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# The D'Alembert Inevitability Theorem

Jonathan Washburn <sup>1</sup>, Milan Zlatanović <sup>2,\*</sup> and Elshad Allahyarov <sup>1,3,4,5</sup>

<sup>1</sup> Recognition Physics Institute, Austin, TX 78701, USA; jon@recognitionphysics.org (J.W.); elshad.allakhyarov@case.edu (E.A.)

<sup>2</sup> Faculty of Science and Mathematics, University of Niš, 18000 Niš, Serbia

<sup>3</sup> Institut für Theoretische Physik II: Weiche Materie, Heinrich-Heine-Universität Düsseldorf, 40225 Düsseldorf, Germany

<sup>4</sup> Theoretical Department, Joint Institute for High Temperatures, Russian Academy of Sciences, Moscow 125412, Russia

<sup>5</sup> Department of Physics, Case Western Reserve University, Cleveland, OH 44106-7202, USA

\* Correspondence: zlatmilan@yahoo.com

## Abstract

We study functions satisfying the composition law  $F(xy) + F(x/y) = P(F(x), F(y))$  with a symmetric polynomial combiner  $P$ . We prove that symmetry together with a quadratic degree bound on  $P$  forces a composition law of d'Alembert type. We establish a degree mismatch exclusion criterion showing that symmetric polynomial combiners with  $\deg P(u, v) \geq 3$  do not admit nonconstant continuous solutions, provided the leading term does not cancel (Theorem 1). For continuous nonconstant functions  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  with  $F(1) = 0$  satisfying the composition law with a symmetric polynomial  $P$  of degree at most two, the combiner is necessarily of the form  $P(u, v) = 2u + 2v + cuv$ ,  $c \in \mathbb{R}$  (Theorem 3). The equation reduces in logarithmic coordinates to the classical d'Alembert functional equation. For  $c \neq 0$ , one obtains hyperbolic or trigonometric branches, while  $c = 0$  yields the squared-logarithm family. Under the cost-function assumptions  $F \geq 0$  and convexity, only the hyperbolic branch with  $c > 0$  remains. A unit log-curvature calibration selects the canonical value  $c = 2$ , which yields the canonical reciprocal cost  $F(x) = \frac{1}{2}(x + x^{-1}) - 1$ . For  $c \neq 0$ , the result extends to  $\mathbb{R}_{>0}^n$ : every solution depends only on a single linear combination of coordinate logarithms; for  $c = 0$ , the solution is a general quadratic form  $\sum_{i,j} a_{ij} \ln x_i \ln x_j$ . In either case, nontrivial coordinate-wise separable costs are excluded.

**Keywords:** d'Alembert functional equation; polynomial composition law; reciprocal cost; curvature calibration; separability; rigidity

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## 1. Introduction

Functional equations often arise when one requires that a quantity associated with ratios behaves consistently under multiplicative composition. Such consistency principles appear naturally in many contexts, including the theory of functional equations [1–5], information geometric theory involving multiplicative models, and models of ratio-based costs.

The classical d'Alembert functional equation

$$H(t + u) + H(t - u) = 2H(t)H(u)$$

is one of the central equations in the theory of functional equations. The origin goes back to d'Alembert's derivation of the parallelogram law of forces [6]. Poisson [7] gave a rigorous

treatment of solutions, and Picard [8] studied its relation with non-Euclidean geometry. Its continuous solutions are the cosine-type functions  $\cosh(\alpha t)$  and  $\cos(\alpha t)$  (see [1,5,9]). Many nonlinear functional relations reduce to this equation after suitable transformations.

In [10], a rigidity result for  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  is obtained. Assuming the polynomial composition law

$$F(xy) + F\left(\frac{x}{y}\right) = 2F(x)F(y) + 2F(x) + 2F(y), \tag{1}$$

together with the curvature calibration

$$\lim_{t \rightarrow 0} \frac{2F(e^t)}{t^2} = 1,$$

the function  $F$  is uniquely determined. The unique solution is the canonical reciprocal cost

$$F(x) = \frac{1}{2}(x + x^{-1}) - 1.$$

This raises a natural structural question. Is the composition law (1) merely a modeling assumption, as it appears in various contexts in the literature (see, e.g., [9,11]), or is it forced by more general consistency requirements?

In this paper, we study functional relations of the form

$$F(xy) + F\left(\frac{x}{y}\right) = P(F(x), F(y)), \quad x, y > 0, \tag{2}$$

where  $P$  is a polynomial combiner. We investigate which polynomial laws admit nontrivial continuous solutions.

In applications, one often interprets  $F(x)$  as a cost or penalty associated with a ratio  $x > 0$ . The normalization  $F(1) = 0$  reflects that the identity element carries zero deviation. It does not restrict generality, since any solution can be reduced to this case by subtracting a constant, with a corresponding translation of the variables in  $P$ . More precisely, if  $\tilde{F}(x) := F(x) - F(1)$ , then  $\tilde{F}(1) = 0$  and

$$\tilde{F}(xy) + \tilde{F}\left(\frac{x}{y}\right) = \tilde{P}(\tilde{F}(x), \tilde{F}(y)), \quad \tilde{P}(u, v) := P(u + F(1), v + F(1)) - 2F(1).$$

Thus, the assumption  $F(1) = 0$  is without loss of generality. This condition determines the boundary identities

$$P(u, 0) = 2u, \quad P(0, v) = 2v,$$

which restrict the form of the polynomial combiner. Requiring compatibility with multiplicative composition then leads to a d'Alembert-type functional equation on  $\mathbb{R}_{>0}$ .

We assume that  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  is continuous and nonconstant, and that  $P \in \mathbb{R}[u, v]$  is a symmetric polynomial, where  $\mathbb{R}[u, v]$  denotes the ring of polynomials in two variables with real coefficients. Symmetry is natural since the roles of  $x$  and  $y$  in the left-hand side of (2) are interchangeable. Under symmetry of  $P$ , we derive reciprocity  $F(x) = F(1/x)$ .

Our first result treats the case of higher-degree polynomial combiners. We show that symmetric polynomial combiners of degree  $d \geq 3$  are incompatible with the functional Equation (2). More precisely, if  $P \in \mathbb{R}[u, v]$  is symmetric, satisfies  $P(0, v) = 2v$ , and its leading term does not cancel on the diagonal, then Equation (2) admits no continuous nonconstant solution  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  with  $F(1) = 0$ . Consequently, only polynomial combiners of degree at most two can admit nontrivial continuous solutions. This reduction to the quadratic case is the main structural step of the paper.

Our main structural result shows that in the quadratic case the composition law is completely determined. If  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  is continuous and nonconstant and the combiner  $P$  is a symmetric polynomial of degree at most two, then necessarily

$$P(u, v) = 2u + 2v + c uv, \quad c \in \mathbb{R}.$$

Under the normalization  $F(1) = 0$ , the functional equation therefore reduces to

$$F(xy) + F\left(\frac{x}{y}\right) = 2F(x) + 2F(y) + c F(x)F(y).$$

Passing to logarithmic coordinates reduces this relation to the classical d’Alembert equation

$$H(t + u) + H(t - u) = 2H(t)H(u),$$

whose continuous solutions are well known [1,5,9,11–14]. All continuous solutions of the original equation can therefore be described explicitly.

Convexity and nonnegativity select the hyperbolic branch (Corollary 8), while a curvature normalization determines the distinguished value  $c = 2$ . In this case, the canonical reciprocal cost

$$F(x) = \frac{1}{2}(x + x^{-1}) - 1$$

appears as a structurally determined solution [10].

Finally, we extend the analysis to functions on  $\mathbb{R}_{>0}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0\}$ . In the multidimensional case, passing to logarithmic coordinates reduces the problem to a functional equation on  $\mathbb{R}^n$  involving sums and differences. Such equations are known to exhibit a collapse to one-dimensional dependence, as in the classical case (see, e.g., [9,11]). We show that, for  $c \neq 0$ , solutions depend on  $\mathbf{x}$  only through the scalar quantity  $\boldsymbol{\alpha} \cdot \ln \mathbf{x}$ , where  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}_{>0}^n$  denotes a vector variable, and  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$  is a vector of weights. Thus, the effective dependence remains one-dimensional.

The paper is organized as follows. In Section 2, we study structural consequences of the polynomial composition law. In particular, we prove reciprocity of  $F$  under symmetry of the combiner and derive the boundary identities that restrict the polynomial  $P$ .

Section 3 contains the classification of admissible polynomial combiners. We show that symmetric combiners of degree at least three do not admit nonconstant continuous solutions under a natural non-cancellation assumption. In the quadratic case, we obtain the bilinear form of the combiner. In Section Reduction to Classical D’Alembert, we pass to logarithmic coordinates and reduce the equation to the classical d’Alembert functional equation. Using the known classification of its continuous solutions, we obtain the corresponding families of functions  $F$ .

In Section 4, we consider the multidimensional case. We show that for  $c \neq 0$ , every solution depends only on the scalar quantity  $\boldsymbol{\alpha} \cdot \ln \mathbf{x}$ . Thus, even in dimension  $n$ , the effective dependence is one-dimensional through the quantity  $\boldsymbol{\alpha} \cdot \ln \mathbf{x}$ . We give an explicit 16-dimensional example (Example 2) illustrating the collapse to a single logarithmic direction.

In Section 5, we introduce a normalization based on the log-curvature  $\kappa(F)$ . We show that this calibration fixes the parameter of the bilinear composition law. For convex nonnegative solutions with  $\kappa(F) = 1$ , the parameter is uniquely determined by  $c = 2$ , which yields the canonical reciprocal cost.

**Main contributions of the paper.** The main results of the paper are the following:

- (i) We show that symmetric polynomial combiners of degree  $d \geq 3$  do not admit nonconstant continuous solutions under a natural non-cancellation assumption.

- (ii) We prove that in the case  $\deg P \leq 2$ , the polynomial combiner is of bilinear d’Alembert type, namely  $P(u, v) = 2u + 2v + c uv$ .
- (iii) We reduce the functional equation to the classical d’Alembert equation and give a complete classification: all continuous solutions form three families: the hyperbolic branch  $F(e^t) = \frac{2}{c}(\cosh(\alpha t) - 1)$ , the trigonometric branch  $F(e^t) = \frac{2}{c}(\cos(\alpha t) - 1)$  (for  $c \neq 0$ ), and the quadratic-logarithm family  $F(x) = k(\ln x)^2$  (for  $c = 0$ ).
- (iv) We extend the result to the  $n$ -dimensional case and show that every solution depends only on a single linear combination of the logarithmic variables.

The main difficulty lies in the analysis of the equation with a general polynomial combiner, where a degree-based argument and a non-cancellation condition are used to exclude higher-degree cases.

## 2. Preliminaries

We consider a function  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ . Passing to logarithmic coordinates, we define

$$G(t) := F(e^t), \quad t \in \mathbb{R}. \tag{3}$$

**Definition 1.** A function  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  is said to satisfy a polynomial composition law if there exists a polynomial  $P \in \mathbb{R}[u, v]$  such that

$$F(xy) + F\left(\frac{x}{y}\right) = P(F(x), F(y)) \quad \text{for all } x, y > 0. \tag{4}$$

Such a polynomial  $P$  is called a combiner.

In the following, we assume that (4) holds with a symmetric polynomial  $P \in \mathbb{R}[u, v]$ . In logarithmic coordinates  $x = e^t, y = e^u$ , Equation (4) becomes

$$G(t + u) + G(t - u) = P(G(t), G(u)), \quad t, u \in \mathbb{R}. \tag{5}$$

Note that (4) determines  $P$  only on the subset

$$\{(F(x), F(y)) : x, y \in \mathbb{R}_{>0}\} \subseteq \mathbb{R}^2,$$

that is, on  $\text{Range}(F) \times \text{Range}(F)$ .

We assume that  $F$  is continuous on  $\mathbb{R}_{>0}$  and nonconstant. Then  $\text{Range}(F)$  contains a nontrivial interval. Consequently, if a polynomial  $P(u, v)$  vanishes on  $\text{Range}(F) \times \text{Range}(F)$ , then  $P$  vanishes identically on  $\mathbb{R}^2$ .

**Remark 1.** Continuity is used to ensure that  $\text{Range}(F)$  contains a nondegenerate interval. All results, such as reciprocity of  $F$ , the boundary conditions, and the bilinear form of  $P$ , remain valid if  $\text{Range}(F)$  is dense in some interval. In particular, continuity can be replaced by measurability together with local boundedness, since measurable solutions of the d’Alembert equation are continuous [9,13,15].

### Structural Properties

We now derive structural consequences of the polynomial composition law (4). We establish the equivalence between symmetry of  $P$  and reciprocity of  $F$ . The normalization  $F(1) = 0$  implies that the boundary values of  $P$  satisfy  $P(0, v) = 2v$  and  $P(u, 0) = 2u$ . The symmetry assumption  $P(u, v) = P(v, u)$  is not arbitrary. It is the algebraic counterpart of reciprocity.

As shown in the next lemma, symmetry of the combiner implies reciprocal symmetry of  $F$ . This expresses invariance under inversion of the ratio, a natural structural property for a discrepancy measure.

**Lemma 1.** *If  $F$  satisfies Equation (4) with a symmetric combiner  $P$ , then*

$$F(z) = F\left(\frac{1}{z}\right) \quad \text{for all } z > 0.$$

**Proof.** Writing Equation (4) for  $(x, y)$  and  $(y, x)$  gives

$$F(xy) + F\left(\frac{x}{y}\right) = P(F(x), F(y)),$$

$$F(yx) + F\left(\frac{y}{x}\right) = P(F(y), F(x)).$$

Since  $xy = yx$  and  $P$  is symmetric, subtracting yields

$$F\left(\frac{x}{y}\right) = F\left(\frac{y}{x}\right).$$

If we set  $y = 1$  in the last equation, we obtain  $F(x) = F(1/x)$ , for all  $x > 0$ .  $\square$

**Lemma 2.** *Assume  $F$  satisfies Equation (4) where  $P$  is a polynomial. If  $F$  is continuous, nonconstant, and reciprocal-symmetric, then  $P$  is symmetric.*

**Proof.** Using reciprocity, we obtain

$$P(F(x), F(y)) = P(F(y), F(x)) \quad \text{for all } x, y > 0.$$

Let  $Q(u, v) = P(u, v) - P(v, u)$ . Then  $Q(F(x), F(y)) = 0$ . Since  $\text{Range}(F)$  contains an interval,  $Q$  vanishes on an open subset of  $\mathbb{R}^2$ , hence  $Q \equiv 0$ .  $\square$

**Remark 2.** *Under the polynomial assumption and nondegeneracy of the range (e.g.,  $F$  continuous and nonconstant), symmetry of  $P$  and reciprocity of  $F$  are equivalent.*

With reciprocity established, the next lemma determines the boundary values of  $P$  at  $F(1)$ .

**Lemma 3.** *Let  $F$  be continuous and nonconstant satisfying Equation (4) with symmetric  $P$ . Then*

$$P(F(1), v) = 2v, \quad P(u, F(1)) = 2u \quad \text{for all } u, v \in \mathbb{R}.$$

**Proof.** Setting  $x = 1$  in Equation (4) gives

$$F(y) + F\left(\frac{1}{y}\right) = P(F(1), F(y)).$$

By reciprocity,  $F(y) = F(1/y)$ , hence

$$2F(y) = P(F(1), F(y)).$$

Let us define the polynomial  $q(v) = P(F(1), v) - 2v$ . Then  $q(F(y)) = 0$ . Since the range of  $F$  contains an interval,  $q \equiv 0$ . Symmetry gives the second identity.  $\square$

**Corollary 1.** Under the assumptions of Lemma 3 and the additional condition  $F(1) = 0$ , it follows that

$$P(u, 0) = 2u, \quad P(0, v) = 2v.$$

We now factor out the constraints at  $F(1)$ .

**Lemma 4.** Let  $F$  satisfy Equation (4) with polynomial symmetric  $P$ . Assume  $F$  is continuous and nonconstant. Then there exists  $R \in \mathbb{R}[u, v]$  such that

$$P(u, v) = 2u + 2v - 2F(1) + (u - F(1))(v - F(1)) R(u, v). \tag{6}$$

**Proof.** Let us define polynomial

$$S(u, v) = P(u, v) - 2u - 2v + 2F(1).$$

By Lemma 3, we have

$$S(F(1), v) = 0, \quad S(u, F(1)) = 0.$$

Fix  $v$ . Since  $S(F(1), v) = 0$ , it follows that  $(u - F(1))$  divides  $S(u, v)$ . Hence, there exists a polynomial  $T \in \mathbb{R}[u, v]$  such that

$$S(u, v) = (u - F(1)) T(u, v).$$

Now evaluate  $S(u, v)$  at  $v = F(1)$ . We obtain

$$0 = S(u, F(1)) = (u - F(1)) T(u, F(1)) \quad \text{for all } u.$$

Since this identity holds for all  $u$ , it follows that

$$T(u, F(1)) = 0 \quad \text{for all } u.$$

Thus,  $(v - F(1))$  divides  $T(u, v)$ . Therefore, there exists  $R \in \mathbb{R}[u, v]$  such that

$$T(u, v) = (v - F(1)) R(u, v).$$

Combining the two factorizations yields

$$S(u, v) = (u - F(1))(v - F(1)) R(u, v),$$

which proves Equation (6).  $\square$

**Corollary 2.** Under the assumptions of Lemma 4 and the additional condition  $F(1) = 0$ , there exists a polynomial  $R \in \mathbb{R}[u, v]$  such that

$$P(u, v) = 2u + 2v + uv R(u, v) \quad \text{for all } u, v \in \mathbb{R}.$$

Since  $P$  is symmetric, it follows that  $R$  is also symmetric.

### 3. Polynomial Classification

In this section, we classify the possible polynomial combiners  $P$ .

Using the factorization (6) from Lemma 4, together with the symmetry of  $P$  and the boundary conditions from Section Structural Properties, we conclude that the polynomial  $R$  is symmetric. Indeed, the term  $2u + 2v - 2F(1)$  is already symmetric in  $(u, v)$ , and therefore, the symmetry of  $P$  forces  $R$  in (6) to be symmetric as well.

At this point, the degree of  $R$  is not restricted. If  $R$  has high degree, then the functional equation becomes structurally more complex. Using (6), Equation (5) becomes

$$G(t + u) + G(t - u) = 2G(t) + 2G(u) - 2F(1) + (G(t) - F(1))(G(u) - F(1))R(G(t), G(u)).$$

Assume that  $G$  is smooth in a neighborhood of 0. Then both sides of the above equation admit Taylor expansions at  $(t, u) = (0, 0)$  in the form of convergent power series

$$G(t + u) + G(t - u) = \sum_{k,l \geq 0} A_{kl} t^k u^l,$$

$$2G(t) + 2G(u) - 2F(1) + (G(t) - F(1))(G(u) - F(1))R(G(t), G(u)) = \sum_{k,l \geq 0} B_{kl} t^k u^l.$$

By uniqueness of power series expansions, we obtain  $A_{kl} = B_{kl}$  for all  $k, l \geq 0$ . This yields an infinite system of algebraic relations between the derivatives of  $G$  at 0 and the coefficients of the polynomial  $R$ . Consequently, without imposing a bound on the degree of  $P$ , the classification problem leads to an infinite system of compatibility conditions.

For this reason, we restrict our attention to polynomial combiners of total degree at most two. Since  $(u - F(1))(v - F(1))$  already has degree two, it follows that  $R$  in (6) must be constant.

**Assumption 1.** *The combiner  $P$  has total degree at most two.*

**Theorem 1.** *Let  $P \in \mathbb{R}[u, v]$  be a symmetric polynomial of degree  $d \geq 3$  with  $P(0, v) = 2v$ , and let  $q(y) := P(y, y)$ . Assume that  $\deg_y q = d$ , where the index  $y$  indicates the variable with respect to which the degree is taken, and that no cancellation occurs in the leading term, so that*

$$\deg_y (P(P(q(y), y) - y, y)) = d^3 - 2d^2 + 2d.$$

*Then there is no continuous nonconstant function  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  with  $F(1) = 0$  satisfying the polynomial composition law*

$$F(xy) + F\left(\frac{x}{y}\right) = P(F(x), F(y)).$$

*In particular, the explicit degree-three combiner treated in Example 1 is excluded.*

**Proof.** Set  $G(t) := F(e^t)$ , so that  $G(0) = 0$  and the functional equation becomes

$$G(t + u) + G(t - u) = P(G(t), G(u)), \quad t, u \in \mathbb{R}. \tag{7}$$

Since  $P(0, v) = 2v$  and  $P$  is symmetric, the factored form  $P(u, v) = 2u + 2v + uvR(u, v)$  holds for some symmetric polynomial  $R$  of degree  $d - 2 \geq 1$ .

We observe that, for each integer  $n \geq 1$ , the quantity  $G(ns)$  can be expressed as a polynomial function of  $y = G(s)$ . Setting  $y = G(s)$  and  $q(y) := P(y, y)$ , the composition law (7) gives

$$G(2s) = P(G(s), G(s)) = q(G(s)) = q(y).$$

Setting  $t = 2s, u = s$  in (7), we obtain

$$G(3s) = P(G(2s), G(s)) - G(s) = P(q(y), y) - y.$$

Setting  $t = u = 2s$  in (7), we obtain

$$G(4s) = q(G(2s)) = q(q(y)).$$

Setting  $t = 3s, u = s$  in (7), we obtain the identity

$$G(4s) + G(2s) = P(G(3s), G(s)). \tag{8}$$

Proceeding inductively, each  $G(ns)$  is obtained from  $y$  by finitely many polynomial substitutions involving  $P$ . Therefore, for each fixed  $n$ ,  $G(ns)$  is a polynomial in  $y = G(s)$ .  $\square$

We will now analyze this identity as a polynomial relation in  $y = G(s)$ .

**Lemma 5.** *Let  $P \in \mathbb{R}[u, v]$  be a symmetric polynomial of degree  $d \geq 3$  with  $P(0, v) = 2v$ , and let  $q(y) := P(y, y)$ . Let  $G(3s)$  and  $G(4s)$  be the polynomials in  $y = G(s)$  obtained in the proof of Theorem 1. Then*

- (i)  $\deg_y G(4s) = (\deg_y q)^2$ .
- (ii)  $\deg_y(P(G(3s), G(s))) \leq d^3 - 2d^2 + 2d$ , with equality under the non-cancellation assumption of Theorem 1.
- (iii) If  $\deg_y q = d$  and equality holds in (ii), then the degree difference is  $d(d - 1)(d - 2) \geq 6$  for all  $d \geq 3$ .

**Proof.** (i) Since  $G(4s) = q(q(y))$ , and  $\deg_y q = d$ , we obtain  $\deg_y G(4s) = d^2$ .

(ii) From the identities above, we have

$$G(3s) = P(q(y), y) - y.$$

Write

$$P(u, v) = 2u + 2v + uv R(u, v), \quad \deg R(u, v) = d - 2.$$

The highest-degree contribution in  $P(q(y), y)$  comes from  $q(y) y R(q(y), y)$ . Since the degree of  $R$  in the variable  $u$  is at most  $d - 2$ , and  $\deg_y q = d$ , we have

$$\deg_y R(q(y), y) \leq (d - 2)d = d^2 - 2d.$$

Therefore

$$\deg_y(q(y) y R(q(y), y)) \leq d + 1 + (d^2 - 2d) = d^2 - d + 1.$$

Hence

$$\deg_y G(3s) \leq d^2 - d + 1.$$

Equality holds if the leading coefficient of  $G(3s)$  (as a polynomial in  $y$ ) is nonzero. This can be verified explicitly in the degree-three case.

Now consider the right-hand side of (8). Its terms are of the form

$$a_{ij} [G(3s)]^i y^j, \quad i + j \leq d, \quad i, j \geq 1.$$

The degree of such a term is

$$i \deg_y G(3s) + j.$$

Using the bound for  $\deg_y G(3s)$ , we obtain

$$\begin{aligned} \deg_y(P(G(3s), G(s))) &\leq \max_{1 \leq i \leq d-1} (i(d^2 - d + 1) + (d - i)) \\ &= \max_{1 \leq i \leq d-1} (i(d^2 - d) + d), \end{aligned}$$

and the maximum is achieved at  $i = d - 1$ . Hence

$$\deg_y(P(G(3s), G(s))) \leq d^3 - 2d^2 + 2d.$$

Under the non-cancellation assumption of Theorem 1, this is the degree of the right-hand side.

(iii) By parts (i) and (ii), in the equality case, the degree difference is

$$\begin{aligned} (d^3 - 2d^2 + 2d) - d^2 &= d^3 - 3d^2 + 2d \\ &= d(d - 1)(d - 2). \end{aligned}$$

Since  $d \geq 3$ , all three factors are positive, and therefore

$$d(d - 1)(d - 2) \geq 6.$$

□

**Final Step of the Proof of Theorem 1.** Since  $G$  is continuous and nonconstant, its range  $\text{Range}(G)$  contains a non-degenerate interval  $I \subset \mathbb{R}$ . From (8), we have

$$G(4s) + G(2s) = P(G(3s), G(s)).$$

By substituting  $y = G(s)$ , and using the previous part of the proof, the quantities  $G(2s)$ ,  $G(3s)$ , and  $G(4s)$  are polynomials in  $y$ . Therefore, the left-hand side and the right-hand side of (8) can be written as polynomial functions of  $y$ . More explicitly,

$$A(y) := q(q(y)) + q(y), \quad B(y) := P(P(q(y), y) - y, y)$$

belong to  $\mathbb{R}[y]$ , are independent of  $s$ , and (8) becomes

$$A(y) = B(y) \quad \text{for all } y \in \text{Range}(G).$$

Since  $\text{Range}(G)$  contains the interval  $I$ , it follows that

$$A(y) - B(y) = 0 \quad \text{for all } y \in I.$$

The polynomial  $A(y) - B(y)$  is a polynomial in one variable. A polynomial that vanishes on a non-degenerate interval must vanish identically. Therefore

$$A(y) \equiv B(y) \quad \text{on } \mathbb{R}.$$

However, by Lemma 5, the two sides have different degrees, which is impossible. Hence, no continuous nonconstant function  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  with  $F(1) = 0$  satisfies the polynomial composition law for a symmetric polynomial  $P$  of degree  $d \geq 3$ . □

In Theorem 1, we assume that no cancellation occurs in the leading term. We now make this assumption precise.

**Remark 3.** Let  $P \in \mathbb{R}[u, v]$  be symmetric of degree  $d \geq 3$  and let  $q(y) = P(y, y)$ . We assume that

$$\deg_y q = d$$

and

$$\deg_y(P(P(q(y), y) - y, y)) = d^3 - 2d^2 + 2d.$$

This ensures that the leading term is preserved under composition. Equivalently, there is no cancellation of the highest-degree contribution on the diagonal  $u = v$ . It guarantees that the right-hand side attains the maximal degree required for the degree mismatch argument.

**Remark 4.** The diagonal polynomial  $q(x) = P(x, x)$  may have degree  $k < d$  if the highest-degree terms of  $P$  vanish on  $u = v$ . For example,

$$P(u, v) = 2u + 2v + uv(u - v)^2$$

has degree 4, while  $q(x) = 4x$ .

For the combiner considered in Example 1, we have  $q(x) = 4x + 2x^3$ ; hence,  $\deg_y q = d = 3$ . Thus, the possible degeneration  $\deg_y q < d$  does not occur in the explicit case considered here.

The following corollary follows directly from Theorem 1 under the assumptions clarified above.

**Corollary 3.** Let  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  be continuous and nonconstant, and assume that

$$F(xy) + F\left(\frac{x}{y}\right) = P(F(x), F(y)),$$

where  $P \in \mathbb{R}[u, v]$  is symmetric and  $F(1) = 0$ . Assume the non-cancellation assumption of Theorem 1. Then  $\deg P(u, v) \leq 2$ . Consequently, by Theorem 3,

$$P(u, v) = 2u + 2v + c uv$$

is the unique polynomial composition law admitting nonconstant continuous solutions.

**Proof.** If  $\deg P \geq 3$ , Theorem 1 shows no continuous nonconstant  $F$  with  $F(1) = 0$  can exist. Hence,  $\deg P \leq 2$ . Then Theorem 3 implies  $P(u, v) = 2u + 2v + c uv$ .  $\square$

**Example 1.** Consider the polynomial

$$P(u, v) = 2u + 2v + u^2v + uv^2,$$

which has degree 3 and satisfies  $P(0, v) = 2v$ . Then

$$q(x) = P(x, x) = 4x + 2x^3.$$

Let  $y = G(s)$ . Using the identities derived in the proof of Theorem 1, we obtain

$$\begin{aligned} G(2s) &= 4y + 2y^3, \\ G(3s) &= 9y + 24y^3 + 18y^5 + 4y^7, \\ G(4s) &= 16y + 136y^3 + 192y^5 + 96y^7 + 16y^9. \end{aligned}$$

The identity

$$G(4s) + G(2s) = P(G(3s), G(s))$$

requires the equality of two polynomials in  $y$ . We have

$$\deg_y(G(4s) + G(2s)) = 9, \quad \deg_y(P(G(3s), G(s))) = 15.$$

Thus, the degrees do not match, so the identity cannot hold identically.

Under Assumption 1,  $P$  can be written in the general quadratic form

$$P(u, v) = a + bu + cv + d uv + e u^2 + f v^2, \quad a, b, c, d, e, f \in \mathbb{R}. \tag{9}$$

**Lemma 6.** *If  $P$  is symmetric, i.e.,  $P(u, v) = P(v, u)$ , then  $b = c$  and  $e = f$ . Consequently,*

$$P(u, v) = a + b(u + v) + d uv + e(u^2 + v^2). \tag{10}$$

**Proof.** The symmetry implies equality of coefficients after interchanging  $u$  and  $v$  in (9). Comparing the coefficients of  $u$  and  $v$  gives  $b = c$ , and comparing those of  $u^2$  and  $v^2$  gives  $e = f$ .  $\square$

We now determine the relations among the coefficients imposed by the functional equation.

**Theorem 2.** *Let  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  be continuous and nonconstant satisfying*

$$F(xy) + F\left(\frac{x}{y}\right) = P(F(x), F(y)) \quad \text{for all } x, y > 0,$$

where  $P$  is a symmetric quadratic polynomial of the form

$$P(u, v) = a + b(u + v) + c uv + e(u^2 + v^2).$$

Then  $e = 0$ , and  $b(2 - b) + ac = 0$ .

**Proof.** By Lemma 3, we have  $P(F(1), v) = 2v$  for all  $v \in \mathbb{R}$ . Substituting  $u = F(1)$  in the given form of  $P$ , we obtain

$$a + b(F(1) + v) + cF(1)v + e(F(1)^2 + v^2) = 2v.$$

Since this identity holds for all  $v \in \mathbb{R}$ , we have

$$e = 0, \quad a + bF(1) = 0, \quad b + cF(1) = 2.$$

Eliminating  $F(1)$  from the last two equations gives

$$b(2 - b) + ac = 0.$$

$\square$

We now consider the effect of the normalization at  $x = 1$ .

**Corollary 4.** *According to the assumptions of Theorem 2, if  $F(1) = 0$ , then*

$$a = 0, \quad b = 2,$$

and hence

$$P(u, v) = 2u + 2v + c uv.$$

**Proof.** By Theorem 2, from the relations  $b + cF(1) = 2$  and  $a + bF(1) = 0$ , substituting  $F(1) = 0$  gives  $b = 2$  and  $a = 0$ .  $\square$

Thus, in the case of  $F(1) = 0$ , the composition law reduces to

$$F(xy) + F\left(\frac{x}{y}\right) = 2F(x) + 2F(y) + c F(x)F(y), \quad x, y > 0. \tag{11}$$

This equation will be analyzed in Section Reduction to Classical D’Alembert, where we make explicit its connection with the classical d’Alembert functional equation.

**Theorem 3** (d’Alembert Inevitability Theorem). *Let  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  be continuous and nonconstant satisfying a polynomial composition law with a symmetric combiner of degree at most two. Then  $P$  must be of the form*

$$P(u, v) = a + b(u + v) + c uv$$

with  $b(2 - b) + ac = 0$ , where  $a, b, c \in \mathbb{R}$ . If, moreover,  $F(1) = 0$ , then

$$P(u, v) = 2u + 2v + c uv,$$

and  $F$  satisfies Equation (11).

**Proof.** By Lemma 6, the polynomial  $P$  has the form

$$P(u, v) = a + b(u + v) + d uv + e(u^2 + v^2).$$

By Theorem 2, we have  $e = 0$ ; hence

$$P(u, v) = a + b(u + v) + d uv.$$

Renaming  $d = c$ , we obtain  $P(u, v) = a + b(u + v) + c uv$ . The relation  $b(2 - b) + ac = 0$  follows directly from Theorem 2.

If  $F(1) = 0$ , Corollary 4 gives  $a = 0$  and  $b = 2$ ; hence

$$P(u, v) = 2u + 2v + c uv.$$

□

**Corollary 5.** *Let  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  be continuous and nonconstant. If two polynomials  $P, P' \in \mathbb{R}[u, v]$  satisfy*

$$F(xy) + F\left(\frac{x}{y}\right) = P(F(x), F(y)) = P'(F(x), F(y)) \quad x, y > 0,$$

then  $P = P'$ .

**Proof.** The identity implies  $P(u, v) = P'(u, v)$  for all  $(u, v) \in \text{Range}(F) \times \text{Range}(F)$ . Since  $F$  is continuous and nonconstant, its range contains a nondegenerate interval. Hence,  $P = P'$  on a set containing a rectangle in  $\mathbb{R}^2$ . Therefore, the polynomial  $P - P'$  is identically zero, and thus  $P \equiv P'$  on  $\mathbb{R}^2$ . □

**Remark 5.** *Let  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  be continuous and nonconstant, with  $F(1) = 0$ . If  $F$  satisfies (4) with a symmetric combiner  $P \in \mathbb{R}[u, v]$  of total degree at most one, then  $P(u, v) = 2(u + v)$ , and the composition law coincides with (11) at  $c = 0$ . Consequently, the degree-one case is not a separate family; it is included in Theorem 3 for  $c = 0$ . Thus, degree two is the minimal degree for which a free parameter appears (namely  $c$ ).*

If, in addition,  $F$  is convex, then  $x = 1$  is a global minimum of  $F$ . In this case, the normalization  $F(1) = 0$  corresponds to shifting the minimum to zero.

**Lemma 7.** *Let  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  be continuous and nonconstant, and suppose*

$$F(xy) + F\left(\frac{x}{y}\right) = P(F(x), F(y)), \quad x, y > 0,$$

where  $P \in \mathbb{R}[u, v]$  is symmetric. Assume, in addition, that  $F$  is convex. Then  $x = 1$  is a global minimum of  $F$ , i.e.,

$$F(1) \leq F(x) \quad \text{for all } x > 0.$$

**Proof.** Since  $P$  is symmetric and  $F$  is continuous and nonconstant, we have

$$F(x) = F\left(\frac{1}{x}\right), \quad x > 0.$$

Suppose there exists  $x_0 > 0$  such that  $F(x_0) < F(1)$ . Then, also,  $F(1/x_0) = F(x_0) < F(1)$ .

Since 1 lies between  $x_0$  and  $1/x_0$ , there exists  $\theta \in (0, 1)$  such that

$$1 = \theta x_0 + (1 - \theta) \frac{1}{x_0}.$$

By convexity,

$$F(1) \leq \theta F(x_0) + (1 - \theta) F(1/x_0) = F(x_0) < F(1),$$

a contradiction.  $\square$

**Remark 6.** For every real value of  $c$ , the bilinear Equation (11) reduces, to a d’Alembert Equation (13) (by using Lemma 9). The parameter  $c$  parametrizes the family but does not create new solution types.

*Reduction to Classical D’Alembert*

In this part, we show that the bilinear family (11) reduces, after a change of variables, to the classical d’Alembert equation.

**Lemma 8.** Assume (11) and define  $G$  by (3). Then for all  $t, u \in \mathbb{R}$ ,

$$G(t + u) + G(t - u) = 2G(t) + 2G(u) + c G(t)G(u). \tag{12}$$

**Proof.** Let  $x = e^t$  and  $y = e^u$  in (11). Using  $xy = e^{t+u}$  and  $x/y = e^{t-u}$  and  $G(t) = F(e^t)$  gives (12).  $\square$

**Lemma 9.** Assume (12) for some constant  $c \in \mathbb{R}$ .

(i) If  $c \neq 0$  and

$$H(t) := 1 + \frac{c}{2} G(t),$$

then  $H$  satisfies the classical d’Alembert equation

$$H(t + u) + H(t - u) = 2H(t)H(u). \tag{13}$$

(ii) If  $c = 0$ , then (12) reduces to

$$G(t + u) + G(t - u) = 2G(t) + 2G(u). \tag{14}$$

**Proof.** (i) If  $c \neq 0$ , substituting  $H(t)$  into (13) one obtains

$$H(t + u) + H(t - u) = 2 + \frac{c}{2} (G(t + u) + G(t - u)).$$

From (12), we have

$$2 + \frac{c}{2} (2G(t) + 2G(u) + c G(t)G(u)) = 2 + cG(t) + cG(u) + \frac{c^2}{2} G(t)G(u).$$

On the other hand, we have

$$\begin{aligned} 2H(t)H(u) &= 2\left(1 + \frac{c}{2}G(t)\right)\left(1 + \frac{c}{2}G(u)\right) \\ &= 2 + cG(t) + cG(u) + \frac{c^2}{2}G(t)G(u), \end{aligned}$$

so (13) holds.

(ii) If  $c = 0$ , (12) reduces directly to (14).

□

We now determine the solutions in both cases.

(i) Case  $c \neq 0$ . The function  $H(t)$  satisfies (13). If  $F$  is continuous, then  $H$  is continuous. Since  $F(x) = F(1/x)$ ,  $H$  is even and  $H(0) = 1$ .

Under standard regularity assumptions, (see [1,9,11,13,16]), all even solutions of (13) with  $H(0) = 1$  are

$$H(t) = \cosh(\alpha t) \quad \text{or} \quad H(t) = \cos(\alpha t),$$

for some  $\alpha \in \mathbb{R}$ . Equivalently,

$$H(t) = \frac{e^{\lambda t} + e^{-\lambda t}}{2},$$

with  $\lambda \in \mathbb{C}$ . Substituting this into the definition of  $F$ , we obtain

$$F(e^t) = G(t) = \frac{2}{c}(H(t) - 1).$$

(ii) If  $c = 0$ , then (12) reduces to

$$G(t + u) + G(t - u) = 2G(t) + 2G(u) \quad \text{for all } t, u \in \mathbb{R}. \tag{15}$$

with  $G$  even and  $G(0) = 0$ .

**Theorem 4** ([13]). *Suppose  $G : \mathbb{R} \rightarrow \mathbb{R}$  satisfies*

$$G(t + u) + G(t - u) = 2G(t) + 2G(u).$$

*If  $G$  is continuous, or continuous at a point, bounded on  $[0, \delta)$  for some  $\delta > 0$ , bounded on a set of positive measure, or measurable, then*

$$G(t) = kt^2, \quad k \in \mathbb{R}.$$

Combining both cases yields the full classification.

**Theorem 5.** *The continuous solutions of*

$$F(xy) + F\left(\frac{x}{y}\right) = 2F(x) + 2F(y) + cF(x)F(y)$$

*are given as follows:*

(i) If  $c \neq 0$ ,

$$F(e^t) = \frac{2}{c}(\cosh(\alpha t) - 1) \quad \text{or} \quad F(e^t) = \frac{2}{c}(\cos(\alpha t) - 1), \quad \alpha \in \mathbb{R}, \quad \alpha \neq 0.$$

(ii) If  $c = 0$ ,

$$F(x) = k(\ln x)^2, \quad k \in \mathbb{R}.$$

**Proof.** Set  $G(t) = F(e^t)$  and  $H(t) = 1 + \frac{c}{2}G(t)$ , for  $c \neq 0$ . By Lemma 9(i), the function  $H$  satisfies the classical d’Alembert equation with  $H$  continuous, even, and  $H(0) = 1$ . By the standard classification [1,5,9,11,13,14,16], the solutions are

$$H(t) = \cosh(\alpha t) \quad \text{or} \quad H(t) = \cos(\alpha t),$$

giving the two branches.

If  $c = 0$ , the equation reduces to Lemma 9(ii), and the result follows from Theorem 4.  $\square$

**Proposition 1.** For the hyperbolic branch in Theorem 5(i), we have  $F(x) \geq 0$  for all  $x > 0$  if and only if  $c > 0$ .

**Proof.** Since  $\cosh(\alpha t) - 1 \geq 0$  for all  $t$ ,

$$F(e^t) = \frac{2}{c}(\cosh(\alpha t) - 1)$$

is nonnegative for all  $t$  if and only if  $\frac{2}{c} > 0$ , that is,  $c > 0$ .  $\square$

We now express the hyperbolic branch in  $x$ -coordinates and identify the parameter regime in which the solution admits a natural interpretation as a reciprocal cost function.

**Corollary 6.** Let  $c > 0$  and consider the hyperbolic branch

$$F(e^t) = \frac{2}{c}(\cosh(\alpha t) - 1), \quad \alpha \in \mathbb{R}.$$

Then, in  $x$ -coordinates,

$$F(x) = \frac{1}{c}(x^\alpha + x^{-\alpha} - 2), \quad x > 0.$$

Moreover:

- (i)  $F(x) \geq 0$  for all  $x > 0$ ;
- (ii)  $F(1) = 0$ ;
- (iii)  $F(x) = F(1/x)$ ;
- (iv) if  $\alpha \neq 0$ , then  $F(x) = 0$  if and only if  $x = 1$ .

In particular, for  $c > 0$  and  $\alpha \neq 0$ ,  $F$  defines a reciprocal cost function on  $\mathbb{R}_{>0}$ .

**Corollary 7.** For every  $c \in \mathbb{R}$ , the equation

$$F(xy) + F\left(\frac{x}{y}\right) = 2F(x) + 2F(y) + cF(x)F(y)$$

admits a continuous nonconstant solution  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  satisfying  $F(1) = 0$  and  $F(x) = F(1/x)$ .

**Proof.** The explicit solutions given in Theorem 5 provide such functions for each  $c \in \mathbb{R}$ .  $\square$

**Corollary 8.** Under the assumptions of Lemma 7, assume in addition that  $F$  is convex and that  $c \neq 0$  in (11). Then  $c > 0$ , and only the hyperbolic branch is admissible, i.e.,

$$F(e^t) = \frac{2}{c}(\cosh(\alpha t) - 1).$$

Moreover,  $\alpha \geq 1$ .

**Proof.** By Lemma 7,  $F(1)$  is a global minimum. Since  $F(1) = 0$ , we have  $F(x) \geq 0$  for all  $x > 0$ . Because  $c \neq 0$ , Theorem 5(i) holds. The cosine branch

$$F(e^t) = \frac{2}{c}(\cos(\alpha t) - 1)$$

is not convex on  $\mathbb{R}_{>0}$ , since

$$G''(t) = -\frac{2}{c}\alpha^2 \cos(\alpha t)$$

changes sign. Hence,  $F''$  also changes sign, so  $F$  cannot be convex. Therefore, only the hyperbolic branch remains

$$F(e^t) = \frac{2}{c}(\cosh(\alpha t) - 1).$$

Since  $F(x) \geq 0$  for all  $x > 0$ , Proposition 1 implies  $c > 0$ . Further, we write  $t = \ln x$  and set  $U(t) := \cosh(\alpha t) - 1$ . Then  $F(x) = \frac{2}{c}U(\ln x)$  and

$$F''(x) = \frac{2}{cx^2}(U''(t) - U'(t)) = \frac{2}{cx^2}(\alpha^2 \cosh(\alpha t) - \alpha \sinh(\alpha t)).$$

Convexity of  $F$  on  $\mathbb{R}_{>0}$  means  $F''(x) \geq 0$  for all  $x > 0$ , i.e.,

$$\alpha^2 \cosh(\alpha t) - \alpha \sinh(\alpha t) \geq 0 \quad \text{for all } t \in \mathbb{R}.$$

Since  $\cosh(\alpha t) = \cosh(|\alpha|t)$ , we may assume without loss of generality that  $\alpha > 0$ . (If  $\alpha = 0$ , then  $G \equiv 0$ , contradicting nonconstancy; if  $\alpha < 0$ , replace  $\alpha$  by  $|\alpha|$  since  $\cosh$  is even.) Dividing by  $\alpha > 0$ , we obtain

$$\alpha \cosh(\alpha t) \geq \sinh(\alpha t) \quad \text{for all } t \in \mathbb{R}.$$

For  $t > 0$ , it is equivalent to

$$\alpha \geq \tanh(\alpha t).$$

Since  $\tanh(\alpha t) \rightarrow 1$  as  $t \rightarrow \infty$ , the above inequality implies  $\alpha \geq 1$ . For  $t < 0$ : since  $\alpha > 0$  and  $\sinh(\alpha t) < 0$ , the inequality  $\alpha \cosh(\alpha t) \geq \sinh(\alpha t)$  holds trivially. For  $t = 0$ , both sides vanish. Hence, the condition  $\alpha \geq 1$  is both necessary and sufficient for all  $t \in \mathbb{R}$ .

Conversely, if  $\alpha \geq 1$ , then

$$\alpha \geq \tanh(\alpha t) \quad \text{for all } t,$$

because  $\tanh(\alpha t) < 1$  for every finite  $t$ . Hence,  $\alpha \geq 1$  is both necessary and sufficient for global convexity.  $\square$

#### 4. D'Alembert Inevitability for $n$ -Dimensional Cost

In this part, we extend the inevitability result to functions defined on  $\mathbb{R}_{>0}^n$ . Let  $\mathbf{x} = (x_1, \dots, x_n)$  and  $\mathbf{y} = (y_1, \dots, y_n)$  be elements of  $\mathbb{R}_{>0}^n$ , with

$$\mathbf{x} \cdot \mathbf{y} = (x_1 y_1, \dots, x_n y_n), \quad \mathbf{x}/\mathbf{y} = (x_1/y_1, \dots, x_n/y_n), \quad \mathbf{1} = (1, \dots, 1).$$

We also write  $\ln \mathbf{x} := (\ln x_1, \dots, \ln x_n) \in \mathbb{R}^n$  and, for  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ , use the notation

$$\mathbf{x}^\boldsymbol{\alpha} := \prod_{k=1}^n x_k^{\alpha_k}, \quad \boldsymbol{\alpha} \cdot \ln \mathbf{x} := \sum_{k=1}^n \alpha_k \ln x_k = \ln(\mathbf{x}^\boldsymbol{\alpha}).$$

**Definition 2.** A function  $F : \mathbb{R}_{>0}^n \rightarrow \mathbb{R}$  satisfies an  $n$ -dimensional polynomial composition law if there exists a polynomial  $P \in \mathbb{R}[u, v]$  such that for all  $\mathbf{x}, \mathbf{y} \in \mathbb{R}_{>0}^n$ ,

$$F(\mathbf{x} \cdot \mathbf{y}) + F(\mathbf{x}/\mathbf{y}) = P(F(\mathbf{x}), F(\mathbf{y})). \tag{16}$$

The algebraic classification of the polynomial combiner  $P$  depends only on the functional equation and on the nondegeneracy of the range of  $F$ , and therefore it is independent of the dimension  $n$ .

**Theorem 6.** Let  $F : \mathbb{R}_{>0}^n \rightarrow \mathbb{R}$  be continuous and nontrivial, with  $F(\mathbf{1}) = 0$ , where  $\mathbf{1} = (1, \dots, 1)$ , and suppose (16) holds with a symmetric polynomial combiner  $P$  of total degree at most two. Then there exists  $c \in \mathbb{R}$  such that

$$P(u, v) = 2u + 2v + c uv.$$

**Proof.** Since  $P$  is symmetric, from (16), we get

$$F(\mathbf{x}/\mathbf{y}) = F(\mathbf{y}/\mathbf{x}) \quad (\mathbf{x}, \mathbf{y} \in \mathbb{R}_{>0}^n).$$

By substituting  $\mathbf{y} = \mathbf{1}$ , we obtain the reciprocity  $F(\mathbf{z}) = F(\mathbf{z}^{-1})$  for all  $\mathbf{z} \in \mathbb{R}_{>0}^n$ .  
 Now set  $\mathbf{x} = \mathbf{1}$  in (16). Using  $F(\mathbf{1}) = 0$  and reciprocity, we obtain

$$P(0, F(\mathbf{y})) = F(\mathbf{y}) + F(\mathbf{y}^{-1}) = 2F(\mathbf{y}) \quad (\mathbf{y} \in \mathbb{R}_{>0}^n).$$

Since  $F$  is continuous and nontrivial with  $F(\mathbf{1}) = 0$ , its range contains a nondegenerate interval  $I$  with  $0 \in I$ . Hence, the polynomial  $v \mapsto P(0, v) - 2v$  vanishes on  $I$ , so

$$P(0, v) = 2v \quad \text{for all } v \in \mathbb{R}.$$

By symmetry, also  $P(u, 0) = 2u$  for all  $u \in \mathbb{R}$ .

Let us write a general symmetric quadratic polynomial

$$P(u, v) = a + b(u + v) + c uv + e(u^2 + v^2).$$

Then

$$P(0, v) = a + bv + ev^2 = 2v \quad \text{for all } v,$$

so  $a = 0, b = 2$ , and  $e = 0$ . Therefore

$$P(u, v) = 2u + 2v + c uv$$

This completes the proof.  $\square$

We now classify the corresponding solutions. In logarithmic coordinates  $\mathbf{t} = (\ln x_1, \dots, \ln x_n) \in \mathbb{R}^n$ , define

$$G(\mathbf{t}) = F(e^{t_1}, \dots, e^{t_n}), \quad \mathbf{t} \in \mathbb{R}^n.$$

We treat  $\mathbf{t} \in \mathbb{R}^n$  as a column vector. If  $F(\mathbf{x}) = F(\mathbf{x}^{-1})$ , then  $G$  is even,  $G(-\mathbf{t}) = G(\mathbf{t})$ .

Assume first that  $c \neq 0$  and define  $H(\mathbf{t}) = 1 + \frac{c}{2}G(\mathbf{t})$ . Then  $H$  is continuous and satisfies the  $n$ -dimensional d'Alembert equation

$$H(\mathbf{t} + \mathbf{u}) + H(\mathbf{t} - \mathbf{u}) = 2H(\mathbf{t})H(\mathbf{u}), \quad \mathbf{t}, \mathbf{u} \in \mathbb{R}^n,$$

with  $H(\mathbf{0}) = 1$  and  $H$  even.

In the following theorem, we will classify the solutions.

**Theorem 7.** All continuous solutions of

$$F(\mathbf{x} \cdot \mathbf{y}) + F(\mathbf{x}/\mathbf{y}) = 2F(\mathbf{x}) + 2F(\mathbf{y}) + cF(\mathbf{x})F(\mathbf{y}), \quad F(\mathbf{1}) = 0, \quad (17)$$

are as follows:

(i) If  $c \neq 0$ , then there exists  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$  such that either

$$F(\mathbf{x}) = \frac{2}{c} \left( \cosh \left( \sum_{k=1}^n \alpha_k \ln x_k \right) - 1 \right) = \frac{1}{c} \left( \prod_{k=1}^n x_k^{\alpha_k} + \prod_{k=1}^n x_k^{-\alpha_k} - 2 \right),$$

or

$$F(\mathbf{x}) = \frac{2}{c} \left( \cos \left( \sum_{k=1}^n \alpha_k \ln x_k \right) - 1 \right).$$

(ii) If  $c = 0$ , then

$$F(\mathbf{x}) = \sum_{i,j=1}^n a_{ij} \ln x_i \ln x_j$$

for some symmetric matrix  $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{n \times n}$ .

**Proof.** (i) Case  $c \neq 0$ . Since  $F$  is continuous,  $G(\mathbf{t}) = F(e^{t_1}, \dots, e^{t_n})$  and  $H(\mathbf{t}) = 1 + \frac{c}{2} G(\mathbf{t})$  are continuous with  $H(\mathbf{0}) = 1$ . A direct computation shows that  $H$  satisfies

$$H(\mathbf{t} + \mathbf{u}) + H(\mathbf{t} - \mathbf{u}) = 2H(\mathbf{t})H(\mathbf{u}), \quad \mathbf{t}, \mathbf{u} \in \mathbb{R}^n.$$

Thus,  $H$  satisfies the classical d'Alembert functional equation on  $\mathbb{R}^n$  in the vector notation used in this section. By the known classification of continuous solutions of the  $n$ -dimensional d'Alembert equation on  $\mathbb{R}^n$  (see [1,9,13]), there exists  $\lambda \in \mathbb{C}^n$  such that

$$H(\mathbf{t}) = \frac{1}{2} (e^{\lambda \cdot \mathbf{t}} + e^{-\lambda \cdot \mathbf{t}}) = \cosh(\lambda \cdot \mathbf{t}).$$

If  $H$  is real-valued for all  $\mathbf{t} \in \mathbb{R}^n$ , then  $\lambda$  must be either real or purely imaginary. Indeed, if  $\lambda$  has both nonzero real and imaginary parts, then  $H(\mathbf{t})$  cannot remain real for all  $\mathbf{t} \in \mathbb{R}^n$ . Writing  $\lambda = \alpha \in \mathbb{R}^n$  or  $\lambda = i\alpha$  gives the two real branches

$$H(\mathbf{t}) = \cosh(\alpha \cdot \mathbf{t}) \quad \text{or} \quad H(\mathbf{t}) = \cos(\alpha \cdot \mathbf{t}).$$

Since  $G(\mathbf{t}) = \frac{2}{c}(H(\mathbf{t}) - 1)$  and  $\mathbf{t} = \ln \mathbf{x}$ , we obtain

$$F(\mathbf{x}) = \frac{2}{c} \left( \cosh(\alpha \cdot \ln \mathbf{x}) - 1 \right) = \frac{1}{c} \left( \mathbf{x}^\alpha + \mathbf{x}^{-\alpha} - 2 \right),$$

or

$$F(\mathbf{x}) = \frac{2}{c} \left( \cos(\alpha \cdot \ln \mathbf{x}) - 1 \right).$$

(ii) Case  $c = 0$ . Then the equation for  $G$  reduces to

$$G(\mathbf{t} + \mathbf{u}) + G(\mathbf{t} - \mathbf{u}) = 2G(\mathbf{t}) + 2G(\mathbf{u}), \quad \mathbf{t}, \mathbf{u} \in \mathbb{R}^n.$$

This is a Jensen-type quadratic functional equation on  $\mathbb{R}^n$ . (see [11,13]). By the standard classification of continuous solutions of the quadratic Jensen-type equation on  $\mathbb{R}^n$  (see [11,13]), every solution has the form

$$G(\mathbf{t}) = \mathbf{t}^T \mathbf{A} \mathbf{t}, \quad \mathbf{t} \in \mathbb{R}^n,$$

where  $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{n \times n}$  is a symmetric matrix. Finally, for  $\mathbf{t} = \ln \mathbf{x}$ , we have

$$F(\mathbf{x}) = \sum_{i,j=1}^n a_{ij} \ln x_i \ln x_j,$$

which completes the proof.

□

**Remark 7.** For  $F$  to serve as a cost function (non-negative with  $F(\mathbf{x}) = 0$  only at  $\mathbf{x} = \mathbf{1}$ ) [10], the matrix  $\mathbf{A}$  must be positive definite.

**Remark 8.** In the case  $c \neq 0$ , the cosh branch with  $c > 0$  and  $\boldsymbol{\alpha} \in \mathbb{R}^n$  satisfies  $F(\mathbf{x}) \geq 0$  with equality if and only if  $\mathbf{x} = \mathbf{1}$ . Hence, this branch is compatible with the interpretation of  $F$  as a cost function.

When  $c \neq 0$ , Equation (16) is very restrictive. By Theorem 7, the function  $F$  depends on  $\mathbf{x} \in \mathbb{R}_{>0}^n$  only through the expression  $\boldsymbol{\alpha} \cdot \ln \mathbf{x}$ . Thus, even in dimension  $n$ , the effective dependence is one-dimensional.

In applications, cost functions on  $\mathbb{R}_{>0}^n$  are assumed to be additively separable, reflecting independent contributions of different coordinates. It is therefore natural to ask whether such separable forms are compatible with the composition law (17).

Suppose that  $F$  has a form

$$F(\mathbf{x}) = \sum_{k=1}^n f_k(x_k). \tag{18}$$

If each  $f_k$  satisfies the composition law in one variable, then

$$F(\mathbf{x} \cdot \mathbf{y}) + F(\mathbf{x}/\mathbf{y}) = 2F(\mathbf{x}) + 2F(\mathbf{y}) + c \sum_{k=1}^n f_k(x_k)f_k(y_k).$$

However,

$$P(F(\mathbf{x}), F(\mathbf{y})) = 2F(\mathbf{x}) + 2F(\mathbf{y}) + cF(\mathbf{x})F(\mathbf{y})$$

contains additional mixed terms of the form

$$c \sum_{k \neq j} f_k(x_k)f_j(y_j),$$

which cannot vanish unless at most one component is nontrivial. This indicates that  $F$  given by (18) is incompatible with the composition law (17) when  $c \neq 0$ .

More precisely, the following statement holds.

**Theorem 8.** Let  $c \neq 0$  and  $n \geq 2$ . If  $F : \mathbb{R}_{>0}^n \rightarrow \mathbb{R}$  satisfies

$$F(\mathbf{x} \cdot \mathbf{y}) + F(\mathbf{x}/\mathbf{y}) = 2F(\mathbf{x}) + 2F(\mathbf{y}) + cF(\mathbf{x})F(\mathbf{y}),$$

then  $F$  cannot be written in the form (18) with two or more nonconstant components.

**Proof.** In logarithmic coordinates

$$G(\mathbf{t}) = \frac{2}{c}(\cosh(\boldsymbol{\alpha} \cdot \mathbf{t}) - 1) \quad \text{or} \quad G(\mathbf{t}) = \frac{2}{c}(\cos(\boldsymbol{\alpha} \cdot \mathbf{t}) - 1).$$

We denote by  $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n$  the zero vector. For the hyperbolic branch,  $G(\mathbf{t}) = \frac{2}{c}(\cosh(\boldsymbol{\alpha} \cdot \mathbf{t}) - 1)$ , and for the trigonometric branch,  $G(\mathbf{t}) = \frac{2}{c}(\cos(\boldsymbol{\alpha} \cdot \mathbf{t}) - 1)$ . In either case, a direct computation gives

$$\frac{\partial^2 G}{\partial t_j \partial t_k}(\mathbf{0}) = \pm \frac{2}{c} \alpha_j \alpha_k \quad (j \neq k),$$

where the sign + corresponds to the hyperbolic branch and the sign – to the trigonometric branch.

If  $F(\mathbf{x}) = \sum_k f_k(x_k)$ , then  $G(\mathbf{t}) = \sum_k g_k(t_k)$ , and therefore all mixed partial derivatives vanish. Hence,  $\alpha_j \alpha_k = 0$  for all  $j \neq k$ . Thus, at most one component of  $\alpha$  is nonzero. Consequently,  $F$  depends on at most one coordinate, so a decomposition with two nonconstant components is impossible.  $\square$

**Corollary 9.** For  $c \neq 0$ , no additively separable cost with at least two nonconstant coordinate components is compatible with the bilinear combiner.

In the following example, we provide a realization of the multidimensional rigidity result. We construct a 16-dimensional system depending on two parameters  $(r, s)$  and show that the induced reciprocal cost depends only on a single scalar aggregate.

**Example 2.** Let  $(r, s) \in \Omega \subset \mathbb{R}^2$  and define

$$\Phi(r, s) = (\phi_1(r, s), \dots, \phi_{16}(r, s)) \in \mathbb{R}^{16}$$

by

$$\Phi(r, s) = (r, s, r + s, r - s, rs, r^2, s^2, r^2 - s^2, 2rs, r^3, s^3, r^2s, rs^2, r^4, s^4, r^2s^2).$$

Let  $\alpha \in \mathbb{R}^{16}$  and set

$$S(r, s) = \alpha \cdot \Phi(r, s).$$

Define

$$\mathbf{x}(r, s) = (e^{\alpha_1 \phi_1(r, s)}, \dots, e^{\alpha_{16} \phi_{16}(r, s)}) \in \mathbb{R}_{>0}^{16}.$$

Then

$$\sum_{k=1}^{16} \ln x_k(r, s) = S(r, s), \quad \prod_{i=1}^{16} x_i(r, s) = e^{S(r, s)}.$$

The reciprocal cost on  $\mathbb{R}_{>0}^{16}$  is given by

$$F(\mathbf{x}) = \frac{1}{2} (R + R^{-1}) - 1, \quad R = \prod_{i=1}^{16} x_i.$$

Under the above parametrization, this becomes

$$F(r, s) = \cosh(S(r, s)) - 1.$$

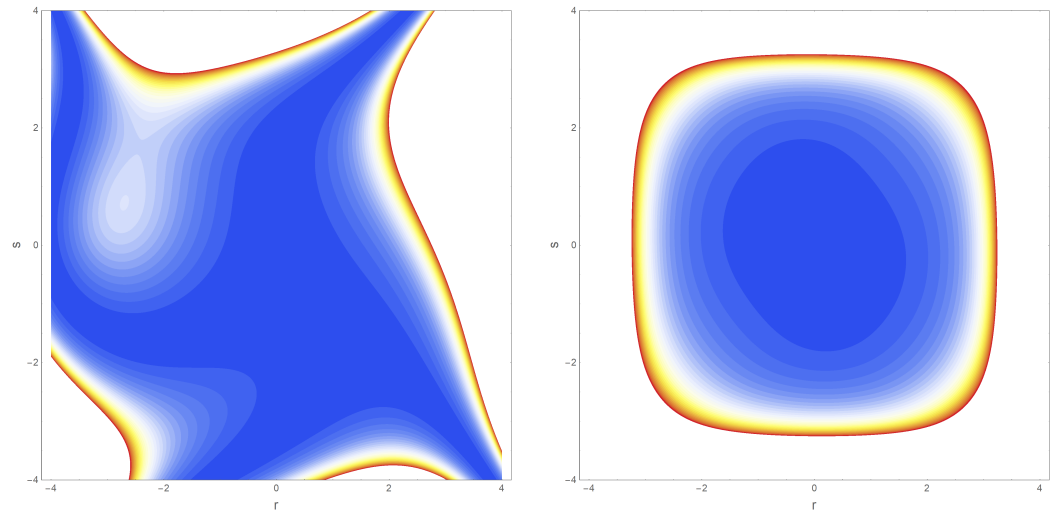
The reciprocal cost depends only on the single scalar quantity

$$S(r, s) = \alpha \cdot \Phi(r, s) = \sum_{k=1}^{16} \ln x_k.$$

By Theorem 7, the system collapses to a logarithmic direction for  $c \neq 0$ . In the case  $c \neq 0$ , the function  $F(r, s)$  depends only on the scalar quantity  $S(r, s) = \alpha \cdot \Phi(r, s)$ . Hence, for any two points  $(r_1, s_1)$  and  $(r_2, s_2)$  such that  $S(r_1, s_1) = S(r_2, s_2)$ , we have  $F(r_1, s_1) = F(r_2, s_2)$ . Therefore,  $F$  is constant along the level sets of  $S$ , and the dependence on  $(r, s)$  reduces effectively to one dimension. Here, the level sets of  $S$  are the sets

$$\{(r, s) \in \Omega : S(r, s) = \gamma\}, \quad \gamma \in \mathbb{R}.$$

The collapse and no-collapse regimes are illustrated in Figure 1.



**Figure 1.** (Left) collapse regime ( $c \neq 0$ ), where  $F$  depends only on  $S(r, s)$  and is constant on its level sets, reducing the dependence to one dimension. (Right) no-collapse regime, where  $F$  depends genuinely on  $(r, s)$ .

**Remark 9.** If we consider a different combiner  $P_k$  in each coordinate,

$$F(\mathbf{x} \cdot_k \mathbf{y}) + F(\mathbf{x} /_k \mathbf{y}) = P_k(F(\mathbf{x}), F(\mathbf{y})),$$

where only the  $k$ -th component is modified, then necessarily

$$P_k(u, v) = 2u + 2v + c_k uv$$

for each  $k$ . The case  $c_1 = \dots = c_n$  reduces to the consideration above. The compatibility of unequal parameters  $c_k$  remains open.

### 5. Canonical Coefficient Selection

The previous sections establish that symmetry and the polynomial composition law uniquely force the one-parameter bilinear family (11) for some real constant  $c \in \mathbb{R}$ .

We now show that a natural local normalization selects one distinguished member of this family.

**Definition 3.** Let  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ . The log-curvature of  $F$ , denoted  $\kappa(F)$ , is defined as

$$\kappa(F) := \lim_{t \rightarrow 0} \frac{2F(e^t)}{t^2},$$

provided this limit exists.

When the limit exists,  $\kappa(F)$  is the quadratic scaling coefficient of  $F(e^t)$  at  $t = 0$ . This does not assume a priori that  $F$  is  $C^2$ . The existence of the limit provides the required regularity.

By the change of variables  $x = e^t$ , the limit exists if and only if

$$\lim_{x \rightarrow 1} \frac{2F(x)}{(\ln x)^2}$$

exists, and in that case, the two limits coincide.

Assume that the limit  $\kappa(F)$  exists. Then necessarily  $F(1) = 0$ , since otherwise the quotient  $\frac{2F(e^t)}{t^2}$  diverges as  $t \rightarrow 0$ . Set  $G(t) = F(e^t)$ . If  $G$  is twice differentiable at 0, then

$$\kappa(F) = G''(0).$$

The calibration condition  $\kappa(F) = 1$  means

$$G(t) = \frac{1}{2}t^2 + o(t^2) \quad (t \rightarrow 0).$$

We now determine how this calibration constrains the parameter  $c$ .

**Theorem 9.** *Let  $F$  be a continuous nonconstant solution of (11) with  $c \neq 0$ . Assume that  $F$  belongs to the hyperbolic branch described in Theorem 5, that is,*

$$F(e^t) = \frac{2}{c}(\cosh(\alpha t) - 1) \quad \text{for some } \alpha > 0.$$

If  $\kappa(F) = 1$ , then  $c = 2\alpha^2$ .

**Proof.** Let  $G(t) = F(e^t)$ . For  $c \neq 0$ , define  $H(t) = 1 + \frac{c}{2}G(t)$ . By Lemma 9(i),  $H$  satisfies the classical d’Alembert Equation (13). By assumption, we are in the hyperbolic branch, so

$$H(t) = \cosh(\alpha t), \quad \alpha > 0.$$

Hence

$$G(t) = \frac{2}{c}(H(t) - 1) = \frac{2}{c}(\cosh(\alpha t) - 1).$$

Using the Taylor expansion at  $t = 0$ ,

$$\cosh(\alpha t) - 1 = \frac{\alpha^2 t^2}{2} + o(t^2),$$

we compute

$$\kappa(F) = \lim_{t \rightarrow 0} \frac{2F(e^t)}{t^2} = \lim_{t \rightarrow 0} \frac{2G(t)}{t^2} = \frac{4}{c} \lim_{t \rightarrow 0} \frac{\cosh(\alpha t) - 1}{t^2} = \frac{4}{c} \cdot \frac{\alpha^2}{2} = \frac{2\alpha^2}{c}.$$

Therefore, the calibration condition  $\kappa(F) = 1$  implies  $c = 2\alpha^2$ .  $\square$

We now combine the d’Alembert Inevitability Theorem 3 with the solution classification in Theorem 5 and the calibration condition.

**Theorem 10.** *Assume:*

- (i)  $F$  is continuous and nonconstant;
- (ii)  $F$  satisfies the bilinear composition law (11);
- (iii)  $F$  is convex and nonnegative on  $(0, \infty)$ ;
- (iv)  $\kappa(F) = 1$ .

Then  $F$  belongs to the hyperbolic family

$$F_\alpha(x) = \frac{1}{\alpha^2}(\cosh(\alpha \ln x) - 1), \quad \alpha \geq 1, \tag{19}$$

and

$$c = 2\alpha^2.$$

**Proof.** Passing to logarithmic coordinates  $G(t) = F(e^t)$  and applying Lemma 9, Equation (11) reduces to the classical d’Alembert equation. By Theorem 5, all continuous solutions are either hyperbolic or trigonometric.

Since  $F$  is convex and nonnegative on  $(0, \infty)$ ,  $F$  must belong to the hyperbolic branch, so that

$$G(t) = F(e^t) = \frac{2}{c} (\cosh(\alpha t) - 1) \quad \text{for some } \alpha > 0.$$

The log-curvature is

$$\kappa(F) = \lim_{t \rightarrow 0} \frac{2F(e^t)}{t^2} = \frac{2\alpha^2}{c}.$$

Using  $\kappa(F) = 1$  gives

$$c = 2\alpha^2.$$

Substituting this relation into the expression for  $F$  yields

$$F_\alpha(e^t) = \frac{1}{\alpha^2} (\cosh(\alpha t) - 1). \tag{20}$$

Finally, global convexity of  $F$  on  $(0, \infty)$  is equivalent to  $\alpha \geq 1$  by Corollary 8.  $\square$

The parameter  $\alpha$  reflects a multiplicative rescaling of the logarithmic coordinate. The representation (20) can be written equivalently as

$$F_\alpha(x) = \frac{1}{\alpha^2} F_1(x^\alpha).$$

Thus,  $\alpha$  does not introduce a new structural type of solution; it corresponds only to a rescaling of the coordinate  $t = \ln x$ . Without loss of generality, we may therefore assume  $\alpha = 1$ .

**Corollary 10.** *Under the assumption of Theorem 10, after normalization of the multiplicative coordinate, the canonical representative is*

$$F(x) = \frac{1}{2}(x + x^{-1}) - 1.$$

**Proof.** If we set  $\alpha = 1$  in (19), we have  $c = 2$  and

$$F(e^t) = \cosh(t) - 1 = \frac{1}{2}(e^t + e^{-t}) - 1.$$

Returning to multiplicative coordinates  $x = e^t$  yields

$$F(x) = \frac{1}{2}(x + x^{-1}) - 1.$$

$\square$

**Remark 10.** *If  $c = 0$ , then the composition law (11) reduces to the additive branch, and the classification yields*

$$F(x) = k(\ln x)^2.$$

*In this case,*

$$\kappa(F) = 2k,$$

*so the normalization  $\kappa(F) = 1$  forces  $k = \frac{1}{2}$ , giving*

$$F(x) = \frac{1}{2}(\ln x)^2.$$

This provides a unit-curvature solution in the additive regime, which lies outside the bilinear ( $c \neq 0$ ) family.

### 6. Conclusions

In this paper, we studied continuous nonconstant functions  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  satisfying a symmetric polynomial composition law

$$F(xy) + F\left(\frac{x}{y}\right) = P(F(x), F(y)).$$

We first considered the case of higher-degree polynomial combiners. Theorem 1 shows that symmetric combiners with  $\deg P \geq 3$  are incompatible with the functional equation under a non-cancellation condition. In particular, the cubic case in Example 1 admits no nonconstant continuous solutions. Consequently, only polynomial combiners of degree at most two can admit nontrivial continuous solutions.

In the quadratic case, the combiner  $P$  is necessarily of the form

$$P(u, v) = 2u + 2v + c uv, \quad c \in \mathbb{R}.$$

We also showed that symmetry of  $P$  is equivalent to reciprocity  $F(x) = F(1/x)$ , and that the normalization  $F(1) = 0$  implies  $P(u, 0) = 2u$  and  $P(0, v) = 2v$ . For a given continuous nonconstant solution  $F$ , the combiner is unique.

Passing to logarithmic coordinates reduces the composition law to the classical d’Alembert functional equation. This gives a complete classification of continuous solutions: the hyperbolic and trigonometric branches, and the quadratic logarithmic case when  $c = 0$ .

In the  $n$ -dimensional case, we showed that the classification of  $P$  remains unchanged. For  $c \neq 0$ , every solution depends only on a single scalar combination  $\alpha \cdot \ln x$ . As a consequence, additive separability  $F(\mathbf{x}) = \sum_k f_k(x_k)$  is impossible for  $n \geq 2$ .

Finally, we introduced the log-curvature calibration  $\kappa(F)$  and proved that, for nonnegative convex solutions with  $c \neq 0$  and  $\kappa(F) = 1$ , the solutions belong to the family

$$F_\alpha(x) = \frac{1}{\alpha^2} (\cosh(\alpha \ln x) - 1), \quad c = 2\alpha^2.$$

After normalization  $\alpha = 1$ , this gives the canonical reciprocal cost

$$F(x) = \frac{1}{2} (x + x^{-1}) - 1.$$

Thus, the canonical reciprocal cost is uniquely determined by the structural constraints.

Several natural questions remain open for further investigation. These include the classification of asymmetric polynomial combiners, the stability of the polynomial composition law in the Hyers–Ulam sense, and the multidimensional case with distinct coordinate parameters.

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