An Eight-Slot Operator Bridge for Reciprocal-Cost Ledgers, with Finite-Amplitude Born Corrections

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(Dated:)

The reciprocal cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1 = \cosh(\log x) - 1$, forced by d'Alembert composition under a unit log-curvature calibration, has a quadratic Hessian that is the Hilbert metric of a finite cyclic operator bridge and a quartic tail that gives a finite-amplitude correction beyond it. We compute both for the eight-slot Gray ledger. With an oriented Gray clock on a three-bit local record, a positive generator ledger, nonmagnetic cotangent completion, and an off-shell closed tick preserving the zero generator fiber, the closed tick is forced to be the pure cotangent lift of the slot shift, and its finite Fourier transform realizes the unitary C_8-representation $\Psi(\widehat{\mathcal{R}}_L s) = P\Psi(s)$ on the non-DC carrier $\mathcal{W} \cong \mathbb{C}^7$. Keeping the quartic term of $\cosh u - 1$ then yields a closed-form, coordinate-dependent correction to tangent Born allocation. For the cost-selected Gray state we compute the full $O(\epsilon^2)$ correction table, an aggregate cyclic-parity shift of $\pm 23/51200\,\epsilon^2$ between odd and non-DC even sectors, and a nonzero Sorkin third-order coefficient $I_3(1,3,5) = 3(1-\sqrt{2})/32768\,\epsilon^4$ in the recognition-cost score, compared with published multipath bounds for scale. The closed-tick rigidity, the correction table, and the Sorkin coefficient are conditional on the primitive package stated below; they are calibration targets, not direct experimental predictions absent a specified path-to-ledger adapter.

Keywords: reciprocal cost; cyclic ledger; cyclic representation; finite Fourier transform; cotangent completion; Hessian metric; operator bridge; projective period; quantum reconstruction

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XIII. Reference-conditioned recognition states

I. INTRODUCTION

Finite cyclic dynamics is mathematically simple: choose a cyclic generator, put a Hermitian metric on the carrier, and the discrete Fourier transform diagonalizes the tick. The question addressed here is more specific. Starting from positive reciprocal ledger variables, what additional structural data must be stated before this standard operator picture is forced, and which later quantum-looking statements still require separate interface assumptions?

The answer is deliberately conditional. The scalar coherent-comparison law determines the reciprocal cost J, but it does not determine the clock, the number of slots, the cyclic order, the cotangent generator coordinate, the state-selection rules, or the measurement interface. The bridge theorem below is therefore relative to a named primitive package. The paper should be read as a dependency-audited finite theorem about cyclic ledgers, neither a reconstruction of quantum theory from ratio reciprocity alone nor a new proof of the spectral theorem for cyclic groups.

a. Position relative to quantum-reconstruction work. Operational reconstructions of quantum theory usually begin from assumptions such as finite information capacity, continuity, local tomography, purification, or operational composition rules; representative examples include Hardy's axiomatizations, the Dakić–Brukner program, Masanes–Müller, and Chiribella–D'Ariano–Perinotti [1–5]. General-probabilistic approaches develop this comparison in a broader convex-operational language [6–8]. Decision-theoretic and operational discussions of Born weights form a separate line of comparison [9–11]. The present manuscript does not compete with those programs by deriving the full operational structure of quantum mechanics. It instead isolates a cyclic-recognition route to one finite operator bridge: reciprocal cost supplies the Hessian, a positive generator ledger supplies the second real coordinate, a cotangent closure supplies the symplectic form, and a chosen cyclic clock supplies the finite characters. Reference-conditioned clock readouts should also be compared with the quantum reference-frame and superselection literature [12]. The symplectic and cotangent language is standard in geometric mechanics and geometric quantization [13–17]; the novelty claimed here is the ledger-cost packaging and the exact finite audit, not the DFT or cotangent formalism themselves.

b. What the reciprocal-cost packaging adds. The reciprocal-cost route also has positive content of its own. The scalar functional equation gives a finite-amplitude nonlinear cost, $\cosh u - 1$, before a Hilbert norm is introduced; the quadratic Hilbert metric is its Hessian germ rather than a starting axiom. The positive generator ledger then gives an explicit second real coordinate with the same nonlinear cost, so the complex carrier is a balanced cotangent/Hessian carrier rather than an arbitrary complexification. Finally, the Gray-ledger example in Section X shows an intrinsic selection mechanism: centered bit-moment constraints choose mixed cyclic parity in the Fourier carrier without hard-wiring the desired Fourier coefficients. Section XI then uses the same exact state to compute the first finite-amplitude correction to Hessian/Born allocation. These are the recognition-cost ingredients not present in the standard reconstruction route, even though the resulting finite operator algebra is itself standard.

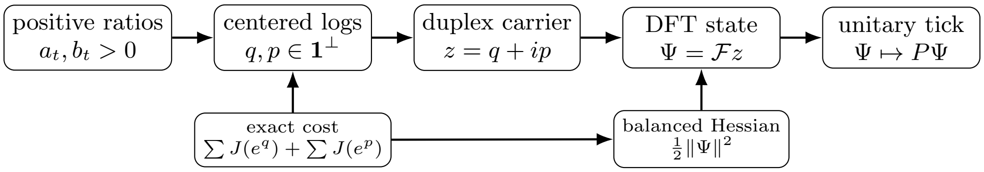


FIG. 1. The bridge pipeline. The exact nonlinear reciprocal cost lives on positive log ratios. The complex Hilbert carrier is the balanced cotangent/Hessian carrier, and the DFT is tied to a chosen cyclic clock.

Route	**Typical primitive inputs**	**Output claimed**
Hardy-type reconstructions	information capacity, continuity, simplicity or related operational axioms	full finite-dimensional Hilbert-space state structure
GPT reconstructions	convex state spaces, effects, composition principles	constraints selecting quantum-like theories from broader operational theories
Geometric mechanics	symplectic manifolds, cotangent bundles, moment maps	phase-space and quantization language
This paper	reciprocal cost, finite local record, chosen cyclic clock, positive generator ledger, off-shell closure	one finite cyclic operator bridge, a dependency audit, and a finite-amplitude correction table

c. Main result. Relative to the structural package, closed recognition has a minimal nonmagnetic cotangent carrier, the closed tick is the pure lift of the cyclic slot shift, and the character transform of the chosen clock turns that lift into the unitary bridge

$$\Psi(\widehat{\mathcal{R}}_L s) = P\Psi(s).$$

Relative to the state-selection package, covariant affine ledger-moment constraints have unique reciprocal-cost minimizers that transport under this bridge. Relative to additional interface assumptions, one obtains balanced Hessian allocation, protected-reference clock states, and optional logarithmic Hamiltonian representatives. These three layers are kept separate throughout.

The local cube record and the cyclic clock are also kept separate. Three binary distinctions give an eight-element $(\mathbb{Z}/2\mathbb{Z})^3$-torsor only after the three-bit local-record primitive is accepted. An oriented Gray-cycle realization is an additional choice that turns that unordered cube into a cyclic C_8-clock. The DFT diagonalizes this cyclic clock, not the three commuting cube involutions.

d. Concrete payoff. The exact Gray-ledger example in Section X is the most concrete test case. It uses intrinsic bit-moment constraints, minimizes the nonlinear reciprocal cost, and produces mixed cyclic odd/even support with exact Fourier coefficients. Section XI then promotes the nonlinear part of the cost from bookkeeping to a calculation: the Gray coefficients yield an explicit table of $O(\epsilon^2)$ corrections to tangent Born weights. That table is the manuscript's falsifiable interface target. Without a laboratory adapter it is not yet an experimental prediction, but it is also no longer routine cyclic representation theory. An "adapter" here means a stated map from the slot/DFT recognition-cost score to a measured detection probability on a specific apparatus; until such a map is fixed by an experiment, the table is a calibration target only.

e. Organization. Section II states the primitive package and the dependency map. Section III proves the reciprocal cost. Section IV separates the local cube record from the cyclic clock. Sections V–VII build the duplex cotangent carrier and DFT bridge. Section IX analyzes vector and ray period, after which Section X gives the exact Gray-ledger example and Section XI computes its finite-amplitude allocation correction. Sections XII–XIII then analyze finite faithfulness and protected references. The remaining measurement, Hamiltonian, and open-channel material is compressed into appendices as interface bookkeeping.

II. PRIMITIVE PACKAGE AND DEPENDENCY MAP

A finite recognition ledger system is a pair

$$(\mathcal{L}, \widehat{\mathcal{R}}_L), \tag{1}$$

Layer	Dependencies
Structural bridge	positive coherent comparison; elementary three-involution local record; chosen oriented Gray-cycle clock; slot-additive cost; internal slot covariance; positive generator ledger; generator-reversal no-spectator nonmagnetic cotangent completion; off-shell closed tick preserving the zero generator fiber.
State selection	predetermined covariant ledger-observable class and nonempty covariant affine moment slices, specified before Fourier support is inspected.
Interface	balanced measurement-interface assumptions, protected-reference convention when used, and angular calibration for optional logarithmic interpolation.

TABLE I. Compact dependency map. The scalar cost is forced, but the operator bridge is conditional on the structural bridge layer. State-selection and interface layers are not used to prove the DFT bridge.

where $\mathcal{L}$ is the set of admissible ledger states and $\widehat{\mathcal{R}}_L : \mathcal{L} \to \mathcal{L}$ is the one-tick recognition update. The construction below derives a spectral representation of the eight-slot shadow of this update. Full faithfulness to $\mathcal{L}$ is addressed later by the finite functional-graph audit.

The following principles are the stated conditional primitive inputs. In particular, the complex Hilbert carrier, the projective anchor, the Born exponent, and the Hamiltonian branch are not structural primitives.

The assumptions naturally separate into three kinds. The structural bridge primitives produce the cotangent carrier and DFT operator bridge. State-selection primitives select particular ledger states inside that carrier. Interface primitives give measurement and Hamiltonian interpretations. The main structural theorem uses only the first block.

The primitive stress test is correspondingly simple. Dropping positive comparison removes the forced cost; dropping the three-involution record removes the eight local slots; dropping the Gray clock removes the C_8-generator; and dropping the positive generator ledger removes the nonlinear fiber cost. On the closure side, dropping generator reversal or nonmagnetism permits vertical magnetic phase; dropping off-shell zero-fiber rigidity permits symplectic shears; dropping covariant moment constraints leaves arbitrary carrier states; and dropping interface assumptions removes measurement weights and Hamiltonian calibration. Thus the conditional nature of the bridge is explicit without treating the dependency map as a second proof.

a. Status of the primitive package. In this manuscript, "dependency-explicit first principles" means the explicit primitive package in Table I. It does not mean derivation from the scalar reciprocal cost alone, and it does not mean derivation from no primitives. Relative to the structural block, the cotangent duplex carrier, nonmagnetic completion, pure lift, DFT bridge, and finite faithfulness audit are derived consequences. Relative to the state-selection block, covariant ledger-moment minimizers obey the bridge. Relative to the interface block, balanced Hessian measurement weights and Hamiltonian branches are calibrated consequences.

b. Status of theorem provenance. The scalar uniqueness theorem (Theorem III.1) is the classical continuous d'Alembert solution applied to the coherent-comparison primitive of Section III [18–20]. The eight-slot DFT diagonalization and the algebraic obstruction $x^2 + 1 = 0$ over $\mathbb{R}$ are standard cyclic character theory [21, 22]. The cotangent-completion, generator-reversal, unified pure-lift, equivariant minimizer-transport, regular-analytic regularity, mixed-cyclic-parity, finite-amplitude correction, finite semiconjugacy, and reference-conditioned theorems are new to the present manuscript with self-contained proofs. The exact Gray coefficients, the finite-amplitude correction table, the cyclic-parity aggregate, and the Sorkin coefficient $I_3(1, 3, 5)$ are exact closed-form values in $\mathbb{Q}[\sqrt{2}]$, independently verified by symbolic computation. Gleason's theorem, used in Theorem A.6 of the appendix, is invoked as a classical external result, not as a consequence of the present framework.

Definition II.1 (Bridge primitive package). *The recognition bridge package is the named dependency object*

$$\mathsf{P}_{\mathrm{bridge}} = (\mathsf{P}_{\mathrm{str}}, \mathsf{P}_{\mathrm{sel}}, \mathsf{P}_{\mathrm{int}}).$$

The structural subpackage $\mathsf{P}_{\mathrm{str}}$ consists of the following hypotheses, stated without reference to later theorem numbers:

(i) positive coherent comparison and slot-additive reciprocal cost;

(ii) an elementary three-involution local record and a chosen oriented Gray-cycle clock τ;

(iii) internal slot covariance under the real cyclic shift induced by τ;

(iv) a positive generator ledger whose centered log coordinate is the conjugate generator coordinate;

(v) generator-reversal, no-spectator, nonmagnetic closed cotangent completion;

(vi) an off-shell closed tick on the completed carrier, locally written after the clock is realized as

$$F(q,p) = (Aq, \Phi(q,p)),$$

where A is the real cyclic shift on the mean-free slot tangent, F preserves the canonical symplectic form on a fiber-convex neighborhood, and the zero generator fiber is preserved:

$$\Phi(q,0) = 0.$$

Exact reciprocal-cost preservation on the zero fiber is an operational sufficient condition for this zero-fiber preservation; full off-shell exact-cost preservation is the stronger finite-amplitude closure version used when nonlinear reciprocal content is discussed.

The state-selection subpackage $\mathsf{P}_{\mathrm{sel}}$ consists of a predetermined covariant ledger-observable class $\mathcal{O}_{\mathrm{ledg}}$ and nonempty covariant affine moment slices $\{\mathcal{C}_s\}$, specified by ledger rules before any desired Fourier support is chosen. The interface subpackage $\mathsf{P}_{\mathrm{int}}$ consists of balanced measurement-interface assumptions, protected-reference conventions when used, and angular calibration for optional logarithmic interpolation.

Thus $\mathsf{P}_{\mathrm{bridge}}$ is not another theorem. It is the complete list of dependencies under which the bridge theorem is proved. The scalar reciprocal-cost theorem determines only the first entry of $\mathsf{P}_{\mathrm{str}}$. It does not determine the clock, the positive generator ledger, the closed-tick zero-fiber condition, the selected moment slices, or the measurement/Hamiltonian interfaces.

> **Off-shell closure hierarchy.** The minimal mathematical rigidity used for the pure lift is symplecticity, coverage of the cyclic base shift, and zero-generator-fiber preservation. Exact reciprocal-cost preservation on the zero fiber is an operational sufficient condition that pins this zero fiber. Full off-shell exact-cost preservation on a fiber-convex domain is a stronger package primitive: it implies the previous conditions and supports finite-amplitude cost statements. The structural bridge theorem needs the minimal rigidity; finite-amplitude reciprocal-cost claims use the stronger closure layer.

Primitive Principle II.2 (Positive coherent comparison)**.** *A recognition comparison depends on two positive quantities only through their ratio $x > 0$. Its cost $J : (0,\infty) \to [0,\infty)$ is reciprocal, has $J(1) = 0$, and its two-comparison polarization is compatible with multiplicative composition:*

$$J(xy) + J(x/y) = 2J(x)J(y) + 2J(x) + 2J(y). \tag{2}$$

The log-cost $u \mapsto J(e^u)$ is continuous on $\mathbb{R}$, twice differentiable at zero, and calibrated by

$$\left.\frac{d^2}{du^2} J(e^u)\right|_{u=0} = 1. \tag{3}$$

Definition II.3 (Elementary closed reciprocal posting)**.** *An elementary closed reciprocal posting is a finite local record equipped with three involutions r, o, c satisfying:*

(i) r exchanges numerator and denominator roles;

(ii) o reverses posting orientation;

(iii) c exchanges the two completion sides;

(iv) r, o, c commute, since they act on distinct recognition coordinates;

(v) the generated action is free and transitive on elementary local slots;

(vi) every local slot distinction of an elementary closed posting is generated by r, o, c.

The last clause is the no-fourth-bit condition. It is a modeling restriction on the elementary local record, not a consequence of the scalar cost theorem: if another independent binary distinction is admitted, the local record belongs to a larger $(\mathbb{Z}/2\mathbb{Z})^n$-torsor rather than to the elementary three-bit case studied here.

Primitive Principle II.4 (Minimal local recognition record)**.** *A local closed ratio recognition act is an elementary closed reciprocal posting in the sense of Definition II.3. Thus the three distinctions r, o, c are not inferred from the scalar cost alone; they are the minimal local-record primitive for the closed reciprocal posting modeled in this paper. The choice $n = 3$ is not claimed to be forced among all possible binary torsors. It is the case in which the local record contains exactly reciprocal role, posting orientation, and completion side; adding a fourth independent binary coordinate gives a different 2^n-slot theory.*

Primitive Principle II.5 (Closed recognition and generator reversal)**.** *Closed recognition does not discard recognized content. At balance its infinitesimal dynamics is reversible and carries a nondegenerate phase pairing between variations and their generators. The conjugate generator posting has a reversal operation, represented on a completed model by* $(q, v) \mapsto (q, -v)$*, and reversal changes the sign of recognized phase. The closed tick is postulated to preserve reciprocal-cost content; infinitesimally it preserves the Hessian content selected by* (3)*. This conservation law is an explicit closure postulate, not a consequence of the scalar cost theorem. Minimal closed completions exclude spectator phase directions.*

Primitive Principle II.6 (Positive generator ledger)**.** *The conjugate generator coordinate of closed recognition is represented by its own positive reciprocal comparison field. Operationally, a generator posting compares the capacity of a slot component to advance or retard the recognized profile in the next closed tick. Its positive ratio* b_t/b_s *compares generator strengths at slots* t *and* s*. Generator reversal exchanges advance with retard and therefore sends this ratio to its inverse, which is represented in centered log coordinates by* $p \mapsto -p$*. Thus the fiber coordinate is not an arbitrary cotangent momentum with an unrelated Legendre-dual cost. It is written*

$$p_t = \log b_t - \frac{1}{8}\sum_s \log b_s, \qquad b_t > 0,$$

and the same coherent comparison law applies to the generator ratios. The exact generator-side cost is therefore $\sum_t J(e^{p_t})$*. The Hessian identification* $Q^* \cong Q$ *explains the local cotangent pairing, while the positive generator ledger supplies the global nonlinear fiber cost.*

> **Positive-generator warning.** The positive generator ledger is not derived from cotangent duality alone. The cotangent theorem supplies a dual phase coordinate. Principle II.6 supplies the additional dimensionless slotwise ratio representation of that coordinate, and therefore licenses the exact nonlinear fiber cost $\sum_t J(e^{p_t})$. Without this primitive the construction still has a local quadratic cotangent/Hessian carrier, but not the finite-amplitude duplex cost used to pin the zero generator fiber or compute nonlinear reciprocal-cost corrections.

Primitive Principle II.7 (Slot-additive reciprocal ledger cost)**.** *Independent slot postings contribute additively to recognized cost. No cross-slot cost is introduced unless a ledger constraint explicitly couples the slots. Thus, for a scale-free profile* $q \in \mathbf{1}^{\perp}$*, the slot action is*

$$G(q) = \sum_{t=0}^{7} J(e^{q_t}),$$

and coupled recognition requirements enter through the admissible constraint set rather than through hidden cross-cost terms.

Primitive Principle II.8 (Off-shell closed tick)**.** *The one-tick closed recognition update is not merely an on-shell rule on already-selected ledger states. It extends to a smooth symplectic transformation of the completed duplex carrier, either globally or locally over a specified base domain. In the local form there is an open connected base domain* $B \subset Q$ *with* $A(B) \subseteq B$*, and an open fiber-convex set* $U \subset B \oplus Q$ *containing* $B \oplus \{0\}$ *and all fiber segments needed in Theorem VI.6. The off-shell map covers the slot shift on the base coordinate and, at minimum, preserves the zero generator fiber. Exact duplex reciprocal-cost preservation on the zero fiber is the operational sufficient condition for that preservation, and full off-shell exact-cost preservation is the stronger finite-amplitude closure version. This is the primitive that lets the pure-lift theorem exclude nonlinear symplectic shears rather than only shears that happen to vanish on selected states.*

> **Off-shell closure assumption.** The operator bridge is rigid only after assuming closed ticks whose completed off-shell action preserves exact reciprocal content on the base-generator neighborhood used by the proof. If recognition closure is imposed only on selected states, the bridge is not rigid: symplectic shears can agree on those states while changing the completed carrier.

c. Load-bearing form of the off-shell principle. The pure-lift proof below is a hierarchy, not one monolithic assumption. First, the minimal rigidity theorem says that symplecticity, coverage of the base shift, and zero-generator-fiber preservation force the pure lift. Second, exact reciprocal-cost preservation on the zero fiber is a recognition-closure condition sufficient to pin that zero fiber. Third, full off-shell exact-cost preservation on a fiber-convex domain is a stronger sufficient condition used when finite-amplitude reciprocal content is part of the package. Thus exact cost is not a hidden substitute for the symplectic argument; it is the operational reason the zero generator fiber is preserved.

Primitive Principle II.9 (Internal slot covariance)**.** *The slots are internal clock labels. For a chosen oriented clock generator τ, a positive slot field a_s transforms by*

$$a_{\widehat{\mathcal{R}}_L s}(\sigma) = a_s(\tau\sigma). \tag{4}$$

The opposite generator is the time-reversed convention.

Primitive Principle II.10 (Recognition measurement interface)**.** *A complete measurement is a decomposition of recognized Hessian content into mutually exclusive refinements. At the balanced Hessian interface, repeatable refinements are represented by stable channel maps satisfying superposition compatibility, repeatability, exclusivity, completeness, and content additivity. The finite-dimensional operator language used later is the representation of this interface, not a consequence of the scalar reciprocal cost alone.*

d. Primitive-package roadmap. The entries in Table I are assumptions of the structural package, not conclusions of the scalar cost theorem. The paper proves their consequences in stages: first the reciprocal cost, then the eight-slot cube record and separate cyclic clock, then the duplex cotangent carrier and pure lift, then covariant state selection, and finally the DFT bridge and balanced measurement interface. This ordering prevents the phrase “first principles” from hiding the extra clock, generator, and interface structure required by the hard-path bridge.

Remark II.11 (No free operator bridge from the scalar cost alone)**.** *For any finite slot set S, the slot-additive functional*

$$G_S(q) = \sum_{s\in S} J(e^{q_s})$$

is invariant under every permutation of S. Therefore the scalar cost alone cannot select $|S| = 8$, cannot choose a cyclic generator, cannot supply a positive generator ledger, cannot exclude off-shell symplectic shears, and cannot define a measurement interface. The bridge below is a theorem of the full package, not of J alone.

III. THE RECIPROCAL COST IS FORCED

The coherent-comparison primitive is stronger than bare reciprocity. The coherent polarization law (2) is the real cost-selection axiom: it asserts that two independent ratio comparisons admit a multiplicative coherent polarization. Under continuity and unit log-curvature, this polarization law is equivalent to d’Alembert’s functional equation and forces the hyperbolic-cosine cost.

It is useful to write

$$C(x) = 1 + J(x). \tag{5}$$

Reciprocity and multiplicative compatibility say that the two-comparison average is polarized by products:

$$C(xy) + C(x/y) = 2C(x)C(y). \tag{6}$$

In log coordinates this is d’Alembert’s equation.

Theorem III.1 (Reciprocal cost)**.** *Principle II.2, including continuity of $u \mapsto J(e^u)$ on $\mathbb{R}$ and twice differentiability at zero with $d^2J(e^u)/du^2|_0 = 1$, forces*

$$J(x) = \frac{1}{2}(x + x^{-1}) - 1 = \cosh(\log x) - 1. \tag{7}$$

Proof. Set $f(u) = 1 + J(e^u)$. With $x = e^t$ and $y = e^u$, equation (2) becomes

$$f(t + u) + f(t - u) = 2f(t)f(u). \tag{8}$$

The calibration gives $f(0) = 1$ and $f''(0) = 1$. By the continuous d’Alembert solution theorem, a continuous solution of (8) with $f(0) = 1$ has the form

$$f(u) = \frac{1}{2}\big(\chi(u) + \chi(-u)\big)$$

for a continuous exponential character $\chi(u) = e^{cu}$, so $f(u) = \cosh(cu)$. The real-valued cosine alternatives are exactly the purely imaginary choices of c. Since $f''(0) = c^2 = 1$ and $J = f - 1 \geq 0$ near zero with positive curvature, the imaginary alternatives are excluded and $c = \pm 1$. Hence $f(u) = \cosh u$, which gives (7). This is the standard continuous d’Alembert solution theorem [18–20]. □

Consequently

$$J(x) = J(x^{-1}), \qquad J(x) \geq 0, \qquad J(x) = 0 \iff x = 1. \tag{9}$$

Near balance,

$$J(e^u) = \frac{u^2}{2} + \frac{u^4}{24} + O(u^6). \tag{10}$$

The quadratic metric used below is therefore the Hessian of the package-selected coherent-comparison cost, not an imported Hilbert metric.

IV. EIGHT SLOTS AND THE CHARACTER TRANSFORM OF AN ORIENTED CLOCK

Proposition IV.1 (Slot-count consequence of the elementary record)**.** *Under the elementary local-record primitive, the local slot set of a closed reciprocal recognition act is a torsor for*

$$(\mathbb{Z}/2\mathbb{Z})^3. \tag{11}$$

Consequently it has eight slots.

Proof. By Definition II.3, the three involutions r, o, c commute and square to the identity. Therefore they generate a quotient of the elementary abelian group $(\mathbb{Z}/2\mathbb{Z})^3$. The action is free, so no nontrivial product of r, o, c fixes a slot; hence the quotient has all eight elements. The action is transitive, so the slot set is a torsor for $(\mathbb{Z}/2\mathbb{Z})^3$. The no-fourth-bit clause is exactly the assertion that this elementary slot record has no further independent local binary coordinate. ☐

Remark IV.2 (The 2^n-slot variant)**.** *Nothing in the scalar cost theorem singles out three binary coordinates. If a local record is instead postulated to have n independent commuting binary distinctions acting freely and transitively, then its slot set is a $(\mathbb{Z}/2\mathbb{Z})^n$-torsor with 2^n slots. Supplying an oriented Hamiltonian cycle on that hypercube gives a cyclic C_{2^n}-clock, and the corresponding character transform diagonalizes that chosen cycle. The present paper specializes to $n = 3$ because the intended elementary closed reciprocal record has exactly the three distinctions r, o, c. Thus the eight-slot case is the package's elementary local-record case, not a consequence of J alone.*

binary coordinates	**slots**	**cyclic clock if supplied**	**comment**
2	4	C_4 Hamiltonian cycle on a square	too small for the duplex eight-phase example; projective periods collapse quickly
3	8	C_8 Gray cycle on the cube	elementary package used here: four cyclic-odd modes and three non-DC cyclic-even modes
4	16	C_{16} Hamiltonian cycle on a four-cube	possible higher-resolution variant requiring additional local-record data not used in this paper

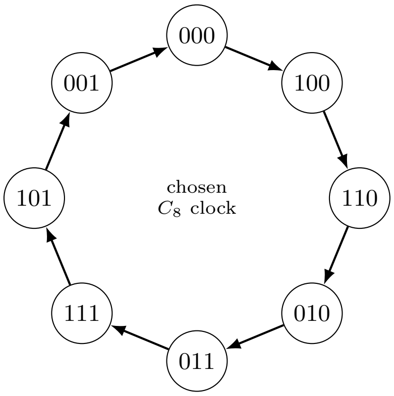
The torsor is not yet a cyclic clock. This distinction is load-bearing. The three involutions define a local cube group; a cyclic clock realization is an additional oriented Hamiltonian cycle on that cube. The DFT below diagonalizes the cyclic clock, not the three commuting involutions themselves; this is the standard character theory of finite cyclic groups [21, 22]. The unordered cube fixes the number of slots, while a cyclic ordering is supplied by an oriented complete atomic realization.

Definition IV.3 (Local cube structure and cyclic clock structure)**.** *The local cube structure is the $(\mathbb{Z}/2\mathbb{Z})^3$-torsor generated by the primitive involutions r, o, c. The cyclic clock structure is a chosen generator τ of an oriented Hamiltonian cycle on the same eight slots. The DFT diagonalizes τ, not the three commuting coordinate involutions. Its odd/even split is therefore cyclic parity under τ^4, not a Walsh split of the local cube.*

Definition IV.4 (Clock realization)**.** *Let S be the eight-element slot torsor. A clock realization is a bijection*

$$\gamma : \mathbb{Z}/8\mathbb{Z} \to S \tag{12}$$

whose successive values are adjacent vertices of the three-cube and whose orientation chooses the forward tick.



The cube supplies eight slots; the oriented Gray cycle supplies the cyclic order.

FIG. 2. Cube record versus cyclic clock. The DFT below diagonalizes the oriented cycle, not the unordered cube group.

Theorem IV.5 (Eight-slot clock)**.** *The local slot torsor admits an atomic complete oriented realization by a Hamiltonian cycle of Q_3. One realization is*

$$000 \to 100 \to 110 \to 010 \to 011 \to 111 \to 101 \to 001 \to 000. \tag{13}$$

Proof. Atomicity allows one changed binary coordinate per tick, so adjacent slots must be cube-adjacent. A complete closed act visits all eight vertices before returning, hence requires a Hamiltonian cycle. The displayed Gray cycle is such a cycle, so this oriented complete realization is an eight-tick clock; see [23] for the broader combinatorics of Gray cycles. □

Theorem IV.5 is an existence statement for a complete atomic oriented clock; it does not claim that the unordered cube itself chooses a unique cyclic order. The additional alternating-polarity rule below is the clock-realization primitive that selects the displayed order up to the stated symmetries.

Definition IV.6 (Alternating-polarity Gray clock realization)**.** *An alternating-polarity Gray clock realization is a Hamiltonian cycle of the local cube in which the primary polarity coordinate toggles on alternating edges, while the two secondary completion coordinates toggle in the intervening gaps to exhaust the cube and close the cycle. This rule selects a cyclic order; it does not identify the cyclic half-turn with the primitive reciprocal involution.*

Proposition IV.7 (Selected Gray cycle under the alternating-polarity clock rule)**.** *Under Definition IV.6, the toggle word is, up to origin, reversal, and exchange of the two secondary labels,*

$$1, 2, 1, 3, 1, 2, 1, 3.$$

With the initial vertex 000*, this gives the displayed Gray cycle* (13)*.*

Proof. The primary polarity coordinate must toggle on every other edge, hence four times in an eight-edge closed cycle. The remaining four edges must toggle the two secondary coordinates. To visit all four secondary states and return to the initial secondary state after eight steps, each secondary coordinate must toggle twice. The two secondary toggles must alternate: if two identical secondary toggles occurred in adjacent secondary gaps of the pattern $1, \_, 1, \_, 1, \_, 1, \_$, then the subword $1, a, 1, a$ would return the walk to a previously visited vertex before all eight cube vertices had been visited. A Hamiltonian realization cannot have such an early repeat. Therefore the four secondary gaps contain two toggles of each secondary coordinate in alternating order. This gives the toggle word above, up to the stated symmetries, and hence (13). The conclusion is a selected-cycle result after the alternating-polarity clock-realization rule has been imposed; it is not a canonical cyclic order of the unordered cube alone. □

a. Operational meaning of the alternating-polarity rule. The toggle word is not meant as an arbitrary Hamiltonian cycle chosen after the fact. It represents a closed reciprocal posting in which the primary numerator/denominator polarity alternates with the two completion substages needed to finish the local record. One convenient reading is:

Bit	Recognition meaning	Operational toggle
1	reciprocal polarity	toggles on alternating edges as numerator and denominator roles are exchanged
2	orientation/completion substage	toggles in the first intervening gap to record oriented posting completion
3	completion side/substage	toggles in the second intervening gap to close the reciprocal record

Thus the word $1, 2, 1, 3, 1, 2, 1, 3$ is the operational clock-realization primitive. The proof above shows uniqueness after this primitive is imposed; it does not claim that the unordered cube chooses the same cycle by itself.

Proposition IV.8 (Gray half-turn obstruction)**.** *Let τ be any Hamiltonian cycle of the cube Q_3, viewed as a permutation of its eight vertices. Then τ^4 preserves the bipartition parity of the cube. Every elementary coordinate involution flips the bipartition parity. Therefore τ^4 cannot equal any one of the three primitive coordinate involutions.*

Proof. A Hamiltonian cycle in the cube moves along cube edges, and every cube edge changes the parity of the bit sum. Hence τ reverses bipartition parity and τ^4 preserves it. A coordinate flip changes exactly one bit, hence reverses parity. A parity-preserving map cannot equal a parity-reversing map. ☐

Remark IV.9 (Two slot structures)**.** *There are two different structures on the same eight slots. The involutions r, o, c give the local $(\mathbb{Z}/2\mathbb{Z})^3$-torsor. The chosen Gray cycle τ gives a cyclic C_8-clock. The DFT is the character transform of C_8, not the Walsh character transform of the cube group. Accordingly, the Fourier split $\mathcal{W} = \mathcal{V}_{\mathrm{cyc},-} \oplus \mathcal{V}_{\mathrm{cyc},+}$ is parity under the cyclic half-turn τ^4. It is not parity under the primitive reciprocal involution r. Whenever the primitive reciprocal complement is needed, it must be written as $r(s)$ or $r(\sigma_t)$, not as $t + 4$ unless a separate non-Gray clock with $\tau^4 = r$ is chosen.*

Proposition IV.10 (Cycle covariance)**.** *The normalized DFT below is canonical for a clock realization (S, γ), not for the unordered torsor S alone. Replacing $\gamma(t)$ by $\gamma(t + r)$ multiplies the k-th Fourier coefficient by ζ^{kr}. Reversing the orientation sends the coefficient sequence to its conjugate-reversed time convention.*

Proof. A cyclic origin shift changes a profile x_t to x_{t+r}, and the character identity gives $(\mathcal{F}x')_k = \zeta^{kr}(\mathcal{F}x)_k$. Reversal changes the character ζ^{-kt} to ζ^{kt}, which is complex conjugation on real profiles and time reversal on complex profiles. ☐

Thus the clock choice is explicit: the local record supplies eight slots, and the oriented Hamiltonian cycle supplies the C_8-torsor whose characters are used below. The DFT is the character transform of the chosen cyclic clock realization, not the Walsh character transform of the underlying $(\mathbb{Z}/2\mathbb{Z})^3$-torsor. The oriented Hamiltonian cycle is part of the clock realization data. Different Hamiltonian cycles correspond to different DFT coordinate conventions unless an application supplies and verifies an equivalence. This paper fixes one oriented clock realization.

Choose an oriented slot labeling $t \in \mathbb{Z}/8\mathbb{Z}$. Let

$$(P_{\mathbb{R}}x)_t = x_{t+1} \tag{14}$$

be the real one-tick shift. Put $\zeta = e^{2\pi i/8}$. The normalized character transform of the oriented clock is

$$(\mathcal{F}x)_k = \frac{1}{\sqrt{8}} \sum_{t=0}^{7} x_t \zeta^{-kt}, \qquad k = 0, \ldots, 7. \tag{15}$$

With respect to the Fourier basis e_k, define the spectral shift by

$$Pe_k = \zeta^k e_k. \tag{16}$$

Then

$$\mathcal{F}P_{\mathbb{R}} = P\mathcal{F}. \tag{17}$$

The transform is unitary by orthogonality of characters of C_8.

V. POSITIVE SLOT COVARIANCE AND THE REAL FOURIER FORM

For a positive slot field define

$$a_t(s) = a_s(\sigma_t), \qquad \ell_t(s) = \log a_t(s). \tag{18}$$

Principle II.9 becomes

$$\ell(\widehat{\mathcal{R}}_L s) = P_{\mathbb{R}}\ell(s). \tag{19}$$

Remove scale and split by the cyclic half-turn of the chosen clock:

$$\bar{\ell} = \frac{1}{8}\sum_{t=0}^{7} \ell_t, \tag{20}$$

$$u_t^- = \frac{1}{2}(\ell_t - \ell_{t+4}), \tag{21}$$

$$u_t^+ = \frac{1}{2}(\ell_t + \ell_{t+4}) - \bar{\ell}. \tag{22}$$

Then

$$\ell - \bar{\ell}\mathbf{1} = u^- + u^+, \qquad u_{t+4}^- = -u_t^-, \qquad u_{t+4}^+ = u_t^+, \qquad \sum_t u_t^+ = 0. \tag{23}$$

The cyclic-odd part is the cyclic-antipode log-ratio:

$$\log \frac{a_t}{a_{t+4}} = 2u_t^-. \tag{24}$$

The cyclic-even part is the non-DC internal clock-reference content. By Proposition IV.8, the ratio a_t/a_{t+4} is a cyclic-antipode ratio, not generally the primitive reciprocal ratio $a_t/a_{r(t)}$.

Define

$$\mathcal{V}_{\mathrm{cyc},-} = \mathrm{span}_{\mathbb{C}}\{e_1, e_3, e_5, e_7\}, \qquad \mathcal{V}_{\mathrm{cyc},+} = \mathrm{span}_{\mathbb{C}}\{e_2, e_4, e_6\}, \qquad \mathcal{W} = e_0^{\perp} = \mathcal{V}_{\mathrm{cyc},-} \oplus \mathcal{V}_{\mathrm{cyc},+}. \tag{25}$$

For a real profile the Fourier image is conjugate symmetric:

$$\mathcal{O}_{\mathbb{R}} = \{c_1 e_1 + c_3 e_3 + \overline{c_3} e_5 + \overline{c_1} e_7 : c_1, c_3 \in \mathbb{C}\}, \tag{26}$$

$$\mathcal{E}_{\mathbb{R}} = \{d_2 e_2 + d_4 e_4 + \overline{d_2} e_6 : d_2 \in \mathbb{C},\ d_4 \in \mathbb{R}\}. \tag{27}$$

These are real forms, not full complex subspaces.

Proposition V.1 (Single-profile cyclic support)**.** *For every real slot profile,*

$$\mathcal{F}u^- \in \mathcal{O}_{\mathbb{R}} \subset \mathcal{V}_{\mathrm{cyc},-}, \qquad \mathcal{F}u^+ \in \mathcal{E}_{\mathbb{R}} \subset \mathcal{V}_{\mathrm{cyc},+}. \tag{28}$$

Proof. Since $P_{\mathbb{R}}^4 u^- = -u^-$ and $P_{\mathbb{R}}^4 u^+ = u^+$, after applying $\mathcal{F}$ one obtains $P^4\mathcal{F}u^- = -\mathcal{F}u^-$ and $P^4\mathcal{F}u^+ = \mathcal{F}u^+$. Because $P^4 e_k = (-1)^k e_k$, the first vector has only odd Fourier support and the second only even support. The mean-free condition removes e_0. Reality gives conjugate symmetry. □

A single positive profile therefore does not produce arbitrary complex amplitudes. It produces the real Fourier form of the positive ledger. The full complex carrier appears only after closed recognition forces the reversible cotangent completion.

VI. DUPLEX AS MINIMAL REVERSIBLE COTANGENT COMPLETION

Let

$$M = (0,\infty)^8/\mathbb{R}_{>0} \tag{29}$$

be the positive slot ledger modulo global scale. At the balanced point, its tangent space is

$$Q = T_{[1]}M \cong \mathbf{1}^{\perp} \subset \mathbb{R}^8, \qquad \dim Q = 7. \tag{30}$$

The Hessian of $\sum_t J(e^{q_t-\bar{q}})$ at balance is

$$g_0(q,q') = \sum_{t=0}^{7} q_t q'_t, \qquad q, q' \in Q. \tag{31}$$

Lemma VI.1 (The positive tangent alone is not a closed phase space)**.** *The seven-dimensional space Q cannot carry a nondegenerate alternating phase form. Hence a closed reversible recognition dynamics satisfying Principle II.5 cannot live on the positive log coordinate alone.*

Proof. Let A be the matrix of such a form. Skew-symmetry gives $A^T = -A$, so $\det A = \det A^T = \det(-A) = (-1)^7 \det A = -\det A$; hence $\det A = 0$. □

Definition VI.2 (Weak closed phase completion)**.** *A weak closed phase completion of Q is a tuple (E, ω, ι, π), where (E, ω) is a finite-dimensional real symplectic vector space, $\iota : Q \hookrightarrow E$ is an embedding, $\pi : E \to Q$ is a linear projection with $\pi \circ \iota = \mathrm{id}_Q$, the base $\iota(Q)$ is isotropic, and the pairing*

$$Q \times \ker \pi \to \mathbb{R}, \qquad (q,v) \longmapsto \omega(\iota(q), v) \tag{32}$$

separates both factors. This separation condition is the no-spectator condition: it rules out extra phase directions not paired with the positive slot tangent. It is nonmagnetic if

$$\omega|_{\ker \pi} = 0, \tag{33}$$

so the conjugate generators carry no autonomous generator-generator phase term.

The dimension doubling below is the linear algebra behind the standard cotangent model of phase space [13–15]. It is forced here by the nondegenerate base–fiber phase pairing. The separation clause excludes spectator phase directions. The nonmagnetic clause excludes generator–generator phase that is not paired with a positive slot variation.

Theorem VI.3 (Nondegenerate closed phase pairing gives cotangent dimension)**.** *Every weak closed phase completion has*

$$\dim E = 2 \dim Q, \tag{34}$$

so the base $\iota(Q)$ is Lagrangian. If the completion is nonmagnetic, then $\ker \pi$ *is also Lagrangian and the completion is symplectomorphic, over Q, to the cotangent completion*

$$Q \oplus Q^* \tag{35}$$

with canonical symplectic form

$$\omega_{\mathrm{can}}\big((q,\alpha),(q',\alpha')\big) = \alpha'(q) - \alpha(q'). \tag{36}$$

Using the Hessian metric g_0 to identify $Q^* \cong Q$, write a point as $(q,p) \in Q \oplus Q$. The canonical symplectic form and compatible complex structure are

$$\omega((q,p),(q',p')) = g_0(q,p') - g_0(p,q'), \qquad I(q,p) = (-p,q). \tag{37}$$

The complex coordinate is

$$z = q + ip \in Q \otimes_{\mathbb{R}} \mathbb{C}. \tag{38}$$

Proof. Let $V = \ker \pi$. Since $\pi \circ \iota = \mathrm{id}$, every $x \in E$ decomposes uniquely as

$$x = \iota(\pi x) + \big(x - \iota(\pi x)\big), \qquad x - \iota(\pi x) \in V. \tag{39}$$

Thus $E = \iota(Q) \oplus V$, and $\dim E = \dim Q + \dim V$. The pairing (32) gives injections $V \hookrightarrow Q^*$ and $Q \hookrightarrow V^*$, so $\dim V = \dim Q$. Hence $\dim E = 2 \dim Q$. Since $\iota(Q)$ is isotropic and half-dimensional, it is Lagrangian. If the completion is nonmagnetic, then V is also isotropic and half-dimensional, hence Lagrangian. The map

$$E \longrightarrow Q \oplus Q^*, \qquad x \longmapsto (\pi x, q \mapsto \omega(\iota(q), x - \iota(\pi x))) \tag{40}$$

is then an isomorphism sending $\iota(q)$ to $(q,0)$, sending V to the cotangent fiber, and transporting ω to (36). The Hessian g_0 is positive and nondegenerate on Q, so it identifies Q^* with Q, yielding (37) and $I^2 = -1$. □

Theorem VI.4 (Generator reversal excludes vertical magnetic phase)**.** *Let $E = Q \oplus V$ be a weak closed phase completion with Q isotropic and with nondegenerate base–fiber pairing. Suppose generator reversal is represented by*

$$\rho(q, v) = (q, -v)$$

and is anti-symplectic:

$$\rho^* \omega = -\omega.$$

Then $\omega|_V = 0$. Hence the completion is nonmagnetic, and by Theorem VI.3 it is symplectomorphic over Q to T^*Q.

Proof. Decompose ω into its base–fiber part and its vertical part $B = \omega|_V$. The base–fiber part changes sign under $v \mapsto -v$. For vertical vectors v_1, v_2, however,

$$(\rho^* B)(v_1, v_2) = B(-v_1, -v_2) = B(v_1, v_2).$$

Anti-symplecticity requires $\rho^* B = -B$. Therefore $B = -B$, so $B = 0$. Thus the fiber is isotropic, and the completion is nonmagnetic. ☐

Remark VI.5 (Why the nonmagnetic clause is stated)**.** *Without $\omega|_{\ker \pi} = 0$, a weak completion may contain an additional vertical two-form, analogous to a magnetic term on a cotangent bundle. Such a term is extra generator-generator structure not fixed by the positive reciprocal ledger. The nonmagnetic condition is therefore an independent no-autonomous-generator-phase convention: generator–generator phase not paired with a positive slot variation is not recognized content in this model. Principle II.5 excludes it by the no-autonomous-generator-phase requirement.*

The quadrature coordinate p is therefore not an arbitrary second copy of the positive ledger. It is the conjugate generator coordinate of the no-spectator nonmagnetic closed phase completion. Since every real p_t can be represented by a positive ratio $b_t = e^{p_t}$, the cotangent coordinate admits the same positive-ledger representation as the original coordinate.

Why the generator coordinate is also a positive reciprocal ledger

The cotangent theorem gives a dual coordinate $p \in Q^*$, and the Hessian metric identifies $Q^* \cong Q$. That linear statement alone does not provide a nonlinear generator-side cost. The extra operational input is Principle II.6: generator comparisons are themselves positive scale comparisons. A generator posting compares slotwise capacity to advance or retard the recognized profile; reversal exchanges the two capacities and sends the generator ratio to its inverse. In centered log form this is $p \mapsto -p$. Thus one may write

$$p_t = \log b_t - \frac{1}{8} \sum_s \log b_s, \qquad b_t > 0,$$

and because these generator comparisons obey the same coherent comparison law, their exact cost is

$$\sum_t J(e^{p_t}).$$

So the cotangent theorem supplies the dual phase coordinate, while the positive generator-ledger primitive supplies the nonlinear reciprocal cost on that coordinate. If the positive generator-ledger primitive is dropped, the construction still has a local cotangent/Hessian phase space, but it no longer has the exact nonlinear duplex cost used below to pin the zero generator fiber, select the pure lift, or discuss finite-amplitude reciprocal-cost corrections.

The remaining question is the lift of the slot shift to the cotangent completion. Let

$$A = P_{\mathbb{R}}|_Q. \tag{41}$$

A general linear lift covering the base update can include a shear,

$$M(q, p) = (Aq, Cq + Dp). \tag{42}$$

> **Off-shell closure hypothesis.** The next theorem is the load-bearing operator-bridge step. The closed tick covers the cyclic base shift, is symplectic on a fiber-convex neighborhood of the completed carrier, and preserves exact duplex reciprocal cost at least strongly enough to preserve the zero generator fiber. Under this hypothesis the pure lift is forced; the DFT bridge is then the character representation of that selected tick. If this hypothesis is weakened to on-shell preservation only, the bridge is no longer rigid: Proposition VI.7 gives the counterexample mechanism.

Theorem VI.6 (Unified pure-lift selection)**.** *Let $A : Q \to Q$ be orthogonal, and let*

$$F(q,p) = (Aq, \Phi(q,p))$$

be a C^1 symplectic map on a fiber-convex domain $U \subset Q \oplus Q$ for the form (37). If the zero generator fiber is preserved,

$$\Phi(q,0) = 0,$$

then

$$\Phi(q,p) = Ap$$

wherever the fiber segment from $(q,0)$ to (q,p) lies in U. In particular the closed tick is the pure cotangent lift $(q,p) \mapsto (Aq, Ap)$.

The zero-fiber hypothesis follows from exact reciprocal-cost preservation on the zero fiber:

$$G(Aq) + G(\Phi(q,0)) = G(q) + G(0), \qquad G(u) = \sum_t J(e^{u_t}),$$

because A is a slot permutation and $G(v) = 0$ only at $v = 0$. In the linear case $M(q,p) = (Aq, Cq + Dp)$, symplecticity and exact duplex-cost preservation force $C = 0$ and $D = A$. For the quadratic Hessian content $H(q,p) = \|q\|^2 + \|p\|^2$, a linear symplectic lift covering A and preserving H likewise has $C = 0$ and $D = A$.

Proof. All adjoints are taken with respect to the Hessian metric g_0. Apply symplecticity to a base variation $(\xi, 0)$ and a vertical variation $(0, \eta)$. Since

$$dF(\xi,0) = (A\xi, D_q\Phi(q,p)\xi), \qquad dF(0,\eta) = (0, D_p\Phi(q,p)\eta),$$

the canonical form gives

$$\langle A\xi, D_p\Phi(q,p)\eta\rangle = \langle \xi, \eta\rangle \quad \text{for all } \xi, \eta.$$

Thus

$$A^T D_p\Phi(q,p) = I.$$

Since A is orthogonal, $D_p\Phi(q,p) = A$. Integrating along a fiber segment gives

$$\Phi(q,p) - \Phi(q,0) = \int_0^1 D_p\Phi(q,sp)p\,ds = Ap.$$

The zero-fiber condition gives $\Phi(q,p) = Ap$. If exact cost holds on the zero fiber, then $G(Aq) = G(q)$ and the displayed identity implies $G(\Phi(q,0)) = 0$, hence $\Phi(q,0) = 0$. In the linear case the same zero-fiber argument gives $C = 0$; the symplectic block condition then gives $A^T D = I$, so $D = A$. In the Hessian case, preservation of H says the block matrix $\left(\begin{smallmatrix} A & 0 \\ C & D \end{smallmatrix}\right)$ is orthogonal for the product metric; together with symplecticity this gives $D = A$ and $C^T C = 0$, hence $C = 0$. ☐

The exact-cost condition in Theorem VI.6 is deliberately off-shell when finite-amplitude closure is claimed. If preservation is imposed only on selected ledger states, then many symplectic shears can be invisible on the selected set. To select the operator bridge rather than an arbitrary on-shell representation, closed recognition is required to extend to the completed carrier and preserve the zero generator fiber on a fiber-convex neighborhood.

Proposition VI.7 (On-shell preservation does not force the pure lift)**.** *If preservation is required only on a finite selected orbit, or only on the zero section, the pure lift is not forced. There exist smooth symplectic maps that agree with the pure lift on that selected set but differ away from it.*

Proof. A finite selected orbit has a complement containing open neighborhoods in the completed carrier. Choose a compactly supported Hamiltonian whose value and first derivative vanish on the selected set but whose Hamiltonian vector field is nonzero elsewhere. Its time-one flow is symplectic and fixes the selected set while changing off-shell points. Composing the pure lift with this flow preserves the selected orbit but is not the pure lift. Thus off-shell zero-fiber rigidity is a load-bearing closed-tick principle. ☐

Applying Theorem VI.6 to $A = P_{\mathbb{R}}|_Q$, the complex coordinate obeys

$$z(\widehat{\mathcal{R}}_L s) = P_{\mathbb{R}} z(s). \tag{43}$$

In positive ratios this means

$$q_t = \log a_t - \frac{1}{8}\sum_s \log a_s, \qquad p_t = \log b_t - \frac{1}{8}\sum_s \log b_s, \qquad a_t, b_t > 0. \tag{44}$$

The duplex ledger is thus the positive representation of the cotangent coordinate.

VII. THE DUPLEX DFT BRIDGE AND UNITARITY

Define

$$\Psi(s) = \mathcal{F}z(s) \in \mathcal{W} = e_0^{\perp}. \tag{45}$$

Theorem VII.1 (Duplex operator bridge)**.** *For every ledger state whose positive slot data obey internal covariance and whose closed tick is the selected pure cotangent lift,*

$$\Psi(\widehat{\mathcal{R}}_L s) = P\Psi(s). \tag{46}$$

Consequently the one-tick spectral update is unitary in the cost-Hessian metric.

Proof. Equation (43) gives $z(\widehat{\mathcal{R}}_L s) = P_{\mathbb{R}} z(s)$. Applying the DFT and using (17),

$$\Psi(\widehat{\mathcal{R}}_L s) = \mathcal{F}P_{\mathbb{R}} z(s) = P\mathcal{F}z(s) = P\Psi(s). \tag{47}$$

In the Fourier basis, P is diagonal with unit-modulus entries ζ^k. Since the DFT is unitary and the metric is the Hessian metric transported by the DFT, $P^*P = I$ on $\mathcal{W}$. □

Theorem VII.2 (Kinematic realization of the complex carrier)**.** *Every vector $\psi \in \mathcal{W}$ is realized by positive duplex slot data. If $z = \mathcal{F}^{-1}\psi$, set*

$$q = \operatorname{Re} z, \qquad p = \operatorname{Im} z, \qquad a_t = e^{q_t}, \qquad b_t = e^{p_t}. \tag{48}$$

Then $q, p \in \mathbf{1}^{\perp}$, $a_t, b_t > 0$, $\prod_t a_t = \prod_t b_t = 1$, and $\mathcal{F}(q + ip) = \psi$.

Proof. Since $\psi \in e_0^{\perp}$, the inverse DFT has zero mean. Decompose it into real and imaginary parts. Exponentiation gives positive slot ratios, and applying the DFT recovers ψ. □

Theorem VII.2 is kinematic. It says positive duplex coordinates can parametrize the Hessian carrier. It does not by itself select which states are admissible. The next theorem gives the missing state-selection mechanism: covariant ledger constraints and uniqueness of reciprocal-cost minimizers force the selected states to obey the bridge covariance.

Lemma VII.3 (Existence and uniqueness of affine reciprocal-cost minimizers)**.** *Let $C \subset Q \oplus Q$ be a nonempty closed affine constraint set. The exact duplex reciprocal cost*

$$\mathcal{A}_{\mathrm{dup}}(q, p) = \sum_t (\cosh q_t - 1) + \sum_t (\cosh p_t - 1)$$

has a unique minimizer on C.

Proof. The function $u \mapsto \cosh u - 1$ is strictly convex and coercive on $\mathbb{R}$. Therefore $\mathcal{A}_{\mathrm{dup}}$ is strictly convex and coercive on the finite-dimensional space $Q \oplus Q$. Its restriction to a nonempty closed affine set is again strictly convex and coercive. A continuous coercive function attains a minimum on such a set, and strict convexity makes that minimizer unique. □

Definition VII.4 (Ledger-generated covariant observable class and moment constraints)**.** *A ledger-generated observable class is a predetermined real subspace*

$$\mathcal{O}_{\mathrm{ledg}}(s) \subset Q$$

of mean-free slot functions assigned before Fourier analysis and before any selected state is inspected. It is covariant when

$$\mathcal{O}_{\mathrm{ledg}}(\widehat{\mathcal{R}}_L s) = P_{\mathbb{R}} \mathcal{O}_{\mathrm{ledg}}(s).$$

For the cube ledger, a canonical source of such observables is the real algebra generated by the centered bit functions

$$h_a(t) = 2(g_t)_a - 1, \qquad a = 1, 2, 3,$$

projected to Q, or any declared τ-stable subspace of that algebra. The important restriction is temporal: $\mathcal{O}_{\mathrm{ledg}}$ is fixed from the finite posting record and transported by the clock; it is not chosen after inspecting $\mathcal{F}(q + ip)$. In applications, *the declared observable class must be specified by a finite ledger rule independent of the desired Fourier support. The bridge theorem is not predictive if the observable class or target moments are chosen by solving backward from a desired spectral state.*

A ledger-generated covariant moment family consists of normals

$$h_i^s, k_j^s \in \mathcal{O}_{\mathrm{ledg}}(s)$$

with

$$h_i^{\widehat{\mathcal{R}}_L s} = P_{\mathbb{R}} h_i^s, \qquad k_j^{\widehat{\mathcal{R}}_L s} = P_{\mathbb{R}} k_j^s.$$

Given tick-invariant target moments $a_i, b_j \in \mathbb{R}$*, the affine slice at state* s *is*

$$\mathcal{C}_s = \{(q,p) \in Q \oplus Q : \langle q, h_i^s \rangle = a_i,\ \langle p, k_j^s \rangle = b_j\}.$$

Thus "ledger-generated" is a formal class restriction, not a synonym for arbitrary affine constraints. The worked Gray model uses declared centered bit observables of the finite ledger rather than Fourier-engineered normals. This anti-engineering condition is part of $\mathsf{P}_{\mathrm{sel}}$*, not a consequence of the minimizer theorem.*

Theorem VII.5 (Equivariant reciprocal-cost selection)**.** *Let*

$$T = P_{\mathbb{R}} \oplus P_{\mathbb{R}}$$

act on $Q \oplus Q$*, and let*

$$\mathcal{A}_{\mathrm{dup}}(q,p) = \sum_t J(e^{q_t}) + \sum_t J(e^{p_t}).$$

For every ledger state s*, let* $\mathcal{C}_s \subset Q \oplus Q$ *be a nonempty closed affine constraint set. Assume:*

(i) covariance of constraints:

$$\mathcal{C}_{\widehat{\mathcal{R}}_L s} = T\mathcal{C}_s;$$

(ii) tick invariance:

$$\mathcal{A}_{\mathrm{dup}}(Tx) = \mathcal{A}_{\mathrm{dup}}(x).$$

Let x_s *be the unique minimizer of* $\mathcal{A}_{\mathrm{dup}}$ *on* $\mathcal{C}_s$*, whose existence and uniqueness are supplied by Lemma VII.3. Then*

$$x_{\widehat{\mathcal{R}}_L s} = Tx_s.$$

Consequently, with $z_s = q_s + ip_s$ *and* $\Psi_s = \mathcal{F}z_s$*,*

$$\Psi_{\widehat{\mathcal{R}}_L s} = P\Psi_s.$$

Proof. Since $x_s \in \mathcal{C}_s$, covariance gives $Tx_s \in \mathcal{C}_{\widehat{\mathcal{R}}_L s}$. Since $\mathcal{A}_{\mathrm{dup}}(Tx_s) = \mathcal{A}_{\mathrm{dup}}(x_s)$, the candidate Tx_s has the transported cost. If some $y \in \mathcal{C}_{\widehat{\mathcal{R}}_L s}$ had lower cost, then $T^{-1}y \in \mathcal{C}_s$ would have lower cost than x_s, contradicting minimality. Thus Tx_s is a minimizer in $\mathcal{C}_{\widehat{\mathcal{R}}_L s}$. By uniqueness, $x_{\widehat{\mathcal{R}}_L s} = Tx_s$. The DFT bridge follows from $\mathcal{F}P_{\mathbb{R}} = P\mathcal{F}$. ☐

Theorem VII.6 (Ledger-moment selection)**.** *Let* $h_i^s, k_j^s \in Q$ *be ledger-generated covariant observables in the sense of Definition VII.4, and let the target moments be tick-invariant. If each affine moment slice* $\mathcal{C}_s$ *is nonempty, then the exact reciprocal cost has a unique minimizer on it, and the minimizers obey*

$$x_{\widehat{\mathcal{R}}_L s} = Tx_s, \qquad T = P_{\mathbb{R}} \oplus P_{\mathbb{R}}.$$

Consequently the selected DFT states satisfy $\Psi_{\widehat{\mathcal{R}}_L s} = P\Psi_s$*.*

Proof. The covariance equations for h_i^s and k_j^s, together with tick-invariant target moments, imply $\mathcal{C}_{\widehat{\mathcal{R}}_L s} = T\mathcal{C}_s$. Lemma VII.3 gives a unique minimizer on every nonempty slice. The result is then Theorem VII.5. ☐

Definition VII.7 (Regular analytic affine constraint family)**.** *A family of affine ledger constraints on a connected open parameter set $U \subset \mathbb{R}^m$ is regular analytic if it has the form*

$$\mathcal{C}_\alpha = \{x \in Q \oplus Q : L(\alpha)x = b(\alpha)\}, \qquad \alpha \in U,$$

where $L(\alpha)$ and $b(\alpha)$ depend real-analytically on α, every $\mathcal{C}_\alpha$ is nonempty, and $L(\alpha)$ has constant rank on U. Equivalently, after deleting redundant equations locally, the active equality constraints remain independent.

Theorem VII.8 (Generic cyclic mixed support under regular analytic cost selection)**.** *Let $\mathcal{C}_\alpha$ be a regular analytic affine constraint family in the sense of Definition VII.7. Let $x_*(\alpha) = (q_*(\alpha), p_*(\alpha))$ be the unique minimizer of the exact reciprocal cost*

$$\mathcal{A}_{\mathrm{dup}}(q,p) = \sum_t (\cosh q_t - 1) + \sum_t (\cosh p_t - 1)$$

on $\mathcal{C}_\alpha$. Then $x_*(\alpha)$, and hence every Fourier coefficient of

$$\mathcal{F}(q_*(\alpha) + ip_*(\alpha)),$$

is real-analytic in α, locally on U. If such a coefficient is nonzero at one parameter value, its zero set has empty interior unless the coefficient vanishes identically on the connected component under consideration.

Proof. After restricting to a neighborhood if necessary, constant rank lets us write the constraint equations with independent rows. The constrained minimizer satisfies the Karush–Kuhn–Tucker system

$$\nabla \mathcal{A}_{\mathrm{dup}}(x) + L(\alpha)^T \lambda = 0, \qquad L(\alpha)x = b(\alpha).$$

The Hessian $H(x) = \nabla^2 \mathcal{A}_{\mathrm{dup}}(x)$ is positive definite because each coordinate contribution has second derivative $\cosh u > 0$. The Jacobian of the KKT system in (x, λ) is

$$\begin{pmatrix} H(x) & L(\alpha)^T \\ L(\alpha) & 0 \end{pmatrix}.$$

This matrix is invertible: if $H\xi + L^T\eta = 0$ and $L\xi = 0$, then taking the inner product with ξ gives $\langle \xi, H\xi \rangle = 0$, hence $\xi = 0$, and then $L^T\eta = 0$, so $\eta = 0$ because the rows are independent. The analytic implicit-function theorem therefore gives real-analytic dependence of $(x_*(\alpha), \lambda(\alpha))$ on α. Fourier coefficients are complex-valued linear functions of $x_*(\alpha)$, hence their real and imaginary parts are real-analytic. If a complex coefficient vanishes on a nonempty open set, both real-analytic parts vanish there and therefore vanish identically on the connected component. This gives the stated zero-set conclusion. ☐

Corollary VII.9 (Structural stability of the Gray mixed-cyclic-parity clock)**.** *If one regular covariant Gray-ledger moment target produces nonzero cyclic-odd and cyclic-even non-DC Fourier support, then all sufficiently small regular perturbations outside the corresponding analytic zero sets also produce mixed cyclic parity and hence exact ray period eight.*

Admissible recognition states are therefore not arbitrary points of $\mathcal{W}$. The kinematic carrier is broad, while the admissible family is selected by ledger constraints and reciprocal-cost minimization. The worked model in Section X instantiates this theorem.

VIII. EXACT LOG BRIDGE VERSUS HESSIAN QUANTUM CARRIER

The exact bridge is exact as a log-coordinate transport law. The Hilbert metric and Born allocation are exact only as the Hessian theory of reciprocal cost at the balanced point, or as the tangent-ray limit of that theory. Away from balance, the primary content functional is the nonlinear reciprocal cost $\cosh u - 1$, and projective Hilbert probabilities are not asserted without a balanced-interface approximation or calibration.

Three levels of the construction should be distinguished.

(i) **Exact positive layer.** The finite nonlinear duplex cost is

$$\mathcal{A}_{\mathrm{dup}}(q,p) = \sum_t J(e^{q_t}) + \sum_t J(e^{p_t}). \tag{49}$$

Cyclic relabeling preserves this cost exactly, because the tick only permutes slot labels.

(ii) **Exact log bridge.** The transport equation $z(\widehat{\mathcal{R}}_L s) = P_{\mathbb{R}} z(s)$ and the DFT intertwining $\mathcal{F} P_{\mathbb{R}} = P\mathcal{F}$ give the bridge law $\Psi(\widehat{\mathcal{R}}_L s) = P\Psi(s)$ exactly in log coordinates.

(iii) **Hessian quantum carrier.** The complex Hilbert metric is the second-order Hessian theory of the reciprocal cost at balance. Its quadratic content is $\mathsf{C}_2(\Psi) = \|\Psi\|^2$. Born weights allocate this Hessian content over orthogonal recognition channels.

Thus the reciprocal cost supplies the exact positive ledger and exact log-coordinate transport. The Hilbert carrier is the balanced Hessian carrier obtained after the minimal cotangent completion and pure lift selection. The finite nonlinear cost is not replaced by the quadratic norm away from balance.

IX. VECTOR CLOCKS, RAY CLOCKS, AND THE MIXED-CYCLIC-PARITY THEOREM

For $w = \sum_{k=1}^{7} c_k e_k \in \mathcal{W} \setminus \{0\}$, define

$$S(w) = \{k : c_k \neq 0\}. \tag{50}$$

Proposition IX.1 (Vector period)**.** *The exact vector period of w under P is*

$$T_{\mathrm{vec}}(w) = \mathrm{lcm}_{k\in S(w)} \frac{8}{\gcd(8,k)}. \tag{51}$$

Thus any cyclic-odd Fourier support gives vector period eight. Support in $\{2,6\}$, including either pure mode, has vector period four; pure e_4 has vector period two; mixed even support has the least common multiple of the corresponding orders.

Proof. The condition $P^m w = w$ is equivalent to $\zeta^{km} = 1$ for every $k \in S(w)$. The order of ζ^k is $8/\gcd(8,k)$. The common return time is the least common multiple. ☐

Theorem IX.2 (Minimal ray-faithful cyclic eight-clock)**.** *The exact projective period of $[w] \in \mathbb{P}(\mathcal{W})$ is*

$$T_{\mathrm{proj}}(w) = \min\{m \geq 1 : \zeta^{(k-\ell)m} = 1 \text{ for all } k,\ell \in S(w)\}. \tag{52}$$

Equivalently,

$$T_{\mathrm{proj}}(w) = 8 \iff \exists k,\ell \in S(w) \text{ with } \gcd(8,k-\ell) = 1. \tag{53}$$

On the non-DC carrier $\mathcal{W}$, this is equivalent to mixed cyclic-odd/cyclic-even support. Therefore the cyclic-odd sector $\mathcal{V}_{\mathrm{cyc},-}$ has vector period eight but ray period at most four, and an exact ray-level eight-clock requires a relative phase between cyclic-odd content and cyclic-even non-DC clock-reference content.

Proof. A projective return means $P^m w = \lambda w$ for some nonzero scalar λ. Therefore all occupied Fourier components must acquire the same phase, which is exactly (52). The return time is eight precisely when at least one phase difference has order eight, equivalently when $\gcd(8, k-\ell) = 1$. In $\{1,\ldots,7\}$, this happens precisely when one occupied index is odd and another is cyclic-even. If all support indices have the same cyclic parity, all differences are even and $[P]^4$ is already the identity on the ray. ☐

For a mixed-cyclic-parity state choose an odd k and a cyclic-even non-DC j with $c_k c_j \neq 0$. The ray-level clock coordinate is

$$\Theta_{k,j}([w]) = \arg\left(\frac{c_k}{c_j}\right) \pmod{2\pi}. \tag{54}$$

Under one tick,

$$\Theta_{k,j}([Pw]) = \Theta_{k,j}([w]) + \frac{2\pi(k-j)}{8} \pmod{2\pi}. \tag{55}$$

The mixed-support domain is invariant under $[P]$, since the update multiplies each occupied Fourier coefficient by a nonzero phase and therefore does not change the support set.

Comparison with general 2^n-slot clocks

The eight-slot case is not selected by cyclic representation theory alone. For $N = 2^n$, let $\zeta_N = e^{2\pi i/N}$, let $P_N e_k = \zeta_N^k e_k$, and work on the non-DC carrier

$$\mathcal{W}_N = \bigoplus_{k=1}^{N-1} \mathbb{C} e_k .$$

If $w = \sum_{k=1}^{N-1} c_k e_k \neq 0$, then

$$T^{(N)}_{\text{vec}}(w) = \text{lcm}_{k\in S(w)} \frac{N}{\gcd(N,k)}, \tag{56}$$

$$T^{(N)}_{\text{proj}}(w) = \min\{m \geq 1 : \zeta_N^{(k-\ell)m} = 1 \text{ for all } k,\ell \in S(w)\}. \tag{57}$$

For $N = 2^n$, the ray period is exactly N if and only if $S(w)$ contains two indices of opposite cyclic parity. Equivalently, in the non-DC carrier, an exact ray-level 2^n-clock needs at least one odd mode and at least one even non-DC mode. The case $n = 1$ has no even non-DC reference. The case $n = 2$ has only the half-turn reference e_2. The case $n = 3$ is the first three-bit local-record case and is the one worked out below; the theorem itself extends unchanged to $n \geq 4$. Thus the mathematics of the cyclic bridge does not by itself select $n = 3$. The three-bit/eight-slot choice is the elementary local-record primitive; the Gray model shows that this primitive is sufficient to generate mixed cyclic parity from intrinsic bit moments.

X. WORKED GRAY-LEDGER MODEL AND COST-SELECTED PROFILES

Let b_1, b_2, b_3 be the standard basis of $\mathbb{Z}^3$. Use the Gray cycle

(58)

t	g_t	$\Delta_t = g_{t+1} - g_t$
0	$(0,0,0)$	b_1
1	$(1,0,0)$	b_2
2	$(1,1,0)$	$-b_1$
3	$(0,1,0)$	b_3
4	$(0,1,1)$	b_1
5	$(1,1,1)$	$-b_2$
6	$(1,0,1)$	$-b_1$
7	$(0,0,1)$	$-b_3$

The postings close, $\sum_t \Delta_t = 0$, and the balance state is $B_t = g_t$.

The same eight vertices carry two structures. In the table below, the cyclic half-turn is $\tau^4(g_t) = g_{t+4}$, while the primitive reciprocal complement is the coordinate flip $r(g_t)$ in the first bit:

(59)

t	g_t	$\tau^4(g_t) = g_{t+4}$	$r(g_t)$
0	$(0,0,0)$	$(0,1,1)$	$(1,0,0)$
1	$(1,0,0)$	$(1,1,1)$	$(0,0,0)$
2	$(1,1,0)$	$(1,0,1)$	$(0,1,0)$
3	$(0,1,0)$	$(0,0,1)$	$(1,1,0)$
4	$(0,1,1)$	$(0,0,0)$	$(1,1,1)$
5	$(1,1,1)$	$(1,0,0)$	$(0,1,1)$
6	$(1,0,1)$	$(1,1,0)$	$(0,0,1)$
7	$(0,0,1)$	$(0,1,0)$	$(1,0,1)$

Thus $t + 4$ is the cyclic antipode of the chosen clock, not the primitive reciprocal-bit complement. The Fourier cyclic-odd/cyclic-even decomposition in this paper is cyclic parity for τ^4.

A parametrized mixed-cyclic-parity check

Choose potentials $\lambda, \eta \in \mathbb{R}^3$ and even mean-free anchors $\mu, \nu \in \mathbb{R}^8$ satisfying $\mu_{t+4} = \mu_t$, $\nu_{t+4} = \nu_t$, and $\sum_t \mu_t = \sum_t \nu_t = 0$. Define mean-free profiles by

$$Q_t = \frac{1}{2}\lambda \cdot (g_t - g_{t+4}) + \mu_t, \qquad R_t = \frac{1}{2}\eta \cdot (g_t - g_{t+4}) + \nu_t. \tag{60}$$

In state s_m, set

$$q_j(s_m) = Q_{m+j}, \qquad p_j(s_m) = R_{m+j}, \qquad a_j(s_m) = e^{q_j(s_m)}, \qquad b_j(s_m) = e^{p_j(s_m)}. \tag{61}$$

Then $z(s_{m+1}) = P_{\mathbb{R}} z(s_m)$ and

$$\Psi(s_m) = \mathcal{F} z(s_m) = P^m \Psi(s_0). \tag{62}$$

For example, with

$$\begin{aligned} \lambda = (1,2,3), \qquad \eta &= (2,-1,1), \\ \mu &= (0,1,2,-3,0,1,2,-3), \\ \nu &= (2,0,1,-3,2,0,1,-3), \end{aligned} \tag{63}$$

the anchors are half-turn even and mean-free. With the DFT convention (15), one obtains

$$c_0 = 0, \qquad |c_1| \approx 4.619, \qquad |c_2| \approx 2.236. \tag{64}$$

Thus the state has nonzero cyclic-odd and cyclic-even non-DC support, so Theorem IX.2 gives exact ray period eight. This first calculation is a parametrized ledger check; the anchors fix the profiles and therefore do not by themselves demonstrate cost selection.

Gray-intrinsic reciprocal-cost selected duplex profile

To avoid Fourier-shaped constraints, use the centered bit observables of the Gray ledger itself:

$$h_i(t) = 2(g_t)_i - 1, \qquad i = 1,2,3. \tag{65}$$

Minimize the exact reciprocal log cost

$$\sum_{t=0}^{7} (\cosh Q_t - 1) \tag{66}$$

over $Q \in \mathbf{1}^{\perp}$ subject to the Gray-bit moment constraints

$$\sum_t Q_t h_1(t) = 2, \qquad \sum_t Q_t h_3(t) = 1. \tag{67}$$

The strict convexity and coercivity of $\cosh u - 1$ on the affine constraint set gives a unique minimizer. Since the Gray-bit functions are orthogonal and have $\sum_t h_i(t)^2 = 8$, the candidate

$$Q_t = \frac{1}{4} h_1(t) + \frac{1}{8} h_3(t) \tag{68}$$

obeys the constraints, namely

$$Q = \left(-\frac{3}{8}, \frac{1}{8}, \frac{1}{8}, -\frac{3}{8}, -\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, -\frac{1}{8} \right). \tag{69}$$

The Euler–Lagrange equation has the required multiplier form because

$$\begin{aligned} \sinh Q_t &= \sinh\left(\frac{1}{4} h_1(t) + \frac{1}{8} h_3(t) \right) \\ &= \sinh\left(\frac{1}{4}\right) \cosh\left(\frac{1}{8}\right) h_1(t) + \cosh\left(\frac{1}{4}\right) \sinh\left(\frac{1}{8}\right) h_3(t). \end{aligned} \tag{70}$$

Thus Q is the unique least-cost profile selected by Gray-intrinsic ledger moments.

Now impose equally intrinsic generator-side constraints and minimize

$$\sum_{t=0}^{7}(\cosh R_t - 1) \tag{71}$$

over $R \in \mathbf{1}^{\perp}$ subject to

$$\sum_t R_t h_2(t) = 2, \qquad \sum_t R_t h_3(t) = -1. \tag{72}$$

The unique minimizer is

$$R_t = \frac{1}{4}h_2(t) - \frac{1}{8}h_3(t), \tag{73}$$

namely

$$R = \left(-\frac{1}{8}, -\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}, \frac{1}{8}, -\frac{3}{8}, -\frac{3}{8}\right). \tag{74}$$

Indeed,

$$\begin{aligned} \sinh R_t &= \sinh\left(\frac{1}{4}h_2(t) - \frac{1}{8}h_3(t)\right) \\ &= \sinh\left(\frac{1}{4}\right)\cosh\left(\frac{1}{8}\right)h_2(t) - \cosh\left(\frac{1}{4}\right)\sinh\left(\frac{1}{8}\right)h_3(t), \end{aligned} \tag{75}$$

so the Euler–Lagrange equation has the multiplier form for the generator-side constraints.

Set $z = Q + iR$. With the normalized DFT convention (15), the Fourier coefficients $c_k = (\mathcal{F}z)_k$ are exactly

$$\begin{aligned} &c_0 = 0, \qquad c_4 = 0, \\ &c_1 = \frac{1}{8} + \frac{\sqrt{2}}{8} - \frac{i}{8}, \\ &c_2 = -\frac{\sqrt{2}}{4}(1+i), \\ &c_3 = \frac{1}{8} - \frac{\sqrt{2}}{8}(2+i) + \frac{3i}{8}, \\ &c_5 = -\frac{1}{8} + \frac{\sqrt{2}}{8} + \frac{i}{8}, \\ &c_6 = -\frac{\sqrt{2}}{4}(1-i), \\ &c_7 = -\frac{1}{8} - \frac{3i}{8} - \frac{\sqrt{2}}{8}(2+i). \end{aligned} \tag{76}$$

Consequently

$$\begin{aligned} |c_1|^2 &= \frac{1}{16} + \frac{\sqrt{2}}{32}, & |c_2|^2 &= \frac{1}{4}, \\ |c_3|^2 &= \frac{5}{16} - \frac{5\sqrt{2}}{32}, & |c_5|^2 &= \frac{1}{16} - \frac{\sqrt{2}}{32}, \\ |c_6|^2 &= \frac{1}{4}, & |c_7|^2 &= \frac{5}{16} + \frac{5\sqrt{2}}{32}. \end{aligned} \tag{77}$$

As a Parseval check,

$$\sum_{k=1}^{7}|c_k|^2 = \frac{5}{4} = \|Q\|^2 + \|R\|^2. \tag{78}$$

These exact formulas prove nonzero cyclic-odd support c_1, c_3, c_5, c_7 and nonzero cyclic-even non-DC support c_2, c_6. The constraint functions $h_i(t) = 2(g_t)_i - 1$ are the centered bit observables of the finite Gray ledger, not Fourier modes. We verified equations (69)–(77), the Parseval identity $\sum_k |c_k|^2 = 5/4$, the entire correction table of Section XI, the parity aggregate $\pm 23/51200$, and the Sorkin coefficient $3(1-\sqrt{2})/32768$ by a short independent numerical check in double precision; all values agree to floating precision. The selected duplex state has nonzero cyclic-odd and cyclic-even non-DC Fourier support. Therefore the mixed-cyclic-parity theorem gives exact ray period eight. By Corollary VII.9, this mixed support is stable under small perturbations of the Gray-bit moment targets outside the corresponding analytic zero sets. The point is not that these four bit moments are universal; it is that reciprocal-cost selection can generate a nonzero duplex mixed-cyclic-parity state from intrinsic finite-ledger observables rather than from directly prescribed Fourier coefficients.

Equivariant cost-selection pipeline

The intrinsic minimizer above can be transported through the whole finite Gray ledger by using covariant constraints. Let

$$T = P_{\mathbb{R}} \oplus P_{\mathbb{R}}$$

on $Q \oplus Q$, and let $\mathcal{C}_{s_0}$ be the affine constraint set defined by

$$\sum_t q_t h_1(t) = 2, \qquad \sum_t q_t h_3(t) = 1, \qquad \sum_t p_t h_2(t) = 2, \qquad \sum_t p_t h_3(t) = -1.$$

For $s_m = \widehat{\mathcal{R}}_L^m s_0$, define

$$\mathcal{C}_{s_m} = T^m \mathcal{C}_{s_0}.$$

These are Gray-intrinsic constraints because the bit observables are transported with the ledger clock. The exact reciprocal-cost minimizer in $\mathcal{C}_{s_0}$ is the duplex profile $x_{s_0} = (Q, R)$ from (69) and (74). By Theorem VII.5,

$$x_{s_m} = T^m x_{s_0}.$$

Consequently, with $z_{s_m} = q_{s_m} + i p_{s_m}$,

$$\Psi_{s_{m+1}} = \mathcal{F} z_{s_{m+1}} = \mathcal{F} P_{\mathbb{R}} z_{s_m} = P \Psi_{s_m}.$$

The exact coefficient formulas (76)–(77) show that Ψ_{s_0} has mixed cyclic-odd/cyclic-even non-DC support, so the selected state has exact ray period eight. This gives the complete pipeline:

finite Gray ledger → covariant constraints → reciprocal-cost minimizer
→ DFT bridge → mixed-cyclic-parity ray clock.

More generally, impose fewer than seven independent linear ledger constraints. Let $A : Q \to \mathbb{R}^r$ and $B : Q \to \mathbb{R}^s$, with $r, s < 7$, and define a covariant family

$$\mathcal{C}_s = \{(Q, R) \in Q \oplus Q : A_s Q = \rho_s,\ B_s R = \sigma_s\}, \qquad \mathcal{C}_{\widehat{\mathcal{R}}_L s} = T \mathcal{C}_s.$$

If each $\mathcal{C}_s$ is nonempty, the exact reciprocal cost has a unique minimizer on it because $\cosh u - 1$ is strictly convex and coercive on finite-dimensional affine slices. The Euler–Lagrange equations have the form

$$\sinh Q_{*,t} = (A_s^T \alpha)_t + \gamma, \qquad \sinh R_{*,t} = (B_s^T \beta)_t + \delta,$$

where γ, δ enforce mean-free constraints. Theorem VII.5 then transports these minimizers through the bridge.

XI. FINITE-AMPLITUDE RECIPROCAL-COST ALLOCATION IN THE GRAY MODEL

The structural bridge uses the Hessian carrier at balance. At that level an orthogonal decomposition allocates content by the usual quadratic weights. The exact recognition cost, however, is not the quadratic norm; it is the nonlinear slot cost $\cosh u - 1$. Keeping the nonlinear cost gives a small-amplitude correction to Hessian allocation. This is where the reciprocal cost produces a concrete quantity beyond textbook cyclic representation theory.

Let $\mathcal{K} \subseteq \mathcal{W}$ be a chosen complex measurement carrier with orthogonal decomposition $\mathcal{K} = \bigoplus_j \Pi_j \mathcal{K}$. For $\phi \in \mathcal{K}$, write

$$\mathcal{F}^{-1}\phi = q_\phi + ip_\phi, \qquad q_\phi, p_\phi \in \mathbf{1}^\perp,$$

and define

$$M_4(\phi) = \sum_{t=0}^{7} q_{\phi,t}^4 + \sum_{t=0}^{7} p_{\phi,t}^4.$$

For $\epsilon > 0$, the exact reciprocal-cost allocation over the fixed frame is

$$W_j(\epsilon, \psi) = \frac{\mathcal{A}_{\mathrm{dup}}(\epsilon \Pi_j \psi)}{\sum_m \mathcal{A}_{\mathrm{dup}}(\epsilon \Pi_m \psi)}.$$

The next proposition is the central correction result of this section; the Gray table below is its concrete one-state specialization.

Proposition XI.1 (Finite-amplitude reciprocal-cost correction)**.** *Let* $B_j = \|\Pi_j \psi\|^2 / \|\psi\|^2$. *If* $\Pi_j \psi \neq 0$, *then*

$$W_j(\epsilon, \psi) = B_j + \frac{\epsilon^2}{12} B_j \left(\frac{M_4(\Pi_j \psi)}{\|\Pi_j \psi\|^2} - \frac{\sum_m M_4(\Pi_m \psi)}{\|\psi\|^2} \right) + O(\epsilon^4). \tag{79}$$

If $\Pi_j \psi = 0$, *then* $W_j(\epsilon, \psi) = 0$. *The correction is coordinate- and interface-dependent; it is not a universal Born-rule correction.*

Proof. Use

$$J(e^u) = \cosh u - 1 = \frac{u^2}{2} + \frac{u^4}{24} + O(u^6)$$

in each slot. Hence

$$\mathcal{A}_{\mathrm{dup}}(\epsilon \phi) = \frac{\epsilon^2}{2} \|\phi\|^2 + \frac{\epsilon^4}{24} M_4(\phi) + O(\epsilon^6).$$

Expanding the quotient gives (79). □

For the exact Gray minimizer (76), take the fixed Fourier-frame projectors Π_k onto $\mathbb{C}e_k$. Write

$$W_k(\epsilon) = B_k + \alpha_k \epsilon^2 + O(\epsilon^4).$$

The exact Born weights $B_k = |c_k|^2 / \sum_m |c_m|^2$ and first finite-amplitude correction coefficients are:

k	B_k	α_k in $W_k(\epsilon) = B_k + \alpha_k \epsilon^2 + O(\epsilon^4)$
1	$\frac{2+\sqrt{2}}{40}$	$-\frac{208+89\sqrt{2}}{2457600}$
2	$\frac{1}{5}$	$-\frac{23}{102400}$
3	$\frac{2-\sqrt{2}}{8}$	$\frac{152-151\sqrt{2}}{491520}$
5	$\frac{2-\sqrt{2}}{40}$	$\frac{-208+89\sqrt{2}}{2457600}$
6	$\frac{1}{5}$	$-\frac{23}{102400}$
7	$\frac{2+\sqrt{2}}{8}$	$\frac{152+151\sqrt{2}}{491520}$

The coefficients sum to zero, as they must after normalization.

Calculation. For a single Fourier mode $c_k e_k$, the inverse transform is

$$(\mathcal{F}^{-1} c_k e_k)_t = \frac{c_k}{\sqrt{8}} \zeta^{kt}.$$

Using the coefficients (76), one obtains

$$\begin{aligned}
\frac{M_4(c_1 e_1)}{|c_1|^2} &= \frac{3}{512} + \frac{3\sqrt{2}}{1024}, & \frac{M_4(c_2 e_2)}{|c_2|^2} &= \frac{1}{64}, \\
\frac{M_4(c_3 e_3)}{|c_3|^2} &= \frac{15}{512} - \frac{15\sqrt{2}}{1024}, & \frac{M_4(c_5 e_5)}{|c_5|^2} &= \frac{3}{512} - \frac{3\sqrt{2}}{1024}, \\
\frac{M_4(c_6 e_6)}{|c_6|^2} &= \frac{1}{64}, & \frac{M_4(c_7 e_7)}{|c_7|^2} &= \frac{15}{512} + \frac{15\sqrt{2}}{1024}.
\end{aligned}$$

Moreover

$$\sum_{k=1}^{7} M_4(c_k e_k) = \frac{149}{4096}, \qquad \frac{\sum_k M_4(c_k e_k)}{\|\psi\|^2} = \frac{149}{5120}, \qquad \|\psi\|^2 = \frac{5}{4}.$$

Substitution in Proposition XI.1 gives the table. □

The aggregate cyclic-parity channels have an especially simple correction:

$$W_{\mathrm{odd}}(\epsilon) = \frac{3}{5} + \frac{23}{51200}\epsilon^2 + O(\epsilon^4), \qquad W_{\mathrm{even}}(\epsilon) = \frac{2}{5} - \frac{23}{51200}\epsilon^2 + O(\epsilon^4). \tag{80}$$

Thus the nonlinear cost shifts a small amount of normalized finite-amplitude weight from the cyclic-even reference side into the cyclic-odd content side for this Gray-minimized state. This sign and magnitude are properties of the chosen Gray ledger, clock, and frame; they are not invariant under arbitrary basis changes.

Relation to higher-order interference tests

Sorkin's hierarchy characterizes theories by which multi-path inclusion-exclusion terms vanish [24, 25]. The present finite-amplitude score is not, by itself, a path probability measure. It is a recognition-cost score in the slot/DFT interface. Nevertheless it identifies the scale at which the exact reciprocal cost can depart from the Hessian/Born allocation once an apparatus supplies path projectors and a readout map.

Let $\mu_\epsilon(A) = \mathcal{A}_{\mathrm{dup}}(\epsilon \Pi_A \psi)$, where Π_A is the sum of the fixed-frame projectors in a subset A. For three singleton alternatives with real slot components (a_t, α_t), (b_t, β_t), and (c_t, γ_t) in the q- and p-coordinates, the third Sorkin difference has the leading term

$$\begin{aligned}
I_3(A,B,C) &= \mu_\epsilon(A \cup B \cup C) - \mu_\epsilon(A \cup B) - \mu_\epsilon(A \cup C) - \mu_\epsilon(B \cup C) \\
&\quad + \mu_\epsilon(A) + \mu_\epsilon(B) + \mu_\epsilon(C) \\
&= \frac{\epsilon^4}{2} \sum_t \Big[a_t b_t c_t (a_t + b_t + c_t) + \alpha_t \beta_t \gamma_t (\alpha_t + \beta_t + \gamma_t) \Big] + O(\epsilon^6).
\end{aligned} \tag{81}$$

Indeed, the third finite difference of u^4 is $12abc(a+b+c)$, and the coefficient of u^4 in $\cosh u - 1$ is $1/24$. Thus the tangent Hessian theory is Sorkin-level two, while the finite reciprocal-cost score can generate higher-order terms at order ϵ^4 in a fixed recognition-coordinate interface. For the Gray Fourier-frame alternatives $\{1, 3, 5\}$, direct substitution gives

$$I_3(1,3,5) = \frac{3(1-\sqrt{2})}{32768}\epsilon^4 + O(\epsilon^6), \tag{82}$$

which is nonzero.

Triple-slit and multipath tests of Born-rule additivity bound higher-order interference in optical and matter-wave implementations. The Sinha–Couteau–Jennewein–Laflamme–Weihs experiment bounded three-path interference below roughly 10^{-2} of the relevant two-path interference [26]; subsequent three-path and five-path interferometric work gives representative bounds at or beyond the 10^{-4} scale, with Kauten et al. providing a five-path optical bound and Cotter et al. giving a large-molecule multipath search [27–29]. These bounds are quoted only to set scale. The recognition-cost score is not itself a path probability, and the numerical thresholds in the cited papers become predictions about ϵ only once a path-to-ledger adapter is specified and validated; absent such an adapter, the comparison in the next paragraph is illustrative.

For the Gray single-mode allocation above,

$$\alpha_{\max} = \alpha_7 = \frac{19}{61440} + \frac{151\sqrt{2}}{491520} \approx 7.44 \times 10^{-4}.$$

If a laboratory adapter identified the normalized finite-amplitude deviation with an experimentally bounded probability deviation δ, the small-amplitude estimate would be

$$\epsilon \lesssim \sqrt{\delta/\alpha_{\max}}.$$

For $\delta = 10^{-2}$ this gives no small-amplitude constraint; for $\delta = 10^{-4}$ it gives $\epsilon \lesssim 0.37$. These numbers are calibration examples, not direct experimental bounds, because this paper does not specify a slit apparatus or a path-to-ledger adapter. The point is narrower but important: the nonlinear reciprocal cost supplies a computable, falsifiable correction target once such an adapter is specified.

XII. FINITE FAITHFULNESS AUDIT

A finite ledger update $R : X \to X$ is a functional graph: every connected component contains one directed cycle with transient trees possibly feeding into it.

Theorem XII.1 (Finite semiconjugacy into a free period-n action). *Let X be finite and let $T : Y \to Y$ be a bijection on a nonempty set such that every point of Y has exact period n. A map $F : X \to Y$ satisfying*

$$F(Rx) = TF(x) \tag{83}$$

exists if and only if every directed cycle of R has length divisible by n. Such an F is injective if and only if every component of R is an exact n-cycle and Y contains enough pairwise disjoint T-orbits to assign one orbit to each component.

Proof. If x lies on a directed cycle of length m, then $F(x) = F(R^m x) = T^m F(x)$. Exact T-period n forces $n \mid m$. Conversely, if every cycle length is divisible by n, first treat a directed cycle $x_0 \mapsto x_1 \mapsto \cdots \mapsto x_{m-1} \mapsto x_0$ by setting $\varphi(x_j) = j \pmod n$. This is consistent at return precisely because $m \equiv 0 \pmod n$. On the transient trees feeding into the cycle, define φ by subtracting transient depth modulo n. Then $\varphi(Rx) = \varphi(x) + 1 \pmod n$. Choosing a seed $y_C \in Y$ for each component and setting $F(x) = T^{\varphi(x)} y_C$ gives (83). Injectivity forbids transients, branching, and cycle lengths larger than n, and it requires distinct components to use disjoint target orbits. These conditions are also sufficient. □

Theorem XII.2 (Eight-cycle bridge audit). *For a finite recognition ledger:*

- *(a) a nonzero vector bridge into $\mathcal{V}_{\mathrm{cyc},-} \setminus \{0\}$ exists precisely when every directed cycle length is divisible by eight;*
- *(b) vector faithfulness is possible precisely when every component is an exact eight-cycle and the chosen seed vectors have pairwise disjoint P-orbits;*
- *(c) bare cyclic-odd core projective bridges exist for every finite map because the completely nonfaithful map collapsing all states to a fixed eigenray satisfies the projective bridge equation; faithful cyclic-odd core projective behavior is limited to exact projective periods 1, 2, and 4;*
- *(d) projective bridges into the mixed-cyclic-parity domain*

$$\Omega_{\mathrm{mix}} = \{[w] \in \mathbb{P}(\mathcal{W}) : S(w) \textit{ has odd and even support}\} \tag{84}$$

obey the exact-eight audit of Theorem XII.1.

Proof. Parts (a) and (b) apply Theorem XII.1 with $n = 8$, $Y = \mathcal{V}_{\mathrm{cyc},-} \setminus \{0\}$, and $T = P$, using Proposition IX.1. Part (c) follows because each eigenray $[e_k]$ is fixed by $[P]$, while Theorem IX.2 shows the cyclic-odd core has ray period at most four. On Ω_{mix}, Theorem IX.2 makes $[P]$ a free period-eight action, so Theorem XII.1 applies. □

The eigenray bridges in part (c) are completely noninformative: they carry no finite-ledger faithfulness and no observable eight-clock. They are included only to separate bare existence of a projective semiconjugacy from meaningful clock representation.

Example XII.3 (Concrete failure cases)**.** *A ledger fixed point cannot have a nonzero vector bridge into the cyclic-odd period-eight carrier. If* $Rx = x$ *and* $F(Rx) = PF(x)$*, then* $F(x) = PF(x)$*; but* P *has no eigenvalue* 1 *on* $\mathcal{V}_{\mathrm{cyc},-}$*, so* $F(x) = 0$*. Likewise, a directed* 12*-cycle cannot have a nonzero bridge into a free period-eight target: the cycle condition would force* $T^{12}F(x) = F(x)$*, while exact target period eight requires* $8 \mid 12$*, which is false. By contrast, a* 16*-cycle can have a nonfaithful period-eight vector shadow by mapping two ledger points to each target phase, but it cannot be faithful because injectivity would require the ledger cycle itself to have exact length eight. These examples show that the audit is not a universal-encoding trick; it gives real representational obstructions.*

The audit records what the spectral state remembers. A bridge into the full non-DC carrier $\mathcal{W}$ does not by itself force period eight: a selected vector records an eight-phase vector shadow exactly when its support contains an order-eight mode, and a selected ray records an exact eight-clock exactly when its support has mixed cyclic parity. A bridge can collapse transients and long cycles into the same selected spectral orbit. Full faithfulness to the finite ledger is possible if and only if the functional graph consists of exact eight-cycles with separated target orbits.

a. Carrier taxonomy. The notation below uses several carriers. The following table records their roles and failure modes.

Carrier	**Role**	**Warning**
$\mathcal{V}_{\mathrm{cyc},-}$	cyclic-odd vector-period-eight core	rays have period at most four
$\mathcal{V}_{\mathrm{cyc},+}$	non-DC cyclic-even clock-reference sector	not primitive reciprocal-bit parity
$\mathcal{W} = \mathcal{V}_{\mathrm{cyc},-} \oplus \mathcal{V}_{\mathrm{cyc},+}$	full non-DC duplex Hessian carrier	full-content measurements disturb all occupied sectors
$\mathfrak{R}$	protected reference-conditioned state space	not an ordinary projective Hilbert space

XIII. REFERENCE-CONDITIONED RECOGNITION STATES

Full-content versus reference-conditioned measurement. In the full-content interface, $w = v + r \in \mathcal{W}$ is one Hilbert vector: the even reference contributes to norm, Born weights, Hamiltonian spectrum, and disturbance. In the reference-conditioned interface, r is a protected phase standard. Content probabilities are conditional on the cyclic-odd content norm and instruments act only on v. This is not a projective decomposition of $\mathcal{W}$.

The mixed-cyclic-parity theorem has a direct consequence for the physical carrier. The cyclic-odd sector $\mathcal{V}_{\mathrm{cyc},-}$ supplies vector period eight, but its rays have period at most four. A ray-observable eight-clock therefore uses a mixed-cyclic-parity pair: cyclic-odd content plus a cyclic-even non-DC reference. The reference-conditioned quotient $\mathfrak{R}$ below is not a Hilbert projective space; content Born weights live on $\mathcal{V}_{\mathrm{cyc},-}$, not on $\mathfrak{R}$ as a whole.

Definition XIII.1 (Protected clock-reference interface)**.** *A reference-conditioned experiment consists of a cyclic-odd content carrier* $\mathcal{V}_{\mathrm{cyc},-}$*, a cyclic-even non-DC reference carrier* $\mathcal{V}_{\mathrm{cyc},+}$*, and an allowed instrument class whose content operations act on* $v \in \mathcal{V}_{\mathrm{cyc},-}$ *while preserving the reference coordinate* $r \in \mathcal{V}_{\mathrm{cyc},+}$*. A general allowed content instrument is a finite family of complex-linear maps*

$$A_j : \mathcal{V}_{\mathrm{cyc},-} \to \mathcal{V}_{\mathrm{cyc},-}, \qquad \sum_j A_j^* A_j = I_{\mathcal{V}_{\mathrm{cyc},-}},$$

with branch weights

$$p_j^A(v \mid r) = \frac{\|A_j v\|^2}{\|v\|^2}$$

and nonzero post-branches

$$[(v, r)] \longmapsto [(A_j v, r)].$$

If $A_j v = 0$*, the branch has zero conditional weight and no normalized post-state. Sharp repeatable refinements are the special case* $A_j = \Pi_j^{\mathcal{V}_{\mathrm{cyc},-}}$ *with mutually orthogonal projectors. Clock readouts are relative phases between occupied cyclic-odd and cyclic-even components. Projectors on the ambient carrier* $\mathcal{W}$ *are not automatically allowed reference-conditioned instruments. This is an operational restriction on instruments, not a theorem about all projectors on* $\mathcal{W}$*: if an experiment allows arbitrary projectors on* $\mathcal{W}$*, the even reference is measured content and can be disturbed.*

This protected-reference convention is an interface assumption analogous in spirit to reference-frame and superselection analyses in quantum information [12]. It is not an ordinary projective state space on $\mathcal{W}$, and it should not be read as a claim that arbitrary projectors on $\mathcal{W}$ preserve the clock reference.

Definition XIII.2 (Reference-conditioned recognition state)**.** *The reference-conditioned state space is the two-sorted quotient*

$$\mathfrak{R} = (\mathcal{V}_{\mathrm{cyc},-} \setminus \{0\}) \times (\mathcal{V}_{\mathrm{cyc},+} \setminus \{0\})/\mathbb{C}^{\times},$$

where common complex scaling identifies

$$(v, r) \sim (\lambda v, \lambda r), \qquad \lambda \in \mathbb{C}^{\times}.$$

A state is denoted $[(v, r)]$*. The clock update is*

$$[(v, r)] \longmapsto [(Pv, Pr)].$$

Sharp reciprocal-content measurements are represented by orthogonal projectors $\Pi_j^{\mathcal{V}_{\mathrm{cyc},-}}$ *on* $\mathcal{V}_{\mathrm{cyc},-}$*. The corresponding normalized repeatable content instrument is partial:*

$$\mathcal{I}_j[(v, r)] = [(\Pi_j^{\mathcal{V}_{\mathrm{cyc},-}} v, r)]$$

when $\Pi_j^{\mathcal{V}_{\mathrm{cyc},-}} v \neq 0$*. If* $\Pi_j^{\mathcal{V}_{\mathrm{cyc},-}} v = 0$*, the branch has zero conditional weight and no normalized post-state. The conditional weights are*

$$p_j(v \mid r) = \frac{\|\Pi_j^{\mathcal{V}_{\mathrm{cyc},-}} v\|^2}{\|v\|^2}.$$

More general protected content instruments use the maps A_j *of Definition XIII.1; they still leave* r *outside the measured content alternatives. Clock observables are relative phases*

$$\Theta_{k,j}([(v, r)]) = \arg\left(\frac{c_k(v)}{c_j(r)}\right), \qquad k \text{ odd}, \quad j \text{ even}, \quad c_k(v) c_j(r) \neq 0.$$

The reference-conditioned interface is not the same experiment as the full-content interface, and the quotient $\mathfrak{R}$ is not an ordinary projective Hilbert space. It assumes that the cyclic-even non-DC component is maintained as a protected phase standard and is not included among reciprocal-content alternatives.

Theorem XIII.3 (Reference-conditioned clock and content)**.** *For every reference-conditioned state* $[(v, r)]$*, the quotient* $\mathfrak{R}$ *is not assigned a Hilbert norm or a projection lattice; all content weights below are computed in the content carrier* $\mathcal{V}_{\mathrm{cyc},-}$ *conditional on the protected reference. With that convention:*

(a) if v *has nonzero cyclic-odd support and* r *has nonzero cyclic-even non-DC support, then some relative phase clock has exact period eight in the quotient* $\mathfrak{R}$*;*

(b) the sharp reciprocal-content weights $p_j(v \mid r)$ *are independent of the normalization and phase of* r*;*

(c) for every protected instrument $\{A_j\}$*, the weights* $p_j^A(v \mid r) = \|A_j v\|^2/\|v\|^2$ *sum to one and are independent of the normalization and phase of* r*;*

(d) under one tick, the content measurement frame transforms covariantly by

$$\Pi_j^{\mathcal{V}_{\mathrm{cyc},-}} \longmapsto P \Pi_j^{\mathcal{V}_{\mathrm{cyc},-}} P^*;$$

(e) an allowed protected content instrument acts only on v *and leaves* r *untouched, so it preserves the clock reference as a reference. A generic projector on* $\mathcal{W}$ *need not do so and can destroy the mixed-cyclic-parity ray clock.*

Proof. For (a), after m ticks the pair returns in $\mathfrak{R}$ precisely when there is a common scalar $\lambda \in \mathbb{C}^\times$ such that

$$P^m v = \lambda v, \qquad P^m r = \lambda r.$$

Choose an occupied odd k of v and an occupied cyclic-even non-DC j of r. The common-scalar condition gives $\zeta^{km} = \lambda = \zeta^{jm}$, hence $\zeta^{(k-j)m} = 1$. Because $k - j$ is odd, this forces $8 \mid m$. Since $P^8 = I$, the period is exactly eight.

Part (b) is immediate because the sharp weights depend only on v and are homogeneous of degree zero. For (c),

$$\sum_j p_j^A(v \mid r) = \frac{\sum_j \langle A_j v, A_j v\rangle}{\|v\|^2} = \frac{\langle v, (\sum_j A_j^* A_j) v\rangle}{\|v\|^2} = 1,$$

and the expression is independent of r. Part (d) follows from $P\mathcal{V}_{\mathrm{cyc},-} = \mathcal{V}_{\mathrm{cyc},-}$ and covariance of projectors under unitary change of time frame. For (e), every allowed protected content branch has the operational rule

$$[(v, r)] \mapsto [(A_j v, r)]$$

when the numerator is nonzero, so the even reference persists as a reference coordinate. The sharp projector formula is the repeatable special case $A_j = \Pi_j^{\mathcal{V}_{\mathrm{cyc},-}}$. A general projector on $\mathcal{W}$ can remove the even component or map a mixed-cyclic-parity vector outside the mixed-cyclic-parity domain, so it need not preserve the clock. ☐

There are therefore two coherent measurement interpretations.

(i) **Full-content interpretation.** Work with $w = v + r \in \mathcal{W}$. The even reference is measured content. It contributes to norm, Born weights, Hamiltonian spectrum, and possible measurement disturbance.

(ii) **Reference-conditioned interpretation.** Work with $[(v, r)]$. The cyclic-even sector is a protected phase standard; reciprocal-content measurements act on $\mathcal{V}_{\mathrm{cyc},-}$ and use $p_j(v)$, while clock observables use relative phases between v and r.

These are different operational interfaces. The full-content interface is the default in the ambient Hilbert carrier. The reference-conditioned interface is required if the cyclic-even non-DC component is to be used only as a clock standard.

XIV. EXACT RECIPROCAL COST AND THE HERMITIAN HESSIAN GERM

Definition XIV.1 (Slot-additive reciprocal ledger cost)**.** *For a scale-free slot profile $q \in \mathbf{1}^\perp$, the recognized cost of independent slot postings is*

$$G(q) = \sum_{t=0}^{7} J(e^{q_t}).$$

Coupled recognition information is represented by the admissible affine slice on which G is minimized; it is not inserted as an additional hidden cross-slot term in the cost itself.

For $(q, p) \in Q \oplus Q$, the duplex cost is the slot-additive cost of the base ledger plus the slot-additive cost of the positive generator ledger:

$$\mathcal{A}_{\mathrm{dup}}(q, p) = \sum_{t=0}^{7} J(e^{q_t}) + \sum_{t=0}^{7} J(e^{p_t}), \qquad q, p \in \mathbf{1}^\perp. \tag{85}$$

Proposition XIV.2 (Hermitian metric germ)**.** *As $(q, p) \to 0$,*

$$\mathcal{A}_{\mathrm{dup}}(q, p) = \frac{1}{2}\|q\|^2 + \frac{1}{2}\|p\|^2 + O(\|q\|^4 + \|p\|^4). \tag{86}$$

For $z = q + ip$ and $\Psi = \mathcal{F} z$,

$$\mathcal{A}_{\mathrm{dup}}(q, p) = \frac{1}{2}\|z\|^2 + O(\|z\|^4) = \frac{1}{2}\|\Psi\|^2 + O(\|\Psi\|^4). \tag{87}$$

Proof. Apply (10) term by term. The identity $\|z\|^2 = \|q\|^2 + \|p\|^2$ is the Hermitian norm in the compatible complex structure, and DFT unitarity gives $\|\Psi\| = \|z\|$. ☐

For one real profile ℓ, with $\ell^0 = \ell - \bar{\ell}\mathbf{1} = u^- + u^+$, the corresponding germs are

$$\mathcal{A}_{\mathrm{slot}}(\ell) = \sum_t J(e^{\ell_t - \bar{\ell}}) = \frac{1}{2}\|u^-\|^2 + \frac{1}{2}\|u^+\|^2 + O(\|\ell^0\|^4), \tag{88}$$

$$\mathcal{A}_{\mathrm{rec}}(\ell) = \sum_t J(e^{\ell_t - \ell_{t+4}}) = 2\|u^-\|^2 + O(\|u^-\|^4). \tag{89}$$

Thus the Hilbert metric is the tangent metric of the reciprocal cost. The finite nonlinear cost remains $\cosh u - 1$; the linear spectral carrier, unitarity metric, and Born weights belong to the exact Hessian theory at balance, not to a claim that the nonlinear cost itself equals a quadratic norm away from balance.

XV. SCOPE AND NON-CLAIMS

The result is intentionally dependency-explicit and should be read with the following limits.

(i) The scalar reciprocal cost does not by itself derive the eight-slot clock, the generator ledger, the symplectic completion, or the measurement interface. Those are explicit entries of the named package $\mathsf{P}_{\mathrm{bridge}}$ in Definition II.1.

(ii) The local cube and the cyclic clock are different structures. The Fourier sectors are cyclic sectors of the chosen C_8-clock, not Walsh sectors of the primitive cube involutions.

(iii) The full complex carrier is the duplex cotangent/Hessian carrier. A single positive profile gives only a real Fourier form; the arbitrary complex state requires the positive generator ledger as well.

(iv) The carrier supports an eight-phase spectral clock, but a selected vector records an eight-phase vector shadow only when its support contains an order-eight mode, and a selected ray records an exact eight-clock only when its support has mixed cyclic parity. Full finite-ledger faithfulness is not automatic and is exactly constrained by the finite functional-graph audit.

(v) The off-shell closed-tick principle is the main explicit closure assumption. The minimal mathematical premise is zero-fiber preservation plus symplectic base-shift coverage; exact reciprocal-cost preservation is the operational sufficient condition that pins the zero fiber. Without this rigidity layer, symplectic shears can agree on selected ledger states while differing away from them.

(vi) Born weights are balanced Hessian allocations or tangent-ray limits. The exact finite nonlinear reciprocal content remains $\cosh u - 1$, and Section XI computes the first finite-amplitude correction for the Gray state.

(vii) The minimum-winding Hamiltonian is a branch interpolation of the discrete tick. It is not a prediction of physical sub-tick continuum dynamics without additional calibration.

These boundaries are what make the operator-bridge claim mathematically checkable.

a. Programmatic connection. The present manuscript supplies a finite operator-bridge layer, not a particle-mass calculation. The harmonic-cascade mass proposal of Washburn and Allahyarov uses a separate recognition-scale lattice and should be evaluated on its own assumptions [30]. The role of the present paper is structural: it states which finite ledger primitives are needed before a cyclic recognition update has a balanced operator carrier. The finite-amplitude correction of Section XI is not used in that on-shell mass-lattice calculation; it is a separate measurement-interface target.

XVI. CONCLUSION

The paper's main claim is dependency-explicit. Positive coherent comparison forces

$$J(x) = \frac{1}{2}(x + x^{-1}) - 1.$$

The operator bridge is not claimed to follow from this scalar theorem alone; Remark II.11 makes the no-free-bridge audit explicit. It follows from the named primitive package $\mathsf{P}_{\mathrm{bridge}}$ in Definition II.1 and Table I. The structural inputs are: elementary three-involution local records, a separate cyclic Gray clock realization, slot-additive cost aggregation, internal covariance, a positive generator ledger, generator-reversal cotangent closure, off-shell exact-cost preservation, and covariant ledger-moment state selection. The local cube involutions and the cyclic C_8-clock are distinct structures; the Fourier split is cyclic parity, not primitive reciprocal-bit parity. Under those hypotheses the completed log coordinate $z = q + ip$ obeys

$$z(\widehat{\mathcal{R}}_L s) = P_{\mathbb{R}} z(s),$$

and the character transform of the chosen clock gives the exact vector bridge

$$\Psi(\widehat{\mathcal{R}}_L s) = P\Psi(s), \qquad \Psi = \mathcal{F}z \in \mathcal{W} = e_0^{\perp}.$$

The detailed measurement, Hamiltonian, and open-channel constructions are interface appendices rather than additional structural premises. The exact nonlinear content functional remains $\cosh u - 1$, and the leading nonquadratic term now has an explicit Sorkin finite-difference coefficient in (82). The complex norm, unitary metric, Hamiltonian interpolation, and Born weights are balanced-interface or tangent-ray structures, not replacements for the nonlinear finite-amplitude ledger cost; the Gray correction table in Section XI records the first term lost when one passes directly to the Hessian limit.

Four structural ingredients pin the bridge. Generator reversal removes vertical magnetic phase. The off-shell smooth pure-lift theorem removes nonlinear symplectic shears. The ledger-moment selection theorem shows how covariant finite-ledger observables select transported bridge states. The exact Gray example computes mixed-cyclic-parity Fourier support in closed form, without relying on decimals.

The cyclic-odd sector supplies vector period eight, while observable ray-level period eight requires mixed cyclic-odd/cyclic-even non-DC support or a protected even reference. Full-content and reference-conditioned measurement are different operational interfaces. Finally, the finite graph audit makes the representational limit explicit: a bridge may collapse longer cycles and transients into the same selected period-eight spectral orbit, and it is faithful to the full finite ledger exactly for exact eight-cycle components with separated spectral orbits.

Appendix A: Compressed interface bookkeeping: balanced Hessian allocation

This appendix records optional measurement-interface algebra after the structural bridge. Nothing here is used to prove the reciprocal cost, the cotangent carrier, the pure lift, or the DFT bridge. The fixed-frame formulas below require only an orthogonal decomposition of a chosen complex carrier; the all-projector uniqueness statement uses the usual Gleason hypothesis in complex dimension at least three.

For a complex measurement carrier $\mathcal{K} \subseteq \mathcal{W}$, write the balanced Hessian content as

$$\mathsf{C}_2(\psi) = \|\psi\|^2.$$

Near balance this is twice the quadratic term of the exact duplex cost $\mathcal{A}_{\mathrm{dup}}$, since $\mathcal{A}_{\mathrm{dup}}(q,p) = \frac{1}{2}\|\psi\|^2 + O(4)$. The factor of two cancels in normalized allocations.

Lemma A.1 (Balanced-interface representation lemma). *Suppose a balanced recognition refinement is represented on $\mathcal{K}$ by maps C_j satisfying Hessian-superposition compatibility, immediate repeatability, mutual exclusivity, and exhaustive completeness. In the Hessian carrier this means*

$$C_j \text{ is complex-linear}, \qquad C_j^2 = C_j, \qquad C_iC_j = 0 \ (i \neq j), \qquad \sum_j C_j = I.$$

This lemma merely translates the balanced-interface assumptions into channel algebra; it is not a derivation of measurement theory from scalar reciprocity.

Theorem A.2 (Orthogonal projectors from additive Hessian channels). *Let $C_1, \ldots, C_n$ satisfy the algebra in Lemma A.1. If they split Hessian content for every $\psi \in \mathcal{K}$,*

$$\|\psi\|^2 = \sum_j \|C_j\psi\|^2,$$

then the ranges of the C_j are mutually orthogonal and each C_j is the orthogonal projector onto its range.

Proof. For $x \in \operatorname{ran} C_i$ and $y \in \operatorname{ran} C_j$, $i \neq j$, apply the splitting identity to $x+y$ and use $C_i(x+y) = x$, $C_j(x+y) = y$. The polarization identity gives $\langle x, y\rangle = 0$. Since C_j is an idempotent with range orthogonal to the sum of the other ranges and $\sum_j C_j = I$, it is the orthogonal projector onto its range. □

Theorem A.3 (Tangent Hessian allocation)**.** *Let* $\{\Pi_j\}$ *be an orthogonal decomposition of* $\mathcal{K}$ *and* $\psi \neq 0$*. For* $\epsilon > 0$*, write* $\mathcal{F}^{-1}\phi = q_\phi + ip_\phi$ *and*

$$W_j(\epsilon, \psi) = \frac{\mathcal{A}_{\mathrm{dup}}(\epsilon\Pi_j\psi)}{\sum_m \mathcal{A}_{\mathrm{dup}}(\epsilon\Pi_m\psi)}.$$

Then

$$\lim_{\epsilon\to 0} W_j(\epsilon, \psi) = \frac{\|\Pi_j\psi\|^2}{\|\psi\|^2}.$$

Proof. The germ $\mathcal{A}_{\mathrm{dup}}(\epsilon\phi) = \epsilon^2\|\phi\|^2/2 + O(\epsilon^4)$, together with orthogonality, gives the result after cancelling the common quadratic factor. ☐

a. Finite-amplitude note. The tangent allocation theorem is only the $\epsilon \to 0$ limit. The finite-amplitude reciprocal-cost correction and its exact Gray-ledger table are given in Section XI.

Theorem A.4 (Conditional Hessian allocation uniqueness)**.** *Let* $\dim_{\mathbb{C}} \mathcal{K} \geq 3$*. Suppose an unnormalized score* $S_\psi(\Pi) \geq 0$ *on all orthogonal projectors is orthogonally additive, homogeneous as* $S_{\lambda\psi} = |\lambda|^2 S_\psi$*, normalized by* $S_\psi(I) = \|\psi\|^2$*, and excludes null alternatives* $\Pi\psi = 0 \Rightarrow S_\psi(\Pi) = 0$*. Then*

$$S_\psi(\Pi) = \|\Pi\psi\|^2.$$

Proof. For $\|\psi\| = 1$, Gleason's theorem gives $S_\psi(\Pi) = \mathrm{Tr}(A_\psi\Pi)$ for a positive trace-one A_ψ. Null exclusion gives zero weight to $I - |\psi\rangle\langle\psi|$, so $A_\psi = |\psi\rangle\langle\psi|$. Rescale for arbitrary ψ. ☐

Theorem A.5 (Normalized recognition weights)**.** *For an orthogonal frame* $\{\Pi_j\}$*, the normalized Hessian weights are*

$$w_j(\psi) = \frac{\|\Pi_j\psi\|^2}{\|\psi\|^2}.$$

For fixed frames this follows directly from Theorem A.3; the dimension condition is needed only for all-projector noncontextual uniqueness.

Theorem A.6 (Standard Gleason uniqueness after the measurement interface)**.** *On a complex Hilbert space* $\mathcal{K}$ *with* $\dim_{\mathbb{C}} \mathcal{K} \geq 3$*, any noncontextual frame-additive probability rule on all rank-one projectors, calibrated by unit weight on* $|\widehat{\psi}\rangle\langle\widehat{\psi}|$*, agrees with* $|\langle\phi, \widehat{\psi}\rangle|^2$*.*

Theorem A.6 is the classical Gleason theorem; we invoke it as a standard external result, not as a consequence of the reciprocal-cost or operator-bridge content of this manuscript. The fixed-frame allocation in Theorem A.3 does not require it.

Appendix B: Logarithmic interpolation and open-channel bookkeeping

The discrete tick P admits many continuous interpolations. A logarithmic generator is therefore a branch convention, not a prediction of physical sub-tick dynamics. On a carrier spanned by Fourier modes e_k, choose angles $\theta_k \in (-\pi, \pi]$ with $e^{i\theta_k} = \zeta^k$. After angular calibration, the minimum-winding diagonal representative is

$$H_{\min} = -\frac{\hbar}{\tau_0}\sum_k \theta_k |e_k\rangle\langle e_k|, \qquad e^{-i\tau_0 H_{\min}/\hbar} = P.$$

Other branches add $2\pi n_k\hbar/\tau_0$ to each diagonal quasienergy. On $\mathcal{W}$, the mode e_4 has eigenvalue -1, so this is a calibrated diagonal logarithm rather than an unrestricted principal-log claim.

Projection-like commit operations are not closed unitary ticks. For a sector $\mathcal{S} \subseteq \mathcal{W}$ with projector Π, the unconditional open channel with Kraus operators

$$K_0 = \Pi + \lambda(I - \Pi), \qquad K_1 = \sqrt{1-\lambda^2}(I - \Pi), \qquad 0 \leq \lambda < 1,$$

preserves trace and damps coherence but does not reduce unconditional off-sector population. The postselected branch $K_0\rho K_0^*/\,\mathrm{Tr}(K_0\rho K_0^*)$, when defined, has conditional off-sector defect

$$\frac{\lambda^2 D(\rho)}{1 - (1-\lambda^2)D(\rho)}, \qquad D(\rho) = \mathrm{Tr}((I - \Pi)\rho).$$

For $0 < D(\rho) < 1$ this strictly decreases when the branch probability is nonzero. If $D(\rho) = 1$ and $\lambda > 0$, the normalized defect remains one; if $D(\rho) = 1$ and $\lambda = 0$, the branch probability is zero and no normalized postselected state exists. Thus strict descent is an open or postselected interface effect, not part of the closed operator bridge.

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