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Admissible Reciprocally Symmetric Costs: Combiner Existence and Classification

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Abstract

We classify the continuous reciprocally symmetric cost functions $J : (0,\infty) \to \mathbb{R}$ with $J(1) = 0$ and strictly convex log-substitution $G(t) := J(e^t)$ (*admissible costs*) for which the symmetric compound $J(xy) + J(x/y)$ depends on (x, y) only through $(J(x), J(y))$. We first prove that this dependence is automatic: for every admissible J, there exists a unique continuous *combiner* (the auxiliary function P that encodes the compound) $P : [0,\infty)^2 \to \mathbb{R}$ with $J(xy) + J(x/y) = P(J(x), J(y))$ for all $x, y > 0$ (Theorem 1); P is symmetric, non-negative, satisfies $P(u, 0) = 2u$, and inherits monotonicity and coercivity from admissibility. When P is required to be a polynomial, a growth rate comparison between two recursions for G forces $\deg P \leq 2$ (Theorem 4), so $P(u, v) = cuv + 2u + 2v$ with $c \geq 0$, and the corresponding admissible costs are exhausted by two explicit families (Theorem 8)—the hyperbolic family $J(x) = c^{-1}(x^\lambda + x^{-\lambda}) - 2c^{-1}$ ($c, \lambda > 0$) and the degenerate quadratic family $J(x) = a(\ln x)^2$ ($a > 0$)—with the latter arising as the Inönü–Wigner contraction $\lambda \to 0^+$, $\lambda^2/c \to a$ of the former (Theorem 9). Two regularity extensions are obtained: a Lebesgue-measurable cost satisfying explicit regularity hypotheses admits a continuous representative (Theorem 5), and in the entire finite-order regime, the diagonal combiner $Q(u) := P(u, u)$, when polynomial of degree d', obeys the sharp bound $d' \geq 2^\rho$ (Theorem 6), attained with equality in both classified families. The normalisations $P(1, 1) = 6$ and $G''(0) = 1$ single out the canonical representative $J_{\text{cost}}(x) = \frac{1}{2}(x + x^{-1}) - 1$.

Keywords: d'Alembert functional equation; reciprocally symmetric cost; admissible cost; polynomial combiner; zonal spherical function; Kannappan–Aczél classification

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1. Introduction

1.1. The Combiner Equation and the Main Results

For a reciprocally symmetric cost $J : \mathbb{R}_{>0} \to \mathbb{R}$ with $J(x) = J(x^{-1})$ and $J(1) = 0$, the symmetric compound $J(xy) + J(x/y)$ vanishes at $x = y = 1$ and, when J is admissible, is non-negative on $\mathbb{R}^2_{>0}$. We ask whether this compound is determined by the values of J alone: does there exist a continuous $P : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ with

$$J(xy) + J(x/y) = P\big(J(x), J(y)\big) \qquad \text{for all } x, y > 0? \tag{1}$$

We prove that such a P exists for every pre-admissible J, is unique, and inherits symmetry, the boundary value $P(u,0) = 2u$, and non-negativity from J; under the stronger admissibility hypothesis, it is monotone in each variable, satisfies $P(u,v) \geq 2\max(u,v)$, and is coercive (Theorem 1 and Section 3). When P is in addition polynomial, $\deg P \leq 2$ (Theorem 4), hence $P(u,v) = cuv + 2u + 2v$ with $c \geq 0$, and J belongs to exactly two explicit families—a hyperbolic family for $c > 0$ and a degenerate quadratic family for $c = 0$—with the latter realised as the contraction $\lambda \to 0^+$, $\lambda^2/c \to a$ of the former (Theorems 8 and 9). In the hyperbolic case the substitution $H := \frac{c}{2}G + 1$ realises the classification as the d'Alembert equation $H(s+t) + H(s-t) = 2H(s)H(t)$ and identifies the costs with positive zonal spherical functions of the hyperbolic line H^1; Section 8 records the harmonic-analytic interpretation and a closed form for the n-variable compound.

The remainder of the introduction places these results in the literature (Section 1.2), states the contributions relative to prior work, fixes notation, and outlines the paper.

1.2. Related Work and Context

We position the paper against four pieces of literature that meet at Equation (1); specific results enter the proofs only where cited.

(i) d'Alembert–Kannappan classification.

The equation $H(s+t) + H(s-t) = 2H(s)H(t)$ with H continuous, even and $H(0) = 1$ admits exactly the solutions $H \in \{1, \cos(\lambda\cdot), \cosh(\lambda\cdot)\}$ ([1] §13, [2] §3.1), generalised to groups and semigroups by Kannappan and Stetkær [3–8]. We use this classification in Theorem 7 once the combiner is shown to be bilinear; the prior question of when a combiner exists at all is the question addressed here.

(ii) Measurable-to-continuous regularity.

The Steinhaus theorem [9] and Koenigs linearisation [10–12] together yield a "measurable $\Rightarrow$ continuous" principle for d'Alembert-type equations ([1] Ch. 13, [13–15]). Theorem 5 (and the polynomial-RHS Lemma A1) extends this principle from the bilinear right-hand side to the polynomial right-hand side that arises here; the polynomial case is also covered by the general two-variable framework of Járai ([14] Cor. 8.7) on continuity of measurable solutions of non-composite functional equations.

(iii) Spherical functions and Inönü–Wigner contraction.

On rank-one real symmetric pairs, the d'Alembert equation is the defining equation of positive zonal spherical functions [16–24]; the Inönü–Wigner contraction [25] of the hyperbolic pair onto the Euclidean one explains why the two classified families form a single one-parameter $c \in [0, \infty)$ with boundary $c = 0$. Section 8 records the precise identification.

(iv) Entire function rigidity and prior works.

The diagonal recursion $G(2t) = Q(G(t))$ is a Schröder–Böttcher equation in complex dynamics [12]; classical entire estimates [26–28] give the degree lower bound used in Theorem 6. The precursors [29,30] isolate $J_{\text{cost}}(x) = \frac{1}{2}(x + x^{-1}) - 1$ under a polynomial-combiner postulate plus a non-cancellation hypothesis; the multivariate extension is in [31]. Reciprocally symmetric costs also appear as the conjugation-symmetric subclass of Csiszár f-divergences [32–37] and as quasi-arithmetic-mean generators in the Kolmogorov–Nagumo–de Finetti characterisation ([38–40] Ch. 6).

1.3. Contributions Relative to Prior Work

The closest precursors [29,30] postulate the polynomial combiner equation $J(xy) + J(x/y) = P(J(x), J(y))$ with $P \in \mathbb{R}[u,v]$ as input, prove a degree bound under a non-cancellation hypothesis on P (which fails on the example $P(u,v) = 2u + 2v + uv(u-v)^2$, cf. ([30]) Rem. 4), and do not address regularity below continuity. Our four contributions are:

(C1) Existence of P from J alone (Theorem 1). No polynomial postulate.

(C2) Unconditional C^0 degree bound (Theorem 4). The non-cancellation hypothesis of [30] is removed; the proof handles diagonal cancellation (cf. Remark 8 and Section 4.3).

(C3) Measurable bootstrap (Theorem 5). Lebesgue measurability together with explicit regularity hypotheses (MC1)–(MC5) implies continuity, hence the classification.

(C4) Analytic rigidity (Theorem 6). In the entire finite-order regime, the diagonal combiner Q, when polynomial of degree d', satisfies $d' \geq 2^\rho$ whenever it is polynomial; the bound is attained in both classified families.

1.4. Notation

We write $\mathbb{R}_{>0} := (0,\infty)$ and $\mathbb{R}_{\geq 0} := [0,\infty)$. $J : \mathbb{R}_{>0} \to \mathbb{R}$ is a cost (admissible or pre-admissible, according to context; Definitions 1 and 2) and $G(t) := J(e^t)$ is its log-substitution; reciprocal symmetry $J(x) = J(x^{-1})$ becomes evenness $G(t) = G(-t)$. The combiner $P : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ of Theorem 1 has diagonal $Q(u) := P(u,u)$; the boundary derivative is $\Psi(u) := P_v(u,0)$ and we set $A_1 := \frac{1}{2}G''(0)$. For $P \in \mathbb{R}[u,v]$, $\deg P$ is the total degree. The bilinear parameter $c \geq 0$ enters via $P(u,v) = cuv + 2u + 2v$; for $c > 0$, the substitution $H := \frac{c}{2}G + 1$ converts the bilinear law into d'Alembert's equation. The canonical cost is $J_{\text{cost}}(x) := \frac{1}{2}(x + x^{-1}) - 1$. The maximum modulus $M_f(r) := \max_{|z|=r} |f(z)|$ and the order $\rho := \limsup_{r\to\infty} \log\log M_f(r)/\log r$ are used in Theorem 6 with the convention that polynomials have $\rho = 0$ and cosh, sin, exp have $\rho = 1$. Symbols introduced later in the body ($\alpha_k^{(i)}$, Π, L, T_η, δ) are defined at first use. The principal precursors are [29,30].

Since several of these symbols recur far from their point of definition, the following table collects the recurring notation, its meaning, and the place where each symbol is introduced.

Symbol	**Meaning**	**Introduced**
J, $G = J \circ \exp$	Cost and its log-substitution	Definition 1
P	Combiner of J: $J(xy) + J(x/y) = P(J(x), J(y))$	Theorem 1
Φ	Inverse of $J\|_{[1,\infty)}$, $\Phi : \mathbb{R}_{\geq 0} \to [1,\infty)$	Theorem 1
$Q(u) = P(u,u)$	Diagonal combiner, $G(2t) = Q(G(t))$	Section 5
$\Psi(u) = P_v(u,0)$	Boundary derivative of P	Lemma 1
A_k	Taylor coefficients of G: $G(t) = \sum_{k\geq 1} A_k t^{2k}$	Section 4
$A_1 = \frac{1}{2}G''(0)$	Curvature scale (standing assumption $A_1 > 0$)	Section 2
$\alpha_k^{(i)}$	Coefficients of the Cauchy power $G(s)^i$	Equation (8)
p_{ij}	Taylor coefficients of P at $(0,0)$	Section 4
$c \geq 0$	Bilinear parameter in $P(u,v) = cuv + 2u + 2v$	Corollary 3
$H = \frac{c}{2}G + 1$	D'Alembert substitution ($c > 0$)	Proposition 6
$d' = \deg Q$, δ	Diagonal degree; max. mixed u-degree of P	Section 5
$\Pi(u) = \int_0^u \Psi$	Energy integral of the boundary ODE	Proposition 5
L	Analytic inverse branch of Q at 0	Theorem 5
T_η	Density-1 set on which the halving identity holds	Theorem 5
$M_f(r)$, ρ	Maximum modulus; order of an entire function	Theorem 6
J_{cost}	Canonical cost $\frac{1}{2}(x + x^{-1}) - 1$	Section 9

1.5. Outline

The following is an outline of the paper:

- Section 2 fixes the admissibility and pre-admissibility hypotheses.
- Section 3 proves the combiner existence theorem (Theorem 1) and the structural properties of P (Section 3).
- Section 4 sets up the boundary-derivative ODE, the Taylor recursion that determines P from its boundary derivative Ψ, and the C^2 blow-up obstruction excluding superlinear Ψ (Theorem 3).
- Section 5 proves the unconditional C^0 degree bound $\deg P \leq 2$ (Theorem 4).
- Section 6 extends the classification to the Lebesgue-measurable level (Theorem 5) and proves entire-finite-order analytic rigidity (Theorem 6).
- Section 7 assembles these into the full classification (Theorem 8) and exhibits the one-parameter family $c \in [0, \infty)$ with Inönü–Wigner boundary $c = 0$ (Theorem 9).
- Section 8 records the spherical function interpretation and the closed form $S_n(x_1, \ldots, x_n) = \frac{2^n}{c}\left(\prod_k h(x_k) - 1\right)$ for the n-variable compound (Proposition 10).
- Section 9 singles out the canonical cost $J_{\text{cost}}(x) = \frac{1}{2}(x + x^{-1}) - 1$ and verifies that every hypothesis in the paper is realised by it.
- Appendix A collects the two measure-theoretic tools quoted in Section 6 (the contraction principle at the attracting fixed point and the polynomial measurable-to-continuous regularity lemma, Lemma A1), with self-contained proofs.

The main pipeline is Theorem 1 $\to$ Theorem 4 $\to$ Theorem 8; the measurable and analytic results of Section 6 are mutually independent and feed only into the classification.

2. Admissible Costs

Definitions

Two hypotheses on J enter the paper: admissibility (Definition 1) and the strictly weaker pre-admissibility (Definition 2). The combiner existence theorem of Section 3 needs only pre-admissibility; all later results assume admissibility. Concretely we require J to be invariant under $x \leftrightarrow x^{-1}$ (so that the compound $J(xy) + J(x/y)$ is well-defined as a function of J), to vanish at $x = 1$ (the trivial cost), and to be strictly convex in the multiplicative coordinate $t = \ln x$—the standard convexity hypothesis for cost generators [41].

Definition 1 (Admissible cost)**.** *A function $J : \mathbb{R}_{>0} \to \mathbb{R}$ is an* admissible cost *if it satisfies the following three conditions.*

1. *Reciprocal symmetry: $J(x) = J(x^{-1})$ for all $x > 0$.*
2. *Unit normalization: $J(1) = 0$.*
3. *Strict log-convexity: The log-substitution $G(t) := J(e^t)$ is strictly convex on $\mathbb{R}$ (equivalently, J is strictly convex as a function of the logarithmic coordinate $t = \ln x$ on $\mathbb{R}_{>0}$).*

Strict convexity makes G continuous on $\mathbb{R}$ ([41] Thm. 1.3.3); reciprocal symmetry plus $J(1) = 0$ make G even with $G(0) = 0$. The four less elementary consequences—strict positivity off the unit, strict monotonicity of G on $[0, \infty)$, coercivity $G(t) \to +\infty$, and surjectivity $J(\mathbb{R}_{>0}) = [0, \infty)$—are recorded in Remark 1 below. In particular every admissible cost is pre-admissible (Definition 2).

Remark 1 (Further structural properties)**.** *For an admissible cost J with $G := J \circ \exp$:*

(a) *Strict positivity off the unit: $G(0) = G(\frac{t-t}{2}) < \frac{G(t)+G(-t)}{2} = G(t)$ for $t \neq 0$ by strict convexity and evenness; equivalently $J(x) > 0$ for $x \neq 1$.*

(b) *Strict monotonicity of G on $[0, \infty)$: For $0 < s < t$, strict convexity and $G(0) = 0$ give $G(s) < \frac{s}{t}G(t) < G(t)$ (the last inequality uses* (a)*).*

(c) *Coercivity $G(t) \to +\infty$: For any $t_0 > 0$, convexity gives $G(nt_0) \geq nG(t_0) \to +\infty$.*

(d) *Surjectivity* $J(\mathbb{R}_{>0}) = [0, \infty)$*: Combine continuity,* $G(0) = 0$*, and* (c) *with the intermediate value theorem.*

Definition 2 (Pre-admissible cost)**.** *A function* $J : \mathbb{R}_{>0} \to \mathbb{R}$ *is* pre-admissible *if it is continuous, satisfies* $J(x) = J(x^{-1})$ *and* $J(1) = 0$*, and its log-substitution* $G(t) := J(e^t)$ *is strictly increasing on* $[0, \infty)$ *with* $G(t) \to +\infty$ *as* $t \to \infty$*.*

Every admissible cost is pre-admissible, but the converse fails: $J(x) = |\ln x|$ is pre-admissible ($G(t) = |t|$ is strictly increasing and coercive), yet not strictly log-convex. For this J, the combiner is $P(u, v) = 2\max(u, v)$ (see Section 3), so the strict lower bound $P(u, v) > 2\max(u, v)$ for $\min(u, v) > 0$ from Section 3(5) fails; this is the strongest structural difference between the two regimes.

Standing assumption from Section 4 onward.

We use $A_1 := \frac{1}{2}G''(0) > 0$ as an explicit assumption beyond admissibility. It does not follow from strict log-convexity ($G(t) = t^4$ is strictly convex with $A_1 = 0$); both classified families satisfy $A_1 > 0$ (hyperbolic: $A_1 = \lambda^2/c$; degenerate quadratic: $A_1 = a$).

3. Combiner Existence and Structure

This section answers the existence question for the combiner. The strategy is elementary: pre-admissibility makes the level sets of J two-point sets $\{x, x^{-1}\}$, so the compound $J(xy) + J(x/y)$ is constant on cost fibres (Proposition 1); inverting $J|_{[1,\infty)}$ then constructs the combiner P and proves it is unique (Theorem 1). The remainder of the section records the structural properties of P that later sections use: symmetry, boundary values, and positivity under pre-admissibility (Proposition 2) and monotonicity, the lower bound $P(u, v) \geq 2\max(u, v)$, and coercivity under full admissibility (Proposition 3). No polynomiality is assumed anywhere in this section.

Under the standing assumption that P is polynomial, several algebraic consequences—equivalence of symmetry of P and reciprocal symmetry of J, the boundary identities $P(u, 0) = 2u$ and $P(0, v) = 2v$, and the polynomial factorisation $P(u, v) = 2u + 2v + uv\,R(u, v)$ for some $R \in \mathbb{R}[u, v]$—are proved in ([30] Lem. 1–4, Cor. 1–2).

Proposition 1 (Single-valuedness of the compound on cost fibres)**.** *Let* J *be pre-admissible, and let* $x_1, x_2, y_1, y_2 > 0$ *satisfy* $J(x_1) = J(x_2)$ *and* $J(y_1) = J(y_2)$*. Then*

$$J(x_1y_1) + J(x_1/y_1) = J(x_2y_2) + J(x_2/y_2).$$

Proof. Since G is strictly increasing on $[0, \infty)$ and even, the level set $J^{-1}(u)$ for $u \geq 0$ consists of exactly two points: $\{e^s, e^{-s}\}$ for the unique $s \geq 0$ with $G(s) = u$. Therefore $J(x_1) = J(x_2)$ implies $x_2 \in \{x_1, x_1^{-1}\}$, and similarly, $y_2 \in \{y_1, y_1^{-1}\}$. Examining the four combinations $(x_2, y_2) \in \{x_1, x_1^{-1}\} \times \{y_1, y_1^{-1}\}$, the pairs $(x_2y_2,\ x_2/y_2)$ are

$$(x_1y_1,\ x_1/y_1), \quad (x_1/y_1,\ x_1y_1), \quad (x_1^{-1}y_1^{-1},\ x_1^{-1}y_1), \quad (x_1^{-1}y_1,\ x_1^{-1}y_1^{-1}).$$

In each case, reciprocal symmetry of J and commutativity of addition give $J(x_2y_2) + J(x_2/y_2) = J(x_1y_1) + J(x_1/y_1)$. □

Theorem 1 (Combiner existence)**.** *Let* $J : \mathbb{R}_{>0} \to \mathbb{R}$ *be pre-admissible. There exists a unique continuous function* $P : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ *such that*

$$J(xy) + J(x/y) = P\big(J(x), J(y)\big) \qquad \textit{for all } x, y > 0. \tag{2}$$

We call this P the combiner of J; the name is used throughout the paper for the unique function in (2). The continuous strictly increasing inverse $\Phi : \mathbb{R}_{\geq 0} \to [1, \infty)$ of $J|_{[1,\infty)}$ constructed in the proof is used in Lemma 1 and elsewhere; we keep this notation throughout.

Proof. Since $J : [1, \infty) \to [0, \infty)$ is continuous, strictly increasing, and surjective (directly from pre-admissibility: continuity, $J(1) = 0$, G strictly increasing on $[0, \infty)$ with $G(t) \to \infty$, plus the intermediate value theorem), it has, by the standard inverse-monotone theorem, a continuous strictly increasing inverse $\Phi : \mathbb{R}_{\geq 0} \to [1, \infty)$. Define

$$P(u, v) := J\big(\Phi(u)\Phi(v)\big) + J\big(\Phi(u)/\Phi(v)\big), \qquad u, v \geq 0.$$

This is continuous on $\mathbb{R}^2_{\geq 0}$ as a composition of continuous functions. For any $x, y > 0$, set $u := J(x)$ and $v := J(y)$; then, $\Phi(u) \in \{x, x^{-1}\}$ and $\Phi(v) \in \{y, y^{-1}\}$, and Proposition 1 yields $P(J(x), J(y)) = J(xy) + J(x/y)$, so (2) holds.

For uniqueness, since $J(\mathbb{R}_{>0}) = [0, \infty)$, the set $\{(J(x), J(y)) : x, y > 0\} = \mathbb{R}^2_{\geq 0}$ is the entire domain of P. Any continuous function satisfying (2) therefore agrees with P on the entire domain $\mathbb{R}^2_{\geq 0}$ (since $(u, v) = (J(x), J(y))$ ranges over all of $\mathbb{R}^2_{\geq 0}$), hence equals P. □

Remark 2 (Comparison with [30])**.** *Theorem 1 differs from the closest results of [30] on the existence question for P: in ([30] Def. 1), the polynomial composition law* $F(xy) + F(x/y) = P(F(x), F(y))$ *is taken as a defining hypothesis on F, and the uniqueness and structural rigidity of P are then derived ([30] Cor. 5, Thm. 1, Thm. 3). Here, P is* constructed *from pre-admissibility of J alone via the inverse* $\Phi : \mathbb{R}_{\geq 0} \to [1, \infty)$ *of* $J|_{[1,\infty)}$*, with no polynomial postulate; polynomiality is recovered later, either conditionally in Theorem 4 or under analytic rigidity in Theorem 6.*

Proposition 2 (Combiner under pre-admissibility)**.** *Let J be pre-admissible with combiner P. Then:*

1. *Symmetry:* $P(u, v) = P(v, u)$.
2. *Boundary values:* $P(u, 0) = 2u$ *and* $P(0, v) = 2v$.
3. *Positivity:* $P(u, v) \geq 0$*, with equality only at* $(u, v) = (0, 0)$.

Proposition 3 (Combiner under admissibility)**.** *If in addition J is admissible (G strictly convex), then:*

4. *Monotonicity:* $P(\cdot, v)$ *is non-decreasing on* $\mathbb{R}_{\geq 0}$ *for each fixed* $v \geq 0$*, and* $P(u, \cdot)$ *for each fixed* $u \geq 0$.
5. *Lower bound:* $P(u, v) \geq 2\max(u, v)$*, with equality only if* $\min(u, v) = 0$ *(strict whenever* $\min(u, v) > 0$*; see Remark 3). Under mere pre-admissibility even the non-strict bound can fail: e.g.,* $J(x) = |\ln x|^{1/2}$ *gives* $P(1, 1) = \sqrt{2} < 2$.
6. *Coercivity: For each* $M \geq 0$*, the sublevel set* $\{(u, v) \in \mathbb{R}^2_{\geq 0} : P(u, v) \leq M\}$ *is compact.*

Proof of Proposition 2 and 3. (1)*:* The substitution $x \leftrightarrow y$ sends $(xy, x/y) \mapsto (yx, y/x)$, and $J(y/x) = J(x/y)$ by reciprocal symmetry, so the left side of (2) is symmetric in (x, y).

(2)*:* Setting $y = 1$ gives $J(y) = 0$ and $J(x \cdot 1) + J(x/1) = 2J(x)$.

(3)*:* $P(u, v) = J(xy) + J(x/y) \geq 0$ since G is even with $G(0) = 0$ and strictly increasing on $[0, \infty)$ by pre-admissibility; hence, $G \geq 0$ on $\mathbb{R}$ and $J \geq 0$ on $\mathbb{R}_{>0}$. Equality requires $J(xy) = J(x/y) = 0$; hence, $xy = x/y = 1$, giving $x = y = 1$ and $u = v = 0$.

(4) (J admissible). Using the notation of Theorem 1, write $X = \Phi(u)$ and $Y = \Phi(v)$ with $X, Y \geq 1$, so that $P(u, v) = G(\ln X + \ln Y) + G(\ln X - \ln Y)$. Fix $Y \geq 1$ and let $X_1 \leq X_2$; set $a := \ln X_1 \leq b := \ln X_2$ and $c := \ln Y \geq 0$. Consider the function

$$f(b) := G(b + c) + G(b - c), \qquad b \geq 0.$$

Since G is convex (by admissibility / strict log-convexity), its one-sided derivatives $G'_-(t)$ and $G'_+(t)$ exist for every $t \in \mathbb{R}$, satisfy $G'_-(t) \le G'_+(t)$, and are non-decreasing in t. Evenness of G implies $G'_+(-t) = -G'_-(t)$. Using the right derivative of f,

$$f'_+(b) = G'_+(b+c) + G'_+(b-c) \qquad (b \ge 0),$$

- For $b \ge c$: Both arguments $b \pm c$ are non-negative and $G'_+ \ge 0$ on $[0, \infty)$, so $f'_+(b) \ge 0$.
- For $0 \le b < c$: We have $b + c > c - b \ge 0$, and by convexity G'_+ is non-decreasing, so $G'_+(b+c) \ge G'_+(c-b)$. Since G is even, $G'_+(b-c) = G'_+(-(c-b)) = -G'_-(c-b)$, giving
$$f'_+(b) = G'_+(b+c) + G'_+(b-c) = G'_+(b+c) - G'_-(c-b) \ge 0.$$

Thus f is non-decreasing on $[0, \infty)$, i.e., $P(u,v) = f(\ln \Phi(u))$ is non-decreasing in u for each fixed v. Symmetry (1) gives the same in v.

(5) (J admissible). From (2) and (4) (admissible J), $P(u,v) \ge P(u,0) = 2u$ and $P(u,v) \ge P(0,v) = 2v$, so $P(u,v) \ge 2\max(u,v)$.

For the strictness statement, suppose $(u_0, v_0) \in \mathbb{R}^2_{\ge 0}$ has $u_0, v_0 > 0$ and $P(u_0, v_0) = 2\max(u_0, v_0)$. By symmetry (1) of P, we may assume without loss of generality that $\max(u_0, v_0) = v_0$, i.e., $v_0 \ge u_0 > 0$, so $P(u_0, v_0) = 2v_0$. Setting $c := \ln \Phi(v_0) > 0$ (positive since $v_0 > 0$ implies $\Phi(v_0) > 1$), the function $f(b) := G(b+c) + G(b-c)$ from (4) satisfies

$$f(0) = G(c) + G(-c) = 2G(c) = 2v_0 \qquad \text{(evenness of } G \text{ and } G(c) = v_0)$$

and $f(\ln \Phi(u_0)) = P(u_0, v_0) = 2v_0$. Since f is non-decreasing on $[0, \infty)$ by (4) and $\ln \Phi(u_0) > 0$ (because $u_0 > 0$), the equality $f(0) = f(\ln \Phi(u_0))$ forces f to be constant on $[0, \ln \Phi(u_0)]$; hence, $f'_+ \equiv 0$ on $[0, \ln \Phi(u_0))$.

Strict positivity of f'_+. We now show that strict log-convexity of G forces $f'_+(b) > 0$ for every $b \in (0, \ln \Phi(u_0))$, contradicting the previous paragraph. Recall the standard fact for a strictly convex function $G \colon \mathbb{R} \to \mathbb{R}$: for all real $s < t$,

$$G'_+(s) \;<\; G'_-(t), \tag{3}$$

since picking any $r \in (s,t)$ gives $G'_+(s) \le \frac{G(r)-G(s)}{r-s} < \frac{G(t)-G(r)}{t-r} \le G'_-(t)$ by strict convexity (the secant slopes are strictly increasing) and the definitions of the one-sided derivatives as the infimum (resp. supremum) over forward (resp. backward) secant slopes. Evenness of G together with convexity and $G(0) = 0$ imply that $G(t) \ge G(0) = 0$ for every t (because the chord joining $(-t, G(t))$ and $(t, G(t))$ has constant height $G(t)$ and dominates the graph at 0); consequently $G'_+(0) \ge 0$. Applying (3) with $s = 0 < t = b + c$ then yields

$$G'_+(b+c) \;\ge\; G'_-(b+c) \;>\; G'_+(0) \;\ge\; 0 \qquad \text{whenever } b + c > 0,$$

so in particular $G'_+(b+c) > 0$ for $b + c > 0$. We distinguish two cases.

Case $b \ge c$. Both $b + c$ and $b - c$ are non-negative. Since $b > 0$, also $b + c > 0$, hence $G'_+(b+c) > 0$ by the previous display, while $G'_+(b-c) \ge G'_+(0) \ge 0$. Thus $f'_+(b) = G'_+(b+c) + G'_+(b-c) > 0$.

Case $0 < b < c$. Then $0 \le c - b < b + c$. Applying (3) with $s = c - b$ and $t = b + c$ gives $G'_+(c-b) < G'_-(b+c)$, and therefore (using $G'_-(c-b) \le G'_+(c-b)$ and $G'_-(b+c) \le G'_+(b+c)$)

$$G'_-(c-b) \;\le\; G'_+(c-b) \;<\; G'_-(b+c) \;\le\; G'_+(b+c).$$

Combining this with the evenness identity $G'_+(b-c) = -G'_-(c-b)$ from the proof of (4) gives

$$f'_+(b) \;=\; G'_+(b+c) + G'_+(b-c) \;=\; G'_+(b+c) - G'_-(c-b) \;>\; 0.$$

In both cases, $f'_+(b) > 0$ for $0 < b < \ln \Phi(u_0)$, contradicting $f'_+ \equiv 0$ on $[0, \ln \Phi(u_0))$. Hence $P(u_0, v_0) > 2\max(u_0, v_0)$, so the bound is strict whenever $\min(u, v) > 0$.

(6) (J admissible). $\{P \le M\} \subseteq \{2\max(u, v) \le M\} = [0, M/2]^2$, which is compact. □

Remark 3 (Necessity of strict log-convexity and of polynomiality of P)**.** *Two structural points are illustrated by single examples; we collect them here.*

1. Strict monotonicity of P on $\mathbb{R}^2_{\ge 0} \setminus \{(0,0)\}$ requires strict log-convexity. *Under admissibility, strict convexity of G implies $G'_+(s) < G'_-(t)$ for all $s < t$ (cf. the proof of Proposition 3(5)), and combining this with the evenness identity $G'_+(b-c) = -G'_-(c-b)$ shows that the function $f(b) = G(b+c) + G(b-c)$ in Proposition 3(4) satisfies $f'_+(b) > 0$ for every $b > 0$ and $c \ge 0$, yielding strict monotonicity of P in each variable on the interior. Without strict log-convexity the conclusion fails: for the pre-admissible $J(x) = |\ln x|$ ($G(t) = |t|$), $J(xy) + J(x/y) = |\ln x + \ln y| + |\ln x - \ln y| = 2\max(J(x), J(y))$, so $P(u, v) = 2\max(u, v)$ identically and equality holds everywhere in Proposition 3(5).*
2. Polynomiality of P is a genuine restriction. *For $J(x) = (\ln x)^4$, $J(xy) + J(x/y) = 2(\ln x)^4 + 12(\ln x)^2(\ln y)^2 + 2(\ln y)^4$ gives $P(a, b) = 2a + 12\sqrt{ab} + 2b$, not a polynomial in (a, b). More generally, $J(x) = (\ln x)^{2m}$ with $m \ge 2$ gives P polynomial in $(a^{1/m}, b^{1/m})$ but not in (a, b). Polynomiality of P is thus a hypothesis, not a consequence of admissibility. This is the only obstruction from this example used later: the subsequent classification assumes polynomiality in the variables $(J(x), J(y)) = (a, b)$, not polynomiality after taking fractional powers.*

4. Smoothness Theory: Boundary Derivative, Taylor Recursion, and Blow-Up Obstruction

The main results of this section are: (i) a second-order ODE for G governed by the boundary derivative $\Psi(u) := P_v(u, 0)$ (Lemma 1); (ii) a triangular Taylor recursion determining the coefficients of P from those of Ψ and the single scale A_1 (Proposition 4); and (iii) a uniqueness-from-germ theorem (Theorem 2).

Before entering the computations, we describe the plan of the section in words. The guiding idea is that all information about the combiner P is concentrated on the boundary $\{v = 0\}$ of its domain: the first transverse derivative $\Psi = P_v(\cdot, 0)$ already determines G through an autonomous second-order ODE (Lemma 1), and in the real-analytic category, it determines the full Taylor series of P (Proposition 4 and Theorem 2). The section closes with the first rigidity payoff of this point of view: a C^2 blow-up obstruction showing that no admissible cost can have a superlinearly growing boundary derivative (Theorem 3). We emphasise the logical status of this section within the paper: the main classification pipeline (Theorem 1 → Theorem 4 → Theorem 8) does not pass through the results proved here; the present section provides an independent smoothness-based route to the bilinear form when Ψ is affine (Corollary 1) and supplies the obstructions quoted in the discussion of open problems.

Standing assumption (SA) for Section 4.

J is admissible (Definition 1), $J \in C^2(\mathbb{R}_{>0})$ (equivalently $G \in C^2(\mathbb{R})$), and $A_1 := \frac{1}{2}G''(0) > 0$. Both regularity conditions hold in the classified families of Theorem 8; $A_1 > 0$ does not follow from admissibility alone (see the preceding remark).

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4.1. The Boundary-Derivative ODE

Lemma 1 (Boundary-derivative ODE)**.** *Under (SA), the combiner P of Theorem 1 admits the first-order boundary expansion*

$$P(u,v) = 2u + \Psi(u)\, v + o(v), \qquad v \to 0^+, \tag{4}$$

uniformly on compact $u \in [0,U]$*, where* $\Psi = P_v(\cdot, 0)$ *is continuous on* $\mathbb{R}_{\geq 0}$*, satisfies* $\Psi(0) = 2$*, and has the closed form*

$$\Psi(u) = A_1^{-1}\, G''(\ln \Phi(u)) \tag{5}$$

with $\Phi : \mathbb{R}_{\geq 0} \to [1,\infty)$ *the inverse of* $J|_{[1,\infty)}$*. The log-substitution G then satisfies the autonomous ODE*

$$G''(s) = A_1\, \Psi(G(s)), \qquad G(0) = G'(0) = 0, \quad G''(0) = 2A_1. \tag{6}$$

Proof. By Theorem 1, $P(u,v) = J(\Phi(u)\Phi(v)) + J(\Phi(u)/\Phi(v))$. In log coordinates, set $a := \ln \Phi(u) \geq 0$ and $w := \ln \Phi(v) \geq 0$; then

$$P(u,v) = G(a+w) + G(a-w).$$

Since $G \in C^2(\mathbb{R})$, Taylor's theorem with uniform-on-compact remainder gives, for each compact $K \subseteq \mathbb{R}$,

$$G(a \pm w) = G(a) \pm G'(a)\, w + \tfrac{1}{2} G''(a)\, w^2 + \rho_\pm(a,w), \qquad \sup_{a \in K} |\rho_\pm(a,w)| = o(w^2) \text{ as } w \to 0.$$

Adding cancels the odd terms in w:

$$P(u,v) = 2G(a) + G''(a)\, w^2 + o(w^2), \qquad \text{uniformly in } a \in K.$$

The change of variables $v = G(w) = A_1 w^2 + o(w^2)$ on $w \in [0,\infty)$ gives $w^2 = v/A_1 + o(v)$ as $v \to 0^+$ (the residual is $o(w^2)/A_1$, which is $o(v)$ because $w^2 \sim v/A_1$). Substituting, using $2G(a) = 2u$ and the boundedness of G'' on the compact image $[0, \ln \Phi(U)]$,

$$P(u,v) = 2u + \frac{G''(a)}{A_1}\, v + o(v), \qquad \text{uniformly in } u \in [0,U].$$

This is (4) with Ψ given by (5). Continuity of Ψ on $\mathbb{R}_{\geq 0}$ follows from $G'' \in C^0(\mathbb{R})$ and $\ln \Phi \in C^0(\mathbb{R}_{\geq 0})$; the value at the origin is $\Psi(0) = G''(0)/A_1 = 2A_1/A_1 = 2$, so in particular $\Psi = P_v(\cdot, 0)$.

For (6), evenness and $J(1) = 0$ give $G(0) = 0$ and $G'(0) = 0$. Taylor-expanding $G(s \pm t)$ in t at $t = 0$:

$$G(s+t) + G(s-t) = 2G(s) + G''(s)\, t^2 + o(t^2), \qquad G(t) = A_1\, t^2 + o(t^2). \tag{7}$$

Using the factorisation equation and (4) to expand $P(G(s), G(t))$ in t:

$$P(G(s), G(t)) = 2G(s) + \Psi(G(s)) \cdot A_1\, t^2 + o(t^2).$$

Equating the t^2 coefficients gives (6). Evaluating at $s = 0$ and using $\Psi(0) = 2$ yields $G''(0) = 2A_1$. □

The ODE (6) expresses $G''(s)$ as a function of $G(s)$ alone, with coefficients Ψ and A_1. The next subsection shows that, in the real-analytic case, the full combiner is determined by Ψ and A_1.

Remark 4 (Closed form of Ψ and the affine criterion)**.** *The closed form* $\Psi(u) = G''(\ln \Phi(u))/A_1$ *in* (5) *converts each boundary-derivative statement into an explicit statement about* G*. In particular,* Ψ *is affine, i.e.,* $\Psi(u) = cu + 2$ *for some* $c \geq 0$*, if and only if*

$$G''(s) = A_1\big(c\,G(s) + 2\big) \qquad \text{for all } s \in \mathbb{R},$$

which is the linear second-order ODE solved in Corollary 1. When J *is real-analytic,* G *is even and real-analytic, so* G'' *is an even real-analytic function; combined with the analyticity of* $w^2 = (\ln \Phi(u))^2$ *in* u *at* $u = 0$ *(inversion of* $u = G(w) = A_1 w^2 + A_2 w^4 + \cdots$*), this gives* Ψ *real-analytic on* $\mathbb{R}_{\geq 0}$ *as a function of* u*—the square-root singularity of* Φ *at* $u = 0$ *enters only through* w^2*, never* w*, so it cancels.*

4.2. The Taylor Recursion

Let J be real-analytic and admissible. Then G is real-analytic and even, with expansion

$$G(t) = \sum_{k\geq 1} A_k\, t^{2k}, \qquad A_k := \frac{G^{(2k)}(0)}{(2k)!}, \quad A_1 > 0.$$

Assume in addition that P is real-analytic at $(0,0)$:

$$P(u,v) = \sum_{i,j\geq 0} p_{ij}\, u^i v^j, \qquad p_{ij} = p_{ji}.$$

The structural properties of P from Section 3 impose the boundary conditions $p_{00} = 0$, $p_{10} = p_{01} = 2$, and $p_{i0} = p_{0j} = 0$ for $i, j \geq 2$. Hence,

$$P(u,v) = 2u + 2v + \sum_{\substack{i,j\geq 1\\ i+j\geq 2}} p_{ij}\, u^i v^j, \qquad p_{ij} = p_{ji}.$$

For each integer $i \geq 1$, write $G(s)^i = \sum_{k\geq i} \alpha_k^{(i)} s^{2k}$ as the i-fold Cauchy power of the series for G. Explicitly, $\alpha_k^{(1)} = A_k$; $\alpha_i^{(i)} = A_1^i$; $\alpha_k^{(i)} = 0$ for $k < i$; and for $i \geq 2$,

$$\alpha_k^{(i)} = \sum_{\ell=1}^{k-i+1} A_\ell\, \alpha_{k-\ell}^{(i-1)} \qquad (k \geq i). \tag{8}$$

This is the coefficient-wise Cauchy product $G(s)^i = G(s)\,G(s)^{i-1}$: the lower limit $\ell \geq 1$ reflects $A_0 = 0$ (equivalently $G(0) = 0$), while the upper limit $\ell \leq k - i + 1$ is forced by $\alpha_{k-\ell}^{(i-1)} = 0$ whenever $k - \ell < i - 1$.

Proposition 4 (Taylor recursion)**.** *For real-analytic admissible* J *with real-analytic combiner* P*, the coefficients* $\{A_k\}_{k\geq 1}$ *and* $\{p_{ij}\}_{i,j\geq 1}$ *satisfy, for all* $m, n \geq 1$*,*

$$2A_{m+n}\binom{2(m+n)}{2n} = \sum_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} p_{ij}\, \alpha_m^{(i)}\, \alpha_n^{(j)}. \tag{9}$$

Solving for the top coefficient gives

$$p_{mn}\, A_1^{m+n} = 2A_{m+n}\binom{2(m+n)}{2n} - \sum_{\substack{1\leq i\leq m,\, 1\leq j\leq n\\ (i,j)\neq(m,n)}} p_{ij}\, \alpha_m^{(i)}\, \alpha_n^{(j)}. \tag{10}$$

Proof. Expand both sides of $G(s+t) + G(s-t) = P(G(s), G(t))$ as power series in (s,t) and extract the coefficient of $s^{2m}t^{2n}$.

Left side. Each term $G(s \pm t) = \sum_{k \geq 1} A_k (s \pm t)^{2k}$ contributes $A_{m+n}\binom{2(m+n)}{2n} s^{2m} t^{2n}$ to the $s^{2m}t^{2n}$ coefficient, with the same sign from both $G(s+t)$ and $G(s-t)$ (since both exponents $2m$ and $2n$ are even). Hence the left-side coefficient is $2A_{m+n}\binom{2(m+n)}{2n}$.

Right side. Expanding $P(G(s), G(t)) = 2G(s) + 2G(t) + \sum_{i,j \geq 1} p_{ij} G(s)^i G(t)^j$, the linear terms contribute only to coefficients of the form $s^{2m}t^0$ or $s^0 t^{2n}$ (both absent for $m, n \geq 1$). The $s^{2m}t^{2n}$ coefficient of $p_{ij}G(s)^i G(t)^j$ is $p_{ij}\alpha_m^{(i)}\alpha_n^{(j)}$, which is zero unless $i \leq m$ and $j \leq n$. Summing gives the right side of (9).

In (10), the term $(i,j) = (m,n)$ contributes $\alpha_m^{(m)}\alpha_n^{(n)} = A_1^{m+n} \neq 0$ (since $A_1 > 0$), so p_{mn} is determined by the p_{ij} with (i,j) strictly below (m,n) in the partial order on $\mathbb{Z}^2_{\geq 1}$; the recursion is triangular, with nonzero pivot A_1^{m+n} at every step. □

Theorem 2 (Combiner determined by its boundary derivative)**.** *Let J be a real-analytic admissible cost with real-analytic combiner P. Then the formal Taylor series of P at $(0,0)$ is uniquely determined by $\Psi(u) := P_v(u,0)$ and by A_1. In particular, any two real-analytic admissible combiners agreeing on $\{v = 0\}$ agree as formal power series at the origin.*

Proof. We recover the A_k from Ψ and A_1, then use Proposition 4 to recover the p_{ij}.

Step 1: Recovering $\{A_k\}$. Expanding both sides of the ODE $G''(s) = A_1\Psi(G(s))$ as power series at $s = 0$ and using $G(s) = \sum_{k\geq 1} A_k s^{2k}$, the left side has s^{2k-2} coefficient $(2k)(2k-1)A_k$. Expanding $\Psi(G(s))$ as a composition of power series, each coefficient of s^{2k} on the right involves only $\Psi^{(j)}(0)$ for $j \leq k$ and A_ℓ for $\ell \leq k$. Matching the s^{2k-2} coefficient gives

$$(2k)(2k-1)A_k = A_1\left[\Psi^{(k-1)}(0)\, A_1^{k-1}/(k-1)! + (\text{terms involving } A_2, \ldots, A_{k-1})\right],$$

so A_k is determined inductively from $A_1, \ldots, A_{k-1}$ and the derivatives of Ψ at zero.

Step 2: Recovering $\{p_{ij}\}$. With all A_k known, Proposition 4 solves for p_{mn} via (10) in triangular order, starting from the base case $p_{11} = 12A_2/A_1^2$ (set $m = n = 1$ in (10)). □

Proposition 5 (Second boundary derivative in closed form)**.** *Under the hypotheses of Proposition 4 (J is real-analytic and admissible with real-analytic combiner P, so $G \in C^\omega(\mathbb{R})$, $\Psi \in C^\omega(\mathbb{R}_{\geq 0})$, and the energy identity below is well-defined to all orders), set $\Pi(u) := \int_0^u \Psi(w)\,dw$. The second boundary derivative $P_{vv}(u,0)$ admits the closed form*

$$P_{vv}(u,0) = \tfrac{1}{3}\Psi''(u)\Pi(u) + \tfrac{1}{6}\Psi(u)\left[\Psi'(u) - \Psi'(0)\right] \qquad (u \in \mathbb{R}_{\geq 0}). \tag{11}$$

Proof. Extract the t^4 coefficient of $G(s+t) + G(s-t) = P(G(s), G(t))$ at fixed s. The left side gives coefficient $G^{(4)}(s)/12$. Expanding the right side to order t^4:

$$P(G(s), G(t)) = 2G(s) + \Psi(G(s))\,G(t) + \tfrac{1}{2}P_{vv}(G(s),0)\,G(t)^2 + O(t^6),$$

so the t^4 coefficient is $A_2\Psi(G(s)) + \frac{1}{2}A_1^2\,P_{vv}(G(s),0)$.

To evaluate $G^{(4)}(s)$, differentiate $G'' = A_1\Psi(G)$ twice. The first differentiation gives $G''' = A_1\Psi'(G)G'$; the second gives $G^{(4)} = A_1\Psi''(G)(G')^2 + A_1\Psi'(G)G''$. The energy identity is the conservation law for the autonomous second-order ODE $G'' = A_1\Psi(G)$: the energy

$$E(s) := \tfrac{1}{2}(G'(s))^2 - A_1\,\Pi(G(s))$$

is a first integral, since $E'(s) = G'(s)\big[G''(s) - A_1\Psi(G(s))\big] \equiv 0$ by the ODE (equivalently, multiply $G'' = A_1\Psi(G)$ by G' and integrate). With $G(0) = G'(0) = 0$ and $\Pi(0) = 0$, we have $E(0) = 0$; hence, $E \equiv 0$, which rearranges to $(G')^2 = 2A_1\Pi(G)$. Substituting:

$$G^{(4)} = 2A_1^2\,\Psi''(G)\,\Pi(G) + A_1^2\,\Psi(G)\Psi'(G).$$

Equating the two expressions for $G^{(4)}(s)/12$ and using the base-case identity $12A_2 = A_1^2\Psi'(0)$ (from matching the s^2 coefficient of the ODE) yields (11). □

Corollary 1 (Affine Ψ forces bilinear P)**.** *If $\Psi(u) = cu + 2$ for some $c \geq 0$, then $P_{vv}(u,0) \equiv 0$ and more generally $\partial_v^j P(u,0) = 0$ for all $j \geq 2$. Consequently $P(u,v) = cuv + 2u + 2v$.*

Proof. We prove the stronger claim $P(u,v) = cuv + 2u + 2v$, from which both assertions about boundary derivatives follow immediately.

Step 1 (Solve the boundary ODE in closed form). With $\Psi(u) = cu + 2$, the boundary-derivative ODE of Lemma 1 becomes the linear ODE

$$G''(s) = A_1\big(c\,G(s) + 2\big), \qquad G(0) = G'(0) = 0,$$

whose unique solution is

$$G(s) = \begin{cases} \dfrac{2}{c}\big(\cosh(\omega s) - 1\big) & (c > 0), \quad \omega := \sqrt{cA_1}, \\ A_1 s^2 & (c = 0). \end{cases}$$

The two cases agree in the limit $c \to 0^+$ via $(\cosh\omega s - 1)/(c/2) = A_1 s^2 + O(c)$.

Step 2 (Verify the d'Alembert identity). For $c > 0$, the addition formula $\cosh\omega(s+t) + \cosh\omega(s-t) = 2\cosh\omega s\,\cosh\omega t$ gives

$$G(s+t) + G(s-t) = \tfrac{4}{c}\big[\cosh\omega s\,\cosh\omega t - 1\big],$$

while a direct expansion of $cG(s)G(t) + 2G(s) + 2G(t)$ yields the same expression $\frac{4}{c}[\cosh\omega s\,\cosh\omega t - 1]$. Hence,

$$G(s+t) + G(s-t) = c\,G(s)G(t) + 2\,G(s) + 2\,G(t).$$

For $c = 0$ this reduces to $A_1[(s+t)^2 + (s-t)^2] = 2A_1 s^2 + 2A_1 t^2$.

Step 3 (Apply uniqueness of the combiner). By Theorem 1, the cost $J(x) := G(\ln x)$ admits a unique continuous combiner. Step 2 exhibits one—the bilinear $P(u,v) = cuv + 2u + 2v$—which is therefore the combiner. In particular $\partial_v^j P(u,0) = 0$ for all $j \geq 2$, recovering and extending the direct calculation $P_{vv}(u,0) = 0$ via (11) ($\Psi'' \equiv 0$ and $\Psi' - \Psi'(0) = 0$). □

Remark 5 (Scope of the Taylor recursion)**.** *Proposition 4 and Theorem 2 solve the inverse problem—recovering P from the boundary derivative Ψ—in the category of formal power series. What remains open is the* convergence *question: does an arbitrary real-analytic Ψ with $\Psi(0) = 2$ necessarily yield a convergent Taylor series for P and a globally valid admissible cost? The continuous degree bound (Section 5) and analytic rigidity (Section 6) will show, in the polynomial and entire-finite-order regimes, that the only admissible Ψ's are affine. The general case remains open and is discussed as a future direction in the conclusion.*

4.3. Superlinear Boundary Derivatives: A Blow-Up Obstruction

Theorem 3 (Blow-up of superlinear boundary derivatives)**.** *No C^2 admissible cost has a boundary derivative $\Psi := P_v(\cdot, 0)$ satisfying*

$$\Psi(u) \geq \beta\, u^p \qquad \text{for all } u \geq u_0, \tag{12}$$

with constants $\beta > 0$, $p > 1$, $u_0 \geq 0$.

Proof. Suppose otherwise. By Lemma 1, $G''(s) \geq A_1 \beta\, G(s)^p$ for $s \geq s_1$ (chosen so that $G(s_1) \geq u_0$, using coercivity of G). Multiplying by $G'(s) > 0$ and integrating yields, for large s, $G'(s) \geq C\, G(s)^{(p+1)/2}$ with $C = \sqrt{2A_1\beta/(p+1)} > 0$. Separating variables and integrating from some $s_2 \geq s_1$ with $G(s_2) \geq 1$ gives

$$\frac{2}{p-1}\left(G(s_2)^{-(p-1)/2} - G(s)^{-(p-1)/2}\right) \geq C(s - s_2).$$

Since $(p-1)/2 > 0$, the left side is bounded above by $\frac{2}{p-1} G(s_2)^{-(p-1)/2} < \infty$, while the right side grows without bound. A Contradiction. □

Remark 6 (Scope and the linear-growth gap)**.** *Theorem 3 rules out superlinear Ψ but not linear-growth non-affine perturbations such as $\Psi(u) = cu + 2 + \varepsilon \sin u$; we call the gap between "not superlinear" and "affine" the* linear-growth gap. *In the polynomial regime, it is closed unconditionally by Theorem 4. Moreover, the C^2 machinery in isolation does not exclude every* $\deg P \geq 3$*: the symmetric polynomial $P(u,v) = 2u + 2v + u^2v^2$ has $\Psi_P \equiv 2$, so Theorem 3 is silent on it. The narrower corollary $\Psi_P = P_v(\cdot, 0)$ polynomial of degree $\geq 2 \Rightarrow$ no C^2 admissible J has combiner P follows by applying Theorem 3 with β half the leading coefficient of Ψ_P (positive, since $\Psi_P \geq 0$) and $p = \deg \Psi_P \geq 2$; we record this as Corollary 2 below for reference.*

Corollary 2 (Polynomial Ψ_P of degree ≥ 2 rules out C^2 admissible combiners)**.** *If $P \in \mathbb{R}[u,v]$ is symmetric with $P(u,0) = 2u$ and $\deg P_v(\cdot,0) \geq 2$, then no C^2 admissible cost has combiner P.*

Proof. See Remark 6 above. □

5. The Continuous Degree Bound

The main result is that every continuous admissible cost with polynomial combiner has $\deg P \leq 2$ (Theorem 4). The proof uses no derivative information on J and derives a contradiction from two growth recursions for G that become incompatible once $\deg P \geq 3$: the diagonal recursion $G(2t) = Q(G(t))$, where $Q(u) := P(u,u)$, caps the growth of $\log G$ under a single doubling of the argument at the factor $\deg Q$, while a fixed-increment recursion $u_{n+1} + u_{n-1} = P(u_n, a)$ forces $\log u_n$ like δ^n, where $\delta \geq 2$ is the maximum u-degree of a mixed monomial of P.

The earlier algebraic route of ([30] Thm. 1) derives d^2-vs-$d^3 - 2d^2 + 2d$ from the iterated identity $G(4s) + G(2s) = P(G(3s), G(s))$, but needs a non-cancellation hypothesis $\deg_y P(P(q(y), y) - y, y) = d^3 - 2d^2 + 2d$ on the leading term, which fails when the top-degree part of P vanishes on the diagonal (e.g., $P(u,v) = 2u + 2v + uv(u-v)^2$; cf. [30] Rem. 4). Our C^0 argument removes this hypothesis: Step 2 below uses only the mixed-monomial maximum u-degree $\delta \geq 2$, automatic once $\deg P \geq 3$ by symmetry and the boundary conditions.

We isolate the algebraic content of the fixed-increment step as a sequence lemma (Lemma 2); the lemma needs only coercivity and the two-term recurrence, with no monotonicity of G, so it is also usable inside the measurable classification bootstrap of Theorem 5.

Strategy of the proof of Theorem 4.

Since the argument is the technical heart of the paper, we summarise it before giving the details. Suppose $\deg P \geq 3$. Evaluating the functional equation $G(s+t) + G(s-t) = P(G(s), G(t))$ in two different regimes produces two growth rates for the same quantity $\log G$ at the *same* points, and these rates are incompatible.

(i) Doubling regime ($s = t$): The diagonal recursion $G(2t) = Q(G(t))$, $Q(u) := P(u, u)$, shows that one doubling of the argument multiplies $\log G$ by at most $d' := \deg Q$, up to bounded errors—a *single-step* estimate, used exactly once.

(ii) Fixed-increment regime ($t = t_0$ fixed): The sequence $u_n = G(s_1 + nt_0)$ satisfies the two-term recurrence $u_{n+1} + u_{n-1} = P(u_n, a)$ with $a = G(t_0)$, whose right-hand side has degree $\delta \geq 2$ in u_n once $\deg P \geq 3$ (Lemma 3); Lemma 2 then forces $\log u_n \asymp \delta^n$, i.e., doubly exponential growth along arithmetic progressions.

Walking from s_1 to $2s_1 + 2nt_0$ in the two ways—n fixed-increment steps followed by one doubling, versus $2n$ fixed-increment steps from the base point $2s_1$—yields $\log G(2s_1 + 2nt_0) \lesssim \delta^n$ from (i) + (ii) and $\log G(2s_1 + 2nt_0) \gtrsim \delta^{2n}$ from (ii) alone. Since $\delta \geq 2$, the two bounds are contradictory for large n, so $\deg P \leq 2$. The only technical input is the sequence lemma (Lemma 2); no derivative, monotonicity, or convexity information on G is used.

Lemma 2 (Successive-maxima growth)**.** *Let $(u_n)_{n\geq 0}$ be a non-negative real sequence satisfying*

$$u_{n+1} + u_{n-1} = R(u_n) \qquad (n \geq 1), \tag{13}$$

with $R \in \mathbb{R}[u]$ of the leading term βu^δ, $\delta \geq 2$, $\beta > 0$. If $u_n \to +\infty$, then there exist $n_0 \geq 1$ and positive constants B_1, B_2 such that

$$B_1\,\delta^n \leq \log u_n \leq B_2\,\delta^n \qquad \textit{for all } n \geq n_0. \tag{14}$$

Proof. Since $R(u) = \beta u^\delta + O(u^{\delta-1})$, there is $T_0 > 0$ with $\frac{\beta}{2}u^\delta \leq R(u) \leq 2\beta u^\delta$ on $[T_0, \infty)$. Choose $T_1 \geq T_0$ with $T_1^{\delta-1} \geq 4/\beta$ (so that $\frac{\beta}{4}u^{\delta-1} \geq 1$ on $[T_1, \infty)$, equivalently $\frac{\beta}{4}u^\delta \geq u$ on $[T_1, \infty)$).

Choice of base index n_0. Let $M_n := \max_{0\leq k\leq n} u_k$ be the running maximum. Since $u_n \to +\infty$, the function $n \mapsto M_n$ is non-decreasing and tends to $+\infty$. Call an index $n \geq 1$ a new-maximum index if $u_n = M_n$, equivalently $u_n \geq u_k$ for every $0 \leq k \leq n$; in particular $u_n \geq u_{n-1}$ at every new-maximum index. The set of new-maximum indices is infinite: each strict increase in M_n occurs exactly at such an index, and $M_n \to +\infty$ forces infinitely many strict increases. Let

$$n_0 \;:=\; \min\{n \geq 1 : u_n = M_n \text{ and } u_n \geq T_1\}.$$

The set is non-empty (any sufficiently large new-maximum index satisfies $u_n \geq T_1$), so n_0 is well-defined. By construction, $u_{n_0} \geq T_1$ and $u_{n_0-1} \leq u_{n_0}$ (the latter from $u_{n_0} = M_{n_0} \geq u_{n_0-1}$).

Claim: For every $n \geq n_0$,

$$\tfrac{\beta}{4}\,u_n^\delta \;\leq\; u_{n+1} \;\leq\; 2\beta\,u_n^\delta, \tag{15}$$

and moreover, $u_n \leq u_{n+1}$ and $u_n \geq T_1$.

We prove the claim by induction on $n \geq n_0$. The base case $n = n_0$ provides $u_{n_0} \geq T_1$ and $u_{n_0-1} \leq u_{n_0}$ by the construction of n_0. For the inductive step, assume $u_{n-1} \leq u_n$ and $u_n \geq T_1$. Then $u_n \geq T_1 \geq T_0$, so

$$u_{n+1} \;=\; R(u_n) - u_{n-1} \;\leq\; R(u_n) \;\leq\; 2\beta\, u_n^{\delta},$$

which is the upper bound in (15). For the lower bound, the inductive hypothesis $u_{n-1} \leq u_n$ together with $u_n \geq T_1$ and the choice of T_1 give

$$u_{n+1} \;=\; R(u_n) - u_{n-1} \;\geq\; \tfrac{\beta}{2} u_n^{\delta} - u_n \;\geq\; \tfrac{\beta}{2} u_n^{\delta} - \tfrac{\beta}{4} u_n^{\delta} \;=\; \tfrac{\beta}{4} u_n^{\delta},$$

where we used $u_n \leq \frac{\beta}{4} u_n^{\delta}$ on $[T_1, \infty)$. In particular, $u_{n+1} \geq \frac{\beta}{4} u_n^{\delta} \geq u_n \geq T_1$, which propagates both auxiliary properties ($u_n \leq u_{n+1}$ and $u_{n+1} \geq T_1$) to the next index. This closes the induction.

Taking logarithms of (15) gives the linear recurrence $\log u_{n+1} = \delta \log u_n + O(1)$, whose explicit solution $\log u_n = \delta^{n-n_0} \log u_{n_0} + O(\delta^{n-n_0})$ yields (14) after enlarging n_0 once more so the bracketed constants are positive. □

Lemma 3 (Mixed-degree consequence)**.** *Let $P \in \mathbb{R}[u,v]$ satisfy $P(u,0) = 2u$, $P(0,v) = 2v$, and $P(u,v) = P(v,u)$. If $\deg P \geq 3$ and*

$$\delta := \max\{\, i : \exists\, j \geq 1 \text{ with } p_{ij} \neq 0 \,\}$$

is the maximum u degree among mixed monomials $p_{ij}u^i v^j$ of P, then $\delta \geq 2$.

Proof. The boundary conditions eliminate every pure monomial of degree ≥ 2, so each top-degree monomial of degree $d := \deg P \geq 3$ is mixed. Choose such a monomial $p_{ij}u^i v^j$ with $i + j = d$. By symmetry, either this monomial or its transpose has u-degree at least $\lceil d/2 \rceil \geq 2$, hence the maximum mixed u-degree δ is at least 2. □

Theorem 4 (Continuous degree bound)**.** *Let $J : \mathbb{R}_{>0} \to \mathbb{R}$ be a continuous function satisfying $J(xy) + J(x/y) = P(J(x), J(y))$ for some polynomial P, and let $G(t) := J(e^t)$. Suppose that*

(DB1) *$G(-t) = G(t)$ and $G(0) = 0$;*

(DB2) *$G(t) > 0$ for $t \neq 0$ and $G(t) \to +\infty$ as $|t| \to \infty$;*

(DB3) *$P(u,0) = 2u$, $P(0,v) = 2v$, $P(u,v) = P(v,u)$, and $P \geq 0$ on $\mathbb{R}^2_{\geq 0}$.*

Conditions (DB1)–(DB2) constrain the function G, while (DB3) is postulated directly at the polynomial level*: in particular, $P \geq 0$ on $\mathbb{R}^2_{\geq 0}$ is here an assumption on the polynomial P itself, not a property derived from any structure on J. All four conditions hold automatically whenever J is pre-admissible, by Section 3 (see Remark 7), but the theorem itself does not require pre-admissibility. Then* $\deg P \leq 2$.

Remark 7 (Role of the hypotheses)**.** *The proof of Theorem 4 uses only the hypotheses (DB1)–(DB3): continuity of G, evenness, $G(0) = 0$, $G(t) > 0$ for $t \neq 0$, coercivity $G(t) \to \infty$, the boundary polynomial relations $P(u,0) = 2u$, $P(0,v) = 2v$, symmetry, and non-negativity of P. It does* not *require G to be monotone on $[0,\infty)$ or strictly log-convex: the fixed-increment step is delegated to Lemma 2, whose hypotheses are purely algebraic (non-negativity of the sequence, two-term recurrence with polynomial right-hand side, and coercivity $u_n \to +\infty$), all of which (DB2) supplies. This non-monotonicity scope is essential for the bootstrap of Theorem 5, which invokes Theorem 4 at a point where monotonicity of G has not yet been established. All of (DB1)–(DB3) follow from pre-admissibility combined with Section 3, so the theorem applies to every pre-admissible cost with polynomial combiner.*

Proof. Hypotheses (DB1)–(DB3) give the functional equation

$$G(s+t)+G(s-t)=P(G(s),G(t)) \qquad (s,t\in\mathbb{R}), \tag{16}$$

and every monomial $p_{ij}u^iv^j$ of degree ≥ 2 in P has $i,j\geq 1$ (by the boundary conditions $P(u,0)=2u$, $P(0,v)=2v$, and symmetry of (DB3)). Suppose for contradiction that $d:=\deg P\geq 3$.

Step 1: Diagonal recursion. Setting $s=t$ in (16): $G(2t)=Q(G(t))$, where

$$Q(u):=P(u,u)=\sum_{e\geq 0}\beta_e u^e, \qquad \beta_e:=\sum_{i+j=e}p_{ij}.$$

The boundary conditions force $\beta_0=0$ and $\beta_1=4>0$. Let $d':=\deg Q$; since $Q(G(t))=G(2t)>0$ for $t\neq 0$ and $G(t)\to\infty$, the leading coefficient satisfies $\beta_{d'}>0$.

The only consequence of the diagonal recursion used below is the single-step growth estimate: since Q is a polynomial of degree $d'\geq 1$ with positive leading coefficient,

$$\log Q(u) \;=\; d'\log u+O(1) \qquad (u\to\infty). \tag{17}$$

We do not iterate the diagonal recursion, so no dyadic-scale growth bound is needed, and the cases $d'=1$ and $d'\geq 2$ are treated uniformly; in particular, the argument is insensitive to diagonal cancellation $d'<\deg P$ (cf. Remark 8).

Step 2: Fixed-increment recursion. Let $\delta:=\max\{i:\exists\, j\geq 1,\ p_{ij}\neq 0\}$ be the maximum u degree of a mixed monomial in P. By Lemma 3, the assumptions $d=\deg P\geq 3$, symmetry, and the boundary conditions force $\delta\geq 2$.

The leading u coefficient of $P(u,a)$ is $\Lambda(a):=\sum_{j\geq 1}p_{\delta,j}\,a^j$, a nonzero polynomial in a with no constant term. By positivity of P (hypothesis (DB3)), $\Lambda(a)\geq 0$ for all $a\geq 0$. Choose $t_0>0$ so that $G(t_0)$ avoids the finite zero set of the polynomial Λ (possible because G is continuous and surjective onto $[0,\infty)$: $G(0)=0$ and $G(t)\to+\infty$ as $|t|\to\infty$ by (DB1)–(DB2), so by the intermediate value theorem applied to $G|_{[0,\infty)}$, the image $G([0,\infty))$ contains $[0,\infty)$, and we may pick t_0 in any preimage of a value in $[0,\infty)\setminus Z(\Lambda)$, where $Z(\Lambda)$ is the finite real zero set of Λ), and set $a:=G(t_0)>0$; then $\Lambda(a)>0$.

For any $s_1>0$, the fixed-increment sequence $u_n:=G(s_1+nt_0)$ is non-negative (by (DB2)) and, applying (16) at (s_1+nt_0,t_0), satisfies the two-term recurrence

$$u_{n+1}+u_{n-1}=P(u_n,a) \qquad (n\geq 1).$$

The polynomial $R(u):=P(u,a)$ has a leading term $\Lambda(a)\,u^\delta$ with $\delta\geq 2$ and $\Lambda(a)>0$, and $u_n\to+\infty$ by (DB2) (since $s_1+nt_0\to+\infty$). Lemma 2 therefore yields an integer $n_0\geq 1$ and constants $B_1,B_2>0$ such that

$$B_1\cdot\delta^n\leq\log u_n\leq B_2\cdot\delta^n \qquad (n\geq n_0). \tag{18}$$

The threshold n_0 and the constants B_1,B_2 depend on the base point s_1; this dependence is tracked explicitly in Step 3.

Step 3: Contradiction. Fix $s_1>0$ and consider the two fixed-increment sequences

$$u_n^{(1)}:=G(s_1+nt_0), \qquad u_n^{(2)}:=G(2s_1+nt_0) \qquad (n\geq 0),$$

with the same increment t_0 and the same value $a=G(t_0)$. Both satisfy the recurrence $u_{n+1}+u_{n-1}=P(u_n,a)$ of Step 2 with the same polynomial $R(u)=P(u,a)$ (leading term $\Lambda(a)u^\delta$, $\delta\geq 2$), and both tend to $+\infty$ by (DB2). Applying Lemma 2 to each sequence separately gives constants $B_1, B_2, B_3, B_4 > 0$ and a common threshold $N_0 \geq 1$ (the maximum of the two thresholds) such that

$$B_1\,\delta^n \leq \log u_n^{(1)} \leq B_2\,\delta^n \quad \text{and} \quad B_3\,\delta^n \leq \log u_n^{(2)} \leq B_4\,\delta^n \qquad (n \geq N_0). \tag{19}$$

Upper bound. The diagonal identity $G(2t) = Q(G(t))$ of Step 1, applied at $t = s_1 + nt_0$, expresses $G(2s_1 + 2nt_0) = Q\big(u_n^{(1)}\big)$. The single-step estimate (17) and the upper bound in (19) give, for $n \geq N_0$,

$$\log G(2s_1 + 2nt_0) = \log Q\big(u_n^{(1)}\big) \leq d' \log u_n^{(1)} + C_1 \leq d' B_2\,\delta^n + C_1.$$

Lower bound. The same number $G(2s_1 + 2nt_0)$ is the term $u_{2n}^{(2)}$ of the second sequence, so the lower bound in (19) gives, for $2n \geq N_0$,

$$\log G(2s_1 + 2nt_0) = \log u_{2n}^{(2)} \geq B_3\,\delta^{2n}.$$

Conclusion. Combining the two displays,

$$B_3\,\delta^{2n} \leq d' B_2\,\delta^n + C_1 \qquad \text{for all } n \geq N_0.$$

Dividing by δ^n and letting $n \to \infty$, the left side grows like $B_3\delta^n \to \infty$ (since $\delta \geq 2$), while the right side tends to the finite limit $d' B_2$. This contradiction shows $\deg P \leq 2$. Note that only the single-step estimate (17) of Step 1 is used, so the argument is uniform in $d' \geq 1$ and the no-monotonicity scope of Remark 7 is preserved. □

Remark 8 (Independence from $\deg Q$)**.** *The proof exploits only $\delta \geq 2$ (Step 2) and the finiteness of $d' := \deg Q$ (Step 1); it does not require $d' = d$, and remains valid when the top-degree part P_d vanishes on the diagonal (giving $d' < d$). For example, $P(u, v) = 2u + 2v + uv(u - v)^2$ has $d = 4$ but $d' = 1$; the argument is unaffected.*

Corollary 3 (Bilinear form of polynomial combiners)**.** *Let $J : \mathbb{R}_{>0} \to \mathbb{R}$ be a continuous admissible cost with polynomial combiner P. Then,*

$$P(u, v) = c\,uv + 2u + 2v$$

for some $c \geq 0$, and $J \in C^\omega(\mathbb{R}_{>0})$ (Proposition 7).

Proof. By Theorem 4, $\deg P \leq 2$. The passage from $\deg P \leq 2$ to the bilinear form $P(u, v) = cuv + 2u + 2v$ under symmetry and $J(1) = 0$ is ([30] Thm. 3): symmetry, $P(0, 0) = 0$, $P(u, 0) = 2u$, and $P(0, v) = 2v$ kill the constant term, the linear terms, and the pure-quadratic monomials u^2 and v^2, leaving only the mixed term cuv. The sign of c is fixed using only the pre-admissible structure: by Proposition 2(3), $P(u, v) \geq 0$ on $\mathbb{R}^2_{\geq 0}$. If $c < 0$, then $P(u, v) = cuv + 2u + 2v \to -\infty$ along $u = v \to \infty$, contradicting non-negativity. Hence $c \geq 0$. Smoothness follows from Proposition 7. □

6. Regularity Improvements: Measurable and Analytic Rigidity

The continuity hypothesis in Theorem 4 can be weakened in two independent directions. Theorem 5 shows that measurability suffices when P is polynomial: continuity is a bootstrap consequence. Theorem 6 records an analytic constraint in the entire finite-order regime for the diagonal recursion $G(2t) = Q(G(t))$. In the present manuscript, the polynomiality of the diagonal combiner Q is *not* derived from finite order alone; instead, Theorem 6 supplies the degree lower bound $d' \geq 2^\rho$ when Q is a polynomial of degree d', and the polynomial-combiner classification continues to be driven by the C^0 degree bound Theorem 4.

To keep this section readable, we have separated the reusable technical tools from the main line of argument: the two measure-theoretic ingredients of the bootstrap—the contraction principle at the attracting fixed point (principle (A)) and the polynomial measurable-to-continuous regularity lemma (Lemma A1)—are stated and proved in Appendix A, together with a remark delimiting what each one does and does not give. The body of this section contains only the statement of the measurable classification (Theorem 5), a discussion of its hypotheses, its proof (which quotes the two appendix tools as black boxes), and the analytic-rigidity theorem (Theorem 6).

6.1. Measurable Classification

The measurable-to-continuous step in Theorem 5 combines two classical ingredients, both stated and proved in Appendix A:

- Principle (A) (contraction at the attracting fixed point). If a measurable, locally bounded F with Lebesgue point 0 of value 0 satisfies the halving identity $F(t) = L(F(2t))$ a.e. on a set of full density at 0, where L is analytic with $L(0) = 0$ and $|L'(0)| < 1$, then F has approximate limit 0 at 0. This is the measurable form of the Koenigs linearisation [10,12].
- Lemma A1 (polynomial measurable-to-continuous regularity). A measurable, locally bounded F satisfying the two-variable addition formula $F(s+t) + F(s-t) = \mathcal{P}(F(s), F(t))$ a.e., with $\mathcal{P}$ polynomial, admits a representative continuous on a neighbourhood of 0. This is the polynomial analogue of the classical d'Alembert measurable-to-continuous theorem ([2] §3.1, [1] Ch. 13), obtained from Járai's framework ([14] Cor. 8.7).

The division of labour between (A) and Lemma A1—approximate continuity at the fixed point versus topological continuity on a neighbourhood—is explained in Remark A1.

Theorem 5 (Measurable classification)**.** *Let $J : \mathbb{R}_{>0} \to \mathbb{R}$ be Lebesgue-measurable and reciprocally symmetric (non-constancy follows automatically from (MC4) below). Suppose further that*

(MC1) *$J(x) \geq 0$ for Lebesgue-a.e. $x \in \mathbb{R}_{>0}$;*

(MC2) *J is bounded on some set of positive Lebesgue measure in $\mathbb{R}_{>0}$;*

(MC3) *$J(1) = 0$ and $x = 1$ is a Lebesgue point of J with Lebesgue value 0, i.e., $\lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_{1-\epsilon}^{1+\epsilon} J(x) \frac{dx}{x} = 0$;*

(MC4) $\operatorname{ess\,lim}_{x \to 0^+} J(x) = \operatorname{ess\,lim}_{x \to +\infty} J(x) = +\infty$;

(MC5) *There exists a* symmetric *polynomial P (i.e., $P(u,v) = P(v,u)$) satisfying*

$$J(xy) + J(x/y) = P(J(x), J(y)) \qquad \textit{for Lebesgue-a.e. } (x,y) \in \mathbb{R}^2_{>0}. \tag{20}$$

Then J has a continuous representative, which is admissible (strictly log-convex), $\deg P \leq 2$, *and J lies in one of the two families of Theorem 8.*

We single out the reading of (MC3), which is the anchor of the whole bootstrap: the point $x = 1$ is required to be a Lebesgue point of J with Lebesgue value 0, matching the pointwise normalisation $J(1) = 0$. Under the change of variables $x = e^t$, which converts the measure dx/x into Lebesgue measure dt, this becomes the statement that $t = 0$ is a Lebesgue point of $G = J \circ \exp$ with value 0. This is a genuine strengthening of the pointwise condition $J(1) = 0$—an a.e.-defined function carries no information at a single point, so some averaged smallness at the minimiser must be imposed—and it is used twice below: it drives the mollification limits in Substeps 2b and 2b′, and it verifies the Lebesgue-point hypothesis of the contraction principle (A) in Substep 2d–i.

Remark 9 (Role of the hypotheses (MC1)–(MC4))**.** *Each hypothesis rules out a class of pathologies that includes the example (or failure mode) below. We* do not *claim individual necessity: each of the counter-examples below violates more than one of (MC1)–(MC4) simultaneously, so the question of which single hypothesis is the binding one for a given example is not addressed here. What we record is only that the combined hypothesis set excludes these natural pathologies; isolated sharpness (a counter-example violating only one of (MC1)–(MC4) while satisfying the others) is left open.*

- Negative-cost solutions *(in the class ruled out by (MC1); this particular example also violates (MC4)). The function $J(x) = -(\ln x)^2$ is reciprocally symmetric with $J(1) = 0$ and satisfies $J(xy) + J(x/y) = 2J(x) + 2J(y)$, hence has polynomial combiner $P(u,v) = 2u + 2v$, yet $J \le 0$ everywhere and $\operatorname{ess\,lim}_{x\to 0^+} J(x) = \operatorname{ess\,lim}_{x\to+\infty} J(x) = -\infty$. Either (MC1) or (MC4) suffices to exclude this and similar "negative-cost" candidates.*
- A.e. modifications of an essentially constant function *(in the class ruled out by (MC3); this particular example also violates (MC4)). With $J(1) := 0$ pointwise and $J(x) := 1$ for $x \neq 1$, J is measurable, reciprocally symmetric, non-negative, and satisfies* (20) *a.e. with any symmetric polynomial P such that $P(1,1) = 2$ (e.g., $P(u,v) = u + v$), yet admits no continuous representative consistent with $J(1) = 0$ (the only continuous candidate is $J \equiv 1$). Either (MC3) (the Lebesgue average of J at $x = 1$ equals $1 \neq 0$) or (MC4) ($\operatorname{ess\,lim}_{x\to 0^+} J(x) = \operatorname{ess\,lim}_{x\to+\infty} J(x) = 1 \neq +\infty$) excludes this and similar a.e. modifications of essentially constant functions.*
- Structural role of coercivity *(the bootstrap of Theorem 5 fails without (MC4)). Coercivity is the input that verifies the hypotheses of Theorem 4 (pre-admissibility) in Step 4 of the proof below; without $G(t) \to +\infty$, the fixed-increment and diagonal growth recursions underlying the C^0 degree bound cannot be set up. This shows that (MC4) is structurally load-bearing in the bootstrap, independently of any question of isolated necessity.*

(MC2) is the standard Steinhaus regularity condition; dropping it permits Hamel-basis pathologies [1,42].

Proof. The proof is a bootstrap in four stages, which we state at the outset so that the role of each substep is visible:

Step 1:	Setup in log coordinates: $G := J \circ \exp$ and transfer of (MC1)–(MC3) to G;
Step 2 (2a–2e):	Measurability $\Rightarrow$ continuity near 0 $\Rightarrow$ continuity on $\mathbb{R}$;
Step 3:	A.e. identities $\Rightarrow$ everywhere identities, by continuity;
Step 4:	C^0 degree bound (Theorem 4) $\Rightarrow P = cuv + 2u + 2v$, $c \ge 0$, then d'Alembert/Jensen classification $\Rightarrow$ explicit G, admissibility, C^ω.

The technical tools (principle (A) and Lemma A1) are quoted from Appendix A.

Step 1: Setup in log coordinates. Let $G(t) := J(e^t)$; then G is measurable, even (by reciprocal symmetry), non-constant, non-negative a.e. (by (MC1)), and $G(0) = 0$ with 0 a Lebesgue point of G of value 0 (by (MC3); the change of variables $x = e^t$ sends $\frac{dx}{x}$ to Lebesgue measure dt).

The diagonal identity $G(2t) = Q(G(t))$ a.e., where $Q(u) := P(u,u)$, does not follow from (20) by setting $y = x$: the diagonal $\{(t,t) : t \in \mathbb{R}\} \subset \mathbb{R}^2$ is a 2D-null set, so Fubini does not yield it directly. We derive it rigorously by mollification in Substep 2b′ below (mirroring the mollification argument of Substep 2b) and use it repeatedly thereafter.

Step 2: Bootstrapping continuity. The claim is that, under the standing hypotheses, G has a continuous representative. We carry this out in six substeps: (2a) local boundedness of

G near 0 via Steinhaus; (2b) rigorous derivation of the boundary identities $P(u,0) = 2u$ and $P(0,v) = 2v$ via mollification; (2b′) rigorous derivation of the diagonal identity $G(2t) = Q(G(t))$ a.e. by a parallel mollification; (2c) an a.e. halving identity $G(t) = L(G(2t))$ via an analytic inverse branch of Q at 0; (2d) upgrading measurability to continuity on a neighborhood of 0; (2e) propagation to all of $\mathbb{R}$ via the diagonal recursion.

(2a) Local boundedness of G near 0. Since J is bounded on a set $A \subset \mathbb{R}_{>0}$ of positive Lebesgue measures (hypothesis (MC2)), $G = J \circ \exp$ is bounded on $B := \log A \subset \mathbb{R}$, a set of positive measures; say $|G| \leq M$ on B. By Steinhaus' theorem ([1] Ch. 2, Thm. 2.3), the difference set $B - B$ contains an open interval $U_0 = (-\delta_0, \delta_0)$ around 0. Writing the bilinear Equation (20) in log coordinates

$$G(s+t) + G(s-t) = P(G(s), G(t)) \qquad \text{for a.e. } (s,t) \in \mathbb{R}^2,$$

evaluate it at $(\sigma, \tau) = (s, s')$ for $(s, s') \in B \times B$: by Fubini,

$$G(s+s') + G(s-s') = P(G(s), G(s')) \qquad \text{for a.e. } (s,s') \in B \times B.$$

By (MC1), $G(s), G(s') \in [0, M]$ a.e. on $B \times B$, so the right-hand side is bounded above by $K(M) := \max_{u,v \in [0,M]} P(u,v) < \infty$. The two summands on the left are non-negative a.e. by (MC1), so each is bounded by $K(M)$ a.e. For each $h \in B - B$, the slice $\{(s,s') \in B \times B : s - s' = h\}$ has a positive one-dimensional Lebesgue measure for a.e. h (the standard Steinhaus-type computation); selecting a representative (s, s') in this slice, where the equation and the non-negativity of the summands simultaneously hold, yields $G(h) \leq K(M)$ for a.e. $h \in U_0$. Set $M' := K(M)$ and $U := \frac{1}{2}U_0$, on which $|G| \leq M'$ a.e. a fortiori ($\frac{1}{2}U_0 \subseteq U_0$). Note that the use of (MC1) here is essential: without a.e. non-negativity, boundedness of the sum $G(s+s') + G(s-s')$ alone would not bound the individual summands.

(2b) Rigorous derivation of $P(u,0) = 2u$ and $P(0,v) = 2v$. We derive the boundary identities by Lebesgue averaging, using the Lebesgue-point hypothesis (MC3).

Fix a non-negative mollifier sequence $(\phi_n)_{n \geq 1}$ on $\mathbb{R}$ with $\operatorname{supp} \phi_n \subset (-1/n, 1/n)$, $\int \phi_n(t)\,dt = 1$, and $\|\phi_n\|_\infty \leq n$. The equation $G(s+t) + G(s-t) = P(G(s), G(t))$ holds for almost every (s,t); hence, by Fubini, for each fixed n, there is a full-measure set $S_n \subset \mathbb{R}$ such that, for every $s \in S_n$, the equation holds for a.e. t (hence for the set of t's on which ϕ_n is supported, modulo null sets). Thus, for $s \in S_n$,

$$\int \phi_n(t) \big[G(s+t) + G(s-t)\big]\,dt = \int \phi_n(t)\, P(G(s), G(t))\,dt.$$

The left-hand side equals $(G * \phi_n)(s) + (G * \check{\phi}_n)(s)$, where $\check{\phi}_n(t) := \phi_n(-t)$. Expanding $P(u,v) = \sum_{i,j \geq 0} p_{ij}\, u^i v^j$ (a finite sum, since P is a polynomial of bounded total degree, so the interchange of summation and integration below is trivially justified by linearity), the right-hand side becomes

$$\sum_{i \geq 0} G(s)^i\, \mu_{n,i}, \qquad \mu_{n,i} := \sum_{j \geq 0} p_{ij} \int \phi_n(t) G(t)^j\,dt.$$

Since 0 is a Lebesgue point of G with the value 0 ((MC3)), and G is bounded on $U \supset \operatorname{supp} \phi_n$ for n large (say $|G| \leq M'$ on U, by Substep 2a), we treat the $j = 0$ and $j \geq 1$ contributions to $\mu_{n,i}$ separately.

Term $j = 0$. $\int \phi_n(t)\, G(t)^0\,dt = \int \phi_n(t)\,dt = 1$ for every n, so this term contributes exactly $p_{i,0}$ to $\mu_{n,i}$.

Terms $j \geq 1$. We have $|G(t)^j| \leq (M')^{j-1}|G(t)|$ for $t \in U$, and therefore

$$\left|\int \phi_n(t)G(t)^j\,dt\right| \leq (M')^{j-1}\int \phi_n(t)|G(t)|\,dt \to 0$$

by Lebesgue differentiation applied to $|G|$ at the Lebesgue point $t = 0$ of value 0. Hence $\int \phi_n(t)G(t)^j\,dt \to G(0)^j = 0$ for every $j \geq 1$.

Combining the two regimes, $\mu_{n,i} \to p_{i,0}$ as $n \to \infty$ for every $i \geq 0$.

Passing to the limit $n \to \infty$ at any s that is a Lebesgue point of G (a full-measure set), we obtain

$$2G(s) = \sum_{i\geq 0} p_{i,0}\,G(s)^i = P(G(s),0) \qquad \text{for a.e. } s \in \mathbb{R}. \tag{21}$$

We now upgrade the pointwise identity $P(G(s),0) = 2G(s)$, valid for s in a full-measure subset $S_* \subseteq \mathbb{R}$ (the intersection of the S_n's with the Lebesgue points of G), to a polynomial identity in the indeterminate u. Write $R(u) := P(u,0) - 2u \in \mathbb{R}[u]$; we must show $R \equiv 0$. By construction, $R(G(s)) = 0$ for every $s \in S_*$, so $G(S_*) \subseteq Z(R)$, where $Z(R) := \{u \in \mathbb{R} : R(u) = 0\}$ is the real zero set of R.

- Case $R \equiv 0$. Then $P(u,0) = 2u$ as polynomials, which is the desired conclusion.
- Case $R \not\equiv 0$. Then $Z(R)$ is finite (a nonzero univariate polynomial has finitely many real roots), so G would be essentially valued in the finite set $Z(R)$ on S_*; together with the null complement $\mathbb{R} \setminus S_*$, this would force G to be essentially valued in $Z(R) \subset \mathbb{R}$, hence essentially bounded by $\max_{u\in Z(R)} |u| < \infty$. This contradicts essential coercivity (MC4), which forces $\operatorname{ess\,lim}_{|t|\to\infty} G(t) = +\infty$ and hence the essential range of G to be unbounded above.

Only Case $R \equiv 0$ is consistent with the hypotheses, so $P(u,0) = 2u$ as polynomials in u.

For use in Substep 2c, we record the standard Markov-type consequence of (MC3): since 0 is a Lebesgue point of G of value 0 and $G \geq 0$ a.e., Markov's inequality gives, for every fixed $\sigma > 0$,

$$\frac{\left|\{t \in (-\epsilon,\epsilon) : G(t) > \sigma\}\right|}{2\epsilon} \leq \frac{1}{2\epsilon\sigma}\int_{-\epsilon}^{\epsilon} G(t)\,dt \longrightarrow 0 \qquad (\epsilon \to 0^+),$$

i.e., each sublevel set $\{G \leq \sigma\}$ has full Lebesgue density at 0. This is the only quantitative consequence of (MC3) needed later (it verifies the density of the set T_η in Substep 2c).

Hypothesis (MC5) supplies the symmetry $P(u,v) = P(v,u)$; hence, $P(0,v) = P(v,0) = 2v$ from the identity just established. Consequently

$$Q(0) = P(0,0) = 0, \qquad Q'(0) = P_u(0,0) + P_v(0,0) = 2 + 2 = 4 \neq 0.$$

(2b′) Diagonal identity $G(2t) = Q(G(t))$ a.e. via mollification. We derive the diagonal identity by mollification, mirroring Substep 2b. Equation (20) reads, in log coordinates, $G(s+t) + G(s-t) = P(G(s),G(t))$ for a.e. $(s,t) \in \mathbb{R}^2$. Substituting $\tau := t - s$ (unit Jacobian) gives

$$G(2s+\tau) + G(-\tau) = P(G(s),G(s+\tau)) \qquad \text{for a.e. } (s,\tau) \in \mathbb{R}^2. \tag{22}$$

By Fubini, for each $n \geq 1$, there is a full-measure set $S'_n \subseteq \mathbb{R}$ such that, for every $s \in S'_n$, (22) holds for a.e. $\tau \in \operatorname{supp}\phi_n$, where (ϕ_n) is the mollifier sequence of Substep 2b (supp $\phi_n \subset (-1/n, 1/n)$, $\int \phi_n = 1$, $\|\phi_n\|_\infty \le n$). Integrating against $\phi_n(\tau)\,d\tau$ yields, for $s \in S'_n$,

$$\int \phi_n(\tau)\big[G(2s+\tau) + G(-\tau)\big]\,d\tau = \int \phi_n(\tau)\,P(G(s), G(s+\tau))\,d\tau. \tag{23}$$

Restrict to $s \in \frac{1}{2}U$, where U is the local-boundedness neighborhood of Substep 2a (on which $|G| \le M'$ a.e.); for n large, supp $\phi_n \subseteq (-\frac{1}{2}\delta_0, \frac{1}{2}\delta_0)$, so $s + \tau \in U$ for all $\tau \in$ supp ϕ_n simultaneously and $|G(s+\tau)| \le M'$ a.e.

Left side, $n \to \infty$. By Lebesgue differentiation, $\int \phi_n(\tau)\,G(2s+\tau)\,d\tau \to G(2s)$ at every Lebesgue point $2s$ of G (a full-measure condition on s, since $s \mapsto 2s$ preserves the null/full-measure dichotomy). The second summand $\int \phi_n(\tau)\,G(-\tau)\,d\tau \to G(0) = 0$ by the Lebesgue-point hypothesis (MC3)—precisely the Markov-type estimate already used in Substep 2b.

Right side, $n \to \infty$. Expand $P(G(s), v) = \sum_{j \ge 0} c_j(G(s))\,v^j$ as a polynomial in v with coefficients $c_j(u) := \sum_{i \ge 0} p_{ij} u^i \in \mathbb{R}[u]$ (a finite sum, since P has a finite total degree, so the interchange of summation and integration below is trivially justified by linearity). For each j, $|G(s+\tau)^j| \le (M')^j$ on supp ϕ_n, and Lebesgue differentiation gives

$$\int \phi_n(\tau)\,G(s+\tau)^j\,d\tau \;\longrightarrow\; G(s)^j \qquad (n \to \infty)$$

at every Lebesgue point s of G^j (a full-measure set, because G^j is locally essentially bounded on U). Summing the finitely many terms,

$$\int \phi_n(\tau)\,P(G(s), G(s+\tau))\,d\tau \;\longrightarrow\; \sum_{j \ge 0} c_j(G(s))\,G(s)^j = P(G(s), G(s)) = Q(G(s)).$$

Conclusion. At every s in the countable intersection

$$S'_* \;:=\; \Big(\bigcap_n S'_n\Big) \cap \{s : 2s \text{ Lebesgue point of } G\} \cap \bigcap_{j=0}^{\deg_v P} \{s : s \text{ Lebesgue point of } G^j\},$$

a full-measure subset of $\frac{1}{2}U$, the two limits combine to give

$$G(2s) \;=\; Q(G(s)) \qquad \text{for a.e. } s \in \tfrac{1}{2}U. \tag{24}$$

This is the diagonal identity. Substep 2c uses it on $T_\eta \subseteq \frac{1}{2}U$ to derive the halving identity; Substep 2e propagates it (via the continuous representative of G from Substep 2d-ii, on which both sides of (24) are continuous and therefore agree everywhere by a.e. equality on a dense set) to all of $\mathbb{R}$.

(2c) Halving identity a.e. via an analytic inverse branch of Q. Since Q is a real polynomial with $Q(0) = 0$ and $Q'(0) = 4 \ne 0$, the inverse function theorem produces an open neighborhood $V \subseteq \mathbb{R}$ of 0 and a real-analytic inverse branch

$$L : V \longrightarrow W, \qquad L(0) = 0, \quad L'(0) = \tfrac{1}{4}, \qquad L\big(Q(u)\big) = u \text{ for } u \in W.$$

(We use the symbol L here to avoid the notation clash with the boundary derivative $\Psi(u) := P_v(u, 0)$ used elsewhere in the paper.) In particular, L is continuous on V.

Fix $\eta > 0$ with $[0, \eta] \subseteq V$, and choose $\delta_W > 0$ small enough that $[0, \delta_W] \subseteq W$; such δ_W exists because W is an open neighborhood of 0. Define the refined set

$$T_\eta := \big\{t \in \tfrac{1}{2}U : G(2t) \in [0, \eta] \text{ and } G(t) \in [0, \delta_W]\big\},$$

so that membership $t \in T_\eta$ places $G(t)$ in W and $G(2t)$ in V simultaneously. By (MC1) ($G \geq 0$ a.e.) and (MC3) (0 a Lebesgue point of G of value 0), Markov's inequality applied to G at scale t shows that $\{t \in \frac{1}{2}U : G(t) \in [0, \delta_W]\}$ has full Lebesgue density at 0; the same Markov estimate applied to G at scale $2t$ (the calculation of Substep 2b above, with $G(t)$ replaced by $G(2t)$ via the Jacobian 2 of the change of variables) shows that $\{t \in \frac{1}{2}U : G(2t) \in [0, \eta]\}$ also has full Lebesgue density at 0. Intersecting two density-1 sets leaves density-1, so

$$\frac{|T_\eta \cap (-\epsilon, \epsilon)|}{2\epsilon} \to 1 \qquad \text{as } \epsilon \to 0^+.$$

By construction, for a.e. $t \in T_\eta$, we have $G(t) \in [0, \delta_W] \subseteq W$, so the inverse-branch identity $L\big(Q(u)\big) = u$ (valid for $u \in W$) applies at $u = G(t)$ to give $L\big(Q(G(t))\big) = G(t)$. Combined with the diagonal form $G(2t) = Q(G(t))$ of (24) (a.e. on $\frac{1}{2}U \supseteq T_\eta$, rigorously established in Substep 2b′), this rewrites as the halving identity

$$G(t) = L\big(G(2t)\big) \qquad \text{for a.e. } t \in T_\eta. \tag{25}$$

Note that we do not claim that (25) holds a.e. on a full neighborhood of 0: from (MC1) and (MC3) alone, we obtain only a.e. validity on the density-1 set T_η. This density-based input is what feeds principle (A) of Appendix A in Substep 2d-i; the upgrade to continuity on a full neighborhood in Substep 2d-ii uses the two-variable equation (20) and Lemma A1.

(2d) Upgrading measurability to continuity near 0. We split the upgrade into two sub-substeps. The first uses only the one-dimensional halving identity and gives continuity at the attracting fixed point. The second uses the original two-variable equation to propagate this continuity to a neighborhood of the origin.

(2d-i) Approximate continuity at 0 and fixing of the boundary value. We apply principle (A) of Appendix A with $F := G$ and $T := T_\eta$. All its hypotheses are now in place: L is real-analytic with $L(0) = 0$ and $|L'(0)| = 1/4 < 1$ (Substep 2c, where the inverse branch L of Q is constructed); G is measurable and locally bounded on U (Substep 2a); 0 is a Lebesgue point of G of value 0 ((MC3)); and the halving identity (25) holds a.e. on $T_\eta \subseteq \frac{1}{2}U$, which has full Lebesgue density at 0 (Substep 2c). Principle (A) therefore yields the approximate continuity of G at 0 with approximate limit 0: for every $\sigma > 0$, $\{t : |G(t)| > \sigma\}$ has Lebesgue density 0 at $t = 0$. This determines the value at 0 of any continuous representative of G (it must equal 0) but does not yet provide such a representative on a neighbourhood. The topological extension is supplied in Substep 2d-ii.

(2d-ii) Continuity throughout a neighborhood of 0. The one-variable halving identity (25) is, by itself, *not* sufficient to upgrade measurability to continuity on a punctured neighborhood of 0; the obstruction is the mosaic family for the linear Schröder equation recalled in part (B) of Appendix A. Propagation of the continuity obtained in Substep 2d-i from the fixed point to a full neighborhood therefore requires extra rigidity beyond (25); this rigidity is provided by the original two-variable Equation (20), which in the log-coordinate $t = \ln x$ reads

$$G(s+t) + G(s-t) = P\big(G(s), G(t)\big) \qquad \text{for Lebesgue-a.e. } (s,t) \in \mathbb{R}^2, \tag{26}$$

a cosine-type addition formula with polynomial P.

We apply Lemma A1 to (26). Its hypotheses are now verified: G is measurable and locally bounded on U (Substep 2a), 0 is a Lebesgue point of G of value 0 ((MC3)), the right-hand side is polynomial and satisfies the boundary identity $P(0, v) = 2v$ (Substep 2b), and (26) holds a.e. on a neighborhood of $(0, 0) \in \mathbb{R}^2$ by hypothesis (MC5). In the normal form (A6), the affine inner maps $(x + y)/2$, $(x - y)/2$, and y have nonzero derivative with respect to the auxiliary variable, so the rank condition in Járai's Corollary 8.7 is satisfied. Hence G has a representative that is continuous on a neighborhood $U_* \subseteq \frac{1}{2}U$ of 0. Its value at $t = 0$ is forced to be 0 by the approximate-continuity output of Substep 2d-i: the representative agrees with G a.e. near 0; hence, its essential range on $(-\rho, \rho)$ collapses to a single point as $\rho \to 0$, and that point is 0 by Substep 2d-i.

(2e) Continuity on all of $\mathbb{R}$. For arbitrary $t \in \mathbb{R}$, choose $k \in \mathbb{N}$ with $2^{-k}t \in U_*$. By Substep 2d-ii, G is continuous at $2^{-k}t$. The forward diagonal recursion

$$G(t) = Q\big(Q(\cdots Q(G(2^{-k}t))\cdots)\big) \quad (k \text{ iterations})$$

expresses $G(t)$ as a polynomial in a continuously varying argument, so G is continuous at t. Non-constancy excludes the trivial branch $G \equiv 0$.

Step 3: Extending the equation to $\mathbb{R}^2$. Once G has a continuous representative, both sides of $G(s+t) + G(s-t) = P(G(s), G(t))$ are continuous in (s,t), and since they agree a.e., they agree everywhere on $\mathbb{R}^2$. Similarly, $G(t) \geq 0$ extends from a.e. t to all $t \in \mathbb{R}$ by continuity (hence, (MC1) gives $G \geq 0$ pointwise). Coercivity ((MC4)) likewise upgrades from essential limits to genuine limits: $G(t) \to +\infty$ as $|t| \to \infty$.

Step 4: Degree bound and classification. Hypotheses (DB1)–(DB3) of Theorem 4 are now verified: G is even with $G(0) = 0$ (reciprocal symmetry + unit normalisation); $G(t) \to +\infty$ as $|t| \to \infty$ and $G > 0$ off 0 (Step 3 combined with the diagonal recursion $G(2^k t) = Q^{\circ k}(G(t))$ and $Q(0) = 0$); and the polynomial identities $P(u,0) = 2u$, $P(0,v) = 2v$, $P(u,v) = P(v,u)$, $P \geq 0$ on $\mathbb{R}^2_{\geq 0}$ follow from Substep 2b and the polynomial-combiner hypothesis (MC5). Theorem 4 therefore yields $\deg P \leq 2$.

Algebraic identification of P. The corollary statement of Corollary 3 formally assumes a continuous admissible cost J (i.e., G is strictly convex); at this point in the proof, we have not yet established admissibility, so we cannot quote Corollary 3 directly. Instead we rerun its purely algebraic step here, which uses only $\deg P \leq 2$, symmetry, the boundary conditions $P(u,0) = 2u$ and $P(0,v) = 2v$, and non-negativity $P \geq 0$ on $\mathbb{R}^2_{\geq 0}$ (all of which have already been verified above). Writing the general degree ≤ 2 symmetric polynomial as $P(u,v) = \alpha + \beta(u+v) + \gamma(u^2+v^2) + c\,uv$ and inserting $P(u,0) = \alpha + \beta u + \gamma u^2 = 2u$ for all $u \geq 0$ forces $\alpha = 0$, $\beta = 2$, $\gamma = 0$ (by comparison of coefficients on a Zariski-dense set), so $P(u,v) = 2u + 2v + c\,uv$. If $c < 0$, then $P(u,u) = 4u + cu^2 \to -\infty$ as $u \to \infty$, contradicting $P \geq 0$ on $\mathbb{R}^2_{\geq 0}$; hence, $c \geq 0$.

Identification of G and admissibility. From the polynomial combiner $P(u,v) = cuv + 2u + 2v$, we obtain the functional equation

$$G(s+t) + G(s-t) \;=\; c\,G(s)G(t) + 2G(s) + 2G(t) \qquad (s,t \in \mathbb{R}) \tag{27}$$

for the continuous function $G\colon \mathbb{R} \to \mathbb{R}$ established in Step 3. We do not invoke Proposition 6 or Proposition 7 at this stage, since both are stated for admissible J. Instead we run the substitution directly.

Case $c > 0$. Set $H(t) := \frac{c}{2}G(t) + 1$, which is continuous and even (since G is) with $H(0) = 1$. A direct substitution of $G = \frac{2}{c}(H-1)$ into (27) yields d'Alembert's equation

$$H(s+t) + H(s-t) \;=\; 2H(s)H(t) \qquad (s,t \in \mathbb{R});$$

this is a purely algebraic rearrangement and does not use admissibility. By Theorem 7 (the Kannappan–Aczél classification of continuous even solutions with $H(0) = 1$), H is one of $H \equiv 1$, $\cosh(\lambda t)$ ($\lambda > 0$), or $\cos(\lambda t)$ ($\lambda > 0$). Coercivity of G forces H to be coercive, which excludes $H \equiv 1$ and $H(t) = \cos(\lambda t)$ (both are bounded). Hence $H(t) = \cosh(\lambda t)$ for some $\lambda > 0$, and

$$G(t) \;=\; \tfrac{2}{c}\big(\cosh(\lambda t) - 1\big).$$

This function is strictly convex on $\mathbb{R}$ (its second derivative $\frac{2\lambda^2}{c}\cosh(\lambda t)$ is strictly positive), so $J(x) = G(\ln x)$ has strictly convex G and is therefore admissible.

Case $c = 0$. Equation (27) reduces to the Jensen equation $G(s+t) + G(s-t) = 2G(s) + 2G(t)$. By the classical Jensen classification ([1] Ch. 13), its continuous solutions with $G(0) = 0$ are quadratic: $G(t) = at^2$ for some $a \in \mathbb{R}$. Coercivity of G forces $a > 0$, so G is strictly convex and J is admissible.

Smoothness. In both cases J is a continuous admissible cost satisfying the bilinear combiner law with $c \geq 0$; Proposition 7 therefore applies (its hypotheses are now fully verified) and gives $J \in C^\omega(\mathbb{R}_{>0})$. The two explicit forms above are exactly the hyperbolic family (for $c > 0$) and the degenerate quadratic family (for $c = 0$) of Theorem 8. □

Remark 10 (Sharpness of the hypotheses)**.** *The boundedness condition (MC2) is the standard Steinhaus regularity condition for Cauchy-type functional equations and cannot be dropped without admitting pathological solutions constructed via Hamel bases [1,42]. The additional hypotheses (MC1)–(MC4) jointly correspond to the natural notion of a coercive non-negative "cost" that is continuous at its minimiser, and each of them rules out a class of pathologies that includes a concrete counter-example, as recorded in Remark 9. We do* not *claim that each of (MC1)–(MC4) is individually necessary in isolation: the counter-examples in Remark 9 each violate more than one of these hypotheses simultaneously, and isolated sharpness (a counter-example violating only one of (MC1)–(MC4) while satisfying the other two and (MC2)) is left as an open question.*

6.2. Analytic Rigidity: A Finite-Order Constraint

Theorem 6 (Analytic rigidity in the finite-order regime)**.** *Let J be an admissible cost whose log-substitution $G = J \circ \exp$ extends to an entire function of finite positive order $\rho \in (0, \infty)$, and suppose the diagonal combiner $Q(u) := P(u, u)$ also extends to an entire function. If Q is a polynomial of degree d', then*

$$d' \geq 2^\rho. \tag{28}$$

The bound is attained with equality in both classified families (hyperbolic: $d' = 2$, $\rho = 1$; degenerate quadratic: $d' = 1$, $\rho = 0$, handled by Remark 11). If in addition $P \in \mathbb{R}[u, v]$ is a polynomial, then Corollary 3 gives $P(u, v) = cuv + 2u + 2v$ with $c \geq 0$ independently of the finite-order hypothesis, and J lies in the classified list of Theorem 8.

Remark 11 (Polynomial-G degenerate case)**.** *In the polynomial case $\rho = 0$ ($G(t) = at^2$ for some $a > 0$, corresponding to the degenerate quadratic family $J(x) = a(\ln x)^2$), the hypothesis $\rho \in (0, \infty)$ of Theorem 6 is not satisfied, but its conclusions hold by direct verification: $Q(u) = P(u, u) = 4u$ is a polynomial of degree $d' = 1 = 2^0$, and the bound $d' \geq 2^\rho$ is attained with equality. Thus the C^ω discussion covers both $\rho = 0$ (by this direct check) and $\rho \in (0, \infty)$ (by Theorem 6).*

Proof. Let $M_f(r) := \max_{|z|=r} |f(z)|$ denote the maximum modulus. Since G is entire of order ρ, the definition of order gives

$$\log M_G(r) \leq r^{\rho+\varepsilon} \quad \text{for every fixed } \varepsilon > 0 \text{ and all } r \geq r_0(\varepsilon). \tag{29}$$

In particular, since $G(r) \leq M_G(r)$ for real $r \geq 0$,

$$\log G(r) \leq r^{\rho+\varepsilon} \qquad (r \geq r_0(\varepsilon)). \tag{30}$$

Step 1: Maximum modulus comparison. By the identity principle applied to the entire functions $Q \circ G$ and $G(2\,\cdot)$, which coincide on $\mathbb{R}$, the equation $Q \circ G = G(2\,\cdot)$ extends to all of $\mathbb{C}$. The maximum modulus inequality therefore gives

$$M_G(2r) = \max_{|z|=r} |Q(G(z))| \leq M_Q(M_G(r)) \qquad (r \geq 0). \tag{31}$$

Step 2: Degree lower bound when Q is polynomial. Assume Q is a polynomial and let $d' := \deg Q$. For a polynomial Q of degree d', $\log M_Q(R) = d' \log R + O(1)$ as $R \to \infty$. Combining this with (31) yields

$$\log M_G(2r) \leq d' \log M_G(r) + O(1). \tag{32}$$

Iterating k times from a fixed base point $r_1 > 0$ and absorbing the bounded contribution of $M_G(r_1)$ into the constant,

$$\log M_G(2^k r_1) \leq C\,(d')^k, \qquad k \geq 0.$$

For arbitrary $r \geq r_1$, write $r = 2^k r_0$ with $k := \lfloor \log_2(r/r_1) \rfloor$ and $r_0 \in [r_1, 2r_1)$; then

$$\log M_G(r) \leq C\,(d')^k = C\,(d')^{\log_2(r/r_1)+O(1)} \leq C'\, r^{\log_2 d'},$$

so $\log\log M_G(r) \leq (\log_2 d') \log r + O(1)$. By the definition of order, $\rho = \limsup_{r\to\infty} \log \log M_G(r) / \log r$; therefore

$$\rho \leq \log_2 d' \qquad \text{i.e.,} \qquad d' \geq 2^\rho. \tag{33}$$

The reverse inequality is not needed for the remainder of the proof (or for any other result in this paper): when P itself is polynomial, the C^0 degree bound (Theorem 4, via Corollary 3) yields $\deg P \leq 2$ regardless of the precise value of d'. We therefore content ourselves with (33) and note that equality $d' = 2^\rho$ holds in each classified case (hyperbolic: $d' = 2$, $\rho = 1$; degenerate quadratic: $d' = 1$, $\rho = 0$), providing the consistency check anticipated in the statement of the theorem.

If P itself is polynomial, the bilinear form follows separately from Corollary 3; this does not use the finite-order hypothesis and is not part of the analytic-rigidity argument. □

Remark 12 (On the form of the proof)**.** *The proof uses the maximum modulus inequality on $\mathbb{C}$ ((31)) together with the definition of order of an entire function to bound the degree of Q in the case that Q is a polynomial. The separate passage to* $\deg P \leq 2$ *recorded at the end of the proof uses only that P is polynomial (so that Theorem 4 applies), and is independent of the finite-order hypothesis.*

Remark 13 (Completing the linear-growth-but-not-affine regime)**.** *Theorem 3 rules out superlinear growth of $P_v(u, 0)$, leaving open the regime in which Ψ has linear but non-affine growth. Theorem 6 rules this regime out under the additional hypothesis that G is entire of finite positive order. The remaining open case (G real-analytic on $\mathbb{R}$ but either not entire on $\mathbb{C}$ or entire of infinite order) is discussed as a future direction in the conclusion.*

Remark 14 (Scope of the finite-order hypothesis)**.** *The finite-order hypothesis on G in Theorem 6 is a hypothesis, not a derivation. Both classified families satisfy it: the hyperbolic family has G entire of order $\rho = 1$, and the degenerate quadratic family has G entire of order $\rho = 0$ (in fact, G is a polynomial). The theorem therefore covers the classified list. It does* not *cover admissible costs for which G is real-analytic on $\mathbb{R}$, but either fails to extend to an entire function on $\mathbb{C}$, or extends to an* *entire function of infinite order; these cases remain open and are discussed as a future direction in the conclusion.*

7. Classification of Admissible Costs with Polynomial Combiner

We now combine the bilinear form $P(u,v) = cuv + 2u + 2v$ from Corollary 3 with the Kannappan–Aczél classification of d'Alembert's equation to obtain Theorem 8.

We use two prior/classical facts in the proof.

Proposition 6 ([30], Lem. 8–9)**.** *For continuous admissible J with $G := J \circ \exp$ and $c \geq 0$, the bilinear law $J(xy) + J(x/y) = cJ(x)J(y) + 2J(x) + 2J(y)$ is equivalent to: for $c > 0$, $H := \frac{c}{2}G + 1$ is a continuous even solution of d'Alembert's equation $H(s+t) + H(s-t) = 2H(s)H(t)$ with $H(0) = 1$; for $c = 0$, G satisfies the Jensen equation $G(s+t) + G(s-t) = 2G(s) + 2G(t)$.*

Proof. Substitute $G = \frac{2}{c}(H-1)$ into the bilinear law and collect; $c = 0$ is direct. See ([30] Lem. 8–9). □

Theorem 7 (Kannappan–Aczél)**.** *The continuous even solutions of $H(s+t) + H(s-t) = 2H(s)H(t)$ with $H(0) = 1$ are exactly $H \equiv 1$, $H(t) = \cosh(\lambda t)$ ($\lambda > 0$), and $H(t) = \cos(\lambda t)$ ($\lambda > 0$).*

Proof. Classical; see ([2] §3.1, [1] Thm. 13.4.4). □

7.1. Smoothness Bootstrap

Proposition 7 (Smoothness bootstrap)**.** *Let J be a continuous admissible cost satisfying $J(xy) + J(x/y) = cJ(x)J(y) + 2J(x) + 2J(y)$ for some $c \geq 0$. Then $J \in C^{\omega}(\mathbb{R}_{>0})$.*

Proof. For $c > 0$, the substitution $H(t) := \frac{c}{2}G(t) + 1$ converts the bilinear law into the d'Alembert equation $H(s+t) + H(s-t) = 2H(s)H(t)$ with H continuous, even, and $H(0) = 1$ (Proposition 6). Using the Kannappan–Aczél classification in the precise forms ([2] §3.1) and ([1] Thm. 13.4.4), the only continuous even solutions with $H(0) = 1$ are

$$H \equiv 1, \qquad H(t) = \cosh(\lambda t), \qquad H(t) = \cos(\lambda t) \qquad (\lambda > 0),$$

each of which is real-analytic on $\mathbb{R}$ (the constant function trivially, and cosh, cos as entire functions of t). Hence H, and therefore $G = \frac{2}{c}(H-1)$ and $J(x) = G(\ln x)$, is real-analytic. (No circularity arises: Theorem 7 is proved by external citation to the Kannappan–Aczél literature and does not invoke Proposition 7 or its consequences.)

For $c = 0$, the bilinear law degenerates to the Jensen equation $G(s+t) + G(s-t) = 2G(s) + 2G(t)$, whose continuous solutions with $G(0) = 0$ are quadratic ($G(t) = at^2$ for some $a \in \mathbb{R}$; classical, see, e.g., [1] Ch. 13); these are real-analytic, so $J(x) = a(\ln x)^2$ is real-analytic on $\mathbb{R}_{>0}$. □

Remark 15 (Position in the logical chain)**.** *The proof above is independent of Theorem 8, so the smoothness clause of Corollary 3 no longer relies on the explicit two-family classification. An equivalent alternative proof would be: by Theorem 8 below, J is either $\frac{1}{c}(x^{\lambda} + x^{-\lambda}) - \frac{2}{c}$ (for $c > 0$) or $a(\ln x)^2$ (for $c = 0$), and both are real-analytic; we record this as a sanity check rather than as the primary route.*

Proposition 8 ([30], Cor. 8)**.** *If J is a continuous admissible cost with bilinear combiner $P(u,v) = cuv + 2u + 2v$, then $c \geq 0$: the constant and cosine branches of Theorem 7 are excluded by strict log-convexity (see [30], pp. 10–11).*

Theorem 8 (Classification of continuous admissible costs with polynomial combiner, [30])**.** *Let $J : \mathbb{R}_{>0} \to \mathbb{R}$ be a continuous admissible cost with polynomial combiner. Then $P(u,v) = cuv + 2u + 2v$ with $c \geq 0$, and exactly one of the following holds:*

(i) *Hyperbolic family: there exist $c > 0$ and $\lambda > 0$ with*

$$J(x) = \tfrac{1}{c}\left(x^{\lambda} + x^{-\lambda}\right) - \tfrac{2}{c}.$$

(ii) *Degenerate quadratic family: there exists $a > 0$ with $J(x) = a(\ln x)^2$.*

The combiner-value identity $P(1,1) = c + 4$ in case (i) and $P(1,1) = 4$ in case (ii) is recorded in Section 9, where it is used to fix c via the calibration $P(1,1) = 6$.

Proof. By Corollary 3, $P(u,v) = cuv + 2u + 2v$ with $c \geq 0$. For $c > 0$, Proposition 6 and Theorem 7 give $H = \frac{c}{2}G + 1 \in \{1, \cosh(\lambda\cdot), \cos(\lambda\cdot)\}$; the constant branch (forcing $G \equiv 0$) and the cosine branch (forcing $G < 0$ somewhere) are excluded by strict log-convexity, leaving $H = \cosh(\lambda t)$ and hence $J(x) = \frac{1}{c}(x^{\lambda} + x^{-\lambda}) - \frac{2}{c}$. For $c = 0$, the functional equation reduces to Jensen's quadratic equation $G(s+t) + G(s-t) = 2G(s) + 2G(t)$, whose continuous solutions with $G(0) = 0$ are $G(t) = at^2$ ([1] Ch. 13); strict convexity gives $a > 0$. □

7.2. The Parameter $c \in [0,\infty)$ and the Inönü–Wigner Boundary

Theorem 9 (Compactified parameter space)**.** *The rescaling group generated by the amplitude rescaling $J \mapsto \alpha J$ ($\alpha > 0$) and the log-axis rescaling $J(x) \mapsto J(x^{\lambda'})$ ($\lambda' > 0$) acts transitively on the hyperbolic 2-parameter family $\{J_{c,\lambda} : (c,\lambda) \in (0,\infty)^2\}$ of Theorem 8 by $(c,\lambda) \mapsto (c/\alpha, \lambda\lambda')$ (see the proof). Fixing any curvature calibration $A_1 := \frac{1}{2}G''(0) = A_* \in (0,\infty)$ (e.g., $A_* = 1/2$, giving the canonical normalisation $G''(0) = 1$ of Section 9) cuts the hyperbolic family down to the one-parameter cross-section $\{J_{c,\lambda(c)} : c \in (0,\infty)\}$ with $\lambda(c) := \sqrt{cA_*}$, which extends continuously to the (calibrated) degenerate quadratic family at the boundary $c = 0$ via*

$$\lim_{\substack{\lambda \to 0^+ \\ \lambda^2/c \to a}} J_{c,\lambda}(x) = a(\ln x)^2 \qquad \text{uniformly on compact subsets of } \mathbb{R}_{>0},$$

where $a = A_*$ along the calibrated cross-section. The calibrated parameter space is therefore the half-closed line $c \in [0,\infty)$.

Proof. Action of the rescalings on (c,λ). For $\alpha > 0$, direct computation gives $\alpha J_{c,\lambda} = J_{c/\alpha,\lambda}$, so the amplitude rescaling acts by $(c,\lambda) \mapsto (c/\alpha,\lambda)$. For $\lambda' > 0$, $J_{c,\lambda}(x^{\lambda'}) = \frac{1}{c}(x^{\lambda\lambda'} + x^{-\lambda\lambda'}) - \frac{2}{c} = J_{c,\lambda\lambda'}(x)$, so the log-axis rescaling acts by $(c,\lambda) \mapsto (c,\lambda\lambda')$. The combined action $(c,\lambda) \mapsto (c/\alpha, \lambda\lambda')$ is transitive on $(0,\infty)^2$ (given any source and target, set $\alpha = c_{\mathrm{src}}/c_{\mathrm{tgt}}$ and $\lambda' = \lambda_{\mathrm{tgt}}/\lambda_{\mathrm{src}}$). The curvature $A_1 = \lambda^2/c$ transforms as $A_1 \mapsto A_1 \cdot \alpha(\lambda')^2$, so the constraint $A_1 = A_*$ cuts out the one-parameter cross-section $\lambda = \sqrt{cA_*}$, parametrised by $c \in (0,\infty)$.

Contraction limit. Expanding $x^{\lambda} + x^{-\lambda} - 2 = \lambda^2(\ln x)^2 + O(\lambda^4)$ gives $J_{c,\lambda}(x) = \frac{\lambda^2}{c}(\ln x)^2 + O(\lambda^4/c)$; the hypothesis $\lambda^2/c \to a$ identifies the leading term as $a(\ln x)^2$, while the error $\lambda^4/c = (\lambda^2/c)\cdot\lambda^2 \to a \cdot 0 = 0$. Uniform convergence on compact subsets of $\mathbb{R}_{>0}$ follows from the uniform convergence of $\lambda\,(\ln x) \to 0$ on compact sets. Along the calibrated cross-section $\lambda = \sqrt{cA_*}$, this limit is realised at $a = A_*$ as $c \to 0^+$. □

Remark 16 (Inönü–Wigner contraction)**.** *The limit $\lambda \to 0^+$, $\lambda^2/c \to a$ is an Inönü–Wigner contraction [25] of the hyperbolic symmetric pair onto the Euclidean pair $(\mathbb{R}, \{e\})$; the spherical function picture of Section 8 makes this manifest.*

8. Spherical Function Interpretation and the *n*-Variable Compound

The substitution $H(t) := \frac{c}{2}G(t) + 1$ converts the bilinear combiner equation into the standard d'Alembert form $H(s+t) + H(s-t) = 2H(s)H(t)$. Two consequences follow. First, an interpretation: the classified $c > 0$ family $\cosh(\lambda t)$ is the positive zonal spherical function of the hyperbolic line H^1 [19,21], and the $c = 0$ family $1 + at^2$ is its Inönü–Wigner contraction onto the Euclidean pair $(\mathbb{R}, \{e\})$ [25]. Second, an identity: the n-variable compound $S_n(x_1, \ldots, x_n)$ admits the closed form $\frac{2^n}{c}\left(\prod_k h(x_k) - 1\right)$, where $h(x) := \frac{c}{2}J(x) + 1$ (Proposition 10 and its iterated form). The interpretation is a corollary of the classification; the identity is the genuinely new structural content of this section.

Remark 17 (Spherical function identification)**.** *Theorem 8, combined with Proposition 6 and Theorems 7 and 9, identifies the two classified families via the substitution* $H := \frac{c}{2}G + 1$*: for* $c > 0$*, the hyperbolic family becomes* $H(t) = \cosh(\lambda t)$*, a positive zonal spherical function of the hyperbolic line* H^1 *[19,21]; for* $c = 0$*, the substitution* $H := 1 + at^2$ *realises the degenerate quadratic family as the Inönü–Wigner contracted kernel on the Euclidean pair* $(\mathbb{R}, \{e\})$ *[25]—the* $\lambda \to 0^+$*,* $\lambda^2/c \to a$ *limit of* $\cosh(\lambda t)$*, satisfying Jensen's quadratic equation rather than the strict d'Alembert equation.*

8.1. Closed Form for the n-Variable Compound

For $c > 0$, the substitution $H := \frac{c}{2}G + 1$ factors the bilinear combiner into a shifted product of H-values, which extends to a closed form for every n-variable compound.

Proposition 9 (Shifted product representation)**.** *For* $c > 0$*, with* $H_u := \frac{c}{2}u + 1$ *and* $H_v := \frac{c}{2}v + 1$*, the combiner factors as*

$$P(u,v) = \tfrac{4}{c}(H_u H_v - 1), \tag{34}$$

by direct expansion of the right-hand side: $\frac{4}{c}\left(\frac{c^2}{4}uv + \frac{c}{2}u + \frac{c}{2}v\right) = cuv + 2u + 2v = P(u,v)$*.*

The coalgebraic interpretation is separated from the closed-form compound identities and recorded after their proof below.

In the limit $c \to 0^+$ with $\lambda^2/c \to a$ (Theorem 9), the substitution h trivialises to $h \equiv 1$ and the multiplicative identity reduces to $2 = 2$, while the underlying J equation degenerates to Jensen's quadratic equation $G(s+t) + G(s-t) = 2G(s) + 2G(t)$ for $G = J \circ \exp$. Thus the multiplicative form is genuinely a $c > 0$ feature; the $c = 0$ family realises the additive (Jensen) form.

8.2. Higher Compounds: Closed Form for Three Variables

We record a closed-form expression for the three-variable compound, derived directly from the multiplicative form $h(xy) + h(x/y) = 2h(x)h(y)$ obtained by setting $h(x) := \frac{c}{2}J(x) + 1$ in the bilinear law (Proposition 6). For $x, y, z \in \mathbb{R}_{>0}$, define

$$S_3(x,y,z) := \sum_{\varepsilon_2, \varepsilon_3 \in \{\pm 1\}} J\big(x\, y^{\varepsilon_2} z^{\varepsilon_3}\big) = J(xyz) + J(xy/z) + J(xz/y) + J\big(x/(yz)\big). \tag{35}$$

This is the $n = 3$ analogue of the two-variable compound $S_2(x,y) := J(xy) + J(x/y)$ in the defining functional equation.

Proposition 10 (Closed form of the three-variable compound)**.** *Let* J *be a continuous admissible cost with polynomial combiner* $P(u,v) = cuv + 2u + 2v$*,* $c > 0$*, and set* $h(x) := \frac{c}{2}J(x) + 1$*. Then*

$$S_3(x,y,z) \;=\; \tfrac{8}{c}\,\big(h(x)\,h(y)\,h(z) - 1\big). \tag{36}$$

Equivalently, $\frac{c}{8}S_3(x,y,z)+1=h(x)h(y)h(z)$: in the h variable, the compound is the triple tensor product.

Proof. Let $s := \log x$, $t := \log y$, $u := \log z$ and write $G := J \circ \exp$, $H(t) := h(e^t) = \frac{c}{2}G(t)+1$. From (35),

$$S_3(x,y,z) = G(s+t+u) + G(s+t-u) + G(s-t+u) + G(s-t-u).$$

Substitute $G = \frac{2}{c}(H-1)$:

$$S_3 = \frac{2}{c}\Big[H(s+t+u) + H(s+t-u) + H(s-t+u) + H(s-t-u) - 4\Big]. \tag{37}$$

Apply d'Alembert's equation $H(\alpha+\beta)+H(\alpha-\beta)=2H(\alpha)H(\beta)$ (Proposition 6) to the first pair with $\alpha = s+t$ and $\beta = u$, and to the second pair with $\alpha = s-t$ and $\beta = u$:

$$H(s+t+u)+H(s+t-u) = 2H(s+t)H(u), \quad H(s-t+u)+H(s-t-u) = 2H(s-t)H(u).$$

Summing and applying d'Alembert once more with $\alpha = s$, $\beta = t$:

$$2H(u)\big(H(s+t)+H(s-t)\big) = 2H(u)\cdot 2H(s)H(t) = 4H(s)H(t)H(u).$$

Substituting into (37) gives $S_3 = \frac{2}{c}[4H(s)H(t)H(u)-4] = \frac{8}{c}(h(x)h(y)h(z)-1)$. □

Remark 18 (Coalgebraic origin of Proposition 10)**.** *The identity (36) is the shadow, on the level of costs, of the group-like property $\Delta H = H\otimes H$ (Remark 19) iterated twice to give $(\Delta\otimes \mathrm{id})\Delta H = H\otimes H\otimes H$. Equivalently: the four-term sum (35) has no manifest multiplicative structure in the J variable (indeed, S_3 is a degree-two polynomial in J with cross-terms), yet after the single affine change $J\mapsto h$, it becomes the tensor product of three copies of h, up to an additive constant. This is the consequence of the multiplicative form of Proposition 9; Proposition 10 is its first non-trivial instance.*

Iterating the same argument gives, for every $n\geq 2$, the n-variable compound

$$S_n(x_1,\ldots,x_n) := \sum_{\varepsilon_2,\ldots,\varepsilon_n\in\{\pm 1\}} J(x_1\, x_2^{\varepsilon_2}\cdots x_n^{\varepsilon_n}) = \frac{2^n}{c}\left(\prod_{k=1}^{n} h(x_k) - 1\right),$$

a C^ω multilinear form in the h variables; the induction applies d'Alembert's identity successively, one variable at a time, exactly as in the step $n = 2\to 3$ above.

The following interpretation is not used in the proof of Proposition 10 or in the induction formula; it is included only to record the algebraic meaning of the shifted product.

Remark 19 (Coalgebraic interpretation)**.** *The d'Alembert equation $H(s+t)+H(s-t) = 2H(s)H(t)$ is the defining property of a group-like element (coproduct $\Delta H = H\otimes H$) in the sense of Sweedler ([43] §3), and Proposition 9 realises P as a pullback of this coproduct along the evaluation map; the $c=0$ contracted coalgebra is spanned by $1+at^2$. We do not invoke deeper coalgebra structure (unit, counit, antipode) and use the language only for compact statements.*

Example 1 (Three-variable compound on the canonical cost)**.** *For the canonical cost $J(x) = \frac{1}{2}(x+x^{-1})-1$ of Section 9 (with $c=2$, $\lambda=1$), $h_\star(x) = J(x)+1 = \frac{1}{2}(x+x^{-1}) = \cosh(\ln x)$, and*

$$S_3(x,y,z) = 4\big(h_\star(x)h_\star(y)h_\star(z)-1\big) = \tfrac{1}{2}(x+x^{-1})(y+y^{-1})(z+z^{-1})-4.$$

Expanding the product gives

$$S_3(x,y,z) = \frac{1}{2} \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3 \in \{\pm 1\}} x^{\varepsilon_1} y^{\varepsilon_2} z^{\varepsilon_3} - 4,$$

which agrees with the direct expansion of $J(xyz) + J(xy/z) + J(xz/y) + J(x/(yz))$*, confirming* (36).

9. The Canonical Cost $J_{\rm cost}$

The classification Theorem 8 gives a two-parameter hyperbolic family $\{J_{c,\lambda}\}$ and a one-parameter degenerate family $\{a(\ln x)^2\}$. Two normalisations single out a canonical representative: $P(1,1) = 6$ selects the hyperbolic family at $c = 2$ (since $P(1,1) = c + 4$ in the hyperbolic family and $P(1,1) = 4$ in the degenerate one), and $G''(0) = 1$ then fixes $\lambda = 1$. Equivalently, the curvature calibration $\kappa(J) := \lim_{t \to 0} 2J(e^t)/t^2 = 1$ with the log-axis scaling $\alpha = 1$ of ([30] Thm. 10, Cor. 10).

Corollary 4 (Canonical cost; specialisation of Theorem 8 at $c = 2$, [29,30])**.** *If J is an admissible cost satisfying the bilinear law with $c = 2$ and the calibration $(J \circ \exp)''(0) = 1$, then $J(x) = J_{\rm cost}(x) := \frac{1}{2}(x + x^{-1}) - 1$.*

Proof. Theorem 8 at $c = 2$ gives $J(x) = \frac{1}{2}(x^{\lambda} + x^{-\lambda}) - 1$ for some $\lambda > 0$, and the calibration $G''(0) = \lambda^2 = 1$ fixes $\lambda = 1$ ([30] Cor. 10; [29]). □

Remark 20 (Verification: $J_{\rm cost}$ realises every hypothesis)**.** $J_{\rm cost}(x) = \frac{1}{2}(x + x^{-1}) - 1$ *satisfies all hypotheses of the paper: reciprocal symmetry and $J_{\rm cost}(1) = 0$; continuity and strict log-convexity from $G(t) = \cosh t - 1$, $G''(t) = \cosh t > 0$; the bilinear law (from* $\cosh(s \pm t)$*); the calibration $G''(0) = 1$; and the entire-finite-order hypothesis of Theorem 6, since* $\cosh t$ *is entire of order* 1 *([27] Ch. 2). Equivalently $J_{\rm cost}(x) = \frac{1}{2}(x^{1/2} - x^{-1/2})^2 \geq 0$, with equality iff $x = 1$. In particular, every hypothesis of Definition 1 and Theorems 6 and 8 is non-vacuous.*

Remark 21 (Polynomiality of P is a genuine restriction)**.** *For $m \geq 2$, $J(x) = (\ln x)^{2m}$ is admissible, but its combiner $P(a,b) = 2\sum_{k=0}^{m} \binom{2m}{2k} a^{k/m} b^{(m-k)/m}$ is polynomial in $(a^{1/m}, b^{1/m})$, not in (a,b) (cf. Section 3).*

10. Conclusions

We have shown that for every continuous reciprocally symmetric cost J with strictly convex log-substitution G, the symmetric compound $J(xy) + J(x/y)$ is automatically a continuous function of $(J(x), J(y))$ alone (Theorem 1), and that when the resulting combiner P is polynomial, it must be bilinear of the form $P(u,v) = cuv + 2u + 2v$ with $c \geq 0$ (Theorem 4 and Corollary 3); the admissible costs with the bilinear combiner are exhausted by two explicit families parametrised by $c \in [0, \infty)$, connected by the Inönü–Wigner contraction $c \to 0$ (Theorems 8 and 9). Two regularity extensions persist: Lebesgue-measurable costs satisfying explicit regularity hypotheses (MC1)–(MC5) admit a continuous representative and join the classification (Theorem 5, based on Lemma A1); in the entire-finite-order regime, the diagonal combiner satisfies the sharp degree bound $d' \geq 2^{\rho}$ when Q is polynomial of degree d' (Theorem 6).

The smoothness machinery of Section 4.1 (Lemma 1), the Taylor recursion (Proposition 4) and the C^2 blow-up obstruction (Theorem 3) provide an independent route to the bilinear form whenever the boundary function $\Psi(u) := P_v(u,0)$ is affine (Corollary 1). The two normalisations $P(1,1) = 6$ and $G''(0) = 1$ single out the canonical representative $J_{\rm cost}(x) = \frac{1}{2}(x + x^{-1}) - 1$, equivalently the spherical function $H(t) = \cosh t$ on the hyperbolic line H^1 under $H = G + 1$.

Future directions.

Three open questions remain. (i) Analytic rigidity beyond finite order. Theorem 6 requires G entire of finite positive order; whether every real-analytic admissible cost with real-analytic combiner already lies in the classified list is open. (ii) Characterisation of admissible boundary derivatives. Theorem 2 determines P formally from its boundary derivative Ψ. We conjecture that every $\Psi = P_v(\cdot, 0)$ arising from an admissible cost is affine, supported by: (a) every classified cost has affine Ψ; (b) under C^2 regularity, superlinear Ψ is excluded (Theorem 3); (c) in the polynomial-combiner and entire-finite-order regimes, Ψ is forced to be affine (Theorems 4 and 6). The remaining "linear-growth gap" (Remark 6) would be closed by an obstruction to non-affine $O(u)$ residuals such as $\varepsilon \sin u$. Combined with Corollary 1, a positive answer would resolve (i) as a corollary. (iii) Multi-variable compositionality. The n-variable compound $S_n(x_1, \ldots, x_n)$ admits a unique continuous combiner P_n for every n. In the classified list, P_n is polynomial for all n (Proposition 10 and the iterated form). Whether polynomiality of P_n for some $n \geq 2$ forces J into the classified list is open; the present paper neither proves nor excludes a non-classified admissible cost with non-polynomial P_2 but polynomial P_3.

Beyond the polynomial combiner.

The polynomiality of P is the single hypothesis of the classification that is not of a regularity type, and it is worth recording what is known, and what a relaxation could look like, beyond it. First, the combiner of Theorem 1 exists for every pre-admissible cost, polynomial or not, so the classification problem "which continuous symmetric P with $P(u, 0) = 2u$ arises as combiners of admissible costs?" is well-posed in full generality; the present paper answers it exactly on the polynomial subclass. Second, natural non-polynomial combiners do occur: the admissible costs $J(x) = (\ln x)^{2m}$, $m \geq 2$, have combiners that are polynomial in the fractional powers $(u^{1/m}, v^{1/m})$ but not in (u, v) (Remarks 3 and 21), and the pre-admissible boundary case $J(x) = |\ln x|$ has the piecewise-linear combiner $P(u, v) = 2 \max(u, v)$. These examples suggest a graded extension of the classification, from polynomial to algebraic combiners (those satisfying a polynomial identity $W(u, v, P) = 0$), for which the two-recursion growth method of Section 5 may still apply: both recursions use only the asymptotic growth of $u \mapsto P(u, a)$ and $u \mapsto P(u, u)$, not polynomiality itself, so any combiner class with well-defined growth exponents is a candidate for the same incompatibility argument. Third, on the smoothness side, the boundary-derivative ODE of Section 4 makes no polynomiality assumption at all; the conjecture in (ii) above—every admissible boundary derivative is affine—is precisely the statement that the classified list is complete in the maximal generality of C^2 costs, with no combiner hypothesis whatsoever. We regard the algebraic-combiner extension and the linear-growth gap as the two most tractable next steps.

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Appendix A. Technical Ingredients for the Measurable Classification

This appendix collects the two measure-theoretic tools used in the proof of Theorem 5: the contraction principle (A) at the attracting fixed point and the polynomial measurable-to-continuous regularity lemma (Lemma A1). Both are stated for a general measurable function F and are independent of the cost function context; they are quoted in Section 6 as black boxes. The closing Remark A1 delimits precisely what each tool delivers.

(A) Contraction at the attracting fixed point. Let L be real-analytic near 0 with $L(0) = 0$ and $|L'(0)| = \theta_0 < 1$. If F is measurable and locally bounded near 0, 0 is a Lebesgue point of F of value 0, and $F(t) = L(F(2t))$ a.e. on a measurable set $T \subseteq \mathbb{R}$ of full Lebesgue density at 0, then F has approximate limit 0 at 0: for every $\sigma > 0$, the level set $\{t : |F(t)| > \sigma\}$ has Lebesgue density 0 at $t = 0$. This is the measurable form of the analytic Koenigs linearisation [10,12]; we provide a self-contained proof.

Proof of (A). Fix $\theta \in (\theta_0, 1)$ and choose $\delta > 0$ small enough that

$$|L(u)| \leq \theta|u| \qquad \text{for all } |u| \leq \delta, \tag{A1}$$

which is possible because $L(0) = 0$ and $L'(0) = \pm\theta_0$ imply $L(u) = \theta_0 u + O(u^2)$ near 0; shrinking the neighborhood further if necessary, L maps $[-\delta, \delta]$ into itself.

Step 1 (iteration on a density-1 set). For each $k \geq 1$ set $T_k := T \cap (T/2) \cap \cdots \cap (T/2^{k-1})$. Each factor $T/2^j$ has Lebesgue density 1 at 0 by scale invariance of density (the map $t \mapsto 2^j t$ preserves Lebesgue density at the fixed point 0), and a finite intersection of sets of density 1 at 0 again has density 1 at 0. Hence T_k has full density at 0 for every k. Applying the functional equation k times along orbits of the doubling map,

$$F(t) \;=\; L^{\circ k}\big(F(2^k t)\big) \qquad \text{for a.e. } t \in T_k. \tag{A2}$$

Step 2 (Lebesgue-point density control of $F(2^k t)$). By the Lebesgue-point hypothesis, $\frac{1}{2\rho}\int_{-\rho}^{\rho} |F| \to 0$ as $\rho \to 0^+$, so Markov's inequality gives

$$\frac{|\{t \in (-\rho, \rho) : |F(t)| > \delta\}|}{2\rho} \;\leq\; \frac{1}{2\rho\delta}\int_{-\rho}^{\rho} |F| \;\longrightarrow\; 0 \qquad (\rho \to 0).$$

Equivalently, $D_\delta := \{t : |F(t)| \leq \delta\}$ has density 1 at 0. By scale invariance of density at the fixed point 0, the rescaled set $2^{-k}D_\delta = \{t : |F(2^k t)| \leq \delta\}$ also has density 1 at 0 for every k. Define

$$T_k^\star \;:=\; T_k \cap 2^{-k}D_\delta \;=\; \big\{t \in T_k : |F(2^k t)| \leq \delta\big\},$$

which is a finite intersection of density-1 sets and therefore itself has density 1 at 0.

Step 3 (contractive bound on $T_k^\star$). For $t \in T_k^\star$, Equation (A2) gives $F(t) = L^{\circ k}(F(2^k t))$ with $|F(2^k t)| \leq \delta$. By the contraction (A1) and the fact that $[-\delta, \delta]$ is L-invariant, iteration yields $|L^{\circ k}(u)| \leq \theta^k |u| \leq \theta^k \delta$ for all $|u| \leq \delta$; thus

$$|F(t)| \;\leq\; \theta^k \delta \qquad \text{for a.e. } t \in T_k^\star. \tag{A3}$$

Step 4 (approximate continuity). Fix $\sigma > 0$ and choose $k = k(\sigma) \geq 1$ with $\theta^k \delta < \sigma$. By (A3) (a.e. on $T_k^\star$), $\{|F| > \sigma\} \cap T_k^\star$ has measure 0, so $\{|F| > \sigma\} \subseteq (T_k^\star)^c$ up to a null set. Since $T_k^\star$ has Lebesgue density 1 at 0, its complement has Lebesgue density 0 at 0, and the same therefore holds of $\{|F| > \sigma\}$. As $\sigma > 0$ is arbitrary, F has approximate limit 0 at 0, which is the asserted conclusion. □

The conclusion of (A) is the approximate continuity at 0 with value 0; topological continuity on a full neighbourhood of 0 is not obtained from (A) alone (it is not in fact implied by full density of T), and is supplied by (B) below. See also Remark A1 after Lemma A1.

(B) Addition formula propagation. The bare halving identity is not enough to upgrade measurability to continuity on a punctured neighborhood of 0: the linear Schröder equation $\varphi(2t) = 4\varphi(t)$ admits the mosaic family $\varphi(t) = t^2\omega(\log_2|t|)$ with ω any measurable 1-periodic bounded function. Propagation requires the two-variable addition formula

$$F(s+t) + F(s-t) = \mathcal{P}\big(F(s), F(t)\big) \qquad \text{a.e. on a neighborhood of } (0,0) \in \mathbb{R}^2, \tag{A4}$$

with $\mathcal{P}$ continuous (in our application, polynomial): under (i) F measurable and locally bounded near 0, (ii) 0 a Lebesgue point of F of value 0, and (A4), F has a representative continuous on a neighborhood of 0. For the bilinear case $\mathcal{P}(u,v) = 2uv$, this is the classical d'Alembert measurable-to-continuous regularity theorem (Aczél ([2] §3.1), Aczél–Dhombres ([1] Ch. 13)); the polynomial case used here is the content of Lemma A1 below, recorded as a special case of Járai's general framework for measurable solutions of non-composite functional equations ([14] Cor. 8.7).

Lemma A1 (Polynomial measurable-to-continuous regularity)**.** *Let $F : \mathbb{R} \to \mathbb{R}$ be Lebesgue-measurable and locally bounded near 0, with 0 a Lebesgue point of F of value 0, and let $\mathcal{P}(u,v) = \sum_{i,j\geq 0} \mathcal{P}_{ij} u^i v^j$ be a real polynomial with $\mathcal{P}(u,0) = 2u$ and $\mathcal{P}(0,v) = 2v$. Suppose*

$$F(s+t) + F(s-t) = \mathcal{P}(F(s), F(t)) \qquad \textit{for Lebesgue-a.e. } (s,t) \in \mathbb{R}^2. \tag{A5}$$

Then F admits a representative continuous on a neighborhood of 0.

Proof. We give the reduction to Járai's Corollary 8.7 ([14] Cor. 8.7) and verify each of its hypotheses explicitly.

Step 1: Reduction to non-composite normal form. Fix a small open interval $I \ni 0$ on which F is essentially bounded. Apply the bijective linear change of variables

$$(s,t) \longmapsto (x,y) = (s+t, s-t), \qquad s = \tfrac{x+y}{2}, \qquad t = \tfrac{x-y}{2},$$

which has constant Jacobian $|\det| = 2$, so null sets in (s,t) correspond to null sets in (x,y). Equation (A5) becomes, for Lebesgue-a.e. $(x,y) \in \mathbb{R}^2$,

$$F(x) = \mathcal{P}\Big(F\Big(\tfrac{x+y}{2}\Big), F\Big(\tfrac{x-y}{2}\Big)\Big) - F(y). \tag{A6}$$

This is Járai's non-composite form with one "outer" variable x, one "auxiliary" variable y, three inner maps

$$g_1(x,y) = \tfrac{x+y}{2}, \qquad g_2(x,y) = \tfrac{x-y}{2}, \qquad g_3(x,y) = y,$$

and outer function $h(x,y,z_1,z_2,z_3) = \mathcal{P}(z_1,z_2) - z_3$.

Step 2: Verification of Járai's hypotheses. The hypotheses of ([14] Cor. 8.7) (with notation as in [14] Ch. 8) require:

(J1) Smoothness of inner maps. Each $g_i \in C^1(I \times I, \mathbb{R})$. This is immediate: the g_i are affine, hence C^∞.

(J2) Rank/transversality. For every $x \in I$ and a.e. $y \in I$, the partial derivative $\partial g_i/\partial y$ is nonzero for each i: $\partial_y g_1 = \frac{1}{2}$, $\partial_y g_2 = -\frac{1}{2}$, $\partial_y g_3 = 1$. Equivalently, each map $y \mapsto g_i(x,y)$ is a C^1 diffeomorphism of I onto its image with bounded Jacobian. This is exactly the one-dimensional form of the rank condition $\operatorname{rank}(\partial_y g_i) = \dim(\text{range}) = 1$ required in ([14] Ch. 8, hyp. (R)).

(J3) Continuity of the outer function. $h(x, y, z_1, z_2, z_3) = \mathcal{P}(z_1, z_2) - z_3$ is a polynomial in (z_1, z_2, z_3) (and constant in (x, y)), hence jointly continuous on $\mathbb{R}^5$.
(J4) Measurability and local essential boundedness of F. F is Lebesgue-measurable by hypothesis and locally essentially bounded near 0 by the standing assumption "F locally bounded near 0". In particular, by Lusin's theorem, for every $\varepsilon > 0$, there is a compact set $K_\varepsilon \subset I$ with $|I \setminus K_\varepsilon| < \varepsilon$ on which F is continuous and bounded.
(J5) A.e. validity of the equation. Equation (A6) holds for a.e. $(x, y) \in I \times I$ by the Jacobian computation above.

Step 3: Transversal preservation under the inner maps. The substantive condition in ([14] Cor. 8.7) is that positive-measure sets remain of positive measure after pull-back by the inner maps. Concretely, given $\varepsilon > 0$ small and $K = K_\varepsilon$ from (J4), for each fixed $x \in I$ near 0, the set

$$E_x := \{y \in I : g_i(x, y) \in K \text{ for } i = 1, 2, 3\} = \{y \in I : y \in K\} \cap \{y \in I : \tfrac{x \pm y}{2} \in K\}$$

satisfies $|E_x| \to |I|$ as $\varepsilon \to 0$, uniformly in x in a neighborhood of 0. This is because each affine map $y \mapsto g_i(x, y)$ has bounded C^1 inverse with Jacobian in $\{\frac{1}{2}, 2, 1\}$, so $|\{y \in I : g_i(x, y) \notin K\}| \le 2|I \setminus K| = 2\varepsilon$ for each i; summing over $i \in \{1, 2, 3\}$ gives $|I \setminus E_x| \le 6\varepsilon$. Combined with the a.e. validity from (J5), for any $x \in I$ near 0, the set

$$\widetilde{E}_x := E_x \cap \{y \in I : \text{(A6) holds at } (x, y)\}$$

has positive measure (and full density at $x = 0$ by Fubini), so fixing any $y \in \widetilde{E}_x$ at which all three of $g_1(x, y), g_2(x, y), g_3(x, y)$ lie in K expresses $F(x)$ as a continuous function of x via $F(x) = h(x, y, F(g_1(x, y)), F(g_2(x, y)), F(g_3(x, y)))$, with the inner-map values varying continuously in x along the trajectory $(g_1(x, y_0), g_2(x, y_0), g_3(x, y_0))$ for fixed $y_0 \in \widetilde{E}_0$.

Step 4: Conclusion. The conditions (J1)–(J5) together with the transversal preservation of Step 3 are exactly the hypotheses of ([14] Cor. 8.7). That corollary then provides a representative of F that is continuous on a neighborhood of 0. (The argument is standard: choose $y_0 \in \widetilde{E}_0$ with F continuous at each of $g_i(0, y_0)$, $i = 1, 2, 3$; then $x \mapsto h(x, y_0, F(g_1(x, y_0)), F(g_2(x, y_0)), F(g_3(x, y_0)))$ is continuous in x on a neighborhood of 0 and equals $F(x)$ at every Lebesgue point at which the equation holds with this y_0, hence at almost every x near 0; this continuous function is the desired representative.) □

Remark A1 ((A) versus (B): approximate vs. topological continuity)**.** *The contraction principle (A) above yields approximate continuity of F at the fixed point 0 only, i.e., Lebesgue density 0 of every super-σ level set $\{|F| > \sigma\}$ at $t = 0$. This is sufficient to identify the value of any* topologically *continuous representative of F at 0, but does not by itself construct such a representative on a neighborhood: density zero of level sets does not entail their measure zero in any open interval (the same obstruction is illustrated by the mosaic family $\varphi(t) = t^2\omega(\log_2 |t|)$ recalled in (B) above). The topological upgrade to continuity on a full neighborhood of 0 requires the two-variable rigidity provided by (B), in the precise form of Lemma A1. In the classification proof, both ingredients are used in tandem: (A) fixes the value of the continuous representative at 0 and (B) extends it to a neighborhood.*

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