Article

Rigidity of Conserved Comparison Ledgers: Structural Axioms Force the Scale, the Cost, and the Ratio

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Abstract

A *conserved comparison ledger* is an abstract system in which each state carries a positive ratio $r(s)$, every measurement factors through that ratio, and pairwise comparisons are scored by an admissible cost J. A separate classification fixes the admissible costs with polynomial combiner and, under a units normalization, selects the representative

$$J_{\text{cost}}(x) = \tfrac{1}{2}(x + x^{-1}) - 1.$$

This paper installs that classified cost in a ledger and determines what the ledger assumptions force. The argument has three structural steps and one neutrality axiom. H1 forces the scale ratio to $\sigma = \varphi$. H2, using that classification, forces $J = J_{\text{cost}}$. H3 reads $r(s)$ from a constrained cost minimizer of an internal positive vector, and A3 imposes zero total log-charge; together they force $r \equiv 1$, make the measurement map constant, and collapse the observational quotient to a singleton. The assumptions are also sharp in the following sense: removing H1, H2, H3, or A3 admits an explicit ledger satisfying the remaining meaningful assumptions while losing the corresponding conclusion. The examples verify joint satisfiability, include non-uniform internal data whose variational minimum gives $r = 1$, and show nontrivial sector dynamics when A3 is not imposed.

Keywords: Conserved comparison ledger; Rigidity theorem; Golden ratio; Admissible reciprocally symmetric cost; Jensen inequality; Conserved log-charge; Observational quotient.

1. Introduction

Subject and background

A *conserved comparison ledger* is a minimal setting for one question: can the structural assumptions of a comparison framework determine the framework's own distinguished constants, instead of leaving them as free parameters? This paper answers the question for conserved comparison ledgers: a short list of structural conditions forces the scale ratio, the cost, and the internal ratio to fixed values.

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Related work

The ingredients of this question have long, independent literatures. The admissible costs are reciprocally symmetric solutions of a composition (functional) equation, in the d'Alembert–Wilson–Kannappan tradition of functional equations on groups and semigroups [1–5]. The forced scale is the golden ratio, which arises here as it does in the rigidity of Fibonacci-type recurrences and self-similar sequences [6–10]; the selection of a single limiting cost is, concretely, the collapse of the two-parameter family of admissible combiners $P(u,v) = cuv + 2u + 2v$ to the single representative J_{cost} once the units normalization is imposed: a normalization condition reduces a parameter family of structures to one distinguished member. This is the structural pattern of Inönü–Wigner contraction, where a singular limit of a parameter family of Lie algebras selects a distinguished (degenerate) algebra, and of symmetric-space rigidity, where fixing the normalizing data leaves no nontrivial deformation [11–16]. The variational step uses strict convexity and Jensen's inequality, in the same mathematical setting as quasi-arithmetic means, to identify the constrained minimizer; H3 then reads the ratio from that minimizer, and A3 sets the charge to zero [17–23], and its dynamical form is the standard gradient-flow, Bakry–Émery, and optimal-transport circle of ideas, with information-geometry counterparts [24–37]. Finally, the conserved log-charge and its level sets behave like a conserved charge whose value no measurement can change: it splits the states into classes that no measurement can tell apart, as in Noether's theorem, superselection theory, and the analysis of observational equivalence [38–51]. These literatures supply only the conceptual setting. The proofs use a quadratic recurrence (B1), the discrete Jensen inequality with convex-analytic existence of constrained minimizers (B3) [17,18], a fiber decomposition, and the classification imported from [52]—together with Picard–Lindelöf and Grönwall for the dynamical interpretation only. The remaining citations are conceptual analogues; no result depends on them.

The ledger also sits near three abstract comparison frameworks, sharing tools with them but differing in aim. Foundations of measurement [53,54] and axiomatic utility/decision theory [55,56] ask when a comparison or preference admits a numerical representation and how unique it is; the theory of means and aggregation functions [19,20,57] characterizes how several values combine into one (the source of our Jensen step). All three concern the *existence* and residual freedom of a representation. Here the cost is already classified in [52], and we instead ask when the surrounding structure removes *all* residual freedom — in measurement-theoretic terms, when the ledger axioms collapse the group of admissible transformations to the identity. The conserved-charge and dynamical ingredients have no counterpart in those theories: the conservation law on $\log r$ and the evolution map T_F are additional structure with no analogue in measurement, utility, or aggregation theory.

Prior classification of admissible costs

Reference [52] classifies the admissible costs. An *admissible cost* is a reciprocally symmetric cost J whose *combiner* P aggregates pairwise comparisons (both made precise in Section 2). When the combiner is polynomial, that work proves it is bilinear, classifies the admissible costs into two families, and — under a units normalization — selects the canonical cost

$$J_{\text{cost}}(x) = \tfrac{1}{2}\left(x + x^{-1}\right) - 1$$

[52, Thms. 3.2, 5.2, 7.6 and Cor. 9.1]. That paper is purely functional-equation-theoretic: it selects the cost but does not place it inside any ambient system. Consequently, every cost-forcing result in the present paper is conditional on the four imported theorems from [52]. The present paper does not reprove automatic factorization, the polynomial degree bound, the two-family classification, or canonical-cost uniqueness; it uses them as black-box analytic input and studies what they imply after the cost is installed in a ledger. Two companion papers complement this classification: [58] establishes uniqueness of the canonical reciprocal cost, and [59] proves the d'Alembert inevitability theorem underlying the governing functional equation.

The four imported theorems are stated in full in Section 2, so the present paper can be verified by taking them as axioms.

Contributions of this paper

The present paper installs the cost classified in [52] into a conserved comparison ledger and determines what the ledger's own assumptions force. The analysis rests on three structural hypotheses (H1–H3) and one modeling axiom (A3): H1 fixes the scale geometry, H2 fixes the cost through the combiner classification of [52], H3 ties the ratio to a constrained variational minimum, and A3 imposes charge neutrality.

Under H1 and H2 alone, the scale and the cost are uniquely forced,

$$\sigma = \varphi, \qquad J = J_{\text{cost}}(x) = \tfrac{1}{2}(x + x^{-1}) - 1.$$

Adding H3 and A3 collapses the ratio interface and the observational quotient,

$$r(s) = 1, \qquad \mu_F \text{ constant}, \qquad \mathrm{SQ}(F) \simeq \mathbf{1}.$$

When A3 is dropped, the same hypotheses partition the state space into the level sets of the conserved log-charge $\log r$, each invariant under the evolution; A3 is precisely the condition selecting the neutral level set $\log r = 0$. The condition package is irredundant: dropping any one of H1, H2, H3, A3 admits a ledger satisfying the rest yet violating the corresponding conclusion. Here A3 is a constraint on the internal-vector data introduced by H3, so H3 and A3 are not independent; the precise sense in which each condition is dropped is given in Section 7. Section 9 confirms joint satisfiability with several explicit ledgers, including one in which $r = 1$ is the genuine output of a single variational step on non-uniform data, and one with an injective observable for which the collapse $\mathrm{SQ}(F) \simeq \mathbf{1}$ is non-vacuous.

Scope, novelty, and significance

The novelty is structural, not analytic: we isolate the conserved comparison ledger and show that one short list of independently motivated conditions fixes its three distinguished quantities (σ, J, r) simultaneously. Each step is elementary — a Fibonacci recurrence (B1), substitution into the classification of [52] (B2), and the discrete Jensen inequality (B3); the contribution is precisely that these independently standard ingredients assemble into a sharp, irredundant rigidity theorem — each hypothesis is necessary, and together they fix (σ, J, r) uniquely. The conditions are by design chosen so each controls one output, so the theorem is a *coherence* statement (a small set of jointly irredundant, individually natural requirements determines the object completely) rather than a coincidence. We claim no natural realization: the examples of Section 9 establish only non-vacuity, and whether a non-constructed system satisfies H1–H3 and A3 is the principal open problem (Section 10).

Why these hypotheses?

Each condition in the package is independently motivated; none is chosen retroactively to match a desired conclusion.

H1 (scale geometry). Uniform geometric growth and additive self-similarity are structurally independent properties of a scale sequence. Each alone is compatible with any ratio $\sigma > 1$; jointly, they are compatible only when $\sigma^2 = \sigma + 1$, whose unique positive root greater than 1 is φ. The golden ratio is not imposed; it is the only value for which both structural properties of (σ, ℓ) can coexist. The sharpness result confirms that dropping H1 immediately admits $\sigma \neq \varphi$.

H2 (polynomial compositionality and calibration). Polynomial compositionality is a convenient sufficient condition rather than a regularity assumption: it says the cost's comparative information is aggregated by a finite algebraic rule, which lets the classification of [52] be imported as a black box (Section 5). Its normalization and calibration are unit conventions that pin that classification to the canonical representative J_{cost}. Each sub-condition is load-bearing: dropping any one leaves a free parameter — relaxing the normalization alone, for instance, permits $J(x) = (\ln x)^2$.

H3 (variational origin of the ratio). H3 distinguishes an *equilibrium* ratio — one that minimizes cost subject to a fixed log-charge constraint — from an arbitrary assignment. Without this mechanism, r carries no variational content and cannot be controlled by A3; a ledger satisfying H1 and H2 may then carry $r \equiv 2$ simply because r was assigned by hand. As motivation (not a theorem): once J governs all comparisons, a natural permutation-symmetric, cost-based way to summarize a multi-channel state $x(s) \in \mathbb{R}^{N_s}_{>0}$ by one scalar is to replace it by the least-cost configuration at fixed log-charge, the constrained minimizer $T_{N_s}(x(s))$. Other summaries are possible; the point is that this one is permutation-symmetric and cost-based, and it turns out to be the uniform configuration. Aggregation-function theory [20,57] suggests one could go further and *derive* H3 from such aggregation axioms (symmetry, cost-monotonicity, and a consistency condition under refinement); we take the constrained-minimizer form as a hypothesis here and record the axiomatic derivation as a natural strengthening (Section 10).

A3 (charge neutrality). The log-sum-zero condition $\Lambda(x(s)) = 0$ is the constraint used here; it places the Jensen equilibrium at the uniform configuration $(1, \ldots, 1)$ and hence forces $r(s) = 1$. Any nonzero log-sum Q places the equilibrium at $(e^{Q/N}, \ldots, e^{Q/N})$ with $r(s) = e^{Q/N} \neq 1$. Section 8 describes the resulting level-set structure in full.

Notation

We fix the following notation, used throughout the paper. We write $\mathbb{R}$ for the reals, $\mathbb{C}$ for the complex numbers, $\mathbb{R}_{>0} := (0, \infty)$, $\mathbb{R}_{\geq 0} := [0, \infty)$, and $\mathbb{N} := \{0, 1, 2, \ldots\}$ (including 0). The golden ratio is $\varphi := \frac{1+\sqrt{5}}{2}$, the unique positive real root greater than 1 of $\sigma^2 = \sigma + 1$. A *conserved comparison ledger* $F = (S_F, T_F, O_F, \mu_F, r, \bar{\mu}_F, J)$ consists of a countable state space S_F, an evolution map $T_F \colon S_F \to S_F$, an observable set O_F, a measurement map $\mu_F \colon S_F \to O_F$ factoring as $\mu_F = \bar{\mu}_F \circ r$ through the ratio interface $r \colon S_F \to \mathbb{R}_{>0}$, and an admissible cost $J \colon \mathbb{R}_{>0} \to \mathbb{R}$; $\mathrm{LogCharge}(s) := \log r(s)$ is the *state-level log-charge*, conserved by T_F. For a vector $c = (c_1, \ldots, c_N) \in \mathbb{R}^N_{>0}$ we write $\Lambda(c) := \sum_{i=1}^{N} \log c_i$ for its *vector-level log-charge*; under hypothesis H3 the two are related by $\mathrm{LogCharge}(s) = \Lambda(x(s))/N_s$. The log-substitution of the cost is $G(t) := J(e^t)$, with evenness $G(t) = G(-t)$ from reciprocal symmetry; $P \colon \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ is the unique continuous *combiner* of J delivered by [52], and $J_{\text{cost}}(x) := \frac{1}{2}(x + x^{-1}) - 1$ is its distinguished representative. A hierarchical structure is (σ, ℓ); under H1, $\sigma = \varphi$.

Method and outline

It is convenient to keep the three forcing steps separate, since each draws on different structural input. *Step B1* fixes the scale from H1 alone and is purely algebraic (Section 4). *Step B2* fixes the cost from H2 together with the classification imported from [52], and is the only step that depends on it (Section 5). *Step B3* fixes the ratio interface from H3 and A3 through a single constrained-minimization (Jensen) argument (Section 6). The dependence is one-directional: B3 uses the cost fixed in B2, whereas B1 and B2 neither use nor are affected by H3 or A3. This layering is what lets the scale-and-cost rigidity and the full collapse stand as separate theorems, and what makes the A3-free analysis of Section 8 a clean restriction of the same argument rather than a different one.

The paper is laid out along the proof sequence introduced above: inheritance from [52], ledger setup, the three rigidity layers B1/B2/B3, the main rigidity theorems, the no-A3 alternative, and the examples. Section 2 records the precise analytic input from [52], together with a basic nonnegativity lemma that justifies the domain of the combiner. Section 3 defines frameworks, ledgers, the observational quotient, and the foundational axioms. Section 4 establishes scale forcing under H1 in both exact and asymptotic forms. Section 5 applies the imported classification to force the canonical cost under H2. Section 6 develops the Jensen minimizer and the collapse $r \equiv 1$; the Lyapunov and gradient-flow refinements are deferred to Section A. Section 7 assembles the A3-free and full rigidity theorems and proves the condition package irredundant (dropping any one condition breaks a specific conclusion). Section 8 develops the level-set structure of $\log r$ when A3 is dropped and identifies A3 as the condition picking the level set $\log r = 0$. Section 9 gives several examples: a canonical static ledger, an N-channel ledger with non-uniform internal data, an explicit $N = 2$ cost-drop calculation, and a ledger with an injective observable exhibiting both observational collapse under A3 and a nontrivial multi-sector quotient when A3 is dropped; a further example exhibits non-trivial conservative dynamics that exercise the conservation law and evolution map. A final finite-sector example shows that nontrivial invariant level sets can already occur with two states and period-two evolution. Section 10 recapitulates the proof sequence layer-by-layer, restates the three forced quantities and their two corollaries, and situates the present paper relative to the classification of [52].

2. Imported Analytic Input from Prior Work

This section records the analytic input taken from [52]. It states, without proof, the four results from [52] that the present paper uses as black-box inputs: they form a logical chain — automatic factorization produces the combiner, the polynomial degree bound forces it to a bilinear form, the two-family classification identifies all admissible costs compatible with that bilinear combiner, and canonical-cost uniqueness selects a single representative once normalization and calibration are imposed. The four theorems below are imported from [52]; Theorem 2.3 is an elementary consequence proved here. Later sections also use standard external facts, cited locally, such as existence of constrained convex minimizers, Jensen's inequality, and Grönwall's estimate.

Definition 2.1 (Admissible cost, [52, Def. 2.1])**.** *A function $J\colon \mathbb{R}_{>0} \to \mathbb{R}$ is an* admissible cost *if it satisfies the following three conditions.*

1. *Reciprocal symmetry: $J(x) = J(x^{-1})$ for all $x > 0$.*
2. *Unit normalization: $J(1) = 0$.*
3. *Strict log-convexity: the log-substitution $G(t) := J(e^t)$ is strictly convex on $\mathbb{R}$ (equivalently, J is strictly convex as a function of the logarithmic coordinate $t = \ln x$ on $\mathbb{R}_{>0}$).*

Remark 2.2 (Continuity is automatic)**.** *Continuity of J on $\mathbb{R}_{>0}$ is automatic and is therefore not listed as an axiom: convexity of G implies continuity of G on $\mathbb{R}$ ([18, Thm. 1.3.3]; [52, Remark following Def. 2.1]), hence $J(x) = G(\ln x)$ is continuous on $\mathbb{R}_{>0}$.*

Before introducing the combiner, we record a basic consequence of admissibility that underpins the rest of the section: every admissible cost takes nonnegative values. This fact justifies the domain $[0, \infty)^2$ of the combiner P in the automatic factorization theorem below, and will be used again in Section 6.

Lemma 2.3 (Nonnegativity of admissible costs)**.** *If J is an admissible cost and $G := J \circ \exp$, then G is even and*

$$G(t) \geq 0 \text{ for all } t \in \mathbb{R}, \qquad G(t) > 0 \text{ for } t \neq 0.$$

Equivalently,

$$J(x) \geq 0 \text{ for all } x > 0, \qquad J(x) = 0 \iff x = 1.$$

Proof. Evenness follows from the symmetry axiom: $G(-t) = J(e^{-t}) = J\big((e^t)^{-1}\big) = J(e^t) = G(t)$. Also $G(0) = J(1) = 0$. For $t \neq 0$, strict convexity applied to the distinct points t and $-t$ gives

$$G(0) = G\left(\frac{t + (-t)}{2}\right) < \frac{G(t) + G(-t)}{2} = G(t),$$

so $G(t) > 0$ for $t \neq 0$ and hence $G(t) \geq 0$ for all t. Since $J(x) = G(\log x)$, the stated properties of J follow. □

Theorem 2.4 (Automatic factorization, [52, Automatic Factorization Theorem, Thm. 3.2])**.** *For every admissible cost J there is a unique continuous function $P\colon [0,\infty)^2 \to \mathbb{R}$ such that*

$$J(xy) + J(x/y) = P\big(J(x), J(y)\big) \qquad (x, y > 0).$$

The combiner P is symmetric and satisfies $P(u,0) = 2u$ and $P(0,v) = 2v$.

Theorem 2.5 (Polynomial combiner classification, [52, Continuous degree bound, Thm. 5.2; bilinear-form corollary, Cor. 5.5])**.** *If an admissible cost has a polynomial combiner P, then*

$$P(u,v) = cuv + 2u + 2v$$

for some $c \geq 0$.

Theorem 2.6 (Two-family classification, [52, Full classification, Thm. 7.6])**.** *Let J be an admissible cost with polynomial combiner. Then exactly one of the following holds:*

(i) Hyperbolic family:

$$J(x) = \frac{1}{c}(x^{\lambda} + x^{-\lambda}) - \frac{2}{c} \qquad (c, \lambda > 0).$$

(ii) Degenerate quadratic family:

$$J(x) = a(\ln x)^2 \qquad (a > 0).$$

Theorem 2.7 (Canonical cost uniqueness, [52, Canonical cost, Cor. 9.1])**.** *If an admissible cost satisfies $P(u,v) = 2uv + 2u + 2v$ and the calibration convention $(J \circ \exp)''(0) = 1$, then*

$$J(x) = J_{\text{cost}}(x) := \tfrac{1}{2}(x + x^{-1}) - 1.$$

3. Frameworks, Ledgers, and Axioms

Having recorded the analytic input from [52], we now establish the abstract objects in which that input will be installed. We begin with the general notion of a framework, then specialize to the conserved comparison ledger, which adds four further requirements: countability of the state space, a ratio interface r through which all observables are read, an admissible cost governing comparisons, and a conservation law on the log-charge. The foundational axioms (A1)–(A2) assert that the state set is nonempty and that the object satisfies the ledger definition. The structural hypotheses H1–H3 and the modeling axiom A3 are introduced separately, in the sections where they are used. Keeping them out of this section—where the definitions fix only the ambient object—prevents the main theorem from being built into the definition of a ledger.

Definition 3.1 (Framework)**.** *A* framework *is a tuple $F = (S_F, T_F, O_F, \mu_F)$ where S_F is a set of states, $T_F \colon S_F \to S_F$ is an evolution map, O_F is a set of observables, and $\mu_F \colon S_F \to O_F$ is a measurement map.*

Two states are observationally indistinguishable when the measurement map assigns them the same value. The observational quotient encodes this identification globally.

Definition 3.2 (Observational quotient)**.** *For a framework F, define $s_1 \sim_F s_2$ if $\mu_F(s_1) = \mu_F(s_2)$. The* observational quotient *is*

$$\mathrm{SQ}(F) := S_F/{\sim_F}.$$

We say the quotient collapses *if $\mathrm{SQ}(F)$ is a singleton.*

Lemma 3.3 (Constant measurement gives a singleton quotient)**.** *If $S_F \neq \emptyset$ and μ_F is constant on S_F, then $\mathrm{SQ}(F) \simeq \mathbf{1}$ (i.e., $\mathrm{SQ}(F)$ contains exactly one element).*

Proof. A constant measurement map places every pair of states in the same equivalence class under $\sim_F$, so $S_F/{\sim_F}$ is a singleton. ☐

Remark 3.4 (Notation $\mathrm{SQ}(F) \simeq \mathbf{1}$)**.** *Throughout the paper, $\mathbf{1}$ denotes a fixed one-element set, and $\simeq$ denotes bijection.*

The ledger specializes the framework by requiring that all observables be read through a single positive ratio variable, that a cost function govern comparisons, and that the log-charge of the ratio variable be preserved by the dynamics.

Definition 3.5 (Conserved comparison ledger)**.** *A* conserved comparison ledger *is a tuple $F = (S_F, T_F, O_F, \mu_F, r, \bar{\mu}_F, J)$ in which (S_F, T_F, O_F, μ_F) is a framework and the additional data satisfy:*

1. *S_F is countable;*
2. *$r \colon S_F \to \mathbb{R}_{>0}$ is a distinguished* ratio interface *and $\bar{\mu}_F \colon \mathbb{R}_{>0} \to O_F$ is a distinguished* reduced measurement map, *with $\mu_F = \bar{\mu}_F \circ r$;*
3. *$J \colon \mathbb{R}_{>0} \to \mathbb{R}$ is a chosen admissible cost;*
4. *for every $s \in S_F$,*

$$\mathrm{LogCharge}\big(T_F(s)\big) = \mathrm{LogCharge}(s), \qquad \mathrm{LogCharge}(s) := \log r(s).$$

The ratio interface r and the reduced measurement map $\bar{\mu}_F$ are part of the ledger's data, not merely asserted to exist; all subsequent ledger-level statements refer to this distinguished pair.

Condition (1) (countability of S_F) is a modeling convention, not an analytic ingredient: the proofs in Section 6 operate on the finite-dimensional internal vector $x(s) \in \mathbb{R}^{N_s}_{>0}$ state by state and never sum or integrate over S_F. Concretely, the proofs are pointwise in s and use only the fiber decomposition of $s \mapsto \log r(s)$; the sole place cardinality enters a *conclusion* is the countably infinite quotient of Theorem 9.8, a property of that example. The invariant $\mathrm{LogCharge}(s) = \log r(s)$ fixed by condition (4) is called the *log-charge* of s.

Definition 3.6 (Foundational axioms)**.** (A1) *F is inhabited: $S_F \neq \emptyset$.*
(A2) *F is a conserved comparison ledger (i.e. satisfies all four conditions of Theorem 3.5).*

Note that (A2) is not circular: Theorem 3.1 defines frameworks, which are not ledgers; (A2) asserts that F is specifically a ledger in the sense of Theorem 3.5.

Remark 3.7 (Hypotheses act on disjoint data)**.** *The three structural hypotheses act on disjoint pieces of the ledger data. H1 constrains only the hierarchical scale pair (σ, ℓ). H2 constrains only the admissible cost J through its combiner. H3 augments the ledger with an internal vector $x(s)$ (defined precisely in Theorem 6.4) from which the ratio interface is recovered by one Jensen step; A3 (Theorem 6.5) then imposes the log-sum-zero condition $\Lambda(x(s)) = 0$ on that data. Each step can therefore be proved independently, and the three conclusions are combined in Section 7 without further argument.*

4. Step B1: Hierarchical Forcing

The first structural hypothesis concerns the scale geometry of the ledger. If the ledger carries a sequence of scales that is simultaneously uniformly geometric — each term a fixed multiple of the previous — and additively self-similar — the third term equal to the sum of the first two — then the two requirements are jointly compatible only at the golden ratio. This section proves that conclusion in exact and limiting-ratio form.

Definition 4.1 (Hierarchical structure on a ledger)**.** *Let F be a conserved comparison ledger. A* hierarchical structure *on F is a pair (σ, ℓ), designated as additional data of F, with $\sigma > 1$ and $\ell \colon \mathbb{N} \to \mathbb{R}_{>0}$ satisfying:*

1. Uniform scaling: *$\ell(k+1) = \sigma\, \ell(k)$ for all $k \geq 0$;*
2. Additive composition: *$\ell(2) = \ell(1) + \ell(0)$.*

We say F carries a hierarchical structure *if such a pair is specified as part of F's data.*

Theorem 4.2 (Scale fixed by exact hierarchy)**.** *If F carries a hierarchical structure (σ, ℓ), then $\sigma = \varphi := \frac{1+\sqrt{5}}{2}$.*

Proof. Uniform scaling gives $\ell(1) = \sigma\ell(0)$ and $\ell(2) = \sigma^2\ell(0)$. Additive composition gives $\ell(2) = \ell(1) + \ell(0) = (\sigma+1)\ell(0)$. Since $\ell(0) > 0$, we obtain $\sigma^2 = \sigma + 1$, i.e. $\sigma^2 - \sigma - 1 = 0$. The two roots are $(1 \pm \sqrt{5})/2$; since $\sigma > 1$, the root $(1-\sqrt{5})/2 < 0$ is excluded, leaving $\sigma = (1+\sqrt{5})/2 =: \varphi$. □

Although condition (2) of Theorem 4.1 constrains only the triple $(0, 1, 2)$, uniform scaling propagates it: $\ell(k) = \sigma^k \ell(0)$ with $\sigma^2 = \sigma + 1$ gives $\ell(k+2) = \ell(k+1) + \ell(k)$ for all $k \geq 0$, the full Fibonacci recurrence.

Remark 4.3 (The golden-ratio characterization is classical)**.** *We claim no originality for Theorem 4.2: that $\sigma > 1$ with $\sigma^2 = \sigma + 1$ is the golden ratio is a classical fact about Fibonacci-type recurrences [6–8], recalled only to fix the form of H1. The one mildly novel point is the independence of the two asymptotic conditions (Theorem 4.5). The exact case reduces to the defining relation $\sigma^2 = \sigma + 1$ of φ; the contribution of layer B1 is the asymptotic forcing of Theorem 4.6, which tolerates perturbations of the exact Fibonacci relation.*

The same equation $\sigma^2 = \sigma + 1$ is recovered when the hierarchical identities hold only in the limit, as the next definition and theorem make precise.

Definition 4.4 (Asymptotic hierarchical structure on a ledger). *Let F be a conserved comparison ledger. An* asymptotic hierarchical structure *on F is a pair (σ, ℓ), designated as additional data of F, with $\sigma > 1$ and $\ell \colon \mathbb{N} \to \mathbb{R}_{>0}$ such that*

1. $\lim_{k\to\infty} \dfrac{\ell(k+1)}{\ell(k)} = \sigma;$
2. $\lim_{k\to\infty} \dfrac{\ell(k+2)}{\ell(k+1)+\ell(k)} = 1.$

Remark 4.5 (Strength of the asymptotic additivity assumption). *Condition (2) in Theorem 4.4 is independent of the existence of the limiting ratio in condition (1). A sequence may satisfy $\ell(k+1)/\ell(k) \to \sigma > 1$ without satisfying $\ell(k+2)/(\ell(k+1)+\ell(k)) \to 1$. Thus the asymptotic version of H1 is not intended to follow from ratio convergence alone; it is a strong asymptotic additivity requirement, useful for allowing perturbations of the exact Fibonacci-type relation while preserving the same limiting forcing equation.*

Theorem 4.6 (Asymptotic hierarchical forcing). *If F carries an asymptotic hierarchical structure (σ, ℓ) with limiting ratio $\sigma > 1$, then $\sigma = \varphi$.*

Proof. Define $q_k := \ell(k+1)/\ell(k)$ and $a_k := \ell(k+2)/(\ell(k+1)+\ell(k))$. By assumption, $q_k \to \sigma$ and $a_k \to 1$. Dividing the identity $\ell(k+2) = a_k(\ell(k+1)+\ell(k))$ by $\ell(k+1)$ gives

$$q_{k+1} = a_k\Big(1 + q_k^{-1}\Big).$$

Since $\{q_{k+1}\}$ is the sequence $\{q_k\}$ shifted by one index, $q_{k+1} \to \sigma$ as $k \to \infty$. Taking limits on both sides and using $a_k \to 1$, $q_k^{-1} \to \sigma^{-1}$ yields $\sigma = 1 \cdot (1+\sigma^{-1})$, hence $\sigma^2 = \sigma + 1$. Since $\sigma > 1$, the unique solution is φ. □

Example 4.7 (A genuinely asymptotic hierarchical scale). *The Fibonacci scale $\ell(k) = F_{k+1}$ (with $F_1 = F_2 = 1$, so $\ell(0), \ell(1), \ell(2), \ldots = 1, 1, 2, 3, 5, 8, \ldots$) is* not *exactly geometric: the ratios $\ell(k+1)/\ell(k) = 1, 2, \frac{3}{2}, \frac{5}{3}, \ldots$ are never constant, so no (exact) hierarchical structure of Theorem 4.1 describes it. It does, however, carry an asymptotic hierarchical structure with $\sigma = \varphi$: condition (2) of Theorem 4.4 holds exactly, since $\ell(k+2) = \ell(k+1) + \ell(k)$, and condition (1) holds in the limit, since $\ell(k+1)/\ell(k) \to \varphi$. Thus Theorem 4.6 forces $\sigma = \varphi$ for a scale beyond the reach of the exact theorem, which is why the asymptotic disjunct is retained in H1.*

Remark 4.8 (Hypothesis H1 and its scope). *In the rest of the paper, hypothesis* (H1) *means the following disjunction: F carries either a hierarchical structure (Theorem 4.1) or an asymptotic hierarchical structure (Theorem 4.4); Theorem 4.7 exhibits a scale of the second kind. In either form, Theorems 4.2 and 4.6 force $\sigma = \varphi$. Because (σ, ℓ) is designated as part of F's data, the forced value $\sigma = \varphi$ is a property of F itself, not of an extraneous auxiliary object. H1 is a structural hypothesis imposed on the ledger; the paper does not assert that every ledger must carry such a structure.*

5. Step B2: Cost Forcing from the Imported Classification

The second structural hypothesis constrains the admissible cost installed in the ledger. Reference [52] has classified all admissible costs with polynomial combiner; the task here is to identify the unique cost selected by the specific normalization and calibration imposed by H2. The "uniqueness" asserted below is uniqueness after fixing units: H2(1) forces only the hyperbolic family, while the normalization H2(2) and calibration H2(3) are unit conventions selecting the representative J_{cost} within it (Theorem 5.5).

Definition 5.1 (Polynomial compositionality). *An admissible cost satisfies* polynomial compositionality *if its combiner P is a polynomial.*

Definition 5.2 (Hypothesis H2)**.** *A ledger satisfies* (H2) *if its cost satisfies the following three conditions:*

1. *polynomial compositionality;*
2. *the canonical normalization* $P(1,1) = 6$*;*
3. *the log-substitution* $G := J \circ \exp$ *is twice differentiable in a neighborhood of* 0*, and the calibration convention* $(J \circ \exp)''(0) = 1$.

The three conditions act in sequence: condition (1) yields a polynomial—hence, by Theorem 2.5, bilinear—combiner, condition (2) fixes its single free parameter, and condition (3) selects the canonical member of the resulting family. The arithmetic is carried out once, in the proof of Theorem 5.4 below.

The polynomial-combiner condition H2(1) is a hypothesis of the present theorem, not a conclusion of the ledger axioms. It is a convenient sufficient condition: it lets the classification of [52] be imported as a single black box (Theorems 2.5 and 2.6), replacing a continuous bootstrap by a one-line algebraic reduction. The same canonical cost J_{cost} is reached without it — under continuity, reciprocal symmetry, unit normalization, and the same curvature calibration, the classical continuous d'Alembert–Wilson–Kannappan theory [1,2,4] isolates the same hyperbolic family — so the polynomial form is an algebraic convenience for a self-contained import.

Remark 5.3 (Condition H2(3) is well-posed)**.** *The expression* $(J \circ \exp)''(0)$ *in condition (3) presupposes twice-differentiability, which admissibility alone does not provide (Theorem 2.1 requires only strict convexity of* $G = J \circ \exp$*). It is, however, automatic once condition (1) holds: by Theorem 2.5 the combiner is bilinear, so Theorem 2.6 places* J *in either the hyperbolic family* $\frac{1}{c}(x^{\lambda} + x^{-\lambda}) - \frac{2}{c}$ *or the degenerate family* $a(\ln x)^2$*, both of which are* C^{∞} *on* $\mathbb{R}_{>0}$*. Hence,* G *is smooth and* $G''(0)$ *exists, so condition (3) is well-posed.*

Theorem 5.4 (Cost forcing under H2)**.** *Let* J *be an admissible cost satisfying* (H2) *(Theorem 5.2). Then,* $J = J_{\text{cost}}$.

Proof. By Theorem 2.5, the polynomial combiner takes the form $P(u,v) = cuv + 2u + 2v$ for some $c \geq 0$. Evaluating at $(u,v) = (1,1)$ gives $P(1,1) = c + 4$, and the normalization $P(1,1) = 6$ forces $c = 2$. Since $c = 2 > 0$, the degenerate quadratic family of Theorem 2.6 (for which $c = 0$ and $P(1,1) = 4$) is excluded: J belongs to the hyperbolic family. Applying Theorem 2.7 with $c = 2$ and the calibration $(J \circ \exp)''(0) = 1$ then yields $J = J_{\text{cost}}$. □

Remark 5.5 (What the normalization $P(1,1) = 6$ does and does not do)**.** *We do not claim* 6 *is singled out by a first principle. By Theorem 2.5 the combiner is* $P(u,v) = cuv + 2u + 2v$ *with* $c = P(1,1) - 4$*, so fixing* $P(1,1)$ *only fixes* c*. The normalization-independent content of B2 is the dichotomy of Theorem 2.6: hyperbolic (*$c > 0$*,* $P(1,1) > 4$*) versus degenerate (*$c = 0$*,* $P(1,1) = 4$*). Any* $P(1,1) > 4$ *gives the hyperbolic family, and after the calibration* $(J \circ \exp)''(0) = 1$ *gives a cost of* $\frac{1}{2}(x^{\lambda} + x^{-\lambda})$*-type; the value* 6 *(*$c = 2$*) is the units convention normalizing the representative to* J_{cost}*. A reader objecting to* 6 *may replace H2(2) by "*$P(1,1) > 4$*" and read the conclusion as "*J *is a calibration of* J_{cost}*."*

With $\sigma = \varphi$ forced by B1 and $J = J_{\text{cost}}$ forced by B2, only the ratio interface remains free. The next section imposes H3 and A3 to determine it.

6. Step B3: Constrained Minimizer and Its Dynamics

The third structural hypothesis augments the ledger with an internal vector at each state, from which the ratio interface is recovered by a single constrained-minimum step.

Only the resulting static minimizer is used in the main text; its dynamical refinements are deferred to Section A, where they give the equilibrium a dynamical rather than merely algebraic interpretation. This section establishes existence and uniqueness of the cost minimizer under a fixed charge constraint: strict convexity and coercivity of $G = J \circ \exp$ guarantee a unique minimizer, and Jensen's inequality identifies it as the uniform configuration.

Lemma 6.1 (Coercivity of G and existence of a unique constrained minimizer)**.** *Let J be an admissible cost and $G := J \circ \exp$. Then G is even, continuous, strictly convex on $\mathbb{R}$, with $G(0) = 0$, $G(t) > 0$ for $t \neq 0$, and $G(t) \to \infty$ as $|t| \to \infty$. Consequently, for every $N \geq 1$, every $Q \in \mathbb{R}$, and every $c \in \mathbb{R}^N_{>0}$ with $\sum_i \log c_i = Q$, the minimization problem*

$$\min\Big\{\sum_{i=1}^{N} J(c_i') \;:\; c' \in \mathbb{R}^N_{>0},\ \sum_{i=1}^{N} \log c_i' = Q\Big\}$$

admits a unique minimizer.

Proof. Evenness of G follows from $J(x) = J(x^{-1})$; strict convexity and $G(0) = J(1) = 0$ are clauses of admissibility, and continuity follows from Theorem 2.2. Moreover, by Theorem 2.3, $G(t) \geq 0$ for all t and $G(t) > 0$ for $t \neq 0$.

For coercivity, suppose for contradiction that a sequence $t_n \to \infty$ satisfies $G(t_n) \leq M < \infty$. For any fixed $s > 0$ and all n large enough that $t_n > s$, global convexity of G on $\mathbb{R}$ gives (writing $s = (s/t_n) \cdot t_n + (1 - s/t_n) \cdot 0$)

$$G(s) \leq \tfrac{s}{t_n} G(t_n) + \Big(1 - \tfrac{s}{t_n}\Big) G(0) \leq \tfrac{sM}{t_n} \xrightarrow{n\to\infty} 0,$$

contradicting $G(s) > 0$. Coercivity as $t \to -\infty$ follows by evenness.

For well-posedness, pass to log-coordinates $\xi_i := \log c_i'$. The functional $\xi \mapsto \sum_i G(\xi_i)$ is continuous and strictly convex on the closed affine hyperplane $H_Q := \{\xi \in \mathbb{R}^N : \sum_i \xi_i = Q\}$. Coercivity of G together with Theorem 2.3 ($G \geq 0$ pointwise) implies coercivity of the sum on H_Q: if $\sum_{j=1}^N G(\xi_j) \leq M$, then $G(\xi_i) \leq M$ for each i (by nonnegativity of the remaining summands), hence each $|\xi_i|$ is bounded by coercivity of G, hence $\|\xi\|$ is bounded. By contraposition, $\sum_j G(\xi_j) \to \infty$ as $\|\xi\| \to \infty$ on H_Q. The infimum is therefore attained at a unique point [17, Thm. 27.3]; exponentiating the coordinates returns the unique minimizer $c' \in \mathbb{R}^N_{>0}$. ☐

Definition 6.2 (N-channel variational update)**.** *Fix an admissible cost J and an integer $N \geq 1$. The* N-channel variational update *is the map $T_N \colon \mathbb{R}^N_{>0} \to \mathbb{R}^N_{>0}$ that sends a configuration $c = (c_1, \dots, c_N) \in \mathbb{R}^N_{>0}$ to the unique minimizer c' (well-defined by Theorem 6.1) of*

$$\mathcal{D}(c') := \sum_{i=1}^{N} J(c_i')$$

subject to the charge constraint

$$\Lambda(c') = \Lambda(c), \qquad \Lambda(c) := \sum_{i=1}^{N} \log c_i.$$

The vector-level log-charge $\Lambda(c) := \sum_i \log c_i$ ($c \in \mathbb{R}^N_{>0}$) and the state-level log-charge $\mathrm{LogCharge}(s) := \log r(s)$ *($s \in S_F$) are distinct quantities. Under hypothesis H3 they are linked by* $\mathrm{LogCharge}(s) = \Lambda(x(s))/N_s$ *(Theorem 6.4); the two values coincide exactly when $N_s = 1$ or, more generally, when $\Lambda(x(s)) = 0$ (axiom A3).*

Theorem 6.3 (Unique constrained minimizer of the cost sum)**.** *Let $c \in \mathbb{R}^N_{>0}$ and write $Q := \Lambda(c)$. Then*

$$T_N(c) = \big(e^{Q/N}, \dots, e^{Q/N}\big),$$

and T_N is idempotent.

Proof. Setting $\xi_i := \log c'_i$ and $G := J \circ \exp$, the objective becomes $\mathcal{D}(c') = \sum_{i=1}^{N} G(\xi_i)$ under the affine constraint $\sum_i \xi_i = Q$. Since G is strictly convex, the discrete Jensen inequality (see e.g. [18, Cor. 1.3.4]) gives

$$\frac{1}{N}\sum_{i=1}^{N} G(\xi_i) \;\geq\; G\Big(\frac{Q}{N}\Big),$$

with equality if and only if all $\xi_i = Q/N$. The unique minimizer therefore has every component equal to $e^{Q/N}$; applying T_N to this output leaves it unchanged, confirming idempotency. □

The static minimizer Theorem 6.3 is all that the main rigidity results require. Two supplementary layers — a one-step Lyapunov contraction (Theorem A1) and exponential convergence of a constrained gradient flow (Theorem A2) — give the equilibrium a dynamical rather than merely algebraic interpretation; they are collected in Section A so as not to interrupt the main line of argument.

Hypothesis H3, axiom A3, and the collapse $r \equiv 1$.

We now introduce the structural hypothesis and axiom that connect the variational update to the ratio interface.

Definition 6.4 (Positive-vector representation of r; hypothesis H3)**.** *A ledger F satisfies hypothesis* (H3) *if it is equipped with the following additional data: for every state $s \in S_F$, a designated integer $N_s \geq 1$ and a designated vector*

$$x(s) = \big(x_1(s), \dots, x_{N_s}(s)\big) \in \mathbb{R}^{N_s}_{>0},$$

(the internal vector *of s), such that the ratio interface is reproduced by one application of the variational update:*

$$r(s) = \big[\, T_{N_s}\big(x(s)\big) \,\big]_1.$$

The internal vector is a genuine structural addition to the ledger: it is not derived from r, and it may be non-uniform. The first coordinate is used only to extract a scalar ratio from the post-update vector. Once Theorem 6.3 applies, all post-update coordinates are equal, so the choice of coordinate carries no hidden asymmetry. The internal vector is retained rather than replaced by a single scalar log-charge $Q(s)$ because the scalar description $r(s) = \exp\big(\frac{1}{N_s}\Lambda(x(s))\big)$ becomes equivalent to it only after the Jensen step has symmetrized the data (Theorem 6.3). Keeping the vector is therefore what lets a state carry genuine pre-equilibrium *asymmetry; Theorem 6.6 records how A3 then acts on that data.*

Definition 6.5 (Axiom A3)**.** *Assume F satisfies* (H3)*, and let $x(s)$ denote the internal vector of each state. Axiom* (A3) *asserts that for every $s \in S_F$,*

$$\Lambda\big(x(s)\big) := \sum_{i=1}^{N_s} \log x_i(s) = 0.$$

Remark 6.6 (A3 as a condition on internal vector)**.** *Axiom A3 is imposed on the internal vector $x(s)$ supplied by H3, not on the ratio interface $r(s)$ directly. This distinction is mathematically* *substantive: the internal data may be genuinely non-uniform, and A3 constrains only its total log-charge to vanish. The pre-equilibrium witness in Theorem 9.3 illustrates this precisely: the internal data has non-uniform coordinates, yet the equilibrium value is uniform because the charge is neutral.*

Corollary 6.7 ($r \equiv 1$ under H3 and A3)**.** *(See Theorem 6.8.) Let F satisfy* (A2), (H3), *and* (A3). *Then,* $r(s) = 1$ *for all* $s \in S_F$.

Proof. Fix $s \in S_F$ and let $x(s) \in \mathbb{R}^{N_s}_{>0}$ be the internal vector supplied by H3. Set $Q := \Lambda\big(x(s)\big)$. By Theorem 6.3 applied to $x(s)$,

$$T_{N_s}\big(x(s)\big) = \big(e^{Q/N_s}, \dots, e^{Q/N_s}\big).$$

Axiom A3 gives $Q = 0$, so $T_{N_s}(x(s)) = (1, \dots, 1)$. By the ratio-reproduction clause of H3,

$$r(s) = \big[\, T_{N_s}\big(x(s)\big) \,\big]_1 = 1. \quad \square$$

Remark 6.8 (Content of the H3–A3 collapse)**.** *The mathematical content of Theorem 6.7 is the strict-convexity/Jensen statement that, at fixed vector-level log-charge Q, the constrained cost minimizer is the uniform vector* $(e^{Q/N}, \dots, e^{Q/N})$. *Hypothesis H3 is the modeling link that reads the ledger ratio from that minimizer, and A3 is the neutrality condition* $Q = 0$. *Thus the conclusion* $r \equiv 1$ *is not an additional analytic classification theorem; it is the ledger-level consequence of combining the Jensen minimizer with the H3 ratio-reading rule and the A3 zero-charge constraint.*

7. Main Rigidity Theorems

Because the three steps act on disjoint data (Theorem 3.7), the two main theorems below are assembled by straightforward combination. Theorem 7.1 is the A3-free version ($\sigma = \varphi$ and $J = J_{\text{cost}}$); Theorem 7.2 adds H3 and A3 to give the singleton observational quotient $\mathrm{SQ}(F) \simeq \mathbf{1}$.

Theorem 7.1 (Scale and cost rigidity)**.** *Let F satisfy* (A2).

Assume:

- (H1) *F carries a hierarchical structure (Theorem 4.1) or an asymptotic hierarchical structure (Theorem 4.4);*
- (H2) *the cost satisfies Theorem 5.2.*

Then $\sigma = \varphi$ *and* $J = J_{\text{cost}}$.

Proof. If H1 is given in exact form, the scale claim follows from Theorem 4.2; if H1 is given in asymptotic form, it follows from Theorem 4.6. The cost claim is Theorem 5.4. $\square$

Theorem 7.2 (Full observational rigidity)**.** *Let F satisfy* (A1) *and the hypotheses of Theorem 7.1. Assume additionally:*

- (H3) *positive-vector representation of r (Theorem 6.4);*
- (A3) *the log-sum-zero condition on the internal vector (Theorem 6.5).*

For the three collapse conclusions below, the proof uses only the non-emptiness of S_F, *the factorization* $\mu_F = \bar{\mu}_F \circ r$, *H3, A3, and Theorem 3.3. The evolution map, the conservation law, and countability are part of the ledger structure but are not needed for this static collapse; they are used in the no-A3 level-set analysis of Section 8. In addition to* $\sigma = \varphi$ *and* $J = J_{\text{cost}}$ *from Theorem 7.1:*

1. $r(s) = 1$ *for all* $s \in S_F$;
2. *the measurement map* μ_F *is constant on* S_F;
3. $\mathrm{SQ}(F) \simeq \mathbf{1}$.

Proof. $\sigma = \varphi$ and $J = J_{\text{cost}}$ follow from Theorem 7.1. Item (1) is Theorem 6.7. For item (2): since $\mu_F = \bar{\mu}_F \circ r$ by axiom (A2) and $r(s) = 1$ for every s by item (1), the measurement map equals $\bar{\mu}_F(1)$ identically, hence is constant. Since (A1) gives $S_F \neq \emptyset$, item (3) then follows from Theorem 3.3. □

Sharpness: irredundancy of the condition package

The two main theorems are tight: dropping any single condition from the package (H1), (H2), (H3), (A3) allows a ledger that satisfies all remaining conditions yet violates the corresponding conclusion. The three propositions and the following A3 remark make this precise for the triple of forced quantities $\sigma = \varphi$, $J = J_{\text{cost}}$, $r \equiv 1$. The four conditions are not on equal logical footing: H1 and H2 act on disjoint data (the scale pair and the cost) and are mutually independent, while A3 is a constraint on the internal-vector data introduced by H3. Accordingly, “dropping H3” means deleting that internal-vector data (Theorem 6.4), which also voids the statement of A3, whereas “dropping A3” retains H3 but releases the log-sum-zero condition. With this reading, each of the four can be removed to break a specific conclusion.

Proposition 7.3 (H1 is necessary for $\sigma = \varphi$)**.** *There exists a ledger $\mathcal{G}_1$ satisfying* (A1), (A2), (H2), (H3), (A3) *that carries a geometric scale sequence with ratio $\sigma = 2$ but does not satisfy* (H1) *(so all remaining axioms and conditions needed after dropping H1 remain in force). Its scale is not forced to φ.*

Proof. Define $\mathcal{G}_1$: $S_F = \mathbb{D}_{>0}$ (positive dyadic rationals), $T_F = \text{id}$, $O_F = \{*\}$, $\mu_F \equiv \bar{\mu}_F \equiv *$, $J = J_{\text{cost}}$, $r \equiv 1$, $N_s = 1$, $x(s) = (1) \in \mathbb{R}^1_{>0}$, and designated scale data $(\sigma, \ell) = (2, k \mapsto 2^k)$.

The sequence $\ell(k) = 2^k$ satisfies uniform scaling $\ell(k+1) = 2\ell(k)$, but $\ell(2) = 4$ while $\ell(1) + \ell(0) = 3 \neq 4$: the additive-composition condition of Theorem 4.1 fails, so the exact hierarchical structure does not hold. The asymptotic form (Theorem 4.4) fails for this same designated data as well: although $\ell(k+1)/\ell(k) = 2 \to 2$ satisfies condition (1) with $\sigma = 2$, one has

$$\frac{\ell(k+2)}{\ell(k+1) + \ell(k)} = \frac{2^{k+2}}{2^{k+1} + 2^k} = \frac{4}{3} \not\to 1,$$

so condition (2) fails. With its designated scale data, $\mathcal{G}_1$ therefore satisfies neither disjunct of (H1) (Theorem 4.8), so (H1) does not hold. Direct verification: (A1) $S_F \neq \emptyset$; (A2) $J = J_{\text{cost}}$ is admissible, $r \equiv 1$ is conserved under $T_F = \text{id}$, and $\mu_F = \bar{\mu}_F \circ r$; (H2) $J = J_{\text{cost}}$ satisfies Theorem 5.2, as verified in Theorem 9.2; (H3) $r(s) = [T_1((1))]_1 = 1$; (A3) $\Lambda((1)) = \log 1 = 0$. Yet the designated scale is $\sigma = 2 \neq \varphi$. □

Proposition 7.4 (H2 is necessary for $J = J_{\text{cost}}$)**.** *There exists a ledger $\mathcal{G}_2$ satisfying* (A1), (A2), (H1), (H3), (A3) *whose cost is $J(x) = (\ln x)^2 \neq J_{\text{cost}}$ (so all remaining axioms and conditions needed after dropping H2 remain in force).*

Proof. Define $\mathcal{G}_2$: same data as $\mathcal{G}_1$ except $J(x) := (\ln x)^2$ and hierarchical data $(\varphi, k \mapsto \varphi^k)$.

The cost $J(x) = (\ln x)^2$ is admissible: $J(x^{-1}) = (\ln x)^2 = J(x)$; $J(1) = 0$; $G(t) = t^2$ is strictly convex. Its combiner is $P(u, v) = 2u + 2v$, the $c = 0$ degenerate case of Theorem 2.5, giving $P(1, 1) = 4 \neq 6$; (H2) fails. The hierarchical structure $(\varphi, k \mapsto \varphi^k)$ satisfies both conditions of Theorem 4.1 ($\varphi^2 = \varphi + 1$), so (H1) holds. Conditions (A1), (A2), (H3), (A3) are verified exactly as in Theorem 7.3, replacing $J = J_{\text{cost}}$ by $J(x) = (\ln x)^2$; admissibility of $(\ln x)^2$ is checked by the same three-axiom verification. The cost $(\ln x)^2 \neq J_{\text{cost}}$. □

Proposition 7.5 (H3 is necessary for $r \equiv 1$)**.** *There exists a ledger* $\mathcal{G}_3$ *satisfying* (A1), (A2), (H1), (H2) *(with neither H3 nor A3 imposed) whose ratio interface satisfies* $r \equiv 2$ *(so the remaining meaningful assumptions after dropping H3, namely A1, A2, H1, and H2, remain in force).*

Proof. Define $\mathcal{G}_3$: $S_F = \mathbb{D}_{>0}$, $T_F = \mathrm{id}$, $O_F = \{*\}$, $\mu_F \equiv \bar{\mu}_F \equiv *$, $J = J_{\mathrm{cost}}$, $r(s) := 2$ for all s, hierarchical data $(\varphi, k \mapsto \varphi^k)$.

The log-charge $\mathrm{LogCharge}(s) = \log 2$ is constant, hence conserved under $T_F = \mathrm{id}$. (A1) and (A2) hold. (H1) holds as in Theorem 7.4. (H2) holds since $J = J_{\mathrm{cost}}$ satisfies Theorem 5.2, as verified in Theorem 9.2. No hypothesis (H3) is imposed: there is no internal vector, and r is assigned directly without any variational mechanism. The ratio interface $r \equiv 2 \neq 1$ is constant but not 1; neither the Lyapunov step nor the Jensen machinery of Section 6 applies to control r. □

Remark 7.6 (A3 is necessary; Section 8 gives the full picture)**.** *Dropping only* (A3) *while retaining* (H3) *leaves the log-charge* $\Lambda(x(s))$ *unconstrained, so* $r(s) = \exp(\Lambda(x(s))/N_s)$ *can take any positive value. Section 8 gives a complete description: the state space partitions into the level sets* $F_Q = \{s : \log r(s) = Q\}$*, each preserved by the dynamics; Theorem 8.3 shows that (A3) is exactly the condition that forces all states into* F_0*. Multiple level sets may coexist whenever (A3) fails, confirming that A3 is not redundant.*

Together, Theorems 7.3–7.6 establish irredundancy in the following precise sense. We use "irredundant" in a deliberately modest sense: each condition is individually necessary for its own conclusion (H1 for $\sigma = \varphi$, H2 for $J = J_{\mathrm{cost}}$, H3 for the collapse mechanism, A3 for $r \equiv 1$ given H3), so no condition can be deleted without losing a forced output — the *independence* of the package. We do *not* claim the stronger property that no logically weaker reformulation forces the same triple.

8. Structure Without A3

The full rigidity theorem requires all four structural assumptions H1, H2, H3, and A3 *in addition to the foundational axioms (A1) and (A2)*. When A3 is dropped, H1 and H2 still force $\sigma = \varphi$ and $J = J_{\mathrm{cost}}$, but r is no longer forced to equal 1. The results here are deliberately elementary. Since $\log r$ is conserved by T_F and every observable factors through r, its level sets form an evolution-invariant partition of S_F on which μ_F is constant — the standard fact that the fibers of a conserved quantity form an invariant partition. The purpose is narrow: to pin down that A3 is exactly the selection of the neutral fiber $\log r = 0$ (Theorem 8.3), so that dropping A3 yields a clean decomposition into charge sectors rather than disorder.

Definition 8.1 (Level-set label and level sets of $\log r$)**.** *For a ledger* F*, the* level-set label *of a state* $s \in S_F$ *is* $Q_s := \log r(s)$*. For* $Q \in \mathbb{R}$*, the* Q-level set *of* F *is*

$$F_Q := \{s \in S_F : \log r(s) = Q\}.$$

Theorem 8.2 (Partition by level sets)**.** *Let* F *satisfy* (A2)*. Then:*

1. $S_F = \bigsqcup_{Q \in \log r(S_F)} F_Q$*;*
2. *each level set* F_Q *is preserved by the evolution* T_F*;*
3. *the measurement map is constant on each level set:* $\mu_F(s) = \bar{\mu}_F(e^Q)$ *for all* $s \in F_Q$*.*

Proof. Statement (1) is the fiber decomposition of the function $s \mapsto \log r(s)$ over its image. Statement (2) follows from log-charge conservation: $\log r(T_F(s)) = \log r(s)$, so T_F maps F_Q into itself. For statement (3), if $s \in F_Q$ then $r(s) = e^Q$, so $\mu_F(s) = \bar{\mu}_F(r(s)) = \bar{\mu}_F(e^Q)$. □

Theorem 8.3 (A3 picks the level set $\log r = 0$)**.** *Let F satisfy* (A1), (A2), *and* (H3). *Then:* (A3) *holds if and only if every state lies in $F_0 = \{s : \log r(s) = 0\}$.*

Proof. Let $\Lambda_s := \Lambda(x(s))$. Under H3, Theorem 6.3 gives $r(s) = \exp(\Lambda_s/N_s)$, so $\log r(s) = \Lambda_s/N_s$. Since $N_s \geq 1$, condition A3 ($\Lambda_s = 0$) is equivalent to $\log r(s) = 0$, i.e. $s \in F_0$. ☐

Corollary 8.4 (Partition of S_F without A3)**.** *Let F satisfy the hypotheses of Theorem 7.1 and also* (H3), *but do not assume* (A3). *Then:*

1. *the forced structural outputs remain $\sigma = \varphi$ and $J = J_{\mathrm{cost}}$;*
2. *the state space decomposes as $S_F = \bigsqcup_{Q \in \log r(S_F)} F_Q$, and A3 is exactly the condition populating only the neutral set $F_0 = \{s : \log r(s) = 0\}$ (Theorem 8.3).*

Proof. The scale and cost are forced by Theorem 7.1 independently of A3. The level-set partition is Theorem 8.2, and the A3 equivalence is Theorem 8.3. ☐

9. Examples

The ledgers below are all *constructed* to satisfy the axioms; their role is to prove *non-vacuity* (H1–H3 and A3 are jointly satisfiable, so the main theorem is non-empty) and to exhibit specific phenomena: direct satisfiability ($\mathcal{F}_0$), the variational origin of $r = 1$ from genuinely non-uniform data ($\mathcal{F}_N$), an explicit cost drop ($N = 2$), a non-vacuous observational collapse and multi-sector quotient (injective observable), and free conservative dynamics. They are existence witnesses, not evidence of a non-designed realization; finding such a realization is the principal open problem (Section 10).

Example 9.1 (Canonical static ledger)**.** *Define the ledger $\mathcal{F}_0$ by:*

- *$S_F = \mathbb{D}_{>0}$, the positive dyadic rationals; (any nonempty countable set serves; $\mathbb{D}_{>0}$ is a concrete choice.)*
- *$T_F = \mathrm{id}_{S_F}$;*
- *$O_F = \{*\}$ and $\mu_F(s) = *$, $\bar{\mu}_F(x) = *$ for all $s \in S_F$, $x \in \mathbb{R}_{>0}$;*
- *ratio map $r(s) = 1$ for every s;*
- *cost $J = J_{\mathrm{cost}}$;*
- *hierarchical structure $(\sigma, \ell) = (\varphi, k \mapsto \varphi^k)$;*
- *internal channel structure $N_s = 1$, $x(s) = (1) \in \mathbb{R}^1_{>0}$ for every s.*

Proposition 9.2 (Axioms hold on $\mathcal{F}_0$)**.** *The canonical ledger $\mathcal{F}_0$ satisfies* (A1), (A2), (A3), *and* (H1)–(H3).

Proof. The state space $\mathbb{D}_{>0}$ is nonempty and countable, giving (A1) and the countability part of (A2). The distinguished ratio interface is $r \equiv 1$, and $\mu_F = \bar{\mu}_F \circ r \equiv *$. Local conservation is automatic since $T_F = \mathrm{id}$. The cost J_{cost} is admissible by direct verification of Theorem 2.1 (J_{cost} is reciprocally symmetric, $J_{\mathrm{cost}}(1) = 0$, and $G(t) = \cosh t - 1$ is strictly convex), completing the verification of (A2).

For (H1): the sequence $\ell(k) = \varphi^k$ satisfies uniform scaling with ratio φ and additive composition $\varphi^2 = \varphi + 1$. For (H2): the cost J_{cost} satisfies H2 by direct verification: its combiner is $P(u,v) = 2uv + 2u + 2v$ (polynomial, satisfying $P(1,1) = 6$), and $G(t) = \cosh t - 1$ gives $G''(0) = 1$. For (H3): the internal vector $x(s) = (1) \in \mathbb{R}^1_{>0}$ gives $T_1(x(s)) = (1)$, so $[T_1(x(s))]_1 = 1 = r(s)$. Finally, (A3) holds because $\Lambda(x(s)) = \log 1 = 0$. ☐

Example 9.3 (N-channel example with non-uniform internal data)**.** *Fix $N \geq 2$ and let*

$$\Theta_N := \{(\theta_1, \ldots, \theta_N) \in \mathbb{D}^N : \theta_1 + \cdots + \theta_N = 0\},$$

where $\mathbb{D}$ denotes the dyadic rationals. (The dyadic choice is only a convenience; any countable dense additive subgroup of $\mathbb{R}$ would serve.)

(Θ_N contains $(0,\dots,0)$ and, for $N \geq 2$, non-uniform elements such as $(1,-1,0,\dots,0) \in \mathbb{D}^N$; the latter has coordinates not all equal, so $x(s) = (e, e^{-1}, 1, \dots, 1)$ is genuinely non-uniform.) Define the ledger $\mathcal{F}_N$ by:

- $S_F = \{s_{\boldsymbol{\theta}} : \boldsymbol{\theta} \in \Theta_N\}$;
- $T_F = \mathrm{id}_{S_F}$;
- $O_F = \{*\}$ *with* $\mu_F \equiv *$ *and* $\bar{\mu}_F \equiv *$;
- *cost* $J = J_{\mathrm{cost}}$;
- *hierarchical structure* $(\sigma, \ell) = (\varphi, k \mapsto \varphi^k)$;
- *internal channel structure* $N_{s_{\boldsymbol{\theta}}} = N$ *and*
$$x(s_{\boldsymbol{\theta}}) := \left(e^{\theta_1}, \dots, e^{\theta_N}\right) \in \mathbb{R}^N_{>0};$$
- *ratio map defined by the variational equilibrium:*
$$r(s_{\boldsymbol{\theta}}) := \left[\, T_N\left(x(s_{\boldsymbol{\theta}})\right) \right]_1.$$

Proposition 9.4 (Axioms hold on $\mathcal{F}_N$; $r = 1$ from non-uniform internal data)**.** *For every $N \geq 2$, the ledger $\mathcal{F}_N$ satisfies* (A1), (A2), (A3), *and* (H1)–(H3)*. Moreover, the value $r(s) = 1$ is a consequence of one Jensen step applied to a genuinely non-uniform internal vector, not a stipulation of the ratio interface.*

Proof. Fix $s_{\boldsymbol{\theta}} \in S_F$. The internal vector $x(s_{\boldsymbol{\theta}}) = (e^{\theta_1}, \dots, e^{\theta_N})$ is non-uniform whenever the coordinates of $\boldsymbol{\theta}$ are not all equal. Its log-charge is
$$\Lambda\left(x(s_{\boldsymbol{\theta}})\right) = \theta_1 + \cdots + \theta_N = 0,$$
since $\boldsymbol{\theta} \in \Theta_N$, so (A3) holds. By Theorem 6.3, the variational update maps $x(s_{\boldsymbol{\theta}})$ to the uniform equilibrium $(1, \dots, 1)$, so
$$r(s_{\boldsymbol{\theta}}) = \left[T_N\left(x(s_{\boldsymbol{\theta}})\right)\right]_1 = 1,$$
verifying (H3) and the induced form of the ratio interface. The remaining ledger conditions are checked as in Theorem 9.2: S_F is countable (indexed by $\Theta_N \subset \mathbb{D}^N$), conservation is automatic under $T_F = \mathrm{id}$, and $\mu_F = \bar{\mu}_F \circ r$ by construction. The hierarchical structure is the same φ-geometric one as in $\mathcal{F}_0$, giving (H1). Hypothesis (H2) holds since $J = J_{\mathrm{cost}}$; the three sub-conditions of Theorem 5.2 were verified explicitly in the proof of Theorem 9.2. □

Example 9.5 (Explicit $N = 2$ calculation)**.** *Take $N = 2$ and $\boldsymbol{\theta} = (1, -1) \in \Theta_2$. The internal vector is $x(s_{\boldsymbol{\theta}}) = (e, e^{-1}) \in \mathbb{R}^2_{>0}$, which is genuinely non-uniform ($e \neq e^{-1}$). Its log-charge is $\log e + \log e^{-1} = 1 + (-1) = 0$, confirming (A3). By Theorem 6.3 with $Q = 0$ and $N = 2$,*
$$T_2\left(e, e^{-1}\right) = \left(e^{0/2}, e^{0/2}\right) = (1, 1).$$
Hence $r(s_{\boldsymbol{\theta}}) = [T_2(e, e^{-1})]_1 = 1$. The cost drop is
$$\mathcal{D}(e, e^{-1}) - \mathcal{D}(1,1) = 2J_{\mathrm{cost}}(e) - 2J_{\mathrm{cost}}(1) = 2\left(\tfrac{e+e^{-1}}{2} - 1\right) - 0 = e + e^{-1} - 2 \approx 1.086 > 0,$$
quantifying the strict decrease from a non-uniform input to the uniform equilibrium. This illustrates Theorem A1: the cost strictly decreases in one variational step whenever the input is not already at the equilibrium.

Example 9.6 (Injective observable: non-vacuous collapse)**.** *The preceding examples use the one-point observable set* $O_F = \{*\}$*, so the singleton conclusion* $\mathrm{SQ}(F) \simeq \mathbf{1}$ *holds automatically and does not test the collapse in Theorem 7.2. To show that the collapse is a genuine consequence of* $r \equiv 1$ *rather than an artefact of a trivial observable, we fix an* injective *reduced measurement map.*

Fix the data $O_F = \mathbb{R}_{>0}$*,* $\bar{\mu}_F = \mathrm{id}_{\mathbb{R}_{>0}}$ *(so* $\mu_F = r$*),* $T_F = \mathrm{id}$*,* $J = J_{\mathrm{cost}}$*, and hierarchical structure* $(\varphi, k \mapsto \varphi^k)$*, and take the states and internal vectors of* $\mathcal{F}_N$ *(Theorem 9.3). By Theorem 9.4 the internal data is charge-neutral, so* $r \equiv 1$ *and* $\mu_F \equiv \bar{\mu}_F(1) = 1$*, whence* $\mathrm{SQ}(F) \simeq \mathbf{1}$*. Here* $\bar{\mu}_F = \mathrm{id}$ *is injective and would separate any two states with distinct ratios; the singleton quotient is therefore forced by r being driven to* 1*, not by the observable set. The complementary no-A3 regime, in which this same injective observable yields a nontrivial multi-sector quotient, is the* $T_F = \mathrm{id}$ *special case of Theorem 9.8 below.*

Proposition 9.7 (Axioms for Theorem 9.6)**.** *The ledger of Theorem 9.6 satisfies* (A1)*,* (A2)*,* (A3)*, and* (H1)–(H3)*, with* $\mathrm{SQ}(F) \simeq \mathbf{1}$*.*

Proof. Every axiom except the choice of O_F and $\bar{\mu}_F$ is verified exactly as in Theorem 9.4; the factorization $\mu_F = \bar{\mu}_F \circ r$ holds by construction and $\mathrm{id}_{\mathbb{R}_{>0}}$ is a legitimate reduced map into $O_F = \mathbb{R}_{>0}$. Since $r \equiv 1$, the map $\mu_F \equiv \bar{\mu}_F(1) = 1$ is constant, so Theorem 3.3 gives $\mathrm{SQ}(F) \simeq \mathbf{1}$. □

Example 9.8 (Non-trivial conservative dynamics within sectors)**.** *All earlier examples take* $T_F = \mathrm{id}$*, so the conservation law (Theorem 3.5, condition (4)) and the evolution map are never exercised. The following ledger* $\mathcal{F}_{\mathrm{dyn}}$ *activates both: its evolution is fixed-point-free yet conserves the log-charge, permuting states within each level set of* $\log r$*.*

Fix $O_F = \mathbb{R}_{>0}$*,* $\bar{\mu}_F = \mathrm{id}_{\mathbb{R}_{>0}}$ *(so* $\mu_F = r$*),* $J = J_{\mathrm{cost}}$*, and hierarchical structure* $(\varphi, k \mapsto \varphi^k)$*. Let* $S_F = \{s_{q,n} : q \in \mathbb{Q},\ n \in \mathbb{Z}\}$*, and put*

$$N_{s_{q,n}} = 1, \qquad x(s_{q,n}) = (e^q) \in \mathbb{R}^1_{>0}, \qquad r(s_{q,n}) = \big[T_1(x(s_{q,n}))\big]_1 = e^q,$$

with evolution given by the index shift $T_F(s_{q,n}) := s_{q,n+1}$*. Taking instead the trivial evolution* $T_F = \mathrm{id}$ *(equivalently, dropping the n index) gives the static special case: states indexed by* $\mathbb{Q}$ *with* $r(s_q) = e^q$*, on which the injective observable* $\bar{\mu}_F = \mathrm{id}$ *yields the countably infinite multi-sector quotient* $\mathrm{SQ}(F) \simeq \{e^q : q \in \mathbb{Q}\}$*, realizing Theorems 8.2 and 8.4.*

Proposition 9.9 (Axioms for Theorem 9.8)**.** *The ledger* $\mathcal{F}_{\mathrm{dyn}}$ *satisfies* (A1)*,* (A2)*,* (H1)*,* (H2)*, and* (H3) *but not* (A3)*. Its evolution* T_F *is a fixed-point-free bijection that preserves every level set* $F_q = \{s_{q,n} : n \in \mathbb{Z}\}$ *and conserves the log-charge; and* $|\mathrm{SQ}(F)| = \infty$*.*

Proof. The state space $\mathbb{Q} \times \mathbb{Z}$ is countable, giving (A1) and the countability clause of (A2). Since $r(s_{q,n}) = e^q$ depends only on q and T_F fixes q, we have $\log r(T_F(s_{q,n})) = q = \log r(s_{q,n})$, so the conservation law holds; with $\mu_F = \bar{\mu}_F \circ r$ by construction this gives (A2). Conditions (H1) and (H2) hold as in Theorem 9.2 (the φ-geometric scale and $J = J_{\mathrm{cost}}$), and (H3) holds with $N = 1$ because the only point of $\mathbb{R}^1_{>0}$ with log-charge q is (e^q), so $T_1((e^q)) = (e^q)$ and $r(s_{q,n}) = e^q$. Axiom A3 fails for every $q \neq 0$ since $\Lambda(x(s_{q,n})) = q$. The shift $T_F(s_{q,n}) = s_{q,n+1}$ is a bijection with no fixed point and maps each F_q onto itself, realising the T_F-invariance of Theorem 8.2 by a free $\mathbb{Z}$-action. Finally $\bar{\mu}_F = \mathrm{id}$ is injective and $\mu_F(s_{q,n}) = e^q$ takes one value per q, so the observational quotient is $\mathrm{SQ}(F) \simeq \{e^q : q \in \mathbb{Q}\}$, which is countably infinite. □

Example 9.10 (Finite nontrivial sectors)**.** *The previous dynamical example uses infinite level sets. The same sector mechanism already appears in a finite form. Let*

$$S_F = \{s_{q,\varepsilon} : q \in \{-1,0,1\},\ \varepsilon \in \{0,1\}\}, \qquad O_F = \mathbb{R}_{>0}, \qquad \bar{\mu}_F = \mathrm{id}_{\mathbb{R}_{>0}},$$

take $J = J_{\mathrm{cost}}$*, hierarchical structure* $(\varphi, k \mapsto \varphi^k)$*, and set*

$$N_{s_{q,\varepsilon}} = 1, \qquad x(s_{q,\varepsilon}) = (e^q), \qquad r(s_{q,\varepsilon}) = e^q, \qquad \mu_F(s_{q,\varepsilon}) = e^q.$$

Define the evolution by the two-cycle

$$T_F(s_{q,0}) = s_{q,1}, \qquad T_F(s_{q,1}) = s_{q,0}.$$

Then each level set

$$F_q = \{s_{q,0}, s_{q,1}\}$$

has two states, is preserved by T_F*, and carries a nontrivial period-two action. The ledger satisfies (A1), (A2), (H1), (H2), and (H3), while A3 fails on the* $q = \pm 1$ *sectors. The observational quotient has three classes, represented by* e^{-1}*, 1, and e. This example shows that the no-A3 sector decomposition is not merely a one-state-per-sector phenomenon.*

Together these examples establish that the condition package is jointly satisfiable, that $r \equiv 1$ is a genuine variational output of the Jensen step on non-uniform data (with a quantitatively strict cost drop), and that — with an injective observable — the collapse $\mathrm{SQ}(F) \simeq \mathbf{1}$ is non-vacuous under A3 while a nontrivial multi-sector quotient and fixed-point-free conservative dynamics appear once A3 is dropped (Theorems 9.2, 9.4, 9.7 and 9.9; Theorems 9.8 and 9.10).

10. Conclusion

Summary of results.

Reference [52] classifies all admissible costs with polynomial combiner. Once that classification is placed inside a conserved comparison ledger, the present paper shows that the ledger axioms together with H1–H3 and A3 force the three quantities (σ, J, r) to $\sigma = \varphi$, $J = J_{\mathrm{cost}}$, and $r \equiv 1$, with the two immediate corollaries that the measurement map is constant and the observational quotient is a singleton (Theorems 7.1 and 7.2). The forcing is layered: H1 alone gives $\sigma = \varphi$; H1 and H2 together give $J = J_{\mathrm{cost}}$; adding H3 and A3 then gives $r \equiv 1$, constant measurement, and singleton quotient. The main results have asymptotic and dynamical variants: H1 has an asymptotic form (Theorem 4.6), and the constrained minimizer of B3 is the exponentially attracting fixed point of a gradient flow (Theorems A1 and A2).

Sharpness.

The condition package is irredundant — each of H1, H2, H3, A3 is individually necessary for its forced output ($\sigma = \varphi$, $J = J_{\mathrm{cost}}$, $r \equiv 1$). Dropping H1 allows $\sigma = 2$ (Theorem 7.3); dropping H2 allows the admissible cost $J(x) = (\ln x)^2$ (Theorem 7.4); dropping H3 allows $r \equiv 2$ (Theorem 7.5); dropping A3 produces the level-set decomposition of Section 8 in which r is not forced to 1.

Examples.

Section 9 provides explicit ledgers that establish joint satisfiability of H1–H3 and A3 and the non-vacuity of each conclusion: the variational origin of $r = 1$ from non-uniform data ($\mathcal{F}_N$, with the explicit $N = 2$ cost drop $e + e^{-1} - 2 \approx 1.086$), a non-vacuous observational collapse and a countably infinite multi-sector quotient without A3 (injective observable), and free conservative dynamics within level sets ($\mathcal{F}_{\mathrm{dyn}}$ and a finite two-point sector).

Open problems.

Several directions remain unexplored.

Non-polynomial combiners. H2 requires a polynomial combiner. The classification of [52] is restricted to this class; whether an analogous rigidity theorem holds for non-polynomial (e.g. transcendental) combiners is open. One concrete question: does an admissible cost with a continuous non-polynomial combiner satisfy any natural normalization that forces it to a specific form? Without polynomiality the combiner lives in an infinite-dimensional space and the d'Alembert–Wilson–Kannappan machinery [2,4] no longer yields a finite-parameter family; some regularity (measurability, monotonicity) is likely needed to exclude Hamel-type pathologies. A solution would either enlarge the admissible class (weakening rigidity) or show polynomiality follows from mild regularity (strengthening it).

Infinite-dimensional state spaces. The countability of S_F and the finite-dimensional internal vectors of H3 make the present theory largely combinatorial. An analogue for uncountable state spaces or infinite-dimensional internal data would require a measure-theoretic Jensen argument and a continuous-variable conservation law; whether the rigidity conclusions survive this extension is unknown. Countability of S_F is not used (Theorem 3.5); the substantive issue is infinite-dimensional internal data. The static collapse plausibly survives — the integral Jensen inequality still makes the constant function the constrained minimizer — but the gradient flow of Theorem A2 becomes infinite-dimensional, and its contraction needs a spectral-gap / Bakry–Émery condition [26,27] automatic only in finite dimensions.

Characterization of ledgers realizing the forced values. The forward direction (H1+H2+H3+A3 $\Rightarrow$ forced outputs) is proved here. The converse question — characterize all ledgers that achieve $\sigma = \varphi$, $J = J_{\mathrm{cost}}$, $r \equiv 1$ without necessarily satisfying H1, H2, H3, A3 — is open. In particular: is there an admissible cost $J \neq J_{\mathrm{cost}}$ and a ledger satisfying the ledger axioms alone (A1, A2) such that $r \equiv 1$ holds without any variational mechanism? The trivial direction is clear (assigning $r \equiv 1$ by hand needs no hypothesis), so the interesting question seeks a converse under constraints that exclude hand-assignment — e.g. requiring r to be variationally generated as in H3. This is a uniqueness question for the aggregation rule; the means/aggregation literature [19,57] characterizes quasi-arithmetic selection by symmetry and consistency axioms, and importing such a characterization would turn H3 into a derived consequence — the most promising route to fewer hypotheses.

Natural systems realizing the framework. A natural open problem is whether some physical or combinatorial system satisfies H1–H3 and A3 without being constructed for that purpose; such a non-designed realization would give the main theorem an interpretation beyond its abstract setting. Until one is known, the theorem should be read as a coherence and rigidity result for the stated axioms, not as a prediction about an independently identified object. This is the most important open problem. Its difficulty is that the conditions constrain unrelated features at once — a scale sequence (H1), a comparison cost (H2), and a charge-neutral internal aggregation (H3–A3) — so a natural realization must exhibit all three coincidentally. Two search directions seem promising: self-similar substitution systems and quasicrystals, where φ appears intrinsically [10]; and charge-neutral statistical ensembles whose free-energy minimization realizes H3 with A3 imposed by a conservation law. Settling it either way would determine whether the framework is more than an internally consistent abstraction.

Relation to prior work.

Together with [52], the present paper settles, under the stated hypotheses, the analytic and framework-level rigidity of admissible costs with polynomial combiner. Reference [52] provides the classification; the present paper provides the framework in which that classification becomes a structural constraint. The present development is logically independent of [52] outside the four imported theorems of Section 2.

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Appendix A Dynamical interpretation of the equilibrium

This appendix records two supplementary layers of Step B3 (Section 6). Neither is used by the main rigidity results, which rest only on the static minimizer Theorem 6.3; their role is to exhibit the Jensen equilibrium as a dynamically attracting state in the canonical setting of J_{cost}, for which $G(t) = \cosh t - 1$ satisfies $G''(t) = \cosh t \geq 1$ everywhere.

Layer 2: One-step quantitative contraction.

Lemma A1 (One-step Lyapunov inequality)**.** *Let J be admissible, $G := J \circ \exp$, and $c \in \mathbb{R}^N_{>0}$ with $\xi_i := \log c_i$ and $\bar{\xi} := \frac{1}{N}\sum_i \xi_i$. Assume there is an interval $I \subseteq \mathbb{R}$ with $\bar{\xi} \in I$ and $\xi_1, \dots, \xi_N \in I$ on which G is twice differentiable, and set $\kappa := \inf_{t \in I} G''(t)$. Then,*

$$\sum_{i=1}^{N} J(c_i) - \sum_{i=1}^{N} J\big(T_N(c)_i\big) \;\geq\; \frac{\kappa}{2} \sum_{i=1}^{N} (\xi_i - \bar{\xi})^2.$$

For $J = J_{\text{cost}}$, one may take $I = \mathbb{R}$ and $\kappa = 1$.

Proof. κ-strong convexity of G gives, for each i, $G(\xi_i) \geq G(\bar{\xi}) + G'(\bar{\xi})(\xi_i - \bar{\xi}) + \frac{\kappa}{2}(\xi_i - \bar{\xi})^2$. Summing over i and using $\sum_i(\xi_i - \bar{\xi}) = 0$ to eliminate the linear terms yields

$$\sum_{i=1}^{N} G(\xi_i) \;\geq\; N\,G(\bar{\xi}) + \frac{\kappa}{2} \sum_{i=1}^{N} (\xi_i - \bar{\xi})^2.$$

Since Theorem 6.3 gives $T_N(c) = (e^{\bar{\xi}}, \dots, e^{\bar{\xi}})$, the right-hand minimum is $N\,G(\bar{\xi})$. For $J = J_{\text{cost}}$ one has $G(t) = \cosh t - 1$, whence $G''(t) = \cosh t \geq 1$. ☐

Layer 3: Continuous-time convergence.

Theorem A2 (Continuous-time convergence to the Jensen equilibrium)**.** *Fix an admissible cost J, let $G := J \circ \exp$, and assume G is C^1 with locally Lipschitz derivative (for instance $G \in C^2$) and κ-strongly convex on $\mathbb{R}$ for some $\kappa > 0$. Let $N \geq 2$ and $Q \in \mathbb{R}$. The gradient flow on the affine hyperplane $\{\xi \in \mathbb{R}^N : \sum_i \xi_i = Q\}$,*

$$\dot{\xi}_i = -G'(\xi_i) + \frac{1}{N} \sum_{j=1}^{N} G'(\xi_j), \tag{A1}$$

preserves the mean $\bar{\xi} = Q/N$ *and satisfies*

$$\operatorname{Var}(\xi(t)) := \frac{1}{N}\sum_{i=1}^{N}\big(\xi_i(t) - \bar{\xi}\big)^2 \;\le\; e^{-2\kappa t}\operatorname{Var}(\xi(0)).$$

Consequently, $\xi(t) \to (\bar{\xi}, \ldots, \bar{\xi})$ *and* $c_i(t) = e^{\xi_i(t)} \to e^{Q/N}$ *exponentially, reaching the same equilibrium as the variational update. For* $J = J_{\text{cost}}$*, one may take* $\kappa = 1$.

Proof. Since G' is locally Lipschitz on $\mathbb{R}$, the right-hand side of (A1) is locally Lipschitz on $\mathbb{R}^N$; the Picard–Lindelöf theorem gives a maximal classical solution on some interval $[0, T_{\max})$.

Constraint preservation.

$$\frac{d}{dt}\sum_{i=1}^{N}\xi_i = -\sum_{i=1}^{N}G'(\xi_i) + \sum_{i=1}^{N}\frac{1}{N}\sum_{j=1}^{N}G'(\xi_j) = 0,$$

so $\bar{\xi} = Q/N$ is constant.

Variance decay. Differentiating $\operatorname{Var}(\xi) = \frac{1}{N}\sum_i(\xi_i - \bar{\xi})^2$ and using $\sum_i(\xi_i - \bar{\xi}) = 0$ to replace $G'(\xi_i)$ by $G'(\xi_i) - G'(\bar{\xi})$,

$$\begin{aligned}\frac{d}{dt}\operatorname{Var}(\xi) &= \frac{2}{N}\sum_{i=1}^{N}(\xi_i - \bar{\xi})\dot{\xi}_i \\ &= -\frac{2}{N}\sum_{i=1}^{N}(\xi_i - \bar{\xi})\Big(G'(\xi_i) - G'(\bar{\xi})\Big).\end{aligned}$$

By κ-strong convexity of G, $(\xi_i - \bar{\xi})(G'(\xi_i) - G'(\bar{\xi})) \ge \kappa(\xi_i - \bar{\xi})^2$, so $\frac{d}{dt}\operatorname{Var}(\xi) \le -2\kappa\operatorname{Var}(\xi)$. Grönwall's inequality [60, p. 24] then gives

$$\operatorname{Var}(\xi(t)) \le e^{-2\kappa t}\operatorname{Var}(\xi(0)) \qquad \text{for all } t \in [0, T_{\max}). \tag{A2}$$

Global existence. From (A2), $|\xi_i(t) - \bar{\xi}| \le \sqrt{N\operatorname{Var}(\xi(0))}$ for all i and all $t \in [0, T_{\max})$. Since $\bar{\xi} = Q/N$ is fixed, every component $\xi_i(t)$ remains in the bounded interval $[Q/N - R,\, Q/N + R]$ with $R := \sqrt{N\operatorname{Var}(\xi(0))}$. On this interval, G' is bounded (it is continuous on a compact set), so the right-hand side of (A1) is uniformly bounded. A uniformly bounded derivative cannot produce blow-up in finite time, hence $T_{\max} = \infty$.

Conclusion. Inequality (A2) holds for all $t \ge 0$, giving $\operatorname{Var}(\xi(t)) \to 0$. Since $\bar{\xi}(t) = Q/N$ is constant in t and $(\xi_i(t) - Q/N)^2 \le N\operatorname{Var}(\xi(t))$ for each i, each $\xi_i(t) \to Q/N$, hence $c_i(t) = e^{\xi_i(t)} \to e^{Q/N}$ exponentially. For $J = J_{\text{cost}}$, $G(t) = \cosh t - 1$ is 1-strongly convex ($G''(t) = \cosh t \ge 1$), so $\kappa = 1$ and the rate is e^{-2t}. □

Remark A3 (Role of the continuous-time theorem)**.** *Theorem A2 upgrades the interpretation of Step B3: the Jensen equilibrium is an exponentially attracting fixed point of the natural constrained gradient flow on each level set of* $\log r$*, not merely a one-step minimizer.*

References

1. J. Aczél, *Lectures on Functional Equations and Their Applications*, Mathematics in Science and Engineering, vol. 19, Academic Press, New York, 1966.
2. J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Encyclopedia of Mathematics and Its Applications, vol. 31, Cambridge University Press, Cambridge, 1989. https://doi.org/10.1017/cbo9781139086578
3. P. L. Kannappan, *Functional Equations and Inequalities with Applications*, Springer Monographs in Mathematics, Springer, New York, 2009. https://doi.org/10.1007/978-0-387-89492-8
4. H. Stetkær, *Functional Equations on Groups*, World Scientific, Singapore, 2013. https://doi.org/10.1142/8830

5. M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities: Cauchy's Equation and Jensen's Inequality*, Państwowe Wydawnictwo Naukowe (PWN), Warsaw, 1985. https://doi.org/10.1007/978-3-7643-8749-5
6. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 6th ed., Oxford University Press, Oxford, 2008. https://doi.org/10.1093/oso/9780199219858.001.0001
7. S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*, Ellis Horwood, Chichester, 1989.
8. T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, New York, 2001. https://doi.org/10.1002/9781118033067
9. R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed., Cambridge University Press, Cambridge, 2013. https://doi.org/10.1017/CBO9781139020411
10. D. Damanik, M. Embree, and A. Gorodetski, The Fibonacci Hamiltonian, *Invent. Math.* **206** (2016), 629–692. (Preprint, arXiv:1403.7823, 2014.) https://doi.org/10.1007/s00222-016-0660-x
11. E. Inönü and E. P. Wigner, On the contraction of groups and their representations, *Proc. Natl. Acad. Sci. USA* **39** (1953), no. 6, 510–524. https://doi.org/10.1073/pnas.39.6.510
12. E. J. Saletan, Contraction of Lie groups, *J. Math. Phys.* **2** (1961), 1–21. https://doi.org/10.1063/1.1724208
13. S. Helgason, *Groups and Geometric Analysis: Integral Geometry, Invariant Differential Operators, and Spherical Functions*, Academic Press, Orlando, FL, 1984. https://doi.org/10.1090/surv/083
14. G. D. Mostow, Quasi-conformal mappings in n-space and the rigidity of hyperbolic space forms, *Publ. Math. Inst. Hautes Études Sci.* **34** (1968), 53–104. https://doi.org/10.1007/BF02684590
15. G. A. Margulis, *Discrete Subgroups of Semisimple Lie Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 17, Springer, Berlin, 1991. https://doi.org/10.1007/978-3-642-51445-6
16. G. Prasad, Strong rigidity of $\mathbb{Q}$-rank one lattices, *Invent. Math.* **21** (1973), 255–286. https://doi.org/10.1007/bf01418789
17. R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970. https://doi.org/10.1515/9781400873173
18. C. P. Niculescu and L.-E. Persson, *Convex Functions and Their Applications: A Contemporary Approach*, 2nd ed., CMS Books in Mathematics, Springer, Cham, 2018. https://doi.org/10.1007/978-3-319-78337-6
19. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed., Cambridge University Press, Cambridge, 1952.
20. P. S. Bullen, *Handbook of Means and Their Inequalities*, Kluwer Academic, Dordrecht, 2003. https://doi.org/10.1007/978-94-017-0399-4
21. A. N. Kolmogorov, Sur la notion de la moyenne, *Atti della R. Accademia Nazionale dei Lincei* **12** (1930), 388–391.
22. M. Nagumo, Über eine Klasse der Mittelwerte, *Japanese J. Math.* **7** (1930), 71–79. https://doi.org/10.4099/jjm1924.7.0_71
23. B. de Finetti, Sul concetto di media, *Giornale dell'Istituto Italiano degli Attuari* **2** (1931), 369–396.
24. B. T. Polyak, Gradient methods for the minimization of functionals, *USSR Comput. Math. and Math. Phys.* **3** (1963), no. 4, 643–653. https://doi.org/10.1016/0041-5553(63)90382-3
25. S. Łojasiewicz, Une propriété topologique des sous-ensembles analytiques réels, *Colloques Internationaux du CNRS* **117** (1963), 87–89.
26. D. Bakry and M. Émery, Diffusions hypercontractives, in *Séminaire de Probabilités XIX, 1983/84*, Lecture Notes in Math. **1123**, Springer, Berlin, 1985, pp. 177–206. https://doi.org/10.1007/bfb0075847
27. D. Bakry, I. Gentil, and M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Grundlehren der mathematischen Wissenschaften, vol. 348, Springer, Cham, 2014. https://doi.org/10.1007/978-3-319-00227-9
28. H. Karimi, J. Nutini, and M. Schmidt, Linear convergence of gradient and proximal-gradient methods under the Polyak–Łojasiewicz condition, in *Machine Learning and Knowledge Discovery in Databases (ECML PKDD 2016)*, Lecture Notes in Computer Science, vol. 9851, Springer, Cham, 2016, pp. 795–811. https://doi.org/10.1007/978-3-319-46128-1_50
29. R. Jordan, D. Kinderlehrer, and F. Otto, The variational formulation of the Fokker–Planck equation, *SIAM J. Math. Anal.* **29** (1998), no. 1, 1–17. https://doi.org/10.1137/s0036141096303359
30. F. Otto and C. Villani, Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality, *J. Funct. Anal.* **173** (2000), no. 2, 361–400. https://doi.org/10.1006/jfan.1999.3557
31. L. Ambrosio, N. Gigli, and G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, 2nd ed., Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 2008. https://doi.org/10.1007/978-3-7643-8722-8
32. C. Villani, *Optimal Transport: Old and New*, Grundlehren der mathematischen Wissenschaften, vol. 338, Springer, Berlin, 2009. https://doi.org/10.1007/978-3-540-71050-9
33. I. Csiszár, Information-type measures of difference of probability distributions and indirect observation, *Studia Sci. Math. Hungar.* **2** (1967), 299–318.
34. L. M. Bregman, The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming, *USSR Comput. Math. and Math. Phys.* **7** (1967), no. 3, 200–217. https://doi.org/10.1016/0041-5553(67)90040-7
35. S. Kullback and R. A. Leibler, On information and sufficiency, *Ann. Math. Statist.* **22** (1951), no. 1, 79–86. https://doi.org/10.1214/aoms/1177729694

36. S. Amari and H. Nagaoka, *Methods of Information Geometry*, Translations of Mathematical Monographs, vol. 191, American Mathematical Society, Providence, RI, 2000. https://doi.org/10.1090/mmono/191
37. S. Amari, α-Divergence is unique, belonging to both classes of f-divergence and Bregman divergence, *IEEE Trans. Inform. Theory* **55** (2009), no. 11, 4925–4931. https://doi.org/10.1109/tit.2009.2030485
38. E. Noether, Invariante Variationsprobleme, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* (1918), 235–257.
39. G. C. Wick, A. S. Wightman, and E. P. Wigner, The intrinsic parity of elementary particles, *Phys. Rev.* **88** (1952), 101–105. https://doi.org/10.1103/PhysRev.88.101
40. J. Goldstone, Field theories with "superconductor" solutions, *Nuovo Cimento* **19** (1961), 154–164. https://doi.org/10.1007/bf02812722
41. R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That*, W. A. Benjamin, New York, 1964. https://doi.org/10.1515/9781400884230
42. R. Haag and D. Kastler, An algebraic approach to quantum field theory, *J. Math. Phys.* **5** (1964), no. 7, 848–861. https://doi.org/10.1063/1.1704187
43. S. Doplicher, R. Haag, and J. E. Roberts, Local observables and particle statistics. I, *Comm. Math. Phys.* **23** (1971), 199–230. https://doi.org/10.1007/BF01877742
44. S. Doplicher, R. Haag, and J. E. Roberts, Local observables and particle statistics. II, *Comm. Math. Phys.* **35** (1974), 49–85. https://doi.org/10.1007/bf01646454
45. D. Buchholz and K. Fredenhagen, Locality and the structure of particle states, *Comm. Math. Phys.* **84** (1982), no. 1, 1–54. https://doi.org/10.1007/bf01208370
46. G. W. Mackey, *The Mathematical Foundations of Quantum Mechanics*, W. A. Benjamin, New York, 1963.
47. L. Hardy, Quantum theory from five reasonable axioms, preprint, arXiv:quant-ph/0101012, 2001. https://doi.org/10.48550/arXiv.quant-ph/0101012
48. J. Barrett, Information processing in generalized probabilistic theories, *Phys. Rev. A* **75** (2007), 032304. https://doi.org/10.1103/PhysRevA.75.032304
49. H. Putnam, *Mathematics, Matter and Method: Philosophical Papers, Volume 1*, Cambridge University Press, Cambridge, 1975 (first published 1969). https://doi.org/10.1017/cbo9780511625268
50. B. C. van Fraassen, *The Scientific Image*, Oxford University Press, Oxford, 1980. https://doi.org/10.1093/0198244274.001.0001
51. J. Earman and J. Norton, What price spacetime substantivalism? The hole story, *British Journal for the Philosophy of Science* **38** (1987), no. 4, 515–525. https://doi.org/10.1093/bjps/38.4.515
52. S. Pardo-Guerra, J. Washburn, and E. Allahyarov, *Admissible Reciprocally Symmetric Costs: Combiner Existence and Classification*. Mathematics 2026, 14, 2157. https://doi.org/10.3390/math14122157.
53. D. H. Krantz, R. D. Luce, P. Suppes, and A. Tversky, *Foundations of Measurement, Volume I: Additive and Polynomial Representations*, Academic Press, New York, 1971.
54. F. S. Roberts, *Measurement Theory, with Applications to Decisionmaking, Utility, and the Social Sciences*, Encyclopedia of Mathematics and Its Applications, vol. 7, Cambridge University Press, Cambridge, 1985. https://doi.org/10.1017/cbo9780511759871
55. P. C. Fishburn, *Utility Theory for Decision Making*, Wiley, New York, 1970.
56. G. Debreu, *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*, Cowles Foundation Monograph 17, Yale University Press, New Haven, 1959.
57. M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap, *Aggregation Functions*, Encyclopedia of Mathematics and Its Applications, vol. 127, Cambridge University Press, Cambridge, 2009. https://doi.org/10.1017/cbo9781139644150
58. Washburn, J.; Zlatanović, M. *Uniqueness of the Canonical Reciprocal Cost*. Mathematics 2026, 14, 935. https://doi.org/10.3390/math14060935
59. Washburn, J.; Zlatanović, M.; Allahyarov, E. *The D'Alembert Inevitability Theorem*. Mathematics 2026, 14, 1386. https://doi.org/10.3390/math14081386
60. P. Hartman, *Ordinary Differential Equations*, John Wiley & Sons, New York, 1964. https://doi.org/10.1137/1.9780898719222