Status	Conditions, data; where
An accelerated phase, or a replacement, on any carrier	nothing derived	
Cost of the ratio as a slow-roll potential near its vacuum	excluded	Proposition 10.1; coordinates of the cost, unit normalization
Same potential at any constant normalization	excluded by inference	Remark 10.2; standard slow-roll formulas; not proved
Golden ratio as an inflationary initial value	excluded	Proposition 10.3; plateau potential with $\alpha = \varphi^2$
$n_s = 1 - 2/\varphi^3 = 0.528$; $r = (\varphi - 1)^4 = 0.146$	excluded	by the measured n_s and $r < 0.036$
$N_e = 44$ e-folds	adopted	defined as one less than $45 = D^2(D + 2)$ at $D = 3$; no dynamics selects it
$n_s = 1 - 2/(N_e + 1) = 43/45 = 0.95556$	adopted	a defined formula at the adopted N_e; -2.22σ from Planck 2018
$r = 2/(45\varphi^2) = 0.016976$	adopted	a defined formula; below 0.036
Tilt, amplitude A_s, tensor ratio, duration, reheating	open	
Horizon and flatness	open	
n_s, A_s	measured	0.9649 ± 0.0042, $\ln(10^{10} A_s) = 3.044 \pm 0.014$ (Planck 2018, TT,TE,EE+lowE+lensing) [87]
r	measured	$r_{0.05} < 0.036$, 95% (BICEP/Keck 2018 data, 2021) [7]

A second candidate places the start of inflation at the golden ratio. Take the bounded potential $V = V_0\big((x-1)/(x+1)\big)^2 = V_0 \tanh^2(t/2)$ of the ratio $x = e^t$, with the canonical field ψ related to t by $t = \sqrt{2/3}\,\psi/\varphi$. This is the plateau form of an α-attractor with $\alpha = \varphi^2$ [66], which was proposed for the ratio.

Proposition 10.3 (the golden ratio lies past the end of slow roll)**.** *For this potential and normalization,* $\varepsilon(x) = 16/\big(3\varphi^2(x - x^{-1})^2\big)$ *for* $x > 0$, $x \neq 1$. *At* $x = \varphi$, $x - x^{-1} = 1$ *and* $\varepsilon(\varphi) = 16/(3\varphi^2) = 2.037\ldots > 1$.

Proof. $\mathrm{d}\log V/\mathrm{d}t = \operatorname{sech}^2(t/2)/\tanh(t/2) = 2/\sinh t$, so $\varepsilon = \frac{1}{2}\big(\frac{\mathrm{d}t}{\mathrm{d}\psi}\cdot\frac{2}{\sinh t}\big)^2 = \dfrac{4}{3\varphi^2\sinh^2 t}$, and $\sinh t = (x - x^{-1})/2$. Finally $\varphi - \varphi^{-1} = 1$ and $3\varphi^2 < 8$. □

So a field that starts at the golden ratio has already left slow roll: the fixed point of $r \mapsto 1 + 1/r$ (Chapter 3) cannot serve as an inflationary initial value in this model.

The adopted tilt is the only entry that meets the data at all, and it does so at $(0.95556 - 0.9649)/0.0042 = -2.22$ standard deviations. A measurement of n_s with an uncertainty near 0.002 would decide it either way; agreement would not make it derived.

What is missing. A phase derived on a recognition carrier whose duration is selected by the law, together with the amplitude, tilt and tensor ratio of the fluctuations it leaves and the reheating that ends it in the thermal state of Section 10.2; or a replacement that explains flatness, the causal connection of the observed sky and the spectrum of fluctuations without accelerated expansion.

10.4 The electroweak transition and sphalerons

Near 160 GeV the Higgs field of the Standard Model takes its vacuum value, and the anomalous processes that change baryon plus lepton number, the sphalerons, fall out of equilibrium near 130 GeV. With the measured Higgs mass near 125 GeV this is a smooth crossover, not a phase transition: lattice computations find the crossover at $T_c = 159.5 \pm 1.5$ GeV [35] and the sphaleron freeze-out at 131.7 ± 2.3 GeV [34]. No carrier of this book holds a gauge field, a Higgs field or a fermion of the Standard Model, so everything a recognition account can say about this epoch is conditional on that content.

The one exact relation available is standard. In a plasma whose processes are in chemical equilibrium, the sphalerons convert an asymmetry in $B - L$ into a fixed baryon fraction [55].

Proposition 10.4 (the baryon fraction of $B{-}L$ in a symmetric-phase plasma)**.** *Adopt the particle content of the Standard Model with N generations and m Higgs doublets, in chemical equilibrium in the symmetric phase: chemical potentials $\mu_Q, \mu_u, \mu_d, \mu_L, \mu_e$ (equal across generations) and μ_0 for the Higgs field, with Yukawa equilibrium $\mu_u = \mu_Q + \mu_0$, $\mu_d = \mu_Q - \mu_0$, $\mu_e = \mu_L - \mu_0$, sphaleron equilibrium $3\mu_Q + \mu_L = 0$ and hypercharge neutrality $N(\mu_Q + 2\mu_u - \mu_d - \mu_L - \mu_e) + 2m\mu_0 = 0$. Let $B = N(2\mu_Q + \mu_u + \mu_d)$ and $L = N(2\mu_L + \mu_e)$, in a common unit. If $B - L \neq 0$ and $22N + 13m \neq 0$, then*

$$\frac{B}{B-L} = \frac{4(2N+m)}{22N+13m}, \qquad \text{which is } \frac{28}{79} = 0.35443 \ \text{ at } N = 3,\ m = 1.$$

Proof. Yukawa equilibrium gives $\mu_u + \mu_d = 2\mu_Q$, so $B = 4N\mu_Q$. Sphaleron equilibrium gives $\mu_L = -3\mu_Q$ and then $\mu_e = -3\mu_Q - \mu_0$, so $L = -N(9\mu_Q + \mu_0)$. Substituting into hypercharge neutrality gives $N(8\mu_Q + 4\mu_0) + 2m\mu_0 = 0$, that is $\mu_0 = -4N\mu_Q/(2N + m)$. Hence $B - L = N(13\mu_Q + \mu_0) = N\mu_Q(22N + 13m)/(2N + m)$, which is nonzero exactly when $\mu_Q \neq 0$, and the ratio follows. □

Table 10.4. The electroweak transition and sphalerons.

Item	Status	Conditions, data; where
Gauge fields, Higgs field, fermions on any carrier	nothing derived	
$B/(B-L) = 28/79$	conditional	Proposition 10.4; Standard Model content and symmetric-phase equilibrium adopted
Transition temperature φ^{51} in native units	adopted	the exponent 51 is supplied; no physical unit exists for it (Section 10.2)
Sphaleron rate $\Gamma = \frac{3}{4}\alpha_W^5 T^4$	identification, contradicted	$3/4 = (3 \cdot 4)/2^2$ from Hamiltonian cycles of the tetrahedron; the lattice Standard Model gives $(18 \pm 3)\alpha_W^5 T^4$ in the symmetric phase [34], 24 times larger
A departure from equilibrium at the transition	open	a crossover supplies none [64]

What is missing. The gauge group, the Higgs field and the fermion content derived on a recognition carrier, or adopted openly as inputs; the electroweak scale in physical units, which needs the energy scale of Section 10.2; and a source of departure from equilibrium, since the transition itself supplies none.

10.5 Baryogenesis

The universe holds matter and almost no antimatter. The excess is measured as the baryon-to-photon ratio. From the Planck 2018 baryon density $\omega_b = \Omega_b h^2 = 0.02237 \pm 0.00015$ (TT,TE,EE+lowE+lensing) [87] and the conversion $10^{10}\eta_B = 273.9\,\omega_b$ [116],

$$\eta_B = (6.127 \pm 0.041) \times 10^{-10}. \tag{10.2}$$

Any account of the excess must meet Sakharov's three conditions [101]: processes that violate baryon number, violation of C and CP, and a departure from thermal equilibrium. Earlier work at the institute organized the calculation as a product of three factors,

$$\eta_B = P\,|\varepsilon_{CP}|\,D, \tag{10.3}$$

where P converts an asymmetry in $B-L$ per unit entropy into baryons per photon, ε_{CP} is the CP asymmetry of the source and D is the fraction of the asymmetry that survives washout. The factorization is bookkeeping. Each factor has its own status.

The prefactor is $P = \frac{28}{79}\, s/n_\gamma$, the baryon fraction of Proposition 10.4 times the entropy per photon $s/n_\gamma = \pi^4 g_{*s}/(45\zeta(3))$ with $g_{*s} = 43/11$, the standard count after electron-positron annihilation, which Chapter 9 obtains with the Friedmann equations as hypotheses. With $\zeta(3) \in (1.202042, 1.202065)$, $s/n_\gamma \in (7.0393, 7.0396)$ and $P \in (2.49494, 2.49505)$; to six figures $P = 2.49499$. This factor is conditional on the Standard Model content.

The survival factor $D = \varphi^{-44} = 6.3759 \times 10^{-10}$ rests on a model in which washout acts in 44 successive slots and each slot multiplies the asymmetry by $|1-\varphi| = \varphi^{-1}$, the eigenvalue of smaller modulus of the Fibonacci matrix; 44 is the number 45 of Weyl fermion species of

three Standard Model generations, less one for the direction $B - L$, which sphalerons leave unchanged. Both the per-slot factor and the number of slots are premises of the model, and no washout dynamics that meets them has been constructed. So D is adopted. The count 44 also fixes the e-folds of Table 10.3, as one less than $45 = 3^2(3 + 2)$, the dimension count of that table; the two candidates adopt their counts independently, and no result ties one to the other.

The CP asymmetry is open. Written as $\varepsilon_{CP} = J_{CP}/\mathcal{N}$ with the measured Jarlskog invariant J_{CP}, its normalization $\mathcal{N}$ is fixed by no result. Given the conditional P and the adopted D, the data require $|\varepsilon_{CP}| = \eta_B/(PD) = 0.3852$. That is an inference from the measurement, not a value of ε_{CP}. The value $\varphi^{-2} = 0.38197$ lies near it; nothing derives it.

The sign of the excess deserves its own statement, because the native law answers it. Exchanging debit and credit on an account maps the empty ledger to itself and a post to a post, and the native law cannot tell the difference.

Proposition 10.5 (the native law selects no sign)**.** *On carrier (c), under (P1)–(P4) and the three readings, let g be one of the 48 signed permutations of the three accounts (a permutation of the accounts together with an exchange of debit and credit on any subset of them), acting on imbalances $s \in \mathbb{Z}^3$. Then $K(gs, gt) = K(s, t)$ for all s, t, and the law p_n of the imbalance after n ticks from the empty record satisfies $p_n(gs) = p_n(s)$. In particular $p_n(-s) = p_n(s)$, so every function f on $\mathbb{Z}^3$ with $f(-s) = -f(s)$ and $\sum_s |f(s)|p_n(s) < \infty$ has $\sum_s f(s)p_n(s) = 0$; the mean imbalance of every account is zero at every tick.*

Proof. A signed permutation preserves the level $|s|_1$, hence the golden weight, maps holds to holds and unit posts to unit posts, and fixes the empty record. So the transported law $(s, t) \mapsto K(gs, gt)$ meets (P1)–(P4) with the same readings, and the uniqueness of K (Chapter 4) gives $K(gs, gt) = K(s, t)$. By induction, $p_{n+1}(gt) = \sum_s p_n(s)K(s, gt) = \sum_s p_n(gs)K(gs, gt) = \sum_s p_n(s)K(s, t) = p_{n+1}(t)$, using that g is a bijection and $p_n(gs) = p_n(s)$. For odd f, the terms at s and $-s$ cancel in the absolutely convergent sum. □

Reading a cosmic charge from the imbalances of recognizers would be an identification, and none has been made. Whatever it turns out to be, its sign has to come from something that breaks the 48 symmetries: an adopted bit or a coupling that is not symmetric.

What is missing. A washout model that meets the premises of D, or a derived survival factor; the normalization of ε_{CP}; a process that violates $B - L$; a departure from equilibrium; and the source of the sign.

Table 10.5. Baryogenesis.

Item	Status	Conditions, data; where
A mechanism meeting Sakharov's conditions on any carrier	nothing derived	
Factorization (10.3)	adopted	bookkeeping
$P = \frac{28}{79} s/n_\gamma \in (2.49494, 2.49505)$	conditional	Standard Model content; $g_{*s} = 43/11$ (Friedmann equations as hypotheses, Chapter 9)
$D = \varphi^{-44}$	adopted	44 slots, survival φ^{-1} per slot: premises; no model constructed
$\lvert\varepsilon_{CP}\rvert$	open	normalization $\mathcal{N}$ unfixed; data require 0.3852 given P and D
$\lvert\varepsilon_{CP}\rvert = \varphi^{-1}$ (a source of strength φ^{-45})	excluded	$\eta_B = P\varphi^{-45} = 9.832 \times 10^{-10}$, $+90\sigma$ from (10.2)
$\eta_B = (1-\varphi^{-8})^2\varphi^{-44} = 6.1074 \times 10^{-10}$	adopted candidate	the square and the factor $1-\varphi^{-8}$ were chosen to match; -0.48σ
Sign of the asymmetry from the native law	conditional (negative)	(c); (P1)–(P4) and the three readings; Proposition 10.5
Source of the sign	open	
Departure from equilibrium	open	Section 10.4
A process violating $B-L$	open	
η_B	measured	(10.2): Planck 2018 ω_b [87] with [116]

10.6 Nucleosynthesis

Between about 1 MeV and 0.07 MeV the weak interactions freeze the neutron-to-proton ratio near 1/6, neutron decay lowers it toward 1/7, and nuclear reactions then build deuterium, helium and a trace of lithium. Standard nucleosynthesis with the measured η_B reproduces the helium and deuterium abundances and overproduces lithium-7 about threefold [28]. No step of this is derived on a recognition carrier: there is no weak interaction, no expansion rate and no thermal state from which to compute a freeze-out.

What is missing. Either a derivation of the freeze-out and the reaction network from recognition carriers, or an open decision to present nucleosynthesis as standard physics fed by the measured η_B. Either way the lithium discrepancy remains unaddressed.

Table 10.6. Nucleosynthesis.

Item	Status	Conditions, data; where
Neutron-to-proton freeze-out, weak rates, reaction network	nothing derived	needs the weak interaction, an expansion rate (Section 10.8) and a thermal state (Section 10.2)
Light-element abundances	nothing derived	
Standard nucleosynthesis as an input	adopted if used	Standard Model, general relativity and the measured η_B; it predicts $^7\mathrm{Li/H} = (4.68 \pm 0.67) \times 10^{-10}$ [28]
Helium mass fraction Y_p	measured	0.2449 ± 0.0040 (metal-poor H II regions, 2015) [4]
D/H	measured	$(2.527 \pm 0.030) \times 10^{-5}$ (quasar absorbers, 2018) [26]
^{7}Li/H	measured	$(1.6 \pm 0.3) \times 10^{-10}$ (metal-poor halo stars, 2010) [102]

10.7 Recombination and the cosmic microwave background

When the plasma cooled below about 3000 K, electrons and nuclei combined into atoms, the photons decoupled, and the sound waves of the plasma froze into the acoustic peaks of the microwave sky. The first two peaks of the temperature spectrum sit at the multipoles $\ell_1 = 220.6 \pm 0.6$ and $\ell_2 = 538.1 \pm 1.3$ (Planck 2018) [88], a ratio $\ell_2/\ell_1 = 2.4393 \pm 0.0089$. No recombination epoch, transfer function or acoustic scale is derived on a recognition carrier. One proposal can be tested anyway: that the peaks sit on a golden ladder of wavenumbers, $k_{n+1}/k_n = \varphi$.

Proposition 10.6 (no golden ratio between the first two acoustic peaks)**.** *Let $\ell_1 \in [220, 221.2]$ and $\ell_2 \in [536.8, 539.4]$, the one-standard-deviation ranges of the two measured multipoles. If $\ell_1 = \mathscr{D} k_1$ and $\ell_2 = \mathscr{D} k_2$ for one $\mathscr{D} > 0$ (a common angular-distance projection of two wavenumbers), then $k_2/k_1 > 2.4 > 1.05\,\varphi$; in particular $k_2/k_1 \neq \varphi$.*

Proof. The common factor cancels: $k_2/k_1 = \ell_2/\ell_1 \geq 536.8/221.2 > 2.42$, while $1.05\varphi < 1.05 \times 1.62 < 1.71$. □

What is missing. The thermal state and the expansion of Section 10.2, then a recombination epoch, a transfer of the photon field through it and the acoustic scale; or an open decision to present the microwave background as standard physics.

Table 10.7. Recombination and the cosmic microwave background.

Item	Status	Conditions, data; where
Recombination epoch, transfer function, acoustic scale	nothing derived	
Blackbody coefficient of the photon lattice	conditional	supplied Bose occupation; Table 10.2
Peaks on a golden ladder, $k_{n+1}/k_n = \varphi$	excluded	Proposition 10.6
T_0, spectral distortions	measured	2.72548 ± 0.00057 K [45]; $\lvert y\rvert < 1.5 \times 10^{-5}$, $\lvert\mu\rvert < 9 \times 10^{-5}$ (95%) [44]
First two peak multipoles	measured	220.6 ± 0.6, 538.1 ± 1.3 (Planck 2018) [88]

10.8 Expansion

Everything in standard cosmology after the first second is written in terms of a scale factor $a(t)$ that obeys the Friedmann equations. In this book the Friedmann equations appear only as hypotheses: Chapter 9 uses them to carry entropy through electron-positron annihilation. Given Einstein's field equations for a spatially flat homogeneous isotropic metric, the first Friedmann equation $(\dot{a}/a)^2 = \kappa\rho/3$ is a theorem; the field equations are adopted, not derived on any carrier of this book.

A horizon form of the Friedmann equation connects it to records. It is an identity, and it shows what records alone do not supply.

Proposition 10.7 (the horizon-record form of the Friedmann equation)**.** *Fix $c, \hbar, k_B, \lambda_{\mathrm{rec}} > 0$ and let $G = \lambda_{\mathrm{rec}}^2 c^3/(4 \ln 2\, \hbar)$, as in (N3). For $H > 0$ put $R = c/H$, $N_H = 4\pi R^2/\lambda_{\mathrm{rec}}^2$ (the horizon area counted in cells of area λ_{rec}^2), $T_H = \hbar H/(2\pi k_B)$, $S_H = k_B \ln 2\, N_H$ (one bit per cell) and, for an energy density ρ, $E_H = \frac{4\pi}{3} R^3 \rho$. Then $H^2 N_H = 4\pi c^2/\lambda_{\mathrm{rec}}^2$, and*

$$E_H = T_H S_H \quad \Longleftrightarrow \quad H^2 = \frac{8\pi G}{3c^2}\,\rho.$$

Proof. $H^2 N_H = 4\pi c^2/\lambda_{\mathrm{rec}}^2$ is the definition of N_H. Next $T_H S_H = 2 \ln 2\, \hbar c^2/(H \lambda_{\mathrm{rec}}^2)$ and $E_H = 4\pi c^3 \rho/(3H^3)$, so $E_H = T_H S_H$ exactly when $H^2 = 2\pi c \lambda_{\mathrm{rec}}^2 \rho/(3 \ln 2\, \hbar)$, and the right side is $8\pi G\rho/(3c^2)$ for the stated G. □

The temperature T_H is the Gibbons-Hawking temperature of a de Sitter horizon [49], used here as an input, and the identity is a form of the first law at the cosmological horizon [14]. Three things in it are adopted: that temperature, one bit per cell, and the form (N3) of G. And it gives no expansion: the energy density ρ as a function of a must be supplied, and nothing in the horizon count says whether a charge of the records dilutes as a^{-1} or as a^{-3}.

A second route reads redshift as accumulated recognition: a region's rate of curved steps per tick is compared with the redshift law $1/(1+z)$. Let a supplied history assign each cell a finite list of plaquette steps, each curved or flat; for a finite region R let $n_c(R)$ and $n(R)$ count its

curved steps and all its steps, and define its complexity $Z_R = \pi n_c(R)$, its ticks $t_R = 4n(R)$ and its rate $q_R = Z_R/t_R$ (zero when $t_R = 0$).

Proposition 10.8 (no finite regional count reads the redshift law)**.** *(i)* $q_R \le \pi/4$ *for every region of every supplied history.*

(ii) The ratio q_R/q_S *of two such rates (zero when* $q_S = 0$*) is a rational number. Hence it never equals* $1/\varphi$*, the value of* $1/(1+z)$ *at* $z = \varphi - 1$*.*

(iii) If q *is any function on* $(-1, \infty)$ *with* $q(z) \le \pi/4$ *and* $q_0 > 0$*, then* $q(z)/q_0 = 1/(1+z)$ *fails for some* $z > -1$*.*

Proof. (i) $n_c(R) \le n(R)$. (ii) $q_R/q_S = \big(n_c(R)/n(R)\big)\big/\big(n_c(S)/n(S)\big)$, a ratio of rationals, and φ is irrational. (iii) As $z \to -1^+$, $1/(1+z)$ exceeds $\pi/(4q_0) \ge q(z)/q_0$. □

Part (iii) fails only toward $z \to -1$, the future; on the past, $z \ge 0$, a bounded rate can follow $1/(1+z)$, and there part (ii) is what excludes the finite counts, at the single point $z = \varphi - 1$. A continuum limit of such counts is not excluded. Nothing yet supplies one.

Table 10.8. Expansion.

Item	Status	Conditions, data; where
A scale factor $a(t)$ on any carrier	nothing derived	
First Friedmann equation	conditional	Einstein's field equations for a spatially flat homogeneous isotropic metric adopted
Entropy conservation, $(T_\nu/T_\gamma)^3 = 4/11$, $g_{*s} = 43/11$	conditional	Friedmann equations as hypotheses; Standard Model content; Chapter 9
Horizon-record form of the Friedmann equation	conditional	Proposition 10.7; Gibbons-Hawking temperature, one bit per cell and (N3) adopted; $\rho(a)$ supplied
Spread of imbalance, or births outside the ledger, as expansion	excluded	Table 10.1
Redshift law from finite regional counts	excluded	Proposition 10.8
Total reach of recognition, $\sum_{m\ge0} 8\varphi^{-m} = 8\varphi^2 = 20.944$ cells	conditional	adopted: reach of 8 cells in the first epoch, shrinking by φ in each later one
A derived redshift law; $\rho(a)$	open	
Hubble constant	measured	Section 10.11

What is missing. A scale factor constructed from the records and their growth (Section 10.2); the redshift law $1/(1+z)$ derived from it; the Friedmann equations derived from the recognition account of gravity, rather than adopted; and the energy density $\rho(a)$ of the derived thermal state, with the horizon temperature identified as the ledger's own.

10.9 Static and dynamic dark energy

The expansion of the universe accelerates. In the standard model of cosmology a cosmological constant supplies a fraction $\Omega_\Lambda = 0.6889 \pm 0.0056$ of the critical density (Planck 2018, TT,TE,EE+lowE+lensing+BAO) [87], with equation of state $w = -1$. The baryon acoustic oscillation survey DESI, combined with the microwave background and supernovae, now prefers an evolving dark energy: in the parametrization $w(a) = w_0 + w_a(1 - a)$ [18, 77], the combination with the five-year Dark Energy Survey supernovae gives $w_0 = -0.752 \pm 0.057$ and $w_a = -0.86^{+0.23}_{-0.20}$, and the constant $w = -1$ is disfavoured at 2.8σ to 4.2σ depending on the supernova sample (DESI DR2, 2025) [31]. No recognition carrier yields an energy density of the vacuum, a cosmological constant or an equation of state. What exists are candidate values with named premises, and several refutations.

Table 10.9. Static and dynamic dark energy.

Item	Status	Conditions, data; where
A vacuum energy density, Λ or $w(z)$ on any carrier	nothing derived	
$\Omega_\Lambda = 11/16 - \alpha/\pi = 0.685177$	adopted candidate	seed $11/16$ defined; shape $-\alpha/\pi$ argued; α measured; -0.66σ
Eleven forced by a budget of sixteen	excluded	passive 10 and 12 meet the same premise
$\alpha \to (1 - \varphi^{-1})/2$, giving 0.626708; its rescalings $\varphi^r \cdot 0.626708$	excluded	-11.1σ; every rescaling with $r \in \mathbb{Z}$ misses by at least 10σ
$\rho_\Lambda = \rho_P \varphi^{-2s}$ as a golden prediction	excluded	Proposition 10.9
Λ in native units, $8\pi G/(45c^4) = 2\pi\varphi^5/(45 \ln 2) = 2.234$	adopted	a definition; no bridge to the measured Λ
Smallness of Λ; its link to H_0	open	
$\mathrm{occ}(n) = \varphi^{-n}$	conditional	composition and self-similar attenuation adopted
$A = \frac{1}{2}\varphi^{-7} = 0.0172209$	conditional	plus the four-dimensional dilution identification
The closure from conservation or least cost	excluded	Proposition 10.10
$(w_0, w_a) = (-0.982779, -0.017221)$	conditional	accumulation premise; Proposition 10.11; marginally -4.05σ, $+3.66\sigma$ from DESI DR2 with DES-SN5YR
Evolving dark energy with $w \geq -1$ explaining the Planck residual	excluded by inference	wrong sign; quadrature not proved
A derived amplitude and shape	open	
Ω_Λ	measured	0.6889 ± 0.0056 (Planck 2018, with BAO) [87]
(w_0, w_a)	measured	$(-0.752 \pm 0.057,\ -0.86^{+0.23}_{-0.20})$ (DESI DR2 with CMB and DES-SN5YR, 2025) [31]

10.9.1 The static fraction

The candidate is

$$\Omega_\Lambda = \frac{11}{16} - \frac{\alpha}{\pi} = 0.6875 - 0.0023228 = 0.685177, \tag{10.4}$$

with $\alpha = 7.2973525643 \times 10^{-3}$, the CODATA 2022 value [21], as the one measured input. Against Planck 2018 the deviation is $(0.685177 - 0.6889)/0.0056 = -0.66\sigma$. The seed $11/16$ is a definition: eleven passive modes out of a budget of sixteen. The shape $-\alpha/\pi$ is argued in words from the electromagnetic coupling, not derived. The budget premise does not force eleven: a budget of sixteen split into active and passive modes is met equally by $6 + 10$ and $4 + 12$, so the count 11 needs a premise that accepts it and rejects its neighbours, and no such premise has been found. Replacing α by the number $(1 - \varphi^{-1})/2 = 0.190983$ gives $\Omega_\Lambda = 0.626708$, at -11.1σ, and every rescaling of that value by an integer power of φ also misses by at least 10σ (0.056).

A second static proposal writes the vacuum energy density as $\rho_\Lambda = \rho_P \varphi^{-2s}$, with ρ_P the Planck density and s the golden logarithm of the ratio of a cosmic length L to the Planck length. It has no golden content at all.

Proposition 10.9 (the continuous vacuum track is base-free)**.** *For every $b > 1$ and $L, \ell > 0$, $b^{-2\log_b(L/\ell)} = (\ell/L)^2$.*

Proof. $b^{\log_b x} = x$ for $x > 0$, so $b^{-2\log_b(L/\ell)} = (L/\ell)^{-2}$. □

So $\rho_P \varphi^{-2\log_\varphi(L/\ell_P)} = \rho_P(\ell_P/L)^2$, the familiar holographic estimate [22] written in golden notation, with any base giving the same number. Only an integer exponent fixed independently of L could carry recognition content, and none is fixed.

10.9.2 The dynamic part

The dynamic candidate rests on a dilution premise. Let an occupancy $\mathrm{occ} : \mathbb{N} \to (0, \infty)$ compose across independent dimensions, $\mathrm{occ}(m + n) = \mathrm{occ}(m)\,\mathrm{occ}(n)$, and attenuate self-similarly in one dimension, $\mathrm{occ}(1) = 1/(1 + \mathrm{occ}(1))$. Then $\mathrm{occ}(1)$ is the positive root of $s^2 + s - 1 = 0$, so $\mathrm{occ}(n) = \varphi^{-n}$. With the identification that the present amplitude of dark-energy evolution is the cost $J(\varphi)$ diluted through the four dimensions of spacetime,

$$A = \varphi^{-4} J(\varphi) = \tfrac{1}{2}\varphi^{-7} = \frac{13\sqrt{5} - 29}{4} = 0.0172209, \tag{10.5}$$

using $J(\varphi) = \varphi - \frac{3}{2} = \frac{1}{2}\varphi^{-3}$. Neither the self-similar attenuation nor the four-dimensional dilution is derived, and the closure cannot be obtained from the two principles one might try.

Proposition 10.10 (the closure is neither conserved nor least-cost)**.** *(i) For $s > 0$: $s = 1/(1 + s)$ if and only if $s = \varphi^{-1}$.*

(ii) For every $s \in (0, 1)$, $\sum_{k \geq 0} s(1 - s)^k = 1$: a geometric occupancy with any ratio conserves the total, so conservation selects no s.

(iii) If $s \in (0, 1]$ and $J(s) \leq J(t)$ for every $t \in (0, 1]$, then $s = 1$: least cost selects the trivial ratio.

Proof. (i) $s = 1/(1 + s)$ is $s^2 + s - 1 = 0$, whose positive root is $(\sqrt{5} - 1)/2 = \varphi^{-1}$. (ii) A geometric series with ratio $1 - s \in (0, 1)$. (iii) $J(1) = 0$ and $J(t) > 0$ for $t \neq 1$. □

The shape of the evolution needs a second premise. Call it accumulation: $w+1$ grows linearly with the scale factor, $w(z)+1 = A\,a = A/(1+z)$.

Proposition 10.11 (the equation-of-state point under accumulation)**.** *Let $A \geq 0$ and suppose $(1+z)\big(w(z)+1\big) = A$ for every $z > -1$. Then*

(i) $w(z) = w_0 + w_a\, z/(1+z)$ with $(w_0, w_a) = (-1+A,\ -A)$, so $w_0 + w_a = -1$, and $w(z) \geq -1$ for every $z > -1$;
(ii) under the continuity equation $\mathrm{d}\log \rho_{\mathrm{DE}}/\mathrm{d}z = 3(1+w)/(1+z)$, $\rho_{\mathrm{DE}}(z)/\rho_{\mathrm{DE}}(0) = \exp\big(3Az/(1+z)\big)$, which is at least 1 for $z \geq 0$ and below e^{3A}.

Proof. (i) $w(z) = -1 + A/(1+z) = (-1+A) - A\,z/(1+z)$, and $A/(1+z) \geq 0$. (ii) The right side of the continuity equation is $3A/(1+z)^2$, whose integral from 0 to z is $3Az/(1+z)$, which lies in $[0, 3A)$ for $z \geq 0$. □

At A of (10.5), $(w_0, w_a) = (-0.982779, -0.017221)$, $w(1) = -0.991390$ and ρ_{DE} stays below $e^{3A} \approx 1.05302$ times its present value. The point lies within 0.02 of the constant $w = -1$ in each coordinate. Against the DESI DR2 combination with the five-year supernovae its marginal deviations are $(-0.982779+0.752)/0.057 = -4.05\sigma$ in w_0 and $(-0.017221+0.86)/0.23 = +3.66\sigma$ in w_a. These are marginal numbers on strongly correlated parameters. A joint comparison needs the survey's likelihood, and none has been made. The candidate makes two sharp statements: $w_0 + w_a = -1$, and w never crosses -1. The DESI combinations prefer $w_0 + w_a < -1$, a crossing in the past; if that is confirmed, the accumulation premise is excluded.

The point was written after the DESI DR2 results appeared, so it is not blind to them. One more use of the dynamic part fails. The static value (10.4) lies below the Planck central value, and evolving dark energy with $w \geq -1$ was proposed to account for the difference. By Proposition 10.11(ii) such dark energy was denser in the past. A numerical quadrature gives an effective fraction below the static value rather than above it. So this explanation has the wrong sign. The quadrature is not proved, so we record the exclusion as an inference.

What is missing. A premise or derivation that selects eleven and rejects ten and twelve, and a derivation of the $-\alpha/\pi$ shape with α itself, which Chapter 9 records as a boundary datum today; a natural scale for Λ that is not circular, and its link to H_0; a physical derivation of self-similar attenuation and of four-dimensional dilution; the accumulation premise derived, adopted by an explicit decision, or tested against the survey likelihoods.

10.10 Dark matter

About a quarter of the critical density is matter that does not interact with light: Planck 2018 gives $\omega_c = \Omega_c h^2 = 0.1200 \pm 0.0012$ and $\omega_b = 0.02237 \pm 0.00015$ (TT,TE,EE+lowE+lensing) [87]. No recognition carrier yields a dark component, its density, its clustering or its particle content.

The candidate value is a sum of two terms. The second is a small gap,

$$\delta = \frac{J(\varphi)}{12\varphi} = \frac{\varphi^{-4}}{24} = 0.0060791, \tag{10.6}$$

an identity by $J(\varphi) = \frac{1}{2}\varphi^{-3}$. It is fixed once one adopts a structure of 24 phases that share equally a total occupancy $\mathrm{occ}(4) = \varphi^{-4}$; that total uses the dilution premises of Section 10.9, so the dark-matter gap and the dark-energy amplitude stand or fall together. The candidate is

$$\Omega_{\mathrm{dm}} = \sin\frac{\pi}{12} + \frac{\varphi^{-4}}{24} = 0.258819 + 0.006079 = 0.264898.$$

The term $\sin(\pi/12)$, the additive form and the identification with the observed density are adopted, and the form was selected among about thirty closed forms after comparison with data. At the Planck value $h = 0.6736$, $\Omega_{\mathrm{dm}}h^2 = 0.12019$, which is $+0.16\sigma$ from ω_c. A second adopted ratio, $\Omega_{\mathrm{dm}}/\Omega_b = 2\varphi + 2 = 5.236$, lies -1.99σ from $\omega_c/\omega_b = 5.3643 \pm 0.0646$, with the two uncertainties combined as independent.

Table 10.10. Dark matter.

Item	Status	Conditions, data; where
A dark component on any carrier; its clustering	nothing derived	
$\delta = J(\varphi)/(12\varphi) = \varphi^{-4}/24$	derived identity	(10.6)
δ as the share of one of 24 phases	conditional	24 equal phases and total occ(4) adopted; dilution premises of Section 10.9
$\Omega_{\mathrm{dm}} = \sin(\pi/12) + \varphi^{-4}/24 = 0.264898$	adopted candidate	first term, sum and identification chosen; selected after comparison; $+0.16\sigma$
$\Omega_{\mathrm{dm}}/\Omega_b = 2\varphi + 2 = 5.236$	adopted candidate	-1.99σ
ω_c, ω_b	measured	0.1200 ± 0.0012, 0.02237 ± 0.00015 (Planck 2018) [87]

What is missing. What dark matter is on a recognition carrier; a density derived from it rather than an adopted sum, with the 24 phases and the dilution premises derived or abandoned; and its gravitational clustering, from halos to the rotation of galaxies.

10.11 The Hubble ratio, σ_8 and structure

Two late-time measurements disagree with the standard model's fit to the early universe. The Hubble constant inferred from the microwave background, $H_0 = 67.36 \pm 0.54$ km s^{-1} Mpc^{-1} (Planck 2018, TT,TE,EE+lowE+lensing) [87], differs from the distance-ladder value 73.04 ± 1.04 (SH0ES, 2022) [98] by $5.68/\sqrt{1.04^2 + 0.54^2} = 4.8\sigma$; their ratio is 1.0843 ± 0.0177 with independent errors. The clustering amplitude $S_8 = \sigma_8\sqrt{\Omega_m/0.3}$ is 0.832 ± 0.013 from Planck 2018 and $0.759^{+0.024}_{-0.021}$ from the KiDS-1000 cosmic shear survey (2021) [3], a difference of $0.073/\sqrt{0.013^2 + 0.024^2} = 2.7\sigma$.

No value of H_0, of the ratio, of σ_8 or of S_8 is derived on any recognition carrier, and neither is a growth equation, a transfer function, a matter power spectrum, a reionization redshift or a lensing amplitude. Closed forms have been written for the Hubble ratio; each is a definition

or a fit, they disagree with one another, and we do not use them. One counting route can be closed. It would take the 13 of a ratio 13/12 from the twelve edges of the cube plus the tick, and the 11 of the seed 11/16 from the twelve edges less the tick. Reading the tick as one of the twelve edges gives 12 and 11; reading it as a thirteenth axis gives 13 and 12. No single reading of the tick gives both numbers.

Table 10.11. The Hubble ratio, σ_8 and structure.

Item	Status	Conditions, data; where
H_0; the early-to-late ratio	nothing derived	closed forms proposed for the ratio are definitions or fits
13 and 11 from one reading of the tick	excluded	edge counts $(12, 11)$ or $(13, 12)$
σ_8, S_8, growth, transfer function, power spectrum	nothing derived	
Reionization, lensing, voids, gravitational-wave background	nothing derived	
H_0	measured	67.36 ± 0.54 (Planck 2018) [87]; 73.04 ± 1.04 (SH0ES, 2022) [98]
S_8	measured	0.832 ± 0.013 (Planck 2018) [87]; $0.759^{+0.024}_{-0.021}$ (KiDS-1000, 2021) [3]

What is missing. Everything that turns an expanding background into structure: a derived matter content and its initial fluctuations (Sections 10.3 and 10.10), a growth law, and from them σ_8, the power spectrum and the late-time expansion rate. A ratio of Hubble rates can be judged only once both rates are derived.

10.12 The order in which the gap closes

Every epoch from inflation onward needs the objects of Section 10.2: a space in which distances grow, an interaction that moves energy between recognizers, an energy scale in physical units, a thermal state and a stress. The electroweak epoch, baryogenesis and nucleosynthesis also need the particle content of the Standard Model, derived or openly adopted; the microwave background needs recombination and a transfer function; the late universe needs a dark sector and a growth law. So dependence fixes the order of work: the interaction law and a derived space first; growth and the energy scale next; then a thermal state, its stress and the expansion; and only then the epochs of the standard account, beginning with the first comparison with nature, the temperature T_0. The candidate values of Sections 10.5 to 10.10 become predictions only if the premises that carry them are derived on the way.

Chapter 11 states each open problem of this chapter and of the earlier ones exactly, with what would settle it.

Chapter 11

Open problems, stated exactly

This book stops where its proofs stop. This chapter states each place where a further step is needed as a problem that a mathematician could try to prove or refute, with the result where it arises and what would settle it.

Start with the first post. Under (R1) and (R2), on a carrier on which the post's bond is a bridge of weight one, the first event with its reciprocal costs $\sqrt{5}-2$ (Theorem 3.15). On the possibility graph of the 27-site torus the same bond lies on cycles of length four (Theorem 3.27), and the price of the first tick lies in $(0.025686, 0.025743)$ (Corollary 3.56). Both computations are exact; which carrier holds the field of the first post is open. A theorem tying the bond weights of the field's carrier to recorded posts would give the first price, and one selecting the full neighbour support the second. Every entry below has that shape.

Three kinds of answer settle a problem: a proof, a counterexample, or a proved underdetermination, two models that satisfy every accepted premise and differ on the question, naming the premise still missing (as Theorems 2.18 and 8.6 do for the beginning and the joint law). The problems run in physical order; a correspondence between two carriers is a problem too (Problem 22). The next derivations begin at Problems 13, 34, 36 and 43 (Section 10.12).

11.1 The foundation

Problem 1. *Three dimensions without a supplied linking correspondence.* (a) Construct a spatial realization whose configurations are the pairs of disjoint embedded circles in S^D up to deformation, and prove the linking correspondence on it (by Alexander duality) as a theorem, not as part of the realization. (b) Prove that the physical carrier of recognition is a realization with the linking correspondence and the deformation-erasure condition. *Arises in* Theorem 1.11, whose physical premises these are. *Settled by* (a) the proved construction, which removes one premise from $D = 3$; (b) a theorem identifying the carrier, or a carrier that meets the starting facts and fails one of the conditions, leaving $D = 3$ conditional on them.

11.2 Before the first event

Problem 2. *The books as the tally of a record.* On the account ledger with integer counts, every count is a count of posts made. This is false as a universal statement: the all-(-1) ledger has the balances of the empty ledger and the same posting steps shifted by -1, yet it is the tally of no record. It is equivalent to the opening premise for the actual books, and no shift-invariant condition coincides with it (Theorem 2.18). *Settled by* a law that reads absolute counts and implies the opening premise, derived from accepted premises, or a justified premise that the record is primitive and the books are its tally.

Problem 3. *Which first post, which genesis atom.* At $d = 3$ there are six first posts, and distinct first posts give distinct first records (Proposition 2.19); on the boundary-theory carrier no translation-closed rule selects an atom (Theorem 2.32). *Settled by* a selection principle derived from accepted premises, or by adopting as a stated premise the reading that every lawful history is carried at once (Section 4.1).

Problem 4. *What prepared the start.* On the native history dilation and on the source and recorder carrier: (a) the empty-record state has a reversible predecessor that lies outside the recorded histories (Theorem 2.24); is the physical state the first state of the recorded carrier or its backward extension? (b) Reading out or recording the seed's coordinate raises the energy by exactly $7E_{\mathrm{nat}}$ (Proposition 2.31), no closed unitary returns the carrier to blanks (Theorem 2.28), and the controller's remainder is not attributed to any subsystem (Theorem 6.20): which subsystem pays for the controller, the contact rate, the fresh blanks and the initial state? (c) Equal energy leaves blank readiness open at every time (Theorem 2.28); what selects it? *Settled by* a derivation of each from accepted premises, or two proved models that differ on it.

Problem 5. *A duration between postings.* Bounded continuous unitary groups through the posting unitary of the history dilation exist, and none keeps the recorded histories between postings (Proposition 2.25). Does an unbounded generator, or a dynamics with intermediate states off the recorded carrier, give the step a duration with physical content? *Settled by* such a construction, or a no-go for a named class of generators.

Problem 6. *One law selecting the initial state.* Find one enacted state law that joins retained history, spatial support, local fields, a common clock and identified energy, and selects the initial source, its attachment and its population. Energy alone selects neither a blank (Theorem 2.26) nor the coherent seed (Proposition 2.27), and the source and recorder carrier has no homogeneous normalized source (Proposition 2.29); which carrier (finite volume, another representation, local observables) can host a homogeneous universe is part of the problem. *Settled by* such a law, or a proved underdetermination naming the missing premise.

Problem 7. *The genesis state in boundary theory.* On the boundary-theory carrier of Section 2.9: which genesis state is physical; whether a physical superselection law exists; and whether the event algebra is the algebra of admitted physical operations, an identification made in [93]. Structure cannot select the sector, and the admitted operations cannot detect it. *Settled by* genesis data from a further premise, or by deriving the admitted operations.

11.3 The first event

Problem 8. *The carrier of the first post's field.* Under (R1) and (R2), does the field of the first post live on its record carrier, one bond of weight one, or on a carrier in which that bond lies on a cycle? *Arises in* Remark 3.57. *Settled by* a theorem tying the bond weights of the field's carrier to recorded posts (the price is then $\sqrt{5} - 2$ with no carrier condition left) or one selecting the full neighbour support (the golden price is then excluded, and on the 27-site torus the price is the bracket above). Any proposed carrier must pass the test of Corollary 3.44: the conductance between the post's ends lies in $[1, 1/(2\log\varphi))$.

Problem 9. *Deriving (R1) and (R2).* Derive how a post attaches to a field: the post's count as the source strength of the exact-cost Gauss law and its strain as the drop across its own bond. A competing selection, of the full minimum-cost support, would exclude the golden value of (R2). *Arises in* Section 3.3. *Settled by* a derivation of the attachment from accepted premises,

or a proof that the premises leave it free.

Problem 10. *The whole first action.* Does the action of the physical carrier at the first event equal, at every field, the cost of the first event's record? By Theorem 3.19(iii) the equality would select the pair graph; by Proposition 3.20 it fails for the whole period-three graph. A related problem is to attach the actual first event to a field: its sites, its support and how its post enters the source. On one bond the first tick has net flow zero at every site, while at the golden ratio its field has a unit source; equal scalar costs do not identify these two readings. *Settled by* a derivation of either identification, or two models that differ on it.

Problem 11. *Questions about the exact-cost Gauss law.* On a finite weighted graph with the law $2\sum_j w_{ij}\sinh(\phi_i-\phi_j) = m(\delta_{ia}-\delta_{ib})$: (a) how several posts compose under the source reading (a single bond with $m=2$ has drop arsinh 1, not $2\log\varphi$; Proposition 3.16); (b) whether strengthening a bond can raise the drop on a carrier of at most four sites (it can on six sites and cannot on series-parallel carriers; Theorem 3.48); (c) whether it can at unit strength inside the golden window when every weight is a whole number, as the post counts of a record are; (d) a lower bound on a general carrier that matches the split ceiling (on the 27-site torus the drop lies within 0.000174 of it; Corollary 3.56). *Settled by* proofs or counterexamples.

Problem 12. *The resistance at the centre of a box.* On the cubic box $\{0,\dots,L-1\}^3$ with unit bonds, the effective resistance across the bond at the centre decreases, as L grows, to a limit of at least $\frac{1}{3}$ (Proposition 3.40). Is the limit exactly $\frac{1}{3}$, the value on the infinite lattice? *Settled by* a proof of convergence to the infinite-lattice value, or a bound separating the two.

11.4 Time

Problem 13. *Deriving the premises of the native law.* Derive (P1)–(P4) and the three readings (the state is the imbalance, a tick may hold, the weight depends on the level) from the recognition law. The golden ratio enters only through (P3), each premise is load-bearing (Proposition 4.18), and every result stated under K in Chapters 4, 8 and 9 carries them. *Settled by* a theorem deriving detailed balance against $\varphi^{-|s|_1}$ and holding-and-return balance from accepted premises, or a proved underdetermination naming the premise that is missing.

Problem 14. *The structure of K.* (a) A closed form for the hold amplitude u of Theorem 4.17, or a proof that none of a stated kind exists. (b) Along an axis, do the holds alternate: for odd $n \geq 1$, is $h(ne_1)$ below both $h((n+1)e_1)$ and $h((n+2)e_1)$, with the reverse for even n? Certified enclosures decide the first distances; the general statement is open. (c) The exact spectral gap of K, now in $[0.00836, 0.008398]$ (Theorem 4.22), and its exact long-run rate. (d) The largest information a voxel law can shed in two ticks from balance, which lies strictly below $\log(31-12\varphi)$ (Theorem 4.31). (e) Proofs of the computed first-completion values under K (Table 4.4), such as the mode 27 and the median 33. *Settled by* closed forms, certified sharp values or proofs of the computed ones.

Problem 15. *The register's version.* The amplitude register's tick has eight versions, all obeying every stated law; a full eight-tick return singles out the minimal version and unitarity on balanced registers the full one (Proposition 4.12). *Settled by* a physical requirement, derived from accepted premises, that selects one version.

Problem 16. *A physical clock.* Every clock in this book is an order, a chart, a local reading or a stretched native tick: rank is a chart, there is no global serial clock (Proposition 4.34), refinement fixes increments only up to a base unit (Proposition 4.35), and no clock built from

K fixes a tick length. *Settled by* a derived unit of elapsed time; the premises (N1)–(N3) of Chapter 9 supply one by adoption, which does not settle it.

11.5 The first spacetime cell

Problem 17. *The cell's premises from the ledger.* Derive from the ledger either name-blindness (all four postings read equally) with light along a history, or, equivalently by Theorem 5.10, space from the cost with one voxel per tick. Either pair would make Theorems 5.5 and 5.6 consequences of the recognition account; until then reading single postings as light is an identification. *Settled by* a theorem that the ledger's interval for the cell satisfies them.

Problem 18. *A Lorentzian match that treats the axes unequally.* Is there a quadratic form Q on $\mathbb{R}^4$ such that, for records v, w of the cell, w extends v exactly when $Q(w - v) \leq 0$ and rank $v \leq$ rank w? No name-blind form does (Proposition 5.9). *Settled by* an example or a no-go proof.

Problem 19. *The action on the record interval.* (a) Is flatness a saddle of the Lorentzian Regge action [105] on λQ_1 along the body path, the Lorentzian counterpart of Theorem 5.16? (b) Compute the static operator of Section 5.6 with the time axis on the main diagonal. (c) Derive the gravitational action from recognition and link it to the cost through quantities that take both signs; balanced nonnegative costs are excluded along the body path (Theorem 5.18). (d) On the closed Lorentzian Regge torus, prove that every zero mode of the flat second variation is a constant strain plus a vertex move, the premise of the closed-torus result of Section 5.9. *Settled by* the computations and proofs, or counterexamples.

Problem 20. *Filling the cone.* A law that carries records into moving frames, or displacements of both signs, would let the cell's histories fill its light cone (Proposition 5.9). *Settled by* such a law derived from accepted premises.

Problem 21. *A spacetime of many cells.* Take the empty record through the first post into a spacetime of many record cells with a causal reading. *Arises in* Section 5.10. *Settled by* a construction, which needs Problems 17 and 22.

11.6 The first interacting dynamics

Problem 22. *Correspondences between the carriers.* For a pair of the carriers of Appendix A, a map of one carrier's objects and dynamics onto the other's (an intertwining isometry, or a correspondence of records and steps) with a theorem that carries a stated class of results across; only negative correspondences exist (Propositions 6.5 and 2.30). The pairs the book needs most are: the record cell and the three-account ledger; the record cell's cone and the dispersion of the clock-photon lattice (the lattice photon read on the cell has $\kappa_\gamma = \frac{1}{2}$ and a Euclidean action, and any match at $\kappa = 1$ needs the factor $\sqrt{3}$, an identification; Theorem 5.15); a contact step of the recognizer laws and a step of the clock-photon lattice; and the boundary-theory record and the imbalance carrier, whose two arrows are stated side by side only. *Settled by* such theorems, or no-go results for named classes of maps.

Problem 23. *The atomic source in the cell carrier.* An embedding of the atomic source carrier into the cell carrier of the clock-photon lattice that intertwines the actual source dynamics with the regional dynamics, or a proof that none exists; the fixed-site charts and the marked encoding fail (Proposition 6.5). *Settled by* the embedding or the no-go.

Problem 24. *The reference vector for physical profiles.* The regional Hamiltonians move the clock ground state by an amount whose square grows at least linearly in the number of cells, for the unit-current families (Proposition 6.32, Theorem 6.34), so the reference vector has no limit (Theorem 6.31). Does the divergence hold for every physical source profile? *Settled by* a proof for a stated class of profiles, or a profile for which it fails.

Problem 25. *Work and matching laws.* Identify the work vector $w_B(u, p)$ of Theorem 6.15(iii) with the work exchanged in an actual displacement of the field, and derive the matching laws (vacuum action, posting work) that the theorem assumes; unbounded or domain-changing corrections lie outside the theorem. *Settled by* the derivation.

Problem 26. *Spectators under the full evolution.* The clock term and one hop of the source move amplitude out of the sector in which both spectators sit in the clock ground state (Proposition 6.21). Does the full coupled evolution leave that sector at every small time $t \neq 0$? *Settled by* a proof or a counterexample.

Problem 27. *Placements that use part of a direction.* For an edge assignment that leaves edges of a used direction empty (a sublattice, for instance), do the couplings to infinitely many *occupied* edges survive, and with them Theorems 7.34 and 7.46? *Settled by* a coupling count on the occupied edges or a counterexample.

Problem 28. *Which patterns hide.* Gradient flip patterns have zero transverse profile. Is every finitely supported current with zero transverse profile a gradient, equivalently of zero circulation around every plaquette? Theorem 7.47 does not need it. *Settled by* a proof or a current that is not a gradient and still has zero profile.

Problem 29. *The invariant states of the clock-off dynamics.* With the clock off, τ_t maps $\mathfrak{A}_\pi$ onto itself, and the long-time limit points of $\omega_{\mathrm{ref}} \circ \tau_t$ are invariant states, unitarily implemented, with the register values of Theorem 7.48 (Theorem 7.49). Open: (a) uniqueness of the limit points; (b) their values on the displacements $\widehat{W}(h)$, the long-time limit of the photon field the register clouds radiate; (c) their values on mixed products; (d) strong continuity of $t \mapsto U_t$; (e) whether $\omega_{\mathrm{ref}} \circ \tau_t$ and ω_{ref} are disjoint, which Theorem 7.56 does not decide; (f) all of these, and the invariance of $\mathfrak{A}_\pi$, for $\epsilon > 0$. *Settled by* proofs or counterexamples.

Problem 30. *The clock on.* With the clock on, the evolved reference state exists and moves continuously (Theorem 7.20), and its long-time limit points are states the dynamics preserves on the algebra $\mathfrak{D}_\epsilon$ it generates (Proposition 7.50), but the clock moves registers between occupations and breaks the conservation of gradient patterns. Which states are these, do the time averages of $\omega_{\mathrm{ref}} \circ \tau_t$ converge, and does dephasing survive the flips? *Settled by* proofs, or a counterexample at some clock strength.

Problem 31. *Bare profiles and continuity.* Does a bare current profile z_p fail the test of Definition 7.13, so that $\widehat{W}(z_p)$ has no evolution in norm on the whole lattice (the expected mechanism is the dipolar tail $|x|^{-3}$ of its couplings)? Norm continuity in t fails for Weyl operators; does a resolvent-type algebra [12] carry a continuous dynamics here? *Settled by* proofs or counterexamples.

Problem 32. *Rates.* The register coherences vanish on time average. Determine their decay at each time, the fall-off of the static couplings with distance (expected as the inverse cube, with a dipolar angular factor), and the decay of the register kernel $\mathscr{K}_T(p, q)$ with the distance between edges; exponential decay of the kernel would give the register dynamics a light cone. *Settled by* bounds with proofs.

Problem 33. *The physical state and the supplied data.* Is ω_{ref}, or another state on $\mathfrak{A}_\pi$, the physical state of the lattice, and from which state does the first interacting dynamics start?

The dynamics of Chapters 6 and 7 select no profile, reference state, placement, scale (a, C_E, ϵ), initial data or unit of time; all are supplied. *Settled by* a derivation of each from accepted premises, or a proved underdetermination.

11.7 Many recognizers

Problem 34. *The requirement that selects a coupling.* Find a physical requirement, beyond giving each recognizer the native law K, that fixes the joint law of a finite group of recognizers. A coupling or conserving law breaks joint detailed balance or joint holding-and-return balance (Theorem 8.7); a shared thermal cell selects no coupling (Proposition 9.15); records, energy operators and the cost minimum select none (Section 8.6). Candidates include holding-and-return balance read at each recognizer's own local ticks and a two-tick priced law on walks. Part of the problem is a choice of specification: whether the target is required at every state or only at reached states, since the every-state form excludes every conserving law (Proposition 8.2). *Settled by* a derived requirement that selects one joint law, or a stated family, on every finite set of recognizers.

Problem 35. *Conserving laws on odd numbers.* No conserving law with native marginals exists on one, three, five or seven recognizers (Proposition 8.11, Theorem 8.13), so either no odd number of recognizers carries one or every odd number from some $N \geq 9$ on does. Does any odd number of recognizers carry such a law? *Settled by* a law on nine recognizers, or an exclusion for every odd number.

Problem 36. *Placement, distance and an isotropic cone.* Derive the placement of recognizers, a distance and a light cone from the recognition law, including the touching premise (L). Under a supplied isolated-pair law the cone is exact and octahedral, with axis speed one and body-diagonal speed $1/\sqrt{3}$ (Theorem 8.28); an isotropic cone needs more. *Settled by* a theorem placing recognizers with a cone.

Problem 37. *Free locking with loops.* When every equal-ended contact may lock, the law on a cycle reads arbitrarily far, while on forests with weak locks a cone holds approximately at every time (Section 8.8). Does free locking on contact graphs with loops have a light cone, exact or approximate? *Settled by* a cone bound or a counterexample graph.

11.8 The hot universe

Problem 38. *A countable record for the contact law.* For $b > 0$, is there a countable cell C with energies E_C and $\sum_c e^{-E_C(c)} < \infty$, a real function E on profiles of two recognizers, and a bounded, energy-conserving, one-joint-move operator on $\ell^2(\mathbb{Z}^3 \times \mathbb{Z}^3 \times C)$ whose one-tick read is the contact law R_b? *Arises in* Remark 9.18; no finite cell does (Proposition 9.17). *Settled by* a construction, or a proof that the unbounded cell energies forced by Proposition 9.17 contradict boundedness or $Z_C < \infty$.

Problem 39. *Couplings through heat.* (a) Does a shared thermal cell without a conserved label, or with exchange terms that do not commute, produce a coupling other than a common cause? (b) Does a bounded, non-additive interaction energy U admit a joint law with K marginals in detailed balance against the weights W_U? *Arises in* Section 9.5. *Settled by* (a) a classification of shared-cell reads with K marginals; (b) a construction, or an extension of the finite-support argument of Proposition 9.16 to bounded U.

Problem 40. *Deriving the thermal premises.* Derive the energy reading (the golden weights are Boltzmann factors), the thermal record cell with its per-tick refresh (Problem 41), and with them (Problem 13) the premises of *K*. *Arises in* Sections 9.1 and 9.2. *Settled by* the derivations.

Problem 41. *A closed universe that refreshes its own cells.* The read (9.3) is the law of one tick from the cell's Gibbs state; Chapter 9 makes it the law of every tick by adopting the per-tick refresh. A cell returned to its Gibbs state before every tick is an external bath, and a closed universe has no outside to do the returning. Find a closed carrier containing recognizers and record cells, under a law derived from the recognition law, whose own evolution returns each cell's marginal to its Gibbs state, exactly or within a stated bound, before each tick; or prove that no such carrier exists for the native law, and state which multi-tick thermal statements of Chapter 9 survive without the refresh. *Arises in* Section 9.2. *Settled by* either answer.

Problem 42. *Growth.* Every law that meets the native target acts on a fixed finite set of recognizers, so none adds a recognizer, and none changes a distance, since distance is not yet defined. Derive a lawful process that adds recognizers or increases distances. The spread of an imbalance as it relaxes is not one (Proposition 8.20). *Settled by* such a process, derived from the conditions of Chapters 4 and 8.

Problem 43. *The energy scale and cooling.* Express the level energy ε in joules and derive a law for its change. With the golden weights fixed, the temperature $\varepsilon/\log\varphi$ changes only through ε (Corollary 9.4), so a phase that cools as it grows needs ε to fall. *Settled by* the derivation; it is also the first step of Problem 54.

Problem 44. *A thermal state of the field.* Derive the Bose occupation that Theorem 9.20 supplies, from the dynamics of the infinite photon field: a state outside the reference folium whose restrictions to regions are Gibbs states at one temperature, reached from a physical initial state, with distortions below the measured bounds. No Fock-space density operator reads as thermal (Theorem 9.21). *Settled by* the construction, which needs Problems 29 and 33.

11.9 The gap to the expanding universe

Each entry is a row of Chapter 10 marked "nothing derived", "adopted" or "conditional"; its tables give the data and the excluded routes.

Problem 45. *Stress and expansion.* Couple the stress of the photon field to a Lorentzian lattice action, and construct a scale factor $a(t)$ from records; derive the Friedmann equations, inputs of Proposition 9.22, from the recognition account of gravity, with $\rho(a)$ of the derived thermal state and the redshift law $1/(1+z)$ derived. Every native regional-aging candidate tested so far fails (Proposition 10.8). *Settled by* the derivations, which need Problems 20, 36 and 44.

Problem 46. *Inflation or its replacement.* Derive a phase with a selected duration, amplitude A_s, tilt n_s, tensor ratio r and reheating, or a replacement that accounts for flatness, the horizon and the spectrum. The cost potential does not roll slowly near its vacuum, with $\varepsilon \geq \frac{9}{8}$ for $0 < q \leq 1$ (Proposition 10.1); that no constant normalization rescues it is an inference from standard formulas (Remark 10.2) and is open as a theorem: for $V(q) = J(1+q)$ with canonical field $\psi = cq$, every $c > 0$ and $50 \leq N \leq 60$ e-folds, is $r = 16\epsilon > 0.036$? The adopted tilt $43/45$ lies 2.22 standard deviations below Planck 2018; a measurement of n_s to about 0.002 decides it. *Settled by* the derivation, and the slow-roll statement by a proof.

Problem 47. *The electroweak epoch.* Derive, or adopt openly, the gauge group, the Higgs field and the fermion content; express the electroweak scale in physical units; and find a departure

from equilibrium, which the measured crossover does not supply. *Settled by* the derivations or an explicit adoption with its consequences.

Problem 48. *Baryogenesis.* In $\eta_B = P|\varepsilon_{CP}|\,D$: a washout model meeting the premises under which $D = \varphi^{-44}$, or a derived survival factor; the normalization $\mathcal{N}$ in $\varepsilon_{CP} = J_{CP}/\mathcal{N}$; a process violating $B - L$; a departure from equilibrium; and the source of the sign, which K does not supply, since K gives every odd observable zero mean (Proposition 10.5). *Settled by* derivations of each factor; until then the candidate values of Section 10.5 are not predictions.

Problem 49. *Nucleosynthesis and the microwave background.* Derive the neutron-to-proton freeze-out and the reaction network on a recognition carrier, or present nucleosynthesis as standard physics fed by a measured η_B; the lithium discrepancy stays open either way. Derive the recombination epoch, the transfer function and the acoustic scale, or present them as standard physics; no common projection gives a golden spacing of the first two acoustic peaks (Proposition 10.6). The first comparison with nature is the radiation temperature today, $T_0 = 2.72548 \pm 0.00057$ K, after Problems 43 and 45. *Settled by* derivations or a decision.

Problem 50. *Dark energy.* (a) A premise that selects the static seed $11/16$ over its rivals on the same budget of sixteen channels, and a derivation of the correction $-\alpha/\pi$ and of α. (b) A non-circular scale for Λ and its link to H_0. (c) A physical derivation of the self-similar attenuation and of the four-dimensional dilution behind the amplitude $A = \varphi^{-4} J(\varphi)$, and of the accumulation premise behind the point $(w_0, w_a) = (-0.982779, -0.017221)$ (Proposition 10.11). (d) A joint comparison of that point with the DESI DR2 likelihood; the marginal deviations -4.05 and $+3.66$ standard deviations treat correlated parameters one at a time. (e) A certified evaluation of the effective fraction that evolving dark energy with $w \geq -1$ gives, which by a numerical quadrature lies below the static value and so cannot explain the Planck residual. *Settled by* the derivations, the likelihood evaluation and a proof of the sign.

Problem 51. *Dark matter.* What dark matter is on a recognition carrier; a derived density, with the premises behind the gap $\varphi^{-4}/24$ (twenty-four phases, dilution) derived or abandoned; and clustering from halos to rotation curves. *Settled by* the derivations; until then the candidate sum of Section 10.10 is a model.

Problem 52. *The late universe.* The matter content and the initial fluctuations, a growth law, σ_8, the power spectrum and H_0; a ratio of two expansion rates is meaningful only once both are derived. *Settled by* the derivations.

Problem 53. *The killed expansion routes as theorems.* Births of recognizers on a schedule outside the ledger, read as expansion, gave at most 0.231 e-fold and no exit to deceleration in a recorded run (Table 10.1). *Settled by* a proof of that bound for the stated class of schedules.

11.10 Physical scale

Problem 54. *The absolute scale.* Derive from the recognition law one pure number that ties a laboratory process to the tick. The first target is the electron's golden level $\gamma_e = \log_\varphi(m_e c^2 \tau_0/\hbar)$, which measured values within one standard uncertainty place in $(-106.019763, -106.019703)$ under (N1) and (N3); with the mass law kept the target is $\eta \in (\varphi^{-145}, \varphi^{-144})$. By Proposition 9.23 the derivation needs a premise that is not invariant under rescaling. *Arises in* Remark 9.31. *Settled by* the derivation, which would make G an output.

Problem 55. *The native constants and α.* Derive (N1)–(N3), and the action s per unit of cost and the ratio λ that fix the coupling $s\lambda/(4\pi)$ of a unit charge; the admissible readings leave both open (Proposition 9.30), so α enters as its measured value. *Settled by* the derivations.

Appendix A

Vocabulary and the carriers

Two prices appear early in the book. Under the premises (R1) and (R2), on a graph in which the first post's bond is a bridge of weight one, the first event costs $2J(\varphi) = \sqrt{5} - 2$ (Chapter 3). Reading out or recording the coordinate of the coherent seed raises the energy by exactly $7E_{\mathrm{nat}}$, and the ready branch carries $7E_{\mathrm{nat}} + \ell(\mathrm{rd})$ (Proposition 2.31). Both are prices of starting something, and it is tempting to add them or convert one into the other. Nothing licenses that. The first is a theorem about a weighted graph with a static field equation, in units of cost; the second is a theorem about a Hilbert space carrying a moving charge, four clock registers and a photon field, in native energy units. No theorem maps one construction onto the other, so the two numbers sit side by side and are never chained.

This appendix is the reference for that discipline. Section A.1 fixes the words and flags letters with more than one meaning. Section A.2 defines each carrier by its objects, its dynamics, its supplied data and the chapters that use it. Section A.3 says, pair by pair, what is known about relating two carriers: no proved relation carries results from one carrier to another, except where one carrier is built on the other by definition or a named premise joins them. Section A.4 lists further results that look combinable and are not, and Section A.5 tabulates every premise and identification with the chapter where it enters.

A.1 Words with a fixed meaning

Each chapter keeps the notation of the work it reports, so some letters do double duty. The context fixes the meaning; Table A.1 lists the cases a reader moving between chapters will meet.

Table A.1. Letters with more than one meaning.

Letter	Meanings
κ	the parameter of the record cell's interval λQ_κ (Ch. 5); the contact rate of the recorder (Chs. 2, 6); a constant photon-energy shift and, as κ_N, a coercivity constant (Ch. 6); the register kernel is written $\mathscr{K}_T$ (Ch. 7)
φ, ϕ	φ is the golden ratio; $\varphi_k(L)$ is the balance of account k (Ch. 2, written $x(L)$ in Ch. 3); ϕ is a field on a graph or lattice (Chs. 3, 6)
ϵ, ε	ϵ is the clock strength of the registers (Chs. 2, 6, 7); ε is the energy of one level under the energy reading (Ch. 9) and, as $\varepsilon(g)$, an exponential vector (Ch. 6)
C	the register's clock; the credit counts of a book; the conductance between two sites (Ch. 3); C_E is the photon energy scale
w	the level weight $w(s) = \varphi^{-\lvert s\rvert_1}$; the bond weights w_{ij} (Ch. 3); the profile weight $w(k)$ of a current (Ch. 6), written $\varrho(k)$ in Ch. 7
λ	the lattice symbol $\lambda(k) = 2\sum_d(1-\cos k_d)$ (Ch. 6), written $s(k)$ in Chs. 2 and 7; the scale of the interval λQ_κ (Ch. 5)
K, T	K: the native law; a Hermitian clock matrix of a region (Ch. 6). T: the total of a book; a tick clock (Ch. 5); the swap of a register's lowest bit; a temperature (Ch. 9)
Q, π	Q: a register's occupation; an interval on the record cell. π: the stationary law $\varphi^{-9}\varphi^{-\lvert s\rvert_1}$; an edge assignment (Ch. 7)

Table A.2 gives each term the meaning it keeps throughout the book. The four senses of "nothing" head it because they are four different objects; the book never claims a beginning from nothing, and the start is balance.

Table A.2. Fixed vocabulary, with the place where each term is defined.

Term	Meaning	Defined in
empty type	a type with no element; the floor below which no derivation goes	Def. 2.1
ratio limit	$x \to 0^+$, where J diverges; a limit, not a state	Def. 2.1
void	a nonempty set whose elements are all equal; it hosts no distinction	Def. 2.1
empty record, unity point	the record with no posts and the empty event ledger: cost $0 = J(1)$, balanced, net flow 0	Def. 2.1, Prop. 2.7
post	a pair (k, σ): an account k and a side, debit or credit	Def. 2.8
ledger, book	$L = (D, C) \in \mathbb{Z}^d \times \mathbb{Z}^d$, the debit and credit counts; the tally of a record	Def. 2.8
record	a finite list of posts, oldest first, growing only by appending one post	Def. 2.8
event	a comparison (s, t, r) at ratio $r > 0$, with reciprocal $(t, s, 1/r)$; under (R1) the first event is one post, and the first tick is that event with its reciprocal	Sec. 2.2, Def. 3.7
tick, history	a tick is one step, a post or a hold; a history is a word of unit posts read from the empty ledger (on the record cell, a chain of five records)	Defs. 4.1, 5.1
pass	a cyclic sequence of patterns in $\{0, 1\}^3$ changing one bit per step; complete when it visits all eight; the least complete pass has eight ticks	Def. 4.4
tick order, rank, time	orderings, which are charts and not seconds; "time is the tick count" is an adopted identification wherever it is used; on the record cell every tick clock equals the rank, a theorem about clocks that advance once per posting; seconds need (N1)–(N3)	Secs. 4.6, 5.3, Def. 9.25
cost	$J(x) = \frac{1}{2}(x + 1/x) - 1$ on $x > 0$ and $J_{\log}(t) = \cosh t - 1$; "cost equals posting count" holds by definition where cost is counted in posts, and nowhere else	Sec. 1.3
record entropy	an adopted name for record length, not a thermodynamic entropy	Sec. 2.3
existence	"x exists" read as $x > 0$ and $J(x) = 0$: the law of existence, an adopted identification	Sec. 2.1
recognizer, voxel	one site carrying an imbalance $s \in \mathbb{Z}^3$, the balance $D - C$ of its three accounts, under the native law; its level is $\lvert s \rvert_1$	Sec. 4.4
native	of a form taken from the recognition foundation: J, the photon field and its dispersion, the register with its clock and occupation, and, under its premises, the law K	Sec. 6.1
supplied	taken as given by a theorem and derived nowhere in the book	throughout
premise	a condition a result uses that nothing in the book derives; adopted openly and labelled where it enters	Sec. 1.10
identification	a stated reading of a mathematical object as a physical one; named once, never presented as derived	Sec. 1.10

A.2 The carriers

A carrier is given by the objects statements are made about, the dynamics that moves them where there is one, and the data the theorems take as given. The letters (a)–(j) are those of Table 1.1. Carriers (g), (h) and (i) share two native objects, stated in full in Definitions 6.1–6.3: the photon space $\mathscr{F}$, the symmetric Fock space over the transverse fields $\mathscr{H}_1 \subset L^2(\mathbb{T}^3;\mathbb{C}^3)$ of the cubic lattice, with one-photon energy $\omega(k) = 2C_E \operatorname{arsinh}(\sqrt{\lambda(k)}/2)$, Weyl displacements $W(g)$, sources $S(g) = a^*(hg) + a(hg)$ and unit-current profiles z_e; and the register $\mathbb{C}^8$, with basis the vertices of $\{0,1\}^3$, clock $C = FDF^*$ with eigenvalues $0,\dots,7$ and rank-four occupation $Q = \frac{1}{2}(I - i\Sigma T)$.

Definition A.1 (Carrier (a): the account ledger and its records)**.** *Objects:* d accounts (the book takes $d = 3$); the $2d$ posts; books $L = (D, C)$ with balances $D - C$, total and parity pattern; records and their tallies; event ledgers with cost $\sum_i J(r_i)$ and net flow (Section 2.2). *Dynamics:* posting appends one post and raises one count; the rule allows all $2d$ posts at every step and selects none, and a transition at strain q is priced by (3.1). *Supplied:* d, and the strain of a post, a free unit. *Used in* Chapters 2–4, through each recognizer's ledger in Chapter 8, and for the accumulated cost of Chapter 9.

Definition A.2 (Carrier (b): field graphs of the first event)**.** *Objects:* finite sets of sites with symmetric weights $w_{ij} \geq 0$ (Chapter 3 calls each a carrier of a post's field); fields ϕ, currents $w_{ij}\sinh(\phi_i - \phi_j)$, the action (3.3), the exact-J law (3.4) with source, sink and strength, the drop and the conductance. Instances: the pair graph, the possibility graphs on $T_N = (\mathbb{Z}/N\mathbb{Z})^3$, the record carrier of a history of site-to-site posts, rings, theta carriers, series-parallel networks and the 27-site torus (Definitions 3.26–3.36). *Dynamics:* none; the law is the stationarity condition of the action (Lemma 3.13). *Supplied:* the graph, its weights and the strength, which (R2) sets to one for a unit post. *Used in* Chapter 3, and as a hypothesis of Proposition 8.18(ii).

Definition A.3 (Carrier (c): one recognizer's imbalance)**.** *Objects:* states $s \in \mathbb{Z}^3$ with neighbours $s \pm e_k$, the images of the six unit posts; weight $w(s) = \varphi^{-|s|_1}$; transition laws; the native law K, unique under (P1)–(P4) and the three readings (Theorem 4.17), with stationary law $\pi(s) = \varphi^{-9}\varphi^{-|s|_1}$; beside the ledger, the amplitude register $\mathbb{C}^{\mathbb{Z}/8}$ and its shift (Section 4.3). *Dynamics:* the Markov chain of K; each tick holds or makes one post. *Supplied:* the readings, (P1)–(P4) and the starting state, mostly the empty record $s = 0$. Chapter 9 joins a finite thermal record cell C (Definition 9.8) on $\ell^2(\mathbb{Z}^3 \times C)$ with an energy-conserving exchange; the cell and its per-tick refresh are premises, and the cell is unrelated to carrier (e). *Used in* Chapters 4, 8, 9.

Definition A.4 (Carrier (d): many recognizers)**.** *Objects:* a finite set V of recognizers; the seven local moves (hold, six posts) and the $7^{|V|}$ joint moves; laws of the recognizers, with states that may carry anything, each showing every recognizer's ledger, an eight-slot register and one event journal; reached states; the native target, which asks each recognizer's one-tick marginal to be K at every reached state (Definition 8.1); placements $V \to \mathbb{Z}^3$ with their contact graph (Definition 8.21); light cones with an offset (Definition 8.24); record dilations, which realize a supplied joint law by one bounded self-adjoint operator on joint states with fresh records (Proposition 8.17). *Dynamics:* the joint law, tick by tick. *Supplied:* the joint law, which the target does not fix; the placement; the touching premise (L); the journal clause. *Used in* Chapter 8, for the serial clock of Chapter 4, and for shared heat cells Chapter 9.

Definition A.5 (Carrier (e): the record cell)**.** *Objects:* the sixteen records $x + e_B$, $B \subseteq \{1, 2, 3, 4\}$, ordered by inclusion, with rank $|B|$; the twenty-four histories, chains of five records; displacements in $\mathbb{R}^4$; tick clocks; intervals with a linear time function, and $Q_\kappa(y) = \sum_a y_a^2 - \kappa(\sum_a y_a)^2$ (Definitions 5.1, 5.4). *Dynamics:* none beyond the order. *Supplied:* name-blindness and light along a history, or the two readings that replace them (Theorem 5.10). *Used in* Chapter 5.

Chapter 5 also uses two constructions separate from the cell: the lattice electromagnetic field, with real potentials on the links of $\mathbb{Z}^4$ and energy density $D = \frac{1}{2}\sum_{\mu<\nu} F_{\mu\nu}^2$ (Section 5.7); and the periodic Euclidean Regge lattice $\mathbb{Z}_N^4$, each four-cube cut into twenty-four chain simplices, with squared edge lengths as variables and Regge's action (Section 5.8).

Definition A.6 (Carrier (f): the native history dilation)**.** *Objects:* the native cell $\mathbb{C}^{64}$ with basis $|x, i\rangle$, $x \in \{00, 01, 10, 11\}$, $i \in (\mathbb{Z}/4)^2$; the native step, an orthogonal matrix with fifteen integer current labels; the sectors $\mathscr{H}_n = \ell^2(\mathbb{Z}; \mathbb{C}^{64\times 15^n})$ of histories with n postings, the posting isometries P_n, the space $\mathscr{H}_+ = \bigoplus_n \mathscr{H}_n$ with its shift V, and the minimal unitary extension U on the inductive limit $\mathscr{K}$. The empty record is the sector $\mathscr{H}_0$ (Section 2.6). *Dynamics:* posting, the isometry V, and its reversible extension U. *Supplied:* nothing is selected; no amplitude, profile, generator, energy or elapsed time. *Used in* Chapter 2; its stages underlie carrier (j).

Definition A.7 (Carrier (g): the source and recorder)**.** *Objects:* the source space $\mathcal{S} = \ell^2(\mathbb{Z}^3; \mathbb{C}^{4096} \otimes \mathscr{F})$, a negative unit charge moving on $\mathbb{Z}^3$ with the four registers of one cell and the photon field, a positive unit charge fixed at the origin; the source Hamiltonian, with photon energy, coupling g along the six edges, clock energy $\epsilon \sum_k C^{(k)}$, the Gauss-law energy of the pair, and hops that displace the field by the profile of the edge crossed; record labels $q \in \mathbb{Z}^3$ with blank 0, the joint carrier $\mathscr{H}_{\mathrm{SR}} = \ell^2(\mathbb{Z}^3 \times \mathbb{Z}^3; \mathbb{C}^{4096} \otimes \mathscr{F})$, the writer W, the contact C_κ and H_κ; the native pair on $\mathscr{F} \otimes \mathbb{C}^{64}$ with its pair clock, coherent zero mode and ready coordinate (Section 2.7). The source alone is the atomic carrier. *Dynamics:* the unitary group of H_κ, self-adjoint with no spatial or photon-number cutoff (Theorem 6.16), and that of the pair-photon Hamiltonian. *Supplied:* a, C_E, g, ϵ, κ; the pair's profiles and longitudinal energies; the blank, the seed, the apparatus, the controller, the resource that funds preparation, and the initial state. *Used in* Chapters 2, 6.

Definition A.8 (Carrier (h): the self-consistent photon lattice)**.** *Objects:* the periodic lattice $\Lambda_N = (\mathbb{Z}/N\mathbb{Z})^3$ and $\bigoplus_{p\in\Lambda_N} \mathscr{F}$, the photon state accompanying one excitation at each site; bond profiles; real fields ϕ and the operators H_ϕ of (6.5); the field cost $\mathscr{A}(\phi) = 2\sum_{p,k} J(e^{\phi_p - \phi_{p+e_k}})$; a reference distribution ϱ of total one and an anchor; the selected field $\phi[x]$, solving the screened Gauss law (6.6); the total energy (Definition 6.6). *Dynamics:* the nonlinear evolution with the field recomputed from the state at every instant. *Supplied:* the period, profiles, reference, anchor, scales and initial state. *Used in* Chapter 6.

Definition A.9 (Carrier (i): the clock-photon lattice)**.** *Objects:* the cells of $\mathbb{Z}^3$, four registers per cell ($8^4 = 4096$ coordinates); the configurations, which differ from the all-zero labelling at finitely many registers; the carrier $\mathscr{H} = \ell^2(\mathscr{X}; \mathscr{F})$ with one shared photon field; register observables and Weyl operators; the uncompensated regional Hamiltonian (7.5); current profiles and injective edge assignments; the reference vector, the reference state ω_{ref} on the quasi-local algebra, and the clock spectral chart (Sections 6.7, 6.8, 7.1). *Dynamics:* the regional unitary groups and, on the whole lattice, the evolution τ_t of the operators that converge

(Definition 7.1). *Supplied:* a, C_E, ϵ, an injective placement of unit edges, zero background, the region, the initial state. The all-zero configuration is a chart origin and a choice of representation, not a selected first state. *Used in* Chapters 6, 7, for the continuum light cone Chapter 5, and for the photon field as light Chapter 9.

Definition A.10 (Carrier (j): the boundary-theory carrier)**.** *Objects:* the stages $\mathscr{H}_k$ and posting isometries of carrier (f); their inductive limit $\mathscr{H}_\infty$; genesis states $v \in \mathscr{H}_0$ and the posted genesis history; events and the closed algebra of admitted operations they generate; atoms, boundaries and lineages (Section 2.9). *Dynamics:* posting through the stages. *Supplied:* the genesis state. *Used in* Chapters 2, 4; the results are proved in [93].

Four smaller constructions serve as foils: configurations $c \in (0,\infty)^N$ with total defect $\sum_i J(c_i)$ and a variational step that keeps the log charge, whose clock is no origin (Definition 2.20); the amplitude register of carrier (c), which shows that writing a record is not a tick (Section 4.3); a cube cell carrying one bit at each vertex, whose record is the closure bit of each face, on which a horizon supplies no orientation (Proposition 4.27); and relations refined depth by depth, on which no increment is fixed (Proposition 4.35).

A.3 Which carriers are related, and how

A correspondence theorem would map one carrier's objects to another's and carry a stated class of results across the map. None has been proved in the positive direction. Table A.3 lists every pair of constructions about which something is proved; for every other pair nothing is known in either direction. A *nesting* means one carrier is built on another, so the inner carrier's results hold inside the outer one by definition. A *shared object* is a common building block, which carries no result about the dynamics across. A *negative* entry is a theorem that a natural map fails.

Remark A.11 (The field cost of carrier (h) is a graph action of carrier (b)). Let $N \geq 2$. The signed possibility graph on $T_N = \Lambda_N$ has weight one on each pair of neighbouring sites for $N \geq 3$ and two for $N = 2$ (Definition 3.26). Its action (3.3), summed over ordered pairs, counts each bond (p,k) twice with the term $\cosh(\phi_p - \phi_{p+e_k}) - 1 = J(e^{\phi_p - \phi_{p+e_k}})$, so it equals the field cost $\mathscr{A}(\phi)$ of Definition 6.6 at every field. This is an identity of two functionals on the same sites. It does not make the selected field of (h), whose source is $\varrho_p - \|x_p\|^2$ and whose bonds are screened by the photon states, the field of a unit post, and it carries no drop or price of Chapter 3 into Chapter 6.

The open problem this section leaves is exact: a correspondence theorem between any two of the carriers (b), (e), (g), (h) and (i), carrying a stated class of results across; and for the atomic and cell carriers, an embedding that intertwines the actual source and regional dynamics, or a proof that none exists. Until one is proved, every result stays on the carrier where it was proved.

Table A.3. What is proved about pairs of carriers. Pairs not listed have no proved relation.

Pair	Kind	What is proved
(a), (b)	premises	Joined only by (R1) and (R2). The pair graph is the only graph with zero diagonal whose action equals, at every field, the cost of the field-read first event with its reciprocal (Thm. 3.19); one recorded cost fixes no graph (Prop. 3.20). Which graph holds the field is open.
(a), (c)	nesting, negative	The imbalance is the ledger's balance $D - C$ by an adopted reading. The balance forgets the record: one posting relation is acyclic on records and cyclic on balances, and no law in detailed balance on the ledger ever posts (Props. 4.25, 4.13).
(a), (d); (c), (d)	nesting	Each state of (d) shows each recognizer's ledger and a journal (the journal clause is adopted). The native target makes K each recognizer's one-tick law, and then each recognizer's whole history has the law of a recognizer alone under every coupling (Def. 8.1, Thm. 8.4).
(c), $\ell^2(\mathbb{Z}^3)$	negative	No orthonormal family on the imbalance alone has the Born weights of a law in detailed balance that leaves the empty record (Prop. 8.16).
(c), space	negative	A law in detailed balance against w, holding or making one post, with equal positive chances for the six posts from balance, is not translation invariant; eight parity patterns name a place only within a block of eight (Props. 8.20, 8.19).
(a), (f)	same form	The opening premise on (a) (Thm. 2.18) and the empty-record boundary on (f) (Thm. 2.24) have the same form; no theorem connects them.
(f), (g)	negative	For every reading of recorder states as recorded histories, a reversible writer is not two postings and a return to blank is not one; for an isometric reading an early signal is not a completed posting (Prop. 2.30).
(f), (j)	shared	The same stages and posting maps, completed differently (orthogonal sum and unitary dilation, against inductive limit), with different admitted operations.
(b), (h)	shared	The field cost of (h) is the action of the signed possibility graph (Remark A.11); sources and weights differ.
(g), (i)	shared, negative	Photon field and register shared. No isometry from the atomic carrier extends the fixed-site embeddings into the cell carrier; the translation-covariant marked encoding is an isometry, but the atomic source action differs from the regional action on marked single-coordinate inputs, and for the clamped source on the photon vacuum also in probability (Prop. 6.5).
(g), (h); (h), (i)	shared	The photon field only (Sec. 6.1).
(i), reference	negative	No vector of the carrier gives ω_{ref} (Prop. 7.21); for the unit-edge family with the clock off the evolved reference state leaves the reference sector at every $t \neq 0$ (Thm. 7.56); no unitary identifies the clock spectral chart with the native representation (Sec. 6.7).
(e), lattice field	negative	Under the energy reading the lattice photon's cone on the cell sits at $\kappa = \frac{1}{2}$, not causal; its action gives a Euclidean field equation; its cone maps onto the cell's cone at $\kappa = 1$ only through $\sqrt{3}$, an identification (Thms. 5.13–5.15).
(e), Regge lattice	negative	The Euclidean form Q_0 is no interval of the records (Prop. 5.7); no affine normalization of a balanced nonnegative cost equals the action along the body path (Thm. 5.18).
(e), (i); (d), (e)	none, open	No theorem connects the record cell's cone with the clock-photon dispersion (Sec. 5.10) or with the octahedral cone of a supplied pair law (Thm. 8.28).
(a), (e)	none	Four accounts posted at most once, against three accounts with two sides and integer counts (Sec. 5.1).

A.4 Results that look combinable and are not

Besides the pairs of Table A.3 and the two prices that open this appendix, these results invite chaining. Each pair may be set side by side; none is chained in this book.

(1) *Three arrows.* The arrow of the record (carrier (a)), the information arrow of K (carrier (c)) and the circulation read by a record on carrier (j) are three results on three carriers (Section 4.5).

(2) *Dephasing and records.* With the clock off, the coherences of each register of carrier (i) that couples to infinitely many others dephase, with no open premise (Theorem 7.33). This says nothing about whether the records of carrier (g) accumulate, which its reversible dynamics does not enforce (Chapter 2). On carrier (i) the statement is per register: flip patterns whose combined current is a gradient are conserved (Proposition 7.38).

(3) *Two energy accounts.* The floor $-2a|R|$ of the uncompensated regional Hamiltonian of the unit-edge family (carrier (i)) and the conserved total energy of the self-consistent field (carrier (h)) belong to different operators.

(4) *Heat and light.* The temperature $\varepsilon/\log\varphi$ of one recognizer under the energy reading (carrier (c)) and the thermal energy of the photon field under a supplied Bose occupation are separate results: no theorem puts the photon field in equilibrium with the recognizers, and ε and C_E are supplied separately (Section 9.6).

(5) *Dilations and Hamiltonians.* The operator of a record dilation on carrier (d) is not the Hamiltonian of carriers (g)–(i) and is not identified with an observed energy (Proposition 8.17).

(6) *Genesis and the first post.* The genesis state of carrier (j) is a vector of the zeroth stage, of which a long record keeps six numbers; it is not a post of carrier (a) or a field of carrier (b).

A.5 Premises and identifications, and where each enters

Table A.4 lists every premise and identification the book uses, with its carrier and the place where it enters and is stated in full; the supplied data of each carrier are in its definition in Section A.2, and measured inputs carry their datasets in Appendix C. Open problems are not premises: the carrier of the first post's field, for instance, enters no theorem as an assumption (Section 3.10). Every row of kind "premise" is a step the recognition account has not yet supplied; Chapter 11 states what would derive each one.

Table A.4. Premises and identifications, with the carrier each concerns and where it enters. P premise, I identification, N adopted name, S supplied, M measured. The supplied data of each carrier are listed in its definition.

Name	Content		Carrier	Enters
opening premise	the books opened empty: every count is nonnegative and was posted	P	(a)	Def. 2.16
counts are posts	every count is a count of posts made; no law blind to a common shift of the counts derives it	I	(a)	Sec. 2.4
law of existence	"x exists" means $x > 0$ and $J(x) = 0$	I	cost	Sec. 2.1
record entropy	the name of record length	N	(a)	Sec. 2.3
variational step	the update rule of configurations; conservation of log charge	P	foil	Def. 2.20
(R1)	the first event is one post	P	(a), (b)	Def. 3.7
(R2)	a post's field obeys the exact-J law on its carrier with unit source and sink at the post's ends; its strain is the drop across its own bond	P	(b)	Sec. 3.3
three readings	the state is the imbalance; a tick may hold; the weight is $\varphi^{-\lvert s\rvert_1}$	P	(c)	Sec. 4.4
(P1)–(P4)	Markov; hold or one post; detailed balance against w; hold-return balance	P	(c)	Def. 4.15
time as tick count	tick order or rank read as elapsed time	I	(a), (c), (e)	Ch. 4, where used
journal clause	a post is booked as a reciprocal pair of events of ratio other than one	P	(d)	Def. 8.1
touching (L)	recognizers act along the contact graph of a placement	P	(d)	Def. 8.21
boundary readings	the events generate the admitted operations; the genesis state read as the ledger's history, its lineages as initial conditions	I	(j)	Sec. 2.9
name-blindness	the cell's interval is unchanged by permuting the four accounts	P	(e)	Sec. 5.3
light along a history	some history direction is null; with causal histories this forces $\kappa = 1$	P	(e)	Sec. 5.4
light by postings	physical light propagates by single postings	I	(e)	Sec. 5.4
two readings	space from the cost Hessian; one voxel per tick	P	(e)	Thm. 5.10
Regge start	the Euclidean signature, the chain subdivision, Regge's action	P	Regge	Sec. 5.8
lattice field	D read as the Hamiltonian; the lattice directions placed on the accounts	I	field	Sec. 5.7
cone map	a linear map of the photon cone onto the record cone, and its scale	I	field, (e)	Thm. 5.15
native forms	the dispersion and the occupation, taken from the foundation	P	(g)–(i)	Sec. 6.1
energy reading	the golden weights are Boltzmann factors, $e^{-\beta(E(s)-E(0))} = w(s)$	P, I	(c)	Def. 9.2
thermal cell	a finite heat cell in its Gibbs state, refreshed before every tick	P	(c)	Def. 9.8
Bose occupation	the thermal occupation of the photon field	S	field	Thm. 9.20
light speed	$c = \ell C_E/\hbar$: the field's long-wave slope read as the speed of light	I	field	Sec. 9.6
(N1)	c is one voxel per tick, an axis speed	P	units	Def. 9.25
(N2)	$\hbar = E_{\mathrm{coh}}\tau_0$ with $E_{\mathrm{coh}} = \varphi^{-5}$, for masses only	P	units	Def. 9.25
(N3)	$G = \lambda_{\mathrm{rec}}^2 c^3/(4 \ln 2\, \hbar)$	P	units	Def. 9.25
Newton's G	the measured value	M	units	App. C

Appendix B

Deferred proofs

B.1 Proofs for Chapter 3

B.1.1 Proof of Theorem 3.28

(i) G, arsinh and sinh are continuous and strictly increasing, so B, A, $A-B$ and S are continuous and strictly increasing in $c \geq 0$, and all vanish at $c = 0$. Since $A \geq c$, $\sinh 2A \geq 2A$ and $\sinh A \geq A$, we get $S(c) \geq 6c$, so $S(1) > 1$ and the root c_* exists and is unique. The law holds at every class of sites. On the slice $p_0 = 2$ the two neighbours across the slices carry opposite values and the in-slice neighbours carry zero. At a site of value c the law reads $G(c) - 2\sinh(B-c) = 0$, the definition of B; at a site of value B it reads $G(B) - \sinh(A-B) + 2\sinh(B-c) = 0$, the definition of A; at the source it reads $2(\sinh 2A + \sinh A + 4\sinh(A-B)) = S(c_*) = 1$. The slice $p_0 = 1$ is the image of the slice $p_0 = 0$ under $p_0 \mapsto 1 - p_0$ with the sign reversed, which exchanges source and sink. Every solution has drop $2A$ by Theorem 3.22.

(ii) The source equation splits into the own-bond current $\sinh 2A$, the current $\sinh A$ toward $(2, 0, 0)$ and the currents $\sinh(A-B)$ toward the four in-slice neighbours. Since $A > 0$ and $A \geq B$ the remainder is positive, so $2\sinh 2A < 1 = 2\sinh\log\varphi$.

(iii) The drop has no closed form. For $0 \leq t \leq 1$ the exponential series gives

$$\Big|\sinh t - \sum_{j=0}^{4} \frac{t^{2j+1}}{(2j+1)!}\Big| \leq \frac{11\,t^{10}}{10 \cdot 10!}, \qquad \Big|\cosh t - \sum_{j=0}^{4} \frac{t^{2j}}{(2j)!}\Big| \leq \frac{11\,t^{10}}{10 \cdot 10!}.$$

Since S and A increase with c, rationals $\ell < u$ with $S(\ell) \leq 1 \leq S(u)$ give $A(\ell) \leq A \leq A(u)$. The values $S(\ell)$, $S(u)$, $A(\ell)$, $A(u)$ are enclosed by applying these bounds to the equations in the variables c, $B - c$ and $A - B$, in exact rational arithmetic, which yields $0.1601 < 2A < 0.1603$. Then $2\sinh 0.1601 > 0.32$ and $2\sinh 0.1603 < \frac{1}{3}$ give the share, and $\log\varphi > 0.4809$ gives $2A < \frac{1}{3}\log\varphi$.

B.1.2 Proof of Theorem 3.37

(i) By induction along the construction. $2w\sinh d$ is continuous, odd, strictly increasing and onto, and so is a sum of two such functions; the inverse of such a bijection is again one, so the series rule preserves the properties.

(ii) Build, by the same induction, a field F on X with $F_a - F_b = d$ that satisfies $2\sum_j I_{ij} = \Psi_X(d)(\delta_{ia} - \delta_{ib})$ at every site. For a bond take $F = (d, 0)$. For a parallel composition place the fields of the parts for the drop d, each vanishing at b; at the shared terminals the strengths add, merged bonds carry the sum of the currents, and every other site belongs to one part. For a series composition put $c = \Psi_X(d)$, give the parts the drops $\Psi_Y^{-1}(c)$ and $\Psi_Z^{-1}(c)$, which

add to d, and shift them to agree at the junction, where Y delivers c as its sink and Z takes c as its source. So F obeys the law with strength m exactly when $\Psi_X(d) = m$, and Theorem 3.41 gives every solution the drop $\Psi_X^{-1}(m)$.

(iii) The carrier is the parallel composition of the bond with X; apply (ii). At $w_0 = m = 1$, $\Psi_X(d) > 0$ for $d > 0$, so the root is below $\operatorname{arsinh} \frac{1}{2} = \log \varphi$.

(iv) Say X' conducts more than X if $\Psi_{X'}(d) > \Psi_X(d)$ for every $d > 0$. A parallel branch adds a positive term; a heavier bond passes $2w' \sinh d > 2w \sinh d$; removing Z from a series composition removes the positive term $\Psi_Z^{-1}(c)$ from $\Psi^{-1}(c)$ for every $c > 0$. The relation passes through every series or parallel composition containing the part, since $\Psi_{Y'}^{-1} < \Psi_Y^{-1}$ on $(0, \infty)$ when Y' conducts more than Y. A left side of the root equation that is larger at every $d > 0$ has a smaller positive root, and the price follows the drop.

(v) For a bond the bounds read $2wd < 2w \sinh d \le 2w \sinh d$. In parallel both bounds add. In series, the bounds for the parts say $\operatorname{arsinh} \frac{c}{2C_Y} \le \Psi_Y^{-1}(c) < \frac{c}{2C_Y}$ for $c > 0$, and likewise for Z; adding gives $\Psi_X^{-1}(c) < \frac{c}{2C_X}$, and $\Psi_X^{-1}(c) \ge \operatorname{arsinh} \frac{c}{2C_Y} + \operatorname{arsinh} \frac{c}{2C_Z} \ge \operatorname{arsinh} \frac{c}{2C_X}$ because arsinh is concave on $[0, \infty)$ and vanishes at 0, hence subadditive there. For the conductance: by (ii) the field F solves the law with strength $\Psi_X(d)$ and drop d, so Theorem 3.43 gives $2Cd < \Psi_X(d) \le 2C \sinh d$; dividing both pairs of bounds by $2d$ and letting $d \to 0$ gives $C = C_X$.

(vi) A chain of L unit bonds has $\Psi^{-1}(c) = L \operatorname{arsinh} \frac{c}{2}$, whose inverse is $2 \sinh(d/L)$, and $C = 1/L$ by the series rule.

B.1.3 Proof of Theorem 3.39

The graph is connected, every bond has weight at least one, and every site has total weight 6. Write $\Delta\phi_{ij} = \phi_i - \phi_j$.

(i) is Theorem 3.41.

(ii) $-\phi$ solves the law at $-m$, so D_N is odd. For solutions ϕ, ψ at strengths m, m', Lemma 3.13 at $v = \phi - \psi$ gives $\sum w_{ij}(\sinh \Delta\phi_{ij} - \sinh \Delta\psi_{ij})(\Delta\phi_{ij} - \Delta\psi_{ij}) = (m - m')(D_N(m) - D_N(m'))$, and the left side is at least $E = \sum w_{ij}(\Delta\phi_{ij} - \Delta\psi_{ij})^2$ since $\cosh \ge 1$. If $m > m'$ and $D_N(m) \le D_N(m')$, then $E = 0$, the fields have the same currents, and the law at a gives $m = m'$; so D_N is strictly increasing. Multiplying the equation of Lemma 3.38 by $2(\phi_i - \psi_i)$ and summing gives $\sum w_{ij} \Delta u_{ij}(\Delta\phi_{ij} - \Delta\psi_{ij}) = D_N(m) - D_N(m')$, and Cauchy-Schwarz with the lemma gives $(D_N(m) - D_N(m'))^2 \le R_N E \le R_N (m - m')(D_N(m) - D_N(m'))$.

(iii) Take $m' = 0$, $\psi = 0$: $\sum w_{ij} \Delta\phi_{ij} \sinh \Delta\phi_{ij} = m D_N(m)$, and $x \sinh x > x^2$ for $x \ne 0$ gives $E < m D_N(m)$ and so $D_N(m) < R_N m$. For the lower end, ϕ_a is the largest and ϕ_b the smallest value (Theorem 3.23 for ϕ and $-\phi$), so each current out of the source is at most $w_{aj} \sinh D_N(m)$, and the law at a gives $m \le 12 \sinh D_N(m)$. Equality would put every neighbour of a at the value ϕ_b; a neighbour $c = a + e_k$ along another axis then has all its currents ≤ 0 summing to 0, so $\phi_c = \phi_a$, against $D_N(m) > 0$.

(iv) By (ii) and (iii) D_N is continuous, strictly increasing, odd and unbounded, so it maps $\mathbb{R}$ onto $\mathbb{R}$ and m_N^* is unique. Then (iii) at m_N^* gives $\log \varphi < R_N m_N^*$ and $m_N^* < 12 \sinh \log \varphi = 6$; and $6 \log \varphi > 1$.

(v) The values of a solution lie between ϕ_b and ϕ_a.

B.1.4 Proof of Theorem 3.48, parts (i) and (ii)

Lemma B.1 (the Wheatstone profile). *For real X, Y let $\phi_a = X + 2Y$, $\phi_{p'} = X + Y$, $\phi_p = X$, $\phi_q = 2Y$, $\phi_{q'} = Y$, $\phi_b = 0$. If*

$$\alpha \sinh Y = \beta \sinh X + \kappa \sinh(X - 2Y), \tag{B.1}$$

then ϕ obeys the exact-J law on $B(\alpha, \beta, \kappa, w_0)$ with strength $m = 2(\alpha \sinh Y + \beta \sinh X + w_0 \sinh(X + 2Y))$ and drop $X + 2Y$. For $X = \log v$, $Y = \log s$ with rational $v, s > 0$, every hyperbolic sine here is rational, since $\sinh \log v = \frac{1}{2}(v - v^{-1})$.

Proof. The field falls by Y across each bond of weight α, by X across pb and aq, by $X - 2Y$ from p to q and by $X + 2Y$ across ab. At p' and q' the two currents cancel; at p and at q the currents sum to zero by (B.1); the currents out of a give the stated strength, and those out of b are their negatives. □

Table B.1. Exact solutions on the Wheatstone carrier. Each row satisfies (B.1) with $X = \log v$, $Y = \log s$; the drop is $\log(vs^2)$. Here $\alpha_W = \frac{3391}{77520}$, $\beta_W = \frac{3391}{156672}$, $\kappa_W = \frac{8811238665448679}{10949496441089888} \approx 0.80472$ and $m_W = \frac{1270924727588341903}{1270924842470889600}$, with $1 - m_W \approx 9.04 \times 10^{-8}$.

Row	α	β	κ	w_0	(v, s)	strength m; drop $\log(vs^2)$
A	4	$\frac{1}{2}$	$\frac{24}{23}$	0	$(\frac{91}{5}, 3)$	$\frac{26944}{1365}$; $\log \frac{819}{5}$
A′	4	$\frac{1}{2}$	$\frac{128}{37}$	0	$(\frac{82}{5}, \frac{16}{5})$	$\frac{1617}{82}$; $\log \frac{20992}{125}$
B	$\frac{49}{12}$	$\frac{1}{2}$	0	0	$(20, \frac{14}{5})$	$\frac{399}{20}$; $\log \frac{784}{5}$
B′	$\frac{49}{12}$	$\frac{1}{2}$	$\frac{23}{18}$	0	$(18, 3)$	$\frac{715}{36}$; $\log 162$
W	α_W	β_W	0	1	$(\frac{19}{15}, \frac{9}{8})$	1; $\log \frac{513}{320}$
W′	α_W	β_W	κ_W	1	$(\frac{333}{263}, \frac{1249}{1110})$	m_W; $\log \frac{1560001}{973100}$

(i) Row A: with $v = \frac{91}{5}$, $s = 3$, $\alpha \sinh Y = \frac{16}{3}$, $\beta \sinh X = \frac{2064}{455}$ and $\kappa \sinh(X - 2Y) = \frac{75072}{94185}$, and $\frac{2064}{455} + \frac{75072}{94185} = \frac{16}{3}$; so Lemma B.1 gives a solution with strength $m_A = \frac{26944}{1365}$ and drop $\log \frac{819}{5}$, which every solution shares. Row A′ is checked the same way: at $\kappa' = \frac{128}{37}$ a solution has strength $\frac{1617}{82} < m_A$ and drop $\log \frac{20992}{125}$. By Theorem 3.46(i) the drop at m_A on the stronger carrier exceeds $\log \frac{20992}{125}$ and exceeds it by at most $(m_A - \frac{1617}{82})/(2C')$, where $C' \geq \frac{4}{5}$ because without the cross bond the carrier is two chains of conductance $\frac{2}{5}$ each. The rise exceeds $\log \frac{20992}{20475} \geq 1 - \frac{20475}{20992} = \frac{517}{20992}$. The second carrier is the same argument with rows B and B′: each chain conducts $\frac{49}{122}$, and the rise exceeds $\log \frac{405}{392} \geq \frac{13}{405}$. For every $\mu > 0$, multiplying both carriers of row A and A′ by μ/m_A keeps the order of the weights and, by Theorem 3.46(iii), the drops.

(ii) Every unit-difference field has quadratic action at least 1 from the post's bond, so $C \geq 1$. The test field equal to 1 at a, $\frac{3}{4}$ at p', $\frac{1}{2}$ at p and q, $\frac{1}{4}$ at q' and 0 at b has action $1 + \frac{\alpha_W}{4} + \frac{\beta_W}{2} = \frac{30415361}{29767680} \approx 1.02176$ on both carriers, since the cross bond carries no difference; and $\frac{30415361}{29767680} < \frac{500}{489} < \frac{1}{2 \log \varphi}$, because $\sinh 0.489 > \frac{1}{2}$. Row W: $\alpha_W \sinh \log \frac{9}{8} = \beta_W \sinh \log \frac{19}{15} = \frac{3391}{656640}$, and the strength is $\frac{3391}{164160} + \frac{160769}{164160} = 1$, so $D_w(1) = \log \frac{513}{320}$. Row W′ gives $D_{w'}(m_W) = \log \frac{1560001}{973100}$ with $m_W < 1$,

and $\frac{1560001}{973100} = \frac{513}{320}\cdot\frac{24960016}{24960015}$. By Theorem 3.46(i), $D_{w'}(1) > D_{w'}(m_W) \ge D_w(1) + \frac{1}{24960016}$, and with $C' \ge 1$, $D_{w'}(1) \le D_{w'}(m_W) + \frac{1-m_W}{2} < D_w(1) + 10^{-7}$.

B.2 Proofs for Chapter 4

B.2.1 Proof of Theorem 4.17

The reduction is proved after the theorem. It remains to show that (4.7) has exactly one positive solution and that it lies in $[1/13, 1]$.

Bounds. Let $u > 0$ solve (4.7). Then $u(s)^2 \le 1$, so $u \le 1$. Each coefficient is at most $\varphi^{1/2} < 2$, so the bracket in (4.7) is at most $u(s) + 12 \le 13$, and $1 \le 13\,u(s)$.

Existence. For $S \ge 0$ let $r(S) = \frac{1}{2}(\sqrt{S^2+4} - S)$, the positive root of $y(y+S) = 1$; it decreases in S and maps $[0, 12]$ into $[1/13, 1]$. On the set $\mathcal{F}$ of functions $x\colon \mathbb{Z}^3 \to [1/13, 1]$, ordered pointwise, let $(Tx)(s) = r\big(\sum_{t\sim s} c(s,t)x(t)\big)$. Then T maps $\mathcal{F}$ into itself and reverses order, so T^2 preserves order; $\mathcal{F}$ is a complete lattice, and T^2 has a fixed point x by Tarski's theorem [119]. Every post changes the parity of $|s|_1$. Put $y = x$ on the states with $|s|_1$ even and $y = Tx$ on the odd ones. At an even s, $(Ty)(s)$ reads only odd neighbours, where $y = Tx$, so $(Ty)(s) = (T^2x)(s) = x(s) = y(s)$; at an odd s it reads only even neighbours, so $(Ty)(s) = (Tx)(s) = y(s)$. Hence $Ty = y$, which is (4.7).

Uniqueness. Let u, v be positive solutions and $q = v/u$. By the bounds q takes values in $[1/13, 13]$, so $B = \sup_s \max\{q(s), 1/q(s)\}$ is finite and $B \ge 1$. Write $a(s) = u(s)^2$ and $b(s,t) = c(s,t)u(s)u(t)$; then $a(s) + \sum_t b(s,t) = 1$ and $a(s) \ge 1/169$. Dividing the equation for v by that for u gives $q(s)\big(a(s)q(s) + \sum_t b(s,t)q(t)\big) = 1$. If $q(s) \ge 1$, then, since $Bq(t) \ge 1$ for every t,

$$B\Big(a(s)q(s) + \sum_t b(s,t)q(t)\Big) \ge a(s)B + \big(1 - a(s)\big) = 1 + a(s)(B-1) \ge 1 + \frac{B-1}{169},$$

so $q(s) \le B/\big(1 + (B-1)/169\big)$. The bound holds trivially where $q(s) < 1$, and, exchanging u and v, for $1/q(s)$. Taking the supremum, $B \le B/\big(1+(B-1)/169\big)$, so $B \le 1$. Hence $v = u$. $\square$

B.2.2 Proof of Theorem 4.22(iii)

Write $\langle X, Y\rangle_\pi = \sum_s X(s)Y(s)/\pi(s)$ and $(XP)(t) = \sum_s X(s)P(s,t)$. A number γ is a *Poincaré gap* of a voxel law P if $\gamma\langle X, X\rangle_\pi \le \langle X, X\rangle_\pi - \langle X, XP\rangle_\pi$ for every finitely supported X with $\sum_s X(s) = 0$.

From a gap to a rate. Let P have every hold at least h and a gap γ with $2h - 1 < 1 - \gamma$ and $(1-\gamma) + (2h-1) \ge 0$. Let T be one tick, $b = 1 - \gamma$, $\alpha = 2h - 1$ and $\mu_k = \langle X, XT^k\rangle_\pi$. By (P3), T is self-adjoint for $\langle\cdot,\cdot\rangle_\pi$, so $\langle XT^j, XT^k\rangle_\pi = \mu_{j+k}$. With $Y = XT - \alpha X$ and $Z = bX - XT$, collecting terms gives the identity

$$(b-\alpha)\big(b^2\mu_0 - \mu_2\big) = \big[b\langle Y, Y\rangle_\pi - \langle Y, YT\rangle_\pi\big] + \big[\langle Z, ZT\rangle_\pi - \alpha\langle Z, Z\rangle_\pi\big] + (b^2 - \alpha^2)\big(b\mu_0 - \mu_1\big).$$

The first bracket is nonnegative by the gap at Y, which is finitely supported with total zero. The second is nonnegative because every hold is at least h: writing $\langle Z, ZT\rangle_\pi$ as a sum over holds and links and bounding each link term by $2zz' \ge -(z^2 + z'^2)$ gives $\langle Z, ZT\rangle_\pi \ge (2h-1)\langle Z, Z\rangle_\pi$.

The last term is nonnegative by the gap at X and the hypotheses. So $\langle XT, XT\rangle_\pi \le b^2\langle X, X\rangle_\pi$. For a start a, the Cauchy-Schwarz inequality gives $d_n(a)^2 \le \chi^2\big(P^n(a,\cdot)\big) = \langle Y_n, Y_n\rangle_\pi$ with $Y_n = P^n(a,\cdot) - \pi$; the contraction applies to Y_n through finitely supported truncations, and χ^2 of the point mass at a is $1/\pi(a) - 1 = \varphi^{9+|a|_1} - 1$. Hence $d_n(a) \le \sqrt{\varphi^{9+|a|_1} - 1}\,(1-\gamma)^n$.

The gap of K. The 48 signed permutations commute with K, so K never mixes the part of a pattern that a symmetry reverses with the part it keeps, and a number is a gap of K exactly when it is a gap on each of three classes of patterns: those that change sign when one account is reflected, those that change sign when two accounts are exchanged, and the symmetric patterns with total zero. A pattern of the first two classes vanishes on a mirror plane that no post crosses, and a positive weight whose one-tick image is everywhere below 0.99161 times itself bounds its quotient (a Schur test [103]); the inequality is checked row by row at the 900 sites of one box, and symmetry and translation reduce every other row to those. For the symmetric class, the same test with another weight covers the states with $|s|_1 > 20$, whose rows stay below 0.991614 times that weight; an exact $LDL^\top$ factorization, with one rank-one correction for the direction of π, covers the 358 symmetry classes with $|s|_1 \le 20$, which stand for the 11,521 states there; and orthogonality to π joins the two parts through the mass of π beyond $|s|_1 = 20$, which is below 0.002911. This gives the gap 0.00836. The ceiling 0.008398 comes from one explicit pattern, odd in the first account, whose quotient $\langle X, XK\rangle_\pi / \langle X, X\rangle_\pi$ is certified to lie between 0.991602 and 0.991608. Every step is a finite computation in exact rational arithmetic on the enclosures of the previous subsection.

The rate. Apply the first part with $\gamma = 0.00836$ and $h = 0.136067$ from Theorem 4.19(iii): $2h - 1 = -0.727866 < 1 - \gamma$ and $(1-\gamma) + (2h-1) = 0.263774 \ge 0$. From the empty record $\sqrt{\varphi^9 - 1} < 9$. □

B.2.3 Proof of Theorem 4.29

Write $q = P^n(a,\cdot)$, supported in $B_{|a|_1+n}$. A path with a zero-chance step has $\mu = \nu = 0$, so sums over X are sums over all paths with present in B_N. From any y with $|y|_1 \le N - k$ every path of k steps ends in B_N, so by (P1) its continuations of k steps carry total weight one. Hence $\sum_x \mu(x) = \sum_s q(s) = 1$, and μ is a probability law on X.

On X the step chances cancel in the ratio: $\mu(x)/\nu(x) = Z\,q(s_0)/\pi(s_0)$, with Z the total stationary weight over X. Summing over the continuations of each start,

$$D(\mu\|\nu) = \sum_x q(s_0) \prod_i P(s_i, s_{i+1}) \log \frac{Z\,q(s_0)}{\pi(s_0)} = \log Z + D_n(a).$$

For the image laws at moment j, let $c(y)$ be the total weight of the continuations $(y, s_{j+1}, \dots, s_m)$ with present in B_N. Summing the first j steps, and using that π is stationary,

$$(r_j)_*\mu(y) = P^{n+j}(a,y)\,c(y), \qquad (r_j)_*\nu(y) = \frac{1}{Z}\,\pi(y)\,c(y).$$

Where $P^{n+j}(a,y) > 0$, $|y|_1 \le |a|_1 + n + j \le N - (m-j)$, so $c(y) = 1$, and

$$D\big((r_j)_*\mu\|(r_j)_*\nu\big) = \sum_y P^{n+j}(a,y) \log \frac{Z\,P^{n+j}(a,y)}{\pi(y)} = \log Z + D_{n+j}(a).$$

Lemma 4.28 gives $L_{r_j}(\mu) = D_n(a) - D_{n+j}(a)$. The cases $j = 0$ and $j = m$ are the first two statements, and the right side does not involve m, which is the third. □

B.2.4 Proof of Theorem 4.31

The closed form of the first tick is proved after the theorem. For the ceiling, let p be a law on a set B and $\pi_B = \pi/\pi(B)$ on B. Then $D(p\|\pi) = D(p\|\pi_B) - \log \pi(B) \geq -\log \pi(B)$, with equality exactly when $p = \pi_B$. Apply this to $p = P^n(a, \cdot)$, supported in $B_{|a|_1+n}$, and subtract from $D_0(a) = (9 + |a|_1)\log\varphi$:

$$D_0(a) - D_n(a) \leq (9 + |a|_1)\log\varphi + \log \pi\big(B_{|a|_1+n}\big),$$

with equality exactly when $P^n(a, \cdot)$ is π conditioned on $B_{|a|_1+n}$. From balance, $\pi(B_n) = \varphi^{-9}\sum_{s\in B_n} w(s)$, which gives Λ_n. With $\varphi^{-1} = \varphi - 1$, $\varphi^{-2} = 2 - \varphi$ and $\varphi^{-3} = 2\varphi - 3$, the sums for $n = 1, 2, 3$ are $1 + 6\varphi^{-1} = 6\varphi - 5$, then $+18\varphi^{-2}$, giving $31 - 12\varphi$, then $+38\varphi^{-3}$, giving $64\varphi - 83$. Since $\sum_s w(s) = \varphi^9$, Λ_n rises to $9\log\varphi$.

At $n = 1$ equality means the law after one tick is π conditioned on B_1, whose chance at 0 is $1/(1 + 6\varphi^{-1}) = h_*$. At $n = 2$ equality would need $P^2(0, 0) = \pi(0)/\pi(B_2) = 1/(31 - 12\varphi) \approx 0.08633$. But the six posts out of 0 are equally likely and (P3) gives $P(e, 0) = \varphi P(0, e)$, so

$$P^2(0,0) = h^2 + \frac{\varphi}{6}(1-h)^2 = \frac{\varphi}{\varphi+6} + \frac{\big((\varphi+6)h - \varphi\big)^2}{6(\varphi+6)} \geq h_* \approx 0.2124,$$

and no voxel law meets the ceiling at $n = 2$. □

B.3 Proofs for Chapter 5

B.3.1 Proof of Theorem 5.13

Write $\langle\cdot,\cdot\rangle$ for the Euclidean product on $\mathbb{R}^4$, $S_v = \sum_a v_a$, and $\mathbf{J}$ for the matrix of ones.

(i) For a time direction $w \neq 0$ with unit vector $n = w/|w|$, the electric part of F is $n \wedge \varepsilon$ with $\varepsilon_a = \sum_b F_{ab} n_b$, and the magnetic part $F - n \wedge \varepsilon$ annihilates w. The two parts are orthogonal, so $D = D_{\mathrm{el}} + D_{\mathrm{mag}}$ for every field. Then $\lambda_e D_{\mathrm{el}} + D_{\mathrm{mag}} = D$ for every field exactly when $(\lambda_e - 1)D_{\mathrm{el}} = 0$ for every field, and the field $F = n \wedge \varepsilon$ with $\varepsilon \perp n$, $\varepsilon \neq 0$, has $D_{\mathrm{el}} > 0$.

(ii) Take $w = (1, 1, 1, 1)$, $n = \frac{1}{2}w$. Since $\langle n, \varepsilon\rangle = 0$ by antisymmetry, $D_{\mathrm{el}} = \frac{1}{2}|\varepsilon|^2 = \frac{1}{8}\sum_a(\sum_b F_{ab})^2$, and the Lagrangian is $\lambda_e D_{\mathrm{el}} - D_{\mathrm{mag}} = (1 + \lambda_e)D_{\mathrm{el}} - D$. On a plane wave with covector k and amplitude A the field strength is $F_{ab} = k_a A_b - k_b A_a$, and

$$\sum_{a<b} F_{ab}^2 = |k|^2|A|^2 - \langle k, A\rangle^2, \qquad \sum_a\Big(\sum_b F_{ab}\Big)^2 = |k|^2 S_A^2 - 2\langle k, A\rangle S_k S_A + |A|^2 S_k^2.$$

Let $h = \alpha I + \beta\mathbf{J}$ with $\alpha = -1$, $\beta = (1 + \lambda_e)/4$. Expanding $h(k,k)h(A,A) - h(k,A)^2$ with these two identities gives

$$8\big((1 + \lambda_e)D_{\mathrm{el}} - D\big)(k \wedge A) = -4\big(h(k,k)\,h(A,A) - h(k,A)^2\big),$$

a Gram determinant of h on the plane of k and A. The matrix h has eigenvalue $\alpha = -1$ on $\{S_y = 0\}$ and $\alpha + 4\beta = \lambda_e$ on the main diagonal, so it is invertible for $\lambda_e \neq 0$. A covector $k \neq 0$ is characteristic when the quadratic form $A \mapsto h(k,k)h(A,A) - h(k,A)^2$ is degenerate beyond

the gauge direction $A = k$. If $h(k,k) \neq 0$, its kernel is the set of A with $h(k,k)\,hA = h(k,A)\,hk$, that is $A \in \mathbb{R}k$; if $h(k,k) = 0$ the form is $-h(k,A)^2$, whose kernel is three-dimensional. So k is characteristic exactly when $h(k,k) = 0$. Since $\mathbf{J}^2 = 4\mathbf{J}$,

$$h^{-1} = \frac{1}{\alpha}\Big(I - \kappa_\gamma \mathbf{J}\Big), \qquad \kappa_\gamma = \frac{\beta}{\alpha + 4\beta} = \frac{1+\lambda_e}{4\lambda_e},$$

and the ray $y = hk$ of k has $y^\top h^{-1} y = h(k,k)$; so the rays of characteristic covectors are exactly the null displacements of Q_{κ_γ}. Real characteristic covectors exist exactly when h is indefinite, $\alpha(\alpha + 4\beta) = -\lambda_e < 0$. At $\lambda_e = 1$, $\kappa_\gamma = \frac{1}{2}$.

(iii) $Q_{1/2}(e_a) = 1 - \frac{1}{2}$. A ray y with $Q_{1/2}(y) = 0$ has $\sum_a y_a^2 = \frac{1}{2}(\sum_a y_a)^2$, hence $Q_1(y) = -\frac{1}{2}(\sum_a y_a)^2$.

(iv) The photon's characteristic covectors are the null set of $I - \frac{1+\lambda_e}{4}\mathbf{J}$, and the cell's wave cone (Proposition 5.12) is the null set of $I + \frac{\kappa}{1-4\kappa}\mathbf{J}$. Both forms are nondegenerate and indefinite for $\lambda_e > 0$ and $\kappa > \frac{1}{4}$, and such a form is determined up to a factor by its null cone, so the cones coincide exactly when $-\frac{1+\lambda_e}{4} = \frac{\kappa}{1-4\kappa}$, that is $\kappa = (1+\lambda_e)/(4\lambda_e) = \kappa_\gamma$. At $\lambda_e = 1$ this is $\kappa = \frac{1}{2}$, where $Q_{1/2}(e_a) > 0$ and (a) of Definition 5.4 fails. The value $\kappa = 1$ needs $4\lambda_e = 1 + \lambda_e$, $\lambda_e = \frac{1}{3}$.

(v) With $w = e_4$ the electric part is $\frac{1}{2}\sum_{i\le 3} F_{i4}^2 = \frac{1}{2}|E|^2$, so the energy reading's Lagrangian is $\frac{1}{2}(|E|^2 - |B|^2)$. On a plane wave, the same Lagrange identity in the form η gives $-\frac{1}{2}\big(\eta(k,k)\eta(A,A) - \eta(k,A)^2\big)$, and the argument of (ii) with $h = \eta$ shows that k is characteristic exactly when $\eta(k,k) = 0$.

(vi) In every frame the action reading is $-D_{\mathrm{el}} - D_{\mathrm{mag}} = -D$, which is the case $\beta = 0$, $h = -I$: then $h(k,k) = -|k|^2 \neq 0$ for every $k \neq 0$, and no covector is characteristic.

B.3.2 Proof of Theorem 5.14

Write $\Delta_\nu f(u) = f(u + e_\nu) - f(u)$ and $\Delta_\nu^- f(u) = f(u) - f(u - e_\nu)$; these differences commute, and the equation reads $\sum_\nu \Delta_\nu^- F_{\rho\nu} = 0$ for every ρ.

(i) Since $F_{\mu\rho} = \Delta_\mu A_\rho - \Delta_\rho A_\mu$, the cyclic identity $\Delta_\nu F_{\mu\rho} + \Delta_\mu F_{\rho\nu} + \Delta_\rho F_{\nu\mu} = 0$ holds. Apply Δ_ν^- and sum over ν:

$$\sum_\nu \Delta_\nu^- \Delta_\nu F_{\mu\rho} = -\Delta_\mu \sum_\nu \Delta_\nu^- F_{\rho\nu} + \Delta_\rho \sum_\nu \Delta_\nu^- F_{\mu\nu} = 0,$$

and $\sum_\nu \Delta_\nu^- \Delta_\nu f(u) = \sum_\nu \big(f(u+e_\nu) - 2f(u) + f(u-e_\nu)\big)$.

(ii) Every component of the field strength of a plane wave is a plane wave $f(u) = P\cos(k\cdot u) + R\sin(k\cdot u)$, and the lattice Laplacian multiplies it by $\sum_\nu (2\cos k_\nu - 2)$. If this factor is nonzero, (i) forces every component to vanish. If it is zero, every $\cos k_\nu = 1$, so $k_\nu \in 2\pi\mathbb{Z}$, the potential is constant on $\mathbb{Z}^4$, and the field strength vanishes. A field with zero field strength solves the equation.

(iii) Put $s = t - z$ and $Dg(s) = g(s+1) - g(s)$. The only nonzero components are $F_{xt}(u) = -Dg(s)$ and $F_{xz}(u) = Dg(s-1)$, up to antisymmetry. For $\rho = x$ the equation is $\Delta_t^- F_{xt} + \Delta_z^- F_{xz} = -2\big(Dg(s) - Dg(s-1)\big) = 0$, the vanishing of the second difference of g; the equations for the

other ρ hold because F_{tx} and F_{zx} do not depend on x. A bounded sequence with zero second difference is constant, so its field vanishes. With the time term weighted -1 the left side is $\big(Dg(s) - Dg(s-1)\big) + \big(Dg(s-1) - Dg(s)\big) = 0$ for every g.

(iv) For $A_x = e^{-\omega t}\cos(qz)$ the equation with $\rho = x$ is $-\big(\Delta_t^- \Delta_t + \Delta_z^- \Delta_z\big)A_x = -\big(2(\cosh\omega - 1) - 2(1 - \cos q)\big)A_x = 0$, and the other equations hold as in (iii). Since $1 - \cos q \in [0, 2]$ and $\cosh\omega - 1$ takes every value in $[0, \infty)$ on $\omega \geq 0$, a rate exists for every q.

B.3.3 Proof of Theorem 5.16

Nondegeneracy. A permutation of the coordinates maps chain simplices to chain simplices and preserves ℓ_q, because the flat value of an edge depends only on the number of postings it spans and the main diagonal is mapped to itself. Every simplex is therefore congruent to the reference simplex $p_0 \cdots p_4$, whose squared edge lengths are $\ell_{ij} = |i - j|$ except $\ell_{04} = 4 + q$. Its Gram matrix agrees with the flat one except in the entries involving p_4; its leading minors of orders one to three are the flat ones, hence positive, and its determinant is $(4 - q^2)/4$. By Sylvester's criterion it is positive definite exactly when $|q| < 2$.

Which triangles change. Only a triangle containing a main diagonal changes its area along the path. Such a triangle is x, $x + e_B$, $x + e_{\{1,2,3,4\}}$ with $\emptyset \neq B \subsetneq \{1, 2, 3, 4\}$. For $N \geq 3$ every simplex containing it is a chain simplex of the four-cube at x whose chain passes through $x + e_B$: two vertices of a chain simplex differ by $\pm e_C$ modulo N, and $\pm e_C \equiv e_{\{1,2,3,4\}}$ forces the plus sign and $C = \{1, 2, 3, 4\}$ when $N \geq 3$, so the two ends of the diagonal are the first and last vertices of the chain. There are $\binom{4}{k}$ such triangles with $|B| = k$ in each four-cube, each in $k!(4 - k)!$ simplices, all congruent by permutations fixing B to the triangle $p_0 p_k p_4$ of the reference simplex. By Schläfli's identity and the area formula,

$$\frac{\mathrm{d}}{\mathrm{d}q} S(\ell_q) = N^4 \sum_{k=1}^{3} \binom{4}{k} \varepsilon_k(q) \frac{\mathrm{d}}{\mathrm{d}q} \frac{\sqrt{4k(4-k) - q^2}}{4} = -N^4 q \sum_{k=1}^{3} \binom{4}{k} \frac{\varepsilon_k(q)}{4\sqrt{4k(4-k) - q^2}},$$

with $\varepsilon_k = 2\pi - k!(4 - k)!\,\theta_k$. The map $y \mapsto e_{\{1,2,3,4\}} - y$ reverses the chain, preserves ℓ_q and carries the triangle with $k = 3$ to the one with $k = 1$, so $\theta_3 = \theta_1$, and the sum is the stated F.

Dihedral angles. The angle at the triangle $p_0 p_k p_4$ is the angle between the components u_i, u_j of p_i, p_j orthogonal to p_k and p_4, where $\{i, j\} = \{1, 2, 3\} \setminus \{k\}$; with $T = \{k, 4\}$ these components have inner products $\langle u_i, u_j \rangle = g_{ij} - g_{iT}\, g_{TT}^{-1}\, g_{Tj}$. A direct computation from the Gram matrix gives, for $k = 2$,

$$\langle u_1, u_3 \rangle = -\frac{2q}{16 - q^2}, \qquad \|u_1\|^2 = \|u_3\|^2 = \frac{8 - q^2}{16 - q^2},$$

and for $k = 1$,

$$\langle u_2, u_3 \rangle = \frac{4 - q^2}{12 - q^2}, \qquad \|u_2\|^2 = \frac{8 - q^2}{12 - q^2}, \qquad \|u_3\|^2 = \frac{2(4 - q^2)}{12 - q^2},$$

which give the stated cosines. At $q = 0$ they give $\theta_1 = \pi/3$ and $\theta_2 = \pi/2$, so $\varepsilon_1(0) = \varepsilon_2(0) = 0$ and $F(0) = 0$.

The limit. $\cos\theta_1$ is an even smooth function of q with values in $(0, 1)$, so θ_1 is even and smooth and $\varepsilon_1(q) = O(q^2)$. Differentiating $\cos\theta_2$ at 0 gives $-\sin\theta_2(0)\,\theta_2'(0) = -\frac{1}{4}$, so $\theta_2'(0) = \frac{1}{4}$ and

$\varepsilon_2(q) = -q + O(q^2)$. Hence

$$\frac{F(q)}{q^2} = -\frac{2\,\varepsilon_1(q)}{q\sqrt{12 - q^2}} - \frac{3\,\varepsilon_2(q)}{2q\sqrt{16 - q^2}} \longrightarrow 0 + \frac{3}{2 \cdot 4} = \frac{3}{8},$$

which also gives $F'(0) = \lim_{q \to 0} F(q)/q = 0$.

B.3.4 Proof of Proposition 5.19

We outline the proof. Let $B_\ell = (-\pi/\ell, \pi/\ell]^3$, $\hat{J}_\ell(p) = \ell^3 \sum_e J_{d_e}(m_e) e^{i m_e \cdot p} e_{d_e}$ over the edges e of direction d_e and midpoint m_e, and $\|v\|_w^2 = \int |p|\,|v(p)|^2 \,\mathrm{d}p$. Substituting $k = \ell p$ in $c(J_\ell, H_\ell, t)/a$ gives $C_\ell(t) = -\frac{i}{(2\pi)^3} \int \langle \hat{J}_\ell, K_\ell \hat{H}_\ell \rangle \,\mathrm{d}p$, with $K_\ell(t,p) = \varepsilon_\ell(p) \sin(t E_\ell(p)) P_\perp(s_\ell(p))$ on B_ℓ and $K_\ell = 0$ outside, where $s_\ell(p)_j = \frac{2}{\ell} \sin \frac{\ell p_j}{2}$ and $\varepsilon_\ell = \ell E_\ell^2 / \big(\sqrt{\lambda(\ell p)} \sqrt{1 + \lambda(\ell p)/4}\big)$; C_0 has $\hat{J}$, $\hat{H}$ and $K_0(t,p) = |p| \sin(t|p|) P_\perp(p)$ in their place.

(i) On B_ℓ, $0 \le |p| - \varepsilon_\ell \le \frac{7}{24} \ell^2 |p|^3$, $|\sin(t E_\ell) - \sin(t|p|)| \le |t| \ell^2 |p|^3/8$ (Proposition 9.19) and $\|P_\perp(s_\ell) - P_\perp(p)\| \le \ell^2 |p|^2/6$; outside it $|p| \ge 2/\ell$. So $\|K_\ell - K_0\| \le \ell^2 (\frac{11}{24}|p|^3 + \frac{|t|}{8}|p|^4)$ everywhere, and $\|K_\ell\|, \|K_0\| \le |p|$. By the Poisson summation formula $\hat{J}_\ell$ is the sum of the aliases $\pm \hat{J}(p + 2\pi n/\ell)$, $n \in \mathbb{Z}^3$; on B_ℓ those with $n \neq 0$ add to at most $c_J \ell^4$, with c_J fixed by the fourth derivatives of J. Since B_ℓ has volume $(2\pi/\ell)^3$ and $|p| \le \sqrt{3}\pi/\ell$ on it, $D_J = 1_{B_\ell}(\hat{J}_\ell - \hat{J})$ has $\|D_J\|_w \le \alpha_J \ell^2$, $\alpha_J = (8\sqrt{3}\pi^4)^{1/2} c_J$. Writing $\hat{J}_\ell = \hat{J} + D_J$ and $\hat{H}_\ell = \hat{H} + D_H$ on B_ℓ, the Cauchy-Schwarz inequality in $\|\cdot\|_w$ gives (i) with $(2\pi)^3 C(T) = \frac{11}{24} M_3 + \frac{T}{8} M_4 + \alpha_J \|\hat{H}\|_w + \alpha_H \|\hat{J}\|_w$ and $(2\pi)^3 D = \alpha_J \alpha_H$, where $M_n = \int |p|^n |\hat{J}|\,|\hat{H}| \,\mathrm{d}p$ is finite because $\hat{J}$ and $\hat{H}$ decay faster than every power.

(ii) Since $|p| \sin(t|p|) P_\perp(p) = (|p|^2 I - p p^{\top}) \sin(t|p|)/|p|$, $C_0(t)$ pairs J with second derivatives of H convolved with the wave propagator, which is supported on the sphere $|x| = |t|$. So $C_0(t) = 0$ when the supports are more than $|t|$ apart, and (i) with $\ell^4 \le \ell^2$ gives the bound.

(iii) $0 < E_\ell(p) < |p|$ follows from $\lambda(k) \le |k|^2$ and $\operatorname{arsinh} u < u$ for $u > 0$. Then $\beta > E_\ell(p)/|p|$ makes the boosted energy $(E_\ell(p) - \beta|p|)/\sqrt{1 - \beta^2}$ negative; boosts preserve $E^2 - |p|^2$, and every lattice mode has $E_\ell \ge 0$.

B.4 Proofs for Chapter 6

B.4.1 Proof of Theorem 6.7

(i) V_ϕ is bounded and symmetric: the Weyl operators are unitary, the diagonal is real, and the two hopping terms of (6.5) are adjoint to each other. A bounded symmetric perturbation of the self-adjoint operator $\bigoplus_p H_0$ is self-adjoint on the same domain [95].

(ii) For $N = 1$ the anchored field is zero and every statement is trivial, so let $N \ge 2$; then no bond closes on itself. Put $s_p = \varrho_p - \|x_p\|^2$, so $\sum_p s_p = 0$. For a bond joining $p \neq p'$ and $\|x\| = 1$, $\|x_p - W x_{p'}\|^2 \le 2(\|x_p\|^2 + \|x_{p'}\|^2) \le 2$, so $\beta_{p,k} \in [0,1]$. Writing $c_{p,k} \|x_p - W(z_{p,k}) x_{p+e_k}\|^2 = c_{p,k}(2 - 2\beta_{p,k})$ gives

$$\mathscr{J}(\phi; x) = \sum_{p,k} \big(2 - 2\beta_{p,k} \cosh(\phi_p - \phi_{p+e_k})\big) + \sum_p s_p \phi_p,$$

a concave function of ϕ whose partial derivative in ϕ_p is the right side of (6.6) minus its left side. Its stationary points are exactly the solutions of (6.6), and each is a maximum.

Weak bonds. Call a bond weak if $\beta < \frac{1}{4}$, that is $\|x_p - W x_{p'}\|^2 > \frac{3}{2}$. Since W is unitary, $\|x_p - W x_{p'}\| \le \|x_p\| + \|x_{p'}\|$, so a weak bond has $(\|x_p\| + \|x_{p'}\|)^2 > \frac{3}{2}$ and carries mass $\|x_p\|^2 + \|x_{p'}\|^2 > \frac{3}{4}$. Two weak bonds with no common end would carry mass above $\frac{3}{2}$. Suppose two weak bonds share the end p and have other ends $q \neq q'$. With $\alpha = \|x_p\|$ and $b^2 = (\|x_q\|^2 + \|x_{q'}\|^2)/2 \le (1 - \alpha^2)/2$, the smaller of $\|x_q\|, \|x_{q'}\|$ is at most b, and $(\alpha + b)^2 \le \frac{3}{2}(\alpha^2 + 2b^2) \le \frac{3}{2}$ by the Cauchy-Schwarz inequality, so that bond is not weak. Hence all weak bonds join one pair of neighbouring sites. Deleting every bond between one pair of neighbours leaves Λ_N connected, because the pair lies on a square of bonds in another direction. Let $\lambda_0 > 0$ be the least, over anchors and over the finitely many graphs obtained from Λ_N by deleting the bonds of at most one pair of neighbours, of the smallest eigenvalue of the graph Laplacian form on fields vanishing at the anchor; it is positive because each graph is connected. Then (6.10) holds with $\kappa_N = \lambda_0/4$.

Existence and uniqueness. Along an anchored direction v the second derivative of $\mathcal{J}$ is $-2\sum \beta \cosh(\Delta\phi)(\Delta v)^2 \le -2\kappa_N \|v\|^2$, so $\mathcal{J}$ is strictly concave on anchored fields. Using $\cosh t \ge 1 + t^2/2$, $\mathcal{J}(\phi; x) \le 2\sum(1 - \beta) - \kappa_N\|\phi\|^2 + (\sum_p |\varrho_p| + 1)\|\phi\| \to -\infty$. An anchored maximizer therefore exists and is unique; it is stationary, so it solves (6.6), and every anchored solution is a stationary point, hence that maximizer.

Bound. At $\phi = \phi[x]$, $\mathcal{J}(\phi; x) \ge \mathcal{J}(0; x)$ and $2(\cosh t - 1) \ge t^2$ give

$$\sum_{p,k} \beta_{p,k}(\phi_p - \phi_{p+e_k})^2 \le \sum_p s_p \phi_p \le \Big(\sum_p |\varrho_p| + 1\Big) \max_p |\phi_p|.$$

With (6.10), $\kappa_N \max_p |\phi_p|^2 \le \kappa_N \sum_p \phi_p^2$, so $\max_p |\phi[x]_p| \le (\sum_p |\varrho_p| + 1)/\kappa_N$.

Budget. $E_{\mathrm{nf}}(x) = \mathcal{J}(\phi[x]; x) \ge \mathcal{J}(0; x) = \mathcal{B}_0(x) \ge 0$, and $E_{\mathrm{ph}} \ge 0$ because $H_0 \ge 0$, so $E_{\mathrm{ph}} = E - E_{\mathrm{nf}} \le E$.

(iii) By the implicit function theorem applied to (6.6), whose derivative in the field is the negative definite anchored Hessian of (ii), the selected field depends locally Lipschitz on the normalized state; extended by $x \mapsto \phi[x/\|x\|]$, it depends locally Lipschitz on x near the unit sphere, and by (ii) it is bounded. The interaction-picture vector field $u \mapsto -i\, e^{itH_0} V_{\phi[e^{-itH_0}u]} e^{-itH_0} u$ is therefore continuous in t, locally Lipschitz in u near the unit sphere and bounded there. The Picard-Lindelöf theorem gives local solutions, unique by Gronwall's inequality. Since V_ϕ is symmetric, $\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 = 2\,\mathrm{Re}\langle \psi, -iV_\phi \psi\rangle = 0$, so solutions stay on the sphere and, the vector field being bounded, exist for all real times. For the continuity law, e^{itH_0} acts site by site, so $\frac{\mathrm{d}}{\mathrm{d}t}\|\psi_p\|^2 = 2\,\mathrm{Im}\langle \psi_p, (V_\phi \psi)_p\rangle$. The diagonal term is real. The forward hop contributes $-2c_{p,k}\,\mathrm{Im}\langle \psi_p, W(z_{p,k})\psi_{p+e_k}\rangle = -j_{p,k}$, and the backward hop $-2c_{p-e_k,k}\,\mathrm{Im}\langle \psi_p, W(z_{p-e_k,k})^* \psi_{p-e_k}\rangle = +j_{p-e_k,k}$, which is (6.11).

(iv) A Weyl operator maps the domain of the photon number N_{ph} into itself, with $\|(N_{\mathrm{ph}} + 1)W(z)\psi\| \le c(\|z\|)\|(N_{\mathrm{ph}} + 1)\psi\|$, and the free group preserves the graph norm of N_{ph}. The argument of (iii) therefore runs in that graph norm, and the solution stays in the number domain, which lies in $\mathcal{D}$ because $H_0 \le \omega_{\max} N_{\mathrm{ph}}$. There the interaction-picture equation is equivalent to (6.7). For conservation, by (6.8) the right side $\langle x, H_\phi x\rangle + \sum_p \varrho_p \phi_p - \mathcal{A}(\phi)$ equals $E_{\mathrm{ph}}(x) + \mathcal{J}(\phi; x)$, whose derivative in ϕ vanishes at $\phi[x]$. Differentiating along the solution,

the motion of the field contributes nothing, and the remaining term is $2\,\mathrm{Re}\langle -iH_\phi\psi, H_\phi\psi\rangle = 0$. A coherent vector has finite photon moments of every order, so the coherent start is covered.

B.4.2 Proof of Theorem 6.15(iv)

First, the complex span of the profiles of single edges is dense in $\mathscr{H}_1$. Let $h \in \mathscr{H}_1$ be orthogonal to all of them. Since $P_T(k)$ is an orthogonal projection and $P_T h = h$, the profile of the unit current on the edge (x,d) gives $0 = \langle z_{(x,d)}, h\rangle = -i\int_{\mathbb{T}^3} w(k)e^{-ix\cdot k}h_d(k)\,\mathrm{d}\mu(k)$. On $\mathbb{T}^3$, $1-\cos t \ge 2t^2/\pi^2$ gives $\lambda(k) \ge 4|k|^2/\pi^2$, so $w(k)^2 \le \pi a/(4C_E|k|)$ and $w \in L^2(\mu)$. Then $wh_d \in L^1(\mu)$, all its Fourier coefficients vanish, and $wh_d = 0$ almost everywhere [67]. Since $w > 0$ for $k \neq 0$, $h_d = 0$ for every d, and $h = 0$.

Second, let $\mathscr{K}$ be the closure of the set A_{nat} of finite sums of the vectors $e^{-it\omega}g$. Because $-g$ is an integer-current profile whenever g is, A_{nat} is an additive group, and so is $\mathscr{K}$; both are invariant under $U_t = e^{-it\omega}$. For $v \in \mathscr{K}$, real c and an integer $n \ge 1$, the vector $n(U_{c/n}v - v)$ lies in $\mathscr{K}$, and $|n(e^{-icy/n}-1) + icy| \le c^2y^2/(2n)$ for real y, so $-ic\,\omega v \in \mathscr{K}$. Fix an integer-current profile g and let $\mathscr{S}_g$ be the closed real-linear span of the vectors $U_t g$, $t \in \mathbb{R}$. The last step puts $i\omega\mathscr{S}_g$ in $\mathscr{K}$. The real combinations of the operators U_t include $i\sin(t\omega) = (U_{-t} - U_t)/2$, so by Fejér's theorem $\mathscr{S}_g$ is invariant under $i\theta(\omega)$ for every real continuous θ on $[0,\omega_{\max}]$ with $\theta(0) = 0$: extend θ to an odd continuous function vanishing at $\pm 2\omega_{\max}$ and approximate it uniformly by sine polynomials [67]. Take $\theta_n(y) = n^2 y$ for $y \le 1/n$ and $\theta_n(y) = 1/y$ for $y \ge 1/n$. For $h \in \mathscr{S}_g$, $-i\theta_n(\omega)h \in \mathscr{S}_g$ and $i\omega(-i\theta_n(\omega)h) = \omega\theta_n(\omega)h \to h$ by dominated convergence, because $\omega > 0$ almost everywhere; so $\mathscr{S}_g \subset \mathscr{K}$. With $\theta_n(y) = \min(ny, 1)$ the same argument gives $i\theta_n(\omega)h \to ih$, so $\mathscr{S}_g$ is a complex subspace. Hence $\mathscr{K}$ contains every complex multiple of every integer-current profile and, being a closed group, the closure of their complex span, which is $\mathscr{H}_1$ by the first step.

B.4.3 Proof of Proposition 6.36

(i) Put $u = he^{ith}z_f$. Then $c(f,\delta_e,t) = 2i\,\mathrm{Im}\langle u, hz_{\delta_e}\rangle$, and $\langle u, hz_{\delta_e}\rangle$ is, up to a factor of modulus one, the Fourier coefficient at the edge e of $\omega w P_T u$. Parseval's identity, $\omega w^2 \le a/2$ (proof of Lemma 6.26), $\omega \le \omega_{\max}$ and the unitarity of e^{ith} give $\sum_e |\langle u, hz_{\delta_e}\rangle|^2 \le \frac{a}{2}\omega_{\max}\|hz_f\|^2$, hence the bound; the tails of a convergent series tend to zero.

(ii) For real currents $F_f(-k) = \overline{F_f(k)}$, while ω and w are even and $P_T(-k) = \overline{P_T(k)}$, so the two terms of c combine into $-2i\int \omega^2 w^2 \sin(t\omega)\langle F_f, P_T F_g\rangle\,\mathrm{d}\mu$. With $u = \sqrt{\lambda}/2$, $\omega w^2 = \frac{a}{2}\mathrm{D}(u)(1+u^2)^{-1/2}$ and $\omega = 2C_E u\,\mathrm{D}(u)$, so $\omega^2 w^2 \sin(t\omega) = \lambda\,\varrho_t(\lambda)$, and $\lambda(k)P_T(k)F_g(k) = F_{Lg}(k)$, where $L = \Delta - \nabla\,\mathrm{div}$ with the positive lattice Laplacian Δ, $(\mathrm{div}\,f)(x) = \sum_d [f(x,d) - f(x-e_d,d)]$ and its adjoint ∇. Each factor of ϱ_t after the constant has modulus at most one.

(iii) On $[0,\infty)$ the derivatives obey $|\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{D}(\sqrt{x}/2)| \le \frac{1}{24}$, $|\frac{\mathrm{d}}{\mathrm{d}x}(1+x/4)^{-1/2}| \le \frac{1}{8}$ and $|\frac{\mathrm{d}}{\mathrm{d}x}\,\mathrm{sinc}(t\omega(x))| \le \frac{t^2C_E^2}{6}$, with $\omega(x) = 2C_E\,\mathrm{arsinh}(\sqrt{x}/2)$. The first follows from $\mathrm{arsinh}\,u - u(1+u^2)^{-1/2} = \int_0^u s^2(1+s^2)^{-3/2}\,\mathrm{d}s \le u^3/3$; the last from $|\sin u - \sin v| \le |u^2 - v^2|/6$ and $0 \le \frac{\mathrm{d}}{\mathrm{d}x}\omega(x)^2 \le C_E^2$. Hence ϱ_t is Lipschitz on $[0,12]$ with constant $\frac{a|t|C_E^2}{2}(\frac{3}{24} + \frac{1}{8} + \frac{t^2C_E^2}{6}) \le K_t$, the cube of D contributing $3\cdot\frac{1}{24}$. Bernstein's estimate gives $|B_N\varrho_t(x) - \varrho_t(x)| \le K_t\,\mathbb{E}|12X/N - x| \le 6K_t/\sqrt{N}$ for X binomial with parameters N and $x/12$. A polynomial of degree N in $\lambda(k)$

is a trigonometric polynomial of range N, and L adds range two. Since $|F_f| \le M(f)$ and $|F_{Lg}| \le M(Lg)$, the error is at most $2 \cdot 6K_t N^{-1/2} M(f) M(Lg)$.

B.5 Proofs for Chapter 7

The notation is that of Chapter 7.

B.5.1 Proof of Lemma 7.5(ii)

For the edge $e = (x, d)$, $F_{1_e}(k) = e^{ix\cdot k} e_d$, so by (7.9) $P_{1_e,g}(t)$ is the Fourier coefficient $\widehat{\phi_{t,d}}(x) = \int e^{-ix\cdot k} \phi_{t,d}(k)\, \mathrm{d}\mu(k)$ of the d-th component of

$$\phi_t(k) = \big(t\omega(k) - \sin(t\omega(k))\big)\, \varrho(k)^2\, P_T(k) F_g(k).$$

Two factors are singular at $k = 0$: ϱ^2 grows like $|k|^{-1}$, and P_T has no limit there. Together with the time factor they form a smooth function. Put $u = \sqrt{s}/2$, so that $\omega = 2C_E \operatorname{arsinh} u$ and $\varrho^2 = a/(4C_E u\sqrt{1+u^2})$, and write $y - \sin y = y^3 \Psi(y)$ with the even entire function $\Psi(y) = \sum_{m\ge 0} (-1)^m y^{2m}/(2m+3)!$. For $s > 0$,

$$(t\omega - \sin t\omega)\, \varrho^2 = s\, \kappa_t(s), \qquad \kappa_t(s) = \frac{a C_E^2 t^3}{2} \Big(\frac{\operatorname{arsinh} u}{u}\Big)^3 \frac{\Psi(2C_E t \operatorname{arsinh} u)}{\sqrt{1+u^2}}.$$

Each factor of κ_t is an even real-analytic function of u, hence a real-analytic function of $s = 4u^2$ on a neighbourhood of $[0, \infty)$, jointly with t. By the projection formula, $s P_T F_g = s F_g - \big(\sum_{d'} r_{d'} F_{g,d'}\big)\rho$ is a trigonometric polynomial. Hence $\phi_t = \kappa_t(s)\, s P_T F_g$ off $k = 0$, and the right side is infinitely differentiable on $\mathbb{R} \times \mathbb{T}^3$. With Δ_k the Laplacian in k, $\psi_{t,d} = (1 - \Delta_k)\phi_{t,d}$ satisfies $|\psi_{t,d}| \le C_T$ on the compact set $[-T, T] \times \mathbb{T}^3$ for one constant C_T and all three d. Two integrations by parts on the torus give $\widehat{\psi_{t,d}}(x) = (1 + |x|^2)\widehat{\phi_{t,d}}(x)$, and Parseval's identity gives $\sum_x (1 + |x|^2)^2 |\widehat{\phi_{t,d}}(x)|^2 \le C_T^2$. For any set X of sites the Cauchy-Schwarz inequality then gives

$$\sum_{x \in X} |\widehat{\phi_{t,d}}(x)| \le C_T \Big(\sum_{x \in X} (1 + |x|^2)^{-2} \Big)^{1/2},$$

and $\sum_{x \in \mathbb{Z}^3} (1 + |x|^2)^{-2} < \infty$ in three dimensions. This gives the absolute sum. Choose a finite set X_η with $3C_T \big(\sum_{x \notin X_\eta} (1 + |x|^2)^{-2}\big)^{1/2} \le \eta$ and put $E_\eta = X_\eta \times \{0, 1, 2\}$; summing over the three directions gives the tail bound for every $|t| \le T$.

B.5.2 Proof of Lemma 7.7(ii)

The argument of the previous proof applies to the sine kernel. Let $\alpha(s) = \operatorname{arsinh}(\sqrt{s}/2)/(\sqrt{s}/2)$, real-analytic on a neighbourhood of $[0, \infty)$. Then $\omega^2 = C_E^2 s\, \alpha(s)^2$, $\omega \varrho^2 = \frac{a}{2} \alpha(s) / \sqrt{1 + s/4}$ and $\omega \sin(u\omega) = u\, \omega^2 \operatorname{sinc}(u\omega)$ with the even entire function $\operatorname{sinc} y = \sin y / y$, so

$$\omega^2 \sin(u\omega)\, \varrho^2 = s\, \lambda_u(s), \qquad \lambda_u(s) = \frac{a C_E^2 u}{2}\, \alpha(s)^3\, \frac{\operatorname{sinc}(u\omega)}{\sqrt{1 + s/4}},$$

real-analytic in s near $[0,\infty)$, jointly with u. As before, $\psi_u(k) = \lambda_u(s(k))\,s(k)P_T(k)F_g(k)$ is infinitely differentiable on $\mathbb{R}\times\mathbb{T}^3$, and $S_{1_{(x,d)},g}(u)$ is the Fourier coefficient $\widehat{\psi_{u,d}}(x)$. The same two integrations by parts and the Cauchy-Schwarz inequality give a constant C with $\sum_x |\widehat{\psi_{u,d}}(x)| \le C$ for all $|u| \le T$, all d, and $g = 1_{(0,d')}$ for each d'. For an edge $e' = (x', d')$, $F_{1_{e'}} = e^{ix'\cdot k}e_{d'}$, so the coefficients for e' at x are those for $(0,d')$ at $x - x'$. Hence $\sum_{e\in\mathscr{E}} |S_{1_e,1_{e'}}(u)| \le 3C$ for every edge e' and $|u| \le T$. Since π is injective, $\sum_{q\in F} |S_{1_{\pi(q)},1_{\pi(p)}}(u)| \le 3C$ for every finite set F of registers, and integrating over u gives $\sum_{q\in F}\mathscr{K}_T(p,q) \le 3CT^2$. Take $K_T = 3CT^2$.

B.5.3 Proof of Lemma 7.8

By the Kato-Rellich theorem B is self-adjoint on the domain of A. Put $\Gamma(s) = e^{isB}e^{-isA}$ and $\beta_s(Y) = e^{isB}Ye^{-isB}$. For ψ in the domain of A, $\frac{\mathrm{d}}{\mathrm{d}s}\Gamma(s)\psi = ie^{isB}(B-A)e^{-isA}\psi = ie^{isB}Ve^{-isA}\psi$. Integrating and extending by continuity, $\Gamma(s) = I + i\int_0^s e^{ixB}Ve^{-ixA}\,\mathrm{d}x$ on the whole space, so Γ is strongly differentiable with $\Gamma'(s) = i\beta_s(V)\Gamma(s)$, and $\Gamma(s)^*$ with derivative $-i\Gamma(s)^*\beta_s(V)$. The function

$$F(s) = e^{isB}e^{i(t-s)A}Xe^{-i(t-s)A}e^{-isB} = \Gamma(s)\,e^{itA}Xe^{-itA}\,\Gamma(s)^*$$

is therefore strongly differentiable with $F'(s) = i[\beta_s(V), F(s)] = \beta_s\big(i[V, e^{i(t-s)A}Xe^{-i(t-s)A}]\big)$. Since $F(0) = e^{itA}Xe^{-itA}$ and $F(t) = e^{itB}Xe^{-itB}$, integrating from 0 to t gives the formula.

B.5.4 Proof of Theorem 7.10

Write $K = K_T$ and $\mathscr{K} = \mathscr{K}_T$, and let λ, $\mathscr{K}'$, μ, σ_R and D_R be as in (7.11).

Step 1: the weight. By Lemma 7.7(ii) the rows of $\mathscr{K}'$ sum to at most $1+K$, so $\sum_q (\mathscr{K}'^{\,n})_{p_0 q} \le (1+K)^n$, $\sum_q \mu_q \le \sum_n \big((1+K)/(2+K)\big)^n = 2+K$, and $\mu_{p_0} \ge 1$. All terms are nonnegative, so for every register q

$$\sum_p \mu_p\,\mathscr{K}'(p,q) = \sum_{n\ge 0} \lambda^{-n}\big(\mathscr{K}'^{\,n+1}\big)_{p_0 q} \le \lambda\,\mu_q. \tag{B.2}$$

Consequently $D_R \le \sum_{q\notin R_4}\sum_p \mu_p\mathscr{K}'(p,q) \le \lambda\sum_{q\notin R_4}\mu_q$, which decreases with R and tends to zero because $\sum_q \mu_q < \infty$.

Step 2: majorants. Fix $R \subseteq S$ with $r \in R$. Nonnegative continuous functions β_p on $[0,T]$, one for each $p \in R_4$, form a *majorant* if

$$\big\|\tau_\sigma^S\big(\tau_\nu^{R,0}(X)\big) - \tau_\sigma^R\big(\tau_\nu^{R,0}(X)\big)\big\| \le \beta_p(|\sigma|)\,\|X\|$$

for every $p \in R_4$, every block X at p, and all real σ, ν with $|\sigma| + |\nu| \le T$. The evolutions are conjugations by unitaries, so $\beta_p \equiv 2$ is a majorant.

Step 3: one Volterra step. If β is a majorant, so is

$$\beta_p'(s) = a_p + \int_0^s \Big(b_p + \sum_{q\in R_4} L_{pq}\big(\beta_q(y) + \beta_p(y)\big)\Big)\,\mathrm{d}y, \tag{B.3}$$

with $a_p = 4\sigma_R(p)$, $b_p = 2\gamma\sigma_R(p)$ and $L_{pq} = 2\epsilon c_1\mathscr{K}'(p,q)$. Fix p, X, σ, ν, put $Y = \tau_\nu^{R,0}(X)$, $\mathscr{A} = (S\setminus R)_4$, $Y_x^R = \tau_{\sigma-x}^{R,0}(Y)$ and $Y_x^S = \tau_{\sigma-x}^{S,0}(Y)$, and recall $|\varphi_q(u)| \le 2\mathscr{K}(p,q)$ for $|u| \le T$.

Lemma 7.8 for $H_S = H_S^0 + \epsilon C_S$ and for $H_R = H_R^0 + \epsilon C_R$ gives

$$\tau_\sigma^S(Y) - \tau_\sigma^R(Y) = \big[\tau_\sigma^{S,0}(Y) - \tau_\sigma^{R,0}(Y)\big] + \int_0^\sigma \Big(\tau_x^S\big(i\epsilon[C_S, Y_x^S]\big) - \tau_x^R\big(i\epsilon[C_R, Y_x^R]\big)\Big)\,\mathrm{d}x,$$

and for x between 0 and σ all times stay in the window $|\sigma| + |\nu| \le T$.

The bracket. By Lemma 7.9(iii) it equals $\tau_\sigma^{R,0}(Y)\big(\Pi_{\mathscr{A}}(\varphi(\sigma+\nu)-\varphi(\nu)) - I\big)$; each phase difference is at most $4\mathscr{K}(p,q)$, so by (7.8) its norm is at most $\|X\| \sum_{q\in\mathscr{A}} 4\mathscr{K}(p,q) \le a_p\|X\|$.

The integrand. By Lemma 7.9(iii), $Y_x^S = Y_x^R \Pi_x$ with $\Pi_x = \Pi_{\mathscr{A}}(\varphi(\sigma - x + \nu) - \varphi(\nu))$, and $C_S = C_R + C_{\mathscr{A}}$ with $C_{\mathscr{A}} = \sum_{q\in\mathscr{A}} O_q(C)$. The clock C_R commutes with Π_x, and every block of $C_{\mathscr{A}}$ commutes with Y_x^R by Lemma 7.9(ii), since its register lies outside R_4. Hence $[C_S, Y_x^S] = [C_R, Y_x^R]\Pi_x + Y_x^R[C_{\mathscr{A}}, \Pi_x]$, and with $Z = i\epsilon[C_R, Y_x^R]$ the integrand equals

$$\tau_x^S(Z) - \tau_x^R(Z) + \tau_x^S\big(Z(\Pi_x - I)\big) + \tau_x^S\big(i\epsilon\, Y_x^R[C_{\mathscr{A}}, \Pi_x]\big).$$

Split the clock of each register q into its blocks $c_{q,g} = O_q(Q^{g_1} C Q^{g_2})$. For $q \ne p$, Lemma 7.9(ii) gives $[c_{q,g}, Y_x^R] = (\zeta - 1) Y_x^R c_{q,g}$ with $|\zeta - 1| \le 2\mathscr{K}(p,q)$; for $q = p$, $\|[c_{p,g}, Y_x^R]\| \le 2\|c_{p,g}\|\|X\|$. Hence $\|Z\| \le \epsilon(2 + 2K)c_1\|X\|$, and with $\|\Pi_x - I\| \le 4\sigma_R(p)$ the third term is at most $4\epsilon(2 + 2K)c_1\sigma_R(p)\|X\|$. By Lemma 7.9(i) each block of $C_{\mathscr{A}}$ passes Π_x at a phase within $4\mathscr{K}(p,q)$ of one, so the fourth term is at most $4\epsilon c_1\sigma_R(p)\|X\|$. Together these are at most $4\epsilon c_1(3+2K)\sigma_R(p)\|X\| = b_p\|X\|$.

For the first term, $Z = i\epsilon \sum_{q\in R_4}\sum_g [c_{q,g}, Y_x^R]$. Both $Y_x^R = \tau_{\sigma-x+\nu}^{R,0}(X)$ and $c_{q,g} = \tau_0^{R,0}(c_{q,g})$ are clock-free evolved blocks to which the majorant applies at level $|x|$, since $|x| + |\sigma - x + \nu| \le T$. The evolutions are multiplicative, so $\tau_x^S(UU') - \tau_x^R(UU') = (\tau_x^S U - \tau_x^R U)\,\tau_x^S U' + \tau_x^R U(\tau_x^S U' - \tau_x^R U')$. For $q \ne p$ this gives $\|\tau_x^S([c_{q,g}, Y_x^R]) - \tau_x^R([c_{q,g}, Y_x^R])\| \le 2\mathscr{K}(p,q)\big(\beta_p(|x|) + \beta_q(|x|)\big)\|X\|\|c_{q,g}\|$, and for $q = p$, applied to $c_{p,g}Y_x^R$ and to $Y_x^R c_{p,g}$, at most $2\big(\beta_p(|x|) + \beta_p(|x|)\big)\|X\|\|c_{p,g}\|$. Multiplying by ϵ and summing over g and q, the first term is at most $\|X\| \sum_{q\in R_4} L_{pq}\big(\beta_q(|x|) + \beta_p(|x|)\big)$. Adding the bracket and integrating proves the step.

Step 4: iteration. Starting from $\beta^0 \equiv 2$, Step 3 gives majorants β^n. Put $\Phi_n(s) = \sum_{p\in R_4} \mu_p \beta_p^n(s)$, so that $\sum_p \mu_p a_p = 4D_R$ and $\sum_p \mu_p b_p = 2\gamma D_R$. By (B.2), $\sum_{p\in R_4} \mu_p L_{pq} \le 2\epsilon c_1 \lambda \mu_q$, and $\sum_{q\in R_4} L_{pq} \le 2\epsilon c_1(1 + K)$. Multiplying (B.3) by μ_p and summing, with $2\epsilon c_1(\lambda + 1 + K) = \gamma$,

$$\Phi_{n+1}(s) \le 4D_R + 2\gamma D_R s + \gamma \int_0^s \Phi_n(y)\,\mathrm{d}y.$$

Since $\Phi_0 \le 2(2 + K)$, induction gives $\Phi_n(s) \le D_R(4 + 2\gamma s)e^{\gamma s} + 2(2 + K)(\gamma s)^n/n!$, using $\int_0^s \gamma(4 + 2\gamma y)e^{\gamma y}\,\mathrm{d}y \le (4 + 2\gamma s)e^{\gamma s} - 4 - 2\gamma s$. For the block X at p_0, with $\sigma = t$, $\nu = 0$ and $\mu_{p_0} \ge 1$, $\|\tau_t^S(X) - \tau_t^R(X)\| \le \beta_{p_0}^n(|t|)\|X\| \le \Phi_n(|t|)\|X\|$. Letting $n \to \infty$ gives part (i), and Step 1 gives $D_R \to 0$.

Step 5: the limit. Fix $O = O_{p_0}(A)$, $T \ge 0$ and $\eta > 0$, and let $C_A = \sum_{\beta_1,\beta_2} \|Q^{\beta_1} A Q^{\beta_2}\|$. By Step 1 there is a finite region $R_0 \ni r$ with $C_A D_R(4 + 2\gamma T)e^{\gamma T} \le \eta/2$ for every $R \supseteq R_0$. For $R_0 \subseteq R \subseteq S$, summing part (i) over the four blocks of O gives $\|\tau_t^S(O) - \tau_t^R(O)\| \le \eta/2$ for $|t| \le T$, and any two regions containing R_0 are compared through their union.

B.5.5 Proof of Theorem 7.15

Fix $R \subseteq S$, put $Y_\nu = \tau^{R,0}_\nu(\widehat{W}(y))$ and $\varphi_q(\nu) = 2\xi(z_q, y, \nu)$. By Theorem 7.12 with $b = 0$, $Y_\nu = \widehat{W}(e^{i\nu h}y)\,\Pi_{R_4}(\varphi(\nu))$, a unitary. As in Lemma 7.9: (a) $\tau^{S,0}_{t_1}(Y_{t_2}) = Y_{t_1+t_2}\Pi_{(S\setminus R)_4}\big(\varphi(t_1+t_2)-\varphi(t_2)\big)$; (b) a block c at q, with indices g_1, g_2, satisfies $cY_\nu = \zeta Y_\nu c$ with $\zeta = e^{i(g_2-g_1)\varphi_q(\nu)}$ if $q \in R_4$ and $\zeta = 1$ otherwise; (c) $|\varphi_q(\nu)| \le 2u(q)$ for $|\nu| \le T$.

Call a nonnegative continuous β on $[0,T]$ a majorant if $\|\tau^S_\sigma(Y_\nu) - \tau^R_\sigma(Y_\nu)\| \le \beta(|\sigma|)$ whenever $|\sigma|+|\nu| \le T$; the constant 2 is one. With $A = A_{R,S}$, $B' = \epsilon\big((2c_1U + c_1)A + \Delta_{R,S}\big)$ and $L = 2\epsilon c_1 U$, if β is a majorant so is $\beta'(s) = A + \int_0^s (B' + L\beta(x))\,\mathrm{d}x$. Indeed, with $Y = Y_\nu$, $Y^R_x = Y_{\sigma-x+\nu}$, $Y^S_x = \tau^{S,0}_{\sigma-x}(Y)$ and $\mathscr{A} = (S\setminus R)_4$, Lemma 7.8 gives the same decomposition as in Step 3 above. By (a) the bracket is $Y_{\sigma+\nu}\big(\Pi_{\mathscr{A}}(\varphi(\sigma+\nu)-\varphi(\nu)) - I\big)$, of norm at most A by (c). By (a) and (b), $[C_S, Y^S_x] = [C_R, Y^R_x]\Pi_x + Y^R_x[C_{\mathscr{A}}, \Pi_x]$ with $\Pi_x = \Pi_{\mathscr{A}}(\varphi(\sigma-x+\nu)-\varphi(\nu))$. By (b) and (c), $[c_{q,g}, Y^R_x] = (\zeta-1)Y^R_x c_{q,g}$ with $|\zeta-1| \le 2u(q)$, so $\|[C_R, Y^R_x]\| \le 2c_1U$, and with $\|\Pi_x - I\| \le A$ the term with $\Pi_x - I$ is at most $2\epsilon c_1 UA$; each block of $C_{\mathscr{A}}$ passes Π_x at a phase within $4u(q)$ of one, so the term with $[C_{\mathscr{A}}, \Pi_x]$ is at most $\epsilon c_1 A$. For the remaining term, $Z = i\epsilon\sum_{q\in R_4}\sum_g(\zeta_{q,g}-1)Y^R_x c_{q,g}$, and since Y^R_x is unitary and $|x| + |\sigma - x + \nu| \le T$,

$$\big\|\tau^S_x(Y^R_x c_{q,g}) - \tau^R_x(Y^R_x c_{q,g})\big\| \le \beta(|x|)\|c_{q,g}\| + \big\|\tau^S_x(c_{q,g}) - \tau^R_x(c_{q,g})\big\|.$$

Summing with the weights $2u(q)$, it is at most $\epsilon\big(2c_1U\beta(|x|) + \Delta_{R,S}\big)$. The terms together are at most $B' + L\beta(|x|)$.

Starting from $\beta^0 = 2$, induction gives majorants with $\beta^n(s) \le (A + B's)e^{Ls} + 2(Ls)^n/n!$, since $\int_0^s L(A + B'x)e^{Lx}\,\mathrm{d}x \le (A+B's)e^{Ls} - A - B's$. Taking $\sigma = t$, $\nu = 0$ and letting $n \to \infty$ gives the bound of the theorem, since $e^{L|t|} \le e^{2\epsilon c_1 UT}$.

For the limit, fix T, $\eta > 0$ and small $\vartheta, \vartheta' > 0$, and choose a finite set F of registers with $\sum_{q\notin F} u(q) \le \vartheta$. Each clock block $c_{q,g}$ is a register observable, so by Theorem 7.10 there is a finite region beyond which its evolutions over any two nested regions differ by at most ϑ' for $|x| \le T$. Let R_0 contain the cells of F and these regions for all $q \in F$ and all g. For $R_0 \subseteq R \subseteq S$, the registers of $(S\setminus R)_4$ lie outside F, so $A_{R,S} \le 4\vartheta$, and with $\|\tau^S_x(c) - \tau^R_x(c)\| \le 2\|c\|$ outside F one may take $\Delta_{R,S} = 8U\vartheta' + 4c_1\vartheta$. For small ϑ, ϑ' the bound is at most $\eta/2$, and any two regions containing R_0 are compared through their union. The clouds are test profiles by Proposition 7.14(ii).

B.5.6 Proof of Lemma 7.44

The equation. Write $\varpi(L) = 2\mathcal{S}(L)/\sqrt{L^2+4L}$ with $\mathcal{S}(L) = \operatorname{arsinh}(\sqrt{L}/2)$, since $\sqrt{L}\sqrt{1+L/4} = \tfrac{1}{2}\sqrt{L^2+4L}$. For $L > 0$, $\mathcal{S}'(L) = 1/\big(4\sqrt{L}\sqrt{1+L/4}\big) = 1/\big(2\sqrt{L^2+4L}\big)$, so

$$\varpi'(L) = \frac{1}{L^2+4L} - \frac{(L+2)\,\varpi(L)}{L^2+4L},$$

which is (7.15).

No rational solution. Suppose $\varpi D = N$ on (α,β) with $D \neq 0$. Differentiating, $\varpi' D + \varpi D' = N'$; multiplying by D and using $\varpi D = N$ gives $\varpi' D^2 = N'D - ND'$. Multiplying (7.15) by D^2, the

polynomial

$$E(N,D) = (N'D - ND')\,X(X+4) + (X+2)\,ND - D^2$$

vanishes on (α,β), hence identically. We show $E(N,D) \neq 0$ for every $D \neq 0$, by induction on the degree of D.

If $D = d$ is a nonzero constant, $E = d(G-d)$ with $G = N'X(X+4) + (X+2)N$. For $N = 0$, $E = -d^2 \neq 0$. For $N \neq 0$ of degree n with leading coefficient c, the coefficient of X^{n+1} in G is $nc + c = (n+1)c \neq 0$, so G is not constant and $G - d \neq 0$.

Let D have positive degree and a root $c \in \mathbb{C}$. If $N(c) = 0$, write $N = (X-c)N_1$ and $D = (X-c)D_1$; a direct computation gives $E(N,D) = (X-c)^2 E(N_1,D_1)$, which is nonzero by induction. If $N(c) \neq 0$, write $D = (X-c)^{m+1}D_1$ with $m \geq 0$ and $D_1(c) \neq 0$. Then $N'D - ND' = (X-c)^m\big[(X-c)(N'D_1 - ND_1') - (m+1)ND_1\big]$, and $E = (X-c)^m G$ with

$$G = \big[(X-c)(N'D_1 - ND_1') - (m+1)ND_1\big]X(X+4) + (X+2)N(X-c)D_1 - (X-c)^{m+2}D_1^2.$$

So $G(c) = -(m+1)N(c)D_1(c)\,c(c+4)$, which is nonzero unless $c \in \{0,-4\}$. If $c = 0$, $G(0) = 0$ and the coefficient of X in G is $-4(m+1)N(0)D_1(0) + 2N(0)D_1(0) = -(4m+2)N(0)D_1(0) \neq 0$. If $c = -4$, expanding in $Y = X+4$, the coefficient of Y is $4(m+1)N(-4)D_1(-4) - 2N(-4)D_1(-4) = (4m+2)N(-4)D_1(-4) \neq 0$. In every case $E(N,D) \neq 0$, a contradiction.

B.5.7 Proof of Theorem 7.45

Suppose only finitely many edges (y,d) have $V(z_{(y,d)},w) \neq 0$. By Lemma 7.43 the bounded function $N_{J,d}$ has only finitely many nonzero Fourier coefficients, so it equals almost everywhere the trigonometric polynomial P with these coefficients. $N_{J,d}$ is continuous on $\mathbb{T}^3 \setminus \{0\}$, where $s > 0$, and two functions continuous on an open set and equal almost everywhere on it agree there. Hence

$$2a\,\varpi(s(k))\,T_d(k) = s(k)\,P(k) \qquad (k \neq 0). \tag{B.4}$$

Restrict to the line $k(u) = (u, k_{0,1}, k_{0,2})$ through k_0 along the first axis. There $s(k(u)) = 2(1-\cos u) + \sigma$ with $\sigma = 2(1-\cos k_{0,1}) + 2(1-\cos k_{0,2}) \geq 0$. Every function $u \mapsto e^{i(mu+b)}$, $m \in \mathbb{Z}$, $b \in \mathbb{R}$, has the form $A(\cos u) + \sin u\,B(\cos u)$ with complex polynomials A, B, by the Chebyshev identity $e^{imu} = T_m(\cos u) + i \sin u\,U_{m-1}(\cos u)$ for $m \geq 1$ and its conjugate for $m < 0$; such functions form an algebra, since $\sin^2 u = 1 - \cos^2 u$. The polynomials T_d and P are finite sums of products of such exponentials along the line, so $T_d(k(u)) = A(\cos u) + \sin u\,B(\cos u)$ and $P(k(u)) = A'(\cos u) + \sin u\,B'(\cos u)$.

For $u \in (0,\pi)$, $L(u) = 2(1-\cos u) + \sigma > 0$, so $k(u) \neq 0$ and $k(-u) \neq 0$. Evaluating (B.4) at u and at $-u$, where the cosine is the same and the sine changes sign, and adding and subtracting (with $\sin u \neq 0$), gives

$$2a\,\varpi(L)\,A(\cos u) = L\,A'(\cos u), \qquad 2a\,\varpi(L)\,B(\cos u) = L\,B'(\cos u) \qquad (0 < u < \pi).$$

At $u = k_{0,0}$, $T_d(k_0) = A(\cos k_{0,0}) + \sin k_{0,0}\,B(\cos k_{0,0}) \neq 0$, so A or B is a nonzero polynomial; call it Q and its partner Q'. For $L \in (\sigma, \sigma+4)$, $u = \arccos\big(1 - (L-\sigma)/2\big)$ lies in $(0,\pi)$ and $\cos u = \ell(L) = 1 + \sigma/2 - L/2$. Hence $\varpi(L) \cdot 2aQ(\ell(L)) = LQ'(\ell(L))$ on $(\sigma,\sigma+4) \subseteq [0,\infty)$. Since ℓ is affine and not constant and $a > 0$, $D = 2aQ \circ \ell$ is a nonzero polynomial, and with $N = X \cdot (Q' \circ \ell)$ this contradicts Lemma 7.44.

B.5.8 Proof of Theorem 7.49

(i) Let $\mathfrak{A}_\pi^{\mathrm{loc}}$ be the algebra generated by the register observables and the test displacements; $\mathfrak{A}_\pi$ is its norm closure. With no background, Theorem 7.4 writes the regional evolution of a block of a register whose cell lies in R as a phase times the block, the displacement $\widehat{W}\big(\delta(e^{ith}z_p - z_p)\big)$, whose profile is a test profile by Proposition 7.14(ii), and the finite product $\Pi_{R_4}(\varphi)$, whose factors $\Pi_q(\varphi_q) = O_q\big(I + (e^{i\varphi_q} - 1)Q\big)$ are register observables; a register observable is the sum of its four blocks. Theorem 7.12 writes $\tau_t^R(\widehat{W}(y))$ as $\widehat{W}(e^{ith}y)\,\Pi_{R_4}(\varphi(t))$, and $e^{ith}y$ is a test profile with y (Proposition 7.14(i)). So for every generator X, $\tau_t^R(X) \in \mathfrak{A}_\pi^{\mathrm{loc}}$ once R contains the cell of X, and the limit $\tau_t(X)$ lies in the norm closure $\mathfrak{A}_\pi$. By Theorem 7.16, τ_t is linear and multiplicative on $\mathfrak{B} \supseteq \mathfrak{A}_\pi$, so it maps $\mathfrak{A}_\pi^{\mathrm{loc}}$ into $\mathfrak{A}_\pi$; it is isometric and $\mathfrak{A}_\pi$ is closed, so it maps $\mathfrak{A}_\pi$ into itself. Every $X \in \mathfrak{A}_\pi$ equals $\tau_t(\tau_{-t}(X))$ with $\tau_{-t}(X) \in \mathfrak{A}_\pi$, so the map is onto. The remaining properties are Theorem 7.16(ii).

(ii) For $X \in \mathfrak{A}_\pi$, $t \mapsto \omega_{\mathrm{ref}}(\tau_t(X))$ is continuous (Theorem 7.20) and bounded by $\|X\|$, so $\omega_T(X)$ is defined and $|\omega_T(X)| \le \|X\|$. The functionals bounded in this way form the product $\prod_{X \in \mathfrak{A}_\pi}\{c \in \mathbb{C} : |c| \le \|X\|\}$, which is compact for pointwise convergence by Tychonoff's theorem, so the net $(\omega_T)_{T\to\infty}$ has cluster points. Every ω_T is linear, has $\omega_T(I) = 1$, and is nonnegative on X^*X, because $\tau_t(X^*X) = \tau_t(X)^*\tau_t(X)$ and ω_{ref} is a state (Theorem 7.19); these properties survive pointwise limits, so every cluster point ϕ is a state. For real s, $\tau_s(X) \in \mathfrak{A}_\pi$ by (i), and $\omega_{\mathrm{ref}}(\tau_t(\tau_s(X))) = \omega_{\mathrm{ref}}(\tau_{t+s}(X))$, so

$$\omega_T(\tau_s(X)) - \omega_T(X) = \frac{1}{T}\Big(\int_T^{T+s} - \int_0^s\Big)\omega_{\mathrm{ref}}(\tau_t(X))\,\mathrm{d}t, \qquad \big|\omega_T(\tau_s(X)) - \omega_T(X)\big| \le \frac{2|s|\|X\|}{T}.$$

Along a subnet that converges to ϕ the right side tends to zero, so $\phi(\tau_s(X)) = \phi(X)$. The reference state is not invariant by Proposition 7.25.

(iii) By (i) and invariance, $\phi(\tau_t(X)^*\tau_t(Y)) = \phi(\tau_t(X^*Y)) = \phi(X^*Y)$. So $\|\Gamma_\phi(\tau_t(X))\Omega_\phi\| = \|\Gamma_\phi(X)\Omega_\phi\|$: the map is well defined and isometric on the dense subspace $\Gamma_\phi(\mathfrak{A}_\pi)\Omega_\phi$, it is onto that subspace because τ_t is onto $\mathfrak{A}_\pi$, and it extends to a unitary U_t of $\mathscr{H}_\phi$. From $\tau_t(I) = I$, $U_t\Omega_\phi = \Omega_\phi$; from the group law, $U_tU_s = U_{t+s}$; and $U_t\Gamma_\phi(Z)\Gamma_\phi(X)\Omega_\phi = \Gamma_\phi(\tau_t(Z))\Gamma_\phi(\tau_t(X))\Omega_\phi = \Gamma_\phi(\tau_t(Z))U_t\Gamma_\phi(X)\Omega_\phi$ gives $U_t\Gamma_\phi(Z) = \Gamma_\phi(\tau_t(Z))U_t$.

(iv) Evaluation at X is continuous for pointwise convergence, so a cluster point takes the limit of $\omega_T(X)$ wherever that limit exists. Theorem 7.48 gives the limit on products of register observables. On an assignment whose edges have one direction and fill it, every flip pattern uses one direction, so the proof of (7.16) in Proposition 7.51 applies word for word. For $F = Q\chi\chi^*(I - Q)$, $E(F) = 0$, while $\omega_{\mathrm{ref}}(O_p(F)) = \langle\chi, F\chi\rangle = \frac{1}{4}$ (Proposition 7.25).

B.6 Proofs for Chapter 8

B.6.1 Proof of Theorem 8.13

Let V have seven recognizers and let a law meet the native target with every move of nonzero weight from a reached state split into pairs. Write $\sigma(x) = (s_v(x))_{v\in V}$ for the list of imbalances at a state x. We derive a contradiction in four steps.

Step 1: two tests on lists. By Lemma 8.12, every reached list σ is *balanced*: for each account k,

$$D_k(\sigma) = \sum_v \big(K(s_v, e_k) - K(s_v, -e_k)\big) = 0.$$

By the second claim of the lemma, every reached list is also *extendable*: for every recognizer v and every post o there is a split move μ with $\mu_v = o$ whose successor list $\sigma + \mu$ is again reached, hence balanced. Both tests are conditions on the list alone. Both are unchanged by the 48 signed permutations of the accounts, which preserve K and the set of split moves, and by reordering the recognizers.

Step 2: exact arithmetic. The enclosures of K give, at each of the 129 imbalances with $|s|_1 \le 4$ (the points of the ball $|s|_1 \le 4$), integer lower and upper bounds, in units of 2^{-24}, on the three drifts $K(s, e_k) - K(s, -e_k)$. A list is *proved unbalanced* when for some account the sum of the lower bounds is positive or the sum of the upper bounds is negative. A search that discards only lists proved unbalanced keeps a superset of the balanced lists, which is the safe direction.

Step 3: the search. Starting from the list of seven zeros, enumerate the split moves, keep the successor lists not proved unbalanced, and reduce each modulo signed permutations and reordering. A list is discarded as not extendable when some recognizer has a post all of whose completions to a split move lead to lists proved unbalanced. The search finds that after one tick the reached list is, up to symmetry, one of the five lists of Table B.2; after two ticks one of 33 lists; 52 further lists are discarded on the way as not extendable; and after three ticks every list that survives has a recognizer at imbalance 0. Each step of the search is a finite check on integers.

Table B.2. The lists of imbalances that can follow the first tick of a conserving law with native marginals on seven recognizers, up to the 48 signed permutations of the accounts and reordering. Every other list after one tick fails the extendability test.

posting recognizers	list of imbalances	recognizers at 0
0	$(0, 0, 0, 0, 0, 0, 0)$	7
2	$(0, 0, 0, 0, 0, e_1, -e_1)$	5
4	$(0, 0, 0, e_1, -e_1, e_1, -e_1)$	3
4	$(0, 0, 0, e_1, -e_1, e_2, -e_2)$	3
6	$(0, e_1, -e_1, e_2, -e_2, e_3, -e_3)$	1

Step 4: counting returns. By Step 3, at every state reached after three ticks at least one recognizer has imbalance 0, so the expected number N_3 of balanced recognizers after three ticks satisfies $N_3 \ge 1$. By Theorem 8.4, each recognizer's imbalance after three ticks has the law of a single recognizer under K from the empty record, so $N_3 = 7\, P_K(s_3 = 0 \mid s_0 = 0)$. The enclosures of K bound the three-tick return chance $P_K(s_3 = 0 \mid s_0 = 0)$ below $1/7$ (its value is about 0.1015). Hence $N_3 < 1$, a contradiction.

For the consequence, if an odd number n carries such a law, so does $n + 2$ (Proposition 8.11); one, three, five and seven carry none. So the odd numbers that carry one, if any, are exactly those from some odd $N \ge 9$ on. □

B.7 Proofs for Chapter 9

B.7.1 Proof of Theorem 9.10

The cell. Fix integers $R \geq 2/h$ and $N \geq 14/h$; the construction takes $R = \lceil 2/h \rceil$ and $N = \lceil 14/h \rceil$. The cell has layers $i = 0, \dots, R$, layer i holding $n_i = \lceil N\varphi^i \rceil$ states of energy $i \log \varphi$. With $Z_C = \sum_i n_i \rho^i$, a state on layer i has Gibbs weight ρ^i / Z_C, and since $N\varphi^i \leq n_i < N\varphi^i + 1$,

$$N \leq n_i \rho^i < N + 1 \qquad (0 \leq i \leq R),$$

so every layer carries nearly the same Gibbs mass and $Z_C \leq (R+1)(N+1)$.

Blocks. A post changes the level by exactly one. Take an edge $\{s, t\}$ with $|t|_1 = |s|_1 + 1$. For $1 \leq i \leq R$, a joint state (s, c) with c on layer i and a joint state (t, c') with c' on layer $i - 1$ have the same total energy $(|s|_1 + i) \log \varphi$. For each such i choose $m_i = \lceil P(s,t) Z_C \varphi^i / R \rceil$ states of layer i beside s and as many of layer $i - 1$ beside t, and pair them one to one. By (P3), $P(t, s) = \varphi P(s, t)$, so also $m_i = \lceil P(t,s) Z_C \varphi^{i-1} / R \rceil$: on any layer j, an edge from x to a neighbour y uses either no states beside x or $\lceil P(x,y) Z_C \varphi^j / R \rceil$ of them.

Room. Different edges at one state must use different cell states. At x and layer j the six edges use at most $\sum_y (P(x,y) Z_C \varphi^j / R + 1) \leq (1-h) Z_C \varphi^j / R + 6$ states, because the posts out of x add to $1 - P(x,x) \leq 1 - h$. From $Rh \geq 2$ and $Nh \geq 14$, $(1-h)(R+1) \leq (1 - \frac{h}{2})R$ and $(1 - \frac{h}{2})(N+1) + 6 \leq N$. With $Z_C \leq (R+1)(N+1)$ this gives $(1-h) Z_C / R \leq N - 6$, so the count is at most $(N-6)\varphi^j + 6 \leq N\varphi^j \leq n_j$. The blocks fit, every cell state beside x serving at most one edge.

The exchange. Let $A_e = \sum_{i=1}^{R} m_i \rho^i / Z_C$ be the Gibbs mass of the edge's states beside s. Since $m_i \geq P(s,t) Z_C \varphi^i / R$, $A_e \geq P(s,t)$, so there is $\theta_e \in [0, \pi/2]$ with $A_e \sin^2 \theta_e = P(s,t)$. Let

$$H = \sum_e \sum_{(x,x') \text{ paired on } e} \theta_e \big(|x\rangle\langle x'| + |x'\rangle\langle x| \big).$$

Each basis state lies in at most one pair, so H is real, symmetric and bounded with norm at most $\pi/2$; it conserves total energy and moves the recognizer by at most one post. On each pair e^{-iH} is the rotation of Example 9.7, $|x\rangle \mapsto \cos\theta_e |x\rangle - i \sin\theta_e |x'\rangle$, and it fixes every unpaired state.

The read. For a neighbour t of s, $\langle t, c' | e^{-iH} | s, c \rangle$ is $-i \sin\theta_e$ when (s, c) is paired with (t, c') and zero otherwise, so the read (9.3) gives $\sum_{c \text{ paired on } e} \pi_C(c) \sin^2 \theta_e = A_e \sin^2 \theta_e = P(s,t)$. Beside t the paired states sit one layer lower and carry Gibbs mass $\sum_i m_i \rho^{i-1} / Z_C = \varphi A_e$, so the same computation gives $\varphi P(s,t) = P(t,s)$. Between states that are not neighbours both laws vanish. Both have rows summing to one, by (P1) and Theorem 9.9, and agree off the diagonal, so they agree on it. For K, $h = c_0^2$ gives $2/h = 14.699$ and $14/h = 102.890$, so $R = 15$ and $N = 103$. □

B.7.2 Proof of Proposition 9.15

(i) The proof of Theorem 9.10 uses only these features of the lattice: finitely many moves at each state, each with a reverse, each changing the level by at most a fixed number, and an orientation of the level-keeping moves that the reverse flips. Profiles under the joint moves

other than the all-hold have them, with level λ, maximal change $|V|$ and the orientation given by the first recognizer that posts. A move that changes the level by g pairs layer i beside its lower end with layer $i-g$ beside its upper end, which conserves total energy, and the room argument goes through with $R=\lceil 2|V|/h\rceil$ and $N=\lceil 2\cdot 7^{|V|}/h\rceil$. The product of K meets the hypotheses because K meets (P1)–(P3) recognizer by recognizer, with $h=c_0^{2|V|}$ by (F4).

(ii) Put $b=|a|^{1/|V|}\le f$ and $K_\pm(s,s+o)=K(s,s+o)\pm b\,\eta(s,o)/w(s)$, with η the sign function of the unit square (written σ in Proposition 8.8). Since $b\le f\le w(s)K(s,s+o)$ on the square, $K_\pm\ge 0$; η sums to zero over the moves, so the rows of $K_\pm$ sum to one; $\eta(s+o,-o)=\eta(s,o)$, so $w(s)K_\pm(s,t)=w(t)K_\pm(t,s)$; and η vanishes on holds, so $K_\pm$ has K's holds, at least c_0^2. By Theorem 9.10 both are reads of one cell C, whose size depends only on the floor. Let ς be the sign of a. The shared cell is a label $\epsilon\in\{+1,-1\}^V$ with $\prod_v\epsilon_v=\varsigma$, of energy zero (so uniform over the $2^{|V|-1}$ such labels in the Gibbs state), together with one copy of C per recognizer; H_v applies the pairs of K_{ϵ_v} to recognizer v and its own copy. The H_v act on different factors and preserve the label, so they commute and $e^{-i\sum_v H_v}=\prod_v e^{-iH_v}$. With $k_v=K(\sigma_v,\sigma_v+\mu_v)$ and $d_v=\eta(\sigma_v,\mu_v)/w(\sigma_v)$ the read is

$$\frac{1}{2^{|V|-1}}\sum_{\epsilon}\prod_{v}(k_v+\epsilon_v b\,d_v)=\sum_{T\subseteq V}b^{|T|}\Big(\prod_{v\notin T}k_v\Big)\Big(\prod_{v\in T}d_v\Big)\frac{1}{2^{|V|-1}}\sum_{\epsilon}\prod_{v\in T}\epsilon_v.$$

The last average is 1 for $T=\emptyset$ and ς for $T=V$; for any other T pick $u\in T$ and $u'\notin T$: flipping ϵ_u and $\epsilon_{u'}$ keeps $\prod_v\epsilon_v$ and flips $\prod_{v\in T}\epsilon_v$, so the average is 0. The read is $\prod_v k_v+a\prod_v d_v=P_a$. Its marginals are K's rows since η sums to zero over any one recognizer's moves and V has a second recognizer; detailed balance against $\prod_v w(\sigma_v)$ and the positive hold follow from the exchange as in Theorem 9.9; P_a differs from the product at the empty record when every recognizer posts e_1, by $a\neq 0$. Joint holding-and-return balance would force the product, by (iii).

(iii) A read of a shared cell is in detailed balance against ρ^λ by the argument of Theorem 9.9 with the summed level; with joint holding-and-return balance, Theorem 8.7 gives the product. □

B.7.3 Proof of Proposition 9.16(iii) and (iv)

(iii) Put $h=e^{-U}$ and fix $p,q\in\mathbb{Z}^3$. For a local move o, the marginal of the second recognizer of R at $(p,q+o)$ for the move $-o$ is $K(q+o,q)$; multiplying by $W_U(p,q+o)$ and applying detailed balance to each term,

$$W_U(p,q+o)K(q+o,q)=\sum_{o'}W_U(p+o',q)\,R\big((p+o',q),(-o',o)\big).$$

Summing over o and using the first recognizer's marginal at $(p+o',q)$ for the move $-o'$,

$$\sum_{o}W_U(p,q+o)K(q+o,q)=\sum_{o'}W_U(p+o',q)K(p+o',p).$$

By (P3), $w(q+o)K(q+o,q)=w(q)K(q,q+o)$ and $w(p+o')K(p+o',p)=w(p)K(p,p+o')$; dividing by $w(p)w(q)$,

$$\sum_{o}K(q,q+o)\,h(p,q+o)=\sum_{o'}K(p,p+o')\,h(p+o',q)\qquad\text{for all } p,q.\qquad(*)$$

Rows of K sum to one, so $g = h - 1$ also satisfies $(*)$, and g vanishes outside a finite set. Suppose $g \neq 0$ somewhere; write $z^{(1)}$ for the first coordinate. Among the profiles (x, y) with $g(x, y) \neq 0$ choose one with the largest $x^{(1)} - y^{(1)}$, and among those one with the largest $x^{(1)}$. Apply $(*)$ at $p = x + e_1$, $q = y$. On the left, $(x + e_1, y + o)$ has difference above the largest when $o^{(1)} \leq 0$, and equal to it with the larger first coordinate $x^{(1)} + 1$ when $o = e_1$, so g vanishes there. On the right, $(x + e_1 + o', y)$ has difference above the largest unless $o' = -e_1$. So $0 = K(x + e_1, x)\, g(x, y)$, and since the posts of K are positive, $g(x, y) = 0$, a contradiction. Hence $U = 0$.

(iv) Sum the second display over q; the sums converge because W_U is positive and summable. On the left, substituting $q' = q + o$ and using $\sum_o K(q', q' - o) = 1$ gives $m(p) = \sum_{q'} W_U(p, q')$; on the right, $\sum_{o'} m(p + o')K(p + o', p)$. So m is an invariant measure of K with total Z_U. All posts of K are positive, so K is irreducible, and π is an invariant probability law by (P3); hence K is positive recurrent and every invariant measure is a multiple of π [83]. So $m = Z_U \pi$. Summing over p gives the statement for the other recognizer. □

B.7.4 Proof of Theorem 9.20, the limit (9.9)

Let $T_n > 0$ tend to 0; the limit along every such sequence is the limit $T \to 0^+$. Put $a = T/C_E$ and substitute $k = ap$. The window $W_a = \{p : ap \in (-\pi, \pi]^3\}$ grows to $\mathbb{R}^3$ as $a \to 0$, and $\omega(ap) = T v_a(p)$ with $v_a(p) = \frac{2}{a} \operatorname{arsinh}(\sqrt{\lambda(ap)}/2)$, so with $f(x) = x/(e^x - 1)$

$$\frac{u_1(T)}{T^4} = \frac{1}{C_E^3} \int_{W_a} f\big(v_a(p)\big) \frac{\mathrm{d}^3 p}{(2\pi)^3}.$$

The bound of the proof of Proposition 9.19, applied to $k = ap$ and divided by a, gives $0 \leq |p| - v_a(p) \leq a^2 |p|^3/8$, so $v_a(p) \to |p|$ at every p. For a dominating function, $|\sin(x/2)| \geq |x|/\pi$ for $|x| \leq \pi$ gives $\sqrt{\lambda(ap)}/2 \geq a|p|/\pi$ on W_a, and $\operatorname{arsinh} u \geq u/2$ for $0 \leq u \leq \sqrt{3}$ (arsinh is concave and $\operatorname{arsinh}\sqrt{3} > \sqrt{3}/2$), so $v_a(p) \geq |p|/\pi$ on W_a. Since $f(x) \leq (1 + x)e^{-x}$ and $(1 + x)e^{-x}$ decreases on $[0, \infty)$, the integrand is at most $(1 + |p|)e^{-|p|/\pi}$, which is integrable. By dominated convergence

$$\lim_{a \to 0} \int_{W_a} f(v_a) \frac{\mathrm{d}^3 p}{(2\pi)^3} = \int_{\mathbb{R}^3} \frac{|p|}{e^{|p|} - 1} \frac{\mathrm{d}^3 p}{(2\pi)^3} = \frac{1}{2\pi^2} \int_0^\infty \frac{\varrho^3 \, \mathrm{d}\varrho}{e^\varrho - 1} = \frac{\Gamma(4)\zeta(4)}{2\pi^2} = \frac{\pi^2}{30}. \qquad \square$$

Appendix C

Numbers and their comparison data

Chapter 10 prints a candidate value for the fraction of the critical density held by the cosmological constant, $\Omega_\Lambda = 11/16 - \alpha/\pi = 0.685177$. Against the Planck 2018 value obtained with baryon acoustic oscillations, 0.6889 ± 0.0056, it lies 0.66 standard deviations low. Against the Planck 2018 value from the microwave background with lensing alone, 0.6847 ± 0.0073, it lies 0.07 standard deviations high. Both data are correct, so a deviation means something only when its datum is named and the same datum is used every time. This appendix fixes one datum for each measured quantity the book quotes, recomputes every deviation from it, and lists every other number the book prints with its formula and conditions.

Table C.1. The pinned measurements, all quoted in Chapter 10; T_0, the distortion bounds, G and m_e also in Chapter 9. Uncertainties are one standard deviation unless a confidence level is given.

Quantity	Value	Dataset, year
radiation temperature today T_0	2.72548 ± 0.00057 K	COBE FIRAS, recalibrated with WMAP, 2009 [45]
spectral distortions	$\lvert y\rvert < 1.5 \times 10^{-5}$, $\lvert\mu\rvert < 9 \times 10^{-5}$ (95%)	COBE FIRAS full data set, 1996 [44]
scalar tilt n_s	0.9649 ± 0.0042	Planck 2018, TT,TE,EE+lowE+lensing [87]
scalar amplitude	$\ln(10^{10} A_s) = 3.044 \pm 0.014$	Planck 2018, same combination [87]
tensor-to-scalar ratio	$r_{0.05} < 0.036$ (95%)	BICEP/Keck through 2018 with Planck and WMAP, 2021 [7]
baryon density	$\omega_b = \Omega_b h^2 = 0.02237 \pm 0.00015$	Planck 2018, TT,TE,EE+lowE+lensing [87]
baryon-to-photon ratio	$\eta_B = (6.127 \pm 0.041) \times 10^{-10}$	from ω_b with $10^{10}\eta_B = 273.9\,\omega_b$ [116]
dark-energy fraction	$\Omega_\Lambda = 0.6889 \pm 0.0056$	Planck 2018, TT,TE,EE+lowE+lensing+BAO [87]
cold dark matter density	$\omega_c = \Omega_c h^2 = 0.1200 \pm 0.0012$	Planck 2018, TT,TE,EE+lowE+lensing [87]
Hubble constant, early	$H_0 = 67.36 \pm 0.54$ km s^{-1} Mpc^{-1} ($h = 0.6736$)	Planck 2018, TT,TE,EE+lowE+lensing [87]
Hubble constant, late	$H_0 = 73.04 \pm 1.04$ km s^{-1} Mpc^{-1}	SH0ES distance ladder, 2022 [98]
clustering amplitude S_8	0.832 ± 0.013; $0.759^{+0.024}_{-0.021}$	Planck 2018 [87]; KiDS-1000 cosmic shear, 2021 [3]
dark-energy evolution	$w_0 = -0.752 \pm 0.057$, $w_a = -0.86^{+0.23}_{-0.20}$	DESI DR2 BAO with CMB and DES-SN5YR supernovae, 2025 [31]
first two TT peaks	$\ell_1 = 220.6 \pm 0.6$, $\ell_2 = 538.1 \pm 1.3$	Planck 2018 [88], Table 5
constants	$\alpha = 7.2973525643(11) \times 10^{-3}$; $G = 6.67430(15) \times 10^{-11}$ m^3 kg^{-1} s^{-2}; $m_e = 9.1093837139(28) \times 10^{-31}$ kg, $m_e/m_P = 4.1855 \times 10^{-23}$	CODATA 2022 [21]; the relative uncertainty of G is 2.2×10^{-5}

The conventions are these. An exact value is printed exactly ($\sqrt{5} - 2$, $38 + 17\sqrt{5}$, $81/26$); a decimal after it is the exact value rounded to the digits shown. A bracket $[a, b]$ or (a, b) after a quantity is an enclosure: the quantity is proved to lie in it, and we never print more digits than the enclosure certifies. The conditions column names the adopted premises, the supplied

data and the carrier. A deviation is $(x-\mu)/\sigma$, where x is the printed value, μ the central value of the datum and σ its quoted one-standard-deviation uncertainty; for an asymmetric uncertainty we use the side facing x, and for a ratio or a difference of two data we combine the two uncertainties in quadrature as if independent. An upper limit is compared by stating whether the value lies below it. None of the measured numbers is computed in this book.

C.1 The pinned measurements

The Planck 2018 values of Table C.1 are those of the baseline combination TT,TE,EE+lowE+lensing of [87], Table 2, except Ω_Λ, for which the book uses the column with baryon acoustic oscillations added, the stricter of the two for the candidate above. The Hubble constant appears twice because Chapter 10 compares two measurements of it.

Chapter 10 also quotes the light-element abundances and three lattice results, each once: the helium mass fraction $Y_p = 0.2449 \pm 0.0040$ (metal-poor H II regions, 2015) [4]; $\mathrm{D/H} = (2.527 \pm 0.030) \times 10^{-5}$ (quasar absorbers, 2018) [26]; $^7\mathrm{Li/H} = (1.6 \pm 0.3) \times 10^{-10}$ (metal-poor halo stars, 2010) [102], against $(4.68 \pm 0.67) \times 10^{-10}$ from standard nucleosynthesis at the measured η_B [28], a factor 2.9; and, from lattice computations of the Standard Model at the measured Higgs mass, the symmetric-phase sphaleron rate $(18 \pm 3)\alpha_W^5 T^4$ and the sphaleron freeze-out temperature 131.7 ± 2.3 GeV [34], and the crossover temperature $T_c = 159.5 \pm 1.5$ GeV [35].

C.2 Every comparison the book makes

Table C.2 lists each value the book sets against a measurement, with its status (the status words of Section 10.1) and the deviation recomputed from Table C.1. No value in it is derived on a recognition carrier from the starting facts alone.

Table C.2. Comparisons with measurement, recomputed. A dagger marks a deviation the chapter states only in words; we print the number.

Value and formula	Status	Datum	Deviation
$n_s = 1-2/(N_e+1) = 43/45 = 0.95556$, with $N_e = 44$ defined	adopted	n_s	-2.22σ
$r = 2/(45\varphi^2) = 0.016976$	adopted	$r_{0.05}$	below 0.036
$n_s = 1 - 2/\varphi^3 = 0.528$	excluded	n_s	$-104\sigma^\dagger$
$r = (\varphi-1)^4 = \varphi^{-4} = 0.146$	excluded	$r_{0.05}$	4.05 times the limit†
$\eta_B = P\varphi^{-45} = 9.832 \times 10^{-10}$, $P \in (2.49494, 2.49505)$	excluded	η_B	$+90\sigma$
$\eta_B = (1-\varphi^{-8})^2\varphi^{-44} = 6.1074 \times 10^{-10}$	adopted candidate, chosen to match	η_B	-0.48σ
sphaleron rate $\frac{3}{4}\alpha_W^5 T^4$	identification, contradicted	lattice $(18 \pm 3)\alpha_W^5 T^4$	24 times smaller; $-5.8\sigma^\dagger$
$\Omega_\Lambda = 11/16 - \alpha/\pi = 0.685177$	adopted candidate; α measured	Ω_Λ	-0.66σ
Ω_Λ with $\alpha \to (1-\varphi^{-1})/2$: 0.626708	excluded	Ω_Λ	-11.1σ; every $\varphi^r \cdot 0.626708$, $r \in \mathbb{Z}$, at least 10σ off

$(w_0, w_a) = (-0.982779, -0.017221)$	conditional (accumulation premise)	DESI DR2 (w_0, w_a)	-4.05σ, $+3.66\sigma$, marginal only
$\Omega_{\mathrm{dm}} = \sin(\pi/12) + \varphi^{-4}/24 = 0.264898$; $\Omega_{\mathrm{dm}}h^2 = 0.12019$ at $h = 0.6736$	adopted candidate, selected after comparison	ω_c	$+0.16\sigma$
$\Omega_{\mathrm{dm}}/\Omega_b = 2\varphi + 2 = 5.23607$	adopted candidate	$\omega_c/\omega_b = 5.3643 \pm 0.0646$	-1.99σ
$\lvert\varepsilon_{CP}\rvert = \varphi^{-2} = 0.38197$	no derivation	$\eta_B/(PD) = 0.3852 \pm 0.0026$	$-1.2\sigma^{\dagger}$
peak spacing $k_2/k_1 = \varphi$	excluded (Proposition 10.6)	$\ell_2/\ell_1 = 2.4393 \pm 0.0089$	$-92\sigma^{\dagger}$
the electron at a whole golden level, $\gamma(m_e, G) = -106$ (Theorem 9.28)	excluded under (N1), (N3)	m_e, G	$\gamma \in (-106.019763, -106.019703)$: 0.0197 outside, about 330 widths
early against late H_0	two data	67.36 ± 0.54, 73.04 ± 1.04	4.8σ; ratio 1.0843 ± 0.0177
S_8, microwave background against cosmic shear	two data	0.832 ± 0.013, $0.759^{+0.024}_{-0.021}$	2.7σ

The value $\Omega_{\mathrm{dm}}h^2$ uses the central h and ignores its uncertainty, as Chapter 10 does; the ratio ω_c/ω_b does not depend on h. The radiation temperature T_0 and the distortion bounds are compared with nothing, since the book computes no temperature in kelvin (Chapter 9); they are the first comparison the account will face. Table C.3 lists the values Chapter 10 sets against no measurement.

Table C.3. Values of Chapter 10 compared with nothing: conditional, adopted, or part of an exclusion.

Number	Formula and conditions	Where
$\varepsilon(q) \geq \frac{9}{8}$ on $0 < q \leq 1$, $\eta(t) > 1$; $16/(3\varphi^2) = 2.037$	cost potential at unit normalization; plateau with $\alpha = \varphi^2$	Props. 10.1, 10.3
$N_e = 44 = 45 - 1$, $45 = D^2(D+2)$; φ^{51}	adopted; exponent supplied	Tables 10.3, 10.4
$\frac{28}{79} = 0.35443$	$B/(B-L)$, Standard Model equilibrium	Prop. 10.4
$P \in (2.49494, 2.49505)$; $\varphi^{-44} = 6.3759 \times 10^{-10}$	conditional; adopted	Sec. 10.5
$A = \frac{1}{2}\varphi^{-7} = \frac{13\sqrt{5}-29}{4} = 0.0172209$; $w(1) = -0.991390$; $e^{3A} = 1.0530$	dilution premises; accumulation premise	Prop. 10.11
$2\pi\varphi^5/(45 \ln 2) = 2.234$; $\frac{\varphi^{-4}}{24} = 0.0060791$; $\sin\frac{\pi}{12} = 0.258819$	Λ in native units, a definition; the gap identity; the adopted term	(10.6)
$8\varphi^2 = 20.944$ cells; ≤ 0.231 e-fold; $q_R \leq \frac{\pi}{4}$	adopted reach; births on a schedule; regional aging	Tables 10.8, 10.1; Prop. 10.8

C.3 The numbers of the book, chapter by chapter

The remaining tables collect the numbers that the chapters derive or compute and compare with nothing: the number, the formula or definition it comes from, the conditions under which the chapter states it, and where it is proved. A row is never stronger than its source: where the chapter states a number under (R1) and (R2), or under (P1)–(P4), so does the row. Exact counts are direct exact computations over the stated finite families; long lists stay in the chapter's own table, which the row names.

Table C.4. The cost and the golden ratio (Chapter 1).

Number	Formula and conditions	Where
$J(1) = 0$, the only zero; $J(2) = J(\frac{1}{2}) = \frac{1}{4}$; $J(e^t) = \cosh t - 1$	$J(x) = \frac{1}{2}(x + x^{-1}) - 1$, $x > 0$: composition law and unit curvature at balance	Thm. 1.1
$F_2(2) = J(4) = \frac{9}{8}$, $F_2(4) = J(16) = \frac{225}{32}$	$F_c(x) = J(x^c)$ solves the composition law for every $c > 0$; curvature κ gives $J(x^{\sqrt{\kappa}})$	Prop. 1.2
$\varphi = (1 + \sqrt{5})/2 = 1.6180340$	positive root of $r^2 = r + 1$; the levels $s_n = \varphi^n s_0$	Thm. 1.7
$2^3 = 8$ patterns; complete pass $T \geq 8$, attained; $D = 3$	three binary distinctions; the linking conditions	Thms. 1.12, 1.11

Table C.5. Before the first event (Chapter 2). Carriers (a), (f), (g) and (j).

Number	Formula and conditions	Where
cost 0, net flow 0, record length 0, counts $(0, 0, 0)$; $6 = 2 \cdot 3$ first posts	the empty ledger, $d = 3$, the unique least tally	Table 2.1
$T(L + c) = T(L) + 2dc$	a common shift of every count; posting and costs of differences are blind to it	Lem. 2.17
$\mathscr{H}_n = \ell^2(\mathbb{Z}; \mathbb{C}^{64 \times 15^n})$	the history dilation after n recorded steps	Sec. 2.6
$8^4 = 4096$ codes; pair space $\mathbb{C}^{64}$	the source and recorder carrier	Sec. 2.7
readout and recording $7E_{\mathrm{nat}}$ each; ready branch $1/32$; flow energy $7E_{\mathrm{nat}} + \ell(\mathrm{rd})$; pair diagonal $14E_{\mathrm{nat}}$	native energy units; seed $\Omega \otimes \zeta$, supplied ready state	Prop. 2.31
six numbers of genesis; $\rho_* \in (0.7792, 0.7793)$, the real root of $8x^5 + 4x^4 - 2x^2 - 2x - 1$; error $C_v C_w 2^{-\lfloor n/8 \rfloor}$	a unit genesis state on the boundary-theory carrier (that program's results)	Thm. 2.32

Table C.6. The first event (Chapter 3). Carriers (a) and (b).

Number	Formula and conditions	Where
$\log \varphi = \operatorname{arsinh} \frac{1}{2} = 0.481212$; $J(\varphi) = \varphi - \frac{3}{2} = 0.118034$; $J(\varphi^2) = \frac{1}{2}$	identities; φ the unique positive fixed point of $r \mapsto 1 + 1/r$, attracting every seed	Prop. 3.17
6 first posts; 3 corners, two posts each	one post flips one parity bit	Cor. 3.3
first tick $2J(r)$, net flow 0; price $2(\cosh q - 1)$ at every strain	ratio r supplied; the price under (R1)	Props. 3.6, 3.8
post prices $\cosh 1 - 1 = 0.5431$, $\varphi - \frac{3}{2}$, $\varphi^{-5} = 0.0902$; $\operatorname{arcosh}(1 + \varphi^{-5}) = 0.4215$	three unit readings $q = 1$, $\log \varphi$, $\operatorname{arcosh}(1 + \varphi^{-5})$; pairwise distinct	Prop. 3.11
$q = \log \varphi$, ratio φ, price $2J(\varphi) = \sqrt{5} - 2 = 0.236068$	(R1), (R2), the first post's bond a bridge of weight one	Thm. 3.15
$\operatorname{arsinh} \frac{1}{2w}$; $\operatorname{arsinh} \frac{1}{4} = 0.247466$; $\log(1 + \sqrt{2})$, price $\sqrt{2} - 1$	(R2): bridge of weight w; reciprocal as a second post; two units on one bond	Table 3.1
$1 \leq C < 1/(2 \log \varphi) = 1.039043$; $\operatorname{arsinh} \frac{m}{2C} \leq D(m) < \frac{m}{2C}$, widths 0.0188, 0.0007	any carrier with conductance C; unit drop $\log \varphi$ for the window; one bond and the torus for the widths	Thm. 3.43, Cor. 3.44
triangle drop $(0.328865, 0.328866)$, price $(0.109130, 0.109132)$, about 46% of $\sqrt{5} - 2$	(R1), (R2), unit weights; rings and thetas with $C = \frac{12}{11}, \frac{6}{5}, \frac{4}{3}, \frac{3}{2}, 2, \frac{17}{6}$ in Table 3.2	Thm. 3.35
$C = 81/26$; drop $2A \in (0.1601, 0.1603)$, below $D^S(1) \in (0.160273, 0.160274)$ and below $\frac{1}{3} \log \varphi = 0.16040$; $[\operatorname{arsinh} \frac{13}{81}, \frac{13}{81}) \subset [0.15981, 0.16050)$; price $(0.025686, 0.025743)$	(R2), possibility graph on the 27-site torus, unit source; share in $(0.32, \frac{1}{3})$; $e^{2A} \approx 1.1737$ and the shares 0.322, 0.160, 0.129 numerical	Thm. 3.28, Cor. 3.56
$1 < e^{2A} < \varphi$	the whole signed period-3 graph	Cor. 3.32

$R_N = (1 - N^{-3})/6$; $\frac{13}{81}$, shares $\frac{26}{81}$, $\frac{55}{81}$; box $\geq \frac{1}{3}$, corner $\geq \frac{2}{5}$	the linear law on T_N ($N = 3$) and on open boxes	Prop. 3.40
$2.998 < m_3^* < 6$; $0.0832 < \operatorname{arsinh} \frac{1}{12} < D_N(1) < R_N$	golden strength in $(6 \log \varphi/(1 - N^{-3}), 6)$; unit-post drop	Thm. 3.39
rises $> \frac{517}{20992} > 0.0246$, $> \frac{13}{405} > 0.0320$; $\log \frac{513}{320}$ rising by $(\frac{1}{24960016}, 10^{-7})$; cap $\frac{1}{2} - \log \varphi \approx 0.018788 < 0.0191$	Wheatstone carriers: a stronger bond can raise the drop; at unit strength with the post's bond of weight at least 1, never above $\log \varphi$	Thm. 3.48
drops $(0.471668, 0.476543)$, price < 0.231424	two loops sharing only their ends	Prop. 3.50
$2A < 0.176062$, price < 0.031079; loop ceilings $> 2A + 0.015761$; split ceiling within 0.000174	five bond-disjoint loops on the torus; linear port $\frac{7}{54}$ against $\frac{26}{243}$	Thm. 3.54
$J(2) = \frac{1}{4}$; $r = 1$	two distinct next events at one price; a frozen golden field through a reciprocal second event	Props. 3.58, 3.59

Table C.7. Time, the native law and the arrow (Chapter 4). Carriers (a) and (c); every row with K assumes (P1)–(P4) and the three readings.

Number	Formula and conditions	Where
6 successors; $\binom{n+5}{5}$ ledgers, 6^n histories after n posts	one post per tick from the empty record; Table 4.1	Prop. 4.2
balanced after 0, 2, 4, 6, 8 posts: 1, 6, 90, 1860, 44,730; $N_8(0,1,5) = N_8(2,3,3) = 560$	$N_{2j}(0) = \sum (2j)!/(a!\,b!\,c!)^2$; none after odd n	Thm. 4.3
$8 = 2^3$	least complete pass; least period of a state-local completing process	Thm. 4.8
12 tours; 3,072 of $6^8 = 1{,}679{,}616$ complete at tick 8; 288 balanced, 96 in one cell	direct exact computation	Thm. 4.9
first completions at even ticks ≥ 8; $N_{24} = 354{,}514{,}338{,}330{,}968{,}064$	$N_n = 2^n A_n$; Table 4.2	Thm. 4.10
$\pi(0) = \varphi^{-9} = 0.0131556$; $4k^2 + 2$ states on level k; mean level 6; entropy $15 \log \varphi$	$\pi(s) = \varphi^{-9}\varphi^{-\lvert s\rvert_1}$, reading (c)	Lem. 4.14
$h_0 \in (0.1839, 0.1840)$ (computed 0.1839463127); exit $(0.8160, 0.8161)$; posts in two, three ticks $(1.6494, 1.6495)$, $(2.4876, 2.4877)$; root $\frac{1}{13} \leq u \leq 1$	the native law K; Table 4.3	Thms. 4.17, 4.19
least hold $(0.136067, 0.1360695)$ at $(1,1,1)$, every hold $> \frac{1}{169}$; $h(s) + \frac{1}{75} < h_0$; $\nu \in (0.85, 0.856)$	K; holds by class in Table 4.3	Thm. 4.19
$0.06325 < \lvert D_i(s)\rvert < 0.09368$	drift toward balance, $s_i \neq 0$	Thm. 4.21
gap 0.00836, none above 0.008398; $d_n(0) \leq 9(1 - 0.00836)^n$; rate in $(0.879, 0.99164)$	K, from the empty record	Thm. 4.22
$d_n(0) < 1$, $\frac{1}{2}$, $\frac{1}{10}$, $\frac{1}{1000}$ at ticks 9, 26, 99, 456; 12.03 to 15.81 ticks per unit	evaluated from proved enclosures; not bounds	Thm. 4.22
$\varphi^9 = 38 + 17\sqrt{5} = 76.013$ ticks; half by tick 8, 99% by 717	mean return to balance (Kac); the last two evaluated	Cor. 4.23
$F_8 = 0.000409204$, $C_{16} = 0.0567528$; mode 27, median 33; $(0.3506, 0.3507)$	first pass under K, Table 4.4; weight of the 96 one-cell passes among 288	Sec. 4.4
$9 \log \varphi = 4.3309$ nats $= 6.248$ bits	information to be lost from the empty record	Thm. 4.30
$h_* = \varphi/(\varphi + 6) = 0.21240$; $\log(6\varphi - 5) = 1.549307$; $\log(31 - 12\varphi) = 2.44959$; $\log(64\varphi - 83) = 3.02306$	loss ceilings over one, two, three ticks, every voxel law	Thm. 4.31

loss (1.54679, 1.54682), 35.715% to 35.716%, (0.00248, 0.00252) short; $D_1(0) \in (2.78408, 2.78412)$; second tick < 0.90281	the first tick under K	Cor. 4.32
$\frac{51}{100}\lvert V\rvert\alpha_0 \le 1$	one post per tick at rates $\ge \alpha_0$	Prop. 4.34
$\frac{23}{24}$, $\frac{20}{21}$; $30 \cdot 4096\,(\frac{41}{48})^{\lfloor s/2 \rfloor}$	boundary-theory carrier (that program's results)	Sec. 4.5

Table C.8. The record cell (Chapter 5). Carrier (e); the Regge row is on the periodic Euclidean Regge lattice.

Number	Formula and conditions	Where
16 records, 24 histories, no chain of six; 15 edge classes, 65 edges; 6 histories per account and step	Kuhn subdivision of the four-cube	Thm. 5.2
$\lambda(\sum_a y_a^2 - \kappa(\sum_a y_a)^2)$, $\lambda > 0$, $\kappa \ge 1$; lengths $0, -2, -6, -12$ at $\kappa = 1$, $-1, -6, -15, -28$ at $\kappa = 2$	name-blind interval, rank as time; $m = 1, \dots, 4$ postings	Thms. 5.5, 5.6
$c \ge 4$; $c = 2$	needed for $\sum y_a^2 - c(w \cdot y)^2$ to make histories causal; a Wick rotation	Prop. 5.7
$\kappa = \frac{3+\alpha^2}{4\alpha^2}$ ($\alpha = 3$: $\frac{1}{3}$); speed $\le \frac{1}{3}$, velocity $-\frac{3}{5}u_1$	hyperdiamond family; order cone against light cone	Props. 5.8, 5.9
$\lambda = \frac{4}{3}$, $T = \frac{1}{4}(1,1,1,1)$; $w = (\frac{1}{2}, -\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6})$	space from cost, one voxel per tick; the tilted counterexample	Thm. 5.10
$\kappa \ne \frac{1}{4}$; $\kappa_\gamma = \frac{1}{2}$; $\sqrt{3}$ at $\kappa = 1$, rational at $\kappa = \frac{5}{4}$	wave-equation cone; lattice photon (acausal); cone map $4\kappa - 1 = (c^2/\det L)^2$	Thms. 5.13, 5.15
$(4 - q^2)/4$; $F/q^2 \to \frac{3}{8}$; $S - S_0 = N^4(\frac{q^3}{8} + o(q^3))$	Euclidean Regge action, body path, $N \ge 3$, $\lvert q\rvert < 2$; the plotted curve is numerical	Thm. 5.16

Table C.9. The first interacting dynamics (Chapters 6 and 7). Carriers (g), (h), (i), with a, C_E, ϵ and the placement supplied.

Number	Formula and conditions	Where
$\omega_{\max} = 2C_E \operatorname{arsinh}\sqrt{3} \approx 2.6339158\,C_E$; $0 \le \lambda \le 12$	$\omega(k) = 2C_E \operatorname{arsinh}(\sqrt{\lambda(k)}/2)$	Def. 6.1
$\frac{1}{3}$; 4096 per cell; $C_{10} = -\frac{1}{2} + \frac{i}{2}(1 + \sqrt{2})$	own-edge pairing; register coordinates; spectator leakage	Lem. 6.4, Prop. 6.21
$2 - 2e^{-\lVert f\rVert^2/2} < 2$; $\varrho_p = \frac{1}{27}$; $k \cdot 2J(r) < 2$	homogeneous vacuum on the 27-site period; no fixed conversion	Thm. 6.10
shells $1, 6, 12, 8$; $324V = -33, -7, -2, 0$; drops $\operatorname{arsinh}(-\frac{13}{81})$, $\operatorname{arsinh}(-\frac{5}{324})$, $\operatorname{arsinh}(-\frac{1}{162})$; energy ≥ 6	localized vacuum, direct exact computation	Table 6.1
floor $-2a\lvert R\rvert$; $\lVert\sqrt{h}\, z_f\rVert^2 \le \frac{a}{2}\sum f(e)^2$	$a \ge 0$, $C_E > 0$, $\epsilon \ge 0$, no background, injective unit edges	Thm. 6.27
mean $\frac{1}{2}$, variance $\frac{1}{4}$; $V_R \ge 4\lvert R\rvert\, \nu_{\min}$	occupation in the reference vector; source variance	Prop. 6.28, Thm. 6.34
$\langle \chi, Q\chi\rangle = \frac{1}{2}$; $\omega_{\mathrm{ref}}(O(E_0)) = \frac{1}{8}$; $c(0) = \frac{1}{4}$, $\lvert c(t)\rvert < \frac{1}{4}$ at $t \ne 0$	reference state; flip coherence, clock off	Prop. 7.25
$924/4096 \approx 0.2256$; $\binom{2n}{n}^2(2n+1) \le 16^n$	six registers at equal rates	Lem. 7.27, Thm. 7.28
$0 \le n_d \le \frac{a}{2}$, 0 on the axis, $\ge \frac{a}{8}$ across; 0.3116	photon density of the static couplings	(7.14)
4^{-6} at every t; $\Delta_t = \frac{1}{4}(1 - e^{-m_t^2/2}) > 0$	gradient star of six flips; folium exit at $t \ne 0$	Ex. 7.39, Thm. 7.56

Table C.10. Many recognizers (Chapter 8). Carrier (d); every row assumes K.

Number	Formula and conditions	Where
$k_0 \in (0.13600, 0.13602)$; posting chance > 0.8160	$k_0 = (1 - h_0)/6$ under K; $h_0 < 0.1840$, $h(s) < 1/6$ for $s \neq 0$	(8.1)
$h_0^2 \in (0.03381, 0.03386)$; $k_0^2 \in (0.018496, 0.018501)$; $2p(1-p) \in (0.39603, 0.39607)$; > 0.85	first-tick joint chances, product law (mirror law: h_0, k_0 or 0)	Table 8.1
$7^{\lvert V\rvert}$ moves; $c = 7^{\lvert V\rvert}\varphi^{\lvert V\rvert/2}$	joint balance forces the product	Thm. 8.7
$7(1 - h_0) \in (5.712, 5.7127)$; $P_K(s_3 = 0 \mid s_0 = 0) < \frac{1}{7}$ (about 0.1015)	no conserving law on $1, 3, 5, 7$ recognizers	Thm. 8.13
$S = \frac{233}{125} + \frac{8(1-h_0)}{6} \in (2.9520, 2.9522) > 2\sqrt{2}$; $h_0 + \varphi(1 - h_0) \in (1.504, 1.505)$	CHSH value; no reversible law on the imbalance	Props. 8.15, 8.16
$\lvert x\rvert + \lvert y\rvert + \lvert z\rvert = n$; speeds 1, $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{3}}$	supplied isolated-pair rule, locality (L)	Thm. 8.28

Table C.11. The hot universe and the physical scale (Chapter 9).

Number	Formula and conditions	Where
$T = \varepsilon/\log\varphi$; $1/\log\varphi = 2.0781$	the energy reading	(9.1)
$\varphi^9 = 76.0132$; $U = 2.8873$; $S = 7.2182$; $F = -4.3309$; mode 4; $\varphi^3 \approx 4.236$	same at $T = 1$, Table 9.1; $J(\varphi^2)/J(\varphi)$	Prop. 9.6
$\lceil 103\varphi^i \rceil$ states, $i \leq 15$; $c_0 = \frac{6188671}{16777216}$; $\frac{2}{h} = 14.699$, $\frac{14}{h} = 102.890$	thermal record of K, $h = c_0^2$	Thm. 9.10
$(0.39266, 0.39272)$; entropy $(1.93948, 1.93952)$	first-tick heat; per-tick refresh	Thm. 9.13
$g_n \in (\log\varphi, 2\log\varphi)$; 0.6931, 0.6389, 0.5928 at $b = 1$	locked-pair energy steps	Prop. 9.17
$\frac{c\hbar^2\lvert p\rvert^3}{8}$; $\frac{\pi^2}{30C_E^3}$	light at long wavelength; blackbody limit, supplied Bose occupation	Prop. 9.19, Thm. 9.20
$g_* = 106.75$, $g_{*s} = \frac{43}{11}$, $\frac{4}{11}$; $\zeta(3) \in (1.202042, 1.202065)$; $s/n_\gamma \in (7.0393, 7.0396)$; 86.25, 61.75, 17.25, 10.75	Standard Model content, Friedmann equations and measured thresholds as inputs	Prop. 9.22
$\hbar = \varphi^{-5}$, $G = \frac{\varphi^5}{4\ln 2}$; determinant -2	(N1)–(N3); dimensions of c, $\hbar$, G	Def. 9.25
$1.6651\, t_P = 8.977 \times 10^{-44}$ s; 2.691×10^{-35} m; 1.450×10^{-7} kg; 1.175×10^{9} J	(N1), (N3), measured G; (N2) for mass; relative uncertainty 1.1×10^{-5}	Table 9.2
$\gamma \in (-106.019763, -106.019703)$; period $(\varphi^{109}, \varphi^{110})$ ticks	electron level, (N1), (N3), CODATA 2022	Thm. 9.28
9.924×10^7, 5.657×10^6; $\eta \in (\varphi^{-145}, \varphi^{-144})$; $> 10^{59}$ bits	electron numbers of the mass law, not derived	Thm. 9.29
$\lambda = 6\varphi - 9 \approx 0.7082$ or $\frac{24}{7}(\varphi - \frac{3}{2}) \approx 0.4047$	candidate electric scales, not determined	Prop. 9.30

Appendix D

Refuted routes

Several natural ways from the starting facts to a hot universe do not work, and in each case we know why. Each entry names one route, the result that closes it (a counterexample, a no-go theorem under stated hypotheses, a measurement, or, where marked, an inference) with its carrier (lettered as in Appendix A) and hypotheses, and where it is proved; the two proofs no chapter needs are Propositions D.1 and D.2. A refutation closes a route at the scope of its hypotheses and no further, and the entry names what stays open. A proposal that changes a named hypothesis is a different route; one that only renames the objects of an entry is the same route.

D.1 Before the first event

Route 1. *The posting laws force a beginning.* On account books with integer counts (carrier (a)), posting commutes with adding a constant to every count. The all-(-1) book has the balances and the shifted posting steps of the empty book, yet it tallies no record and is not reached from the empty book; no shift-invariant condition is nonnegativity of the counts or reachability, and without the opening premise every book has predecessors at every depth. A beginning enters only as the opening premise, inside which the empty book is the unique first moment. (Chapter 2, Lemma 2.17 and Theorem 2.18.)

Route 2. *A past hypothesis at the cost minimum.* On configurations $c \in (0,\infty)^N$ with total defect $\Phi(c) = \sum_i J(c_i)$ and the adopted variational step at fixed log charge, the configuration $\mathbf{1}$ is the unique zero of Φ, the successor of every configuration of log charge 0 is $\mathbf{1}$, and a trajectory that reaches $\mathbf{1}$ never leaves it. The minimum is where the dynamics ends, not a state from which it starts. (Chapter 2, Proposition 2.22.)

Route 3. *Defect as the currency of time.* On the same carrier, a cost is coupled to a record (every fall of cost changes the record) exactly when it is a function of the record; the total defect is coupled to no countable record; no variational trajectory posts at every step while each posting is paid by a fall of cost of at least a fixed $q > 0$; and every fall of cost after the first step is zero. (Chapter 2, Proposition 2.23.)

Route 4. *A global minimization step as the source of primordial fluctuations.* The same variational step erases every fluctuation of the logarithms about their mean in one step, in every ensemble.

Proposition D.1 (one step erases every fluctuation)**.** *Let $N \geq 1$, let c be a configuration with log charge $\eta(c) = \sum_i \log c_i$, and let c' be its variational successor. Then $\log c_i' = \eta(c)/N$ for every i. Hence $\sum_i f_i \log c_i' = 0$ for every $f \in \mathbb{R}^N$ with $\sum_i f_i = 0$; and for every finite family of configurations $c^{(a)}$ with successors $c'^{(a)}$, all real weights w_a and all $f, g \in \mathbb{R}^N$ with zero sums,*

$$\sum_a w_a \Big(\sum_i f_i \log c_i'^{(a)} \Big) \Big(\sum_i g_i \log c_i'^{(a)} \Big) = 0.$$

Proof. With $x_i = \log c_i$, $\Phi = \sum_i(\cosh x_i - 1)$ and $\eta = \sum_i x_i$. Since cosh is strictly convex, $\sum_i \cosh x_i \geq N\cosh(\eta/N)$, with equality only when every $x_i = \eta/N$; so the successor is the uniform configuration with entries $e^{\eta/N}$ (Chapter 2). Then $\sum_i f_i \log c_i' = (\eta/N)\sum_i f_i = 0$, and every term of the weighted sum vanishes. □

This holds for every distribution of initial configurations, so fluctuations that seed structure need another source. The variational step itself is adopted (Chapter 2).

Route 5. *A forward record clock with a bounded generator.* On the native history dilation (carrier (f)) no bounded self-adjoint G with $e^{-iG} = U$ has e^{-itG} keep the embedded recorded histories $\iota\mathscr{H}_+$ for every $t \geq 0$. Open: unbounded generators, intermediate states off the carrier, other clocks. (Chapter 2, Proposition 2.25.)

Route 6. *A source spread evenly over the infinite lattice.* On the source and recorder carrier (g) no unit vector has a site probability independent of the site, at any time of the evolution, and no unit source vector is carried by every lattice translation into a phase multiple of itself. Open: homogeneous local observables, finite volumes, other representations. (Chapter 2, Proposition 2.29.)

Route 7. *Selecting the first state by least energy.* On carrier (g) with $a, C_E > 0$ and contact strength $\kappa \neq 0$, a blank record whose source has weight off the origin is not an eigenvector, and mixing in the matching record lowers its energy. With $C_E > 0$ the coherent seed is not an eigenvector, a lower-energy unit vector exists, and its one-photon component is nonzero at times arbitrarily close to 0 on both sides. Energy selects neither the blank nor the seed. (Chapter 2, Theorem 2.26 and Proposition 2.27.)

D.2 The first event

Route 8. *A cascade of seeds that crystallizes into φ.* Under $T(r) = 1 + 1/r$ on $r > 0$, $T^n(r) = \varphi$ for some n only if $r = \varphi$, and the seeds 2 and 3 both converge to φ without ever meeting. The golden ratio is the fixed point that attracts every positive seed; no finite history reaches it and attraction selects no seed. (Chapter 3, Propositions 3.17 and 3.18.)

Route 9. *The golden strain on the possibility graph or on any torus neighbour graph.* On the torus T_N, $N \geq 2$, every neighbour bond of every possibility graph lies on a cycle of length four, so under (R2) the flux of every post divides and the condition for the golden price fails for every post at every strain. For the first post on the 27-site torus the possibility graph gives a drop below $\frac{1}{3}\log\varphi$, while the record carrier of the same post gives $\log\varphi$. (Chapter 3, Theorem 3.27 and Corollary 3.32.)

Route 10. *The golden drop, or the pair action, on the whole native graph.* Under (R2), on the whole signed possibility graph at period three the unit source has $1 < e^{\text{drop}} < \varphi$. The action of that whole graph is not, at every field, the cost of the field-read first event between two neighbours, and no carrier's action equals at every field the cost of a fixed ledger of positive cost (an action vanishes at zero field). (Chapter 3, Corollary 3.32 and Proposition 3.20.)

Route 11. *Ranking carriers by inclusion.* Strengthening a bond can raise the drop: explicit six-site bridges do so at every strength, one of them inside the golden window. At unit strength, with the post's bond of weight at least one, the rise is below $\frac{1}{2} - \log\varphi < 0.0191$ and never lifts a drop above $\log\varphi$; on a series-parallel network raising a weight always lowers the drop. Two loops that share only the ends of the post's bond do not together cap the drop at the value of the bond beside both. Open: which carrier holds the first post's field. (Chapter 3,

Theorem 3.48 and Proposition 3.50.)

Route 12. *Selecting the second event by a frozen golden field, by least cost, or by the recognizer's own history.* After the reciprocal post the record carrier is one bond of weight 2 with cancelling sources, and the field $(\log r, 0)$ is stationary only at $r = 1$. Under K, the all-hold law (the least-cost successor) misses the native target, since K posts with chance above 0.8160 at every imbalance; a potential of a recognizer's own walk prices a law meeting the target exactly when it is a constant plus $(\log \varphi/\beta)$ times the net imbalance of the walk, so it prices every such law or none. On every balanced ledger two distinct completable events of cost $J(2) = \frac{1}{4}$ can open next. (Chapter 3, Propositions 3.58 and 3.59; Chapter 8, Proposition 8.18.)

D.3 Time and the arrow

Route 13. *The Gray tour as the history of the first events.* On the event ledger (carrier (a)), posting the eight edges of the Gray tour of the cube, or the edges of two disjoint square faces, as events of ratio φ with their reciprocals gives two ledgers with 16 events each, all eight labels, only cube edges, the same cost and net flow 0 at every label; only the first crosses between the faces. Count, cost and balance do not select one covering tour. (Chapter 4, Proposition 4.11.)

Route 14. *One global serial clock.* For a finite set V of recognizers, $|V| \geq 2$, whose own moves have the chances of K, every state gives positive chance to a move in which two recognizers post at once. A law with at most one post per tick that keeps each recognizer's posting rate at least $\alpha_0 > 0$ needs $\frac{51}{100}|V|\alpha_0 \leq 1$, so every rate must fall as recognizers are added, which needs a number no single recognizer holds. (Chapter 4, Proposition 4.34; Chapter 8, Proposition 8.3.)

Route 15. *Cost as the direction of time.* $J(x) = J(1/x)$, and the cost of every change of a ledger equals the cost of its reverse, so cost orders no pair of moments. The direction of a post lies in its legality: counts never decrease. (Chapter 4, Proposition 4.26.)

Route 16. *A horizon as the direction of time.* A seam whose recorded count is the sum of the two sides' counts names no side. On a cell whose record is its six face bits, every achievable record has an achievable complement, so every direction read from the number of posted bits is reversed by a symmetry of the achievable records. A count bounded below and unbounded above, such as the length of a record, admits no order-reversing relabelling. (Chapter 4, Proposition 4.27.)

D.4 The record cell and the lattice action

Route 17. *The gravitational action of the lattice as a balanced recognition cost.* For $N \geq 3$ and every real $C \geq 0$ near 0 with $C(0) = 0$, there are no constants α, β with $S(\ell_q)/N^4 = \alpha + \beta C(q)$ along the body path of the four-dimensional Euclidean lattice, because flatness is a saddle of Regge's action while a balanced nonnegative cost only rises; this includes the recognition cost of postings whose log-potentials balance at $q = 0$. Open: signed differences of cost, the phases of a Lorentzian action, backgrounds away from balance, actions induced by the dynamics of fields. (Chapter 5, Theorem 5.18.)

D.5 The first interacting dynamics

Route 18. *Fixed conversions between field energy and recognition cost.* On the finite-period photon lattice (carrier (h)) at period 3, with the uniform reference and axial profiles, the energy of the homogeneous orbit equals $k \cdot 2J(r)$ only if $k \cdot 2J(r) < 2$, so no fixed $k > 0$ converts it, or the photon energy it makes, into the paired cost $2J(r)$ for all $r > 0$; a profile equals the cut profile of the field it selects along the orbit only if it is zero; and the localized vacuum lies more than 4 above the homogeneous one, which the orbit never reaches. At period 2 a nonzero source, twice the divergence of a field spike's current, has zero cut profile in every Gauss solution. (Chapter 6, Propositions 6.12, 6.14 and 6.13(iii).)

Route 19. *The vacuum response as a determination of the operator.* The first response of the localized vacuum reads the bond profile, but adding $\kappa\Pi_n$ (Π_n the projection on n-photon states, $n \geq 1$) keeps the Hamiltonian self-adjoint and every vacuum output, and changes it on n-photon states exactly when $\kappa \neq 0$. The response fixes the profile, not the operator; matching the posting work on a dense set as well, a premise, fixes a bounded correction to zero. (Chapter 6, Theorem 6.15.)

Route 20. *An infinite-volume limit of the centered reference-vector action.* On the clock-photon lattice (carrier (i)) with $C_E > 0$, along every family of finite regions whose source variance V_R tends to infinity, the vectors $(\widetilde{H}_R - c_R)\Omega$ have no norm limit, whatever the scalars c_R. For the unit-edge family with no background $V_R \geq 4|R|\,\nu_{\min}$, so this holds for every edge assignment. (Chapter 6, Theorems 6.31 and 6.34.)

Route 21. *The marked encoding as a correspondence of the atomic and cell carriers.* No linear isometry from the atomic carrier (g) into the cell carrier (i) extends the fixed-site embeddings. The marked encoding is a translation-covariant isometry, but the regional Hamiltonian has zero amplitude between configurations that differ at two cells, while the atomic source with coupling $g \neq 0$ moves a marked source one step with nonzero overlap; for the clamped source on the photon vacuum the Born weights differ at every small $t \neq 0$. (Chapter 6, Proposition 6.5.)

Route 22. *The joint long-time state as the product of the one-register states.* For the unit-edge family with $\epsilon = 0$ and an edge assignment onto all edges, a finite set of registers and matrices A_p have long-time average 4^{-6} where the occupation-dephased product gives 0. Each register whose direction is filled dephases alone to its occupation-diagonal state; the joint statement is false, and the pointer basis holds per register only. (Chapter 7, Theorems 7.40 and 7.34.)

Route 23. *The reference representation as the carrier of the dynamics.* For the unit-edge family with $\epsilon = 0$, at every $t \neq 0$ the evolved reference state differs by at least $\Delta_t > 0$ from every local excitation of ω_{ref} on an observable of norm at most $\frac{1}{2}$. It is the state of no vector of the reference representation, and no unitary of that representation implements the dynamics. A physical or thermal state of the infinite field has to lie outside the reference folium. (Chapter 7, Theorem 7.56 and Remark 7.57.)

D.6 Many recognizers

Route 24. *A reversible quantum evolution of one recognizer on its imbalance alone.* For a transition law P on $\mathbb{Z}^3$ meeting (P1) and (P3) with $P(0,0) < 1$, no orthonormal family in $\ell^2(\mathbb{Z}^3)$ has the Born weights of P; in particular no unitary has the weights of K (carrier (c)).

A record cell enlarges the carrier, and then every such law has a reversible picture, which selects nothing. (Chapter 8, Propositions 8.16 and 8.17.)

Route 25. *A coupling selected by balance rules, by fresh records or by one energy operator.* On finite sets of recognizers (carrier (d)) under K, joint detailed balance against the product weight with joint holding-and-return balance forces the product of the rows of K; each balance alone admits a continuum of coupled laws; a fresh record with one bounded self-adjoint operator realizes every joint law; and a shared thermal record cell admits coupled laws, and forces the product again under joint holding-and-return balance. (Chapter 8, Theorem 8.7 and Propositions 8.8, 8.17; Chapter 9, Proposition 9.15.)

Route 26. *Imbalance as position, and its spread as expansion.* A transition law on $\mathbb{Z}^3$ that holds or makes one post, has detailed balance against $w(s) = \varphi^{-|s|_1}$, makes the six posts from balance equally likely and posts from balance with positive chance is not invariant under the translations of $\mathbb{Z}^3$. The imbalance is a ledger coordinate, not a place, and its relaxation is not motion in a space. (Chapter 8, Proposition 8.20.)

D.7 Heat

Route 27. *The cost J as the energy behind the golden weights.* No real β satisfies both $e^{-\beta J(\varphi)} = \varphi^{-1}$ and $e^{-\beta J(\varphi^2)} = \varphi^{-2}$, since $J(\varphi^2)/J(\varphi) = \varphi^3 \neq 2$. A constant factor in front of J is absorbed into β, so no energy unit rescues the reading. (Chapter 9, Proposition 9.5.)

Route 28. *Accumulated cost, a fixed price per post, or averaging over the clock origin as heat.* Posting an event of ratio $r \neq 1$ with its reciprocal n times gives cost $2nJ(r)$ at net flow zero, so accumulated cost is not a function of the net flows and no bounded sequence equals it. No function on $\mathbb{Z}^3$ rises by a fixed $q > 0$ at every post, since a post and its reverse return to the start. Averaging over the eight origins of a clock removes coherences between different clock labels and keeps every diagonal entry, so it moves no energy. (Chapter 9, Proposition 9.14.)

Route 29. *A finite record cell for the contact law.* Under (P1)–(P4) and $b > 0$, no real energy on the profiles of two recognizers, finite thermal cell and energy-conserving operator moving one joint step reads the contact law R_b on every joint move; a cell with finite partition function that does is infinite, with energies unbounded above. Open: a countable cell. (Chapter 9, Proposition 9.17 and Remark 9.18.)

Route 30. *An exact T^4 law at every temperature.* On the photon field of carrier (i) with $C_E > 0$ and a supplied Bose occupation, the thermal energy per polarization lies between $Te^{-\omega_{\max}/T}$ and T, so for every $B > 0$ some temperature has $u_1(T) < BT^4$. The quartic law holds as the limit $T \to 0$, with coefficient $\pi^2/(30C_E^3)$. (Chapter 9, Theorem 9.20.)

Route 31. *A thermal reading from a photon state of finite trace.* With $C_E > 0$, for every finite set of source states and every joint state, some one-photon probe reads 0 while the heat form $\Theta(v) = \theta\langle v, e^{-\tau H_0} v\rangle$ reads it positive; the uniform gap is at least $\theta e^{-\tau\omega_{\max}}$; no sequence of such states converges to Θ on every probe; and Θ is the reading of no density operator. A thermal photon field must be read in finite regions or modes, or be a state outside the Fock space. (Chapter 9, Theorem 9.21.)

D.8 Toward the hot, expanding universe

Route 32. *The cost of a ratio as an inflaton potential, and φ as its initial value.* In the coordinates of the cost, $V(q) = J(1+q)$ has $\varepsilon(q) \geq \frac{9}{8}$ on $0 < q \leq 1$ and, in $t = \log(1+q)$, $\eta(t) > 1$ for $t \neq 0$. The plateau potential $V_0 \tanh^2(t/2)$ with $t = \sqrt{2/3}\,\psi/\varphi$ has $\varepsilon(\varphi) = 16/(3\varphi^2) = 2.037 > 1$: a field starting at the golden ratio has already left slow roll. That no constant normalization rescues the cost potential is an inference from standard formulas, not a proved result. (Chapter 10, Propositions 10.1, 10.3 and Remark 10.2.)

Route 33. *Golden values of the primordial spectrum* (measurement). The tilt $n_s = 1 - 2/\varphi^3 = 0.5279$ lies 104 standard deviations below the Planck 2018 value 0.9649 ± 0.0042 [87], and $r = (\varphi - 1)^4 = 0.1459$ exceeds the BICEP/Keck bound $r_{0.05} < 0.036$ (95%) [7]; the data are pinned in Appendix C. (Chapter 10, Table 10.3.)

Route 34. *A bare CP source of strength φ^{-45}* (measurement). In $\eta_B = P|\varepsilon_{CP}|\,D$ with the conditional $P \in (2.49494, 2.49505)$ and the adopted $D = \varphi^{-44}$, $|\varepsilon_{CP}| = \varphi^{-1}$ gives $\eta_B = P\varphi^{-45} = 9.832\times10^{-10}$, about 90 standard deviations above the pinned $(6.127\pm0.041)\times10^{-10}$. (Chapter 10, Table 10.5.)

Route 35. *A golden ladder of acoustic peaks.* If $\ell_1 \in [220, 221.2]$ and $\ell_2 \in [536.8, 539.4]$, the one-standard-deviation ranges of the first two measured peaks, are projections $\ell_i = \mathscr{D} k_i$ of two wavenumbers by one $\mathscr{D} > 0$, then $k_2/k_1 > 2.4 > 1.05\,\varphi$. (Chapter 10, Proposition 10.6.)

Route 36. *Births outside the ledger, read as expansion.* Recognizers added on a schedule, with a chosen sign pattern, are not lawful events: every admissible joint law acts on a fixed finite set of recognizers and defines no distance (Chapter 8). Computed runs of such a schedule give at most 0.231 e-fold and no exit to deceleration; that bound is a computation, not a theorem. (Chapter 10, Table 10.1.)

Route 37. *The redshift law from finite regional counts.* For a supplied history of curved and flat plaquette steps, a region's rate $q_R = \pi n_c(R)/(4n(R))$ is at most $\pi/4$; a ratio of two such rates is rational, never $1/\varphi$, the value of $1/(1+z)$ at $z = \varphi - 1$; and no function bounded by $\pi/4$ with $q_0 > 0$ has $q(z)/q_0 = 1/(1+z)$ for all $z > -1$ (this part fails only toward $z \to -1$). Open: a continuum limit of such counts. (Chapter 10, Proposition 10.8.)

Route 38. *Eleven forced by a budget of sixteen.* In the candidate $\Omega_\Lambda = 11/16 - \alpha/\pi$, a budget of sixteen modes split into active and passive is met equally by $6+10$ and $4+12$, so the budget does not force eleven passive modes. (Chapter 10, Section 10.9.)

Route 39. *A golden number in place of α* (measurement). Replacing α by $(1 - \varphi^{-1})/2 = 0.190983$ gives $\Omega_\Lambda = 0.626708$, 11.1 standard deviations below the pinned 0.6889 ± 0.0056, and every rescaling of the value 0.626708 by an integer power of φ misses by at least 0.056, ten standard deviations. (Chapter 10, Section 10.9.)

Route 40. *A vacuum density $\rho_P\varphi^{-2s}$ as a golden prediction.* For every base $b > 1$ and $L, \ell > 0$, $b^{-2\log_b(L/\ell)} = (\ell/L)^2$: with $s = \log_\varphi(L/\ell_P)$ the form is the holographic estimate $\rho_P(\ell_P/L)^2$ in golden notation and carries no golden content. (Chapter 10, Proposition 10.9.)

Route 41. *The dark-energy closure $s = 1/(1+s)$ from conservation or least cost.* Its positive root is φ^{-1}, but every geometric occupancy $s(1-s)^k$ with $0 < s < 1$ conserves the total, and least cost on $(0, 1]$ selects the trivial ratio $s = 1$. (Chapter 10, Proposition 10.10.)

Route 42. *Evolving dark energy with $w \geq -1$ explaining the Planck residual* (inference). Under the accumulation premise such dark energy was denser in the past (Proposition 10.11), and a numerical quadrature, not proved, gives an effective fraction below the static value: the

wrong sign. (Chapter 10, Section 10.9.)

Route 43. *The Hubble thirteen and the dark-energy eleven from one reading of the tick.* Reading the tick as one of the twelve cube edges gives 12 and 11; reading it as a thirteenth axis gives 13 and 12; no single reading gives 13 and 11. (Chapter 10, Section 10.11.)

Route 44. *A gap term* $1/(8\log\varphi)$ *in the dark-matter fraction.* In the adopted candidate $\Omega_{\mathrm{dm}} = \sin(\pi/12) + \delta$ of Chapter 10 (Section 10.10) the gap is $\delta = \varphi^{-4}/24 = 0.0060791$. The term $1/(8\log\varphi) = 0.259761$, about 43 times δ (the sum would be 0.518580), cannot take its place.

Proposition D.2 (the gap term is not small)**.** $1/(8\log\varphi) > 0.17$. *Hence* $1/(8\log\varphi) > 27{\cdot}\varphi^{-4}/24$ *and* $\sin(\pi/12) + 1/(8\log\varphi) > 0.42$.

Proof. $1 < \varphi < 2$ gives $0 < \log\varphi < \log 2 < 0.6932$, so $8\log\varphi < 5.546$ and $1/(8\log\varphi) > 1/5.546 > 0.18$. Next $\varphi^{-4}/24 < 0.0061$ and $27 \cdot 0.0061 < 0.17$; and $\sin(\pi/12) = (\sqrt{6} - \sqrt{2})/4 > 0.2588$. □

Route 45. *A sign of the matter excess from cost or from the native law.* Cost is even (Route 15), so it orders neither side of a swap that leaves it unchanged. On carrier (c), under (P1)–(P4) and the three readings, K is invariant under the 48 signed permutations of the accounts; from the empty record every imbalance is exactly as likely as its reverse at every tick, and every odd observable has mean zero. A sign needs an input that breaks these symmetries: an adopted bit or a coupling that is not symmetric. (Chapter 4, Proposition 4.26; Chapter 10, Proposition 10.5.)

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