Article

Uniqueness of the Canonical Reciprocal Cost

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Abstract

We study a rigidity problem for functions $F : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ that penalize deviation of a positive ratio from equilibrium $x = 1$. Assuming (i) a d'Alembert-type composition law on $\mathbb{R}_{>0}$, and (ii) a single quadratic calibration at the identity (in logarithmic coordinates), we prove that F is uniquely determined. The composition law implies the normalization $F(1) = 0$. The unique solution is called the canonical reciprocal cost, namely the difference between the arithmetic and geometric means of x and its reciprocal. Our proof uses the logarithmic coordinates $H(t) = F(e^t) + 1$, where the composition law becomes d'Alembert's functional equation on $\mathbb{R}$. The calibration provides the minimal regularity needed to invoke the classical classification of continuous solutions and fixes the remaining scaling freedom, selecting the hyperbolic-cosine branch. We also establish the necessity of each assumption: without calibration the composition law admits a continuous one-parameter family; without the composition law the calibration does not determine the global form; and without regularity the composition law admits pathological non-measurable solutions. Finally, we establish a stability estimate for approximate solutions under bounded defect and characterize some properties of the canonical cost.

Keywords: d'Alembert functional equation; reciprocal cost function; quadratic calibration; rigidity; hyperbolic cosine; Bregman divergence

MSC: 39B52, 39B05, 39B82.

1. Introduction

In the paper, we consider a function $F : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ with a minimum at $x = 1$. Reciprocity $F(x) = F(x^{-1})$ is natural for ratio costs, as it penalizes reciprocal deviations equally.

Our central assumption is a d'Alembert-type composition law on $\mathbb{R}_{>0}$ (simply called the composition law),

$$F(xy) + F\left(\frac{x}{y}\right) = 2F(x)F(y) + 2F(x) + 2F(y).$$

Under this equation, reciprocity $F(x) = F(x^{-1})$ and the normalization $F(1) = 0$ follow.

Passing to logarithmic coordinates $H(t) := F(e^t) + 1$, the composition law becomes d'Alembert's functional equation

$$H(t + u) + H(t - u) = 2H(t)H(u).$$

The theory of d'Alembert's equation is that classical, continuous solutions on $\mathbb{R}$ are classified [1–7], discrete restrictions produce Chebyshev structure [8], and additional regu-

larity assumptions yield strong rigidity [9,10]. We use a single local calibration at equilibrium

$$\lim_{t \to 0} \frac{2F(e^t)}{t^2} = 1,$$

which supplies the needed regularity and fixes the remaining scaling freedom.

Assuming a d'Alembert-type composition law on $\mathbb{R}_{>0}$ and a local quadratic calibration at the identity (in logarithmic coordinates), we prove that F is uniquely determined and equals the canonical reciprocal cost

$$J(x) = \frac{1}{2}(x + x^{-1}) - 1.$$

The paper is organized as follows. Section 2 states the main theorem, introduces the logarithmic coordinates, and recalls the relevant facts about d'Alembert's equation. Section 3 proves the rigidity theorem and demonstrates the necessity of each assumption via explicit counterexamples and pathological solutions. Section 4 establishes a quantitative consistency estimate for approximate solutions to the d'Alembert equation with bounded defect. Section 5 presents further structural properties of J, including its Bregman divergence, induced metric, and Chebyshev structure.

2. Definitions and Basic Properties

In this section we give the basic definitions and notation, state the main rigidity theorem, and rewrite the problem in logarithmic coordinates. The proof of the rigidity theorem is given in Section 3.

2.1. Logarithmic Coordinates

For any $F : \mathbb{R}_{>0} \to \mathbb{R}$, we introduce logarithmic coordinates $t = \ln x$ and define the associated functions

$$G(t) := F(e^t), \qquad H(t) := G(t) + 1 = F(e^t) + 1, \qquad t \in \mathbb{R}. \tag{1}$$

If $F(1) = 0$ then $G(0) = 0$ and $H(0) = 1$.

Lemma 1. *Let $F : \mathbb{R}_{>0} \to \mathbb{R}$ satisfy the composition law*

$$F(xy) + F\left(\frac{x}{y}\right) = 2F(x)F(y) + 2F(x) + 2F(y).$$

Then either $F \equiv -1$, or $F(1) = 0$.

Proof. Set $y = 1$. Then

$$2F(x) = 2F(x)F(1) + 2F(x) + 2F(1),$$

hence $0 = 2F(1)(F(x) + 1)$ for all $x > 0$. Thus either $F(1) = 0$ or $F(x) = -1$ for all $x > 0$. $\square$

2.2. The Canonical Reciprocal Cost

Definition 1. *The function $J : \mathbb{R}_{>0} \to \mathbb{R}$ defined by*

$$J(x) := \frac{x + x^{-1}}{2} - 1 \tag{2}$$

is called the canonical reciprocal cost *function.*

The function J satisfies:

(i)   Normalization: $J(1) = 0$;

(ii) Reciprocity: $J(x) = J(x^{-1})$;

(iii) Non-negativity: for all $x \in \mathbb{R}_{>0}$, $J(x) = \frac{(x-1)^2}{2x} \geq 0$.

In logarithmic coordinates $t = \ln x$, one has $J(e^t) = \cosh(t) - 1$.

2.3. Calibration

Definition 2. *Let $F : \mathbb{R}_{>0} \to \mathbb{R}$. The* log-curvature *of F, denoted $\kappa(F)$, is defined as*

$$\kappa(F) := \lim_{t \to 0} \frac{2\,F(e^t)}{t^2},$$

provided this limit exists.

When the limit exists, $\kappa(F)$ represents the quadratic coefficient in the local behavior of $F(e^t)$ at $t = 0$. This definition does not assume a priori that F is C^2; the existence of the limit provides the necessary quadratic behavior near $t = 0$.

By the change of variables $x = e^t$, the limit exists if and only if

$$\lim_{x \to 1} \frac{2\,F(x)}{(\ln x)^2}$$

exists, and in that case the two limits are equal.

The condition $\kappa(F) = 1$ excludes the constant solution $F \equiv -1$ and forces $F(1) = 0$.

2.4. Main Result

We now state the main theorem, which establishes the uniqueness of the canonical reciprocal cost under the composition law and a single quadratic calibration.

Theorem 1. *Let $F : \mathbb{R}_{>0} \to \mathbb{R}$. Assume that F satisfies:*

(i)   Composition law on $\mathbb{R}_{>0}$: for all $x, y > 0$,

$$F(xy) + F\left(\frac{x}{y}\right) = 2\,F(x)\,F(y) + 2\,F(x) + 2\,F(y); \tag{3}$$

(ii) Unit log-curvature (Definition 2): $\kappa(F) = 1$, i.e.,

$$\lim_{t \to 0} \frac{2\,F(e^t)}{t^2} = 1. \tag{4}$$

Then for all $x > 0$,

$$F(x) = \frac{x + x^{-1}}{2} - 1 = J(x).$$

Definition 3. *A function $F : \mathbb{R}_{>0} \to \mathbb{R}$ is called a* reciprocal cost *if*

$$F(x) = F(x^{-1}) \qquad \text{for all } x > 0.$$

It is normalized if $F(1) = 0$.

Remark 1. *Reciprocity is a natural symmetry for ratio costs. In Proposition 1, we show that reciprocity is implied by the composition law.*

Lemma 2. *If F is reciprocal then G and H given by (1) are even.*

Proof. Since $e^{-t} = (e^t)^{-1}$ and $F(x) = F(x^{-1})$, we have

$$G(-t) = F(e^{-t}) = F((e^t)^{-1}) = F(e^t) = G(t).$$

Further, for $H(-t)$, we have

$$H(-t) = G(-t) + 1 = G(t) + 1 = H(t).$$

□

2.5. The d'Alembert Functional Equation

The key structural identity in this paper is d'Alembert's functional equation (also called the cosine equation), which is equivalent to our composition law on $\mathbb{R}_{>0}$ (Lemma 7).

Definition 4. *A function $H : \mathbb{R} \to \mathbb{R}$ is said to satisfy the d'Alembert functional equation if, for all $t, u \in \mathbb{R}$,*

$$H(t + u) + H(t - u) = 2 H(t) H(u). \tag{5}$$

The equation goes back to d'Alembert [11] and was further studied by Poisson [12] and Picard [13], and many others. The equation plays an important role in determining the sum of two vectors in various Euclidean and non-Euclidean geometries.

Substituting $t = u = 0$ in (5) gives $H(0) \in \{0, 1\}$. If $H(0) = 0$ then (5) forces $H \equiv 0$. In our case $H(t) = F(e^t) + 1$ with $F(1) = 0$; hence, $H(0) = 1$ and the trivial branch $H \equiv 0$ are excluded.

Theorem 2 ([3]). *If $H : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies (5) then either $H \equiv 0$ or $H \equiv 1$, or*

$$H(t) = \cos(kt), \qquad or \qquad H(t) = \cosh(kt),$$

for some real constant k.

More generally, the following holds:

Theorem 3 ([14]). *The general complex-valued solutions of d'Alembert's functional Equation (5) on the Cartesian square of an abelian group G are given by*

$$H(x) = \frac{h(x) + h(-x)}{2},$$

where $h : G \to \mathbb{C}$ satisfies $h(x + y) = h(x)h(y)$, $x, y \in G$.

In this paper we consider d'Alembert's functional equation on the group $G = (\mathbb{R}, +)$ for functions $H : \mathbb{R} \to \mathbb{R}$.

If one assumes continuity of H (as we will later obtain from the curvature calibration) then the multiplicative solutions h have the classical exponential form $h(t) = e^{ct}$ with $c \in \mathbb{C}$; hence,

$$H(t) = \frac{e^{ct} + e^{-ct}}{2}.$$

For real-valued H, this reduces to the standard cosine/hyperbolic-cosine classification (cf. [1–3]).

Lemma 3. *If H satisfies Definition 4 then H is even.*

Proof. Fix $u \in \mathbb{R}$ and apply the d'Alembert Equation (5) with $t = 0$:

$$H(u) + H(-u) = 2\,H(0)\,H(u) = 2H(u),$$

so $H(-u) = H(u)$. $\quad\square$

Lemma 4. *If H satisfies Definition 4 then for all $t, u \in \mathbb{R}$,*

$$H(t + u)\,H(t - u) = H(t)^2 + H(u)^2 - 1.$$

Proof. Apply (5) with $a = t + u$ and $b = t - u$:

$$H((t + u) + (t - u)) + H((t + u) - (t - u)) = 2H(t + u)H(t - u),$$

so

$$H(2t) + H(2u) = 2H(t + u)H(t - u).$$

Using

$$H(2t) = 2H(t)^2 - 1$$

(obtained from (5) with (t, t) and $H(0) = 1$), and similarly for u, yields the claim. $\quad\square$

Lemma 5. *If H satisfies Definition 4 then for all $t, u \in \mathbb{R}$,*

$$\left(H(t + u) - H(t - u)\right)^2 = 4\left(H(t)^2 - 1\right)\left(H(u)^2 - 1\right).$$

Proof. Let $A := H(t + u)$ and $B := H(t - u)$. Then $A + B = 2H(t)H(u)$ by Definition 4, and $AB = H(t)^2 + H(u)^2 - 1$ by Lemma 4. Hence,

$$
\begin{aligned}
(A - B)^2 &= (A + B)^2 - 4AB \\
&= 4H(t)^2 H(u)^2 - 4(H(t)^2 + H(u)^2 - 1) \\
&= 4(H(t)^2 - 1)(H(u)^2 - 1).
\end{aligned}
$$

$\quad\square$

Lemma 6. *If H satisfies Definition 4 and if $\lim_{t \to 0} 2(H(t) - 1)/t^2$ exists then H is continuous on $\mathbb{R}$.*

Proof. The limit assumption implies that $\lim_{t \to 0} H(t) = 1$. Since $H(0) = 1$, it follows that H is continuous at 0.

Fix $t \in \mathbb{R}$. For $u \to 0$, Equation (5) gives

$$\lim_{u \to 0}\left(H(t + u) + H(t - u)\right) = 2H(t)\lim_{u \to 0} H(u) = 2H(t).$$

By Lemma 5, we have

$$\lim_{u \to 0}\left(H(t + u) - H(t - u)\right)^2 = 4\left(H(t)^2 - 1\right)\lim_{u \to 0}\left(H(u)^2 - 1\right) = 0,$$

hence,

$$\lim_{u \to 0}\left(H(t + u) - H(t - u)\right) = 0.$$

Moreover,

$$H(t + u) = \frac{H(t + u) + H(t - u)}{2} + \frac{H(t + u) - H(t - u)}{2}.$$

Taking limits as $u \to 0$ and using

$$\lim_{u \to 0} \big(H(t+u) + H(t-u)\big) = 2H(t), \qquad \lim_{u \to 0} \big(H(t+u) - H(t-u)\big) = 0,$$

we obtain

$$\lim_{u \to 0} H(t+u) = H(t).$$

Similarly, $\lim_{u \to 0} H(t-u) = H(t)$. Therefore, H is continuous at every $t \in \mathbb{R}$. □

Let H satisfy (5) and set $G := H - 1$. In this case, a direct calculation shows that G satisfies

$$G(t+u) + G(t-u) = 2\,G(t)\,G(u) + 2\,G(t) + 2\,G(u).$$

For example (see [3]), the function

$$H(t) = J(\mathrm{e}^t) + 1 = \cosh(t)$$

satisfies the d'Alembert Equation (5), and the corresponding $G(t) = \cosh(t) - 1$ satisfies the identity above.

2.6. Composition Law on $\mathbb{R}_{>0}$

Although our main theorem is stated on $\mathbb{R}_{>0}$, the composition law is most transparent after passing to logarithmic coordinates, where it reduces exactly to d'Alembert's equation for $H(t) = F(e^t) + 1$.

Definition 5. *A function $F : \mathbb{R}_{>0} \to \mathbb{R}$ satisfies the* composition law *on $\mathbb{R}_{>0}$ if for all $x, y > 0$,*

$$F(xy) + F\Big(\frac{x}{y}\Big) = 2\,F(x)\,F(y) + 2\,F(x) + 2\,F(y). \tag{6}$$

Lemma 7. *Let $F : \mathbb{R}_{>0} \to \mathbb{R}$, and $H : \mathbb{R} \to \mathbb{R}$, such that $H(t) = F(\mathrm{e}^t) + 1$. Then F satisfies Definition 5 if and only if H satisfies the d'Alembert Equation (5).*

Proof. Assume F satisfies Definition 5. Let $t, u \in \mathbb{R}$ and set $x = \mathrm{e}^t$, $y = \mathrm{e}^u$. Then

$$\begin{aligned}
H(t+u) + H(t-u) &= \big(F(\mathrm{e}^{t+u}) + 1\big) + \big(F(\mathrm{e}^{t-u}) + 1\big) \\
&= \Big(F(xy) + F\Big(\frac{x}{y}\Big)\Big) + 2 \\
&= \big(2F(x)F(y) + 2F(x) + 2F(y)\big) + 2 \\
&= 2\big(F(x) + 1\big)\big(F(y) + 1\big) = 2H(t)H(u),
\end{aligned}$$

so H satisfies (5).

Conversely, if H satisfies (5), by reverse calculation with $x = \mathrm{e}^t$, $y = \mathrm{e}^u$, we obtain that F satisfies Definition 5. □

3. Main Results

This section contains the rigidity argument for Theorem 1. Using logarithmic coordinates $H(t) = F(e^t) + 1$, the composition law on $\mathbb{R}_{>0}$ becomes d'Alembert's Equation (5). We verify the assumptions for the canonical cost J, classify the solutions, and obtain uniqueness. We also briefly discuss which assumptions are essential.

Lemma 8. *Let $J(x) = \frac{1}{2}\big(x + x^{-1}\big) - 1$, $x > 0$. Then the following properties hold:*

(i) J is reciprocal and normalized: $J(x) = J(x^{-1})$ for all $x > 0$ and $J(1) = 0$.

(ii)   J satisfies the composition law on $\mathbb{R}_{>0}$ (Definition 5).
(iii)   J has unit log-curvature: $\kappa(J) = 1$.

Proof. (i) Reciprocity follows directly from the definition of $J(x)$. Also $J(1) = \frac{1}{2}(1 + 1) - 1 = 0$.

(ii) Let $H(t) = J(e^t) + 1$, $t \in \mathbb{R}$. Let us first compute H:

$$H(t) = \left(\frac{1}{2}\left(e^t + e^{-t}\right) - 1\right) + 1 = \frac{1}{2}\left(e^t + e^{-t}\right) = \cosh(t).$$

Hence $H(t) = \cosh(t)$ for all $t \in \mathbb{R}$.

The function cosh satisfies the d'Alembert equation

$$H(t + u) + H(t - u) = 2\,H(t)\,H(u) \qquad \text{for all } t, u \in \mathbb{R}. \tag{7}$$

Let $x, y > 0$ and

$$t = \ln x, \qquad u = \ln y.$$

Then $e^t = x$, $e^u = y$, and, consequently,

$$e^{t+u} = xy, \qquad e^{t-u} = \frac{x}{y}.$$

Using the definition of H, we can rewrite (7) as

$$\left(J(e^{t+u}) + 1\right) + \left(J(e^{t-u}) + 1\right) = 2\left(J(e^t) + 1\right)\left(J(e^u) + 1\right).$$

Substituting $e^{t+u} = xy$, $e^{t-u} = x/y$, $e^t = x$, and $e^u = y$, we obtain

$$\left(J(xy) + 1\right) + \left(J\!\left(\frac{x}{y}\right) + 1\right) = 2\left(J(x) + 1\right)\left(J(y) + 1\right).$$

Therefore, J satisfies the composition law on $\mathbb{R}_{>0}$ given by Definition 5:

$$J(xy) + J\!\left(\frac{x}{y}\right) = 2\,J(x)\,J(y) + 2\,J(x) + 2\,J(y), \qquad x, y > 0.$$

(iii) Using $J(e^t) = \cosh(t) - 1$ and the Taylor expansion $\cosh(t) = 1 + t^2/2 + o(t^2)$ as $t \to 0$, we have

$$\lim_{t \to 0} \frac{2J(e^t)}{t^2} = \lim_{t \to 0} \frac{2(\cosh(t) - 1)}{t^2} = 1,$$

so $\kappa(J) = 1$. $\square$

The next theorem shows that a calibrated d'Alembert solution is forced onto a single cosine/hyperbolic-cosine branch, with the calibration determining the parameter.

Theorem 4. *Let $H : \mathbb{R} \to \mathbb{R}$ satisfy the d'Alembert Equation (5). Assume the following limit exists:*

$$\kappa_H := \lim_{t \to 0} \frac{2\,(H(t) - 1)}{t^2} \in \mathbb{R}. \tag{8}$$

Then:

1.   *If $\kappa_H > 0$ then $H(t) = \cosh(\sqrt{\kappa_H}\, t)$ for all $t \in \mathbb{R}$.*
2.   *If $\kappa_H < 0$ then $H(t) = \cos(\sqrt{-\kappa_H}\, t)$ for all $t \in \mathbb{R}$.*
3.   *If $\kappa_H = 0$ then $H(t) = 1$ for all $t \in \mathbb{R}$.*

In particular, if $\kappa_H = 1$ then $H(t) = \cosh(t)$ for all $t \in \mathbb{R}$.

Proof. By Lemma 6, the existence of

$$\lim_{t \to 0} \frac{2\left(H(t) - 1\right)}{t^2} \in \mathbb{R}$$

implies that H is continuous on $\mathbb{R}$. Hence we will apply the classical classification of continuous real-valued solutions of the d'Alembert Equation (5) (see, for example, ref. [1]):

$$H(t) \equiv 1, \quad H(t) = \cos(kt) \quad \text{or} \quad H(t) = \cosh(kt), \qquad k \in \mathbb{R}.$$

If $H(t) \equiv 1$ then $H(t) - 1 \equiv 0$, so $\kappa_H = 0$, and the conclusion in (3) holds.

If $H(t) = \cosh(kt)$ for some $k \in \mathbb{R}$ then using the Taylor expansion $\cosh(t) = 1 + t^2/2 + o(t^2)$ as $t \to 0$, we have

$$\cosh(kt) - 1 = \frac{(kt)^2}{2} + o(t^2), \qquad t \to 0.$$

Therefore,

$$\kappa_H = \lim_{t \to 0} \frac{2(\cosh(kt) - 1)}{t^2} = \lim_{t \to 0} \frac{2\left(\frac{k^2 t^2}{2} + o(t^2)\right)}{t^2} = k^2 \geq 0.$$

If $\kappa_H > 0$ then $k \neq 0$ and $|k| = \sqrt{\kappa_H}$; hence,

$$H(t) = \cosh(kt) = \cosh(\sqrt{\kappa_H}\, t), \qquad t \in \mathbb{R},$$

since cosh is even. If $k = 0$ then $H \equiv 1$ and $\kappa_H = 0$, which is already covered by the first case.

Similarly, if $H(t) = \cos(kt)$ for some $k \in \mathbb{R}$ then the Taylor expansion $\cos(kt) - 1 = -k^2 t^2/2 + o(t^2)$ yields $\kappa_H = -k^2$, and the claim in (2) follows. $\square$

Corollary 1. *Let $F : \mathbb{R}_{>0} \to \mathbb{R}$. Assume F satisfies the composition law on $\mathbb{R}_{>0}$ and has unit log-curvature $\kappa(F) = 1$. Then*

$$F(x) = \frac{x + x^{-1}}{2} - 1 \qquad \text{for all } x > 0.$$

Proof. Let $H(t) = F(e^t) + 1$. By Lemma 7, H satisfies d'Alembert Equation (5), and moreover,

$$\kappa_H = \lim_{t \to 0} \frac{2(H(t) - 1)}{t^2} = \lim_{t \to 0} \frac{2F(e^t)}{t^2} = \kappa(F) = 1.$$

From Theorem 4, $H(t) = \cosh(t)$; hence, $F(e^t) = \cosh(t) - 1 = J(e^t)$. For $x > 0$ and $x = e^{\ln x}$, we have $F(x) = J(x)$. $\square$

Remark 2. *Together with the codomain restriction $F : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ and normalization $F(1) = 0$ in Theorem 1, this yields the stated uniqueness.*

For completeness, and because the results below are used in the stability analysis (Section 4), we record an alternative characterization: the existence of the curvature limit implies sufficient smoothness to differentiate (5) and obtain a linear ODE for H. These lemmas provide an independent way to the classification, although Theorem 4 as stated follows directly from the classical classification.

Lemma 9. *(c.f. [15]) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Fix $T > 0$ and define the central second difference*

$$D_h f(t) := \frac{f(t+h) - 2f(t) + f(t-h)}{h^2} \qquad (|t| \leq T, \ h \neq 0).$$

If there is a continuous function $L : [-T, T] \to \mathbb{R}$ such that

$$\lim_{h \to 0} \sup_{|t| \leq T} |D_h f(t) - L(t)| = 0$$

then $f \in C^2([-T, T])$ and $f''(t) = L(t)$ for all $|t| \leq T$.

Lemma 10. *Let $H : \mathbb{R} \to \mathbb{R}$ satisfy the d'Alembert Equation (5). Suppose the following limit exists*

$$\kappa_H := \lim_{t \to 0} \frac{2 (H(t) - 1)}{t^2} \in \mathbb{R}.$$

Then $H \in C^2(\mathbb{R})$ and

$$H''(t) = \kappa_H H(t) \qquad \text{for all } t \in \mathbb{R}.$$

Proof. By Lemma 6, H is continuous. Fix $T > 0$. Let $t \in [-T, T]$ be arbitrary and let $h \in \mathbb{R}$ with $0 < |h| \leq 1$. Since H is defined on $\mathbb{R}$ then $H(t \pm h)$ are well-defined. The d'Alembert equation yields

$$H(t+h) + H(t-h) = 2H(t)H(h).$$

Rearranging the above, we obtain

$$\frac{H(t+h) - 2H(t) + H(t-h)}{h^2} = 2H(t) \frac{H(h) - 1}{h^2}.$$

Define

$$q(h) := \frac{2(H(h) - 1)}{h^2}.$$

By assumption, $q(h) \to \kappa_H$ as $h \to 0$.

Since H is continuous on $[-T - 1, T + 1]$, there exists $M_T > 0$ such that

$$|H(t)| \leq M_T \qquad \text{for all } |t| \leq T.$$

Hence,

$$\sup_{|t| \leq T} \left| \frac{H(t+h) - 2H(t) + H(t-h)}{h^2} - \kappa_H H(t) \right| = \sup_{|t| \leq T} |H(t)| \, |q(h) - \kappa_H|$$

$$\leq M_T \, |q(h) - \kappa_H| \xrightarrow[h \to 0]{} 0.$$

By Lemma 9 (applied with $f = H$ and $L(t) = \kappa_H H(t)$), it follows that $H \in C^2([-T, T])$ and

$$H''(t) = \kappa_H H(t) \qquad \text{for all } t \in [-T, T].$$

Since $T > 0$ is arbitrary, the conclusion holds for all $t \in \mathbb{R}$. $\quad \square$

Lemma 11. *Let $H \in C^2(\mathbb{R})$ satisfy the d'Alembert Equation (5). Then for all $t \in \mathbb{R}$,*

$$H''(t) = H''(0) \, H(t).$$

Proof. Given that H satisfies

$$H(t + u) + H(t - u) - 2H(t)H(u) = 0.$$

Taking the second derivative with respect to u for fixed t, we get

$$H''(t + u) + H''(t - u) - 2H(t)H''(u) = 0.$$

Therefore, evaluating at $u = 0$, we have

$$2H''(t) - 2H(t)H''(0) = 0,$$

so $H''(t) = H''(0)H(t)$ for all t.　□

Lemma 12. *Let $H \in C^1(\mathbb{R})$ be even. Then $H'(0) = 0$.*

Proof. Since H is even, $H(t) = H(-t)$ for all t. Then

$$H'(0) = \lim_{t \to 0} \frac{H(t) - H(0)}{t} = \lim_{t \to 0} \frac{H(-t) - H(0)}{t} = -\lim_{t \to 0} \frac{H(t) - H(0)}{t} = -H'(0);$$

hence, $H'(0) = 0$.　□

Lemma 13. *Let $\kappa > 0$ and let $f \in C^2(\mathbb{R})$ satisfy $f''(t) = \kappa f(t)$ for all t, with $f(0) = 0$ and $f'(0) = 0$. Then $f(t) = 0$ for all t.*

Proof. Let $\lambda := \sqrt{\kappa}$. Define $g(t) := f'(t) - \lambda f(t)$ and $h(t) := f'(t) + \lambda f(t)$. Then

$$g'(t) = f''(t) - \lambda f'(t) = \kappa f(t) - \lambda f'(t) = \lambda(\lambda f(t) - f'(t)) = -\lambda g(t),$$
$$h'(t) = f''(t) + \lambda f'(t) = \kappa f(t) + \lambda f'(t) = \lambda(\lambda f(t) + f'(t)) = \lambda h(t).$$

Since $g(0) = f'(0) - \lambda f(0) = 0$ and $h(0) = f'(0) + \lambda f(0) = 0$, the unique solutions are $g(t) = g(0)e^{-\lambda t} \equiv 0$ and $h(t) = h(0)e^{\lambda t} \equiv 0$. Hence, $f' = \frac{1}{2}(g + h) \equiv 0$, so f is constant, and with $f(0) = 0$ we conclude $f \equiv 0$.　□

Lemma 14. *Let $\kappa < 0$ and let $f \in C^2(\mathbb{R})$ satisfy $f''(t) = \kappa f(t)$ for all t, with $f(0) = 0$ and $f'(0) = 0$. Then $f(t) = 0$ for all t.*

Proof. Let $\kappa = -\lambda^2$ with $\lambda = \sqrt{-\kappa} > 0$. Define the energy function

$$E(t) := f'(t)^2 + \lambda^2 f(t)^2 \geq 0.$$

Then $E \in C^1(\mathbb{R})$ and, using $f'' = -\lambda^2 f$,

$$E'(t) = 2f'(t)f''(t) + 2\lambda^2 f(t)f'(t) = 2f'(t)\big(f''(t) + \lambda^2 f(t)\big) = 0.$$

Hence, E is constant. Since $E(0) = f'(0)^2 + \lambda^2 f(0)^2 = 0$, we have $E(t) = 0$ for all t. Therefore $f'(t) = 0$ and $f(t) = 0$ for all t.　□

In terms of logarithmic coordinates $H(t) = F(e^t) + 1$, the rigidity problem can be summarized as follows: once the d'Alembert Equation (5) holds and the quadratic calibration (8) exists, the solution is forced into one of the standard cosine/hyperbolic-cosine families, with the parameter fixed by κ_H. In our case, the normalization $F(1) = 0$ forces $H(0) = 1$, and the unit calibration in Theorem 1 forces $\kappa_H = 1$; hence, $H(t) = \cosh(t)$ and $F = J$.

Theorem 1 is intentionally minimal: each assumption plays a distinct role. The following propositions show what fails when individual assumptions are removed.

Proposition 1. *Let $F : \mathbb{R}_{>0} \to \mathbb{R}$ satisfy (6) and $F(1) = 0$. Then $F(x) = F(x^{-1})$ for all $x > 0$.*

Proof. Plug $x = 1$ into Definition 5 to obtain

$$F(y) + F\left(\frac{1}{y}\right) = 2F(1)F(y) + 2F(1) + 2F(y) = 2F(y);$$

hence, $F\left(\frac{1}{y}\right) = F(y)$ for all $y > 0$. $\square$

Proposition 2. *Assume there is a function $W : \mathbb{R}_{>0} \to (0, \infty)$ satisfying*

$$W(xy) = W(x)W(y) \qquad x, y > 0.$$

Define $F_W : \mathbb{R}_{>0} \to \mathbb{R}$ by
$$F_W(x) := \frac{W(x) + W(x)^{-1}}{2} - 1.$$

Then $F_W(1) = 0$, and F_W satisfies the $\mathbb{R}_{>0}$ composition law (Definition 5). If, in addition, W is continuous then there exists $\lambda \in \mathbb{R}$ with $W(x) = x^\lambda$ for all $x > 0$; hence,

$$F_W(x) = \cosh(\lambda \ln x) - 1.$$

Proof. First, $W(1) = W(1 \cdot 1) = W(1)^2$ and $W(1) > 0$ force $W(1) = 1$, so $F_W(1) = \frac{1}{2}(1 + 1) - 1 = 0$. In addition,

$$W(x)W(x^{-1}) = W(xx^{-1}) = W(1) = 1,$$

so $W(x^{-1}) = W(x)^{-1}$ and, hence, $F_W(x) = F_W(x^{-1})$.

Define $H(t) := F_W(e^t) + 1 = \frac{1}{2}(W(e^t) + W(e^t)^{-1})$. Let $t, u \in \mathbb{R}$. Then

$$
\begin{aligned}
H(t + u) + H(t - u) &= \frac{W(e^{t+u}) + W(e^{t+u})^{-1} + W(e^{t-u}) + W(e^{t-u})^{-1}}{2} \\
&= \frac{(W(e^t) + W(e^t)^{-1})(W(e^u) + W(e^u)^{-1})}{2} = 2H(t)H(u),
\end{aligned}
$$

so H satisfies d'Alembert. By Lemma 7, F_W satisfies the $\mathbb{R}_{>0}$ composition law.

Finally, if W is continuous, define $w(t) := \ln W(e^t)$. Then $w : \mathbb{R} \to \mathbb{R}$ is additive: using $W(e^{t+u}) = W(e^t)W(e^u)$, we get $w(t + u) = w(t) + w(u)$. Continuity of W implies continuity of w; hence, $w(t) = \lambda t$ for some $\lambda \in \mathbb{R}$. Therefore $W(e^t) = e^{\lambda t}$, i.e., $W(x) = x^\lambda$ for $x > 0$, and $F_W(e^t) = \cosh(\lambda t) - 1$; equivalently, $F_W(x) = \cosh(\lambda \ln x) - 1$. $\square$

Corollary 2. *For any $\lambda > 0$, define*

$$F_\lambda(x) := \cosh(\lambda \ln x) - 1 \qquad (x > 0).$$

Then F_λ is continuous on $\mathbb{R}_{>0}$, satisfying $F_\lambda(1) = 0$, satisfying the composition law (6), and has $\kappa(F_\lambda) = \lambda^2$. In particular, the unit calibration $\kappa(F) = 1$ fixes $\lambda = 1$.

Proof. Take $W(x) = x^\lambda$ in the previous proposition, so $F_\lambda = F_W$ and (6) holds. Moreover, since $\cosh(\lambda t) - 1 = \frac{\lambda^2 t^2}{2} + o(t^2)$ as $t \to 0$, we get

$$\kappa(F_\lambda) = \lim_{t \to 0} \frac{2(\cosh(\lambda t) - 1)}{t^2} = \lambda^2.$$

□

The following proposition shows that normalization and unit calibration alone do not imply the composition law on $\mathbb{R}_{>0}$.

Proposition 3. *Let us define $F(x) := \frac{1}{2}(\ln x)^2$ on $\mathbb{R}_{>0}$. Then F is continuous, satisfying $F(1) = 0$, and we have $\kappa(F) = 1$, but it does not satisfy the composition law on $\mathbb{R}_{>0}$.*

Proof. Continuity and $F(1) = 0$ are immediate. Moreover, $F(e^t) = t^2/2$, so

$$\kappa(F) = \lim_{t \to 0} \frac{2(t^2/2)}{t^2} = 1.$$

Define $H(t) := F(e^t) + 1 = 1 + t^2/2$. Then for $t, u \in \mathbb{R}$,

$$H(t + u) + H(t - u) = 2 + \frac{(t + u)^2 + (t - u)^2}{2} = 2 + t^2 + u^2,$$

while

$$2H(t)H(u) = 2\left(1 + \frac{t^2}{2}\right)\left(1 + \frac{u^2}{2}\right) = 2 + t^2 + u^2 + \frac{t^2 u^2}{2}.$$

The last two expressions differ when $tu \neq 0$; hence, H does not satisfy d'Alembert's functional equation. By Lemma 7, F does not satisfy the composition law on $\mathbb{R}_{>0}$.

□

The following proposition shows that, without a regularity assumption, the composition law on $\mathbb{R}_{>0}$ (6) admits pathological solutions.

Proposition 4. *There exist functions $F : \mathbb{R}_{>0} \to \mathbb{R}$ satisfying $F(1) = 0$ and the composition law on $\mathbb{R}_{>0}$ (6) that are not measurable (and, hence, not continuous).*

Proof. It is a consequence of the existence of a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ that there exists an additive function $a : \mathbb{R} \to \mathbb{R}$, $a(t + u) = a(t) + a(u)$, which is not measurable (and, hence, not continuous).

Let us define
$$H(t) := \cosh(a(t)), \qquad t \in \mathbb{R}.$$

We claim that H is not measurable. Suppose for contradiction that H is measurable. Since $\mathrm{arcosh} : [1, \infty) \to [0, \infty)$ is continuous and $\cosh(a(t)) \geq 1$, the composition

$$|a(t)| = \mathrm{arcosh}(H(t))$$

is measurable. For each $n \in \mathbb{N}$, define the measurable sets

$$E_n := \{t \in \mathbb{R} : |a(t)| \leq n\}.$$

Then $\bigcup_{n=1}^\infty E_n = \mathbb{R}$, so at least one E_n has a positive Lebesgue measure. On that set, a is bounded. It is a classical fact that an additive function bounded on a set of positive

measure must be linear, hence continuous, contradicting the choice of a. Therefore H is not measurable.

Using the additivity of a and the identity $\cosh(x + y) + \cosh(x - y) = 2\cosh(x)\cosh(y)$, we compute for all $t, u \in \mathbb{R}$,

$$
\begin{aligned}
H(t + u) + H(t - u) &= \cosh(a(t) + a(u)) + \cosh(a(t) - a(u)) \\
&= 2\cosh(a(t))\cosh(a(u)) = 2H(t)H(u).
\end{aligned}
$$

Thus H satisfies d'Alembert's functional equation and $H(0) = \cosh(a(0)) = 1$.

Finally, we define

$$
F(x) := H(\ln x) - 1, \qquad x > 0.
$$

Then $F(1) = 0$, and by Lemma 7 F satisfies the composition law on $\mathbb{R}_{>0}$ (6). If F were measurable then $H(t) = F(e^t) + 1$ would be measurable as a composition of measurable functions, contradicting the non-measurability of H. Hence, F is not measurable. $\square$

4. Consistency estimate under bounded d'Alembert defect

In this part, we give a local stability result. If the d'Alembert equation holds with a uniform defect on a compact set and the function is sufficiently smooth then the solution is close to the hyperbolic cosine case.

Definition 6. *For $H : \mathbb{R} \to \mathbb{R}$, we define the* d'Alembert defect

$$
\Delta_H(t, u) := H(t + u) + H(t - u) - 2H(t)H(u). \tag{9}
$$

Theorem 5. *Fix $T > 0$. Let $H \in C^3([-T, T])$ be even with $H(0) = 1$, and set $a := H''(0)$. Assume $a > 0$. Let*

$$
\varepsilon := \sup_{|t| \leq T, \, |u| \leq T} |\Delta_H(t, u)|, \qquad B := \sup_{|t| \leq T} |H(t)|, \qquad K := \sup_{|t| \leq T} |H^{(3)}(t)|.
$$

Then for every h with $0 < h \leq T$ and every t with $|t| \leq T - h$,

$$
\left| H(t) - \cosh(\sqrt{a}\, t) \right| \leq \frac{\delta(h)}{a} \left(\cosh(\sqrt{a}\, |t|) - 1 \right),
$$

where

$$
\delta(h) := \frac{\varepsilon}{h^2} + \frac{(1 + B)K}{3} h.
$$

Proof. Fix $0 < h \leq T$ and $|t| \leq T - h$.

Using the integral form of Taylor's theorem, we have

$$
H(t + h) = H(t) + hH'(t) + \frac{h^2}{2}H''(t) + \int_0^h \frac{(h - s)^2}{2} H^{(3)}(t + s)\, ds,
$$

$$
H(t - h) = H(t) - hH'(t) + \frac{h^2}{2}H''(t) - \int_0^h \frac{(h - s)^2}{2} H^{(3)}(t - s)\, ds.
$$

Adding the last two equations and bounding $|H^{(3)}| \leq K$ yields

$$
\left| H(t + h) + H(t - h) - 2H(t) - h^2 H''(t) \right| \leq \frac{K}{3} h^3. \tag{10}
$$

Since H is even, $H'(0) = 0$, and the integral form at 0 gives

$$\left|H(h) - 1 - \frac{a}{2}h^2\right| \leq \frac{K}{6}h^3. \tag{11}$$

Now, by definition of Δ_H (9), we have for (t, h)

$$H(t + h) + H(t - h) = 2H(t)H(h) + \Delta_H(t, h).$$

Subtracting $2H(t) + ah^2 H(t)$ from both sides, we obtain

$$h^2\big(H''(t) - aH(t)\big) = \big(H(t + h) + H(t - h) - 2H(t) - h^2 H''(t)\big)$$
$$+ \Delta_H(t, h) + 2H(t)\big(H(h) - 1 - \frac{a}{2}h^2\big).$$

Taking absolute values in the last equation and using (10), (11), $|H(t)| \leq B$, and $|\Delta_H(t, h)| \leq \varepsilon$, we obtain

$$h^2|H''(t) - aH(t)| \leq \frac{K}{3}h^3 + \varepsilon + 2B \cdot \frac{K}{6}h^3 \leq \varepsilon + \frac{(1 + B)K}{3}h^3.$$

Dividing by h^2 yields the uniform bound

$$|H''(t) - aH(t)| \leq \delta(h) \qquad (|t| \leq T - h). \tag{12}$$

Let $y(t) := \cosh(\sqrt{a}\,t)$, so $y'' = ay$, $y(0) = 1$, and, since H is even, $H'(0) = 0 = y'(0)$. Define $e(t) := H(t) - y(t)$. Then $e \in C^2([-T + h, T - h])$, $e(0) = e'(0) = 0$, and

$$e''(t) - ae(t) = H''(t) - aH(t),$$

so, by (12), $|e''(t) - ae(t)| \leq \delta(h)$ for $|t| \leq T - h$. For $t \in [0, T - h]$, the equation for $e'' = ae + r$ with zero initial condition gives

$$e(t) = \int_0^t \frac{1}{\sqrt{a}}\sinh(\sqrt{a}(t - s))\,r(s)\,ds,$$

where $r(s) := e''(s) - ae(s)$. Hence,

$$|e(t)| \leq \delta(h)\int_0^t \frac{1}{\sqrt{a}}\sinh(\sqrt{a}(t - s))\,ds = \frac{\delta(h)}{a}\big(\cosh(\sqrt{a}\,t) - 1\big).$$

Since e is even as the difference of even functions, this bound holds for negative t, yielding the inequality for all $|t| \leq T - h$. $\square$

Remark 3. *The C^3 smoothness is natural when H arises as an approximate d'Alembert solution with controlled defect; if H additionally satisfies $H \in C^2(\mathbb{R})$ and (5) then Lemma 11 gives $H'' = H''(0)H$, which implies $H \in C^\infty$.*

Corollary 3. *Let the assumptions of Theorem 5 hold and define*

$$F(x) := H(\ln x) - 1, \qquad x \in \mathbb{R}_{>0}.$$

Then for every $x \in \big(e^{-(T-h)}, e^{T-h}\big)$,

$$\left|F(x) - \big(\cosh(\sqrt{a}\,\ln x) - 1\big)\right| \leq \frac{\delta(h)}{a}\Big(\cosh\big(\sqrt{a}\,|\ln x|\big) - 1\Big).$$

In particular, for every $S \in (0, T - h)$ the function F is uniformly close to $\cosh(\sqrt{a} \ln x) - 1$ on the compact interval $[e^{-S}, e^{S}]$.

If, moreover, a is close to 1 and $\delta(h)$ is small then F is uniformly close to $J(x) = \cosh(\ln x) - 1$ on $[e^{-S}, e^{S}]$.

In the special case $a = 1$, the estimate simplifies to

$$\left| F(x) - J(x) \right| \leq \delta(h) \, J(x), \qquad x \in \left(e^{-(T-h)}, e^{T-h} \right),$$

since $\cosh(|\ln x|) - 1 = \cosh(\ln x) - 1 = J(x)$.

Proof. Apply Theorem 5 with $t = \ln x$. Since $F(x) = H(t) - 1$, we obtain

$$\left| F(x) - (\cosh(\sqrt{a} \, t) - 1) \right| \leq \frac{\delta(h)}{a} \left(\cosh(\sqrt{a} \, |t|) - 1 \right),$$

which is the required inequality after substituting $t = \ln x$.

The case $a = 1$ follows from $\cosh(|\ln x|) - 1 = \cosh(\ln x) - 1 = J(x)$. $\quad \square$

5. Properties of Canonical Reciprocal Cost

Having shown that the canonical reciprocal cost

$$J(x) = \frac{1}{2}(x + x^{-1}) - 1$$

is uniquely determined by the composition law on $\mathbb{R}_{>0}$ and the unit log-curvature calibration, we now give some properties of J that make it interpretable and useful.

5.1. Arithmetic and Geometric Means

The function $J(x)$ can be written as

$$J(x) = \mathrm{AM}\left(x, \frac{1}{x} \right) - \mathrm{GM}\left(x, \frac{1}{x} \right),$$

since $\mathrm{GM}(x, 1/x) = 1$ and $\mathrm{AM}(x, 1/x) = \frac{1}{2}(x + x^{-1})$. In particular, $J(x) \geq 0$ for all $x > 0$, with equality if and only if $x = 1$.

5.2. Bregman Divergence

The logarithmic expression of the canonical reciprocal cost is

$$G(t) = J(e^{t}) = \cosh t - 1.$$

Let $\Phi : \mathbb{R} \to \mathbb{R}$ be defined by $\Phi(t) = \cosh t$. Then G coincides with the Bregman divergence generated by Φ at the point 0, namely

$$G(t) = D_{\Phi}(t, 0) = \Phi(t) - \Phi(0) - \Phi'(0)(t - 0),$$

since $\Phi(0) = 1$ and $\Phi'(0) = \sinh 0 = 0$.

Since $\Phi \in C^{2}(\mathbb{R})$ and

$$\Phi''(t) = \cosh t > 0 \qquad \text{for all } t \in \mathbb{R},$$

the function Φ is strictly convex. Consequently,

$$D_{\Phi}(t, 0) > 0 \quad \text{for } t \neq 0, \qquad D_{\Phi}(0, 0) = 0,$$

and 0 is the unique global minimizer of Φ.

Thus, in logarithmic coordinates the canonical cost J corresponds to the Bregman divergence $G(t) = D_\Phi(t, 0)$. The strict convexity of Φ implies non-negativity and a unique minimum at $t = 0$, while the Taylor expansion of cosh at 0 gives the required quadratic behavior.

5.3. Metric Associated with J

Starting from the canonical reciprocal cost function

$$J(x) = \frac{1}{2}\left(x + \frac{1}{x}\right) - 1,$$

we introduce logarithmic coordinates $t = \ln x$ and set

$$G(t) = J(e^t) = \cosh t - 1, \qquad \Phi(t) = G(t) + 1 = \cosh t.$$

The function Φ is smooth and strictly convex on $\mathbb{R}$, since

$$\Phi''(t) = \cosh t > 0 \qquad \text{for all } t \in \mathbb{R}.$$

The Hessian metric induced by Φ on $\mathbb{R}$ is

$$ds^2 = \Phi''(t)\, dt^2 = \cosh t\, dt^2.$$

In x-coordinates, we get

$$ds^2 = \cosh(\ln x)\frac{dx^2}{x^2} = \frac{x^2 + 1}{2x^3}\, dx^2.$$

The corresponding Riemannian distance is

$$d_J(x, y) = \left|\int_{\ln x}^{\ln y} \sqrt{\cosh u}\, du\right| = \left|\int_x^y \sqrt{\frac{\xi^2 + 1}{2\xi^3}}\, d\xi\right|.$$

Since cosh is an even function, the distance is reciprocally symmetric, i.e.,

$$d_J(x, y) = d_J\left(\frac{1}{x}, \frac{1}{y}\right).$$

Near $x = 1$ (that is, $t = 0$), one has $\cosh t = 1 + O(t^2)$, and, hence, $d_J(x, y)$ is locally equivalent to the logarithmic distance $|\ln y - \ln x|$. More precisely, there exist constants $c, C > 0$ and a neighborhood U of 1, such that

$$c\,|\ln y - \ln x| \leq d_J(x, y) \leq C\,|\ln y - \ln x|, \qquad x, y \in U.$$

Globally, the growth of d_J is qualitatively different from the logarithmic distance because the metric weight $\sqrt{\cosh u}$ grows exponentially as $|u| \to \infty$.

As $|t| \to \infty$, one has $\cosh t \sim \frac{1}{2}e^{|t|}$, and therefore

$$d_J(1, R) \sim \sqrt{2}\, R^{1/2} \qquad \text{as } R \to \infty,$$

reflecting the exponential growth of $\Phi''(t)$ for large $|t|$.

5.4. Chebyshev Structure of J

It is known that normalized solutions of d'Alembert equation

$$H(m + n) + H(m - n) = 2H(m)H(n), \qquad m, n \in \mathbb{Z},$$

can be expressed in terms of Chebyshev polynomials (see, for example, ref. [8]). In particular, such solutions satisfy

$$H_{n+1} = 2H_1 H_n - H_{n-1} \tag{13}$$

and can be written in the form $H_n = T_n(H_1)$, where T_n denotes the Chebyshev polynomials.

We now show how this discrete Chebyshev structure is realized by the canonical reciprocal cost. Let $H(t) = J(e^t) + 1 = \cosh t$, which satisfies the continuous d'Alembert Equation (5). Fix $t = \ln x$ and define the discrete sequence $H_n := H(nt)$ for $n \in \mathbb{Z}$. Then (H_n) satisfies the recursion (13), and

$$J(x^n) + 1 = \cosh(n \ln x) = T_n(\cosh(\ln x)) = T_n(J(x) + 1);$$

that is,

$$J(x^n) = T_n(J(x) + 1) - 1, \qquad n \in \mathbb{Z}.$$

Since $H_1 = \cosh(\ln x) \geq 1$, the Chebyshev identity $T_n(\cosh s) = \cosh(ns)$ applies (with $s = \ln x$), yielding $H_n = \cosh(n \ln x)$. In particular, for $x \neq 1$ we have $H_1 > 1$, so the oscillatory cosine branch (which takes values in $[-1, 1]$) is not compatible with the canonical cost J.

Thus, the Chebyshev recursion associated with the discrete d'Alembert equation is realized explicitly by the functional J.

5.5. Golden Ratio

A natural connection with the golden ratio follows from the fixed-point structure associated with the canonical reciprocal cost. Consider the Möbius transformation $T(x) = 1 + \frac{1}{x}$ on $\mathbb{R}_{>0}$. Its fixed points satisfy $x = T(x)$, i.e., $x^2 - x - 1 = 0$, and the unique positive fixed point is

$$\varphi = \frac{1 + \sqrt{5}}{2}.$$

Since $J(x) = J(x^{-1})$, the reciprocal pair (φ, φ^{-1}) is assigned the same cost, and explicitly

$$J(\varphi) = J(\varphi^{-1}) = \varphi - \frac{3}{2}.$$

The convergence $x_{n+1} = T(x_n) \to \varphi$ for any $x_0 > 0$ is valid. For a related "golden ratio via means" perspective, see [16].

5.6. Energy Interpretation

Although J is characterized here only by a functional equation, it is natural to view it heuristically as an energy-like penalty for multiplicative imbalance. Indeed, $J \geq 0$ with a unique minimizer at equilibrium $x = 1$, and in logarithmic coordinates $x = e^t$ one has

$$J(e^t) = \cosh t - 1 = \frac{t^2}{2} + O(t^4) \qquad (t \to 0),$$

which matches the standard second-order behavior near a stable equilibrium in many variational models. This interpretation should be read as an analogy: J is not claimed to be the Hamiltonian energy of a specific dynamical system.

6. Conclusions

This paper studied the rigidity problem for non-negative functions $F : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ normalized by $F(1) = 0$. The problem addressed in this paper was to determine which structural assumptions force F to have a unique functional form. Under the composition law (6) on $\mathbb{R}_{>0}$ and the unit quadratic calibration (4), F is uniquely determined. The resulting function is the canonical reciprocal cost

$$J(x) = \frac{1}{2}(x + x^{-1}) - 1;$$

equivalently, $J(e^t) = \cosh(t) - 1$ in logarithmic coordinates.

The composition law becomes d'Alembert's functional Equation (5) on $\mathbb{R}$, and the calibration becomes a curvature condition at $t = 0$. The calibration plays two distinct roles: it provides minimal regularity so that classical classification results for d'Alembert's equation apply, and it fixes the remaining scale parameter that otherwise yields the family $H(t) = \cosh(kt)$ or $H(t) = \cos(kt)$. As a result, the calibrated solution is forced onto the single branch $H(t) = \cosh(t)$, hence $F = J$.

Section 3 also established the minimality of the assumptions. The composition law together with $F(1) = 0$ implies reciprocity $F(x) = F(x^{-1})$. Without fixing the calibration, one gets a one-parameter family of continuous solutions $F_\lambda(x) = \cosh(\lambda \ln x) - 1$. On the other hand, the normalization $F(1) = 0$ together with the calibration does not imply the composition law (6) (see Proposition 3). Finally, even under the normalization $F(1) = 0$, the composition law (6) admits non-measurable solutions (see Proposition 4). Hence, some additional regularity assumption is necessary.

Several directions remain for further study. An open question is whether the composition law (6) can be derived from weaker assumptions, rather than being imposed explicitly. Another direction is to consider alternative calibration conditions and to classify other polynomial-type relations between $F(xy) + F(x/y)$ and the pair $(F(x), F(y))$.

It would also be of interest to extend the analysis to other multiplicative structures, such as positive definite matrices [17] or more general structures.

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