Two Records of One Claim

Disagreement, rewriting, and the bystander in networks of bilateral obligations

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Abstract

Models of obligation networks carry one number per link. Real books carry two: the creditor's record of the claim and the debtor's record of the liability, each kept on its own books, with no rule that they coincide. We take that second record as the primitive and ask what it forces. Three results are the core. First, any instrument that reads the network through its exposures matrix (who owes whom how much) or through net positions is blind to a datum that decides who bears the cost of rewriting a cycle of obligations at fixed positions: two networks with identical exposures and positions can differ on which parties the cheapest rewrite touches and on whether a party with nothing outstanding consents to it. Second, for every separable, differentiable, strictly convex cost of open positions that is minimised on empty lines, the cheapest rewrite at fixed positions is unique, is determined by the cut space of the graph of standing relationships, and moves onto a relationship carrying nothing exactly when the marginal costs circulate around a cycle that relationship closes; a party who owed nothing and was owed nothing is then charged. Third, a fixed cost of opening a position (a kink at rest, of which the gross-notional objective used in portfolio compression is the extreme case) suppresses this effect whenever the circulation is smaller than the kink, and a one-sided line suppresses it outright; so the compression literature's finding that fixed-net compression does not reach uninvolved parties is a property of its objective and feasible set, not of the network. Around these we prove what two records force with no cost at all (a disagreement is conserved by every move the two parties make together and is closed only by a revaluation whose two halves sum to the gap; the books contain exactly two figures for a disputed claim, and settlement costs neither party more than the gap exactly on the interval between them), and what they leave free (shares of a cost, the split of a revaluation, outside options, rent schedules, which relationships are kept). An axiomatic section characterises the disagreement index that is additive over relationships, orientation-free and scale-free as $J(x) = \frac{1}{2}(x + 1/x) - 1$ of the ratio of the two records, and shows every objective defined on positions or exposures is cycle-blind. A final section gives a located-unit model in which the two records arise: a transfer is recorded by the two books in different periods, so matched books are kept records with different stamps, and each party's free-account balance is forced to be minus its net position.

1 Introduction

A network of bilateral obligations is usually written as a matrix: entry (i, j) is what i is owed by j, or as a vector of net positions. This is the input of the clearing model of Eisenberg and Noe [1], of the contagion models that followed it [2, 3, 4], and of the compression problem of D'Errico and Roukny [5]. It is a natural input, and it discards something that every real system of accounts records: a claim is booked twice, once by the creditor as an asset and once by the debtor as a liability, on two separate sets of books, and the two entries need not agree. National accounting calls the paired entry quadruple entry [9]; every firm's receivables ledger and its counterparty's payables ledger are the two records.

This paper takes the two records as primitive and asks what they force. We write the smallest calculus in which a claim has two records (Section 2), prove what it forces with no cost function at all (Section 3), and then place a cost on open positions and study the cheapest rewriting of a cycle of obligations that holds every party's net position fixed (Section 4). That operation is portfolio compression as it is run by TriOptima and the central counterparties [6, 5], and the question of whether it can harm a third party is the question of Veraart [7] and Schuldenzucker and Seuken [8].

The answer is the paper's core, and it has three parts.

(1) *The cut-space datum.* For every separable, differentiable, strictly convex cost minimised at a rest value, the cheapest rewrite at fixed positions is unique and depends on the network only through the cut space of the graph of standing relationships, including the relationships that carry nothing. Adding a relationship at rest that is a bridge never moves the rewrite. Adding one that closes a cycle moves it exactly when the marginal costs at the old optimum have nonzero circulation around that cycle. When the cost is minimised on empty lines, a party who owed nothing and was owed nothing is then charged (Theorems 4.4, 4.6).

(2) *Exposure functionals are blind to it.* Any functional of the exposures matrix, or of net positions, takes the same value on two networks that differ on who the rewrite reaches and on who consents to it. On a fixed chart, or on a forest, amounts determine the rewrite outright; the datum they miss is the set of open relationships carrying no balance, and it matters exactly through the cycles those relationships close (Theorems 5.1, 5.2).

(3) *A kink at rest hides it.* If opening a position carries a fixed cost (the cost has a kink at rest), the rewrite stays off a relationship at rest whenever the circulation is smaller than the kink; a line that may carry a claim in one direction only is a kink of infinite size. The gross-notional objective of compression in practice is the first case, and conservative compression in the sense of D'Errico and Roukny is the second. Whether fixed-net compression reaches uninvolved parties is therefore decided by the objective and the feasible set, not by the network (Proposition 4.7, Remark 4.8).

Around the core we prove what two records force without any cost. A counterpart gap, the difference between the two records of one claim, is unchanged by every move the two parties make together and by every internal entry either makes on its own free accounts; it is closed only by a revaluation on a designated account, and any pair of revaluations that closes it sums to the gap (Theorem 3.1). For a disputed claim the books contain exactly two figures; for money-monotone parties a settlement is a concession by at least one of them, and costs neither more than the gap exactly on the interval between the two figures. In the bilateral case this is the core of the assignment game of Böhm-Bawerk and Shapley and Shubik [14, 15], and what the two records add is where the endpoints come from (Theorem 3.2). The cost of a rewrite has a forced total and free shares (Theorem 4.2); consolidation is worth exactly the carrying cost it frees (Theorem 6.3); the coalition game induced by fixed-net rewriting has a books' allocation outside its core whenever the rewrite charges anyone (Theorem 6.2); a pruning dynamics on kept relationships stops exactly at the pairwise-stable sets of Jackson and Wolinsky [16] and is history dependent (Theorem 6.5); a rent is a contract term the books do not fix (Theorem 6.7).

Section 7 is axiomatic. A disagreement index that is additive over relationships, indifferent to which party is called creditor, and governed by the same law at every scale is $\frac{1}{2}(x + 1/x) - 1$ of the ratio of the two records, up to a unit and an exponent (Theorem 7.2); the additivity axiom is what excludes maximin and Nash-product objectives, and any objective defined on positions or exposures is cycle-blind by Theorem 5.1. Section 8 asks where two records come from and answers with a located-unit model: a transfer between two books is recorded by them in different periods, so a matched pair of books is a relation between kept records, never a simultaneous state; and each party's balance on its free accounts is forced to equal minus its net position, so the buffer that absorbs a revaluation is not a free parameter (Theorems 8.2, 8.3). Section 9 lists what the model leaves to the parties and what remains open.

Every result is proved in the text. Postulates are stated where they are used and nowhere else: Sections 2 to 6 use none; Section 7 states its axioms on the index; Section 8 states the located-unit model as a postulate. The convex cost of an open position that Sections 4 to 6 place on the books is a hypothesis of each theorem, and the theorems hold for every cost in the class defined there.

Relation to the literature. The calculus is double-entry bookkeeping made algebraic; Ijiri [10] and Ellerman [11] treat double entry with one set of books and so have no counterpart gap. The fixed-position rewrite is a separable convex flow problem after Rockafellar [12], and its cut-space dependence is the cut-cycle decomposition of edge space [13]. Clearing after Eisenberg and Noe and its descendants take the exposures matrix as input; Section 5 states what that input cannot identify. The compression results of D'Errico and Roukny, Veraart, and Schuldenzucker and Seuken work with notional objectives; Proposition 4.7 locates them inside the class. The two-figure interval is the assignment game [14, 15]; pruning follows Jackson and Wolinsky; the coalition game uses the core of Gillies [17]. The index of Section 7 sits in the axiomatic tradition of Atkinson [18] and Shorrocks [19], with additivity across relationships in the role of decomposability; its functional equation is d'Alembert's [20].

2 The model

Definition 2.1 (Chart)**.** A chart has n parties, k accounts per party, and E relationships. Relationship e runs from a source $s(e)$ to a destination $d(e) \neq s(e)$ and designates an asset account A_e in the source and a liability account L_e in the destination. Distinct relationships leaving one party designate distinct asset accounts; distinct relationships entering one party designate distinct liability accounts; no account is both. An account is *designated* if some relationship designates it and *free* otherwise.

Definition 2.2 (Books)**.** Books on a chart are an array $b_i(a) \in \mathbb{R}$, debit positive. Party i is *balanced* if $\sum_a b_i(a) = 0$. The *counterpart gap* on relationship e is $g_e = b_{s(e)}(A_e) + b_{d(e)}(L_e)$; the relationship is *matched* if $g_e = 0$. Books are *consistent* if every party is balanced and every relationship is matched.

A single-number model of the network is the case $g_e \equiv 0$: the creditor's asset entry and the debtor's liability entry are one number with two signs. The object of this paper is the possibility $g_e \neq 0$.

The running example. Three parties, $0, 1, 2$, each with an asset account, a liability account and one free account (cash). Three standing relationships form a triangle, $0 \to 1$, $1 \to 2$, $2 \to 0$, each running from the party that may be owed to the party that may owe. Party 0 is owed one unit by party 1; the other two relationships are open and carry nothing. Party 2 owes nothing and is owed nothing. In symbols: $b_0(A) = 1$, $b_0(\text{cash}) = -1$, $b_1(L) = -1$, $b_1(\text{cash}) = 1$, all else zero. Every party is balanced and every relationship is matched.

Definition 2.3 (Generators; balanced moves)**.** Two moves generate the calculus.

(i) A *journal entry* at party i with vector $\eta \in \mathbb{R}^k$, $\sum_a \eta_a = 0$, adds η to party i's accounts.

(ii) A *cross posting* of amount x on relationship e through free counter accounts o_S in $s(e)$ and o_D in $d(e)$ makes four entries: $+x$ on A_e, $-x$ on o_S; $-x$ on L_e, $+x$ on o_D.

A *word* is a finite sequence of generators applied first to last. A generator is *balanced* if it is a journal entry vanishing on every designated account of its party, or a cross posting through free counter accounts. A word is balanced if each generator is.

A balanced generator is double entry within each party together with agreement across the relationship: a cross posting writes the same amount to both records at once, and a balanced journal entry moves value among a party's free accounts without touching anything a counterparty has a record of. The moves outside the balanced fragment include every unilateral one: an impairment, a change of estimate, a write-down, each altering a designated account on one side only.

Theorem 2.4 (Reachability)**.** *On any chart: (a) every books in which every party is balanced is a word of journal entries applied to the zero books; (b) a books is a balanced word applied to the zero books iff every party is balanced, every relationship is matched, and every relationship carrying a nonzero claim has a free counter account at both ends; (c) a books with an unmatched relationship is not the result of any balanced word applied to the zero books.*

Proof. (a) Each party's account vector sums to zero and is itself a journal entry. (b) Necessity is Theorem 3.1 below together with the observation that a cross posting on e needs a free counter account at each end. Sufficiency: post each claim by one cross posting through the free accounts, then journal each party's remaining balanced residual on its free accounts. (c) is the contrapositive of the matched clause of (b). □

Proposition 2.5 (The calculus is free)**.** *Words under concatenation form the free monoid on the two generators; for any set with an action of journal entries and of cross postings there is exactly one monoid map from words extending the actions, and applying a word to books is that map. On books, a word is undone by its reverse.*

This is the universal property of the free monoid; its content is in the generators. Any system of accounts admitting the two moves is a model. The generating signature (party, account, relationship, the two moves, their composition and reversal, designated account, unit) contains no symbol for preference, utility, equilibrium, money, price, demand or supply. Two results below assume something about parties (a payoff monotone in money: Theorem 3.2 and Remark 3.3) and say so; every other result holds with no assumption about parties.

Definition 2.6 (Exposures; positions)**.** The *exposures matrix* of a chart and books assigns to each ordered pair of parties the total booked claim of the first on the second, summed over the relationships joining them; a party's *position* is its row sum minus its column sum. A functional of (chart, books) is an *exposure functional* if it depends on the data only through the exposures matrix.

A relationship carrying zero balance contributes nothing to the matrix: an open line with nothing outstanding is invisible to every exposure functional. That invisibility is the subject of Section 5.

3 What two records force

Theorem 3.1 (A gap is conserved by every joint move)**.** *Every balanced generator leaves every counterpart gap unchanged. Hence balanced words preserve matched books, and if a relationship carries a gap no balanced word closes it. Closing a gap $g_e = r_0 - r_1$ (source books r_0, destination books $-r_1$) requires a revaluation posted through a designated account on at least one side, and any pair of revaluations (δ_0, δ_1) that closes it satisfies $\delta_0 + \delta_1 = g_e$.*

Proof. A journal entry at party i vanishing on designated accounts changes no $b_{s(e)}(A_e)$ and no $b_{d(e)}(L_e)$. A cross posting of x on e adds x to $b_{s(e)}(A_e)$ and $-x$ to $b_{d(e)}(L_e)$, so g_e changes by 0, and touches no other designated account. Induct along the word. The zero books have every gap zero. A revaluation by δ_0 at the source and δ_1 at the destination changes g_e by $-\delta_0 - \delta_1$. □

Nothing two parties do together, and nothing either does on its own free accounts, removes a disagreement between them. Who revalues and by how much is a choice; that the two revaluations sum to the gap is not.

Theorem 3.2 (The two figures)**.** *On a relationship the books contain exactly two figures: the source's booked claim r_0 and the destination's booked liability r_1. Settling the relationship at a figure p is a revaluation by $r_0 - p$ at the source and $p - r_1$ at the destination, and the two sum to the gap. If each party's payoff is monotone in the money it books, each of these is a concession exactly when it is positive, and p costs neither party more than the gap iff p lies in the closed interval between r_1 and r_0. When $r_0 > r_1$ no figure is a concession for neither party: settlement requires at least one to move off its own record.*

Proof. The only entries on e are $b_{s(e)}(A_e) = r_0$ and $b_{d(e)}(L_e) = -r_1$. Settlement at p sets both to $\pm p$: the source writes down by $r_0 - p$, the destination writes up by $p - r_1$, and $(r_0 - p) + (p - r_1) = r_0 - r_1$ (Theorem 3.1). For a money-monotone party a write-down of its asset or write-up of its liability is a loss. Since the two concessions sum to the gap, one is at most the gap iff the other is nonnegative; so neither exceeds the gap iff both are nonnegative iff $r_1 \le p \le r_0$ (for $r_0 \ge r_1$). Both are nonpositive iff $r_0 \le p \le r_1$, empty when $r_0 > r_1$. □

Remark 3.3 (The assignment market). Take m sources and n destinations with money-monotone payoffs and identify each party's book with its valuation. This is the assignment market of Böhm-Bawerk [14] and Shapley and Shubik [15]: the core is the set of competitive prices, an interval whose endpoints are booked figures. For one source and one destination it is the interval of Theorem 3.2. With $m \neq n$ and homogeneous books every core value equals the book of the long side (two sources booking 1 against one destination booking 4 pin at 1); with heterogeneous books the endpoints are the books of the marginal parties, the last paired and the first unpaired (sources 1 and 3 against a destination 4: $[1, 3]$, both endpoints sources' books); replication leaves the interval unchanged. In every case the endpoints are records some party keeps, never a third number.

Theorem 3.4 (Matched is every gap zero)**.** *A network is matched iff every counterpart gap is zero; the sum of gaps being zero is necessary and not sufficient. A single-number network carries one flow per link; projecting a two-record state onto it and asking for closure does not recover matching.*

Proof. Gaps $+1$ and -1 on two relationships sum to zero and are not matched. A matched relationship booked at r by both parties projects to the single flow $r \neq 0$, so closure of the projection is a different condition. □

Theorem 3.5 (Invariants of the pair)**.** *Let c be any even function of the log-ratio, $c(r_0/r_1) = c(r_1/r_0)$. Then $|r_0 - r_1|$ and $c(r_0/r_1)$ are invariant under relabelling the two parties and reversing the relationship, while the signed gap $r_0 - r_1$ changes sign and any payoff strictly monotone in a party's own money distinguishes the two parties (on books 1 and 4 it takes different values; $c(1/4) = c(4)$). The identity $\delta_0 + \delta_1 = r_0 - r_1$ holds under every relabelling.*

Proof. Relabelling and reversal send r_0/r_1 to r_1/r_0 and $r_0 - r_1$ to $r_1 - r_0$; a strictly monotone u has $u(1) \neq u(4)$. □

The distinction between functions of the unordered pair and functions of one book organises everything below: the results that hold for every party are functions of the pair; the variables the model leaves free are not.

4 Rewriting at fixed positions

Fix a chart and consistent books. Each relationship carries one booked coordinate $u_e \in \mathbb{R}$ (the source's asset entry; on matched books the destination's liability entry is $-u_e$). A negative coordinate is a claim running against the relationship's orientation: a standing relationship is an open line on which either party may end up owed, and the orientation only fixes the sign convention. The *position* of party i is the signed sum of the coordinates on its relationships, outgoing minus incoming; on consistent books it is the total on the party's designated accounts. A *rewrite* is a change $u \mapsto u + t$ that holds every position fixed; the set of such t is the cycle space $\mathrm{cyc}(G)$ of the graph G of standing relationships, so the reachable states form the affine *fibre* $u_0 + \mathrm{cyc}(G)$. Portfolio compression that may create exposure in either direction on an open line is a rewrite in this sense [5]; compression restricted to one direction per line is treated in Remark 4.8.

The two-record structure enters this section at one point, and it is the point Section 5 turns on. A standing relationship is a pair of designated accounts, one in each party's books, and it is an object of the chart whether or not it carries a balance. The graph G therefore includes the open lines that carry nothing, and $\mathrm{cyc}(G)$ depends on them. In a single-number model a link that carries nothing is not there.

Two readings of the coordinate. The calculus does not say what u_e measures, and two readings are used, never mixed. In the *amount reading* u_e is the money booked on e, a zero coordinate is nothing outstanding, and a position is a money position. In the *log reading* u_e is the logarithm of the booked figure relative to a per-relationship reference (par), a zero coordinate is par, and the fixed quantity at each party is the product of its ratios to par. The theorems below hold in both readings because they hold for every cost in the following class.

The hypothesis. The books here are consistent and u_e is a single agreed figure. We place on it a *carrying cost* of keeping the position open. That an open position carries a cost, and its form, are the hypotheses of every quantitative result of this section; the results are stated for the whole class so that nothing rests on the form except the numbers computed at particular members.

Definition 4.1 (Admissible costs; incident charge; shares)**.** A *per-relationship cost* is a function $c_e : I_e \to \mathbb{R}$ on an open interval $I_e \subseteq \mathbb{R}$, differentiable and strictly convex, with a *rest value* $\rho_e \in I_e$ at which $c_e(\rho_e) = 0$ is its minimum, and $c_e(u) \to \infty$ as u approaches either end of I_e. The cost of a state $u \in \prod_e I_e$ is $\sum_e c_e(u_e)$. Three members are used: the *quadratic cost* $c_e(u) = u^2/2$ on $\mathbb{R}$, rest 0 (amount reading); the *ratio cost* $c_e(u) = J(e^u) = \cosh u - 1$ on $\mathbb{R}$, rest 0 (log reading), where $J(x) = \frac{1}{2}(x + 1/x) - 1$; and $J(a/\rho_e)$ on $(0, \infty)$ with rest at par $\rho_e > 0$, which reads J on amounts. A *share vector* assigns the cost on each relationship to its two ends with shares summing to one; the *incident charge* of a party is the sum of its shares; *half shares* give each end one half.

A cost is *aligned* with given books if every relationship carrying nothing sits at its rest value and every relationship carrying a claim sits off it. In the amount reading alignment is the convention $\rho_e = 0$; in the log reading it is that par is the empty state. The rest values are otherwise free, and a cost whose rest on the debt line is the booked figure itself makes the booked state optimal and every result below trivial; the theorems that speak of empty relationships and bystanders are about aligned costs and say so. The third member above, $J(a/\rho_e)$ with par $\rho_e > 0$, is not aligned in the amount sense: a relationship at par carries ρ_e, and the party "at rest" there holds offsetting claims, not nothing.

Theorem 4.2 (Shares are free, the total is forced)**.** *For every share vector the sum over parties of the incident charges equals the network cost.*

Proof. $\sum_i \sum_e \alpha_{ie} c_e = \sum_e c_e \sum_i \alpha_{ie} = \sum_e c_e$. $\square$

Theorem 4.3 (One cheapest rewrite)**.** *For every admissible cost the state cost has exactly one minimiser on the part of the fibre inside the domain. The ratio cost dominates the quadratic cost and agrees with it to second order.*

Proof. $F = (u_0 + \mathrm{cyc}(G)) \cap \prod_e I_e$ is convex, nonempty and relatively open in the fibre. The cost is strictly convex on F and tends to ∞ along any sequence leaving every compact subset of F (some coordinate approaches an end of its interval; the other terms are bounded below by 0). So the infimum is attained, at exactly one point. $\cosh u - 1 \geq u^2/2$ with equality to second order is the Taylor expansion. $\square$

Theorem 4.4 (The cut-space datum)**.** *For every admissible cost:*

(a) the minimiser is the unique feasible u^* at which the marginal costs $(c'_e(u^*_e))_e$ lie in the cut space $\mathrm{cut}(G) = \mathrm{cyc}(G)^\perp$; given the books, the costs and their domains, it depends on the chart only through $\mathrm{cut}(G)$;

(b) adding a relationship at rest that is a bridge leaves the minimiser unchanged, for every books;

(c) adding a relationship at rest that closes a cycle C moves the minimiser iff the marginal costs at the old minimiser have nonzero circulation around C, $\sum_{e\in C} \pm c'_e(u^*_e) \neq 0$ with signs along the orientation of C; and for every such relationship there are books on which adding it at rest, which changes no exposure, moves the minimiser;

(d) reversing a relationship at rest keeps every exposure, never moves the minimiser when the relationship is a bridge, and moves it for some books when it closes a cycle.

Proof. (a) On a relatively open convex subset of an affine set the minimiser of a strictly convex differentiable function is where the gradient is orthogonal to the direction space, here $\mathrm{cyc}(G)$. For the quadratic cost $\nabla c(u) = u$ and u^* is the orthogonal projection of u_0 onto $\mathrm{cut}(G)$. (b) A bridge lies on no cycle, so $\mathrm{cyc}(G)$ is unchanged; the new coordinate sits at rest where its derivative vanishes, and the old minimiser still satisfies (a). (c) Adding e closing C enlarges $\mathrm{cyc}(G)$ by the span of the signed indicator χ_C. The old minimiser extended by $u_e = \rho_e$ has $\nabla c \perp \mathrm{cyc}(G_{\mathrm{old}})$ and $c'_e(\rho_e) = 0$, so it satisfies (a) for the new graph iff $\langle \nabla c, \chi_C\rangle = 0$. Existence: $C \setminus e$ is a path, not a cycle, so $\chi_{C\setminus e} \notin \mathrm{cyc}(G_{\mathrm{old}})$ and its projection g onto $\mathrm{cut}(G_{\mathrm{old}})$ is nonzero with $\langle g, \chi_{C\setminus e}\rangle = |g|^2 > 0$. Each $c'_{e'}$ is continuous, strictly increasing and zero at $\rho_{e'}$, so its range contains a neighbourhood of 0; scale g into those ranges and set $u_{e'} = (c'_{e'})^{-1}(g_{e'})$. Then $\nabla c(u) = g \in \mathrm{cut}(G_{\mathrm{old}})$, so u is its own minimiser by (a), and the circulation around C is $|g|^2 + c'_e(\rho_e) \neq 0$. (d) Reversal negates one column of the incidence matrix and changes no exposure; a bridge stays a bridge; a cycle-closing relationship changes $\mathrm{cyc}(G)$ and (c) applies. $\square$

Remark 4.5 (Closing a cycle is necessary, not sufficient). Quadratic cost, coordinates q and $-q$ on two relationships of a triangle, third added at rest: the fibre is $(q+c, -q+c, c)$, the derivative at $c = 0$ is $q - q + 0 = 0$, and the minimiser stays although a cycle is closed. The circulation decides, and here it vanishes.

Theorem 4.6 (The bystander is charged)**.** *On the running triangle, for every admissible cost aligned with the books (party 2's two relationships at rest, the debt relationship $0 \to 1$ off rest), the cheapest rewrite at fixed positions puts a nonrest coordinate on both relationships of party 2, who owed nothing and was owed nothing. Under any share rule giving each end of a relationship a positive share of its cost, party 2 is charged.*

Proof. Coordinates on $(0 \to 1, 1 \to 2, 2 \to 0)$; alignment gives $u_{0,1} \neq \rho_1$ and $u_{1,2} = \rho_2$, $u_{2,0} = \rho_3$. The cycle space is spanned by $(1,1,1)$; the fibre is $\{u_0 + (c,c,c)\}$ and the cost is $\Phi(c) = c_1(u_{0,1} + c) + c_2(\rho_2 + c) + c_3(\rho_3 + c)$. $\Phi'(0) = c'_1(u_{0,1}) \neq 0$, since a strictly convex differentiable function has derivative zero only at its minimum. So $c^* \neq 0$ and both of party 2's relationships leave rest, each with positive cost. This is Theorem 4.4(c): the circulation around the triangle at the booked position is $c_1'(u_{0,1}) \neq 0$. $\square$

The numbers, at half shares. Quadratic cost (amount reading, unit debt): $c^* = -\frac{1}{3}$, one third of the debt is routed the long way round, and party 2's charge is $2 \cdot \frac{1}{2} \cdot \frac{1}{2}(\frac{1}{3})^2 = \frac{1}{18}$. J on amounts, par 1, amounts $(4, 1, 1)$: minimise $J(4 + u) + 2J(1 + u)$ on $u > -1$; stationarity $3 = (4 + u)^{-2} + 2(1 + u)^{-2}$, root $u^* = -0.174$, charge $J(0.826) = 0.018$ (here party 2 stands at par, owing one unit and owed one unit; it is the party at rest, visible in the exposures, not the party with nothing outstanding). Ratio cost, $u_0 = (\log 4, 0, 0)$: $\sinh(\log 4 + c) + 2 \sinh c = 0$ gives $6e^c = \frac{9}{4}e^{-c}$, $e^{c^*} = \sqrt{3/8}$ exactly, every ratio to par is multiplied by $\sqrt{3/8}$, and the charge is $J(\sqrt{3/8}) = 0.123$. Three costs, one sign, one theorem: party 2 is reached because its two relationships close the cycle through which the cheapest rewrite routes part of the debt.

Proposition 4.7 (A kink at rest hides the bystander)**.** *Let the standing graph be a single cycle C and let each c_e be convex with one-sided derivatives at its rest value, $c_e'(\rho_e^+) = \kappa_e \geq 0$ and $c_e'(\rho_e^-) = -\kappa_e$ (a fixed cost κ_e of opening a position; $\kappa_e = 0$ is the differentiable case). Let S be the circulation around C of the marginal costs on the relationships that are off rest (each differentiable at its booked value), and K the sum of κ_e over the relationships at rest. Then the booked state is a cheapest rewrite iff $|S| \leq K$, and when the costs are strictly convex it is the unique one. In particular, on the running triangle with $c_e(u) = 3|u| + u^2$ and $u_0 = (1, 0, 0)$, $S = 5$ and $K = 6$, so party 2 is not reached; and under the gross-notional cost $c_e(u) = |u|$ the booked state is a cheapest rewrite whenever the cycle has at least as many at-rest relationships as off-rest ones.*

Proof. On a single cycle the fibre is one-dimensional, $\Phi(c) = \sum_e c_e(u_e + c)$, convex in c. Its right derivative at 0 is $S + K$ and its left derivative is $S - K$; a convex function of one variable is minimised at 0 iff $\Phi'(0^-) \leq 0 \leq \Phi'(0^+)$, that is $|S| \leq K$; strict convexity makes the minimiser unique. For the example, $c_1'(1) = 3 + 2 = 5$, $\kappa_2 = \kappa_3 = 3$. For $c_e = |u|$, each off-rest relationship contributes ± 1 to S and each at-rest one contributes 1 to K. $\square$

Remark 4.8 (One-sided lines). Restricting a line to carry a claim in one direction only, $u_e \geq 0$, is a domain restriction the admissible class excludes ($[0, \infty)$ is not open and the cost is finite at its end). On the running triangle with party 2's lines so restricted, every $c < 0$ is infeasible and every $c > 0$ raises the cost, so $c^* = 0$ and party 2 is not reached: a one-sided line is a kink of infinite size at rest. Conservative compression in the sense of D'Errico and Roukny [5], which forbids exposure on a line that had none, is this restriction on every empty line.

Compression as it is run minimises gross notional, and conservative compression forbids new exposure; both suppress the bystander by construction, the first by a kink and the second by a barrier. Under a cost that is smooth at rest with two-sided lines, spreading an open position across more relationships is always cheaper at the margin than concentrating it, and the bystander is reached exactly when the marginal costs circulate around a cycle its empty relationships close. Whether compression can harm a third party [7, 8] is therefore a question about the objective and the feasible set, not about the network. For smooth costs on two-sided lines the answer is yes, with the circulation criterion of Theorem 4.4(c) saying when.

5 What amounts cannot see

The pair. Four parties. In both systems party 0 is owed one unit by party 1 and nothing else is outstanding. In A the standing relationships are $0 \to 1$, $1 \to 0$, $2 \to 3$, $3 \to 2$: two two-cycles. In B they are $0 \to 1$, $1 \to 2$, $2 \to 3$, $3 \to 0$: one four-cycle. The exposures matrices are identical (a single 1 at $(0, 1)$) and so are the positions $(1, -1, 0, 0)$. Under the quadratic cost the fixed-position rewrite of A is $(\frac{1}{2}, -\frac{1}{2}, 0, 0)$: the debt is split across the antiparallel pair and parties 2, 3 are untouched. The rewrite of B is $(\frac{3}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})$: a quarter of the debt is routed through 2 and 3. Party 2's half-share charge is $\frac{1}{32}$ in B and 0 in A; it consents (its charge does not rise) in A and not in B. For every admissible cost aligned with the books the two rewrites differ: in B the circulation around the four-cycle at the booked state is $c'_{01}(u_{01}) \neq 0$, so by Theorem 4.4(c) the minimiser leaves party 2's lines off rest, while in A party 2's two-cycle is disconnected from the debt and its lines stay at rest.

Theorem 5.1 (Exposure functionals are cycle-blind)**.** *Every exposure functional takes the same value on any two (chart, books) with the same exposures matrix. On the pair every exposure functional agrees, and for every admissible cost aligned with the books the cheapest rewrite, the set of parties it moves, the half-share bystander charge and the consent verdict all differ. None of the four is an exposure functional, and none is a functional of positions.*

Proof. The pair has the same matrix and the same positions; the four quantities differ on it (computed above for the quadratic cost; for every aligned admissible cost by the circulation argument above). A functional that differs on two inputs with the same projection does not factor through it. ☐

Alignment is necessary here as in Theorem 4.6: a cost whose rest on $0 \to 1$ is the booked figure makes the booked state optimal in both systems and the four quantities agree. The theorem is about how the cost class treats empty lines, not about the class alone.

Theorem 5.2 (Where amounts are enough)**.** *On a fixed chart the exposures matrix determines every position and the fibre, hence the rewrite. On a chart whose standing graph is a forest the rewrite is determined by the matrix outright.*

Proof. Positions are row sums minus column sums. On a fixed chart the matrix determines each booked coordinate up to a redistribution among parallel relationships joining the same ordered pair, and such a redistribution lies in $\mathrm{cyc}(G)$ (two parallel relationships form a cycle); so the fibre $u_0 + \mathrm{cyc}(G)$ is determined, and with it the minimiser. On a forest $\mathrm{cyc}(G) = 0$, the fibre is a point, and the rewrite is the identity whatever the chart. ☐

An instrument that reads amounts is right about everything amounts determine: positions, the fibre, and on a forest the rewrite itself. It is silent about who is reached when a cycle of obligations is rewritten and whether a bystander consents; the datum it cannot record is the set of standing relationships that carry no balance, and Theorem 4.4 says that datum matters through the circulation around the cycles it closes. A model that also observes the standing graph is not blind to it, and the theorem says what that observation buys.

6 Firms, pruning, rent

Definition 6.1 (Coalition value)**.** Fix a network, maintenance $m_e \geq 0$ per relationship, and the quadratic cost. A relationship is *internal* to a coalition S if both ends lie in S. The *value* $v(S)$ is the saving S can realise on its own: the carrying cost of the state restricted to its internal relationships, minus the least carrying cost of any state supported on those relationships with the same positions at every party, minus the maintenance of the internal relationships. The *books' allocation* of the grand coalition gives each party the fall in its half-share carrying cost when every relationship is deleted, minus its half-share of the maintenance on its incident relationships. The *core* of v is the set of allocations summing to $v(N)$ that give every S at least $v(S)$ [17].

Theorem 6.2 (The books' allocation is outside the core)**.** *On the unit triangle with maintenance* $1/36$*,* $v(\{0,1,2\}) = 1/12$*, every pair has value* $-1/36$*, every singleton* 0*, the core is the simplex of nonnegative splits of* $1/12$*, and the books' allocation is* $(2/9,\ 2/9,\ -1/36)$*, outside the core: it sums to* $5/12$*, so is not efficient, and gives party* 2 *less than* $v(\{2\}) = 0$*. In general, whenever* *the fixed-position rewrite puts cost on some party's relationships, the books' allocation is not in the core of v.*

Proof. Grand coalition: internal state $(1,0,0)$ costs $1/2$; the cheapest state with the same positions is $(2/3,-1/3,-1/3)$, cost $1/3$; maintenance $3/36$; $v = 1/2 - 1/3 - 1/12 = 1/12$. A pair has one internal relationship, no rewrite is possible, $v = -1/36$. Books: party 0's half-share carrying cost before deletion is $1/4$, after 0, minus two half-shares of $1/36$: $2/9$; likewise party 1; party 2 carries nothing and pays $1/36$. General clause: the books' allocation sums to the freed carrying cost minus maintenance (Theorem 6.3), which exceeds $v(N)$ by the cost of the cheapest rewrite whenever that is positive. □

Theorem 6.3 (What consolidation is worth)**.** *Consolidating a group deletes its internal relationships. For any per-relationship cost vanishing at rest, the saving from consolidation equals the carrying cost freed on the deleted relationships. The internal-rewrite value of Theorem 6.2 is a different candidate (*$1/12$ *against* $5/12$ *on the triangle). With the ratio cost in place of the quadratic cost the triangle's books' allocation still lies outside the core.*

Proof. Deleting a relationship sets its coordinate to rest, where its cost is zero; the saving is the sum of the deleted costs. The triangle figures are those of Theorem 6.2. Under the ratio cost the rewrite of Theorem 4.6 has positive cost, so the general clause of Theorem 6.2 applies. □

Definition 6.4 (Pruning)**.** Fix positions b, maintenance $m_e \geq 0$ and an admissible cost. A kept-set g of relationships is *admissible* if it can carry b. A party's *burden* in g is its half-share incident cost at the cheapest rewrite on g plus its half-share maintenance. One *pruning step* cuts a relationship whose removal keeps admissibility and lowers an endpoint's burden, or adds one that one endpoint strictly prefers and the other does not oppose. A kept-set is *pairwise stable* [16], with deviations restricted to admissible sets, if no such cut or add exists.

Theorem 6.5 (Pruning)**.** *Which admissible kept-set is used is not determined by the positions: two different kept-sets can carry the same positions. For every admissible cost the pruning dynamics stops exactly at the pairwise-stable kept-sets, and a party with zero position and exactly two relationships on a maintained cycle is always cut. The order of cuts selects which stable kept-set is reached.*

Proof. On the triangle with positions $(1,-1,0)$, the relationship $0 \to 1$ alone, and the other two with coordinate -1 each, both carry the positions. A pruning step exists iff some cut or add of the stated kind exists; the negation is pairwise stability with the same add clause. Pass-through: a party with zero position and two relationships on a maintained cycle carries the same coordinate in and out; cutting one keeps admissibility, removes its share of that relationship's cost and a positive maintenance, and adds nothing. History dependence: two cut orders from the full triangle reach two stable kept-sets on the planted instance. □

Proposition 6.6 (Two counterexamples, quadratic cost)**.** *The inequality "the private kept-set carries at most the cycle content of the efficient one" fails on a four-party instance, and a pairwise-stable forest need not exist even when forests are admissible (five-party instance).*

Proof. Four parties, all six relationships available, positions $(-2,1,-1,2)$, maintenance $(1/100, 1/15, 1/9, 1/7, 1/8, 1/15)$ on $(01, 02, 03, 12, 13, 23)$. The four-cycle $\{01, 03, 12, 23\}$ is pairwise stable with cycle rank 1: its fixed-position rewrite is $(-3/4, -5/4, 1/4, -3/4)$, and cutting 12 raises the burden of both its endpoints ($1303/5600$ to $51/200$ and $877/3360$ to $17/60$) although it lowers the total by $1/56$. The efficient kept-set is a forest, cycle rank 0. So the private set carries more cycle content than the efficient one. The same instance has five pairwise-stable sets (the four-cycle and four forests), and the efficient tree is not among them. Five parties, relationships $\{02, 03, 12, 13, 24\}$, positions $(2, 2, -2, -1, -1)$, maintenance

$(41/75, 31/360, 2/15, 1/20, 86/105)$: party 4 hangs off party 2 by the bridge 24 and the other four sit on the cycle 0-2-1-3-0. The admissible kept-sets are the four spanning trees and the whole graph. Each tree is broken by adding its missing cycle relationship, both endpoints strictly gaining, so no forest is stable; the whole graph is. Every number is a rational computed from the instance; the burdens are the half-share quadratic costs at the certified minimiser plus half-share maintenance. Under the ratio cost the same two instances are set up and the verdicts persist numerically; the sign certificates there are not in closed form. □

Theorem 6.7 (Rent)**.** *A rent on a relationship is a contract term: two schedules on one corridor are both consistent with the books. For every schedule the party stays iff the rent covers its incident charge plus its maintenance; in the log reading the corridor charge on a flow of log-ratio v is $\cosh v - 1$, so the survival condition is $r|v| \geq m + \cosh v - 1$. A nonpositive rent with positive maintenance exits. Nothing in the books bounds the rent by the freed carrying cost.*

Proof. A schedule is a function of the flow the books do not constrain. Staying is defined as rent at least burden; burden is incident charge plus maintenance; at half shares one relationship carrying v charges $\frac{1}{2}(\cosh v - 1)$ at each end, $\cosh v - 1$ for the corridor. With $r \leq 0$, $m > 0$ the condition fails. No theorem above mentions the rent. □

7 The disagreement index

Sections 4 to 6 hold for every admissible cost. This section asks which cost is singled out by three properties one may want of an index of disagreement between two records: it does not care which party is called the creditor; it obeys the same law at every place on the scale; and it adds over relationships. The first two are stated as a functional equation, the third as an axiom on the network index.

Let $F(x)$ be the index assigned to two figures that stand in the ratio $x > 0$: the two records of a disputed claim (Section 3), or an agreed figure against its par (Sections 4 to 6, where par plays the role of the second record). The axioms concern F as a function of the ratio, and the composition they refer to is composition of ratios: if A stands to B in ratio x and B to C in ratio y, then A stands to C in ratio xy when the second comparison runs the same way as the first and x/y when it runs the other way. Nothing in the chart need realise the three figures at once; the axioms are on the index, as in the inequality literature they are on the measure.

Postulate 7.1 (Axioms on the index)**.** (A1) *No orientation.* The index does not know which way a link points, so its law fixes only the sum $F(xy) + F(x/y)$ over both orientations.

(A2) *Same law at every scale.* Writing $x = e^s$, $y = e^t$, the summed effect of a step t from position s factors into a function of position and a function of step: after an affine normalisation $F(e^{s+t}) + F(e^{s-t})$ is a product $H(s)H(t)$. Concretely there is $\kappa > 0$ with

$$F(xy) + F(x/y) = \kappa\, F(x)F(y) + 2F(x) + 2F(y).$$

(A3) *Nondegeneracy and monotonicity.* F distinguishes some two ratios, and $t \mapsto F(e^t)$ is monotone on $[0, \infty)$.

(A4) *Additivity.* The index of a network is the sum over relationships of the index of each.

Theorem 7.2 (The index of a ratio)**.** *Under (A1) to (A3) there is $\lambda \neq 0$ with $\frac{\kappa}{2}F(x) = J(x^\lambda)$, $J(x) = \frac{1}{2}(x + 1/x) - 1$. In particular $F(x) = F(1/x)$, $F(1) = 0$, and $F > 0$ away from 1.*

Proof. Put $G = \frac{\kappa}{2}F$ and $H(t) = 1 + G(e^t)$. Multiplying the law by $\kappa/2$ and adding 2 to both sides gives $H(s+t) + H(s-t) = 2H(s)H(t)$, d'Alembert's equation. Setting $s = t = 0$ gives $H(0) \in \{0, 1\}$; $H(0) = 0$ forces $H \equiv 0$, against nondegeneracy; so $H(0) = 1$, and setting $s = 0$ gives $H(t) = H(-t)$, which is $F(x) = F(1/x)$ and $F(1) = 0$. Regularity: H is monotone on

$[0,\infty)$ and even, hence measurable, and every measurable solution of d'Alembert's equation is continuous [20]. The continuous solutions are $H \equiv 0$, $\cosh(\lambda t)$, $\cos(\lambda t)$; monotonicity on $[0,\infty)$ excludes the cosine for $\lambda \neq 0$ and nondegeneracy excludes $\lambda = 0$. So $G(e^t) = \cosh(\lambda t) - 1 = J(e^{\lambda t})$. ☐

The exponent λ is the scale of the log-ratio coordinate and the constant κ is a unit; neither changes an ordering, a minimiser or a zero set. At $\kappa = 0$ the same law is solved exactly by $F(x) = (\log x)^2$, since $(s+t)^2 + (s-t)^2 = 2s^2 + 2t^2$: the quadratic cost in the log coordinate is the member of the family in which the law has forgotten position, which is why it served as the comparison cost above.

Theorem 7.3 (The network index)**.** *Under (A1) to (A4) the index of a network is $\sum_e J\big((r_0/r_1)_e^\lambda\big)$ up to the unit κ, and it is invariant under relabelling the parties and reversing every relationship.*

Proof. (A4) gives the sum; Theorem 7.2 gives each term; evenness gives the invariance (Theorem 3.5). ☐

Axiom (A4) carries weight and it is worth saying what it excludes: any index that is not a sum over relationships, the position of the worst-off party (maximin), the product of positions (the Nash product), any functional with a term coupling two relationships. Additivity across relationships plays the role that decomposability plays in the inequality literature [18, 19], and it is the axiom a reader who wants those objectives must reject.

Corollary 7.4 (Objectives on positions or exposures are cycle-blind)**.** *Every objective that is a function of the parties' positions or of the exposures matrix is an exposure functional and, by Theorem 5.1, takes the same value on two networks that differ on who is reached by a fixed-position rewrite and who consents to it.*

The index of Theorem 7.3 is not an exposure functional: on the pair of Section 5 it differs after the rewrite. It is the total disagreement of the network read from both records, and it sees the bystander because the bystander's relationships enter it.

8 Where two records come from

Every result above holds on given books. This section gives a model in which the two records, and the free-account balance that absorbs a revaluation, arise rather than being posited. Only the first theorem below uses the model; the buffer and frozen theorems are pure bookkeeping.

Postulate 8.1 (Located units)**.** Value consists of indivisible units, each located at every period in one cell; a book at a cell records the units in that cell. A unit moves at most one cell per period and at most one unit moves per period.

Under the postulate a transfer from the cell of book α to the cell of book β is one unit that α records at period t and β records at period $t+1$: a fall in α's count and a rise in β's across the same period are the same unit, since at most one moves.

Theorem 8.2 (Never simultaneous)**.** *Under Postulate 8.1, at no period do both books record a unit in transfer between them. Read as books on a chart with one relationship $\alpha \to \beta$ and one free account each, the transfer is the one-sided state at t (source recorded, destination not) and the opposite one-sided state at $t+1$; it is never matched, and both states are balanced within each party.*

Proof. At one period a unit is in one cell, so two books that both record it are the same book. With $\alpha \neq \beta$, β does not record at t and α does not at $t+1$. On the one-relationship chart the source's asset carries x and the destination's liability 0 at t, and 0 and $-x$ at $t+1$; neither sums to zero when $x \neq 0$. Each party's free account carries the counter-entry, so each is balanced. ☐

A matched pair of books is therefore a relation between *kept* records with different stamps, never a simultaneous state of the underlying transfers; the two-record structure is the memory of the transfers, and parties must keep records to match at all. The one-sided state of Theorem 2.4(c) is not merely representable: every transfer passes through it.

Theorem 8.3 (The buffer is the residue)**.** *Call the total on a party's designated accounts its* own net*: its own record of what it is owed less what it owes (on consistent books this is its position). For a party whose books balance, the total on its free accounts is minus its own net. A party with a nonzero own net holds a nonzero balance on some free account. When every party balances, the free-account totals of all parties sum to zero exactly when the counterpart gaps sum to zero.*

Proof. Balance makes the free total the negative of the designated total. The designated accounts of a party correspond one to one with the relationships incident to it, an asset account per relationship leaving and a liability account per relationship entering, no account both (Definition 2.1); so summing own nets over parties counts each relationship's two entries once each, which is the sum of counterpart gaps. □

The free-account balance that absorbs a revaluation is not a free parameter of the model: it is fixed by the party's own record of its net, and the buffers of all parties add to zero exactly when the network's gaps cancel in aggregate.

Theorem 8.4 (No free account, no posting)**.** *A party every account of which is designated cannot post: its books are unchanged along every balanced word.*

Proof. A balanced journal entry vanishes on designated accounts, so on such a party it is zero. A cross posting needs a free counter account at each end, so such a party is neither end. Induct along the word. □

Proposition 8.5 (Rate bound)**.** *Under Postulate 8.1, at most one transfer occurs in the whole system per period, and over N periods at most N.*

Proof. At most one unit moves per period. □

The postulate supplies the transfer and its two stamps, not the magnitude of a unit; the size of the first claim is outside the model.

9 What is chosen, and what is open

The variables the model leaves to the parties are listed with the theorem stating what holds for every value of each.

Variable	Status	What holds for every value
Cost of an open position	Hypothesis	Every result of Sections 4 to 6 holds for every admissible cost, aligned with the books where empty lines are spoken of; the index axioms of Section 7 single out J.
Shares of a cost	Free	Theorem 4.2: the total is forced.
Split of a revaluation	Free	Theorem 3.1: the sum is the gap.
Outside options	Free	Theorem 3.2: the books supply two figures; the disagreement outcome is exogenous.
Point in the interval	Free (known)	Remark 3.3.
Characteristic function	Two candidates	Theorems 6.2, 6.3: books' allocation outside the core under both.
Kept relationships	Free	Theorem 6.5: dynamics stops exactly at pairwise-stable sets, for every cost.
Rent schedule	Free	Theorem 6.7.
Free-account balance	Forced	Theorem 8.3: minus the position.

Three things are open. Under the ratio cost the two sign verdicts of Proposition 6.6 are numerical, not closed form. The circulation criterion of Theorem 4.4(c) decides whether the minimiser moves; which parties it moves onto, on a general graph, is determined by the cut space but not yet characterised in closed form beyond the single-cycle case of Proposition 4.7. And the model of Section 8 fixes the form of the first claim and not its size.

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