

Physics-Based Morality

One recognition, written twice: the cost of a wrong,
the record of what happened, and the price of a duty

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Abstract

Two people do one thing together and each writes it down. Where their two records differ there is a gap, and the gap belongs to the relationship, not to either of them. Read as two sides of one recognition, which is the one definition the paper makes, the gap is priced by the same law that prices every imbalance in Recognition Science; the definition is the paper's one modelling commitment, and that the two figures of a real pair of records are such a pair is the open link the paper names and does not close. From the two records and their history, with no moral axiom, follow who wrote, who consented, who was changed, how the cost of the disagreement divides (read from the books alone: equally between the two ends, whoever wrote, for any reciprocal cost), how a gap is closed and by whom, and who drove the records apart. The two records cannot say what is owed: a creditor who delivered and was refused a signature has the same record history as one who invented the claim. What is owed needs a record of what physically happened. A duty appears only for a party that goes on acting from a finite reserve; its price is one premise, stated exactly and proved irreducible on its signature. Of the five places the physics seems to enter, two are physics and three are the mathematics of the ledger, and the paper says which. Every theorem is proved on the page.

1 The question

Two people make an agreement, and each writes it down. If both write the same thing there is one agreement and two copies. If they write different things there is a gap, and the gap belongs to the relationship, not to either person. This paper starts from that object, one act written twice, and asks what follows from it by theorem.

The answer has three parts, and the paper is built around the lines between them. First, a great deal follows. The two records and the line that says who wrote each entry fix who acted, who consented, who was changed, how far the records differ, what the difference costs while it stands and how that cost divides between the two ends, how a gap is closed and by whom, and, from the history, who drove the two records apart. None of this needs a moral axiom. Second, the cost is not a new quantity brought in for ethics. Recognition Science derives one cost of imbalance from its stated premises, $J(x) = \frac{1}{2}(x + x^{-1}) - 1$, the function every other constant of the spine is read from (Section 2 says which, and that none of them is used here); it is a cost the physics already had, not one brought in for ethics. In the spine's own account of the hierarchy a join of two parts is a recognition whose cost is J of the ratio of the parts. Two records of one act are two parts compared. Their disagreement costs J of their ratio because that is what a compared pair costs. Third, two things do not follow, and the paper proves that they do not. What is owed is not in the two records or in their whole history: a record of what

physically happened is needed, and the paper says exactly what it must carry. And no duty follows from the records: a duty appears only for a party that goes on acting from a finite reserve, and the paper prices that premise and proves it cannot be reduced on its signature.

Method. The method is the one used for arithmetic in the δ -calculus paper [1]: fix the smallest object the subject can be stated on; prove which words are forced by structural conditions on that object and which are choices, with a countermodel for every condition dropped; locate the existing theories as instruments each reading one projection of the object; and price exactly the step that cannot be derived. Ethics gets the same treatment, and the same honesty about where the derivation stops.

Where the object already has names. The object is not new to three literatures, and the paper meets each where it stands. In accounting it is intercompany reconciliation: two entities' books of one transaction, reconciled by confirmation from outside both [24, 25]. In contract theory the distance between the two records and the event they record is the distance between what both parties observe and what a third party can verify [28, 29]. In the theory of rights the release that only the creditor can give is a power in Hohfeld's sense, and the claim it extinguishes is Thomson's [31, 32]. In the theory of practical reason the duty of Section 10 has the shape of constitutivism, and its floor theorem is the objection to constitutivism proved as a theorem [33, 34, 35]. Section 14 says how far each reaches.

What the earlier papers established, and what this one adds. The economics paper [7] built the two-level books: parties, accounts, relationships, and the two-entry record of a claim, with the counterpart gap as the object a price is read from. The cost paper [8] proved that a reciprocal, normalized cost of a ratio obeying the composition law and monotone in the imbalance is one member of the family $F_c(x) = \frac{1}{2}(x^c + x^{-c}) - 1$, and that the calibration fixes $c = 1$. The ledger paper [9] priced a network of directed bonds by J and stated, as open, whether its quantities are harm, consent, suffering and obligation. The evidence paper [10] proved that agreement does not settle what is owed. This paper puts the whole structure in one place: the object and why its cost is the physical cost (Section 2); what the two records force (Sections 3 to 5); what the history adds (Section 6); what neither can say and what the third record supplies (Sections 7 to 9); who pays and what a duty costs (Section 10); the three projections and the two carriers as instruments (Sections 11, 12); who can be a patient (Section 13); and, last, exactly where the physics is in all of this and where it is not (Section 14).

The sentence that bounds the paper. The results determine what the two records force, what their history adds, and what is due under a stated exchange rule and an explicit evidence record. The rule's moral authority, the fairness of agreed terms, welfare, rescue, collective responsibility, a duty from physics alone, and which physical systems are parties lie outside the conclusions, and the paper says so where each would enter.

Terms used in this paper

Term	Object	Plain gloss
chart	parties, accounts, relationships with two designated accounts each	who can owe whom, and where each side writes it
books	a real number on every account of every party	everyone's own ledger, side by side
gap	$\gamma_e = x_e - y_e$, source's figure minus destination's	the disputed part of one claim
move	one party's zero-sum write, or a joint record by both ends	the two ways the books can change
author line	the parties on whose rows a move is written	who wrote it
consent	wrote one's own side	the part of agreeing the books record: who wrote; not why
patient	non-author at an end of a changed gap	someone it happened to
compared pair	the two figures (x_e, y_e) read as the two parts of one recognition	the one definition this paper makes
cost	$J(x) = \frac{1}{2}(x + x^{-1}) - 1$ of the ratio x_e/y_e	what a disagreement costs per tick, in the units every imbalance costs
harm	half the change of J on each of one's relationships, signed	the assigned cost of a move at each end; not blame
departed	wrote a move that drove the records apart, no agreement since	who left the last state both had signed
third record	the events: undertaking, crossing, re-payment, release	what happened, apart from what was written
due	the exchange rule's register after the events so far	what is owed, by the rule
ceiling	a bound on a party's lifetime charge	the one hypothesis under which every ratio closes

Contributions

1. The cost of a disagreement is the physical cost. The two figures of a relationship, read as the two parts of one recognition, cost exactly what the spine says a compared pair costs, J of their ratio; equal figures cost nothing and distinct figures cost something. No second cost enters (Section 2).
2. A gap changes only under a move with one author, an end of the relationship; a joint record changes no gap; only an end can close a gap, by one of two exact acts; a third party cannot touch it and pays when the network clears. Consent is the author line and separates from non-worsening in both directions (Section 3).
3. The disputed balance is forced up to a unit; the cost of harm is the hyperbolic family, and read from the books alone (the author line refused) its division between the two ends is equal, whoever wrote, for any reciprocal cost of the ratio, the composition law playing no part in the split; for every condition a reading that fails it alone; consent is not in the books (Section 4).
4. An open gap costs linearly in time and past every bound; repair by either end harms no third party; disagreeing records meet at the geometric mean; under a settlement that holds each party's net reading, the bystander to a triangle bears 0.0600 against the quadratic's 1/18, and confined to the disputed relationship repair costs the bystander nothing (Section 5).

5. Who departed is on the history and in neither the books nor the last move; the honest creditor and the fraudster reach the same books (Section 6).
6. What is owed is not in the two records or their history; it is fixed by a record of what happened, whose physical part is a unit's crossing; the same crossing is a sale or a gift; agreement is not truth, and correcting a false agreement costs (Sections 7 to 9).
7. A duty binds only a party that goes on acting: finite supply with no overdraft forces every ratio to one (the gap may stay); whoever is committed to going on is committed, by the instrumental principle, to not sustaining extraction; the premise is irreducible on its signature and is nobody's preference (Section 10).
8. Three projections of the full reading are each blind to a proved pair; two independently written carriers agree on the reading; a party can be a patient exactly when it has a relationship, whatever its complexity (Sections 11 to 13).
9. Of the five places the physics enters, two are physics and three are mathematics of the ledger, and the dependence test says which is which; of fifty-eight moral words, three strictly and eleven leniently are carried by the two records, and what the rest need is named (Section 14).

2 One recognition, two records

2.1 What the spine supplies

Recognition Science begins from a recognition: one distinction with two sides. From eight stated premises and one cited classical theorem, its front-door theorem concludes the cost of an imbalance,

$$J(x) = \frac{1}{2}(x + x^{-1}) - 1, \quad x > 0, \quad (1)$$

together with the hierarchy ratio φ , three spatial dimensions, the period eight, and the level weights φ^{-n} [1, 8]. The uniqueness of the cost under its hypotheses, the classification of the reciprocal costs around it, the forcing of its composition rule, and the way a conserved ledger fixes its own scale, cost and ratio are in the peer-reviewed record [2, 3, 4, 5]; the eight-tick register and the balanced posting are in [6]. The cost is zero exactly at $x = 1$, is the same for x and $1/x$, and rises on either side. Three of its properties are used here and nothing else about the physics is. First, J is the unique cost of a ratio that is reciprocal, normalized, obeys the composition law and is monotone in the imbalance, with the calibration $c = 1$ fixed upstream; the uniqueness is proved again on the page in Section 4, where it is used, and only the calibration is imported. Second, in the spine's account of the hierarchy, a level is the join of two earlier levels, and a join is a recognition: its two parts are compared, and the cost the join posts is J of the ratio of the parts. Equal parts cost nothing; distinct parts cost J of their ratio, and a join is costly exactly when its parts are distinct. Third, a located unit has a position at every tick and moves one step at a time; the spine's space is a lattice of three dimensions, and Section 8 uses of it only that a unit has a position and that a place is coarser than a position. The dimension does no work in this paper, and Section 14.1 says so.

2.2 The books

A relationship has two designated entries, one in each party's book. A party can change its own entry. A joint record changes both entries by opposite signed amounts, so their sum stays fixed.

Definition 2.1 (Chart). A *chart* has a finite set P of parties, a finite set A of accounts, and a finite set R of relationships. Each relationship e has a source $s(e) \in P$ and a destination

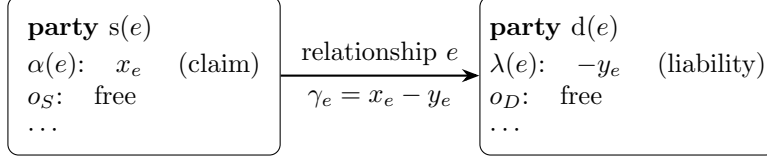


Figure 1: One relationship, two records. The source writes its claim on its own asset account; the destination writes its liability on its own liability account. Neither can write in the other's book. The gap is the claim minus the acknowledged liability.

$d(e) \in P$, with $s(e) \neq d(e)$. It has an asset account $\alpha(e) \in A$ on the source's books and a liability account $\lambda(e) \in A$ on the destination's books. The designated accounts of a party are the $\alpha(e)$ for relationships it sources and the $\lambda(e)$ for relationships it receives. Distinct relationships from the same source have distinct asset accounts; distinct relationships to the same destination have distinct liability accounts; at the same party no account is both an asset account of one relationship and a liability account of another. A party is an *end* of e when it is $s(e)$ or $d(e)$. An account of a party that is not designated is *free*; a party with no free account is *frozen*. The chart is fixed: moves change balances and never create or remove a relationship.

Definition 2.2 (Books, figures, gap). *Books* are a function $B : P \times A \rightarrow \mathbb{R}$, a signed balance on every account of every party, debit positive. On relationship e the *source's figure* is $x_e(B) = B(s(e), \alpha(e))$, what the source books as its claim; the *destination's figure* is $y_e(B) = -B(d(e), \lambda(e))$, the liability the destination books, read as a positive number. The *gap* is

$$\gamma_e(B) = B(s(e), \alpha(e)) + B(d(e), \lambda(e)) = x_e(B) - y_e(B). \quad (2)$$

The two records *agree* on e when $\gamma_e(B) = 0$. The books are *matched* when every gap is zero and *positive* when every x_e and every y_e is strictly positive.

Definition 2.3 (Move). A *move* on a chart is one of two things.

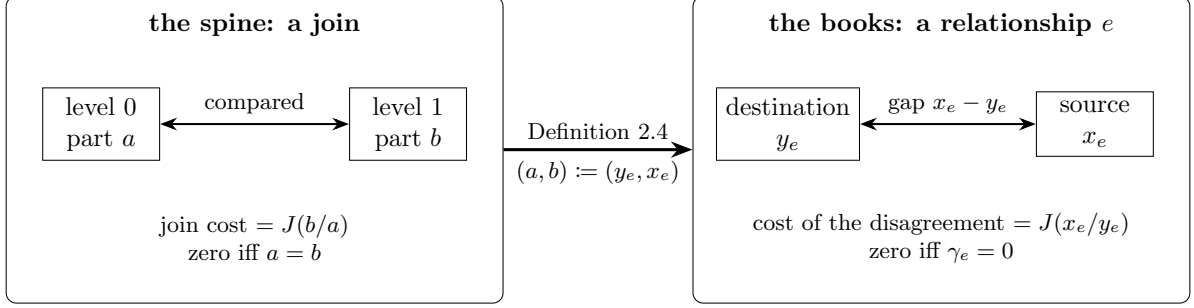
- An *own write* (i, v) : a party i and a vector $v : A \rightarrow \mathbb{R}$ with $\sum_a v(a) = 0$. It sends B to $B + v$ on row i and leaves every other row unchanged. Designated accounts are allowed: a party may write a claim, or a liability, alone. An own write that moves no designated account is *routine*.
- A *joint record* (e, x, o_S, o_D) : a relationship e , an amount x , a free account o_S of the source and a free account o_D of the destination. It adds x to $B(s(e), \alpha(e))$, subtracts x from $B(s(e), o_S)$, subtracts x from $B(d(e), \lambda(e))$, and adds x to $B(d(e), o_D)$. Both rows change.

The *author line* of a move is the set of parties on whose rows it is written: $\text{Wrote}(i, v) = \{i\}$ and $\text{Wrote}(e, x, o_S, o_D) = \{s(e), d(e)\}$. A *history* is a sequence of moves $m_0, m_1, \dots$; the books at tick t are $B_t = m_{t-1}(\dots m_0(B_0))$.

2.3 The one definition: the two figures are a compared pair

Everything moral in this paper rests on one identification, made here and nowhere else; Figure 2 draws it. A recognition has two sides. A relationship has two records. The two figures x_e and y_e are the two parts of one recognition, the same act seen from its two ends.

Definition 2.4 (Compared pair). On positive books, the *compared pair* of relationship e is the two-level sequence with the destination's figure at level 0 and the source's figure at level 1: (y_e, x_e) . Its *ratio* at lag one is x_e/y_e , and its *join cost* is $J(x_e/y_e)$, the cost the spine assigns to a join whose two parts stand in that ratio.



Proposition 2.5: the two costs are the same number, in the same units

Figure 2: The bridge. Left: in the spine a level is a join of two parts, compared at cost J of their ratio. Right: a relationship has two records. The one definition reads the two figures as the two parts; after it, the cost of a disagreement is the spine's join cost, and no second cost enters.

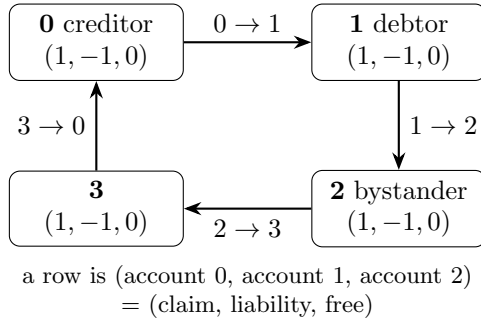
The definition has the standing the spine gives its own identifications of the clock and the hierarchy: a named definition, with every consequence a theorem under it. Nothing derives from the books that their two figures are two sides of one recognition; everything below is a theorem once that is said. It is the place a reader who does not accept the physical reading has to disagree (the exchange rule, the settlement rule and the continuation premise are the paper's other stipulations, each stated where its theorems are), and it introduces no quantity the physics did not already have. What it fixes is that the parts are the figures themselves: not the figures raised to a power, not the figures shifted by a constant, not any other pair built from them. Each of those would be a different pair with a different join cost. The order of the two parts is not among the choices: $J(x_e/y_e) = J(y_e/x_e)$, so which end sits at level 0 changes nothing. Which theorems below depend on the shape of J , and which hold for every reciprocal cost of the ratio, is the subject of Section 14.1.

What the definition does not establish is stated here once. It does not establish that the two figures of any real pair of records are magnitudes the cost law prices, nor that a real disagreement expends J of its ratio in a resource the parties hold. Those are the substrate map: which physical system carries a pair of records, and what its two figures are. The map is open, and Section 13 names the first check; until it is made, every theorem below is a theorem about the object, and its reach into a given pair of real records is the reach of the map. The published ledger construction derives balanced two-party postings from conservation and pairwise locality [6]; a joint record is such a posting, and the own write, one party's zero-sum entry on its own row alone, is what the two-record object adds to it.

Proposition 2.5 (The cost of a gap is the physical cost). *On positive books, for every relationship e :*

- (1) *the cost the ethics charges the disagreement on e , J of the ratio of the two records, equals the spine's join cost of the compared pair at lag one;*
- (2) *the pair costs nothing as a join iff $\gamma_e = 0$, and costs something iff the gap is open;*
- (3) *the join cost is $J(e^{t_e})$ where $t_e = \log(x_e/y_e)$ is the log-ratio reading every later theorem is stated on.*

Proof. (1) Both sides are $J(x_e/y_e)$ by Definition 2.4 and the definition of the join cost. (2) $J(r) = 0$ iff $r = 1$ for $r > 0$, and on positive books $x_e/y_e = 1$ iff $\gamma_e = 0$ by (2); $J(r) > 0$ for $r \neq 1$ since $J(r) = (r - 1)^2/(2r)$. (3) $e^{\log(x_e/y_e)} = x_e/y_e$ on positive books. $\square$



unit claim: 0 writes $(1, 0, -1)$

row 0: $(1, -1, 0) \rightarrow (2, -1, -1)$; row 1 unchanged
 $(x_0, y_0) = (2, 1)$, gap 1

joint unit record: 0 and 1 sign $(0 \rightarrow 1, 1, 2, 2)$

row 0: $(1, -1, 0) \rightarrow (2, -1, -1)$

row 1: $(1, -1, 0) \rightarrow (1, -2, 1)$

$(x_0, y_0) = (2, 2)$, gap 0

debtor's withdrawal: 1 writes $(0, 1, -1)$

row 1: $(1, -2, 1) \rightarrow (1, -1, 0)$; row 0 unchanged

$(x_0, y_0) = (2, 1)$, gap 1

Figure 3: Chart B on the unit books, and the three moves on $0 \rightarrow 1$ used throughout. A party's row is its three account balances. The unit claim moves one row and opens a gap; the joint unit record moves two rows and opens none; the withdrawal, taken after the joint record, moves one row and opens the gap that the claim opened, from the other side.

The proposition is an identity under the definition, and that is its content: after Definition 2.4 no second cost enters, and the paper's load-bearing wall is the definition, not the proposition. Every cost printed later, the quarter at ratio 2, the eighth each end bears, the bystander's 0.0600, is a join cost of a compared pair, in the units every imbalance in the theory costs. Where the paper later needs the composition law, reciprocity or the calibration $c = 1$ (Section 4), they are the spine's, not assumptions added for ethics; and the one result that matters most for how the cost divides, the equal split, does not need the composition law at all.

2.4 The running example

Example 2.6 (Chart B and the unit books). Four parties 0, 1, 2, 3 on a cycle of four relationships $0 \rightarrow 1, 1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 0$; three accounts, asset 0, liability 1, free 2. The *unit books* put 1 on every asset account, -1 on every liability account, and 0 on every free account: every relationship reads $(x_e, y_e) = (1, 1)$, every gap is 0, the books are matched and positive. The agreed unit on every relationship is a convention of the examples, not a quantity the physics supplies: a cost of the ratio needs a positive agreed figure to be defined at all, and every number printed in this paper is a number for disputing one unit against an agreed one. Section 4 says how the cost changes when the agreed figure does. The cast throughout is a creditor (party 0), a debtor (party 1), a bystander (party 2), and a fourth party 3; later an environment party joins. Three moves recur. The *unit claim*: the creditor writes $+1$ on its asset account against its free account, alone; $(x_0, y_0) = (2, 1)$, gap 1, ratio 2, join cost $J(2) = \frac{1}{4}$. The *joint unit record*: both ends write, $(2, 2)$, gap 0, cost 0. The *debtor's withdrawal*: the debtor writes $+1$ on its liability account against its free account, alone, from $(2, 2)$ to $(2, 1)$: the same books as the unit claim, reached by the other end.

The unit claim and the debtor's withdrawal reach the same books with different authors. Almost every separation the paper proves is visible on this pair.

On chart B, 0 is the creditor, 1 the debtor, 2 the bystander to $0 \rightarrow 1$, and 3 closes the cycle. The payer examples in Section 10 use a separate chart with creditor 0, debtor 1, and environment 2. Figure 3 draws the chart and the three moves.

2.5 One unit, eight scenes

Every theorem in the paper lands on one story, and the table below is the paper in miniature. A creditor (party 0) and a debtor (party 1) each keep a book. Their relationship starts matched at $(1, 1)$: one unit already stands on both books as a baseline, gap 0; what each party claims

scene	books on $0 \rightarrow 1$	cost, charge	departed	what happened	due
A. Both sign the unit	(2, 2)	0, 0 each	nobody	unit crossed	1
B. ... then the debtor withdraws its signature	(2, 1)	$\frac{1}{4}, \frac{1}{8}$ each	debtor	unit crossed	1
C. Creditor writes the claim alone; nothing crossed	(2, 1)	$\frac{1}{4}, \frac{1}{8}$ each	creditor	nothing	0
D. Creditor writes the claim alone; unit crossed, debtor silent	(2, 1)	$\frac{1}{4}, \frac{1}{8}$ each	creditor	unit crossed	1
E. Both sign with nothing due; creditor corrects	(2, 2), then (1, 2)	0, then $\frac{1}{4}, \frac{1}{8}$ each	creditor	nothing	0
F. Scene C, creditor pays its charge every tick	(2, 1) held	$\frac{1}{8}$ per tick	creditor	reserve S runs out at tick $8S + 1$	0
G. Scene C, then the debtor signs a joint record with the bystander on $1 \rightarrow 2$	(2, 1) unchanged	$\frac{1}{4}, \frac{1}{8}$ each	creditor	nothing	0
H. Triangle $0 \rightarrow 1 \rightarrow 2 \rightarrow 0$, $0 \rightarrow 1$ at ratio e ; 0 and 1 settle around 2	($e, 1$) to ($e/y^*, 1$)	bystander 2: 0 to 0.0600	0 and 1 wrote; 2 wrote nothing	not asked	not asked

Table 1: One unit through the paper, in eight scenes. Scenes B and C have identical books and are told apart by the history (Section 6). Scenes C and D have identical books and identical histories and are told apart by nothing on the two records (Section 7); the third record does it (Section 8). Scene E is Section 9, scene F is Section 10. Scene G is the yes that leaves a standing claim untouched (Section 3); scene H is the one way a party that wrote nothing is charged (Section 5).

or acknowledges in the story is its figure above that baseline, so at the start nobody claims anything. One unit may or may not cross from creditor to debtor. Table 1 follows the story through eight scenes; the two record columns are what the paper derives from the books alone, the two right columns need the third record. The last two scenes leave the one relationship: a yes given elsewhere, and a settlement around a third party.

3 What the two records force

The four words this section defines, author, consent, patient and gap, are the words every later theorem is stated in. They are defined first and their content is then proved: what a joint record can and cannot do, who can close a gap, how the author is read off the books, and in which directions consent separates from being made better or worse off. The section ends with the two facts about a yes that the books carry, a standing claim the yes does not touch and a crossing shown at one amount and kept at another.

3.1 Author, consent, patient

Definition 3.1 (Consent, patient). A party i *consents* to a move m when $i \in \text{Wrote}(m)$: it wrote its own side. A party i is a *patient* of m on books B when $i \notin \text{Wrote}(m)$ and some relationship e incident to i satisfies $\gamma_e(m(B)) \neq \gamma_e(B)$. It did not write the move, and the move changed a relationship it is an end of.

Consent here is the record of who wrote, and the word is used for the part of agreeing that the books carry. For a one-sided write the author's consent is its own signature; the work the word does is at the other end, where the party a move happened to wrote none of it. The only

place a second signature appears on this object is a joint record, and the sense of consent as a power to permit [16, 17] appears there and only there. Nobody can write in another party's book; what reaches a party is a change on a relationship it shares.

Lemma 3.2 (Gap under an own write). *For an own write (i, v) and any relationship e ,*

$$\gamma_e(B + v \text{ on row } i) = \gamma_e(B) + [s(e) = i] v(\alpha(e)) + [d(e) = i] v(\lambda(e)), \quad (3)$$

where $[\cdot]$ is 1 when the condition holds and 0 otherwise.

Proof. Only row i changes. Of the two entries in (2), the first lies on row $s(e)$ and moves by $v(\alpha(e))$ if that row is i ; the second lies on row $d(e)$ and moves by $v(\lambda(e))$ if that row is i . $\square$

Only the writer's row moves. A relationship can feel that write only at an entry belonging to the writer.

Lemma 3.3 (A joint record changes no gap). *For a joint record (e, x, o_S, o_D) and every relationship e' , $\gamma_{e'}$ is unchanged.*

Proof. For $e' = e$ the two designated entries move by $+x$ and $-x$ and their sum is unchanged. For $e' \neq e$: the entry $B(s(e'), \alpha(e'))$ is on a designated asset account of e' , which at the party $s(e')$ is distinct from $\alpha(e)$ (if $s(e') = s(e)$, by injectivity of asset accounts at a source) and from $\lambda(e)$ (no account is both), and distinct from the free accounts o_S, o_D ; the same holds for $B(d(e'), \lambda(e'))$. $\square$

A joint record changes what both parties acknowledge by the same amount. Their disputed balance stays fixed. Their ratio does not: from $(2, 1)$ a joint unit record gives $(3, 2)$, gap 1 still, and the cost falls from $J(2) = \frac{1}{4}$ to $J(3/2) = \frac{1}{12}$. Under the cost a joint record is not inert. It lowers the cost of a standing dispute by enlarging what is agreed around it, and Section 4 assigns that change to both ends. The $(t + 2, t + 1)$ history of Section 10 is this repeated: two parties who never settle their unit dispute and drive its cost to zero by agreeing more.

Theorem 3.4 (A gap changes only under one author, who is an end). *If a move m changes γ_e , then m is an own write (i, v) with $i \in \{s(e), d(e)\}$. Hence no routine move changes any gap; if neither end of e is in $\text{Wrote}(m)$ then $\gamma_e(m(B)) = \gamma_e(B)$; and if $\gamma_e(B) \neq 0$ and $\gamma_e(m(B)) = 0$ then m is an own write by one end of exactly $-\gamma_e(B)$ on its designated account of e .*

Proof. By Lemma 3.3 m is not a joint record, so it is an own write (i, v) . By Lemma 3.2 the change is $[s(e) = i]v(\alpha(e)) + [d(e) = i]v(\lambda(e))$, which is zero unless i is $s(e)$ or $d(e)$; a routine own write has both terms zero because it avoids the writer's designated accounts. For the closing, since $s(e) \neq d(e)$ exactly one indicator is 1 and the change equals $-\gamma_e(B)$. $\square$

Closing a gap requires one end to change its own entry. A bystander cannot make that correction, and a joint record leaves the gap as it was.

The closing act therefore has exactly two forms, and each has a name in ordinary life. The source writes minus the gap on its asset account, moving its claim to what the destination acknowledges: restitution of an excess claim, or its forgiveness. Or the destination writes minus the gap on its liability account, moving its acknowledgement to what is claimed: acceptance. The two records have no third form of closing, and no third party appears in either.

Proposition 3.5 (The author line is the set of changed rows). *If a move m changes the books at all, then $\text{Wrote}(m)$ is exactly the set of parties whose rows differ between B and $m(B)$. Under an own write (i, v) , x_e changes only if $i = s(e)$ and y_e changes only if $i = d(e)$. Every move m has a reversal $\bar{m}$ with $\bar{m}(m(B)) = B$ and $\text{Wrote}(\bar{m}) = \text{Wrote}(m)$.*

Proof. An own write (i, v) with $v \neq 0$ changes row i and no other. A joint record with $x \neq 0$ changes $B(s(e), \alpha(e))$ by x and $B(d(e), \lambda(e))$ by $-x$, and no other row; $\alpha(e) \neq o_S$ and $\lambda(e) \neq o_D$ since the counter accounts are free, so neither row's change cancels. x_e is an entry on row $s(e)$ and y_e on row $d(e)$. The reversal negates v , or negates x . $\square$

When the books change, the changed rows identify everyone who wrote. The exception is a write of nothing: the books stay fixed even though the move has an author.

Example 3.6 (Consent is in the rows and not in the gaps). On chart B from the zero books, the unit claim by 0 gives $(x_0, y_0) = (1, 0)$; the debtor's own write $(0, -1, 1)$ by party 1, the unit liability recorded alone, gives $(x_0, y_0) = (0, 1)$. The two books have the same absolute gap on every relationship, so any verdict computed from the profile of absolute gaps agrees on them; the pair (x_0, y_0) separates them, and with it the author. Signed gaps do not determine the author either: the debtor's withdrawal $(0, 1, -1)$ gives the same gap $+1$ as the creditor's claim. Consent is therefore a function of no gap profile, and for every move that changes the books it is a function of the books before and after: i consented iff row i changed. For a move that changes nothing it is not: two parties' writes of the zero vector leave every books unchanged and have different author lines.

Proposition 3.7 (Three separations; who is a patient). *Let a party's incident cost be half of $J(x_e/y_e)$ summed over the relationships it is an end of. On the unit books of chart B:*

1. *Party 2 is untouched by the unit claim written by 0: its cost stays 0, and it is neither a patient of the claim nor a consenter to it.*
2. *After the unit claim, $(x_0, y_0) = (2, 1)$; the creditor's reversal returns it to $(1, 1)$ and lowers party 1's cost from $\frac{1}{2}J(2) = \frac{1}{8}$ to 0; 1 is a patient of the reversal and did not consent.*
3. *Party 0 consents to its own unit claim and its cost rises from 0 to $\frac{1}{8}$: it is worsened by a move it consented to.*

An author is never a patient of its own move; a joint record has no patient; a party with no incident relationship is never a patient; the unit claim has exactly one patient, the debtor 1. On any chart with at least two accounts a party can be a patient of some move iff it is an end of some relationship.

Proof. $J(2) = \frac{1}{2}(2 + \frac{1}{2}) - 1 = \frac{1}{4}$. (1) The unit claim writes only on relationship $0 \rightarrow 1$; neither of 2's relationships changes and 2 wrote nothing. (2) The reversal is party 0's write; it changes γ_0 from 1 to 0 and 1 is the destination. (3) 0 wrote the claim, which moves relationship 0 from ratio 1 to ratio 2. The patient facts are the definition and Lemma 3.3. For the last claim let j be the other end of e and let j write 1 on its designated account of e and -1 on any other account; γ_e moves by 1 and i did not write. $\square$

Being left alone is not consenting; being helped is not consenting; and one can consent to something that costs. Being a possible patient is having a relationship. No label fixed on the parties decides this on every chart. If a label admits some party, a chart with no relationship refutes it. If it admits none, one relationship refutes it. A criterion of patienthood by complexity [22, 23] would have to enter as a condition on what counts as a party or a relationship, not as a threshold on the chart.

3.2 A standing claim and a mismatched crossing

Two conditions on a valid yes in ordinary use are that it was free and that it was informed. The object does not carry pressure, options, or knowledge. It carries two facts each of them presupposes.

Proposition 3.8 (The threat gap survives the yes; kept minus shown). *If $\gamma_e(B) \neq 0$ and B' is reached from B by any finite sequence of routine moves, then $\gamma_e(B') \neq 0$. The second object is being shown one amount while another is kept: a presented crossing on e with kept amount k and shown amount s is a pair of own writes. The source writes k on $\alpha(e)$ against a free account, and the destination writes $-s$ on $\lambda(e)$ against a free account. From books matched on e it leaves $\gamma_e = k - s$, matched iff $s = k$. The destination's row carries s and nothing of k .*

Proof. Theorem 3.4 for the first; Proposition 3.5 and (2) for the second. $\square$

On chart B, 0 writes a unit claim on 1 alone; then 1 and 2 write a joint record on $1 \rightarrow 2$; γ_0 is still 1, and 1's consent is a consent to relationship $1 \rightarrow 2$, not to $0 \rightarrow 1$. A one-sided claim can stand against a party while it writes a yes on another relationship. The yes leaves that claim untouched. Whether the claim caused the yes, which makes it a threat [21], is not on the object. Of deception it carries two records of one crossing that disagree, where the destination's record is what it was shown; whether the source intended the difference is not on the object.

4 Is the vocabulary forced?

A *reading* is a function from the object to a verdict. For the disputed balance and for harm, stated conditions determine the reading. Each condition has a counterexample that keeps the others.

4.1 The disputed balance is the gap, up to the unit

The source has written a on its asset account and the destination l on its liability account. A reading of the disputed balance is a number computed from (a, l) . Three things it should do: read zero when the two records match ($l = -a$); add when the source writes more; scale with the unit of account.

Definition 4.1 (Adequate balance reading). A balance reading assigns to each relationship e of a chart with books B a number $D_e(B)$. It is *local* when D_e depends on B only through the two entries $(a, l) = (B(s(e), \alpha(e)), B(d(e), \lambda(e)))$ of e , and is then written $D(a, l)$. An adequate balance reading is local and satisfies

- (i) matched entries have no disputed balance: $D(a, -a) = 0$;
- (ii) a further write by the source adds: $D(a + a', l) = D(a, l) + D(a', 0)$;
- (iii) a claim scales with the unit of account: $D(ca, 0) = c D(a, 0)$ for $c \geq 0$.

Theorem 4.2 (Balance rigidity). *Every reading satisfying (i) and (ii) is a function of the gap alone, $D(a, l) = g(a + l)$ with g additive. Every adequate balance reading is $D(a, l) = c(a + l)$ with $c = D(1, 0)$: a constant times the gap. Condition (iii) can be replaced by monotonicity, $D(a, 0) \leq D(a', 0)$ for $a \leq a'$, with the same conclusion and $c \geq 0$.*

Proof. From (ii) with $a = 0$: $D(a', l) = D(0, l) + D(a', 0)$. Put $g(a) = D(a, 0)$; (ii) with $l = 0$ gives $g(a + a') = g(a) + g(a')$, so $g(0) = 0$ and $g(-a) = -g(a)$. From (i): $0 = D(-l, l) = D(0, l) + g(-l)$, so $D(0, l) = g(l)$ and $D(a, l) = g(a) + g(l) = g(a + l)$ by additivity, which is the first claim and uses no regularity. From (iii): $g(ca) = c g(a)$ for $c \geq 0$, so $g(c) = c g(1)$ for $c \geq 0$ and, by oddness, for all real c . An additive function that is monotone is linear (linear on the rationals by additivity, squeezed between rationals by monotonicity). $\square$

In words: once a reading owes nothing on matched entries and adds up, it can see only the gap; once it also scales, it is the gap times a unit, and the unit is the one thing left to choose. Without (iii) or monotonicity the reading is $g(a + l)$ with g additive but not necessarily linear; such g exist given a basis of the reals over the rationals, and none can be written down without one. Drop (i): the one-record reading $D(a, l) = a$ is additive and scales, but reads 1 on matched entries $(1, -1)$. Drop (ii): the absolute reading $|a + l|$ vanishes on matched entries and scales, but $D(0, 0) = 0 \neq D(1, 0) + D(-1, 0) = 2$. Drop locality: take $\gamma_e(B) \cdot (1 + [\text{some other gap of the chart is nonzero}])$. This satisfies (i) to (iii) with the other entries fixed. On chart B it reads 2 on $0 \rightarrow 1$, whose own gap is 1, when $2 \rightarrow 3$ also carries a unit claim.

4.2 Harm: given the cost law, the division is forced

On the unit books of chart B the creditor writes the unit claim, carrying relationship $0 \rightarrow 1$ from $(1, 1)$ to $(2, 1)$. The relationship's cost went from $J(1) = 0$ to $J(2) = \frac{1}{4}$. Who is charged the quarter? The theorem says that a reading of the books alone must charge each end one eighth, whoever wrote, and says exactly which condition a reader must refuse to charge the author all of it.

The conclusion, in words: the cost of a disagreement is $\frac{1}{2}(x + x^{-1}) - 1$ in the ratio of the two records, up to the base of the reading, and each end is charged half of its change, whoever wrote. Write $t_e = \log(x_e/y_e)$ for the reading of relationship e on positive books, and let a move carry the readings from t to t' . The seven conditions that force the conclusion, in words:

- (0) the same ratios before and after give the same harm, whoever wrote: the quarter opened at $(2, 1)$ is charged the same way whether the creditor wrote $+1$ or the debtor wrote -1 to get there;
- (i) harm adds up over relationships and lands only on their ends: party 2, an end of neither record on $0 \rightarrow 1$, is charged 0;
- (ii) names do not matter, only the side one stands on, who wrote, and the number of parties: the same claim on $2 \rightarrow 3$ charges 2 and 3 as 0 and 1 were charged;
- (iii) each term is a fixed share of a change in a cost of the ratio, and that cost obeys the law for combining ratios and does not fall as the imbalance grows: the quarter is $J(2) - J(1)$, and a second doubling to $(4, 1)$ costs more than the first;
- (iv) some move charges somebody something: the reading that charges everyone 0 always is excluded;
- (v) owing more than is acknowledged and being owed more than is claimed are the same burden: swapping which end of $0 \rightarrow 1$ is source and which is destination changes nobody's charge;
- (vi) the parties together bear the network's cost change exactly once: the two charges on the unit claim sum to the quarter.

Definition 4.3 (Adequate harm reading). A harm reading is one rule for every chart. Given a chart, a move m carrying positive books with readings t to books with readings t' , and a party i , it assigns a number $H(m, t, t', i)$. It is adequate when

- (0) books alone: H depends on m only through t and t' ;
- (i) locality: H is a sum of one term for each relationship e of the chart. Each term depends on i , on e , on (t_e, t'_e) , and on the author line of m . It is zero unless i is an end of e ;
- (ii) invariance: the term depends on i and e only through two facts. They are the side i stands on (s, d, or not an end) and which of the two ends of e are in the author line. It depends on the chart only through its number of parties;

- (iii) ratio law: the term is $w(F(e^{t'e}) - F(e^{te}))$. The weight w does not depend on the readings or on the books. The cost of the ratio F obeys $F(xy) + F(x/y) = 2F(x)F(y) + 2F(x) + 2F(y)$. The function $u \mapsto F(e^u)$ is non-decreasing on $u \geq 0$;
- (iv) nondegeneracy: on some chart, some move is charged to some party;
- (v) orientation symmetry: reversing the orientation of a relationship changes no party's harm. To reverse e , exchange its source and destination. Let the former destination book y_e as its claim on an asset account. Let the former source book x_e as its liability on a liability account. Leave every other figure where it was. The reading of e becomes $-t_e$ and no other reading moves. Any move on the reversed chart with the same author line that carries the transported books to the transported readings is read as the original move was;
- (vi) single charging: $\sum_i H(m, t, t', i) = \sum_e (F(e^{t'e}) - F(e^{te}))$, the change in the network's cost.

Condition (0) says that harm is a fact about the ratios: two moves from books with the same ratios that reach the same ratios on every relationship harm everyone alike, whoever wrote them. Condition (v) says that being owed more than is acknowledged and owing more than is claimed are the same burden. These two are the conditions a reader may refuse, and what refusing each buys is written out below. The composition law is a postulate, taken from the cost paper [8]; its uniqueness theorem and the classification around it are peer-reviewed [2, 3, 4]. With monotonicity, it fixes the cost family through d'Alembert's equation [12]. The conditions on the two records fix how that cost divides between the ends, and they do so for any reciprocal cost of the ratio: the equal split below does not depend on the postulate.

Theorem 4.4 (Harm rigidity). *Every reading satisfying (0) to (iv) is*

$$H(t, t', i) = \sum_{e: s(e)=i} w_s(F_c(e^{t'e}) - F_c(e^{te})) + \sum_{e: d(e)=i} w_d(F_c(e^{t'e}) - F_c(e^{te})), \quad (4)$$

with $F_c(x) = \frac{1}{2}(x^c + x^{-c}) - 1$ for some $c > 0$. For each number of parties there are two constants w_s, w_d . They are not both zero at the number of parties on which (iv) is witnessed. Every reading satisfying (0) to (vi) has $w_s = w_d = \frac{1}{2}$ at every number of parties:

$$H(t, t', i) = \frac{1}{2} \sum_{e \ni i} (F_c(e^{t'e}) - F_c(e^{te})). \quad (5)$$

The split is independent of the composition law: if (iii) is weakened to its form alone, a fixed weight times the change of some cost F of the ratio with $F(1/x) = F(x)$ and $F(e) \neq F(1)$, then (0), (i), (ii), (v), (vi) still give $w_s = w_d = \frac{1}{2}$ with that F in place of F_c . The composition law and monotonicity name the cost; they play no part in dividing it.

Proof. With $G(u) = F(e^u) + 1$ the composition law reads $G(s+u) + G(s-u) = 2G(s)G(u)$. At $u = 0$: $2G(s) = 2G(s)G(0)$, so $G(0) = 1$ unless $G \equiv 0$; $G \equiv 0$ is $F \equiv -1$, whose changes are all zero, excluded by (iv). At $s = 0$: $G(u) + G(-u) = 2G(u)$, so G is even: F is reciprocal and $F(1) = 0$. G is non-decreasing on $[0, \infty)$, so $G \geq 1$ there. Put $S = \sqrt{G^2 - 1} \geq 0$ on $[0, \infty)$. For $0 \leq u \leq s$: the equation at (s, u) gives $G(s+u) + G(s-u) = 2G(s)G(u)$, and at $(s+u, s-u)$ together with $G(2v) = 2G(v)^2 - 1$ (the equation at (v, v)) gives $G(s+u)G(s-u) = G(s)^2 + G(u)^2 - 1$. So $G(s+u)$ and $G(s-u)$ are the two roots of $z^2 - 2G(s)G(u)z + G(s)^2 + G(u)^2 - 1 = 0$, which are $G(s)G(u) \pm S(s)S(u)$, and since $s+u \geq s-u \geq 0$ and G is non-decreasing, $G(s+u) = G(s)G(u) + S(s)S(u)$. Squaring, $S(s+u) = G(s)S(u) + S(s)G(u)$. So $\varphi = G + S$ satisfies $\varphi(s+u) = \varphi(s)\varphi(u)$ for all $s, u \geq 0$, with $\varphi \geq 1$ non-decreasing; $\log \varphi$ is additive and non-decreasing on $[0, \infty)$, so it is cu for a constant $c \geq 0$. Since $(G+S)(G-S) = 1$, $G(u) = \cosh(cu)$ on $[0, \infty)$ and, G being even, everywhere. So $F = F_c$ for some $c \geq 0$, with $F_0 \equiv 0$; (iv) excludes $c = 0$.

By (i) to (iii) the term on e for the party on a given side is $w(n, \text{side}, \pi) (F(e^{t'_e}) - F(e^{t_e}))$, where n is the number of parties and π says which ends of e wrote, and the term of a non-end is zero. Condition (0) removes π : fix n , positive books, a relationship e whose ends each have a free account, and a target reading t'_e . The source's own write of $y_e e^{t'_e} - x_e$ on $\alpha(e)$ against its free account carries t_e to t'_e , keeps the books positive, and moves no other reading; so does the destination's own write of $y_e - x_e e^{-t'_e}$ on $\lambda(e)$ against its free account; and any other move carrying readings t to t' , a joint record among them where one does, is by (0) read exactly as these two are. So $w(n, \text{side}, \pi)$ takes one value for every pattern π any move realizes, and (4) follows, with (iv) giving "not both zero" at the witness's n .

For (v): reverse the orientation of e . The party that was its source is now its destination with reading $-t_e$, and its term becomes $w_d(F_c(e^{-t'_e}) - F_c(e^{-t_e})) = w_d(F_c(e^{t'_e}) - F_c(e^{t_e}))$ by reciprocity. Invariance of its harm for all t_e, t'_e , with F_c not constant, gives $w_s = w_d$ at each n .

For (vi): summing (4) over all parties, each relationship e contributes its change once at $s(e)$ with weight w_s and once at $d(e)$ with weight w_d , and these are distinct parties. So the total is $(w_s + w_d) \sum_e (F_c(e^{t'_e}) - F_c(e^{t_e}))$, and equality with the network change for all t, t' gives $w_s + w_d = 1$. The two ends of a relationship are the two agents of a symmetric cost-sharing problem, and equal division is its symmetric budget-balanced rule [13, 14].

For the last sentence: the (v) and (vi) arguments used of F_c only that $F_c(e^{-u}) = F_c(e^u)$ and that $F_c(e) - F_c(1) \neq 0$, on the one instance where the source writes the matched books $(1, 1)$ to reading 1 and on its image under reversal. The second paragraph, which reduces the term to a side weight times a change of the rule's own cost, used (0), (i), (ii) and the form in (iii) and never the composition law; only the first paragraph, which names the cost, used it. So the last three paragraphs run unchanged for any reciprocal F that separates e from 1. $\square$

With these conditions, each end bears half the change in its relationship's cost. Changing who wrote cannot change that split. The remaining freedom is the base of the cost reading.

The word *harm* names the assigned share of the change of cost at an end and nothing else: not suffering, not fault, not who should pay. That the two ends of a disputed relationship bear the same share says that the cost of their disagreement is one number split by a symmetric rule under conditions that refuse to read the author line; it says nothing about which end is the aggressor. Who drove the records apart is the question of Section 6, and the reading that charges by it, departed-pays, is the one that refuses condition (0) and keeps the rest (Proposition 6.4).

The disputed balance is in units of account and the cost is of the ratio, and the two behave oppositely under a change of unit. Multiply both figures of a relationship by $\lambda > 0$: the gap becomes λ times the gap, and the ratio, hence the cost, is unchanged. A cost that read the gap would change with the currency it was written in; a cost of the ratio is the same in every currency. That is why the cost is of the ratio, and it has the consequence Section 10 returns to: one unit disputed against an agreed hundred costs $J(101/100) = 1/20200$, the same unit disputed against an agreed one costs $\frac{1}{4}$, and the disputed balance reads 1 in both. This has content beyond invariance: one unit disputed against an agreed figure b costs $J((b+1)/b) = 1/(2b(b+1))$, so the same disputed unit costs less inside a larger agreed relationship, and for large agreed figures the cost falls as the square of the agreed figure. The agreed figure b is not a convention of the object: it is read from the books, the amount both records already carry on the relationship, and a relationship on which nothing is agreed, $(0, 0)$, has no ratio and no priced dispute, because there is no recognition in common to disagree about. What is a convention is the examples' $b = 1$, and every printed number is for that b . The cost reads the disagreement against what is agreed, not the amount in dispute. Whether that is what harm is, is exactly what the identification of Section 2.3 decides, and it decides it once.

The base of the reading. $F_c(e^t) = F_1(e^{ct})$: the exponent c is the base in which the reading is taken, and it is the one parameter (0) to (vi) leave free, as the balance theorem leaves $D(1, 0)$ free. The cost paper [8] fixes $c = 1$ from a premise about the carrier of counting that this object does not contain. From here on the paper is stated at $c = 1$, $F_1 = J$; every structural statement holds for every $c > 0$ with F_c in place of J , and the printed numbers are the $c = 1$ values.

Corollary 4.5 (Both ends bear the same). *Under (5), a move that changes one relationship charges its two ends equally, $\frac{1}{2}(J(e^{t'_e}) - J(e^{t_e}))$ each, whichever end wrote it; the charge is positive when the disagreement grows and negative, a benefit, when it shrinks. The unit claim on the unit books charges creditor and debtor $\frac{1}{2}J(2) = \frac{1}{8}$ each.*

Opening the unit claim raises both parties' costs equally. Knowing which party wrote the claim does not alter this reading.

Proposition 4.6 (One reading for each condition). *Each reading below satisfies every condition among (0) to (vi) except the one named, with $F = J$ unless stated. On the unit claim on the unit books (relationship $0 \rightarrow 1$ from reading 0 to $\log 2$, change of cost $\frac{1}{4}$) the forced reading charges creditor and debtor $\frac{1}{8}$ each.*

<i>Fails only</i>	<i>Reading</i>	<i>Unit claim: creditor, debtor</i>
(0)	author pays: weight 1 if i alone wrote, 0 if the other end alone wrote, $\frac{1}{2}$ otherwise	$\frac{1}{4}, 0$
(i)	network change spread equally over all $ P $ parties, ends or not	$\frac{1}{16}$ each, and $\frac{1}{16}$ to party 2
(ii)	weight 0.7 to the lesser-numbered end, 0.3 to the other	$\frac{7}{40}, \frac{3}{40}$
(iii) composition	quadratic $F(x) = \frac{1}{2}(\log x)^2$, weights $(\frac{1}{2}, \frac{1}{2})$; fails at $x = y = e$: $2 \neq \frac{3}{2}$	$\frac{1}{4}(\log 2)^2$ each
(iii) constancy	weights $\frac{1}{2} \pm \frac{1}{4} \tanh(t'_e - t_e)$, the end whose side grew pays more; $\tanh \log 2 = \frac{3}{5}$	$\frac{13}{80}, \frac{7}{80}$
(iii) monotone	$F(x) = \cos(\log x) - 1$, weights $(\frac{1}{2}, \frac{1}{2})$: opening a gap is a benefit	negative each
(iv)	$F_0 \equiv 0$: charges nobody for anything	0, 0
(v)	weights (1, 0), or any $(\lambda, 1 - \lambda)$ with $\lambda \neq \frac{1}{2}$: after reversing $0 \rightarrow 1$, party 0 is charged 0 on the same move	$\frac{1}{4}, 0$
(vi)	weights (1, 1): the parties together are charged twice the network change	$\frac{1}{4}, \frac{1}{4}$
none	F_2 with weights $(\frac{1}{2}, \frac{1}{2})$: the base is not among the conditions	$\frac{9}{16}$ each

Proof. Each row is a direct check of the six other conditions and one evaluation. Author pays is unchanged by reversing orientation (authorship is not orientation) and its two weights sum to 1, so it keeps (v) and (vi); it reads the author line, refusing (0). Equal spread reads the books alone, depends on the chart only through $|P|$, and sums to the network change; it charges party 2 for a move on a relationship it is not an end of. The ordered rule's weights sum to 1 and do not move under reversal; the order on parties is the extra structure (ii) forbids. The quadratic is non-decreasing in $|\log x|$ and fails the composition law at $x = y = e$: $\frac{1}{2}(2^2) + 0 = 2 \neq 2 \cdot \frac{1}{4} + 1 + 1$. The tanh weights sum to 1 and, under reversal, $t'_e - t_e$ changes sign while the ends exchange, so each party keeps its weight. With $G = \cos$ the composition law is the cosine addition formula and $F \leq 0$. F_0 obeys the law ($0 + 0 = 0$), is orientation symmetric, and sums to the network change, which is 0. $(\lambda, 1 - \lambda)$ charges the source $\lambda \cdot \frac{1}{4}$ and after reversal $(1 - \lambda) \cdot \frac{1}{4}$. (1, 1) sums to twice the change. $F_2(2) = \frac{1}{2}(4 + \frac{1}{4}) - 1 = \frac{9}{8}$. $\square$

Reading. Among readings of the books alone, a relationship's cost is charged once and the two ends bear it equally, whoever wrote the change. So the object separates two questions, who wrote and what it costs, and a reading that answers the second from the books alone must answer it symmetrically. The books hold two assertions and no record of which is right. Under (0) to (iv), an unequal split can depend on which end is the creditor. Condition (v) excludes that preference. A harm with a direction needs either the author line, which (0) refuses and author-pays uses, or a record of what is owed, which the books do not have. History can identify departure. What is owed requires information beyond that history.

4.3 Consent is not in the books

Theorem 4.7 (Where consent is, and is not). *Consent is a function of no gap profile: on chart B the unit claim by 0 and the debtor's withdrawal $v = (0, 1, -1)$ by 1 give the same gap on every relationship, $+1$ on $0 \rightarrow 1$ and 0 elsewhere, and 0 consented to the first and not to the second. For every move that changes the books, consent is a function of the books before and after: i consented iff row i changed (Proposition 3.5). For a move that changes nothing, it is not: party 0's write of the zero vector and party 1's write of the zero vector leave every books unchanged, and 0 consented to the first only.*

The two rigidity theorems fix the reading of the books alone; this theorem fixes what the books alone cannot read. A verdict computed from gaps, signed or absolute, cannot see who consented; a verdict that sees which row moved can, whenever anything moved.

5 What a wrong costs, how it grows, what repair does

Section 4 fixed what a disagreement costs and how the cost divides. This section follows the cost through time: it accrues at every tick a gap stands, it is removed by one of two exact acts, either end's repair costs no third party anything, and when two records meet halfway the cheapest meeting place is the geometric mean. The section ends with the one way a party that wrote nothing can be charged: the network is settled around it.

5.1 Standing, repair, meeting

The cost of one relationship is $J(x_e/y_e)$. The network's cost is the sum over relationships. The *standing cost* of books B over N ticks is N times the network cost. A party's *charge* is the sum of half the cost of each relationship it is an end of, as in (5).

Theorem 5.1 (Standing, repair, meeting). *On positive books:*

1. *The standing cost is zero at every tick count iff the books are matched; with one open gap it is positive, linear in the ticks, and exceeds every bound.*
2. *A repair on e is an own write by one end on its designated account of e against a free account of its own. It changes γ_e by the amount written and no other gap. With the right amount it closes e . Under every reading of the form (4), a party that is not an end of e takes zero harm from it.*
3. *If the two records disagree, x against y , and $u > 0$ is a matched figure both could move to, the total concession cost $J(u/x) + J(u/y)$ is least at $u = \sqrt{xy}$ and only there, where both sides concede the same: $J(\sqrt{xy}/x) = J(\sqrt{xy}/y)$.*

Proof. (1) $J(r) \geq 0$ with equality iff $r = 1$, and $x_e/y_e = 1$ iff $\gamma_e = 0$. (2) Lemma 3.2: the write touches a designated account of e only and a free account, so only γ_e moves; every other reading is unchanged and each term of a non-end's harm is zero. (3) $J(u/x) + J(u/y) = \frac{x+y}{2xy} u + \frac{x+y}{2} \frac{1}{u} - 2$,

and $pu + q/u$ is minimized on $u > 0$ uniquely at $u = \sqrt{q/p}$; here $q/p = xy$. At $u = \sqrt{xy}$, u/x and u/y are reciprocals and J is reciprocal. $\square$

An unchanged disagreement costs again at each tick. One end can repair it without charging a bystander. If both concede to a common figure, the geometric mean gives the least total concession cost.

The cost is a function of the ratio, not of the gap: $(2, 1)$ and $(101, 100)$ both have gap 1 and cost $\frac{1}{4}$ and $\frac{1}{20200}$. The meeting point is the equilibration theorem of the ledger paper [9] restated on the two records, an equal-sacrifice point in the sense of Young [11]. It is the point that minimizes the total concession cost, and that is all it is: the theorem says where the two figures meet most cheaply, not that meeting there is fair. The sacrifice function is J applied to the ratio of a party's old figure to its new one, the same function the object forces for the disagreement, used once more for a concession.

The cost has a direction as well as a size. At reading t the cost of a relationship is $J(e^t) = \cosh t - 1$, with derivative $\sinh t$: positive above the match, negative below, zero at it, and on positive books the sign of the reading is the sign of the gap. The cost rises away from balance on both sides. This is a fact about the function, not a claim that anything pushes the figures back; what moves a figure is a write.

5.2 The settlement and the bystander

A gap between two parties can always be closed by the two of them with no charge to anyone else: one end repairs, or both concede to the meeting point. A settlement of a whole cycle does something else: it moves every reading on the cycle while holding each party's net position, the way a clearing house nets a ring of claims. The cheapest settlement of that kind can charge a party that wrote nothing. Here settlement is defined to preserve each party's *net reading*, the sum of the readings on its outgoing relationships less the sum on its incoming ones (a clearing house holds net balances; the quantity held here is a sum of log-ratios, which is a different quantity, and this is one settlement rule among the possible ones); on the triangle $0 \rightarrow 1 \rightarrow 2 \rightarrow 0$ party 0's net reading is $t_0 - t_2$, party 1's is $t_1 - t_0$, party 2's is $t_2 - t_1$. Subtracting one constant ℓ from all three readings leaves every net reading where it was; that is the *constant shift*, and on the triangle it is the only kind of settlement there is. This is an additional condition, and the bystander's 0.0600 below is the price of this rule: a repair confined to the disputed relationship charges the bystander nothing. The books themselves conserve row totals (Section 10). Figure 4 draws the instance.

Proposition 5.2 (The settlement and the bystander). *On the triangle $0 \rightarrow 1 \rightarrow 2 \rightarrow 0$ with readings $t = (1, 0, 0)$, the settlements of t are exactly the constant shifts $t - \ell$. A settlement has the same net reading, sources minus destinations, at every party. On a chart without cycles the only settlement of t is t itself. The total cost along the settlements is $T(\ell) = J(e^{1-\ell}) + 2J(e^{-\ell}) = Ae^\ell + Be^{-\ell} - 3$ with $A = \frac{2e+1}{2e}$, $B = \frac{e+2}{2}$, minimized uniquely at $e^{\ell^*} = y^* = \sqrt{e(e+2)/(2e+1)} = 1.4116$. Party 2, an end of neither record on $0 \rightarrow 1$, then carries reading $-\ell^*$ on both its relationships, and under the forced reading its harm is $J(y^*) = \frac{e^2+4e+1}{2(2e+1)y^*} - 1 = 0.0600$, against the quadratic cost's $1/18 = 0.0556$ on the same instance. Any sequence of moves carrying t to $t - \ell$ with $\ell \neq 0$ opens both of 2's relationships, which by Theorem 3.4 needs an own write by an end of each; if 2 writes nothing it is a patient at least twice. Three own writes by 0 and 1 reach the settlement with 2 in no author line.*

Proof. Net readings on the triangle are $t_0 - t_2$, $t_1 - t_0$, $t_2 - t_1$; a shift by ℓ preserves all three, and equal net readings force $g_1 - g_0 = -1$, $g_2 - g_1 = 0$. On a chart where every nonempty set of relationships has a leaf, a circulation is zero, so the settlement is unique. $J(e^{1-\ell}) = \frac{1}{2}(e e^{-\ell} + e^{-1} e^\ell) - 1$ and $2J(e^{-\ell}) = e^{-\ell} + e^\ell - 2$; $Ae^\ell + Be^{-\ell}$ is minimized where

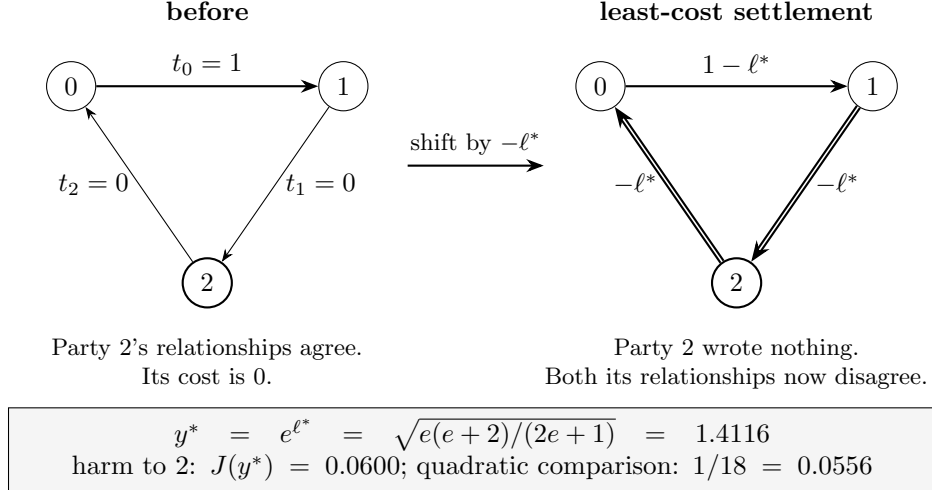


Figure 4: Settlement on the triangle. Before, only $0 \rightarrow 1$ disagrees (reading 1, ratio e). A settlement must leave each party's net reading (outgoing readings less incoming) unchanged, so it can only subtract one constant ℓ from all three readings. The ℓ^* that minimises the total cost leaves $0 \rightarrow 1$ at $1 - \ell^*$ and opens both of party 2's relationships to $-\ell^*$. Party 2 wrote nothing; its harm is $J(y^*) = 0.0600$. A repair confined to $0 \rightarrow 1$ would have left party 2's cost at 0.

$e^{2\ell} = B/A$. Party 2 is the source of $2 \rightarrow 0$ and the destination of $1 \rightarrow 2$, both at reading $-\ell^*$, and half of each of two equal terms is one term; $J(y^*) = \frac{(y^*)^2 + 1}{2y^*} - 1$ with $(y^*)^2 + 1 = \frac{e^2 + 4e + 1}{2e + 1}$. For the inequality, $J(y) = \frac{(y-1)^2}{2y} > \frac{1}{18}$ iff $9y^2 - 19y + 9 > 0$ iff, for $y > 1$, $y > \frac{19 + \sqrt{37}}{18}$; that root is 1.3935, below 1.395, whose square is below 1.95; and $(y^*)^2 > 1.95$ iff $e^2 - 1.9e - 1.95 > 0$, which holds at $e = 2.7$ (value 0.21) and for every larger e . The quadratic shifts every relationship by $\frac{1}{3}$ and charges 2 half of $2 \cdot \frac{1}{2}(\frac{1}{3})^2$. The execution, on the triangle chart with three accounts (every relationship booked on account 0 at its source and account 1 at its destination, account 2 free): from books with $(x_0, y_0) = (e, 1)$ and the other two relationships at $(1, 1)$, party 1 writes $(1/y^* - 1, 0, 1 - 1/y^*)$, party 0 writes $(0, 1 - y^*, y^* - 1)$, party 0 writes $(e/y^* - e, 0, e - e/y^*)$; each is zero-sum, touches one designated account and a free account, keeps the books positive, and together they give readings $(1 - \ell^*, -\ell^*, -\ell^*)$. $\square$

A third party cannot touch the gap between two others. What can happen to it is that its own relationships are re-read when the network is settled around it, by writes of the others at their ends; then it pays, and its consent appears nowhere. Six hundredths of a unit of cost on the unit triangle, where a repair by either end would have cost it nothing.

For a gap of reading ε in place of 1 the same computation gives the settlement shift

$$\ell(\varepsilon) = \frac{1}{2}(\varepsilon + \log(e^\varepsilon + 2) - \log(2e^\varepsilon + 1)), \quad (6)$$

since $A(\varepsilon) = (2e^\varepsilon + 1)/(2e^\varepsilon)$ and $B(\varepsilon) = (e^\varepsilon + 2)/2$; expanding the logarithms, $\ell(\varepsilon) = \varepsilon/3 + \varepsilon^3/81 + O(\varepsilon^5)$. The quadratic is the small-gap limit of J ; its shift $\frac{1}{3}$ is the linear coefficient, and J , being more convex than its quadratic limit, moves the settlement further and charges the bystander more. The quadratic is what a reader who kept only the small-gap limit of the cost would compute; the whole curve charges the bystander more, and the difference between 0.0600 and 0.0556 is the first place in the paper where the shape of J beyond its second derivative reaches a number.

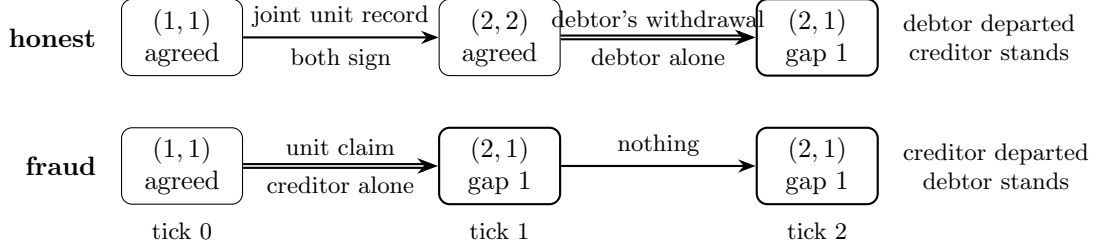


Figure 5: Two histories on relationship $0 \rightarrow 1$, shown as (x_0, y_0) . From tick 2 the books are identical. The double arrow is the write that widened the gap; its author is the party that departed. The books at tick 2 cannot tell the two stories apart; the history can.

6 Who departed

Two stories on chart B, from the unit books. In the honest story the creditor delivers and both sign: the joint unit record, $(x_0, y_0) = (2, 2)$; then the debtor withdraws its acknowledged liability alone: $(2, 1)$. In the fraud story the creditor writes the unit claim from nothing: $(2, 1)$. Same books, same gap, same charge $\frac{1}{8}$ to each end under the forced reading. Every reading of the present books alone gives the same verdict. The earlier writes distinguish the stories. In the honest story the debtor was the last hand to widen the gap; in the fraud story it was the creditor. Figure 5 draws both.

Standing on the record is a definition, and it is stated here once: the end that did not drive the two records apart, while they still differ. It is not a claim about who is in the right; Section 7 proves that no reading of the two records is.

Definition 6.1 (Agreed, departed, standing). Along a history from books B_0 , the two records agree on e at tick u when $\gamma_e(B_u) = 0$. Party i has departed on e by tick t when some tick $s < t$ meets two conditions. First, i wrote the move ($i \in \text{Wrote}(m_s)$) and drove the records further apart: $|\gamma_e(B_s)| < |\gamma_e(B_{s+1})|$. Second, the records have not agreed since: $\gamma_e(B_u) \neq 0$ for all $s < u \leq t$. Party i stands on the record on e at t when it is an end of e , $\gamma_e(B_t) \neq 0$, and it has not departed on e by t .

Narrowing the gap is not departing: a creditor that concedes half of an open claim has not departed, and the debtor whose withdrawal opened the gap still has, through the concession.

Theorem 6.2 (Departure). *Along any history on any chart:*

1. Only an end of e can have departed on e .
2. If the records agreed on e at some tick $s \leq t$ and $\gamma_e(B_t) \neq 0$, some party has departed on e by t .
3. Suppose the records agreed at s , the gap is nonzero at every u with $s < u \leq t$, and the gap does not change between $s + 1$ and t . Then a party has departed on e by t iff it wrote m_s . On a gap opened from agreement by one write and held, the departed party is the author of the opening write.
4. Along the history that writes nothing, nobody departs.

Proof. (1) A write that widened γ_e changed it, and by Theorem 3.4 its author is an end. (2) Let $s_0 \leq t$ be the last tick at or before t at which the gap is zero; $s_0 < t$ since the gap at t is nonzero; at $s_0 + 1$ the gap is nonzero, so $|\gamma|$ strictly grew at s_0 , and by Theorem 3.4 the move m_{s_0} has an author i , an end; the gap is nonzero at every u with $s_0 < u \leq t$ by choice of s_0 ; so i departed. (3) If i departed at some s' then $s' < s$ is impossible (the records agreed at s , between s' and t), $s' > s$ is impossible (the gap does not change there, so it did not widen), so

$s' = s$ and i wrote m_s ; conversely the author of m_s widened the gap from 0 and it has been open since. (4) The zero write has no author who changed anything. $\square$

An open gap that followed agreement has at least one departing author. If one write opened it and the gap then stayed fixed, that writer is the only party that departed.

Theorem 6.3 (The honest and the fraud story). *On chart B from the unit books let the honest history be the joint unit record at tick 0, the debtor's withdrawal at tick 1, and nothing after. Let the fraud history be the unit claim by 0 at tick 0 and nothing after. At every tick $t \geq 2$ both histories have the same books, row 0 equal to $(2, -1, -1)$ and every other row $(1, -1, 0)$, so $(x_0, y_0) = (2, 1)$. In the honest history the debtor has departed and the creditor stands on the record; in the fraud history the creditor has departed and the debtor stands. Consequently no reading of the present books is the standing predicate. Nor is any reading of the last move together with the books before and after it. The two histories have the same books at tick 2. At tick 3 they have the same last move (the zero write m_2) with the same books before and after it. They give opposite verdicts on the creditor at both ticks.*

Proof. The joint unit record takes both rows to $(2, -1, -1)$ and $(1, -2, 1)$; the withdrawal takes row 1 back to $(1, -1, 0)$; the unit claim takes row 0 to $(2, -1, -1)$ directly. In the honest history the gap is 0 at ticks 0, 1 and 1 from tick 2; the withdrawal at tick 1 widened it and the debtor wrote it, so the debtor departed by Theorem 6.2(3); the creditor did not write m_1 , so it stands. In the fraud history the gap is 0 at tick 0 and 1 after; the creditor wrote m_0 . For the last sentence, if a predicate P of the books gave the standing predicate at every history and tick, then $P(B_2, 0 \rightarrow 1, 0)$ would be true along the honest history and false along the fraud history with the same B_2 ; if a predicate of (m_t, B_t, B_{t+1}) gave it at every tick $t + 1$, then at $t = 2$ both histories present the zero write between two copies of the same books, and their verdicts on the creditor at tick 3 differ. $\square$

The same assertions can be reached by opposite departures. Keeping the earlier writes distinguishes the two stories even after both parties stop writing.

Proposition 6.4 (Departed pays). *The departed share of an end i of e at t is determined as follows. It is 1 if i has departed on e and the other end has not, 0 if the other end has and i has not, and $\frac{1}{2}$ otherwise. The departed charge to i is the sum over its relationships of its share times the relationship's cost $J(x_e/y_e)$. The two ends' shares sum to one, so the departed charges over all parties sum to the network's cost: single charging holds. At tick 2 of the two stories, on the same books, the departed charge $J(2) = \frac{1}{4}$ goes to the debtor in the honest story and to the creditor in the fraud story. The forced reading charges $\frac{1}{8}$ to each in both. The departed charge is not a function of the books: it is condition (0) refused, and (vi) kept.*

Proof. The shares are $(1, 0)$, $(0, 1)$ or $(\frac{1}{2}, \frac{1}{2})$. The rest is Theorem 6.3 and $J(2) = \frac{1}{4}$. $\square$

Reading. Who departed is on the history and nowhere shorter. It extends the author line to histories where the author of the opening write is not the last hand on the books, and it gives harm a direction without a preference for creditors or debtors as such. What it does not give is who is owed: the debtor who withdrew may have withdrawn because the unit never came, and the creditor who wrote alone may have delivered and been refused a signature. The two-record history contains no delivery event.

7 What the two records cannot say

What is owed is defined here by a stated exchange rule. The rule is stated, not derived: the theorems and their limit are about that rule, and Section 8 says what any rule must be given.

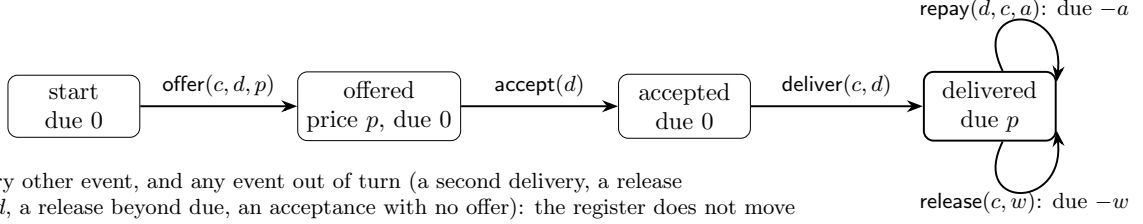


Figure 6: The exchange rule as a machine on its register. Only the delivery makes anything due, and only repayment by the debtor and release by the creditor lower it. Nothing in the machine reads the books.

It is the simplest exchange there is: one party undertakes to deliver one unit against a price, the other accepts, the unit is delivered or not, and what is due is then repaid or released.

The creditor offers the debtor one unit for 1; the debtor accepts; then either the unit crosses or it does not. Meanwhile the creditor writes the unit claim into its book and the debtor writes nothing. Whether the unit crossed, the books look the same.

Definition 7.1 (The exchange rule). Fix a creditor c and a debtor d . An *event* is nothing, $\text{offer}(i, j, p)$, $\text{accept}(j)$, $\text{deliver}(i, j)$, $\text{repay}(j, i, a)$, or $\text{release}(i, w)$. An offer is i 's undertaking to deliver one unit to j against price p . Delivery is the unit's crossing from i to j . Repayment transfers a from j back to i . Release is i 's release of w of what is due. The *register* is a state (price, offered, accepted, delivered, due), initially $(0, \text{no}, \text{no}, \text{no}, 0)$. The rule reads one event and moves the register:

- $\text{offer}(c, d, p)$ when nothing is yet accepted: $\text{price} := p$, offered;
- $\text{accept}(d)$ when offered and not yet accepted: accepted;
- $\text{deliver}(c, d)$ when accepted and not yet delivered: delivered, $\text{due} := \text{price}$;
- $\text{repay}(d, c, a)$ when delivered and $0 \leq a \leq \text{due}$: $\text{due} := \text{due} - a$;
- $\text{release}(c, w)$ when $0 \leq w \leq \text{due}$: $\text{due} := \text{due} - w$;
- any other event, or any event the register does not permit in its present state (a second delivery, a release beyond what is due, a release by anyone but c , an acceptance with no offer), leaves the register unchanged.

The *amount due* at tick t is the due field after the events before t . No books appear anywhere in the rule. A *run* is a history of the books together with a stream of events, one per tick.

Delivery is an event, not a move: the unit can cross without either party writing. In Section 8 its positions at consecutive ticks supply the delivery event. Figure 6 draws the rule.

Definition 7.2 (Faithful). On chart B from the unit books, the creditor's *claim* at tick t is $x_0(B_t) - 1$, its figure above the unit baseline, and the debtor's *acknowledgement* is $y_0(B_t) - 1$. The creditor's record is *faithful* at t when its claim equals the amount due at t , and the debtor's when its acknowledgement does. A party is *right* at t when its record is faithful. The unit baseline is a matched position both parties hold before the exchange begins; it keeps every ratio defined, and the exchange rule prices only what stands above it.

Theorem 7.3 (Identical histories, different amounts due). *Let the terms be $\text{offer}(0, 1, 1)$ at tick 0 and $\text{accept}(1)$ at tick 1. Consider two runs on chart B from the unit books, both with the fraud history (the unit claim by 0 at tick 0, the debtor never writes). The delivered run has the terms and $\text{deliver}(0, 1)$ at tick 2. The invoice run has the terms and nothing at tick 2. Their histories are identical. At tick 3 the amount due is 1 in the delivered run and 0 in the invoice run. So*

		tick 0	tick 1	tick 2	tick 3
one history, both runs	books (x_0, y_0)	(1, 1)	(2, 1)	(2, 1)	(2, 1)
	writes	unit claim by 0	nothing	nothing	nothing
events, one per run	delivered run	offer(0, 1, 1)	accept(1)	deliver(0, 1)	due 1
	invoice run	offer(0, 1, 1)	accept(1)	nothing	due 0

Figure 7: Two runs with the same two-record history. Only the event at tick 2 differs, and it is not a write. At tick 3 the creditor’s claim is 1 in both; it is what is due in one run and not in the other.

there is no function f of the two-record history and the tick with $\text{due}(r, t) = f(\text{history}(r), t)$ for every run r and tick t . There is also no predicate P of the history and the tick with “the creditor is right at t ” $\iff P(\text{history}(r), t)$. In the delivered run the creditor is right ($1 = 1$) and the silent debtor is not ($0 \neq 1$). In the invoice run the debtor is right and the creditor is not. In the run with nothing written and nothing done, nothing is due and both records are faithful at every tick.

Proof. The register after the terms is $(1, \text{yes}, \text{yes}, \text{no}, 0)$. The delivery sets due to the price, 1; nothing at tick 2 leaves due 0. The creditor’s claim along the fraud history is $2 - 1 = 1$ from tick 1 on and the debtor’s acknowledgement is $1 - 1 = 0$. If f existed, $1 = f(h, 3) = 0$. If P existed, $P(h, 3)$ would be true by the delivered run and false by the invoice run. The zero run keeps the unit books and the register at its initial state. $\square$

Reading. Figure 7 shows the two runs. The two records, with their whole history, measure what each party asserted and when. Whether the unit crossed is not written on either row, and no length of history supplies it: the delivered run and the invoice run agree on every entry at every tick. The creditor who delivered without a signature has exactly the history of the creditor who invented the claim. The amount due therefore requires information beyond the two-record history. Under the stated rule, that information is the event stream.

The content of the theorem is the pair of runs, not the clause about functions. That a machine which never reads the books cannot be computed from the books is immediate. What is not immediate is that the two runs it separates are the honest creditor and the fraud: identical books, identical authors, identical departures, at every tick. Anyone who holds that those two are owed different amounts holds that the two records, at any length, are the wrong object for what is owed. The rule is the way of making that difference exact.

8 The third record

The two records cannot say whether the unit crossed. Something else must. This section says what that something is at the smallest scale Recognition Science has: a unit has a position on the lattice of three dimensions at every tick, a place is a block of positions, and a crossing is the unit’s being in one block at one tick and another at the next. That is the second of the two places the physics enters this paper, and the only one that uses space. The crossing is not the whole of the third record: the same crossing is a sale or a gift, and the section ends with the rule that reads the events, proved to fix exactly one amount due.

Delivery is taken at the smallest scale: the unit is somewhere at every tick, and delivery is its moving from the creditor’s place to the debtor’s. A place is a cube of voxels, so that where the unit is inside a place is a finer fact than which place it is in. A delivery event can then

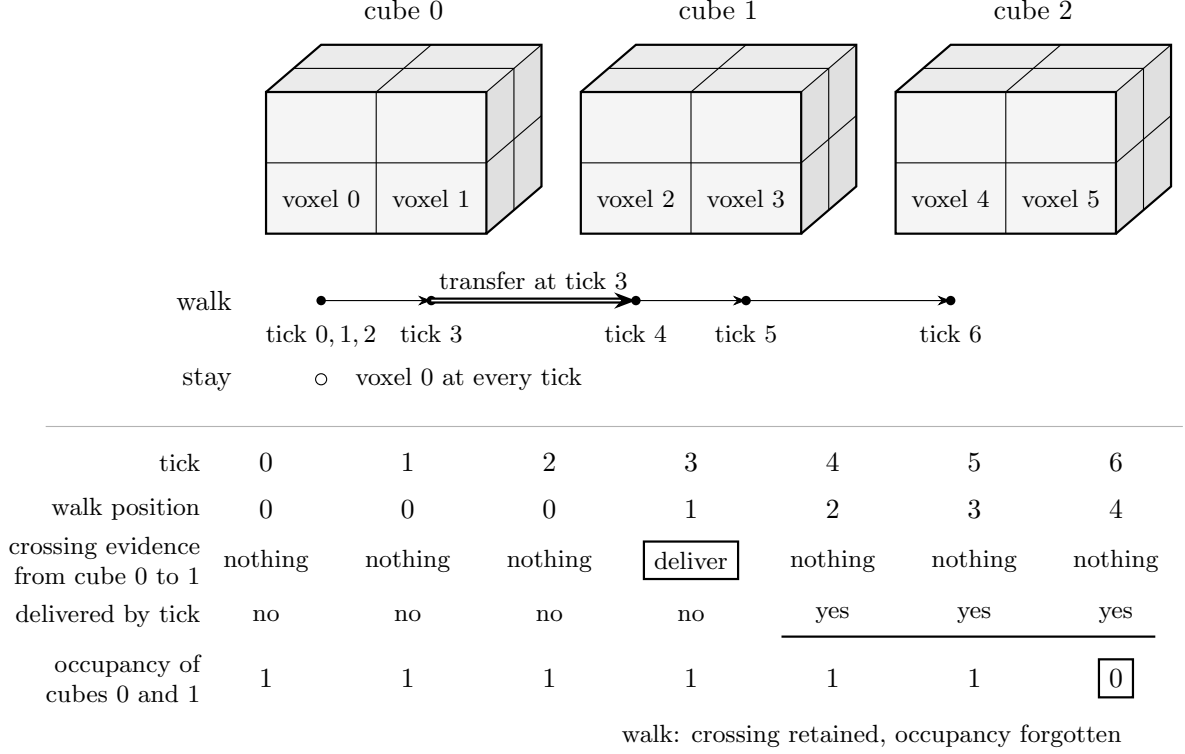


Figure 8: One unit on a line through three cubes, each a $2 \times 2 \times 2$ block of voxels. Walk crosses from cube 0 to cube 1 at tick 3 and is delivered from tick 4; stay never crosses. The crossing record fires once and the delivered record stays true. A count of units in cubes 0 and 1 reads 1 until walk has left both, then 0: at tick 6 it no longer shows that a crossing ever happened.

be read from the unit's position before and after a tick: crossing between two specified cubes supplies the event. The lattice and the cubes are the smallest setting in which three things are exact at once: the unit has a position at every tick, a place is a set of positions coarser than one position, and a crossing is a comparison of two consecutive ticks. Any notion of place coarser than position gives the same theorems; the cubes are there so that the figure can be drawn.

8.1 The crossing

Definition 8.1 (Crossing, delivered). A *unit path* is a position $u(t) \in \mathbb{Z}^3$ at each tick, on the lattice of voxels, with $u(t+1)$ equal to $u(t)$ or to a neighbour of it (one coordinate changed by one). A *cube* is a block of $2 \times 2 \times 2$ voxels; its address is the position with each coordinate halved and rounded down, so a unit at position (x, y, z) is in the cube $(\lfloor x/2 \rfloor, \lfloor y/2 \rfloor, \lfloor z/2 \rfloor)$. A *cross posting* from cube α to cube β at tick t is: the unit is in α at t and in β at $t+1$; a *transfer* is a cross posting between distinct cubes, $\alpha \neq \beta$. The *crossing evidence* of a path for creditor c and debtor d across (α, β) is the event stream that reads `deliver`(c, d) at exactly the ticks at which the path transfers from α to β and nothing otherwise. The unit is *delivered* across (α, β) by tick t when it transferred at some $s < t$.

Take one unit on a line of voxels. Two paths: *walk* rests at the origin for two ticks and then moves one voxel per tick; *stay* never moves. Figure 8 draws them and the two records one could keep of them.

Theorem 8.2 (The crossing is recorded when it happens and retained). A path never transfers from a cube to itself. Once delivered by t , the unit is delivered by every $t' \geq t$; nothing is

delivered by tick 0. Let the path walk stay at the origin for ticks 0, 1 and 2 and then move one voxel per tick along the first axis: positions 0, 0, 0, 1, 2, 3, 4, $\dots$. It is in cube 0 (voxels 0 and 1) at ticks 0 to 3. It is in cube 1 (voxels 2 and 3) at ticks 4 and 5, and in cube 2 (voxels 4 and 5) at ticks 6 and 7. Let stay rest at the origin forever. Walk is delivered from cube 0 to cube 1 by tick 4 and not by tick 3; stay never is. At tick 6 walk has been delivered, and the occupancy count of cubes 0 and 1 (the number of units present in each) is zero in both. An occupancy record reads the present position and forgets the crossing. The crossing record keeps it.

Proof. A transfer requires $\alpha \neq \beta$. Delivered-by is an existential over $s < t$, monotone in t and empty at $t = 0$. Walk is at voxel 1, cube 0, at tick 3 and at voxel 2, cube 1, at tick 4: a transfer at $s = 3 < 4$ and at no earlier tick; stay is always at cube 0. At tick 6 walk is at voxel 4, cube 2. $\square$

At tick 3 the unit leaves one cube for the other. At tick 4 the delivered record becomes true. Later movement can empty both cubes without undoing the recorded delivery.

Theorem 8.3 (The gate on physically grounded runs). *Let the grounded run of a path be the fraud history together with the following events. At a tick where the path transfers across (cube 0, cube 1), use the crossing evidence's $\text{deliver}(0, 1)$. At every other tick, use the terms of Theorem 7.3 (the offer at tick 0, the acceptance at tick 1, nothing after). The grounded runs of walk and stay have identical histories. At tick 4 the amount due is 1 for walk and 0 for stay. Walk is delivered by tick 4 and stay is not. The creditor is right for walk and not for stay.*

Proof. Walk transfers at tick 3 and at no earlier tick, so its events are the offer, the acceptance, nothing, and the delivery; stay never transfers, so its events are the terms alone. Theorem 7.3 with the delivery event supplied by Theorem 8.2 in place of a stipulated one. $\square$

The path supplies the event that the books lack. Walk and stay leave identical assertions, but only walk performs the accepted delivery.

Proposition 8.4 (The same crossing is a sale or a gift). *Let the gift run be the delivered run of Theorem 7.3 with price 0 in the offer. Its delivery events coincide tick for tick with the delivered run's, and its history is the same; at tick 3 the amount due is 1 in the delivered run and 0 in the gift run.*

Proof. The delivery sets due to the price. $\square$

Reading. The physical half of the third record is the unit's own path. A transfer compares where the unit was with where it is. The crossing event is emitted when the transfer happens; the delivered record retains that fact after the unit moves on. In the vocabulary of distributed ledgers it is an oracle, a feed the rule reads that no party writes; what this section fixes is what the feed must carry (the crossing, not the count) and what it is a record of (the unit's path, not anyone's assertion). It is not a book. But counting crossings is not entitlement. The same crossing performs a sale or a gift; terms are separate data, and so, as the next section shows, is the authority to release. The third record is a record of the rule's events, of which the crossing is the physical part.

8.2 What is due

The creditor offers a unit for 1, the debtor accepts, the unit crosses, the debtor repays a half, the creditor forgives a quarter. What is due is $\frac{1}{4}$. A clerk who never saw the rule, and only kept a running tally of what was paid and what was released once the unit had crossed, reads $1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$. The theorem is that these two never disagree, on any stream of events and at any cutoff. Figure 9 draws the clerk's three states.

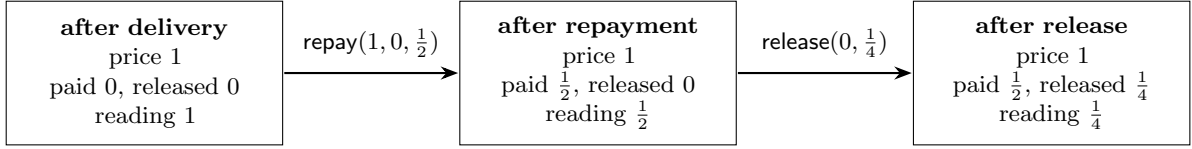


Figure 9: The clerk's tally on the unit exchange. It stores what was paid and what was released, never what is due, and reads price less paid less released once the unit has crossed: 1, then $\frac{1}{2}$, then $\frac{1}{4}$.

Definition 8.5 (The rule as a relation; the tally). The *step relation* holds between σ , an event, and σ' exactly when one clause of Definition 7.1 carries σ to σ' on that event. It has one clause per permitted transition and one per refusal. The *due specification* holds of (t, σ) when σ is reached from the initial state by t steps along the event stream. It is a relation, not a procedure: nothing in its statement says it picks out one state. The *tally* has fields (price, offered, accepted, performed, paid, released), initially zero and no. It records offers and acceptance as the rule does. It marks performed on the first permitted delivery, adds each permitted repayment to paid, and adds each permitted release to released. It checks permission against its own *reading*: price – paid – released once performed, and 0 before. The tally stores no due field.

Theorem 8.6 (What is due is determined, and the tally reads it). *For every event stream and every cutoff t there is exactly one register state satisfying the due specification at t . It is the register of Definition 7.1 after the events before t . The state at t depends only on the events before t . The tally's reading at t equals q iff the unique state satisfying the specification at t has due q . On every event stream and every cutoff, the reader that only tallies what was paid and released reads exactly what the rule says is due.*

Proof. Existence and uniqueness by induction on t : the relation at each step is the graph of the rule's transition, which is a function. Prefix dependence is the same induction. For the tally, the invariant coupling register and tally is: equal price, offered and accepted fields; delivered iff performed; due equal to the reading; paid and released both zero until performed; and performed only if accepted. It holds initially, and each of the six event cases preserves it: an offer or acceptance moves the same fields under the same guard; a permitted delivery sets due to the price on one side and performed on the other, with paid and released still zero so the reading becomes price; a repayment or release is permitted on one side iff on the other (the guard $a \leq \text{due}$ is $a \leq \text{reading}$), and lowers due by a on one side while raising paid or released by a on the other. $\square$

The rule and the tally store the exchange differently but return the same amount. Both decide what counts as repayment or release from the events already recorded.

Theorem 8.7 (Waiver, denial, repayment). *Continue the delivered run of Theorem 7.3 (due 1 at tick 3) with one more event at tick 3.*

1. *Authorized release:* **release**(0, w) with $0 \leq w \leq 1$ leaves due $1 - w$ at tick 4, with no signature from the debtor and no change to the books.
2. *The debtor cannot release:* **release**(1, w) leaves due 1.
3. *Repayment:* **repay**(1, 0, a) with $0 \leq a \leq 1$ leaves due $1 - a$.
4. *Denial changes nothing:* if the debtor writes its withdrawal (0, 1, -1) into the books at tick 1 of the same run, the amount due at tick 4 is still 1.

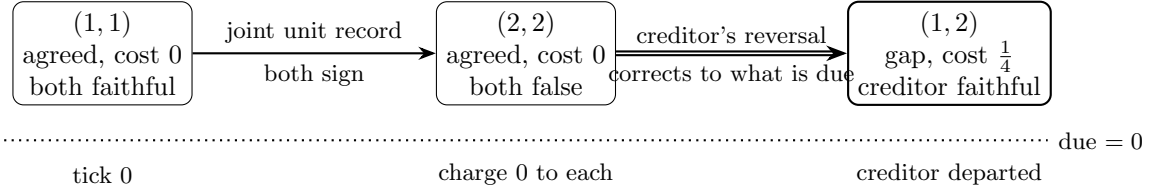


Figure 10: Nothing is due. Signing the unit together costs nothing and makes both records false. Correcting one record to the truth opens the gap, costs a quarter, and the corrector is the party that departed.

5. *The waiving creditor still stands on the record: in the honest history of Theorem 6.3 with $\text{release}(0, w)$ at tick 3, the creditor stands on relationship $0 \rightarrow 1$ at tick 3 and due is $1 - w$ at tick 4. A release is an event; the books do not move.*

Proof. Each is one application of the rule to the register $(1, \text{yes}, \text{yes}, \text{yes}, 1)$: the release clause requires the releasing party to be $c = 0$; the repayment clause subtracts a ; the rule reads no books, so the denial is invisible to it; standing is Theorem 6.3 at tick 3. $\square$

Reading. The rule was stated once, as a relation. The tally was built without a due field. They agree everywhere. That is what it means to say the amount due is determined by the evidence of undertakings, performance, repayment and authorized release, and by nothing that the two parties assert. The debtor's denial is an assertion, and the rule does not read assertions; the creditor's forgiveness is an authorized event, and it changes what is due without either party writing a figure.

9 Agreement is not truth

Both parties can agree on a claim that neither is entitled to record as due. Nothing is due: no offer was made, or the unit never crossed. Both parties nonetheless sign the joint unit record: $(x_0, y_0) = (2, 2)$, agreed, cost 0, charge 0 to each. The creditor then looks at what is due and corrects its own figure to it. Figure 10 draws the three states.

Theorem 9.1 (Correcting a false consensus opens a gap, costs, and departs). *Take chart B from the unit books, with no events (due 0 at every tick). Let the history be the joint unit record at tick 0 and the creditor's reversal $(-1, 0, 1)$ at tick 1. At tick 1 the records agree on $0 \rightarrow 1$ at $(2, 2)$, and neither is faithful: both claim 1 and nothing is due. At tick 2 the books are $(1, 2)$: the creditor's record is faithful, the relationship's cost has gone from 0 to $J(\frac{1}{2}) = \frac{1}{4}$, and for every $c > 0$ the same holds with F_c : 0 to $F_c(\frac{1}{2}) > 0$. The creditor has departed on $0 \rightarrow 1$ by tick 2.*

Proof. The joint record takes rows 0, 1 to $(2, -1, -1)$ and $(1, -2, 1)$; the reversal takes row 0 to $(1, -1, 0)$, so $x_0 = 1$, $y_0 = 2$. Due is 0 throughout, so faithful means figure 1 above baseline equals 0 (false at tick 1 for both) or $0 = 0$ (true at tick 2 for the creditor). $J(1/2) = \frac{1}{2}(\frac{1}{2} + 2) - 1 = \frac{1}{4}$; $F_c(\frac{1}{2}) = \frac{1}{2}(2^c + 2^{-c}) - 1 > 0$ for $c > 0$. The reversal widened $|\gamma_0|$ from 0 to 1, the creditor wrote it, and the gap is -1 , nonzero, at tick 2. $\square$

The correction restores the creditor's record to what is due. It also creates disagreement with the debtor's unchanged record. The forced reading charges each end $\frac{1}{8}$; the history identifies the creditor as the one that departed.

Posting means changing one's own row. Posting is *priced* when it occurs only while a chosen account, less the disagreement charges accumulated so far, is non-negative. If those charges are always zero, a positive bounded account permits continued posting.

Theorem 9.2 (Priced agreement can remain false forever). *On chart B from the unit books, with no events, let the history start with the joint unit record at tick 0. At every odd tick let it make a joint record of $+\frac{1}{2}$ on $0 \rightarrow 1$, and at every even tick ≥ 2 a joint record of $-\frac{1}{2}$. Then for every $t \geq 1$ the records agree on $0 \rightarrow 1$, at $(2, 2)$ or $(\frac{5}{2}, \frac{5}{2})$, and neither is faithful (they claim 1 or $\frac{3}{2}$; nothing is due). Party 0 posts at every tick and is charged nothing at any tick. Its posting is therefore priced and its source bounded in the sense of Section 10, with its claim account as reserve account (balance 2 or $\frac{5}{2}$, charges accrued 0).*

Proof. A joint record changes no gap (Lemma 3.3), so agreement is kept; the figures alternate between 2 and $\frac{5}{2}$, both above the baseline; every relationship of party 0 is matched at a nonzero figure at every tick, so its charge is 0, and the reserve on its claim account is that account's balance, which alternates between 2 and $\frac{5}{2}$. $\square$

Agreement is charged nothing, whatever it agrees to. The cost of Section 4 reads disagreement. In this history it remains zero, and the claim account stays positive and bounded.

Theorem 9.3 (Each part of the evidence is necessary; renaming commutes). *What is due is a function of the event stream: two runs with the same events have the same amount due at every tick. Each part is necessary. Erasing every delivery event makes the delivered run and the invoice run identical in events and history, with due 1 $\neq$ 0. Erasing the price of every offer makes the delivered run and the gift run identical, with due 1 $\neq$ 0. Erasing who released makes the authorized full waiver and the debtor's attempted full release identical, with due 0 $\neq$ 1 at tick 4. And for every permutation π of the parties, the amount due from $\pi(d)$ to $\pi(c)$ along the renamed events equals the amount due from d to c along the original.*

Proof. The rule reads events only. The three erasures are the runs of Theorem 7.3, Proposition 8.4 and Theorem 8.7. The rule compares party names only by equality, so renaming commutes with every clause. $\square$

Reading. The two records measure agreement, not truth. A law that penalizes disagreement, and every reading of the books alone in Section 4 is such a law, penalizes the party that corrects a false consensus, and Section 6's history reading names that party as the one who departed. These readings charge disagreement. The event record supplies the separate test of whether either assertion is true. The party who agrees can go on agreeing, affordably and forever, while nothing it agrees to is owed. What is owed is fixed by the events, each of which (performance, terms, authority) is needed and none of which is a figure either party writes.

10 Who pays, and what a duty costs

Everything above says what the words mean, what a disagreement costs, who departed, and what is owed. None of it says that anyone ought to close a gap. This section says for whom a duty holds and what it rests on, in two layers that share one cost function and one reading, and are otherwise two models. The first layer is on the books: a party that goes on acting from a finite reserve, paying what its disagreements cost, is under one hypothesis, a ceiling on its lifetime charge, and under that hypothesis every ratio on its relationships tends to one. The second layer is on the agent: a bounded agent that holds an imbalance whose upkeep exceeds its budget exhausts its reserve in a computable number of windows, so an agent that continues does not sustain extraction; and an agent committed to its own continuation is, by the instrumental principle, committed to not sustaining extraction. The premise of every duty in this section is the agent's own continuation, stated of the agent in the third person. It is proved irreducible on its signature, and it is nobody's preference.

10.1 The one hypothesis

Definition 10.1 (Posting, ceiling). A party i *posts* at tick t when its own row changes, $B_{t+1}(i, \cdot) \neq B_t(i, \cdot)$. Its *charge* on books B is $\frac{1}{2}J(x_e/y_e)$ summed over the relationships it is an end of. It has a *lifetime ceiling* Λ along a history when $\sum_{s < t} \text{charge}(B_s) \leq \Lambda$ for every t .

Theorem 10.2 (Under a ceiling every ratio tends to one). *Along a history of positive books, suppose i has a lifetime ceiling Λ . Then $\text{charge}(B_s) \rightarrow 0$ and, for every relationship e that i is an end of, $x_e(B_s)/y_e(B_s) \rightarrow 1$. The disagreement vanishes relative to the size of the claim, whichever end opened it. From matched books, the history that writes nothing has ceiling 0. The fraud history of Theorem 6.3 has no ceiling: its charge to the creditor is $\frac{1}{8}$ at every tick from 1.*

Proof. A series of non-negative terms with bounded partial sums converges, so its terms tend to 0; each term is at least $\frac{1}{2}J(x_e/y_e)$ for each incident e , so $J(x_e/y_e) \rightarrow 0$; J is continuous, zero only at 1, and goes to $+\infty$ at 0 and at $+\infty$, so the ratio goes to 1. On matched positive books the charge is 0. Along the fraud history the books are $(2, 1)$ on $0 \rightarrow 1$ from tick 1, so the partial sums exceed $\frac{1}{8}(t - 1)$, past every bound. $\square$

A fixed positive disagreement charge cannot fit under a finite lifetime bound. The relative disagreement must shrink. The disputed balance itself can stay fixed while both recorded figures grow.

One way to a ceiling is affordability. Fix an account ρ of i as its reserve account (in the instances below, a free one). The *reserve* at tick t is $B_t(i, \rho) - \sum_{s < t} \text{charge}(B_s)$: what the account holds, less every charge so far. Posting is *priced* when i posts at t only if its reserve at t is non-negative. There is *no unlimited source* when $B_t(i, \rho)$ is bounded above. Before the proposition, the fraud creditor of scene F with $\frac{3}{8}$ in its reserve account, holding $(2, 1)$ open from tick 1 and posting at every tick:

tick	0	1	2	3	4	5
charge this tick	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
charges so far	0	0	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{2}{4}$	$\frac{5}{8}$
reserve	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{3}{8}$	0	$-\frac{1}{8}$
may post	yes	yes	yes	yes	yes	no

The reserve account never moved; the charges alone drove the reserve to zero at tick 4. A priced party can post at tick 4 and never after, unless its charge has fallen, and the charge falls only as the ratio on $0 \rightarrow 1$ moves toward one. So a party that keeps posting under a bounded reserve account has charges over all time under $\frac{3}{8}$, and every ratio it is an end of tends to one: that is the proposition.

Proposition 10.3 (Priced bounded posting gives the ceiling; both facts are needed; the ratio, not the gap). *Along a history of positive books, let i 's posting be priced with $B_t(i, \rho) \leq \bar{B}$ and let i post at infinitely many ticks. Then i has lifetime ceiling $\bar{B}$, and if i is an end of a relationship held open from the start of the history (both figures constant, positive, unequal) with per-tick charge κ , it does not post at any tick $t > \bar{B}/\kappa$; a relationship opened at tick t_0 shifts the bound to $t > t_0 + \bar{B}/\kappa$, which for the table above ($t_0 = 1$, $\bar{B} = \frac{3}{8}$, $\kappa = \frac{1}{8}$) is tick 4, the last tick it may post. Both facts are needed. Take chart B with four accounts (2 and 3 free, 2 the reserve account), with every relationship other than $0 \rightarrow 1$ at $(1, 1)$. From row 0 equal to $(2, -1, -1, 0)$, a party that alternates ± 1 between accounts 2 and 3 posts at every tick and holds the gap open. Its reserve account is bounded, and its reserve is negative at every tick it posts (unpriced). From row 0 equal to $(2, -1, 0, -1)$, a party that writes $+1$ to account 2 from account 3 at every tick is priced with reserve $7t/8$. It holds the gap open with unbounded reserve*

account. The conclusion is about the ratio. Start from books with row 0 equal to $(2, -1, \frac{1}{4}, -\frac{5}{4})$ and the rest matched at $(1, 1)$, with 2 the reserve account, and let 0 and 1 write the joint record of amount 1 on $0 \rightarrow 1$ with account 3 as the counter account at both ends, at every tick. Then $(x_0, y_0) = (t+2, t+1)$: the ratio tends to 1 while the gap is 1 forever. Party 0 posts at every tick, its reserve account is constant at $\frac{1}{4}$, its charge at tick t is $\frac{1}{4(t+1)(t+2)}$ and its reserve is $\frac{1}{4(t+1)} > 0$: priced, bounded, and the gap never closes. If y_e stays bounded the gap does tend to 0.

Proof. Posting at t under pricing gives $\sum_{s < t} \text{charge}(B_s) \leq B_t(i, \rho) \leq \bar{B}$ at infinitely many t , and the partial sums are non-decreasing. With a held-open relationship the charge is at least κ per tick, so the reserve is at most $\bar{B} - t\kappa < 0$ once $t > \bar{B}/\kappa$. The two countermodels are direct. For the joint history, a joint record changes no gap and moves both figures by 1; $J(\frac{t+2}{t+1}) = \frac{1}{2(t+1)(t+2)}$, whose halves telescope to $\frac{1}{4}(1 - \frac{1}{t+1})$. $\gamma_e = y_e(x_e/y_e - 1)$. $\square$

A fixed disputed balance survives a ceiling only by both parties writing the acknowledged part of the claim up without bound, so the dispute becomes an ever smaller fraction of it. That is what the ceiling forces and all it forces. And the ceiling itself is not free: Theorem 9.2 already showed a priced, bounded party agreeing falsely forever, so the ceiling drives every ratio to one and says nothing about truth.

10.2 Who pays

The charge is a tally computed from the books. Paying it requires an entry. The fraud creditor of Theorem 6.3 holds $(2, 1)$ open and is charged $\frac{1}{8}$ per tick. Suppose it has a reserve of S and must actually pay each tick's charge out of it. After $8S$ ticks the reserve is gone. The questions are what "pay" is on this object, where the charge can go, and what happens if it cannot come back.

Theorem 10.4 (Every move conserves every row total). *For any books B and party i let the row total be $\sum_a B(i, a)$. Every own write and every joint record leaves every row total unchanged. Hence along every history every row total is constant. A party's reserve account can only move against its other accounts: $B_t(i, \rho)$ equals the row total less the rest of the row at every tick.*

Proof. An own write adds a zero-sum vector to one row. A joint record adds x and $-x$ to two accounts of the source's row and $-x$ and x to two accounts of the destination's row. $\square$

So on the two-record object no history pays anything out of a row: a charge can leave the reserve account only for another account of the same party. To have a payer the chart needs a party to pay to. Three things are kept apart below. The *charge* is a number read off the books each tick. *Paying* it is a joint record that moves that amount out of the reserve. Because a row total never changes, paying does not make the payer poorer on its own books: the amount leaves the reserve and appears as the payer's claim on the environment, a claim it never draws on. What is spent is the reserve, and the claim is the record that it was spent. The *ceiling* is a bound on the sum of the charges over all time; it is what a finite reserve imposes when every charge is paid. The ceiling theorems below use only the bound on the reserve; the environment and the dissipation say what paying is on this object, not what the theorems need. A party that writes a charge off its reserve into another of its own accounts satisfies the same bound and reaches the same ceiling (Theorem 10.9(3)); it has settled with itself, and the amount has left no row.

Definition 10.5 (Environment, dissipation). A chart C' *extends* C when there are injections of parties, accounts and relationships of C into C' preserving sources, destinations, asset and

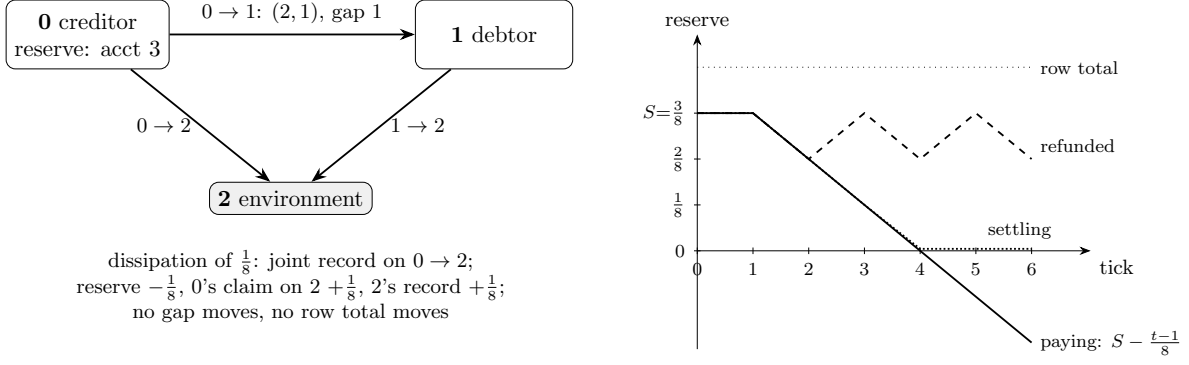


Figure 11: Left: the two-party chart extended by an environment; paying a charge is a joint record on the creditor's relationship to it. Right: the creditor's reserve along the three instances of Theorem 10.9, drawn with supply $S = \frac{3}{8}$. The row total never moves. Paying the standing charge on an open claim runs the reserve below zero; a refund every other tick keeps it up forever and removes the ceiling; settling at tick 3 leaves it at zero and the books matched.

liability accounts. Books on C embed in C' by carrying each balance to its image and setting every new account to 0. An *environment* for the extension is a party of C' outside the image of C with one relationship from every embedded party to it. The *dissipation* of c by party i is the joint record on i 's relationship to the environment, of amount c , with counter account i 's reserve ρ and a free account of the environment.

Proposition 10.6 (Embedding and dissipation). *Under an embedding, every gap of C is preserved on its image relationship. The dissipation of c by i lowers i 's reserve account by c and raises i 's claim on the environment by c . It raises the environment's record of what it has received from i by c . It changes no gap and no row total, and touches no other party's row. An own write by i of c on its asset account to the environment, by contrast, opens a gap of c on that relationship.*

Proof. The embedding carries the two designated entries of e to the two designated entries of its image. The dissipation is a joint record: Lemma 3.3, Theorem 10.4, and the four entries it moves. The own write is Lemma 3.2. $\square$

Paying is therefore a joint record: the party posts its charge from its reserve to its relationship with the environment, and the environment acknowledges it. The party's row total does not fall; its reserve does, and what it holds instead is an acknowledged claim on an environment that, by the next definition, never gives it back. Figure 11 draws the extended chart and the three ways the reserve can go.

Definition 10.7 (Finite supply). Along a history, party i with reserve account ρ has *no overdraft* when $B_t(i, \rho) \geq 0$ for every t . It *carries its charges out of supply S* when $B_t(i, \rho) \leq S - \sum_{s < t} \text{charge}(B_s)$ for every t . Each tick's charge has left the reserve and nothing has come in. Its posting is *priced on the books* when it posts at t only if $B_t(i, \rho) \geq 0$.

Theorem 10.8 (Finite supply forces the ceiling). *Along any history on any chart, suppose i has no overdraft and carries its charges out of supply S . Then i has lifetime ceiling S . On positive books every ratio of the two records on i 's relationships tends to one. The same holds if "no overdraft" is weakened to "priced on the books" and i posts at infinitely many ticks. With supply 0, no overdraft and positive books, every relationship of i is matched at every tick. And for any party carrying its charges out of some supply on positive books: either it overdrafts, or every ratio on its relationships tends to one.*

Proof. $0 \leq B_t(i, \rho) \leq S - \sum_{s < t} \text{charge}(B_s)$ gives the ceiling; Theorem 10.2 gives the ratios. Under pricing the bound holds at the posting ticks and the partial sums are non-decreasing. With $S = 0$ the partial sums of non-negative terms are ≤ 0 , so every charge is 0, so every incident $J(x_e/y_e) = 0$, so $x_e = y_e$. The last sentence is the first with the hypothesis split. $\square$

Each paid charge uses some of the finite supply. With no refill and no overdraft, the total paid cannot exceed that supply. The limit on ratios of positive records follows from this bound. It is a limit of ratios: the disputed balance itself may stand at 1 forever while both figures grow, as in the $(t + 2, t + 1)$ history of Proposition 10.3.

Theorem 10.9 (The instances). *Take the two-party chart (creditor 0, debtor 1, relationship $0 \rightarrow 1$) extended by an environment 2 with relationships $0 \rightarrow 2$ and $1 \rightarrow 2$; four accounts, which on the creditor's row are: account 0 its asset on $0 \rightarrow 1$, account 1 free, account 2 its asset on $0 \rightarrow 2$, account 3 the reserve. (The debtor's liability on $0 \rightarrow 1$ is its account 1; the environment's liabilities are its accounts 2 and 3.) A dissipation of c is the joint record on $0 \rightarrow 2$ moving $+c$ to the creditor's account 2 and $-c$ to its reserve. Start from books with every relationship matched at 1, every row total 0, and supply S on the reserve. At tick 0 the creditor writes the unit claim on $0 \rightarrow 1$, opening gap 1; its charge is $\frac{1}{8}$ per tick from tick 1.*

1. The fraud creditor paying its charge overdrafts. *If it dissipates $\frac{1}{8}$ to the environment at every tick $t \geq 1$, it carries its charges out of S . Its reserve account reads $S - (t - 1)/8$ at every tick $t \geq 1$. This is negative once $t - 1 > 8S$: it has an overdraft, and its posting is not priced on the books.*
2. Inflow removes the theorem. *Suppose instead it dissipates $\frac{1}{8}$ at odd ticks and receives $\frac{1}{8}$ back at even ticks (a dissipation of $-\frac{1}{8}$). With $S \geq \frac{1}{8}$ it never overdrafts and posts at every tick. It has no lifetime ceiling at all: it carries its charges out of no supply.*
3. Settlement. *Suppose it dissipates $\frac{1}{8}$ at ticks 1 and 2. At tick 3 let it write the own move $(-1, 1 + \frac{1}{8}, 0, -\frac{1}{8})$, reversing its claim and writing its third charge off the reserve. Then with $S \geq \frac{3}{8}$ it has no overdraft and carries its charges out of S . It has lifetime ceiling S . The ratio on $0 \rightarrow 1$ tends to 1 (it is 1 from tick 4), and it never posts again.*

Proof. Direct computation on the books; the environment's records rise by $\frac{1}{8}$ per dissipation and its row total stays 0. In (2) the charge is $\frac{1}{8}$ at every tick and the reserve account alternates between S and $S - \frac{1}{8}$. In (3) the books from tick 4 are matched on $0 \rightarrow 1$ and the reserve account reads $S - \frac{3}{8}$. $\square$

Reading. The one hypothesis under which every ratio closes is a ceiling on the lifetime cost of disagreement, and the ceiling is what a party with a finite reserve and no one to refill it is under, provided it actually pays. Paying is a joint record to an environment: the row total never falls, the reserve does, and the environment holds the acknowledgement. The fraud creditor that keeps paying overdrafts. The creditor that settles stops posting. The refunded creditor pays a charge and receives it back, so its reserve does not carry the lifetime charge out of any finite supply. None of this is yet a duty: it is what the books do to a party that goes on posting from a finite supply. Why a party goes on, and what its going on commits it to, is the next subsection.

10.3 The agent that goes on

The books price a held ratio tick by tick. The same function prices it on the agent's side, where the question becomes whether the agent survives holding it. This subsection states that side exactly. Its objects are an agent with a budget, the cost of the imbalance it holds, and the two premises under which a commitment follows. The agent is a second model, tied to the

	on the books (the two subsections above)	on the agent (this subsection)
what is held	a relationship e with records (x_e, y_e)	a reading σ_n
the link	log-ratio $t_e = \log(x_e/y_e)$	$\sigma := t_e$ (the one identification)
its cost	$J(x_e/y_e)$, half charged to each end	upkeep $C(\sigma) = J(e^\sigma) + J(e^{-\sigma})$, both ends' costs
the clock	one tick per move	one window of eight ticks
what is spent	a finite supply S , never refunded	a reserve R_n , renewed by b per window
the result	lifetime ceiling; every ratio tends to 1	exhaustion at $\lceil R_0/d \rceil$; continuing entails not extracting

Table 2: The two models of Section 10. The books have a supply and no renewal; the agent has a renewable budget and pays the whole cost of the ratio it holds from both ends. They share the cost function and the log-ratio, and nothing else is carried across.

books by one identification, its reading with the log-ratio of a relationship; its budget and its two-ended upkeep are its own assumptions, not quantities read off a chart. Table 2 sets the two models side by side before the definition.

Definition 10.10 (Bounded agent, extraction). A *bounded agent* holds a reading $\sigma_n \in \mathbb{R}$ at each eight-tick window n , has a renewable budget $b > 0$ per window and a starting reserve $R_0 \geq 0$. The *upkeep* of a reading σ is the cost of the held ratio counted from both its ends,

$$C(\sigma) = J(e^\sigma) + J(e^{-\sigma}) = 2(\cosh \sigma - 1), \quad (7)$$

zero at $\sigma = 0$ and positive elsewhere. The agent's *reserve* after n windows is $R_0 + \sum_{k < n} (b - C(\sigma_k))$; it is *viable* at n when its reserve is positive, and it *continues* when it is viable at every window. It *sustains extraction* when there is a floor σ_0 with $C(\sigma_0) > b$ and $C(\sigma_k) \geq C(\sigma_0)$ at every window: it holds, at every window, an imbalance whose upkeep exceeds what it can renew.

The reading σ is the log-ratio of Section 4 held by the agent across a window; C is J of that ratio and of its reciprocal, the two ends' costs of one held ratio. The budget and the reserve are the agent's own; the environment of the previous subsection is where the books send what the agent pays.

Theorem 10.11 (Extraction exhausts a bounded agent). *If a bounded agent sustains extraction with floor σ_0 , put $d = C(\sigma_0) - b > 0$. Then its reserve after n windows is at most $R_0 - nd$, and the agent is not viable at any window $n \geq \lceil R_0/d \rceil$. Consequently every agent that continues does not sustain extraction. The converse is not claimed: the theorem says that extraction above the floor ends the agent, not that staying below it keeps the agent going. The agent that holds $\sigma \equiv 0$ with a positive starting reserve continues and does not extract, so not extracting is a condition some continuing agent meets and every continuing agent must. (The reserve must be positive: viability at window 0 is $R_0 > 0$, and the zero-reserve agent, admitted by $R_0 \geq 0$, is not viable at window 0 whatever it holds; Theorem 10.14 uses exactly that agent.)*

Proof. Each window's net change is $b - C(\sigma_k) \leq b - C(\sigma_0) = -d$, so the reserve after n windows is at most $R_0 - nd$, which is ≤ 0 once $n \geq R_0/d$. An agent that continues is viable at every window, so it cannot sustain extraction. For $\sigma \equiv 0$ and $R_0 > 0$ the upkeep is 0 and the reserve is $R_0 + nb > 0$ at every window. $\square$

The theorem is resource accounting with a clock: an agent that spends more than it renews on holding an imbalance runs out, at a window one can compute from its reserve and the excess. Nothing pushes the agent toward balance; it is simply not there after $\lceil R_0/d \rceil$ windows if it holds the imbalance.

Definition 10.12 (Deliberator, practical reason, the premise). A *deliberator* is a bounded agent that at each window has at least two admissible next readings to choose among. A model of *practical reason* is a relation “ D is committed to G ”, for deliberators D and properties G of deliberators, closed under the *instrumental principle*: if D is committed to G and every deliberator with G has H , then D is committed to H . The *continuation premise* for a model of practical reason says that every deliberator is committed to continuing as an agent.

The instrumental principle is the ordinary requirement that a commitment to an end carries to the end’s necessary conditions. The continuation premise is the one substantive input: that a deliberator is committed to its own going on. It is stated of the deliberator, not of any reader of this paper, and it is not derived here.

Theorem 10.13 (What continuing commits an agent to). *In any model of practical reason satisfying the continuation premise, every deliberator is committed to not sustaining extraction. In the coupled setting, where an agent’s per-window cost is its own upkeep plus a fixed positive share K of a neighbour’s, $C(\sigma_k^{\text{self}}) + K C(\sigma_k^{\text{nbr}})$, a neighbour that holds a floor m_0 with $K C(m_0) > b$ exhausts the agent whatever the agent’s own readings; so under the continuation premise every coupled deliberator is committed to its neighbour not sustaining extraction.*

Proof. By the premise D is committed to continuing. By Theorem 10.11 every deliberator that continues does not sustain extraction. The instrumental principle carries the commitment. In the coupled case the reserve after n windows is at most $R_0 - n(KC(m_0) - b)$, so a continuing coupled agent has no such neighbour, and the same principle carries the commitment. $\square$

The coupled commitment is the agent’s, and its content is a state of the neighbour: the agent is committed to its neighbour not sustaining extraction. It is not a duty of the neighbour. Nothing in the model transfers one agent’s necessary condition into another agent’s commitment; what the neighbour is committed to is fixed by the neighbour’s own continuation under the same premise, and by nothing here about the agent it is coupled to.

Theorem 10.14 (The premise is irreducible on its signature). *Over models of practical reason as defined:*

- (1) *The instrumental principle alone does not give the premise: the model in which nothing is committed to anything satisfies the principle and not the premise.*
- (2) *Commitment to deliberating does not give it: there is a model, free of impossible ends, in which every deliberator is committed to having open options at every window and not to continuing.*
- (3) *Commitment to a necessary condition of continuing does not give it: there is a model in which every deliberator is committed to not sustaining extraction and not to continuing.*
- (4) *The premise has a model free of impossible ends: the model in which every deliberator is committed exactly to the properties every continuing deliberator has.*
- (5) *Exact boundary: every deliberator has some committed end that entails continuing iff the model satisfies the premise.*
- (6) *A deliberator committed to an impossible end is committed to everything.*

Proof. (1) The empty relation is closed under the principle. (2) Let D be committed to G iff every deliberator has G ; every deliberator has at least two options, and the deliberator with zero reserve has open options and does not continue, so “continues” is not committed. (3) Let D be committed to G iff every deliberator that does not sustain extraction has G ; the zero-reserve deliberator does not sustain extraction and does not continue. (4) Let D be committed to G iff every continuing deliberator has G ; the agent with $\sigma \equiv 0$ and reserve 1

continues, so the model commits to no impossible end. (5) If each D is committed to some G_D that entails continuing, the principle commits each D to continuing; the converse takes G_D to be continuing itself. (6) Everything follows from an impossible end by the principle. $\square$

The argument has a known shape. Korsgaard and Velleman derive norms from what is constitutive of agency [33, 34]; Enoch’s objection is that an agent may decline the enterprise, be a “shmagent”, and that nothing in agency itself answers it [35]; Ferrero answers that agency cannot be declined from inside [36]. Item (2) is Enoch’s objection as a theorem: being a chooser, with open options at every window, does not commit one to going on. Items (1) and (3) close the two other routes. What the theorems add to that exchange is the floor itself, proved rather than argued, and its exact boundary in item (5). The instrumental principle used is the strong transmission form: a commitment to an end carries to every necessary condition of the end. Broome and Kolodny have argued that the principle, read as a requirement of rationality, is weaker than this [37, 38]; under a weaker principle Theorem 10.13 would need restating, and the floor theorem would stand, since it says that even the strong form does not yield the premise.

Items (1) to (3) are the three ways one might have hoped to derive the premise from less, each with the model that refuses it: from rationality alone, from being a chooser, from already accepting the conclusion. Item (5) says the premise is exactly this: each deliberator has some committed end that entails going on. The quantifier is per deliberator; a model in which one deliberator has such an end and another is committed to nothing satisfies the principle and not the premise. Irreducible on its signature means items (1) to (3) and no more: the premise is not implied by the principle, by being a deliberator, or by commitment to its own consequence. It does not say the premise is the weakest that yields the conclusion; item (3) is a model committed to not extracting without it, so the conclusion can be held on other grounds, and the theorem does not rank them. Nor does the model represent what Enoch’s question is about, whether what is constitutive of agency has authority over the agent: it has commitments and no authority, so it settles what follows from the premise and not whether anyone is bound to hold it. The premise is therefore the floor of the account: what is proved is everything above it, and the paper does not claim it.

10.4 Reading

The two layers say one thing from two sides. On the books, a party that goes on posting from a finite supply, paying its charges to an environment that never refunds them, has a lifetime ceiling, and under a ceiling every ratio on its relationships tends to one; the $(t + 2, t + 1)$ history shows that this is a statement about ratios and not about the disputed balance, which can stand forever. On the agent, holding an imbalance above the budget ends the agent at a computable window, so continuing entails not extracting; and an agent committed to continuing is committed to not extracting. The price of the duty is the continuation premise, stated once, proved irreducible, and asked of no one: it holds of whichever agent is in fact committed to its own going on, and a duty binds only such an agent. What it binds it to is not sustaining an imbalance whose upkeep exceeds its renewal, and, coupled, to the state in which its neighbour does not do so; the neighbour’s own duty comes from the neighbour’s own continuation and from nowhere else in the model. The commitment is conditional twice: on the premise, and on the model of the agent, a renewable budget and an upkeep counted from both ends. An imbalance whose upkeep the budget renews is not extraction, and nothing here speaks against holding it; the false agreement of Section 9 costs nothing at all and no duty reaches it. The paper calls the commitment a duty in this sense only: a condition of continuing, for the agent committed to continue. Nothing here says that going on is good, and nothing here needs it to.

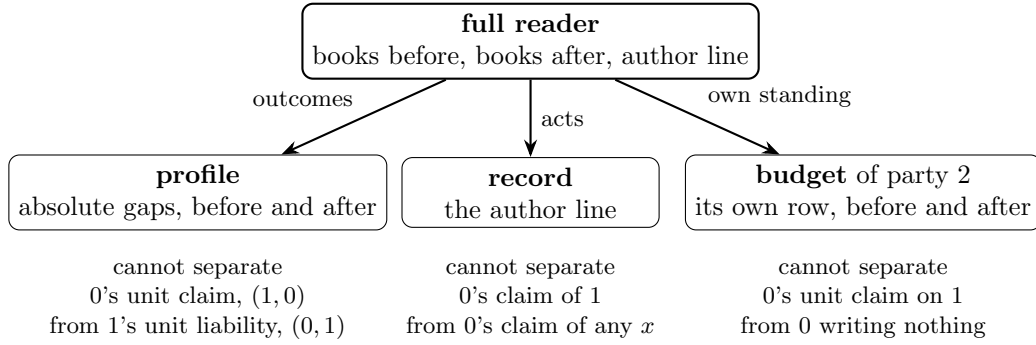


Figure 12: Three projections of the full reading and the pair each one confuses, on chart B from the zero books. Each arrow is a function; none runs backward.

11 Three projections of the full reading

A *situation* is a pair (books before, move). A *reader* is a verdict that depends on the situation through one projection. The *profile* keeps the absolute gap on every relationship before and after. The *record* keeps the author line. The *budget* of a party r keeps its own row before and after. The *full reader* is (books before, books after, author line); on every move that changes the books the third component is a function of the first two (Proposition 3.5), so the act is written into the outcome. Figure 12 draws the three projections and what each fails to separate.

Theorem 11.1 (Three blindnesses, one containment). *Take chart B from the zero books. Every profile reader gives the same verdict on the unit claim by the creditor and on the unit liability recorded alone by the debtor. The full reader separates them by author and patient. Every record reader gives the same verdict on the unit claim and on the claim of x for every x . The full reader separates them for $x \neq 1$. Every budget reader for party 2 gives the same verdict on the unit claim by 0 against 1 and on the zero write by 0. The full reader separates them by whether anyone is a patient. On every chart the profile, the record, and every party's budget factor through the full reader, and none of the three converses holds. The signed gap profile and the profile of costs before and after are blind in the first way as well. On the unit books the creditor's doubling to $(2, 1)$ and the debtor's halving to $(1, \frac{1}{2})$ have the same cost on every relationship and different authors.*

Proof. Example 3.6 for the first; both moves are party 0's own write for the second; row 2 is untouched by both moves for the third. The projections are functions of the full reader by definition. $J(2) = J(\frac{1}{2})$. $\square$

The profile loses the author, the author line loses the amount, and party 2's row misses a claim between the others. Keeping the full reading separates all three pairs.

Where each tradition starts. Theorem 11.1 is about three projections, and each projection is what one moral tradition reads first. A reader of outcomes sees how far the books differ and what that costs, and not who did it or who agreed to it. A reader of acts sees who acted and who agreed, and not what it cost. A reader of the agent's own row sees that agent's standing, and not a wrong between two others. The traditions themselves are not these thin readers: a consequentialist weighs welfare, not gaps [18]; the Formula of Humanity looks at the patient and at consent; contractualism asks what is justifiable to the person who undergoes the move [19], a question that needs at least the patient and the author line, which of the three projections none has and the full reader does; a virtue theorist reads the situation, not one

row [20]. What the theorem locates is the projection each starts from, and the pair it cannot separate from there. And the full reader itself, with its whole history, is blind in the way Section 7 proved: to what is owed. Every tradition that reads the two records and their history without the third record inherits that blindness, and the distinction between a culmination outcome and a comprehensive outcome [15] does not remove it, because the delivery is in neither.

12 Two carriers, one reading

The two-level books of this paper and the bond ledger of the ledger paper [9] were written independently. A bond ledger is a finite set of directed bonds, each with a positive multiplier and two ends; an action strains bonds by log factors and names one acting agent; an agent's burden is the sum of J of the multipliers of its incident bonds, and the harm of an action to an agent is the change in its burden.

Definition 12.1 (Two dictionaries). The *one-record dictionary* sends books B to the bond ledger with multiplier e^{x_e} on bond e . The *two-record dictionary* sends positive books to the bond ledger with multiplier x_e/y_e on bond e . A move that carries positive books B to B' transports to the action that strains bond e by the factor $(x_e/y_e)(B')/(x_e/y_e)(B)$ and, if the move has one author, names that author as its agent.

Proposition 12.2 (The one-record dictionary is blind to the second record). *Books that agree on every source figure give the same bond ledger under the one-record dictionary, whatever the destination figures. On chart B , the unit books and the books where the debtor has doubled its liability give the same bond ledger; their gaps on relationship $0 \rightarrow 1$ are 0 and -1 .*

Under the one-record dictionary, two books that differ only in what the debtor acknowledges are the same ledger. The second record is exactly what that carrier does not see.

Theorem 12.3 (The balance dictionary). *Under the two-record dictionary on positive books, the bond ledger's reading of a bond, the log of its multiplier, is zero iff the gap is zero, has the sign of the gap, and*

$$\gamma_e = y_e (\text{multiplier}_e - 1). \quad (8)$$

The unit claim on the unit books of chart B gives multiplier 2, gap 1, and bond reading $\log 2$. The two books of Theorem 12.2 read 0 and $\log \frac{1}{2}$.

Proof. $\log(x/y)$ has the sign of $x - y$ and is zero iff $x = y$; $x - y = y(x/y - 1)$. □

The signed difference and the scale-free ratio are related exactly through the destination's figure, and each determines the other given y_e .

Theorem 12.4 (The harm dictionary). *Under the two-record dictionary, the bond ledger's harm to a party under the transported action equals the pair-(1,1) reading (4) with $F = J$: twice the forced reading (5).*

Proof. Both are $\sum_{e \ni i} (J(e^{t'_e}) - J(e^{t_e}))$: each incident relationship charged the full change in J of its multiplier. □

The bond ledger charges each incident bond its whole change of cost; the two records charge each end half. The two readings differ by the factor two and in nothing else.

Theorem 12.5 (The consent dictionary and its limit). *For a one-author move, at every party, having written the move holds iff the transported action names that party. A joint record has two authors, both of whom consent, and no action's single agent field agrees with that verdict at both ends.*

Proof. The first is the transport rule. An action names one agent, and $s(e) \neq d(e)$. $\square$

An action has one agent; a joint record has two authors. Wherever a move has one author the two carriers name the same party; where it has two, one carrier has no field for the second.

Locating the bond ledger. The bond ledger with multipliers is the one-author fragment of the two-record object: under the two-record dictionary it sees the disputed balance, sees harm at twice the forced reading, and sees the author of a one-sided write; it is silent on the second author of a joint record; and under the one-record dictionary it is silent on the gap itself. Two independent carriers agree wherever both can see.

13 Who can be a patient

What reaches a party on this object is a one-sided write at a relationship it is an end of. The party at the other end is a patient, and anything the object can say happened to a party happened to a patient. This section says who can be one.

Definition 13.1. A party i can be a patient on books B when some move makes it a patient.

Theorem 13.2 (Being a possible patient is having a relationship). *On any chart with at least two accounts and for every books, a party can be a patient iff it is an end of some relationship.*

Proof. A patient is by definition an end of a changed relationship. For the other direction let i be an end of e and let j be the other end. Let j write 1 on its designated account of e and -1 on any other account (two accounts suffice). By Lemma 3.2 γ_e moves by 1; $i \neq j$ did not write; so i is a patient. $\square$

The write in the proof is on a designated account and needs no free account: a frozen party can still make others patients. Nothing about i 's reserve or history enters, and being a possible patient does not depend on the books.

Labels on parties. Being a possible patient is incidence, so no rule reading a label fixed on the parties decides it on every chart: if the rule admits some party, the chart with no relationship refutes it; if it admits none, one relationship refutes it. On a fixed chart, incidence decides it and any label agreeing with incidence there does too. This is a typing fact, not an argument against any account of moral status. Which physical systems are parties and which stand in relationships is the substrate question below; the object does not answer it, and a criterion of patienthood by complexity [22, 23] would have to enter there, as a condition on what counts as a party or a relationship, not as a threshold on the chart.

13.1 The count of undergoing

Being a possible patient is a yes or no. How much a party undergoes over a stretch of history is a count.

Definition 13.3 (Undergoing). Fix a history and a window of eight ticks. A party's *undergoing* over the window is the number of ticks t in it at which the party is a patient of the move m_t on the books B_t .

Theorem 13.4 (The count and its ends). *A party's undergoing over an eight-tick window is at most 8. A party with no relationship undergoes 0 on every history. On chart B , if the creditor writes the unit claim against the debtor at every one of the eight ticks, the debtor undergoes 8; if it writes on three of the eight ticks and nothing on the other five, the debtor undergoes 3.*

Proof. A count over eight ticks is at most 8. A party with no relationship is never a patient (Theorem 13.2). On chart B the unit claim changes γ_0 and the debtor did not write it, so the debtor is a patient at each tick the claim is written and at no tick where nothing is written. $\square$

The count has the same shape as another count in this programme, the degree of consciousness of a boundary, which is also a number of non-silent ticks in an eight-tick window. The two are different functions on different structures; identifying them would need one object on which both are defined, and no such object is defined here. Nothing in this paper is a claim about the consciousness of any system.

13.2 What is open

The books hold two assertions and no record of what is owed. The history says who departed. The third record says what is due, within a stated rule and an explicit evidence map. What remains open is which physical systems supply the records, and which obligations the stated rule covers.

- **The substrate map.** The crossing record of Section 8 is read off a unit path. Which physical systems are units on paths, which places are their cubes, and which systems are parties on a chart is the substrate question above. The designed first check is the smallest system with two parties and one relationship: two cells joined by one gap junction, each cell's record being what it holds on its side of the junction, the gap the difference, and repair the flux across it. The check passes if who is a patient, what repair costs, and who is charged agree with the observed relaxation; no such check has been run.
- **The coerced yes.** The object carries a standing one-sided claim that a yes does not touch (Proposition 3.8). Whether the claim caused the yes is not on the object. The third record as stated has no event for it.
- **Breach by silence.** An undertaking accepted and not performed leaves due 0 under the rule of Definition 7.1: the rule prices delivery and not its absence. A rule in which non-performance itself has a price is a different rule, and its events are not among the six.
- **Over-acknowledgment.** A debtor that acknowledges more than is due is unfaithful in the direction the rule never penalizes; the gap it opens is read like any other. Whether a gift written as a liability is a wrong is not a question the rule asks.
- **The authority of the rule, fair terms, welfare, rescue, collective responsibility, a duty to act.** The result determines obligations within a stated exchange rule and an explicit evidence map. Why that rule and not another, whether the price both sides accepted was fair, and what a settlement does to anyone's welfare lie outside the conclusion. So does whether a repair not written by a party that could have written one is a wrong. A record with more than two authors also lies outside the conclusion. What a duty costs, and for whom one holds, is Section 10.

14 Where the physics is, and where it is not

The title of this paper is a claim, and this section tests it. Five things in the paper look like physics: the cost law, the unit's path, the conservation of the gap and of row totals, the clock, and the finite budget. For each, the test is the same: replace the physical input by another and see which theorems move. Then the section counts what the two records carry of the moral vocabulary, prints the three theses the paper's own theorems refuse, and locates the literatures that already hold parts of the object.

entry point	cost law J	unit path	conservation	clock, eight	finite budget
what it reaches	the whole curve: marginal cost; 0.0600 vs 0.0556; $\frac{1}{4}$ vs $\frac{1}{20200}$; $\lceil R_0/d \rceil$	delivery is an event no party writes; any place works	gap under joint records; row totals	cost linear in ticks; eight = range of the under- going count	sums with bounded partial sums
verdict	physics	physics	ledger mathematics	ledger mathematics	resource accounting

shaded: a theorem changes when the physical input is replaced; unshaded: none does

Figure 13: The dependence test. Two of the five places where the physics seems to enter are physics: the shape of the cost curve and the existence of a carrier whose crossing no party writes. The other three are mathematics of the ledger in physical dress.

14.1 The dependence test on the five entry points

The cost law. Replace J by any reciprocal cost F of the ratio that separates e from 1. The equal split stands: Theorem 4.4 proved that conditions (0), (i), (ii), (v), (vi) and the form of (iii) give $w_s = w_d = \frac{1}{2}$ for every such F . What moves is everything that reads the curve. The marginal cost $\sinh t$ and its sign; the settlement on the triangle, whose shift $\ell^* = 0.3447$ and bystander charge 0.0600 are read off cosh, against $\frac{1}{3}$ and $\frac{1}{18}$ for the quadratic; the ratio-against-gap numbers $\frac{1}{4}$ and $\frac{1}{20200}$; the meeting point, which uses the convexity of J in the log-ratio; the exhaustion window $\lceil R_0/d \rceil$, whose d is $2(\cosh \sigma_0 - 1) - b$. So the cost law is physics in the exact sense that theorems change when it is replaced, and the one theorem that decides how harm divides is not among them.

The unit path. Replace the lattice of three dimensions by any set of positions and any notion of place coarser than a position. Every theorem of Section 8 holds: a path never transfers from a place to itself, delivery is retained, the occupancy count forgets. The dimension is not used; the cubes are there so that the figure can be drawn. What the physics supplies is that there is a carrier with a position at every tick, so that delivery is a fact about the carrier and not an assertion by a party. Remove the carrier and the third record has no physical part: delivery becomes another thing somebody says. That is why the entry counts as physics: a theorem, the gate on grounded runs, exists only because a path does.

Conservation. A joint record changes no gap, and every move conserves every row total. Both read as conservation laws and both are the definition of a move: an own write is a zero-sum vector on one row, a joint record moves opposite amounts on two designated accounts and their counter accounts. Replace the physics by anything and these stand, because there is no physics in them. They are the mathematics of two-sided writing.

The clock and the eight. Ticks are the order of moves; the standing cost is linear in their number under any clock. The number eight enters once, as the length of the window over which undergoing is counted, and it reaches exactly one statement: the count lies between 0 and 8. No theorem of the paper changes if the window is seven or nine, except the range of that count. The spine's period is borrowed for the window and does no other work here.

The finite budget. The reserve, the supply, the budget per window, the ceiling: these are resource accounting. The theorems are that a series of non-negative terms with bounded

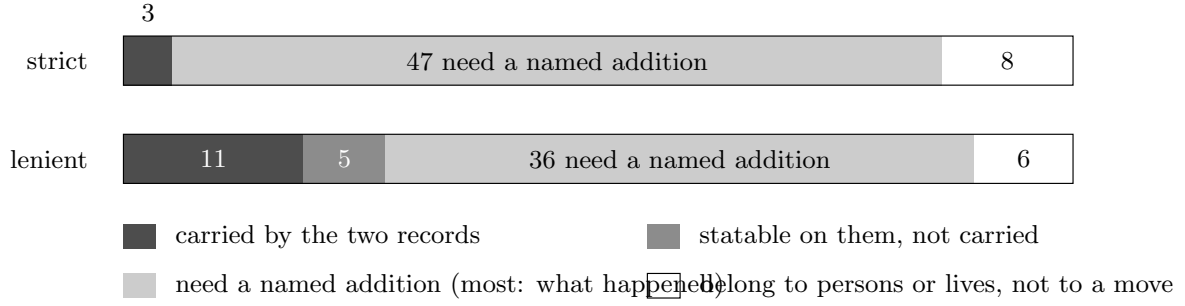


Figure 14: Fifty-eight moral words placed once each, under two readings. Strict: the word keeps its ordinary meaning. Lenient: its record-level sense counts. The two readings disagree only on how much of a word’s ordinary meaning the record-level sense is allowed to stand in for.

partial sums has terms tending to zero, and that a sum that falls by at least d per step is negative after $\lceil R_0/d \rceil$ steps. Nothing thermodynamic is claimed or used; the word does not appear in the theorems because nothing in them is about heat, entropy or equilibrium. The budget is a number a party has, and the clock counts the steps at which it falls.

Verdict. Two of five are physics. The cost that prices a wrong is the cost that prices every imbalance in the theory, and the theorems that read its shape change if it is replaced; and what happened is read off a carrier that has a position, without which nothing is delivered rather than asserted. Three of five are the mathematics of the ledger, and calling them physics would be wrong. The title names the two, and the subtitle bounds them.

14.2 The census

Fifty-eight moral words and open items were placed, each exactly once, in one of four classes: carried by the two records; statable on them as a predicate but not carried; needing a named addition to the object; or belonging to persons, dispositions or whole lives rather than to a move. The placement was done twice, once strictly (the word keeps its ordinary meaning) and once leniently (the word’s record-level sense counts). Figure 14 draws both.

Strictly, three words are carried: *author*, *disputed balance*, and *repair* of a disagreement. Leniently, eleven: those three and *patient*, *consent* (the record of who wrote), *harm* (the assigned change of cost), *restitution* (the exact reversal), *judgment* (the bystander’s charge under a settlement), *deception* (kept minus shown), *coercion* (a standing claim the yes does not touch), and the *meeting point* (the geometric mean; the census’s lenient reading files it against the word *love* under a named identification that the strict reading refuses, and leaves love as an attitude outside; a reader who refuses that row counts ten). Five more are statable on the two records as predicates with a positive and a negative instance each, and carried by none of them in the ordinary sense: *wrong* (a gap opened from agreement), *rescue* (an end closes an open gap), *forgiveness* (the patient’s matching entry), *breach* (departure after a joint record), *loyalty* (standing on a record). Each of the five names a record-level shape that its ordinary word does not reduce to: a corrected false consensus opens a gap from agreement and is not a wrong; a patient’s matching entry is acceptance, repayment or error correction as often as forgiveness.

Of the words that need an addition, the largest group needs what Section 8 supplied: a record of what happened. Under the stated exchange rule the paper now carries *what is due*, *release* as the creditor’s authorized event, the difference between a *sale* and a *gift*, and *who is right* in the exchange sense; an *undertaking* is an offer accepted; *fraud* is the history that

writes a claim no crossing grounds. Those words moved, in this paper, from needing the third record to being carried by it under the rule; *breach* did not, because the rule prices delivery and not its absence. The appendix classifies every word against the two records, so there they stand as additions, with the third record named in the last column as what carries them. The remaining additions are of four kinds and the paper names them without supplying them. A *comparator*: fairness, exploitation, equality, distributive justice, each needing a standard the two records do not carry, since agreed unfair terms are matched books at cost zero. A *response object*: punishment, desert, proportionality, mercy, retributive justice, each needing a sanction related to an offence, and the object has no sanction. A *bearer*: rights, dignity, autonomy, duty and obligation in their ordinary sense, each needing a predicate on parties the chart does not have. A *group rule*: collective author, collective responsibility, complicity, each needing an author line with more than the two ends of one joint record. Eight words, strictly, belong to persons or lives and not to any move: virtue, vice, character, the good life, care, gratitude, love as an attitude, and the reason to act at all, which Section 10 places with the agent and not with the books. The object is not enlarged to reach them.

The census is a count of a requested vocabulary, not of morality; it says which words the two records carry, which the third record carries under a rule, and what kind of thing each of the rest is waiting for. The fifty-eight words and both placements of each are printed in Appendix A, so that the counts can be checked and the classification disputed word by word.

14.3 Three theses that fail

Three sentences might be taken as the thesis of a paper with this title. The paper's own theorems refuse each, and what survives each refusal is the actual result.

All morality from one act recorded twice. Refused by the two runs of Theorem 7.3: the creditor who delivered and was refused a signature and the creditor who invented the claim have the same two records, the same authors, the same departures, at every tick and at every length of history, and are owed 1 and 0. What survives: everything the two records do carry, listed in the census, and the exact statement of what they cannot.

Right and wrong as fidelity to what happened. Refused twice. The same crossing is a sale or a gift (Proposition 8.4): the physical event is identical and what is owed differs by the terms, which are not events of the carrier. And an authorized release lowers what is due with no crossing and no write (Theorem 8.7): what is owed changes while nothing happens to the unit. What survives: the third record is a record of the rule's events, of which the crossing is the physical part; performance, terms and authority are each necessary and none is a figure a party writes.

Morality as the price of continuing to act. Refused twice. The $(t + 2, t + 1)$ history of Proposition 10.3 is priced and bounded, posts at every tick, and holds a disputed balance of 1 forever: continuing to act under a finite reserve forces ratios to one and forces no gap closed. And priced agreement can remain false forever (Theorem 9.2): a party can go on agreeing, affordably, to what is not owed. What survives: a duty holds for the agent that goes on, and it is the duty not to sustain an imbalance whose upkeep exceeds its renewal; it is not a duty to close gaps and not a duty to be right.

14.4 Where the moral traditions reach

Theorem 11.1 located the three thin readings each tradition starts from and the pair each cannot separate; the paragraph after it said where the traditions themselves, read carefully, go

beyond those readings. What this section adds is the reach of the whole: the full reader, with its history, is blind to what is owed, and every tradition that reads the two records without a record of what happened inherits that blindness. The paper does not rank the traditions. It says which projection each begins on, what that projection erases, and what none of them has without the third record.

14.5 Where the ledger literatures reach

Four literatures already hold parts of this object, and each stops at a line the paper crosses.

Double-entry bookkeeping is one party's books. Ellerman showed that a T-account is an ordered pair of non-negative numbers taken modulo $(a, b) \sim (c, d)$ iff $a + d = b + c$, the group-of-differences construction, and that a set of books is a vector of such pairs whose signed entries sum to zero [24]. A row of Definition 2.2 is one such vector, and an own write is Ellerman's balanced transaction on it. Everything here begins at the second vector: a relationship is two entries on two different parties' rows, and the gap is a quantity defined on neither row alone.

Intercompany reconciliation meets the gap daily. Two entities of one group book the same transaction and their balances against each other differ; the difference is worked down by one side adjusting its own entry (Theorem 3.4), and goods shipped by one and not yet booked by the other are scene D of Table 1. What closes a reconciliation is not written on either book: it is the shipping document, the bank statement, the counterparty's confirmation. Auditing has a standard for exactly this. External confirmation [25] asks a third party what happened, because an entity's own record, and an agreement between two entities, is not evidence that the transaction occurred. Theorems 7.3 and 9.2 say why in the smallest case: two records, however long their history and however well they agree, do not fix what is due.

Triple-entry bookkeeping has two meanings, and the object separates them. Ijiri's third entry adds to wealth and income a third layer, momentum, on one party's books [26]; it deepens one record and lies inside one row. Grigg's third entry is a receipt signed by both parties and held by a third [27]; it is a joint record in the sense of Definition 2.3, changes no gap (Lemma 3.3), and by Theorem 9.2 can be signed at every tick while nothing is due. The shared signed receipt is agreement made durable. The third record of this paper is not a third signature. It is a record of the event, and Theorem 9.3 says no signature stands in for its performance part.

Incomplete-contract theory separates what the parties can observe from what a court can verify, and prices the difference [28, 29]. Delivery here is observable by both parties and written by neither; the third record is what makes it verifiable. Theorem 7.3 adds the step that theory does not take: the parties' own two books, kept forever and in full agreement, never close the observable-verifiable gap by themselves. The oracle problem of distributed ledgers [30] is the same gap seen from the ledger's side; Theorem 9.3 says what the oracle must carry.

Hohfeld's correlatives are the two entries of a relationship read as one jural relation: A 's claim against B is B 's duty to A , one fact stated from two sides [31]. Here the two sides can be stated differently, and the gap is the amount by which they are. Hohfeld's scheme has no place for that quantity because it describes the relation, not the two records of it. Thomson's account of a claim as something only its holder can release [32] is Theorem 8.7(1) and (2): the creditor's release lowers what is due and the debtor's does nothing.

15 Scope

The results are about parties in relationship on a fixed chart, with a stated exchange rule and an evidence record whose physical part is a unit path. An act on a party with no relationship does nothing on the books, and nothing here says how relationships come to exist. An act

between two parties enters only as a move on two books or as an event under the rule. A punch, written as a one-sided write on the relationship between two people, returns an author, a patient, a gap, the absence of the patient’s consent, and, from the history, who departed; not which of the two is in the right. If the two had undertaken anything under the rule, it also returns what is due. A wrong that cannot be written as a move or an event is outside these results. Consent is the record of who wrote; of the conditions on a valid yes, the object carries a standing prior claim and a gap of kept minus shown (Proposition 3.8) and nothing of pressure, knowledge, or intent [21, 39]. The complexity of a party plays no role anywhere.

Every theorem in this paper is a statement about finite charts, real-valued books, finite event streams and unit paths on the lattice, and is proved on the page. The base $c = 1$ of the cost reading is the spine’s calibration [8]; every structural statement holds for every $c > 0$ with F_c in place of J , and the printed numbers are the $c = 1$ values. Two things are taken from outside the two records and named where they enter: the cost of a compared pair, which is Recognition Science’s cost of every imbalance, and the lattice a unit moves on. One definition brings the physics in, in Section 2.3: that the two figures of a relationship are the two parts of one recognition. A reader who refuses the cost must refuse it there. The other stipulations of the paper are hypotheses of the theorems that use them, each named where it enters: consent as the author line, the exchange rule, the settlement rule, the unit baseline, the continuation premise. Under them, everything else is a theorem.

A The fifty-eight words

The census of Section 14.2, one row per word, each placed once under each reading. *Carried*: a theorem on the two records carries the word’s sense. *Statable*: a predicate on the two records with a positive and a negative instance, not carried in the ordinary sense. *Addition*: the word needs a named addition to the object; the last column names it, or names what the two records do carry of the word. *Persons or lives*: the word belongs to a person, a disposition or a whole life, not to a move. *Strict*: the word keeps its ordinary meaning. *Lenient*: the word’s record-level sense counts. Totals: strict $3 + 47 + 8$; lenient $11 + 5 + 36 + 6$.

word	strict	lenient	what is carried, or what is needed
1 author	carried	carried	the writer of its own row; recovered from the changed rows of a nonzero move
2 patient	addition	carried	non-author at a relationship whose gap changed; moral status is not included
3 consent	addition	carried	the record of who wrote; validity, capacity and knowledge are not carried
4 harm	addition	carried	the assigned change of cost at an end under the stated conditions; welfare is not carried
5 wrong	addition	statable	a gap opened from agreement; wrongfulness needs a justified claim or norm
6 disputed balance	carried	carried	the gap, up to the unit
7 repair (of a disagreement)	carried	carried	an end’s write that closes the gap; physical injury is not repaired
8 restitution	addition	carried	the exact reversal of the opening write; rightful possession is not carried
9 forgiveness	addition	statable	the patient’s matching entry; the same entry is acceptance or correction
10 judgment	addition	carried	the bystander’s charge under the stated settlement rule; authority is not carried
11 punishment	addition	addition	a sanction related to an offence (response object)
12 desert	addition	addition	a responsibility condition and a merit or blame standard (response object)

	word	strict	lenient	what is carried, or what is needed
13	proportionality	addition	addition	a response and a justified comparison to the offence (response object)
14	promise	addition	addition	an undertaking with future content; the third record carries it under the exchange rule as an offer accepted
15	breach	addition	statable	departure after a joint record; the rule prices delivery, not its absence
16	trust	addition	addition	reliance or expectation directed at a party
17	betrayal	addition	addition	a trust relation and its violation
18	loyalty	addition	statable	standing on a record; the disposition is not carried
19	lie	addition	addition	asserted content, a truth relation, the speaker's belief
20	deception	addition	carried	kept minus shown; intent is not carried
21	coercion	addition	carried	a standing one-sided claim the later yes does not touch; the threat is not carried
22	exploitation	addition	addition	a comparator: agreed unfair terms are matched books at cost zero
23	fairness (incl. agreed terms)	addition	addition	a comparator over terms or claims
24	distributive justice	addition	addition	a comparator: goods, claims, a distribution standard
25	retributive justice	addition	addition	culpability, desert, authority, a sanction (response object)
26	procedural justice	addition	addition	a procedure, participation rights, a decision rule
27	rights	addition	addition	a bearer: directed claims and correlative duties
28	duty (ordinary sense)	addition	addition	a bearer: a required act directed at an agent; Section 10 gives the agent's own
29	obligation (ordinary sense)	addition	addition	what is owed, to whom, how discharged; the third record carries what is due under the rule
30	supererogation	addition	addition	a duty baseline and a good act beyond it
31	rescue	addition	statable	an end closes an open gap; peril and a duty to aid are not carried
32	negligence	addition	addition	a care standard, capability, opportunity, an omission
33	intention	addition	addition	intended content attached to a move
34	excuse	addition	addition	responsibility conditions: capacity, knowledge, control
35	apology	addition	addition	an addressed act acknowledging the wrong
36	mercy	addition	addition	an authorized response and its deliberate reduction (response object)
37	gratitude	persons or lives	persons or lives	a responsive attitude
38	dignity	addition	addition	a bearer: a status predicate on parties
39	equality	addition	addition	equality of what; orientation symmetry is not moral equality
40	autonomy	addition	addition	competent agency and control of choice; own-row authorship is compatible with coercion
41	privacy	addition	addition	observers and an access relation; the books are readable
42	property	addition	addition	title, possession, authorized transfer
43	theft	addition	addition	property and an unauthorized taking
44	fraud	addition	addition	the third record carries it under the exchange rule as a claim no crossing grounds; content and reliance are not carried
45	collective responsibility	addition	addition	a group rule: membership, contribution, aggregation
46	complicity	addition	addition	a group rule: assistance and knowledge
47	virtue	persons or lives	persons or lives	a stable excellence of a person
48	vice	persons or lives	persons or lives	a stable defect of a person
49	character	persons or lives	persons or lives	a persisting organization of dispositions

	word	strict	lenient	what is carried, or what is needed
50	the good life	persons or lives	persons or lives	an evaluation of a whole life
51	love	persons or lives	carried	strictly an attitude; leniently the meeting point (geometric mean) under a named identification
52	care	persons or lives	persons or lives	attending to needs over time
53	why act at all	persons or lives	addition	a reason: Section 10 places it with the agent, not the books
54	substrate map	addition	addition	a map from a physical system to chart, books and moves
55	undergoing and the 0 to 8 dial	addition	addition	the count is carried; its identification with a degree of consciousness is not made
56	who is right (full sense)	addition	addition	the history gives the recorded part; the third record carries the exchange sense under the rule
57	failure to rescue	addition	addition	peril, capability, opportunity, an omission, a duty to aid
58	collective author	addition	addition	a group rule: an author line beyond one writer or the two ends of one joint record

References

- [1] J. Washburn and M. Lj. Zlatanović, The δ -calculus: from distinction to arithmetic, *Advances in Pure Mathematics*, accepted, 2026; arXiv:2607.29349.
- [2] J. Washburn and M. Lj. Zlatanović, Uniqueness of the canonical reciprocal cost, *Mathematics* 14 (2026) 935; arXiv:2602.05753.
- [3] J. Washburn, M. Lj. Zlatanović and E. Allahyarov, The d’Alembert inevitability theorem, *Mathematics* 14 (2026) 1386; arXiv:2603.16237.
- [4] S. Pardo-Guerra, J. Washburn and E. Allahyarov, Admissible reciprocally symmetric costs, *Mathematics* 14 (2026) 2157.
- [5] S. Pardo-Guerra, J. Washburn and E. Allahyarov, A coherence theorem for conserved comparison ledgers: structural axioms pin down the scale, the cost, and the ratio, *Mathematics* 14 (2026) 2672.
- [6] S. Pardo-Guerra, A. Thapa, M. Simons and J. Washburn, Coherent comparison as information cost: axiomatic foundations for discrete ledger dynamics, *Foundations* 6 (2026) 17.
- [7] J. Washburn, *Actual Economics: Separate records, reconciliation, and the standing network*, Recognition Physics Institute, 2026.
- [8] J. Washburn, *Checking the Cost Theorem: An Elementary Descent, Two Redundant Hypotheses, and a Calibration Nothing Satisfies*, Recognition Physics Institute, 2026.
- [9] J. Washburn, *A Cost Calculus on Directed Ledgers, and the Gap to Moral Terms*, Recognition Physics Institute, 31 August 2026.
- [10] J. Washburn, *Agreement, Evidence, and Obligation: Ethics from Two Records*, Recognition Physics Institute, September 2026.
- [11] H. P. Young, *Equity: In Theory and Practice*, Princeton University Press, 1994.

- [12] J. Aczél, *Lectures on Functional Equations and Their Applications*, Academic Press, 1966.
- [13] H. Moulin, Axiomatic cost and surplus sharing, in K. J. Arrow, A. K. Sen, K. Suzumura (eds.), *Handbook of Social Choice and Welfare*, vol. 1, Elsevier, 2002, 289–357.
- [14] H. P. Young, Cost allocation, in R. J. Aumann, S. Hart (eds.), *Handbook of Game Theory with Economic Applications*, vol. 2, Elsevier, 1994, 1193–1235.
- [15] A. Sen, Maximization and the act of choice, *Econometrica* 65 (1997) 745–779.
- [16] H. M. Hurd, The moral magic of consent, *Legal Theory* 2 (1996) 121–146.
- [17] D. Owens, *Shaping the Normative Landscape*, Oxford University Press, 2012.
- [18] D. Parfit, *On What Matters*, Oxford University Press, 2011.
- [19] T. M. Scanlon, *What We Owe to Each Other*, Harvard University Press, 1998.
- [20] R. Hursthouse, *On Virtue Ethics*, Oxford University Press, 1999.
- [21] A. Wertheimer, *Coercion*, Princeton University Press, 1987.
- [22] P. Singer, *Practical Ethics*, 3rd ed., Cambridge University Press, 2011.
- [23] J. McMahan, *The Ethics of Killing*, Oxford University Press, 2002.
- [24] D. P. Ellerman, The mathematics of double entry bookkeeping, *Mathematics Magazine* 58 (1985) 226–233.
- [25] International Auditing and Assurance Standards Board, *International Standard on Auditing 505: External Confirmations*, IFAC, 2009.
- [26] Y. Ijiri, A framework for triple-entry bookkeeping, *The Accounting Review* 61 (1986) 745–759.
- [27] I. Grigg, Triple entry accounting, Systemics Inc., 2005, https://iang.org/papers/triple_entry.html.
- [28] O. Hart and J. Moore, Incomplete contracts and renegotiation, *Econometrica* 56 (1988) 755–785.
- [29] J. Tirole, Incomplete contracts: where do we stand?, *Econometrica* 67 (1999) 741–781.
- [30] G. Caldarelli, Understanding the blockchain oracle problem: a call for action, *Information* 11 (2020) 509.
- [31] W. N. Hohfeld, Some fundamental legal conceptions as applied in judicial reasoning, *Yale Law Journal* 23 (1913) 16–59.
- [32] J. J. Thomson, *The Realm of Rights*, Harvard University Press, 1990.
- [33] C. M. Korsgaard, *The Sources of Normativity*, Cambridge University Press, 1996; *Self-Constitution: Agency, Identity, and Integrity*, Oxford University Press, 2009.
- [34] J. D. Velleman, *The Possibility of Practical Reason*, Oxford University Press, 2000.
- [35] D. Enoch, Agency, shmagency: why normativity won’t come from what is constitutive of action, *Philosophical Review* 115 (2006) 169–198.

- [36] L. Ferrero, Constitutivism and the inescapability of agency, *Oxford Studies in Metaethics* 4 (2009) 303–333.
- [37] J. Broome, *Rationality Through Reasoning*, Wiley-Blackwell, 2013.
- [38] N. Kolodny, Instrumental reasons, in D. Star (ed.), *The Oxford Handbook of Reasons and Normativity*, Oxford University Press, 2018.
- [39] J. Feinberg, *Harm to Others*, Oxford University Press, 1984.