

FROM ONE GENERATOR TO LOOP ORDER: RECOGNITION AND MEMORY ON HYPERCUBES

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ABSTRACT. We study recognition of based closed walks on cube graphs. On the three-cube, every additive one-step reading factors through directed transition counts, whereas the reduced loop word takes values in the free group of rank five. For each $k \geq 1$, we construct two walks of the same length and with the same directed transition counts which every sequential reading with at most k states identifies, although their reduced loop words are not conjugate. Thus even exact transition counts together with any fixed finite memory cannot recover loop order. An unbounded stack recovers the reduced word on every walk.

In every hypercube Q_n , $n \geq 3$, the shortest nontrivial loop with zero abelianization has edge length 14. A fourteen-step witness already has nonzero degree-two commutator information. On the three-cube the reduced word defines a proper recognition quotient, strictly between equality and abelianization. A declared quaternion clock adds a 24-element group reading to transition counts, giving a proper quotient incomparable with the reduced-word quotient. We also prove an exact criterion for every occupation-based constraint to separate into independent state and history conditions: the occupation vector must be constant on the allowed histories. These results distinguish the information retained by a state, a commutative summary, and an ordered reading of a history.

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1. FROM ONE GENERATOR TO LOOP ORDER

A bare distinction $x \neq y$ does not give any arithmetic structure. It gives no zero, no repeatable operation and no induction. We assume finite repeatability, represented by the empty record 0 and the one step extension $t \mapsto St$, where t is a record. In [1], we proved that the generated orbit is a free additive monoid on one generator, isomorphic to $(\mathbb{N}, +, 0)$. A recognizer of a monoid M is a monoid homomorphism

$$r : M \longrightarrow N,$$

where the target monoid N need not be finite. This is more general than the finite monoid convention used in [14]. The relation

$$x \sim_r y \iff r(x) = r(y)$$

is the kernel congruence of r , and $M/\sim_r$ is its recognition quotient.

Under excluded middle, in [1], we classified the recognition quotients of this monogenic monoid: every recognizer is either injective or has a finite index-period quotient $M(i, p)$. Since the carrier is generated by $S0$, every monoid recognizer is determined by the image of this one generator, so there is no independent order information. We now consider a carrier with several independent loops, where order can matter. The foundational classification has a constructive refinement: for a decidable congruence with an explicit collision, its index and period are obtained by bounded searches. Markov's principle suffices when only noninjectivity is given, and excluded middle suffices for an arbitrary congruence. Here the separating walks and their readings are explicit. The question carried forward is which distinctions survive a recognizer, now when the source admits independent composition order.

We take as carrier the monoid $\mathcal{W}$ of based closed walks on the 1-skeleton of the three-dimensional cube Q_3 . The restriction to based closed walks is a convention, not a consequence of recognition. We fix a spanning tree and obtain a homomorphism

$$\rho : \mathcal{W} \longrightarrow F_5,$$

where F_5 is the free group of rank five, which assigns to each closed walk its reduced loop word. For Q_2 , the fundamental group is free of rank one. For Q_3 , it is free of rank five. Thus we first obtain noncommuting reduced loop words on Q_3 .

Definition 1.1. A sequential reading consists of a state set S , an initial state $s_0 \in S$, update maps $T_e : S \rightarrow S$ for directed edges e , a set Y , and an output map $q : S \rightarrow Y$. On a walk whose directed edge sequence is $e_1, \dots, e_n$, its value is

$$q(T_{e_n} \circ \dots \circ T_{e_1}(s_0)).$$

We call such a sequential reading a *clock*. The state set may be finite or infinite. For an additive reading, we take S to be an abelian group and

$$T_e(s) = s + g(e).$$

The edge updates extend to a homomorphism from edge strings into the transformation monoid of S , with multiplication taken in reading order. Evaluation at s_0 and the output map then give the sequential reading. Those last operations need not preserve multiplication, so the resulting output need not be a monoid recognizer. We use *recognition quotient* for the kernel of a monoid recognizer.

Let

$$\text{ab} : F_5 \longrightarrow \mathbb{Z}^5$$

be the abelianization map, and put

$$a = \text{ab} \circ \rho. \tag{1}$$

We construct two closed walks with the same directed transition counts. Every additive one-step reading takes the same value on these walks, while their reduced loop words are different. So, directed transition counts, and more generally any commutative aggregation, do not determine the reduced loop word.

We next hold the directed transition counts fixed while increasing the required memory. Let $A = (000; 0101)$ and let $C = (000; 01020102)$ be the Gray cycle. For $N = k!$, the walks

$$P_k = A^N C A^{3N} C, \quad Q_k = A^{2N} C A^{2N} C$$

both have length $16N + 16$ and transition tally $4N\tau_A + 2\tau_C$. Every sequential reading with at most k states identifies them, but their reduced loop words are not even conjugate. This remains an obstruction when the entire transition tally is supplied alongside the finite state. An unbounded stack computes $\rho(w)$ on every history by cancelling adjacent inverse letters. Theorem 5.5 gives the uniform finite-memory obstruction; Theorem 5.6 gives exact recovery.

The induced congruences satisfy

$$\Delta_{\mathcal{W}} \subsetneq \sim_{\rho} \subsetneq \sim_a,$$

where $\Delta_{\mathcal{W}}$ is equality on $\mathcal{W}$. Thus ρ is not injective, but its kernel is strictly finer than that of a . In particular, ρ defines a proper noncommutative recognition quotient of $\mathcal{W}$.

The shortest based closed walk with trivial abelianization and a nontrivial reduced loop word has length 14. This minimum is already attained outside the third lower-central subgroup. In particular,

$$\min\{|w| : \rho(w) \in \gamma_2(F_5) \setminus \gamma_3(F_5)\} = 14.$$

The same minimum holds in every Q_n , $n \geq 3$. Here $\gamma_1(F_5) = F_5$ and $\gamma_{n+1}(F_5) = [\gamma_n(F_5), F_5]$, so $\gamma_2(F_5) = [F_5, F_5]$. Thus $\rho(w) \in \gamma_2(F_5)$ means that its abelianization is trivial, while $\rho(w) \notin \gamma_3(F_5)$ means that its class in $\gamma_2(F_5)/\gamma_3(F_5)$ is nontrivial.

A declared quarter-turn product, defined by assigning a quaternion factor to each of the three coordinate directions, gives a second proper order-sensitive congruence; see Theorem 8.2. This congruence is incomparable with $\sim_\rho$; see Proposition 8.4.

2. THE THREE-DIMENSIONAL CUBE

We now fix the vertices of the three-dimensional cube Q_3 , the spanning tree, and the five generators associated with the cotree edges. Let

$$V = \{0, 1\}^3$$

be the vertex set of Q_3 . We number the coordinates by 0, 1, 2. A step in direction i changes the i -th coordinate from 0 to 1, or from 1 to 0. We fix the basepoint $o = 000$ and write a walk as

$$w = (o; a_1 a_2 \cdots a_n), \quad a_k \in \{0, 1, 2\}.$$

Let $v_0 = o, v_1, \dots, v_n$ be the associated vertex sequence, where v_k is obtained from v_{k-1} by changing coordinate a_k . The walk w is closed when $v_n = o$. We write

$$e_k = (v_{k-1}, v_k)$$

for the k -th directed edge. Thus the same walk can also be written $w = e_1 \cdots e_n$ when its directed edges are used.

Lemma 2.1. *A walk $w = (o; a_1 \cdots a_n)$ is closed if and only if each direction 0, 1, 2 occurs an even number of times among $a_1, \dots, a_n$.*

Proof. Each step in direction i changes only the i -th coordinate. Hence the i -th coordinate returns to its initial value if and only if direction i is used an even number of times. Therefore $v_n = o$ if and only if each of the three directions occurs an even number of times. $\square$

Two walks are equal when their step sequences coincide. Concatenation of based closed walks is based and closed, and the empty walk $1 = (o;)$ is a neutral element. Thus $\mathcal{W}$ is a monoid. By Lemma 2.1 it is the submonoid of the free monoid on $\{0, 1, 2\}$ consisting of the words in which each letter occurs an even number of times.

2.1. The universal source and its commutative quotient. A nonempty based closed walk is a *first-return loop* if it meets o only at its two ends. Let L be the set of such loops.

Proposition 2.2. *The monoid $\mathcal{W}$ is the free monoid on L . Thus, for every monoid M , a recognizer $\mathcal{W} \rightarrow M$ is determined uniquely by arbitrary values assigned to the members of L . Its universal commutative quotient is the free commutative monoid of finite multisets of L .*

Proof. Cut a walk at every return to o . The resulting factors are first-return loops, and every first-return factorization must use exactly these cuts. This gives the unique factorization into L . An assigned value on each factor extends by ordered multiplication, which proves the first universal property. If the target is commutative, this product depends only on the multiset of factors. Conversely, any assignment into a commutative monoid extends uniquely from those multisets, proving the second property. $\square$

This is the direct successor of the one-generator universal property: the single generator is replaced by the first-return loops. Directed transition counts are a further quotient of their multiset, obtained by summing each factor's edge counts. The quotient is strict. Indeed,

$$u = (000; 01011010), \quad v = (000; 00101101)$$

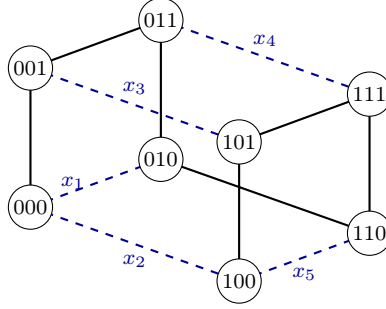


FIGURE 1. The 1-skeleton of Q_3 . Solid edges form the Gray-code spanning tree. Dashed edges are the five cotree edges, labeled by $x_1, \dots, x_5$.

have the same directed transition counts, but their first-return factorizations are respectively

$$(0101)(1010), \quad (00)(101101).$$

The first multiset has two factors of length four; the second has factors of lengths two and six. Thus transition counts are universal for additive edge readings, while first-return multisets are universal for all commutative monoid recognizers.

Raw based-walk composition can already be noncommutative on Q_2 . The distinction special to Q_3 is the first noncommutative *reduced-loop* quotient among hypercubes. Tree cancellation and first-return factorization retain different information.

2.2. The reduced-loop quotient. We fix the Gray-code spanning tree

$$000, 001, 011, 010, 110, 111, 101, 100,$$

where successive vertices differ in one coordinate. The remaining five edges are the cotree edges

$$(0, 2), \quad (0, 4), \quad (1, 5), \quad (3, 7), \quad (4, 6),$$

where a vertex is written $b_2b_1b_0$, with b_i the i -th coordinate, and is read as the integer $4b_2 + 2b_1 + b_0$. Thus $000, \dots, 111$ become $0, \dots, 7$, and a step in direction i changes this integer by $\pm 2^i$. We orient each cotree edge from the smaller integer to the larger and denote them by $x_1, \dots, x_5$.

For a based closed walk w , a forward crossing of the i -th cotree edge contributes x_i , a backward crossing contributes x_i^{-1} , and a tree edge contributes no letter. This gives a word $\ell(w)$. After free reduction we obtain the homomorphism

$$\rho : \mathcal{W} \longrightarrow F_5.$$

Since

$$\ell(uv) = \ell(u)\ell(v),$$

we have

$$\rho(uv) = \rho(u)\rho(v), \quad \rho(1) = 1.$$

Hence ρ is a monoid homomorphism. This is the spanning tree presentation of the fundamental group of a connected graph (see e.g. [2, 3, 4]). In the sense of Definition 1.1, free reduction is a clock whose state is the reduced word.

One transcription fixes the convention. The square walk $(000; 0101)$ passes through the vertices $0, 1, 3, 2, 0$. Its first three steps are tree edges and contribute nothing. The last step crosses the cotree edge $(0, 2)$ from the larger integer to the smaller, so it contributes x_1^{-1} . Hence its reduced loop word is x_1^{-1} .

Proposition 2.3. *The cycle ranks of Q_0, Q_1, Q_2, Q_3 are*

$$0, 0, 1, 5.$$

Hence

$$\pi_1(Q_2) \cong \mathbb{Z}, \quad \pi_1(Q_3) \cong F_5.$$

Thus Q_3 is the first hypercube whose reduced loop words can be noncommutative.

Proof. For a connected graph, the cycle rank is

$$|E| - |V| + 1.$$

The graph Q_0 has one vertex and no edges. For $d \geq 1$, the d -cube has 2^d vertices and $d2^{d-1}$ edges. Hence its cycle rank is

$$d2^{d-1} - 2^d + 1,$$

which gives the corresponding values.

The fundamental group of a connected graph is free of rank equal to its cycle rank [2, 3, 4]. Thus

$$\pi_1(Q_2) \cong F_1 \cong \mathbb{Z}, \quad \pi_1(Q_3) \cong F_5.$$

Since F_1 is abelian and F_5 is nonabelian, the last statement follows. $\square$

2.3. The classical 7 + 5 decomposition. We choose an orientation of the twelve edges and let $C_1(Q_3; \mathbb{R})$ denote the real edge space. Using the standard Euclidean identification of $C^1(Q_3; \mathbb{R})$ with $C_1(Q_3; \mathbb{R})$, let

$$d : C^0(Q_3; \mathbb{R}) \longrightarrow C^1(Q_3; \mathbb{R})$$

be the coboundary map, so that for a vertex potential ϕ ,

$$(d\phi)(u, v) = \phi(v) - \phi(u)$$

on an oriented edge $u \rightarrow v$. Since Q_3 is connected, $\ker d$ consists of the constant potentials. Hence

$$\dim(\operatorname{im} d) = 8 - 1 = 7.$$

Let

$$\partial : C_1(Q_3; \mathbb{R}) \longrightarrow C_0(Q_3; \mathbb{R})$$

be the boundary map. Its kernel is the cycle space. Hence

$$\dim(\ker \partial) = 12 - 8 + 1 = 5.$$

For $\phi \in C^0(Q_3; \mathbb{R})$ and $z \in \ker \partial$, we have

$$\langle d\phi, z \rangle = \langle \phi, \partial z \rangle = 0.$$

Thus $\operatorname{im} d$ and $\ker \partial$ are orthogonal, and their dimensions add to twelve.

Accordingly, the edge space admits the orthogonal decomposition

$$C_1(Q_3; \mathbb{R}) = \operatorname{im} d \oplus \ker \partial, \quad 12 = 7 + 5.$$

The map $a = \operatorname{ab} \circ \rho$ given by (1) records the five cycle coordinates, i.e. the exponent sums of the five generators. More precisely, let C_i be the fundamental cycle associated with the i -th cotree edge, oriented so that this cotree edge has coefficient +1. Then $C_1, \dots, C_5$ form a basis of the integral cycle space. For every based closed walk w , its oriented 1-chain satisfies

$$z(w) = \sum_{i=1}^5 a_i(w) C_i.$$

In particular, $a(w) = 0$ if and only if $z(w) = 0$.

These coordinates do not determine the reduced loop word in F_5 , nor the order of the five cotree crossings.

3. ADDITIVE READINGS

Let $\mathcal{A} = \mathbb{Z}^V$ be the group of integer functions on the vertex set V . For $\phi \in \mathcal{A}$ and a walk $w = (v_0, \dots, v_n)$, define

$$R_\phi(w) = \sum_{k=1}^n (\phi(v_k) - \phi(v_{k-1})). \quad (2)$$

This is the pairing of the oriented 1-chain defined by the walk with the exact cochain $d\phi$.

Theorem 3.1. *For every $\phi \in \mathcal{A}$ and every walk $w = (v_0, \dots, v_n)$, it holds*

$$R_\phi(w) = \phi(v_n) - \phi(v_0).$$

In particular, $R_\phi(w) = 0$ for every closed walk.

Proof. Expanding the sum, all intermediate terms cancel, so

$$\sum_{k=1}^n (\phi(v_k) - \phi(v_{k-1})) = \phi(v_n) - \phi(v_0).$$

□

Thus an exact potential reading depends only on the endpoints of the walk. On $\mathcal{W}$, it is therefore the trivial recognizer.

The potential reading (2) is a special additive one-step reading, with the weight of a directed edge $p \rightarrow q$ given by

$$g(p, q) = \phi(q) - \phi(p).$$

We now allow arbitrary weights on directed steps. Let G be an abelian group and let

$$g : V \times V \longrightarrow G$$

be a weight function. For a walk $w = (v_0, \dots, v_n)$, define

$$R_g(w) = \sum_{k=1}^n g(v_{k-1}, v_k).$$

We call R_g an *additive one-step reading*. Thus $R_g(w)$ is the accumulated weight along the walk. We call g *antisymmetric* if

$$g(q, p) = -g(p, q)$$

for every edge $\{p, q\}$. Then opposite traversals cancel, so R_g depends only on the oriented 1-chain.

For $p, q \in V$, define the directed transition count by

$$\tau_w(p, q) = \#\{k : (v_{k-1}, v_k) = (p, q)\}.$$

Theorem 3.2. *Every additive one-step reading factors through the directed transition counts:*

$$R_g(w) = \sum_{(p,q) \in V \times V} \tau_w(p, q) g(p, q).$$

Consequently, if $\tau_u = \tau_v$ then $R_g(u) = R_g(v)$ for every abelian group G and every weight function g .

Proof. Each directed step (p, q) occurs exactly $\tau_w(p, q)$ times in the walk. By collecting equal terms, we obtain

$$R_g(w) = \sum_{(p,q) \in V \times V} \tau_w(p, q) g(p, q).$$

Hence $R_g(w)$ depends on the directed transition counts and not on their order. □

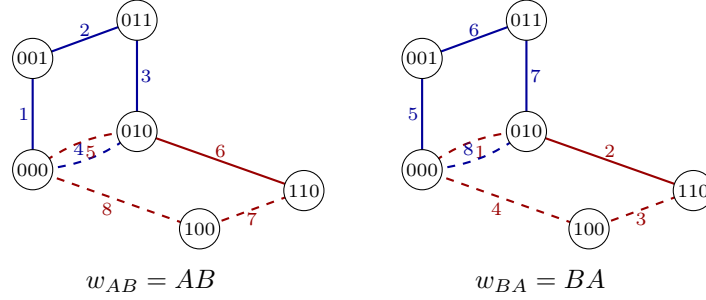


FIGURE 2. The square faces A and B concatenated in both orders, with the step order marked and cotree edges dashed. The two walks traverse the same directed edges, so $\tau_{w_{AB}} = \tau_{w_{BA}}$, while their reduced loop words differ.

Theorem 3.2 applies to one-step readings with values in an abelian group. It does not apply to arbitrary functions of the ordered sequence, nor to noncommutative updates.

Let

$$A = (000; 0101), \quad B = (000; 1212). \quad (3)$$

be two square loops, and define

$$w_{AB} = AB, \quad w_{BA} = BA.$$

Both walks are closed and have length eight.

Example 3.3. The walk A is the square loop in coordinates 0 and 1, and B is the square loop in coordinates 1 and 2, both based at 000. Hence AB and BA traverse the same two loops, but in different order. The corresponding vertex sequences are

$$0, 1, 3, 2, 0, 2, 6, 4, 0 \quad \text{and} \quad 0, 2, 6, 4, 0, 1, 3, 2, 0.$$

Theorem 3.4. The walks w_{AB} and w_{BA} have the same directed transition counts but different reduced loop words:

$$\tau_{w_{AB}} = \tau_{w_{BA}}, \quad \rho(w_{AB}) \neq \rho(w_{BA}).$$

Proof. Transition counts are additive under concatenation, so we have

$$\tau_{AB} = \tau_A + \tau_B = \tau_B + \tau_A = \tau_{BA}.$$

We have

$$\rho(A) = x_1^{-1}, \quad \rho(B) = x_1 x_5^{-1} x_2^{-1}.$$

Therefore, we obtain

$$\rho(AB) = x_5^{-1} x_2^{-1}, \quad \rho(BA) = x_1 x_5^{-1} x_2^{-1} x_1^{-1}.$$

These are distinct reduced words in F_5 . □

Thus no reading determined by the directed transition counts can recover the reduced loop word. Since a walk $w = e_1 \cdots e_n$ records the order of its directed edges, an arbitrary function of w may depend on this order. The claim is only that commutative aggregation does not.

4. THE FOURTEEN-STEP WITNESS

The pair w_{AB}, w_{BA} shows that additive readings do not detect order, but each walk has nonzero abelian loop content. We now give a stronger witness. Its five signed cycle counts are all zero, but it has nontrivial noncommutative information.

Let w be a walk of length n . Write

$$a_i(w) \in \mathbb{Z}, \quad 1 \leq i \leq 5,$$

for the signed traversal count of the i -th cotree edge. Let $s_i^{(k)} \in \{-1, 0, 1\}$ be the contribution of step k to this count, and set

$$a_i^{(0)} = 0, \quad a_i^{(k)} = \sum_{\ell=1}^k s_i^{(\ell)}.$$

Thus $a_i^{(k-1)}$ is the signed count before step k , and $a_i^{(n)} = a_i(w)$. We define the antisymmetric degree two coordinate

$$\mu_{ij}(w) = \sum_{k=1}^n \left(a_i^{(k-1)} s_j^{(k)} - a_j^{(k-1)} s_i^{(k)} \right).$$

We write

$$\mu(w) = (\mu_{ij}(w))_{1 \leq i < j \leq 5} \in \mathbb{Z}^{10}.$$

Thus μ_{ij} records the order interaction between the i -th and j -th cotree crossings: each j -crossing is weighted by the current i -count, and each i -crossing by the current j -count with the opposite sign.

Lemma 4.1. *The coordinates μ_{ij} are invariant under free reduction, so $\mu(w)$ depends only on $\rho(w)$. If $a(w) = 0$, then each $\mu_{ij}(w)$ is even and, under the isomorphism*

$$\gamma_2(F_5)/\gamma_3(F_5) \cong \Lambda^2 \mathbb{Z}^5,$$

where $[x_i, x_j]$ corresponds to $x_i \wedge x_j$, the class of $\rho(w)$ has coefficient $\mu_{ij}(w)/2$ on $x_i \wedge x_j$ for $i < j$. In particular, if $a(w) = 0$, then

$$\rho(w) \in \gamma_3(F_5) \iff \mu(w) = 0.$$

Proof. For invariance, insert an adjacent pair $x_m x_m^{-1}$ into the cotree letter sequence. Fix $i < j$. If $m \notin \{i, j\}$, the inserted letters give no contribution to μ_{ij} . If $m = i$, the two contributions are

$$-a_j^{(k-1)} \quad \text{and} \quad +a_j^{(k-1)},$$

while if $m = j$, they are

$$+a_i^{(k-1)} \quad \text{and} \quad -a_i^{(k-1)}.$$

In both cases they cancel. After the inserted pair, all running counts have their previous values, so no later term is affected. The case $x_m^{-1} x_m$ is identical. Therefore μ is invariant under free reduction, and hence depends only on $\rho(w)$.

For the parity, we write

$$e_{ij} = \sum_{k=1}^n a_i^{(k-1)} s_j^{(k)}.$$

At step k , we have

$$a_i^{(k)} = a_i^{(k-1)} + s_i^{(k)}, \quad a_j^{(k)} = a_j^{(k-1)} + s_j^{(k)}.$$

Hence

$$a_i^{(k)} a_j^{(k)} - a_i^{(k-1)} a_j^{(k-1)} = a_i^{(k-1)} s_j^{(k)} + a_j^{(k-1)} s_i^{(k)} + s_i^{(k)} s_j^{(k)}.$$

One step crosses at most one cotree edge, so

$$s_i^{(k)} s_j^{(k)} = 0 \quad (i \neq j).$$

Summing over k , and using $a_i^{(0)} = a_j^{(0)} = 0$, we obtain

$$e_{ij} + e_{ji} = a_i(w) a_j(w).$$

From the definition, we also have

$$\mu_{ij} = e_{ij} - e_{ji} = 2e_{ij} - a_i(w) a_j(w).$$

If $a(w) = 0$, then $e_{ji} = -e_{ij}$, and therefore

$$\mu_{ij} = 2e_{ij}.$$

In particular, each $\mu_{ij}(w)$ is even.

If $a(w) = 0$, then $\rho(w) \in \gamma_2(F_5)$. By the degree-two case of the Magnus expansion (see e.g. [5, 6, 7, 10]), the coefficient of $x_i \wedge x_j$ in the class of $\rho(w)$ is $e_{ij} = \mu_{ij}(w)/2$. Hence this class is zero if and only if $\mu(w) = 0$. Therefore,

$$\rho(w) \in \gamma_3(F_5) \iff \mu(w) = 0.$$

□

For a graph, the least length of a nontrivial edge-neutral nonbacktracking closed walk is its *abelian girth*. Friedman, Izsak, and Silberman characterize this invariant by connected subgraphs of cycle rank two without leaves, counting each ordinary edge twice and a barbell connector four times [13]. Our length minimum is the hypercube instance of this invariant: deleting reversals preserves edge neutrality, and vertex transitivity removes dependence on the chosen basepoint. We give the elementary cube proof and identify a minimizing walk's nonzero degree-two commutator class.

We measure length in the Q_3 edge metric, before spanning-tree reduction. This differs from shortest word results in the free generator metric [11, 12, 16].

Geometrically, $\mu_{ij}/2$ is a signed area coordinate. For smooth paths the same term appears in the path signature [8, 9]. For a discrete sequence of increments it is the degree-two coordinate of the iterated sums signature [15]. Let us consider the walk

$$W = (000; 01012020101202). \quad (4)$$

Its vertex sequence is

$$0, 1, 3, 2, 0, 4, 5, 1, 0, 2, 3, 1, 5, 4, 0.$$

Theorem 4.2. *The walk W defined by (4) is closed and satisfies*

$$R_\phi(W) = 0 \quad \text{for every } \phi \in \mathcal{A}, \quad a(W) = 0,$$

while

$$\mu_{12}(W) = -2.$$

Its reduced loop word is

$$\rho(W) = x_1^{-1} x_2 x_3^{-1} x_1 x_3 x_2^{-1}.$$

In particular, $\rho(W) \neq 1$ in F_5 .

Proof. If we run the displayed fourteen steps, the walk returns to 0. Its cotree letter sequence is

$$x_1^{-1}, x_2, x_3^{-1}, x_1, x_3, x_2^{-1}.$$

We have that each exponent sum is zero, and the displayed word has no adjacent inverse pair. Hence it is freely reduced, and therefore

$$\rho(W) = x_1^{-1} x_2 x_3^{-1} x_1 x_3 x_2^{-1}.$$

step	1	2	3	4	5	6	7	8	9	10	11	12	13	14
direction	0	1	0	1	2	0	2	0	1	0	1	2	0	2
vertex	1	3	2	0	4	5	1	0	2	3	1	5	4	0
cotree letter				x_1^{-1}	x_2		x_3^{-1}		x_1			x_3		x_2^{-1}
term in μ_{12}				0	-1		0		-1			0		0

FIGURE 3. The fourteen-step walk W of Theorem 4.2: for each step, the direction used, the vertex reached in decimal code, the cotree letter contributed, and its term in μ_{12} . The six letters spell $x_1^{-1}x_2x_3^{-1}x_1x_3x_2^{-1}$, all exponent sums vanish, so $a(W) = 0$ while $\rho(W) \neq 1$ and $\mu_{12}(W) = -2$.

For the pair $(1, 2)$, the six cotree letters contribute, in order,

$$0, -1, 0, -1, 0, 0,$$

so we obtain $\mu_{12}(W) = -2$. Since W is closed, Theorem 3.1 gives

$$R_\phi(W) = 0$$

for every $\phi \in \mathcal{A}$. □

Since W is closed and $a(W) = 0$, Section 2 gives $z(W) = 0$. Hence $R_g(W) = 0$ for every antisymmetric weight g . Thus W is invisible to every antisymmetric additive reading, in particular to every potential reading. It is not invisible to every additive reading: for $G = \mathbb{Z}$, the constant weight $g \equiv 1$ gives $R_g(W) = 14$.

Remark 4.3. The witness is the commutator of two adjacent square faces after cancelling one immediate reversal. Put $A' = (000; 2020)$, so $\rho(A') = x_2x_3^{-1}$, and use the convention $[g, h] = ghg^{-1}h^{-1}$. Then

$$\rho(W) = [\rho(A), \rho(A')] = [x_1^{-1}, x_2x_3^{-1}].$$

The axis string of $AA'A^{-1}(A')^{-1}$ is 0101202010100202; deleting the adjacent 00 gives the displayed fourteen-step walk. The commutator pairing in γ_2/γ_3 is bilinear in the abelianized arguments, so its class is

$$(-e_1) \wedge (e_2 - e_3) = -e_1 \wedge e_2 + e_1 \wedge e_3.$$

Thus $\mu_{12}(W) = -2$, $\mu_{13}(W) = 2$, and every other $\mu_{ij}(W)$, $i < j$, is zero.

Theorem 4.4. *Among based closed walks on Q_3 , the shortest walk with $a(w) = 0$ and $\rho(w) \neq 1$ has length 14. Moreover,*

$$\min\{|w| : a(w) = 0, \rho(w) \neq 1\} = \min\{|w| : \rho(w) \in \gamma_2(F_5) \setminus \gamma_3(F_5)\} = 14.$$

The walk W realizes both minima.

Proof. Theorem 4.2 gives a fourteen-step walk with $a(W) = 0$ and $\mu_{12}(W) = -2$. By Lemma 4.1, its reduced word lies in $\gamma_2(F_5) \setminus \gamma_3(F_5)$.

For the lower bound, let $a(w) = 0$ and $\rho(w) \neq 1$, and let H be the connected subgraph consisting of the edges used by w . The integral edge chain $z(w)$ is zero. If H is a tree, every closed walk in H is null-homotopic. If H has cycle rank one, the abelianization $\pi_1(H) \rightarrow H_1(H; \mathbb{Z})$ is an isomorphism, since both groups are infinite cyclic. The zero chain then again makes w null-homotopic in H , hence in Q_3 . Both cases contradict $\rho(w) \neq 1$. Thus the cycle rank of H is at least two.

Write $v = |V(H)|$ and $e = |E(H)|$. Connectedness gives $e - v + 1 \geq 2$, so $e \geq v + 1$. If $e \leq 6$, then $v \leq 5$. Bipartiteness gives $e \leq \lfloor v^2/4 \rfloor < v + 1$ for $v \leq 4$. The remaining possibility is $v = 5, e = 6$, which forces $H = K_{2,3}$. But two distinct vertices of a hypercube have exactly two common neighbours when their Hamming distance is two, and none otherwise. Hence no hypercube contains $K_{2,3}$, and $e \geq 7$.

Since $z(w) = 0$, each used edge occurs equally often in both directions, at least once in each. Consequently

$$|w| \geq 2e \geq 14.$$

This proves the lower bound for every nontrivial zero-abelian loop, including those whose degree-two information vanishes. The witness attains the bound and has nonzero degree-two information, proving both equalities. $\square$

Corollary 4.5. *Every length-fourteen walk on Q_3 with $a(w) = 0$ and $\rho(w) \neq 1$ traverses each edge of a connected seven-edge, six-vertex support exactly once in each direction. Its support has cycle rank two and bipartition sizes three and three.*

Proof. The lower-bound proof forces exactly seven used edges, each used twice, and at most six vertices. A bipartite graph on at most five vertices has at most six edges, so there are six vertices and cycle rank two. A bipartition of sizes two and four has at most six edges in Q_3 , since every vertex has degree at most three; sizes one and five give still fewer. Thus the sizes are three and three. $\square$

Remark 4.6. The value 14 does not depend on the chosen spanning tree. For two spanning trees T, T' , the corresponding free bases of $\pi_1(Q_3, o)$ differ by an automorphism α of F_5 . Hence

$$\rho_{T'} = \alpha \circ \rho_T.$$

Since the terms of the lower central series are characteristic, we have

$$\rho_T(w) \in \gamma_2(F_5) \setminus \gamma_3(F_5) \iff \rho_{T'}(w) \in \gamma_2(F_5) \setminus \gamma_3(F_5).$$

Also, the condition $a(w) = 0$ is tree-independent, since it is equivalent to $z(w) = 0$. Under this condition, Lemma 4.1 gives that $\mu(w) = 0$ is also tree-independent. The individual coordinates $\mu_{ij}(w)$ may depend on the tree. In particular,

$$\mu_{12}(W) = -2$$

refers to the tree fixed in Section 2.

5. MEMORY AND ORDER RESOLUTION

The previous sections show what is lost by commutative summaries. We now consider finite sequential memory and an unbounded stack.

A k -state fold is a sequential reading in the sense of Definition 1.1 with a state set of at most k elements.

Theorem 5.1. *There is a two-state fold D such that*

$$D(w_{AB}) \neq D(w_{BA}).$$

Proof. The directed edges of A are

$$0 \rightarrow 1, \quad 1 \rightarrow 3, \quad 3 \rightarrow 2, \quad 2 \rightarrow 0,$$

and the directed edges of B are

$$0 \rightarrow 2, \quad 2 \rightarrow 6, \quad 6 \rightarrow 4, \quad 4 \rightarrow 0.$$

We set $e_A = (0, 1)$ and $e_B = (0, 2)$. Then e_A occurs in A only, and e_B in B only, each exactly once.

We take

$$S = \{0, 1\}, \quad s_0 = 0, \quad q(s) = s,$$

and define

$$T_{e_A}(s) = 0, \quad T_{e_B}(s) = 1, \quad T_e(s) = s$$

for every other directed edge e .

Thus the final state is determined by which of e_A and e_B occurs last. Therefore, we obtain

$$D(w_{AB}) = 1, \quad D(w_{BA}) = 0.$$

□

We next show that no fixed finite-state bound recovers the whole reduced loop word. For B as in (3), let B^m denote its m -fold concatenation, a closed walk of length $4m$.

Theorem 5.2. *Let $k \geq 1$. For every k -state fold F ,*

$$F(B^{k!}) = F(B^{2 \cdot k!}),$$

while

$$\rho(B^{k!}) \neq \rho(B^{2 \cdot k!}).$$

Proof. Let S be the state set of F , with $|S| \leq k$, and let $t : S \rightarrow S$ be the state update obtained by reading one copy of B . Starting from the initial state s_0 , consider

$$s_0, t(s_0), \dots, t^k(s_0).$$

Since $|S| \leq k$, there exist $0 \leq i < j \leq k$ such that

$$t^i(s_0) = t^j(s_0).$$

Set $p = j - i$. Then

$$t^{n+p}(s_0) = t^n(s_0) \quad \text{for every } n \geq i.$$

Since $1 \leq p \leq k$, we have $p \mid k!$, and since $i \leq k \leq k!$, it follows that

$$t^{k!}(s_0) = t^{2 \cdot k!}(s_0).$$

Applying the output map gives

$$F(B^{k!}) = F(B^{2 \cdot k!}).$$

On the other hand, we have

$$\rho(B) = x_1 x_5^{-1} x_2^{-1},$$

which is a nontrivial cyclically reduced word. Hence

$$\rho(B^m) = \rho(B)^m$$

is reduced for every $m \geq 1$. Therefore

$$\rho(B^{k!}) \neq \rho(B^{2 \cdot k!}).$$

□

This factorial witness is not purely noncommutative, since

$$a(B) = (1, -1, 0, 0, -1) \neq 0, \quad a(B^m) = m a(B),$$

so abelianization already separates the two powers.

5.1. An equal-count obstruction for every finite memory bound. The power pair above changes multiplicity. The following pair keeps all directed edge counts fixed, so the missing information is order.

Theorem 5.3. *Let $A = (000; 0101)$, $C = (000; 01020102)$, $k \geq 1$, and $N = k!$. Put*

$$U_k = A^N C^N A^{2N}, \quad V_k = A^{2N} C^N A^N.$$

Then

$$|U_k| = |V_k| = 20N, \quad \tau_{U_k} = \tau_{V_k} = 3N\tau_A + N\tau_C,$$

and

$$\rho(U_k) = x_1^{-N} x_2^{-N} x_1^{-2N} \neq x_1^{-2N} x_2^{-N} x_1^{-N} = \rho(V_k).$$

Every sequential reading with at most k states takes the same value on U_k and V_k , from every initial state.

Proof. The square A contributes the single cotree letter x_1^{-1} . The first seven edges of C follow the Gray tree, and its closing edge contributes x_2^{-1} . Thus the displayed words are already reduced. At their $(N+1)$ -st letter one has x_2^{-1} and the other x_1^{-1} , so they differ. The lengths and transition counts follow by addition under concatenation.

Let f and g be the state transformations obtained by reading A and C . For any transformation t of a set of at most k states, the orbit of each state enters a cycle of length at most k after at most $k-1$ steps. Since every such cycle length divides $N = k!$ and $N \geq k$,

$$t^N = t^{2N}$$

as functions on the whole state set. In particular $f^N = f^{2N}$. The two histories act as

$$f^{2N} \circ g^N \circ f^N \quad \text{and} \quad f^N \circ g^N \circ f^{2N},$$

which are equal. Applying any output map proves the claim. $\square$

Corollary 5.4. *No reading consisting of the full directed transition tally together with a fixed finite-state fold determines ρ on all histories.*

Proof. For a state bound k , Theorem 5.3 gives the same tally and final state on U_k, V_k , but different reduced words. Every function of those two recorded quantities therefore fails on at least one member of the pair. $\square$

Theorem 5.5. *For $N = k!$, $k \geq 1$, let*

$$P_k = A^N C A^{3N} C, \quad Q_k = A^{2N} C A^{2N} C.$$

These walks have the same length $16N + 16$, the same directed transition tally $4N\tau_A + 2\tau_C$, and the same final state under every sequential reading with at most k states, from every initial state. Their reduced loop words are not conjugate.

Proof. Let f, g be the state transformations for A, C . The identity $f^N = f^{2N}$ from Theorem 5.3 also gives $f^{3N} = f^N$. Hence both histories act as $g \circ f^N \circ g \circ f^N$. Their tallies and lengths agree by addition.

For nonconjugacy, use the free-group homomorphism to $\text{SL}_2(\mathbb{Z})$ sending x_1^{-1} to X , x_2^{-1} to Y , and the other generators to the identity, where

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

For nonnegative integers r, s , direct multiplication gives

$$\text{tr}(X^r Y X^s Y) = 2 + 2(r + s) + rs.$$

The images of $\rho(P_k)$ and $\rho(Q_k)$ therefore have traces $2 + 8N + 3N^2$ and $2 + 8N + 4N^2$, respectively. Since $N > 0$, the traces differ. Trace is invariant under conjugation, so the original free-group words cannot be conjugate. $\square$

Thus the finite-memory obstruction concerns free homotopy as well as based-loop recovery. In contrast, the earlier pair U_k, V_k of Theorem 5.3 is conjugate, just as AB, BA is.

5.2. The stack recovers the reduced word.

Theorem 5.6. *There is a sequential reading with an unbounded stack which recovers $\rho(w)$ for every $w \in \mathcal{W}$.*

Proof. We take the state set to be the set of finite reduced words in

$$x_1^{\pm 1}, \dots, x_5^{\pm 1},$$

with the empty word as the initial state and the last letter as the top of the stack. The output map is the identity.

A tree edge leaves the state unchanged. If a cotree edge contributes a letter l , we delete the last letter when it is l^{-1} , and otherwise we append l .

By induction on the initial segments of w , after each step the stack contains the freely reduced cotree word of the segment already read. Therefore, after the whole walk is read, the stack contains $\rho(w)$. $\square$

Corollary 5.7. *There is a pair of walks which is not distinguished by any additive reading, but is distinguished by a two-state fold. For every $k \geq 1$, the walks*

$$U_k, \quad V_k$$

have equal directed transition counts and are not distinguished by any k -state fold, while the stack distinguishes them. Hence no fixed finite-state bound recovers ρ on all walks, whereas the stack recovers $\rho(w)$ for every $w \in \mathcal{W}$.

Proof. The first statement follows from Theorems 3.4, 3.2, and 5.1, and the second from Theorems 5.3 and 5.6. $\square$

For each fixed pair of distinct finite histories, a sufficiently large finite-state reading can separate them: it stores the input prefix up to the larger of the two lengths, with one overflow state. The obstruction is uniformity. Every fixed state bound misses some pair; one unbounded stack computes ρ for all pairs.

5.3. States, histories, and readable distinctions. A state description and a history reading answer different questions. The following criterion states exactly when a proposed summary retains the distinctions needed by a task.

Proposition 5.8. *Let $r : \mathcal{W} \rightarrow X$ be a summary and $m : \mathcal{W} \rightarrow Y$ a desired reading, with no homomorphism assumptions. There exists a function $d : r(\mathcal{W}) \rightarrow Y$ such that $m = d \circ r$ if and only if*

$$r(u) = r(v) \implies m(u) = m(v) \quad (u, v \in \mathcal{W}).$$

Proof. The forward implication follows by applying d . Conversely define $d(r(w)) = m(w)$. The stated implication makes this definition independent of the representative, and every point of $r(\mathcal{W})$ has a representative. $\square$

Thus a semantic reading of a sequence is recoverable from a summary exactly when it is constant on that summary's fibres. The examples here make the failure explicit: the same edge multiset supports different loop words, and even a fixed finite memory added to that multiset does not restore all distinctions. If a state evolves by ordered edge updates, it is itself a sequential reading

and must be assessed by those updates. Telescoping potential differences alone describe a particular endpoint reading, not every possible evolving state.

Our reduced words retain a fixed basepoint and a starting point in the history. If only free homotopy is recorded, words are identified up to conjugacy. In particular ab and ba are conjugate, so that reading identifies the AB, BA pair. The fourteen-step witness remains distinct from silence even after conjugacy, since only the identity is conjugate to the identity. A choice of readable distinctions therefore determines which notion of history is required.

6. A PROPER ORDER-SENSITIVE CONGRUENCE

We compare the kernel congruences induced by

$$\rho : \mathcal{W} \longrightarrow F_5 \quad \text{and} \quad a : \mathcal{W} \longrightarrow \mathbb{Z}^5. \quad (5)$$

Recall that $a(w) = (a_1(w), \dots, a_5(w))$ records the signed cotree traversal counts.

Theorem 6.1. *We have*

$$\Delta_{\mathcal{W}} \subsetneq \sim_{\rho} \subsetneq \sim_a.$$

Proof. Since $a = \text{ab} \circ \rho$, we have

$$\rho(u) = \rho(v) \implies a(u) = a(v),$$

and therefore

$$\sim_{\rho} \subseteq \sim_a.$$

The inclusion is strict by Theorem 3.4, since

$$\rho(w_{AB}) \neq \rho(w_{BA}),$$

while

$$a(w_{AB}) = a(A) + a(B) = a(B) + a(A) = a(w_{BA}).$$

For the first inclusion, let

$$T = (000; 00)$$

be the two-step backtrack along a tree edge. Since both steps are tree edges, we have

$$\rho(T) = 1 = \rho(1),$$

while $T \neq 1$ in $\mathcal{W}$, since their step sequences are different. Hence

$$\Delta_{\mathcal{W}} \subsetneq \sim_{\rho}.$$

□

Corollary 6.2. *The recognition quotients determined by ρ and a given by (5) satisfy*

$$\mathcal{W}/\sim_{\rho} \cong F_5, \quad \mathcal{W}/\sim_a \cong \mathbb{Z}^5.$$

Proof. By the first isomorphism theorem for monoids,

$$\mathcal{W}/\sim_{\rho} \cong \text{im } \rho, \quad \mathcal{W}/\sim_a \cong \text{im } a.$$

For each cotree edge, the corresponding based fundamental loop maps to x_i , while its reverse maps to x_i^{-1} . Hence $\rho : \mathcal{W} \rightarrow F_5$ is surjective. Thus,

$$\mathcal{W}/\sim_{\rho} \cong F_5.$$

Since $a = \text{ab} \circ \rho$ and both ρ and ab are surjective, $a : \mathcal{W} \rightarrow \mathbb{Z}^5$ is also surjective. Therefore

$$\mathcal{W}/\sim_a \cong \mathbb{Z}^5.$$

□

Theorem 6.3. *Let M be a commutative monoid and let $f : \mathcal{W} \rightarrow M$ be a homomorphism. Then*

$$f(uv) = f(vu)$$

for all $u, v \in \mathcal{W}$. In particular,

$$f(w_{AB}) = f(w_{BA}).$$

Proof. Since M is commutative, we have

$$f(uv) = f(u)f(v) = f(v)f(u) = f(vu).$$

Setting $u = A$ and $v = B$ we obtain $f(w_{AB}) = f(w_{BA})$. $\square$

Thus a monoid recognizer separating w_{AB} from w_{BA} cannot take values in a commutative monoid. By Theorem 3.4, ρ is such a recognizer.

Remark 6.4. The maps ρ and τ are incomparable. By Theorem 3.4, we have

$$\tau_{w_{AB}} = \tau_{w_{BA}}, \quad \rho(w_{AB}) \neq \rho(w_{BA}),$$

so τ does not determine ρ . Conversely, for $T = (000; 00)$, we have

$$\rho(T) = \rho(1), \quad \tau_T \neq \tau_1,$$

so ρ does not determine τ .

A second proper order-sensitive congruence is constructed in Section 8.

7. THE RECTANGULARITY BOUNDARY

In the previous sections, we studied readings of a history $w \in \mathcal{W}$. We now consider constraints on pairs $(\alpha, w) \in \mathcal{A} \times \mathcal{W}$. We determine when such constraints are rectangular and when they are not.

Let $\mathcal{U} = \mathcal{A} \times \mathcal{W}$ be the product carrier. A constraint $P \subseteq \mathcal{U}$ is called *rectangular*, or *projection-separable*, if there exist subsets $P_{\mathcal{A}} \subseteq \mathcal{A}$ and $P_{\mathcal{W}} \subseteq \mathcal{W}$ such that

$$(\alpha, w) \in P \iff \alpha \in P_{\mathcal{A}} \text{ and } w \in P_{\mathcal{W}}.$$

Equivalently,

$$P = P_{\mathcal{A}} \times P_{\mathcal{W}}.$$

We first work on the full group $\mathcal{A} = \mathbb{Z}^V$. Write

$$\mathcal{A}_0 = \{\alpha \in \mathcal{A} : \sum_{p \in V} \alpha(p) = 0\}$$

for the kernel of the sum map

$$\alpha \mapsto \sum_{p \in V} \alpha(p).$$

Let $N \geq 1$, let

$$F_i : \mathcal{A} \longrightarrow \mathcal{A}, \quad 1 \leq i \leq N,$$

and let

$$\Phi : \mathbb{Z}^N \longrightarrow \{\text{false}, \text{true}\}$$

be a predicate. Define $P \subseteq \mathcal{U}$ by

$$(\alpha, w) \in P \iff \Phi(R_{F_1(\alpha)}(w), \dots, R_{F_N(\alpha)}(w)).$$

Proposition 7.1. *Every such constraint is constant on $\mathcal{U}$, and hence rectangular.*

Proof. Every $w \in \mathcal{W}$ is closed. By Theorem 3.1, we have

$$(R_{F_1(\alpha)}(w), \dots, R_{F_N(\alpha)}(w)) = (0, \dots, 0)$$

for every $(\alpha, w) \in \mathcal{U}$. Hence

$$(\alpha, w) \in P \iff \Phi(0, \dots, 0).$$

Therefore $P = \mathcal{U}$ if $\Phi(0, \dots, 0)$ is true, and $P = \emptyset$ otherwise. In either case, P is rectangular. $\square$

Remark 7.2. Proposition 7.1 uses only that every $w \in \mathcal{W}$ is closed. This is part of the definition of $\mathcal{W}$.

For $w = (v_0, \dots, v_n) \in \mathcal{W}$, define $\text{occ} : \mathcal{W} \rightarrow \mathcal{A}$ by

$$\text{occ}(w)(p) = \#\{k : 0 \leq k < n, v_k = p\}.$$

Thus the closing vertex $v_n = v_0$ is not counted. Define $\mathcal{O} \subseteq \mathcal{U}$ by

$$(\alpha, w) \in \mathcal{O} \iff \alpha = \text{occ}(w).$$

We call $\mathcal{O}$ the *occupation-binding constraint*.

Theorem 7.3. *The occupation-binding constraint $\mathcal{O}$ is not rectangular on $\mathcal{U}$.*

Proof. Choose $u, v \in \mathcal{W}$ with

$$\text{occ}(u) \neq \text{occ}(v),$$

for example the empty walk $u = 1$ and the square $v = (o; 0101)$. Then

$$(\text{occ}(u), u) \in \mathcal{O}, \quad (\text{occ}(v), v) \in \mathcal{O},$$

while

$$(\text{occ}(u), v) \notin \mathcal{O}.$$

Suppose that $\mathcal{O}$ is rectangular. Then

$$\mathcal{O} = P_{\mathcal{A}} \times P_{\mathcal{W}}$$

for some $P_{\mathcal{A}} \subseteq \mathcal{A}$ and $P_{\mathcal{W}} \subseteq \mathcal{W}$. From $(\text{occ}(u), u) \in \mathcal{O}$ we obtain $\text{occ}(u) \in P_{\mathcal{A}}$, and from $(\text{occ}(v), v) \in \mathcal{O}$ we obtain $v \in P_{\mathcal{W}}$. Hence

$$(\text{occ}(u), v) \in \mathcal{O},$$

which is a contradiction. $\square$

Proposition 7.4. *For $(\alpha, w) \in \mathcal{A}_0 \times \mathcal{W}$,*

$$(\alpha, w) \in \mathcal{O} \iff \alpha = 0 \text{ and } w = 1.$$

Proof. For every $w \in \mathcal{W}$, it holds

$$\sum_{p \in V} \text{occ}(w)(p) = |w|.$$

If $(\alpha, w) \in \mathcal{O}$ and $\alpha \in \mathcal{A}_0$, then $\alpha = \text{occ}(w)$, so

$$|w| = \sum_{p \in V} \text{occ}(w)(p) = \sum_{p \in V} \alpha(p) = 0.$$

Hence $w = 1$ and $\alpha = 0$. The converse is immediate. $\square$

Remark 7.5. Thus

$$\mathcal{O} \cap (\mathcal{A}_0 \times \mathcal{W}) = \{0\} \times \{1\}.$$

Hence $\mathcal{O}$ is rectangular on $\mathcal{A}_0 \times \mathcal{W}$; in this case the restriction consists of a single point.

7.1. The exact collapse criterion. A constraint $P \subseteq \mathcal{U}$ is called *occupation-based* if

$$(\alpha, w) \in P \iff \Psi(\alpha, \text{occ}(w))$$

for some predicate

$$\Psi : \mathcal{A} \times \mathcal{A} \longrightarrow \{\text{false}, \text{true}\}.$$

Let $\mathcal{S} \subseteq \mathcal{W}$ be an arbitrary subset.

Here a subset $P \subseteq \mathcal{U}$ is called *rectangular* on $\mathcal{A} \times \mathcal{S}$ if

$$P \cap (\mathcal{A} \times \mathcal{S}) = P_{\mathcal{A}} \times P_{\mathcal{S}}$$

for some $P_{\mathcal{A}} \subseteq \mathcal{A}$ and $P_{\mathcal{S}} \subseteq \mathcal{S}$.

Theorem 7.6. *The following statements are equivalent:*

- (i) *There is $c_0 \in \mathcal{A}$ such that $\text{occ}(w) = c_0$ for every $w \in \mathcal{S}$.*
- (ii) *Every occupation-based constraint is rectangular on $\mathcal{A} \times \mathcal{S}$.*

Proof. First, we assume (i). If

$$(\alpha, w) \in P \iff \Psi(\alpha, \text{occ}(w)),$$

then on $\mathcal{A} \times \mathcal{S}$, we have

$$(\alpha, w) \in P \iff \Psi(\alpha, c_0).$$

Hence

$$P \cap (\mathcal{A} \times \mathcal{S}) = P_{\mathcal{A}} \times \mathcal{S}, \quad P_{\mathcal{A}} = \{\alpha \in \mathcal{A} : \Psi(\alpha, c_0)\}.$$

Conversely, suppose (i) is not valid. Then there exist $u, v \in \mathcal{S}$ with $\text{occ}(u) \neq \text{occ}(v)$. The constraint $\mathcal{O}$ is occupation-based, and the cross-pair argument of Theorem 7.3 shows that $\mathcal{O} \cap (\mathcal{A} \times \mathcal{S})$ is not rectangular in $\mathcal{A} \times \mathcal{S}$. $\square$

For instance, let $\mathcal{S}$ be the class of based closed walks $(v_0, \dots, v_8)$ with $v_0, \dots, v_7$ pairwise distinct. The Gray cycle shows $\mathcal{S} \neq \emptyset$, and $\text{occ}(w)(p) = 1$ for every $w \in \mathcal{S}$ and every $p \in V$, so every occupation-based constraint is rectangular there.

8. A DECLARED QUARTER-TURN CLOCK

The reduced loop word ρ gives one order-sensitive recognizer. We now define a second one by a declared ordered quaternion product on the three coordinate directions of the cube.

Let

$$\mathbb{H}_{\mathbb{Z}} = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : a, b, c, d \in \mathbb{Z}\}$$

be the multiplicative monoid of integer quaternions. The quaternion units satisfy

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j}.$$

We assign to the three coordinate directions

$$q_0 = 1 + \mathbf{i}, \quad q_1 = 1 + \mathbf{j}, \quad q_2 = 1 + \mathbf{k}.$$

The normalized elements

$$\frac{q_0}{\sqrt{2}}, \quad \frac{q_1}{\sqrt{2}}, \quad \frac{q_2}{\sqrt{2}}$$

are unit quaternions representing quarter-turns about the three coordinate axes.

For a walk

$$w = (o; a_1 \cdots a_n),$$

define

$$h : \mathcal{W} \longrightarrow \mathbb{H}_{\mathbb{Z}}, \quad h(w) = q_{a_1} \cdots q_{a_n}.$$

The map h uses the integer elements q_a , not unit quaternions.

Let

$$\mathcal{T} = \{\tau_w : w \in \mathcal{W}\} \subseteq \mathbb{N}^{V \times V},$$

with pointwise addition. Since $\tau_{uv} = \tau_u + \tau_v$ and $\tau_1 = 0$, the set $\mathcal{T}$ is a submonoid of $\mathbb{N}^{V \times V}$.

Write $c(w) = (\tau_w, h(w))$. The following description identifies the finite part of this reading before we compare its kernel with the other recognizers.

Proposition 8.1. *Let*

$$2T = \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\} \cup \left\{ \frac{\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}}{2} \right\},$$

with all sixteen sign choices in the second set. This is a group of 24 unit quaternions. The normalized reading

$$g(w) = 2^{-|w|/2} h(w)$$

is a surjective monoid homomorphism $\mathcal{W} \rightarrow 2T$. Moreover,

$$c(u) = c(v) \iff \tau_u = \tau_v \text{ and } g(u) = g(v).$$

Proof. Quaternion multiplication shows that the displayed set is closed; conjugation gives inverses, and all its elements have norm one. Its eight axial elements form the quaternion group Q_8 . Every product $q_a q_b / 2$ belongs to $2T$: for equal axes it is the corresponding imaginary unit, and for unequal axes it is a half-integral element in the displayed list. Closed cube walks have even length, so pairing successive factors proves $g(w) \in 2T$. Additivity of length makes g a homomorphism.

The closed walks 0011, 1122, and 0101 give respectively $\mathbf{k}$, $\mathbf{i}$, and $t = (-1 + \mathbf{i} + \mathbf{j} + \mathbf{k})/2$. Here $t^3 = 1$, and direct multiplication gives

$$2T = Q_8 \cup Q_8 t \cup Q_8 t^2.$$

The images $\mathbf{i}, \mathbf{k}$ generate Q_8 , and their inverses, as well as t^{-1} , are positive powers. Hence the image of g is all of $2T$. Finally, $|w| = \sum_e \tau_w(e)$, so equality of tallies fixes the scale $2^{|w|/2}$. This proves the last equivalence. $\square$

The noncommutative part of c is therefore finite. At the end of each closed history it has 24 possible values. For individual edge updates the normalized products lie in the 48-element group $2T \cup 2T q_0 / \sqrt{2}$: an odd number of quarter-turn factors lies in the second coset, and an even number lies in $2T$. Thus a 48-state sequential reading computes g on closed walks, while the tally supplies the discarded scale. Corollary 5.4 applies directly to this clock and explains why it cannot recover all reduced loop words.

Theorem 8.2. *The map*

$$c : \mathcal{W} \longrightarrow \mathcal{T} \times \mathbb{H}_{\mathbb{Z}}, \quad c(w) = (\tau_w, h(w)),$$

is a monoid homomorphism and

$$\Delta_{\mathcal{W}} \subsetneq \sim_c \subsetneq \sim_{\tau}.$$

Proof. For $u, v \in \mathcal{W}$, we have

$$h(uv) = h(u)h(v), \quad \tau_{uv} = \tau_u + \tau_v.$$

Hence c is a monoid homomorphism. Since the first component of c is τ , we have

$$\sim_c \subseteq \sim_{\tau}.$$

To show that this inclusion is strict, let us use

$$u = (000; 001122), \quad v = (000; 002211).$$

The two walks have the same directed transition counts, so $\tau_u = \tau_v$. We have

$$\begin{aligned} h(u) &= (1 + \mathbf{i})^2(1 + \mathbf{j})^2(1 + \mathbf{k})^2 = 8\mathbf{ijk} = -8, \\ h(v) &= (1 + \mathbf{i})^2(1 + \mathbf{k})^2(1 + \mathbf{j})^2 = 8\mathbf{ikj} = 8. \end{aligned}$$

Thus $c(u) \neq c(v)$, and therefore

$$\sim_c \subsetneq \sim_\tau.$$

For the first inclusion, take

$$r = (000; 001001), \quad t = (000; 100100).$$

These are distinct based closed walks. Their directed edge sequences are cyclic rotations, hence

$$\tau_r = \tau_t.$$

Also,

$$\begin{aligned} h(r) &= ((1 + \mathbf{i})^2(1 + \mathbf{j}))^2 = (2\mathbf{i}(1 + \mathbf{j}))^2 = (2\mathbf{i} + 2\mathbf{k})^2 = -8, \\ h(t) &= ((1 + \mathbf{j})(1 + \mathbf{i})^2)^2 = ((1 + \mathbf{j})2\mathbf{i})^2 = (2\mathbf{i} - 2\mathbf{k})^2 = -8. \end{aligned}$$

Hence $c(r) = c(t)$, while $r \neq t$. Therefore, we get $\Delta_{\mathcal{W}} \subsetneq \sim_c$. $\square$

Remark 8.3. The factor q_a depends on the axis and not on the direction of traversal. Thus an immediate reversal contributes the same factor, and $(1 + \mathbf{i})^2 = 2\mathbf{i} \neq 1$. The ordered product h therefore does not factor through the fundamental group. The clock retains quaternion values: the normalized witnesses -1 and $+1$ are distinct in the unit quaternions, although they induce the same ordinary spatial rotation. The distinctions proved here belong to the declared quaternion reading.

Proposition 8.4. *The congruences $\sim_c$ and $\sim_\rho$ are incomparable.*

Proof. Let u, v be the walks from Theorem 8.2. Both are concatenations of three two-step backtracks from the origin. Hence

$$\rho(u) = \rho(v) = 1,$$

while $c(u) \neq c(v)$. Therefore, we have $\sim_\rho \not\subseteq \sim_c$.

For the converse, we use the walks A and B from (3). By Theorem 3.4, we have

$$\tau_{AB} = \tau_{BA}, \quad \rho(AB) \neq \rho(BA).$$

Also,

$$(1 + \mathbf{i})(1 + \mathbf{j}) = (1 + \mathbf{j})(1 + \mathbf{k}) = 1 + \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

Hence

$$h(A) = ((1 + \mathbf{i})(1 + \mathbf{j}))^2 = ((1 + \mathbf{j})(1 + \mathbf{k}))^2 = h(B).$$

Therefore

$$h(AB) = h(A)h(B) = h(B)h(A) = h(BA).$$

Together with $\tau_{AB} = \tau_{BA}$, we obtain

$$c(AB) = c(BA).$$

Since

$$\rho(AB) \neq \rho(BA),$$

it follows that $\sim_c \not\subseteq \sim_\rho$. $\square$

The quarter-turn map h is an additional structure on $\mathcal{W}$. It is not determined by the preceding constructions.

9. WHY THE THREE-CUBE

We use the cube Q_3 for two reasons. First, the paper continues the one-generator case of [1]. By Proposition 2.3, Q_3 is the first hypercube with noncommuting loop words.

Second, the edges of Q_3 are divided into three coordinate directions. The quarter-turn clock of Theorem 8.2 uses exactly these three directions. The graph K_4 with one edge removed has four vertices and five edges, hence cycle rank two and fundamental group F_2 . It is a smaller graph with noncommuting loop words, but its edges do not carry this three-coordinate product structure.

Thus Q_3 is not the smallest graph with a nonabelian fundamental group, but it is the first hypercube with this property and with the three coordinate directions used by the quarter-turn clock.

9.1. Higher-dimensional cubes. The 1-skeleton of Q_n has 2^n vertices and $n2^{n-1}$ edges. Hence its cycle rank is

$$r_n = n2^{n-1} - 2^n + 1 = (n - 2)2^{n-1} + 1.$$

As in the proof of Proposition 2.3, the fundamental group of a connected graph is free of rank equal to its cycle rank. Hence

$$\pi_1(Q_n) \cong F_{r_n}.$$

For every $n \geq 3$, a coordinate copy of Q_3 is contained in Q_n . We extend the Gray spanning tree of this subcube to a spanning tree T_n of Q_n . Since the Gray tree already spans the eight vertices of the subcube, no other edge of the subcube belongs to T_n . Thus its five cotree edges remain cotree edges for T_n .

For a closed walk in this subcube, the cotree word computed with T_n is obtained from its Q_3 cotree word by the homomorphism

$$F_5 \longrightarrow F_{r_n}$$

which maps the five generators to distinct members of a free basis of F_{r_n} . This homomorphism is injective, since a subset of a free basis freely generates a free subgroup. Hence AB and BA still have different reduced loop words in Q_n , while their directed transition counts agree. Therefore commutative aggregation remains blind to loop order for every $n \geq 3$.

Theorem 9.1. *For every $n \geq 3$, let ρ_n be the reduced-word reading on based closed walks of Q_n , with values in F_{r_n} , and let $a_n = \text{ab} \circ \rho_n$. Then*

$$\min\{|w| : a_n(w) = 0, \rho_n(w) \neq 1\} = \min\{|w| : \rho_n(w) \in \gamma_2(F_{r_n}) \setminus \gamma_3(F_{r_n})\} = 14.$$

Both equal-count finite-memory obstructions hold on Q_n : the based pair has length $20k!$, and the nonconjugate pair of Theorem 5.5 has length $16k! + 16$.

Proof. The lower-bound proof of Theorem 4.4 uses only bipartiteness and the absence of $K_{2,3}$, which hold in every hypercube. The coordinate copy of Q_3 supplies the fourteen-step witness. Its five free generators form a free factor of F_{r_n} . The retraction onto this factor carries $\gamma_3(F_{r_n})$ into $\gamma_3(F_5)$, so the witness remains outside the third subgroup.

The histories U_k, V_k in the same subcube keep their equal edge counts and distinct reduced words. Restricting any k -state reading on Q_n to the subcube gives a k -state reading on Q_3 , which identifies them by Theorem 5.3. The same restriction applies to P_k, Q_k . Extending their matrix representation by the identity on the additional free generators preserves the trace distinction, hence nonconjugacy. $\square$

10. CONCLUSION

Recognition by a single repeatable generator records multiplicity. Independent loops introduce distinctions of composition order. On cube histories, these distinctions survive in the reduced-word quotient while commutative aggregation identifies explicit ordered pairs. No fixed finite memory repairs this loss, even when its final state is combined with the exact transition tally. The obstruction persists for histories distinct up to free homotopy. A stack gives a single reading that recovers the reduced word on every history.

The shortest zero-abelian nontrivial history has fourteen edges in every hypercube of dimension at least three. Its support must contain two independent cycles, and edge balance forces two traversals of each of at least seven edges. The witness already has nonzero degree-two commutator information.

There are nevertheless several inequivalent ways to recognize order. The reduced word and the declared quaternion clock define incomparable proper quotients. For state-history constraints, the occupation vector gives an exact separation criterion. These results give concrete tests for a proposed reading: which histories it identifies, which desired distinctions survive those identifications, and how much memory exact recovery requires.

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