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Recognition-Operator Dynamics from Reciprocal Cost and an Eight-Tick Kernel

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Abstract

We study a finite-dimensional operator construction selected by two recognition-science inputs: the reciprocal comparison cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ and the three-dimensional eight-tick ledger kernel. One recognition period defines a neutral eight-sample register with cyclic shift P. The odd Fourier sector $\mathcal{V} = \ker(P^4 + I)$ is the quarter-turn core selected by the shift. On $\mathcal{V}$, the one-beat recognition step is exactly unitary. Its principal logarithm yields a distinguished self-adjoint Hamiltonian, and the principal-branch interpolation reproduces the discrete beats. Rank-one recognition defects on ray space reduce to transition-probability defects, so reversible symmetries are Wigner symmetries. A single site has a rigid four-level spectrum. After coupling finitely many sites, local ququart Weyl pairs generate the full finite-dimensional operator algebra, so arbitrary finite-dimensional Hermitian dynamics are representable on invariant subspaces. The same core also carries a reduced first-order action and an exact finite-dimensional time-sliced propagator representation.

Keywords: recognition operator; reciprocal cost; eight-tick kernel; quarter-turn core; Wigner symmetry; principal logarithm; finite-dimensional quantum mechanics

1. Introduction

Recognition Science studies comparison-driven dynamics through a cost on positive ratios and a discrete ledger of updates. Its geometric layer describes observables through recognizers and induced quotients [1]. The present paper uses two inputs from that program. The first is the reciprocal comparison cost fixed by coherent composition [2]. The second is the three-dimensional eight-tick ledger kernel selected by atomic, spatially complete schedules [3]. From these inputs we extract a finite-dimensional reversible core and its induced generator.

The problem addressed here is narrower than a full reconstruction of quantum theory. We do not derive arbitrary Hilbert-space kinematics from general operational axioms. We fix the eight-tick register selected by the recognition kernel and ask which reversible sector and which generator are distinguished inside that carrier. The construction is therefore conditional on the imported recognition-science inputs and finite-dimensional from the outset. The present paper also treats the three-dimensional kernel as an input. Other ledger dimensions would lead to different periods and different local core dimensions, and they are outside the present scope.

Once the carrier is fixed, the mathematical tools are standard. Comparison costs and divergences belong to information theory and information geometry [4–6]. The cost classification reduces to the multiplicative d'Alembert equation [7,8]. The generator uses the

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principal matrix logarithm [9]. Projective closure uses Wigner's theorem and its refinements [10–12]. The many-body operator basis uses finite Weyl systems and Schwinger's unitary bases [13,14]. The discrete-step setting also overlaps with quantum walks and Floquet theory [15–19]. The contribution is the synthesis of these standard ingredients on a carrier selected by reciprocal cost and an eight-tick ledger. It is not a claim of novelty for each ingredient in isolation.

The scope also differs from operational reconstructions of quantum theory [20–23]. Those programs ask how much of quantum theory follows from abstract physical axioms. Here the finite carrier is fixed first. The question is then which reversible core, which ray geometry, and which logarithmic generator are forced inside that carrier.

The paper has four parts. Section 2 recalls the reciprocal cost and the exact recognition action. Section 3 defines the neutral eight-sample register, the quarter-turn core, admissible extensions, and the one-step recognition update. Section 4 derives the principal-branch Hamiltonian, Wigner closure, the reduced first-order action, and the exact time-sliced propagator on the reversible core. Section 5 treats coupled cores and finite-dimensional representability. Section 6 states the operational transient law and the limits of the present construction.

The main conclusions are concise. The odd Fourier sector $\mathcal{V} = \ker(P^4 + I)$ is the distinguished minimal reversible core selected by the eight-tick shift. On $\mathcal{V}$, the one-beat recognition step is exactly unitary and has a distinguished principal logarithm. The induced ray-defect geometry closes under Wigner symmetries. The same core carries a reduced first-order action and an exact finite-dimensional time-sliced propagator representation. A single site has a rigid four-level spectrum. General finite spectra appear only after coupling finitely many local cores and, when needed, restricting to invariant subspaces.

2. Recognition-Science Inputs and the Exact Recognition Action

2.1. Unique Reciprocal Cost from Coherent Comparison

The basic recognition-science dynamical input is the claim that recognition proceeds by coherent comparison of positive ratios. Let $x > 0$ denote such a ratio and let $J(x)$ be the associated recognition cost. The coherence axiom is the multiplicative d'Alembert relation

$$J(xy) + J(x/y) = 2J(x)J(y) + 2J(x) + 2J(y), \qquad x, y > 0. \tag{1}$$

Following Ref. [2], it is natural to impose the normalization

$$J(1) = 0, \qquad \frac{\mathrm{d}^2}{\mathrm{d}u^2} J(\mathrm{e}^u)\Big|_{u=0} = 1. \tag{2}$$

The second condition is the unit log-curvature calibration at equilibrium.

Theorem 1 (Unique reciprocal cost)**.** *Let $J : (0, \infty) \to \mathbb{R}$ be continuous and satisfy Eqs. (1) and (2). Then*

$$J(x) = \frac{1}{2}\left(x + x^{-1}\right) - 1 = \cosh(\ln x) - 1. \tag{3}$$

In particular, $J(x) = J(x^{-1})$, $J(x) \geq 0$, and $J(x) = 0$ iff $x = 1$.

The proof is the classical reduction of Eq. (1) to d'Alembert's functional equation in logarithmic coordinates and is given in Appendix A. Recognition Science uses this theorem as a keystone result: cost is not chosen for convenience but fixed by coherent composition plus a single local calibration [2]. Near equilibrium,

$$J(e^{\eta}) = \cosh\eta - 1 = \frac{1}{2}\eta^2 + \frac{1}{24}\eta^4 + O(\eta^6). \tag{4}$$

The quadratic law is therefore a local quadratic approximation, not the primitive dynamics.

2.2. Eight-Tick Schedule and the Neutral Register

The second recognition-science input comes from the cost-first ledger framework. Ref. [3] derives atomic ticks, balanced postings, discrete potentials, and the minimal schedule period 2^d for spatially complete atomic walks on the hypercube Q_d, where *atomic* means one edge-flip per tick and *spatially complete* means that all 2^d vertices are visited over one period. In the three-dimensional case used throughout the present manuscript, the minimal period is eight and is realized by a Gray cycle. We take that eight-tick period as the finite temporal carrier of a single recognition cycle.

One full cycle is therefore naturally represented by a vector

$$\psi = (\psi_0, \psi_1, \ldots, \psi_7) \in \mathbb{C}^8, \tag{5}$$

and one-beat advance is the cyclic shift. The complex notation is a convenient Fourier carrier for the eight-beat register. Equivalently one may start from the real eight-beat register with its Euclidean inner product and use complex coordinates only to diagonalize the shift. The intrinsic real statement appears below, where $K = P^2|_{\mathcal{V}_{\mathbb{R}}}$ supplies the complex structure on the reversible core.

$$(P\psi)_t := \psi_{t-1 \bmod 8}. \tag{6}$$

Closed-cycle balance motivates removal of the DC mode. We therefore work on the neutral register $\mathcal{W} \subset \mathbb{C}^8$. The shift preserves $\mathcal{W}$.

Let

$$\omega := e^{-i\pi/4} \tag{7}$$

and define the DFT-8 modes

$$e_k(t) = \frac{\omega^{kt}}{\sqrt{8}}, \qquad k = 0, 1, \ldots, 7. \tag{8}$$

Then $Pe_k = \omega^{-k} e_k$, and neutrality removes e_0. Thus

$$\mathcal{W} := \left\{ \psi \in \mathbb{C}^8 : \sum_{t=0}^{7} \psi_t = 0 \right\} = \bigoplus_{k=1}^{7} E_k, \qquad E_k := \mathbb{C}e_k. \tag{9}$$

Throughout the operator sections, $\langle \cdot, \cdot \rangle$ denotes the Hilbert-space inner product, conjugate-linear in the first slot and linear in the second. The odd-mode sector is

$$\mathcal{V} := \ker(P^4 + I) = E_1 \oplus E_3 \oplus E_5 \oplus E_7. \tag{10}$$

Its orthogonal complement inside $\mathcal{W}$ is the even-mode sector

$$\mathcal{E}_{\text{even}} := E_2 \oplus E_4 \oplus E_6 = \mathcal{V}^{\perp} \cap \mathcal{W}. \tag{11}$$

Thus

$$\mathcal{W} = \mathcal{V} \oplus \mathcal{E}_{\text{even}}. \tag{12}$$

The quarter-turn core is therefore distinguished by the eight-tick shift itself. It does not depend on any later structural choice.

2.3. Exact Recognition Action

Let $q(t) = (q^1(t), \ldots, q^n(t))$ be configuration variables and let $\eta^a(q, \dot{q}, t) \in \mathbb{R}$, $a = 1, \ldots, N$, denote local mismatch coordinates. The exact recognition Lagrangian is

$$\mathcal{L}_R(q, \dot{q}, t) := \sum_{a=1}^{N} J\left(e^{\eta^a(q, \dot{q}, t)}\right), \tag{13}$$

and the exact recognition action is

$$\mathcal{A}_R[q] := \int_{t_i}^{t_f} \mathcal{L}_R(q, \dot{q}, t)\, dt. \tag{14}$$

Here t is a path parameter. As written, $\mathcal{A}_R$ is dimensionless. If one wants a quantity with units of action, one may multiply $\mathcal{A}_R$ by any fixed constant A_0 with units of action; the stationary curves are unchanged. Because $J(e^\eta) = \cosh \eta - 1$, the variational calculation is elementary and the Euler-Lagrange equation closes in elementary functions [24].

Theorem 2 (Exact recognition stationary action)**.** *Stationary paths of Eq.* (14) *with fixed endpoints satisfy*

$$\frac{d}{dt}\left(\sum_a \sinh \eta^a \frac{\partial \eta^a}{\partial \dot{q}^i}\right) - \sum_a \sinh \eta^a \frac{\partial \eta^a}{\partial q^i} = 0. \tag{15}$$

The proof is recorded in Appendix B. The result is exact. No small-mismatch expansion is used in deriving Eq. (15).

2.4. Quadratic Approximation and Defect Geometry

This subsection serves one specific purpose. Later sections use a Hilbert-space defect on structured sectors, and the estimate below explains why that defect is the correct local quadratic avatar of the exact reciprocal action near a reversible sector. The canonical reciprocal cost has a quadratic leading term, but the bridge is local and uses real normal coordinates rather than a global identification of variables.

Proposition 1 (Quantitative quadratic approximation)**.** *Fix $\rho > 0$. If $|\eta^a| \le \rho$ for all a, then*

$$0 \le \sum_a J(e^{\eta^a}) - \frac{1}{2}\sum_a (\eta^a)^2 \le \frac{\cosh \rho}{24} \sum_a (\eta^a)^4. \tag{16}$$

In particular,

$$\sum_a J(e^{\eta^a}) = \frac{1}{2}\sum_a (\eta^a)^2 + O\left(\left(\sum_a (\eta^a)^2\right)^2\right) \tag{17}$$

for $\eta \to 0$.

Proof. For one real variable, Taylor's theorem gives

$$\cosh \eta - 1 - \frac{1}{2}\eta^2 = \frac{\cosh \xi}{24}\eta^4 \tag{18}$$

for some ξ between 0 and η. Since $|\xi| \le \rho$ and cosh is even and increasing on $[0, \infty)$,

$$0 \le \cosh \eta - 1 - \frac{1}{2}\eta^2 \le \frac{\cosh \rho}{24}\eta^4. \tag{19}$$

Summing over a yields Eq. (16). The estimate $\sum_a (\eta^a)^4 \le (\sum_a (\eta^a)^2)^2$ gives Eq. (17). $\square$

If $\mathcal{S}$ is any linear subspace of a finite-dimensional Hilbert space, choose a real orthonormal basis $\{n_a\} \subset \mathcal{S}^\perp$ of the normal space and write

$$\psi = \psi_{\mathcal{S}} + \sum_a \xi^a n_a, \qquad \psi_{\mathcal{S}} \in \mathcal{S}, \tag{20}$$

for the corresponding real normal coordinates relative to $\mathcal{S}$. Then the squared-distance defect is

$$\mathcal{D}_{\mathcal{S}}(\psi) := \|(I - \Pi_{\mathcal{S}})\psi\|^2 = \sum_a (\xi^a)^2. \tag{21}$$

Applying Proposition 1 to these real normal coordinates gives the local bridge

$$\sum_a J(\mathrm{e}^{\xi^a}) = \frac{1}{2}\mathcal{D}_{\mathcal{S}}(\psi) + O\Big(\mathcal{D}_{\mathcal{S}}(\psi)^2\Big) \tag{22}$$

in normal coordinates. This is the mathematically correct sense in which the defect geometry gives the local quadratic approximation of the exact recognition action. It is a local coordinate bridge, not an identity between the mismatch variables of $\mathcal{A}_R$ and the Hilbert-space coordinates of the reversible core.

3. Canonical Minimal Core, Structured Extensions, and the One-Step Recognition Map

The finite-dimensional recognition kernel has two layers. The first is canonical and contains no remaining structural ambiguity: the neutral eight-tick register $\mathcal{W}$, the shift P, and the quarter-turn core $\mathcal{V}$. The second is optional and models a larger structured vocabulary. The present paper separates them explicitly.

Definition 1 (Canonical minimal core and admissible structured extension)**.** *The* minimal core singled out by the shift *is the pair $(\mathcal{W}, \mathcal{V})$ with $\mathcal{V}$ given by Eq. (10). An* admissible structured extension *is a subspace $\mathcal{S} \subseteq \mathcal{W}$ such that*

$$\mathcal{V} \subseteq \mathcal{S}, \qquad P\mathcal{S} = \mathcal{S}. \tag{23}$$

For such $\mathcal{S}$, we use the previously introduced squared-distance defect $\mathcal{D}_{\mathcal{S}}(\psi) = \|(I - \Pi_{\mathcal{S}})\psi\|^2$ and define the one-step recognition update by

$$\widehat{\mathcal{R}}_{\mathcal{S}} := \Pi_{\mathcal{S}} P. \tag{24}$$

The minimal update is the special case $\mathcal{S} = \mathcal{V}$:

$$\widehat{\mathcal{R}}_{\mathcal{V}} = \Pi_{\mathcal{V}} P. \tag{25}$$

The point of Definition 1 is that the core $\mathcal{V}$ is no longer hidden inside a larger underspecified sector. Larger structured vocabularies may be considered, but they are now optional extensions of a closed minimal core, not prerequisites for defining the reversible quantum sector.

Proposition 2 (All extension freedom lies outside the quarter-turn core)**.** *Let $\mathcal{S} \subseteq \mathcal{W}$ be an admissible structured extension. Then*

$$\mathcal{S} = \mathcal{V} \oplus \mathcal{M}, \qquad \mathcal{M} := \mathcal{S} \cap \mathcal{V}^\perp, \tag{26}$$

where $\mathcal{M} \subseteq \mathcal{E}_{\mathrm{even}}$ is uniquely determined and P-invariant.

Proof. Because P is unitary and $\mathcal{V}$ is P-invariant, $\mathcal{V}^\perp$ is also P-invariant. Inside $\mathcal{W}$, Eq. (12) identifies $\mathcal{V}^\perp$ with $\mathcal{E}_{\mathrm{even}}$. Since $\mathcal{V} \subseteq \mathcal{S}$, every $\psi \in \mathcal{S}$ decomposes uniquely as $\psi = \psi_{\mathcal{V}} + \psi_\perp$ with $\psi_{\mathcal{V}} \in \mathcal{V}$ and $\psi_\perp \in \mathcal{V}^\perp$. Because $\psi, \psi_{\mathcal{V}} \in \mathcal{S}$, one has $\psi_\perp \in \mathcal{S} \cap \mathcal{V}^\perp = \mathcal{M}$. Hence $\mathcal{S} \subseteq \mathcal{V} \oplus \mathcal{M}$, and the reverse inclusion is immediate. Finally,

$$P\mathcal{M} \subseteq P\mathcal{S} \cap P\mathcal{V}^\perp = \mathcal{S} \cap \mathcal{V}^\perp = \mathcal{M}. \tag{27}$$

Thus $\mathcal{M}$ is P-invariant. □

Proposition 2 shows that any larger structured vocabulary can only add an invariant complement inside the even-mode sector. It cannot change the quarter-turn core or any theorem stated on that core.

3.1. Why Propagation and Commit Must Separate

The recognition step $\widehat{\mathcal{R}}_{\mathcal{S}} = \Pi_{\mathcal{S}} P$ should be read as a composition of two distinct operations: reversible propagation by P, followed by irreversible commit by $\Pi_{\mathcal{S}}$. A finite-dimensional obstruction explains why these roles cannot be merged into a single unitary defect-decreasing map.

Proposition 3 (Trace obstruction: strict descent is incompatible with unitarity)**.** *Let $\mathcal{H}$ be finite-dimensional, let $M \geq 0$ be Hermitian, and define*

$$D(v) := \langle v, Mv \rangle. \tag{28}$$

If U is unitary on $\mathcal{H}$ and

$$D(Uv) \leq D(v) \qquad \text{for all } v \in \mathcal{H}, \tag{29}$$

then in fact

$$D(Uv) = D(v) \qquad \text{for all } v \in \mathcal{H}. \tag{30}$$

Therefore no genuinely defect-decreasing update can be unitary on the same sector.

The proof is the one-line trace argument given in Appendix C. In the present recognition kernel, the natural choice is $M = I - \Pi_{\mathcal{S}}$. Proposition 3 is exactly why recognition dynamics must separate propagation from commit.

Proposition 4 (Propagation-commit identities)**.** *Let $\mathcal{S} \subseteq \mathcal{W}$ be any admissible structured extension. Then*

$$\Pi_{\mathcal{S}} P = P \Pi_{\mathcal{S}}, \tag{31a}$$
$$\mathcal{D}_{\mathcal{S}}(P\psi) = \mathcal{D}_{\mathcal{S}}(\psi), \tag{31b}$$
$$\widehat{\mathcal{R}}_{\mathcal{S}} \psi = P\psi \qquad (\psi \in \mathcal{S}), \tag{31c}$$
$$\mathcal{D}_{\mathcal{S}}(\widehat{\mathcal{R}}_{\mathcal{S}} \psi) = 0. \tag{31d}$$

Proof. Because P is unitary and $P\mathcal{S} = \mathcal{S}$, one also has $P\mathcal{S}^\perp = \mathcal{S}^\perp$. Hence the orthogonal projector $\Pi_{\mathcal{S}}$ commutes with P, proving Eq. (31a). Then

$$\mathcal{D}_{\mathcal{S}}(P\psi) = \|(I - \Pi_{\mathcal{S}}) P\psi\|^2 = \|P(I - \Pi_{\mathcal{S}})\psi\|^2 = \mathcal{D}_{\mathcal{S}}(\psi), \tag{32}$$

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which is Eq. (31b). If $\psi \in \mathcal{S}$, then $P\psi \in \mathcal{S}$, so $\Pi_{\mathcal{S}} P\psi = P\psi$, proving Eq. (31c). Finally, $\widehat{\mathcal{R}}_{\mathcal{S}}\psi \in \mathcal{S}$, so its defect vanishes. □

Equation (31b) is important. The shift itself does not lower the defect. All irreversible decrease is carried by commit. That is the precise recognition-theoretic resolution of the apparent conflict between defect descent and unitary motion.

Proposition 5 (Uniqueness of the symmetry-compatible finite-time commit)**.** *Let $C : \mathcal{W} \to \mathcal{W}$ be a linear map such that*

1. *$C\psi = \psi$ for every $\psi \in \mathcal{S}$;*
2. *C is self-adjoint and positive;*
3. *C commutes with every unitary of the form $U = I_{\mathcal{S}} \oplus U_{\perp}$ on the orthogonal decomposition $\mathcal{W} = \mathcal{S} \oplus \mathcal{S}^{\perp}$.*

Then there is a unique scalar $\lambda \geq 0$ such that

$$C = \Pi_{\mathcal{S}} + \lambda(I - \Pi_{\mathcal{S}}) = C_{\lambda,\mathcal{S}}. \tag{33}$$

If in addition $\mathcal{D}_{\mathcal{S}}(CP\psi) \leq \mathcal{D}_{\mathcal{S}}(\psi)$ for all $\psi \in \mathcal{W}$, then $0 \leq \lambda \leq 1$.

Proof. Because C is self-adjoint and $C|_{\mathcal{S}} = I$, for $s \in \mathcal{S}$ and $w \in \mathcal{S}^{\perp}$ one has

$$\langle s, Cw \rangle = \langle Cs, w \rangle = \langle s, w \rangle = 0, \tag{34}$$

so $C\mathcal{S}^{\perp} \subseteq \mathcal{S}^{\perp}$. Hence $C = I_{\mathcal{S}} \oplus T$ for a self-adjoint positive operator T on $\mathcal{S}^{\perp}$. By assumption, $TU_{\perp} = U_{\perp}T$ for every unitary $U_{\perp}$ on $\mathcal{S}^{\perp}$. Choose an orthonormal eigenbasis $\{e_j\}$ of T, with $Te_j = \mu_j e_j$. For any $j \neq k$, let U_{jk} be the unitary that swaps e_j and e_k and fixes the remaining basis vectors. The commutation relation $TU_{jk} = U_{jk}T$ then gives $\mu_j = \mu_k$. Thus all eigenvalues of T are equal, so $T = \lambda I_{\mathcal{S}^{\perp}}$ for a unique $\lambda \geq 0$. This proves Eq. (33). Moreover,

$$(I - \Pi_{\mathcal{S}})CP = \lambda(I - \Pi_{\mathcal{S}})P, \tag{35}$$

so Proposition 4 gives

$$\mathcal{D}_{\mathcal{S}}(CP\psi) = \lambda^2 \mathcal{D}_{\mathcal{S}}(\psi). \tag{36}$$

If $\mathcal{D}_{\mathcal{S}}(CP\psi) \leq \mathcal{D}_{\mathcal{S}}(\psi)$ for all ψ, then $\lambda^2 \leq 1$. □

The previous proposition shows that the relaxed family is not an ad hoc interpolation. Once one asks for a positive finite-time commit that fixes $\mathcal{S}$ exactly and is isotropic on the orthogonal complement, there is only one possible form, namely Eq. (33), with a single scalar relaxation factor λ.

Proposition 6 (Relaxed commit family)**.** *For $0 \leq \lambda \leq 1$, define*

$$C_{\lambda,\mathcal{S}} := \Pi_{\mathcal{S}} + \lambda(I - \Pi_{\mathcal{S}}), \qquad \widehat{\mathcal{R}}_{\lambda,\mathcal{S}} := C_{\lambda,\mathcal{S}} P. \tag{37}$$

Then $C_{\lambda,\mathcal{S}}$ acts as the identity on $\mathcal{S}$, the projector update $\widehat{\mathcal{R}}_{\mathcal{S}} = \Pi_{\mathcal{S}} P$ is the endpoint $\lambda = 0$, and

$$\mathcal{D}_{\mathcal{S}}(\widehat{\mathcal{R}}_{\lambda,\mathcal{S}}\psi) = \lambda^2 \mathcal{D}_{\mathcal{S}}(\psi) \tag{38}$$

for all $\psi \in \mathcal{W}$. Consequently,

$$\mathcal{D}_{\mathcal{S}}(\widehat{\mathcal{R}}^{n}_{\lambda,\mathcal{S}}\psi) = \lambda^{2n}\mathcal{D}_{\mathcal{S}}(\psi) \tag{39}$$

for all integers $n \geq 0$.

Proof. Because P commutes with $\Pi_{\mathcal{S}}$, it also commutes with $C_{\lambda,\mathcal{S}}$. Moreover,

$$(I - \Pi_{\mathcal{S}})C_{\lambda,\mathcal{S}} = \lambda(I - \Pi_{\mathcal{S}}). \tag{40}$$

Hence

$$(I - \Pi_{\mathcal{S}})\widehat{\mathcal{R}}_{\lambda,\mathcal{S}}\psi = \lambda(I - \Pi_{\mathcal{S}})P\psi, \tag{41}$$

and Proposition 4 gives Eq. (38). Iterating and using commutation yields the geometric decay formula. □

Proposition 6 answers a natural observability concern. The projector commit used in the minimal kernel is the instantaneous endpoint ($\lambda = 0$). A finite-time relaxation onto the structured sector is obtained by any $\lambda \in (0, 1)$, while the reversible core $\mathcal{V} \subseteq \mathcal{S}$ is still left exact.

Theorem 3 (Operational separation of transient relaxation and reversible core motion)**.** *Let*

$$\psi_0 = s_0 + c_0, \qquad s_0 := \Pi_{\mathcal{S}}\psi_0 \in \mathcal{S}, \quad c_0 := (I - \Pi_{\mathcal{S}})\psi_0 \in \mathcal{S}^{\perp}. \tag{42}$$

Under the relaxed update $\psi_n := \widehat{\mathcal{R}}^{n}_{\lambda,\mathcal{S}}\psi_0$, one has

$$\psi_n = P^n s_0 + \lambda^n P^n c_0. \tag{43}$$

Consequently,

$$\langle \psi_n, (I - \Pi_{\mathcal{S}})\psi_n \rangle = \lambda^{2n}\|c_0\|^2. \tag{44}$$

If Q is any observable supported on the structured sector, $Q = \Pi_{\mathcal{S}}Q\Pi_{\mathcal{S}}$, then

$$\langle \psi_n, Q\psi_n \rangle = \langle s_0, P^{-n}QP^n s_0 \rangle. \tag{45}$$

Thus the same trajectory exhibits exact exponential decay of the off-sector population and exact unitary evolution of all in-sector observables.

Proof. Because $C_{\lambda,\mathcal{S}}$ commutes with P, one has

$$\widehat{\mathcal{R}}^{n}_{\lambda,\mathcal{S}} = C^{n}_{\lambda,\mathcal{S}}P^n. \tag{46}$$

Since $C_{\lambda,\mathcal{S}}$ is the identity on $\mathcal{S}$ and multiplication by λ on $\mathcal{S}^{\perp}$,

$$C^{n}_{\lambda,\mathcal{S}} = \Pi_{\mathcal{S}} + \lambda^n(I - \Pi_{\mathcal{S}}), \tag{47}$$

which gives Eq. (43). Equation (44) follows because P is unitary and preserves $\mathcal{S}^{\perp}$. If $Q = \Pi_{\mathcal{S}}Q\Pi_{\mathcal{S}}$, then $QP^n c_0 = 0$ and all cross terms vanish, yielding Eq. (45). □

Theorem 3 is the clean observable signature of the relaxed model. The transient does not disappear after the first tick. Rather, one channel carries a pure exponential relaxation law, while a second channel evolves exactly as the reversible core dynamics.

Definition 2 (Defect proxy and calibration envelope)**.** *Let $\mathcal{U} \subseteq \mathcal{W}$ be a preparation class. A positive observable O is called a* defect proxy *for $\mathcal{D}_\mathcal{S}$ on $\mathcal{U}$ if there exist constants $0 < m_O \leq M_O$ such that*

$$m_O\, \mathcal{D}_\mathcal{S}(\psi) \leq \langle \psi, O\psi \rangle \leq M_O\, \mathcal{D}_\mathcal{S}(\psi) \tag{48}$$

for every nonzero $\psi \in \mathcal{U}$. The pair (m_O, M_O) is the associated calibration envelope. Because both sides are homogeneous of degree two, this is equivalent to checking the inequality only on normalized states.

Proposition 7 (Operational calibration of λ from defect proxies)**.** *Let O be a defect proxy on a P- and $C_{\lambda,\mathcal{S}}$-invariant preparation class $\mathcal{U}$. For*

$$\psi_n := \widehat{\mathcal{R}}^{\,n}_{\lambda,\mathcal{S}}\psi_0 \in \mathcal{U}, \qquad y_n := \langle \psi_n, O\psi_n \rangle, \tag{49}$$

one has

$$m_O \lambda^{2n} \mathcal{D}_\mathcal{S}(\psi_0) \leq y_n \leq M_O \lambda^{2n} \mathcal{D}_\mathcal{S}(\psi_0). \tag{50}$$

Hence, whenever $y_n, y_{n+1} > 0$,

$$\sqrt{\frac{m_O}{M_O}\frac{y_{n+1}}{y_n}} \leq \lambda \leq \sqrt{\frac{M_O}{m_O}\frac{y_{n+1}}{y_n}}. \tag{51}$$

If $O = I - \Pi_\mathcal{S}$, then $m_O = M_O = 1$ and

$$\lambda = \sqrt{\frac{y_{n+1}}{y_n}} \tag{52}$$

exactly.

Proof. Equation (50) follows by combining Definition 2 with Proposition 6. Dividing the inequalities for $n + 1$ and n gives Eq. (51). The exact-projector case is Eq. (44). □

Remark 1 (P-invariance of the defect Hessian)**.** *For every admissible extension,*

$$P^\dagger(I - \Pi_\mathcal{S})P = I - \Pi_\mathcal{S}. \tag{53}$$

Equivalently, the defect Hessian $2(I - \Pi_\mathcal{S})$ is constant and P-invariant. This is immediate from Eq. (31a).

4. Quarter-Turn Core and Recognition-Derived Schrödinger Flow

We now prove that the reversible quantum sector is exactly the quarter-turn core and is independent of the optional extension $\mathcal{S}$.

Theorem 4 (Quarter-turn core and extension-independence of the reversible sector)**.** *On the underlying real space $\mathcal{W}_\mathbb{R}$,*

$$\ker_\mathbb{R}(P^4 + I) = \mathcal{V}_\mathbb{R}. \tag{54}$$

Consequently:

(i) *$\mathcal{V}_\mathbb{R}$ is the unique maximal real P-invariant subspace of $\mathcal{W}_\mathbb{R}$ on which*

$$K := P^2|_{\mathcal{V}_\mathbb{R}} \tag{55}$$

satisfies

$$K^2 = -I. \tag{56}$$

(ii) *If $\mathcal{S} \subseteq \mathcal{W}$ is any admissible structured extension, then the maximal real P-invariant subspace of $\mathcal{S}_{\mathbb{R}}$ on which $P^4 = -I$ is still $\mathcal{V}_{\mathbb{R}}$. In particular, the reversible quantum core is independent of $\mathcal{M}$.*

(iii) *For every admissible $\mathcal{S}$,*

$$\widehat{\mathcal{R}}_{\mathcal{S}}\big|_{\mathcal{V}} = P\big|_{\mathcal{V}}. \tag{57}$$

Thus the one-step recognition update is exactly unitary on $\mathcal{V}$.

The operator $K = P^2|_{\mathcal{V}_{\mathbb{R}}}$ is singled out because it acts on the underlying real space and satisfies $K^2 = -I$: two recognition beats therefore supply the intrinsic complex structure on the reversible core.

Proof. On each Fourier mode E_k,

$$P^4 e_k = \omega^{-4k} e_k = \mathrm{e}^{\mathrm{i}\pi k} e_k = (-1)^k e_k. \tag{58}$$

Hence $(P^4 + I)e_k = 0$ for odd k and $(P^4 + I)e_k = 2e_k$ for even k. Therefore the complex kernel of $P^4 + I$ is exactly $\mathcal{V}$, and the real kernel is $\mathcal{V}_{\mathbb{R}}$, proving Eq. (54). Any real subspace on which $K^2 = -I$ satisfies $P^4 = -I$, hence is contained in $\ker_{\mathbb{R}}(P^4 + I) = \mathcal{V}_{\mathbb{R}}$. This proves maximality and uniqueness in part (i). Part (ii) is immediate because $\mathcal{V} \subseteq \mathcal{S}$. Part (iii) follows from Eq. (31c) of Proposition 4. □

Theorem 4 identifies the quarter-turn core as the canonical maximal sector on which two beats act as the imaginary unit, and that statement is unaffected by any admissible structured extension.

4.1. The Principal Logarithm and the Recognition-Derived Hamiltonian

The odd-mode spectrum of the shift is

$$\sigma\big(P\big|_{\mathcal{V}}\big) \subseteq \Big\{\mathrm{e}^{-\mathrm{i}\pi/4},\ \mathrm{e}^{-3\mathrm{i}\pi/4},\ \mathrm{e}^{3\mathrm{i}\pi/4},\ \mathrm{e}^{\mathrm{i}\pi/4}\Big\}. \tag{59}$$

In particular, $-1 \notin \sigma(P|_{\mathcal{V}})$, so the principal matrix logarithm is well defined [9].

Theorem 5 (Canonical principal-branch generator of the recognition beat)**.** *Let τ_0 denote one recognition beat. Define*

$$H_{\mathrm{eff}} := \frac{\mathrm{i}\hbar}{\tau_0} \log\big(P\big|_{\mathcal{V}}\big). \tag{60}$$

Then H_{eff} is self-adjoint with spectral window

$$\sigma\Big(\frac{\tau_0}{\hbar} H_{\mathrm{eff}}\Big) \subset (-\pi, \pi), \tag{61}$$

and

$$U_t := \exp\Big(-\frac{\mathrm{i}t}{\hbar} H_{\mathrm{eff}}\Big) \tag{62}$$

is the strongly continuous unitary one-parameter group determined by this principal-branch choice, satisfying

$$U_{\tau_0} = P|_{\mathcal{V}} = \widehat{\mathcal{R}}_{\mathcal{S}}|_{\mathcal{V}} \tag{63}$$

for every admissible structured extension $\mathcal{S}$. More generally, for every integer n,

$$U_{n\tau_0} = P^n|_{\mathcal{V}}. \tag{64}$$

Consequently, the discrete recognition-beat sequence $\psi_n := P^n\psi_0$ is reproduced exactly by the continuous interpolation $\psi(t) = U_t\psi_0$, and

$$\mathrm{i}\hbar\,\partial_t\psi(t) = H_{\mathrm{eff}}\psi(t) \tag{65}$$

for every trajectory $\psi(t) = U_t\psi_0$ in $\mathcal{V}$.

Proof. Because $P|_{\mathcal{V}}$ is unitary and its spectrum avoids the branch cut point -1, the principal logarithm exists and is skew-adjoint [9]. The spectrum of $\log(P|_{\mathcal{V}})$ lies on the imaginary strip $\mathrm{i}(-\pi, \pi)$, so Eq. (60) defines a self-adjoint operator and Eq. (61) holds. In finite dimension, U_t is automatically a strongly continuous unitary one-parameter group [25]. At one beat,

$$U_{\tau_0} = \exp\!\left(-\frac{\mathrm{i}}{\hbar}\cdot\frac{\mathrm{i}\hbar}{\tau_0}\log(P|_{\mathcal{V}})\cdot\tau_0\right) = \exp\!\big(\log(P|_{\mathcal{V}})\big) = P|_{\mathcal{V}}, \tag{66}$$

which is also $\widehat{\mathcal{R}}_{\mathcal{S}}|_{\mathcal{V}}$ by Theorem 4. The group property then gives Eq. (64). Differentiating $\psi(t) = U_t\psi_0$ yields Eq. (65). □

Corollary 1 (Exact beat dynamics and finite-difference form)**.** *Let $t_n := n\tau_0$ and $\psi_n := P^n\psi_0 \in \mathcal{V}$. Then*

$$\psi(t_n) = U_{t_n}\psi_0 = \psi_n \tag{67}$$

for all integers n. Equivalently,

$$\psi_{n+1} = \exp\!\left(-\frac{\mathrm{i}\tau_0}{\hbar}H_{\mathrm{eff}}\right)\psi_n, \qquad \frac{\psi_{n+1}-\psi_n}{\tau_0} = \frac{\exp(-\mathrm{i}\tau_0 H_{\mathrm{eff}}/\hbar) - I}{\tau_0}\,\psi_n. \tag{68}$$

Thus Eq. (65) is the generator equation of the principal-branch continuous unitary interpolation of the exact discrete recognition beat. Other continuous interpolations obtained from nonprincipal logarithms exist, but the principal branch is the distinguished choice used here because its spectrum lies in the window (61). A separate refinement family would be required to turn Eq. (68) into a continuum-limit statement across varying beat sizes; that refinement is not constructed here.

This is the precise sense in which the Hamiltonian is recognition-derived within the fixed finite carrier. The self-adjoint generator is not introduced independently. Once the one-beat step and the principal branch are fixed, H_{eff} is fixed by the reversible quarter-turn core.

The restriction to $\mathcal{V}$ is essential. On a larger admissible extension $\mathcal{S} = \mathcal{V} \oplus \mathcal{M}$, the optional even-mode complement $\mathcal{M} \subseteq \mathcal{E}_{\mathrm{even}}$ may contain the -1-eigenspace E_4 of P. In that case the principal logarithm of $P|_{\mathcal{S}}$ is obstructed by the branch-cut point -1. The quarter-turn core is therefore the distinguished sector on which the principal-branch Hamiltonian is unobstructed.

In the Fourier basis, the bare single-register quasienergies are explicit. Writing $|m\rangle := e_{2m+1}$ for $m \in \mathbb{Z}_4$,

$$P|m\rangle = \mathrm{e}^{\mathrm{i}\pi(2m+1)/4}|m\rangle. \tag{69}$$

Thus H_{eff} has eigenvalues

$$\frac{\hbar}{\tau_0}\left\{-\frac{\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{4}\right\} \tag{70}$$

up to ordering by the chosen basis. As in Floquet theory, these quasienergies are defined only modulo $2\pi\hbar/\tau_0$. The principal branch selects the representatives in $(-\pi, \pi)\hbar/\tau_0$. Thus the single-register reversible kernel is rigidly four-level: within the present carrier its spectrum is fixed and evenly spaced in quarter-turn units. The role of Sec. 5 is precisely to show how arbitrary finite spectra re-enter only after coupling finitely many such local cores and, when needed, restricting to invariant subspaces. The finite beat operator is therefore exactly a Floquet step operator, but one whose origin is the recognition-science eight-tick core rather than an externally imposed periodic drive [17–19].

Proposition 8 (Locked quarter-turn quasienergies)**.** *Within the present finite kernel, the quasienergy spectrum of the one-beat unitary step is fixed, up to the usual Floquet-zone identification $\varepsilon \sim \varepsilon + 2\pi\hbar/\tau_0$, to the four odd quarter-turn phases,*

$$\sigma(P|_{\mathcal{V}}) = \left\{e^{\pm i\pi/4}, e^{\pm 3i\pi/4}\right\}. \tag{71}$$

In contrast to generic Floquet systems, this spectrum cannot be tuned continuously without changing either the carrier dimension or the shift period.

Proof. Equation (59) identifies $\sigma(P|_{\mathcal{V}})$ with $\{\omega^{-k} : k \text{ odd}\}$ for $\omega = e^{-i\pi/4}$. The odd values $k = 1, 3, 5, 7$ give the four phases in Eq. (71). Those eigenvalues are fixed by the DFT-8 carrier and cannot be deformed within the present kernel without changing the period or the carrier itself. □

4.2. Ray-Defect Symmetry and the Standard Quantum Pictures

For a ray $[\phi] \in \mathbb{P}(\mathcal{V})$, let $\Pi_{[\phi]}$ denote the orthogonal projector onto that ray. For a second ray $[\psi]$, define the rank-one recognition defect by

$$\mathcal{D}_{[\phi]}([\psi]) := \|(I - \Pi_{[\phi]})\psi\|^2 = 1 - \mathrm{Tr}\big(\Pi_{[\phi]}\Pi_{[\psi]}\big) = 1 - |\langle\phi, \psi\rangle|^2, \tag{72}$$

where ϕ and ψ are any unit representatives of the respective rays. Thus the rank-one defect is simply one minus transition probability.

Definition 3 (Reversible ray-defect symmetry)**.** *A bijection $T : \mathbb{P}(\mathcal{V}) \to \mathbb{P}(\mathcal{V})$ is a* reversible ray-defect symmetry *if*

$$\mathcal{D}_{T[\phi]}(T[\psi]) = \mathcal{D}_{[\phi]}([\psi]) \qquad \text{for all } [\phi], [\psi] \in \mathbb{P}(\mathcal{V}). \tag{73}$$

Theorem 6 (Ray-defect symmetries are Wigner symmetries)**.** *For a bijection $T : \mathbb{P}(\mathcal{V}) \to \mathbb{P}(\mathcal{V})$, the following are equivalent:*

(i) *T is a reversible ray-defect symmetry.*

(ii) *T preserves transition probabilities,*

$$\mathrm{Tr}\big(\Pi_{T[\phi]}\Pi_{T[\psi]}\big) = \mathrm{Tr}\big(\Pi_{[\phi]}\Pi_{[\psi]}\big) \qquad \text{for all } [\phi], [\psi] \in \mathbb{P}(\mathcal{V}). \tag{74}$$

(iii) *There exists a unitary or antiunitary operator $W_T : \mathcal{V} \to \mathcal{V}$, unique up to an overall phase, such that*

$$T([\psi]) = [W_T\psi] \qquad \text{for all } 0 \neq \psi \in \mathcal{V}. \tag{75}$$

Proof. The equivalence of (i) and (ii) is immediate from Eq. (72). The equivalence of (ii) and (iii) is Wigner's theorem for projective Hilbert-space symmetries and its standard linearity refinements [10–12]. □

Theorem 6 identifies the niche of the recognition operator. It is not a rival to the Schrödinger, Heisenberg, or density-matrix pictures at the same conceptual level. Rather, the recognition update fixes the reversible ray geometry and the distinguished time-translation step, while the standard quantum pictures are induced descriptions of that same core. The additional recognition content lies off the reversible core: the exact reciprocal cost, the eight-tick selection rule, and the irreversible commit.

The point of Theorem 6 is therefore not to present Wigner's theorem as new mathematics. Its role is to identify the specific ray-defect geometry selected by the recognition kernel and to show that its reversible symmetry class closes onto the familiar unitary or antiunitary one.

Corollary 2 (Schrödinger, Heisenberg, and von Neumann pictures from the same recognition core)**.** *Let $U_t = \exp(-\mathrm{i}tH_{\mathrm{eff}}/\hbar)$ be the recognition time-translation group of Theorem 5. Then the induced ray flow*

$$T_t([\psi]) := [U_t\psi] \tag{76}$$

is a one-parameter group of reversible ray-defect symmetries on $\mathbb{P}(\mathcal{V})$. For every density matrix ρ and every observable O on $\mathcal{V}$,

$$\rho(t) = U_t\rho(0)U_t^{\dagger}, \qquad \mathrm{i}\hbar\,\dot{\rho}(t) = [H_{\mathrm{eff}}, \rho(t)], \tag{77a}$$

$$O(t) = U_t^{\dagger}O(0)U_t, \qquad \mathrm{i}\hbar\,\dot{O}(t) = [O(t), H_{\mathrm{eff}}]. \tag{77b}$$

Thus the Schrödinger, Heisenberg, and von Neumann pictures are equivalent representations of the same reversible recognition dynamics on $\mathcal{V}$.

Proof. Because U_t is unitary, one has

$$\mathrm{Tr}\big(\Pi_{[U_t\phi]}\Pi_{[U_t\psi]}\big) = \mathrm{Tr}\big(U_t\Pi_{[\phi]}U_t^{\dagger}U_t\Pi_{[\psi]}U_t^{\dagger}\big) = \mathrm{Tr}\big(\Pi_{[\phi]}\Pi_{[\psi]}\big), \tag{78}$$

so T_t preserves the ray defect by Eq. (72). The density-matrix and Heisenberg equations follow by differentiating the conjugation formulas and using Theorem 5. □

4.3. Direct Recognition Variational Principle on the Reversible Core

The continuous unitary dynamics of Theorem 5 is Hamiltonian on the recognition-derived quarter-turn core. This places the reversible sector inside the standard geometric formulation of finite-dimensional quantum mechanics [26,27]. Because $\mathcal{V} \subset \mathbb{C}^8$ is itself the reversible recognition subspace, it carries the inherited real metric and symplectic form

$$g_{\mathrm{R}}(u,v) := 2\,\mathrm{Re}\,\langle u,v\rangle, \qquad \Omega_{\mathrm{R}}(u,v) := 2\,\mathrm{Im}\,\langle u,v\rangle. \tag{79}$$

Define the recognition Hamiltonian on $\mathcal{V}$ by

$$h_{\mathrm{R}}(\psi) := \langle \psi, H_{\mathrm{eff}}\psi\rangle, \tag{80}$$

and the symplectic potential by

$$\Theta_{\mathrm{R}}|_{\psi}(v) := \frac{\mathrm{i}\hbar}{2}\big(\langle\psi,v\rangle - \langle v,\psi\rangle\big) = -\hbar\,\mathrm{Im}\,\langle\psi,v\rangle. \tag{81}$$

Then $\mathrm{d}\Theta_\mathrm{R} = -\hbar\Omega_\mathrm{R}$.

Proposition 9 (Recognition Hamiltonian flow on the quarter-turn core)**.** *For a differentiable curve $\psi(t) \in \mathcal{V}$, the Hamilton equation*

$$\Omega_\mathrm{R}(\dot{\psi}, v) = \frac{1}{\hbar}\,\mathrm{d}h_\mathrm{R}(\psi)[v] \qquad \textit{for all } v \in T_\psi\mathcal{V} \tag{82}$$

is equivalent to the Schrödinger equation

$$\mathrm{i}\hbar\,\dot{\psi} = H_\mathrm{eff}\psi. \tag{83}$$

Proof. Because H_eff is self-adjoint,

$$\mathrm{d}h_\mathrm{R}(\psi)[v] = 2\,\mathrm{Re}\,\langle H_\mathrm{eff}\psi, v\rangle. \tag{84}$$

Also

$$\Omega_\mathrm{R}(\dot{\psi}, v) = 2\,\mathrm{Im}\,\langle \dot{\psi}, v\rangle = 2\,\mathrm{Re}\,\langle \mathrm{i}\dot{\psi}, v\rangle. \tag{85}$$

Hence Eq. (82) holds for every v if and only if

$$\mathrm{i}\dot{\psi} = \frac{1}{\hbar}H_\mathrm{eff}\psi, \tag{86}$$

which is Eq. (83). □

Corollary 3 (Canonical reduced recognition action on the quarter-turn core)**.** *The action*

$$S_\mathrm{R}[\psi] = \int_{t_i}^{t_f}\mathrm{d}t\Big[\Theta_\mathrm{R}|_{\psi(t)}(\dot{\psi}(t)) - h_\mathrm{R}(\psi(t))\Big] = \int_{t_i}^{t_f}\mathrm{d}t\left[\frac{\mathrm{i}\hbar}{2}\big(\langle\psi,\dot{\psi}\rangle - \langle\dot{\psi},\psi\rangle\big) - \langle\psi, H_\mathrm{eff}\psi\rangle\right] \tag{87}$$

has stationary curves exactly the solutions of Eq. (65). Moreover, if Θ'_R is any other one-form on $\mathcal{V}$ with $\mathrm{d}\Theta'_\mathrm{R} = -\hbar\Omega_\mathrm{R}$, then on $\mathcal{V} \cong \mathbb{R}^8$ one has $\Theta'_\mathrm{R} = \Theta_\mathrm{R} + \mathrm{d}f$, so the corresponding action differs from S_R only by the endpoint term $f(\psi_f) - f(\psi_i)$.

The proof is given in Appendix D. This is the strongest honest variational statement available here. The exact reciprocal action $\mathcal{A}_R$ gives the stationary principle and Euler-Lagrange equations for mismatch coordinates. The action S_R is the exact reduced first-order action on the reversible core selected by the recognition kernel. The two functionals are therefore related by reduction, not by literal identity. That distinction is unavoidable: $\mathcal{A}_R$ is built from the positive reciprocal cost on mismatch variables, while S_R is a first-order phase action defined only up to endpoint terms. Their direct bridge is the local defect expansion of Eq. (22); beyond that bridge, S_R should be read as the canonical reduced recognition action on $\mathcal{V}$. Together with Corollary 2, this closes the reversible-picture triangle: Schrödinger evolution, Heisenberg observable flow, von Neumann density-matrix evolution, and the reduced first-order recognition action all describe the same core dynamics.

The propagator on the reversible core also has an exact finite-dimensional time-sliced representation.

Proposition 10 (Exact time-sliced coherent-state representation)**.** *Let $S(\mathcal{V}) := \{\psi \in \mathcal{V} : \|\psi\| = 1\}$, and let $\mathrm{d}\mu_H$ be the $U(4)$-invariant measure on $S(\mathcal{V})$ normalized by*

$$\int_{S(\mathcal{V})} |\psi\rangle\langle\psi|\,\mathrm{d}\mu_H(\psi) = I_\mathcal{V}. \tag{88}$$

Then for every $N \in \mathbb{N}$, with $T := t_f - t_i$, $\Delta t := T/N$, and unit endpoints $\psi_0 = \psi_i$, $\psi_N = \psi_f$,

$$\begin{aligned} K(\psi_f, t_f; \psi_i, t_i) &:= \langle \psi_f, \exp\Big(-\frac{\mathrm{i}T}{\hbar} H_{\mathrm{eff}}\Big)\psi_i\rangle \\ &= \int_{S(\mathcal{V})^{N-1}} \prod_{n=1}^{N-1} \mathrm{d}\mu_H(\psi_n) \prod_{n=0}^{N-1} \langle \psi_{n+1}, \exp\Big(-\frac{\mathrm{i}\Delta t}{\hbar} H_{\mathrm{eff}}\Big)\psi_n\rangle. \end{aligned} \tag{89}$$

Proof. By $U(4)$-invariance, the average rank-one projector over $S(\mathcal{V})$ is proportional to the identity; the normalization in Eq. (88) fixes the constant. Repeatedly inserting Eq. (88) between the N short-time factors of $\exp(-\mathrm{i}\Delta t H_{\mathrm{eff}}/\hbar)$ yields Eq. (89). □

Equation (89) is the exact finite-dimensional path-sum statement familiar from coherent-state time slicing [28,29]. For smooth phase choices and small Δt, it is customarily written in the continuum shorthand

$$K(\psi_f, t_f; \psi_i, t_i) = \int_{\psi(t_i)=\psi_i}^{\psi(t_f)=\psi_f} \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\; \exp\Big(\frac{\mathrm{i}}{\hbar} S_{\mathrm{R}}[\psi]\Big). \tag{90}$$

This is the path-integral connection claimed in the present paper. What remains open is a constructive continuum measure theory and any extension beyond the finite-dimensional unitary sector.

5. Coupled Quarter-Turn Cores and Finite-Dimensional Hamiltonian Synthesis

The preceding section establishes the single-register reversible core and its locked quarter-turn spectrum. The present section answers the complementary question: not how to deform that one-site spectrum, but how coupled local cores generate arbitrary finite-dimensional Hamiltonians through their full tensor-Weyl operator algebra and invariant-subspace embeddings. We now show that finitely many such cores are algebraically universal for finite-dimensional many-body quantum dynamics.

5.1. Local Ququart Structure and the Canonical Weyl Pair

Let Λ be a finite set of sites. At each site $x \in \Lambda$, take a copy $\mathcal{V}_x \cong \mathbb{C}^4$ of the quarter-turn core with odd-mode basis

$$|m\rangle_x := e_{2m+1,x}, \qquad m \in \mathbb{Z}_4. \tag{91}$$

Define the local clock operator by

$$Z_x := \mathrm{e}^{-\mathrm{i}\pi/4} P_x\big|_{\mathcal{V}_x}, \qquad Z_x|m\rangle_x = \mathrm{i}^m|m\rangle_x. \tag{92}$$

Indeed, Eq. (69) gives $P_x|m\rangle_x = \mathrm{e}^{\mathrm{i}\pi/4}\mathrm{i}^m|m\rangle_x$, so the prefactor $\mathrm{e}^{-\mathrm{i}\pi/4}$ is exactly the one that produces the eigenvalue i^m. Let $F_{4,x}$ be the intrinsic four-point Fourier transform

$$F_{4,x}|m\rangle_x := \frac{1}{2}\sum_{n=0}^{3} \mathrm{i}^{mn}|n\rangle_x, \tag{93}$$

and define the local shift operator

$$X_x := F_{4,x}^{\dagger} Z_x F_{4,x}. \tag{94}$$

Then $X_x|m\rangle_x = |m+1 \bmod 4\rangle_x$.

Proposition 11 (Canonical local Weyl pair). *For each site x,*

$$X_x^4 = Z_x^4 = I, \qquad Z_x X_x = \mathrm{i} X_x Z_x. \tag{95}$$

Hence the sixteen monomials

$$W_x(a,b) := X_x^a Z_x^b, \qquad a, b \in \mathbb{Z}_4, \tag{96}$$

form an orthogonal basis of $\mathrm{End}(\mathcal{V}_x)$ *under the Hilbert-Schmidt inner product.*

The proof is given in Appendix E. Here "Weyl pair" simply means the standard finite-dimensional shift/clock pair obeying a phase-twisted commutation relation. Proposition 11 says that a single recognition core is not merely a two-level system in disguise. It is canonically a four-level system, and its local algebra is exactly the standard finite Weyl algebra familiar from finite-dimensional quantum kinematics [13,14].

5.2. Tensor-Weyl Basis, Exact Coefficients, and Locality

The many-body Hilbert space is

$$\mathcal{H}_\Lambda := \bigotimes_{x \in \Lambda} \mathcal{V}_x. \tag{97}$$

For multi-indices $(\mathbf{a}, \mathbf{b}) \in (\mathbb{Z}_4^2)^\Lambda$, define the tensor-Weyl monomials

$$W(\mathbf{a}, \mathbf{b}) := \bigotimes_{x \in \Lambda} W_x(a_x, b_x) = \bigotimes_{x \in \Lambda} X_x^{a_x} Z_x^{b_x}. \tag{98}$$

Their support is

$$\mathrm{supp}(\mathbf{a}, \mathbf{b}) := \{x \in \Lambda : (a_x, b_x) \neq (0,0)\}. \tag{99}$$

Theorem 7 (Tensor-Weyl basis and finite-dimensional universality). *Let Λ be finite.*

(i) *The family $\{W(\mathbf{a}, \mathbf{b})\}$ is an orthogonal basis of* $\mathrm{End}(\mathcal{H}_\Lambda)$ *under the Hilbert-Schmidt inner product.*

(ii) *Every operator $A \in \mathrm{End}(\mathcal{H}_\Lambda)$ has the exact coefficient formula*

$$A = 4^{-|\Lambda|} \sum_{\mathbf{a}, \mathbf{b}} \mathrm{Tr}\big(W(\mathbf{a}, \mathbf{b})^\dagger A\big) W(\mathbf{a}, \mathbf{b}). \tag{100}$$

(iii) *If A is Hermitian, then the coefficients satisfy the usual adjoint-conjugation relation, so A is exactly a recognition Hamiltonian on the coupled bundle.*

(iv) *If the coefficient in Eq. (100) vanishes whenever* $\mathrm{supp}(\mathbf{a}, \mathbf{b})$ *is not contained in a chosen interaction hyperedge $E \subseteq \Lambda$, then A acts trivially outside E. Conversely, every operator acting nontrivially only on E has support contained in E in the Weyl expansion. Therefore locality is tracked exactly by support.*

Proof sketch. Orthogonality factorizes from the one-site basis, and the number of tensor-Weyl monomials is $16^{|\Lambda|} = \dim \mathrm{End}(\mathcal{H}_\Lambda)$, so the family is a basis. The coefficient formula is the standard expansion in an orthogonal basis. Support locality follows because factors outside the support are identities, while Hilbert-Schmidt coefficients against monomials with larger support vanish by trace factorization. Full details are given in Appendix E. □

Theorem 7 completes the finite-dimensional many-body part of the construction: once the local reversible recognition core is present, the coupled many-body algebra is exactly the full finite-dimensional operator algebra. It should be read as a representability statement rather than as a microscopic selection principle.

Corollary 4 (Exact spectral synthesis)**.** *Let $H_\star$ be Hermitian on $\mathbb{C}^D$. Choose N with $D \le 4^N$ and an isometry*

$$V_{\mathrm{enc}} : \mathbb{C}^D \to \mathcal{H}_{\Lambda_N}, \qquad |\Lambda_N| = N. \tag{101}$$

Then there exists a recognition Hamiltonian H_{RS} on $\mathcal{H}_{\Lambda_N}$ such that

$$H_{\mathrm{RS}} = V_{\mathrm{enc}} H_\star V_{\mathrm{enc}}^\dagger \oplus 0_{Q^\perp}, \qquad Q := V_{\mathrm{enc}}\mathbb{C}^D. \tag{102}$$

Hence every finite-dimensional spectrum is realized exactly on an invariant subspace of a coupled recognition bundle. If $D = 4^N$, the global spectrum agrees exactly with that of $H_\star$.

Corollary 4 resolves the finite-dimensional representability question completely. What it does *not* resolve is the dynamical selection question: the theorem says every finite-dimensional Hamiltonian can be represented, not which one a deeper physical recognition law chooses.

5.3. Reversible Many-Body Dynamics and Examples

Once a Hermitian recognition Hamiltonian H_{RS} is fixed on $\mathcal{H}_\Lambda$, the reversible many-body dynamics is standard. If $Q \subseteq \mathcal{H}_\Lambda$ is an invariant subspace, define

$$\widehat{\mathcal{R}}_{Q,\Delta t} := \Pi_Q \exp\!\left(-\frac{\mathrm{i}\Delta t}{\hbar} H_{\mathrm{RS}}\right). \tag{103}$$

If $H_{\mathrm{RS}} Q \subseteq Q$, then $\widehat{\mathcal{R}}_{Q,\Delta t}|_Q$ is exactly the unitary propagator on Q, so Q obeys the Schrödinger equation

$$\mathrm{i}\hbar\,\partial_t \Psi(t) = H_{\mathrm{RS}} \Psi(t). \tag{104}$$

The corresponding action is

$$S[\Psi] = \int \mathrm{d}t \left[\frac{\mathrm{i}\hbar}{2}\big(\langle \Psi, \dot\Psi\rangle - \langle \dot\Psi, \Psi\rangle\big) - \langle \Psi, H_{\mathrm{RS}}\Psi\rangle\right]. \tag{105}$$

Three examples are worth recording.

Encoded qubits. On each site,

$$Q_x := \mathrm{span}\{|0\rangle_x, |2\rangle_x\} \tag{106}$$

is a two-dimensional subfiber on which, in the ordered basis $(|0\rangle_x, |2\rangle_x)$,

$$\sigma_x^z := Z_x\big|_{Q_x}, \qquad \sigma_x^x := X_x^2\big|_{Q_x} \tag{107}$$

act as Pauli matrices, and

$$\sigma_x^y := -\mathrm{i}\sigma_x^z \sigma_x^x \tag{108}$$

completes the Pauli triple on Q_x. Finite spin-1/2 models are therefore contained exactly as encoded sectors.

Clock and ququart models. Without encoding, each site is already a four-state clock degree of freedom. Finite-range clock models, generalized Ising couplings, and arbitrary local ququart interactions appear directly in the tensor-Weyl basis.

Abstract finite spectra. Corollary 4 removes any need to start from a traditional lattice model at all: arbitrary finite spectra are obtained by invariant-subspace embedding.

6. Discussion and Limits

The paper fixes a finite-dimensional operator construction from two imported inputs: the reciprocal cost and the three-dimensional eight-tick kernel. Within that setting, several statements are exact. The quarter-turn sector $\mathcal{V}$ is the unique maximal real invariant subspace on which P^2 squares to $-I$. The one-beat step on $\mathcal{V}$ is unitary. Its principal logarithm gives a self-adjoint generator on that core. Ray-defect symmetries reduce to transition-probability symmetries and therefore close under Wigner's theorem. The same core carries a reduced first-order action and an exact finite-dimensional time-sliced propagator representation. For finitely many coupled sites, tensor-Weyl monomials span the full operator algebra. A single site has a rigid four-level spectrum. General finite spectra appear only after coupling finitely many local cores.

These results locate the recognition operator precisely. It does not replace the Schrödinger, Heisenberg, or von Neumann pictures. It fixes the reversible core and the one-beat step from which those pictures follow. The nonunitary part of the construction is the commit. Proposition 3 shows that strict defect descent cannot be unitary on the same sector. The symmetry-compatible finite-time commit therefore carries the transient content. If a state has a nonzero component in $\mathcal{S}^{\perp}$, then the defect decays exactly as λ^{2n}. A defect proxy converts this law into an operational estimate of λ. The in-sector channel remains unitary at the same time. The resulting signal is therefore two-channel: exponential off-sector decay together with reversible core motion.

The paper remains finite-dimensional. It treats the $d = 3$ eight-tick kernel as an imported input. Other ledger dimensions would lead to different periods and different local core dimensions. The paper also does not derive a microscopic adapter from the eight-tick register to a specific experimental platform. It does not derive a continuum path-integral measure, open-system dynamics, or a laboratory collapse model. Standard leakage-sensitive metrology can serve as the experimental adapter for the defect channel [30–33].

Three problems remain open. First, Theorem 7 proves representability of arbitrary finite-dimensional Hermitian dynamics, but not a coefficient-selection principle for concrete laboratory Hamiltonians. Second, the relaxation factor λ is operationally calibratable, but a microscopic derivation of its value requires additional model input. Third, continuum limits, renormalization, relativistic covariance, and field theory lie outside the present paper. Any broader recognition-operator program has to preserve the finite-dimensional results proved here.

7. Conclusions

This paper studies a finite-dimensional recognition operator selected by reciprocal cost and an eight-tick ledger. The neutral register $\mathcal{W}$ and the quarter-turn core $\mathcal{V}$ are fixed by the shift structure. The reversible step on $\mathcal{V}$ is exact and unitary. Its principal logarithm defines the generator H_{eff} on that core.

The same core supports the standard reversible quantum pictures. Schrödinger, Heisenberg, and von Neumann evolution all descend from the same one-parameter unitary group. The core also carries a reduced first-order action and an exact finite-dimensional time-sliced propagator. For finitely many coupled sites, local ququart Weyl pairs generate the full finite-dimensional operator algebra. A single site gives a rigid four-level quarter-turn spectrum. General finite spectra arise only after coupling finitely many local cores.

The result is conditional and finite-dimensional. It identifies a reversible core and a transient commit law inside the fixed eight-tick carrier. It does not yet select laboratory couplings or establish continuum and field-theoretic limits. Those problems remain open.

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Appendix A. Proof of Theorem 1

Define

$$G(u) := J(\mathrm{e}^u) + 1. \tag{A1}$$

Then Eq. (1) becomes the classical d'Alembert equation

$$G(u+v) + G(u-v) = 2G(u)G(v), \tag{A2}$$

with $G(0) = 1$ by Eq. (2). By the classical classification of continuous real solutions [7,8], one has either

$$G(u) = \cosh(\alpha u) \tag{A3}$$

for some real α, or

$$G(u) = \cos(\beta u) \tag{A4}$$

for some real β. The cosine branch is excluded by the calibration, because

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2}\left(\cos(\beta u) - 1\right)\Big|_{u=0} = -\beta^2 \leq 0,$$

while Eq. (2) requires the value $+1$. Therefore

$$J(\mathrm{e}^u) = \cosh(\alpha u) - 1. \tag{A5}$$

Differentiating twice at $u = 0$ gives

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2} J(\mathrm{e}^u)\Big|_{u=0} = \alpha^2. \tag{A6}$$

The calibration (2) forces $\alpha^2 = 1$. Since cosh is even, this gives

$$J(e^u) = \cosh u - 1, \qquad J(x) = \frac{1}{2}\left(x + x^{-1}\right) - 1. \tag{A7}$$

This is Eq. (3).

Appendix B. Proof of Theorem 2

Because $J(e^\eta) = \cosh \eta - 1$,

$$\frac{\partial \mathcal{L}_R}{\partial q^i} = \sum_a \sinh \eta^a \frac{\partial \eta^a}{\partial q^i}, \qquad \frac{\partial \mathcal{L}_R}{\partial \dot{q}^i} = \sum_a \sinh \eta^a \frac{\partial \eta^a}{\partial \dot{q}^i}. \tag{A8}$$

Substituting these into the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}_R}{\partial \dot{q}^i}\right) - \frac{\partial \mathcal{L}_R}{\partial q^i} = 0 \tag{A9}$$

yields Eq. (15).

Appendix C. Proof of Proposition 3

The hypothesis $D(Uv) \leq D(v)$ for all v is equivalent to

$$U^\dagger M U \leq M \tag{A10}$$

in Loewner order. Taking the trace and using cyclicity gives

$$\mathrm{Tr}(M - U^\dagger M U) = \mathrm{Tr}(M) - \mathrm{Tr}(M) = 0. \tag{A11}$$

Since $M - U^\dagger M U \geq 0$, zero trace implies

$$M - U^\dagger M U = 0. \tag{A12}$$

Therefore $U^\dagger M U = M$, so $D(Uv) = D(v)$ for all v.

Appendix D. Proof of Corollary 3

First, the one-form of Eq. (81) satisfies

$$(\mathrm{d}\Theta_\mathrm{R})_\psi(u, v) = -\hbar\Omega_\mathrm{R}(u, v) \tag{A13}$$

for all $u, v \in T_\psi\mathcal{V} \cong \mathcal{V}$. Indeed,

$$\Theta_\mathrm{R}|_\psi(v) = -\hbar\,\mathrm{Im}\,\langle \psi, v\rangle, \tag{A14}$$

so linearity in ψ gives

$$(\mathrm{d}\Theta_\mathrm{R})_\psi(u, v) = u[\Theta_\mathrm{R}(v)] - v[\Theta_\mathrm{R}(u)] = -2\hbar\,\mathrm{Im}\,\langle u, v\rangle = -\hbar\Omega_\mathrm{R}(u, v). \tag{A15}$$

Now write the action as

$$S_\mathrm{R}[\psi] = \int_{t_i}^{t_f} \mathrm{d}t\,\left[\Theta_\mathrm{R}(\dot{\psi}) - h_\mathrm{R}(\psi)\right]. \tag{A16}$$

Let ψ_ε be a variation with fixed endpoints and variation field $\delta\psi := \partial_\varepsilon \psi_\varepsilon|_{\varepsilon=0}$. The standard first-order variational formula gives

$$\delta S_\mathrm{R} = \int_{t_i}^{t_f} \mathrm{d}t\,\Big[(\mathrm{d}\Theta_\mathrm{R})(\delta\psi, \dot{\psi}) - \mathrm{d}h_\mathrm{R}(\psi)[\delta\psi]\Big] + \Big[\Theta_\mathrm{R}(\delta\psi)\Big]_{t_i}^{t_f}. \tag{A17}$$

The boundary term vanishes because the endpoints are fixed. Using Eq. (A13), stationarity for arbitrary $\delta\psi$ is equivalent to

$$\Omega_{\mathrm{R}}(\dot{\psi}, v) = \frac{1}{\hbar}\,\mathrm{d}h_{\mathrm{R}}(\psi)[v] \qquad \text{for all } v \in T_{\psi}\mathcal{V}, \tag{A18}$$

which is Eq. (82). Proposition 9 then yields Eq. (65).

Equivalently, one may vary Eq. (87) in bra/ket form by treating ψ and $\bar{\psi}$ as independent variables; variation with respect to the bra gives $\mathrm{i}\hbar\dot{\psi} = H_{\mathrm{eff}}\psi$, while variation with respect to the ket gives the adjoint equation.

Finally, if Θ'_{R} is another one-form with $\mathrm{d}\Theta'_{\mathrm{R}} = -\hbar\Omega_{\mathrm{R}}$, then

$$\mathrm{d}(\Theta'_{\mathrm{R}} - \Theta_{\mathrm{R}}) = 0. \tag{A19}$$

Because $\mathcal{V} \cong \mathbb{R}^8$ is contractible, $H^1_{\mathrm{dR}}(\mathcal{V}) = 0$. Hence there exists a smooth function f on $\mathcal{V}$ such that

$$\Theta'_{\mathrm{R}} - \Theta_{\mathrm{R}} = \mathrm{d}f. \tag{A20}$$

The corresponding action therefore differs from S_{R} only by the endpoint term $f(\psi_f) - f(\psi_i)$.

Appendix E. Proofs for the Many-Body Weyl Basis

Appendix E.1. Proof of Proposition 11

Using Eqs. (92) and (93), one computes directly that

$$X_x|m\rangle_x = |m+1 \mod 4\rangle_x. \tag{A21}$$

Therefore $X_x^4 = Z_x^4 = I$. Moreover,

$$\begin{aligned} X_x Z_x|m\rangle_x &= \mathrm{i}^m|m+1\rangle_x, \\ Z_x X_x|m\rangle_x &= \mathrm{i}^{m+1}|m+1\rangle_x, \end{aligned} \tag{A22}$$

so $Z_x X_x = \mathrm{i}X_x Z_x$. The sixteen monomials $W_x(a,b)$ are orthogonal because

$$\mathrm{Tr}\big(W_x(a,b)^{\dagger} W_x(a',b')\big) = 4\,\delta_{a,a'}\delta_{b,b'}. \tag{A23}$$

Since $\dim \mathrm{End}(\mathcal{V}_x) = 16$, they form a basis.

Appendix E.2. Proof of Theorem 7

Hilbert-Schmidt products factor over tensor products, so

$$\mathrm{Tr}\big(W(\mathbf{a},\mathbf{b})^{\dagger} W(\mathbf{a}',\mathbf{b}')\big) = 4^{|\Lambda|}\,\delta_{\mathbf{a},\mathbf{a}'}\delta_{\mathbf{b},\mathbf{b}'}. \tag{A24}$$

Hence the tensor-Weyl monomials are orthogonal. Their number is

$$16^{|\Lambda|} = \dim \mathrm{End}(\mathcal{H}_{\Lambda}), \tag{A25}$$

so they form a basis, proving part (i). Part (ii) is the standard expansion in an orthogonal basis. For Hermitian A, the coefficient of $W(\mathbf{a},\mathbf{b})^{\dagger}$ is the complex conjugate of the coefficient of $W(\mathbf{a},\mathbf{b})$, proving part (iii). For part (iv), if $\mathrm{supp}(\mathbf{a},\mathbf{b}) \subseteq E$, then $W(\mathbf{a},\mathbf{b})$ acts as the identity outside E. Conversely, if A acts trivially outside E, then its Hilbert-

Schmidt coefficient against any monomial with support not contained in E vanishes by trace factorization on at least one outside site.

Appendix F. Imported Recognition-Science Inputs

Only two external recognition-science inputs are used in this paper. The first is the coherent-comparison law with the equilibrium calibration in Eqs. (1) and (2). Theorem 1 is then proved from that input inside the present manuscript. The second is the cost-first ledger result that, in three dimensions, a spatially complete atomic schedule has minimal period eight [3]. Everything else is either a model definition introduced here or a finite-dimensional argument proved here.

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