

# Golden and Metallic Structures on Hessian Manifolds

Jonathan Washburn <sup>1</sup>  and Milan Zlatanović <sup>2,\*</sup> 

<sup>1</sup> Recognition Physics Institute, Austin, TX 78701, USA; jon@recognitionphysics.org (J.W.)

<sup>2</sup> Faculty of Science and Mathematics, University of Niš, 18000 Niš, Serbia

\* Correspondence: zlatmilan@yahoo.com

## Abstract

We consider the reciprocal cost function  $J(x) = \frac{1}{2}(x + x^{-1}) - 1$  and its  $n$ -dimensional extension

$$J(x_1, \dots, x_n) = \frac{1}{2}(R + R^{-1}) - 1, \quad R = \prod_{i=1}^n x_i^{\alpha_i}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \setminus \{0\}.$$

In logarithmic coordinates  $t_i = \log x_i$ , the Hessian of  $J$  has rank one at every point. The associated Hessian geometry is degenerate and does not define a Riemannian metric. To obtain a nondegenerate geometric structure, we introduce a family of Hessian metrics  $h_\lambda$ . Combining the rank-one tensor with the Hessian metric  $h_\lambda$ , we construct a  $(1, 1)$ -tensor field  $A_\lambda$ . Its trace normalization defines a projector  $P_\lambda$ , which induces an almost product structure and the corresponding golden and metallic structures. We study several properties of the projector  $P_\lambda$  and the induced structures, including eigendistributions, parallelism, integrability, and curvature. The construction is given in arbitrary dimension, and explicit formulas are obtained in the two-dimensional case. In particular, we show that the projector  $P_\lambda$  is generally not parallel with respect to either the canonical flat affine connection or the Levi-Civita connection  $\nabla^\lambda$  of the Hessian metric  $h_\lambda$ .

**Keywords:** Hessian geometry, golden structures, metallic structures, projector, reciprocal cost function.

## 1. Motivation

The golden ratio is known since Euclid and appears under different names, such as the golden section, divine ratio, golden mean or golden proportion. It occurs in nature, especially in patterns related to Fibonacci numbers, like phyllotaxis and certain flowers.

It also appears in music, in harmonic relations, and in proportions of the human body. From ancient times, it has played an important role in architecture and art, for example in the proportions of temples, sculptures, and paintings. The golden ratio can be defined geometrically by dividing a segment into two parts such that the ratio of the whole to the larger part equals the ratio of the larger part to the smaller one. This ratio is the positive solution of the equation  $x^2 - x - 1 = 0$ . It appears in geometric figures such as the pentagon, decagon and dodecagon. On the other hand, let us consider the general quadratic equation

$$x^2 - \alpha x - \beta = 0,$$

where  $\alpha$  and  $\beta$  are positive integers. Its positive solution is

$$\sigma_{\alpha,\beta} = \frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2},$$

Received:

Revised:

Accepted:

Published:

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which defines the *metallic means family*. This family includes, for instance, the golden mean, the silver mean, the subtle mean, etc., and was introduced by Spinadel [17,18].

The golden ratio appears in quasicrystals, dynamical systems, and certain models in mathematical physics (see e.g. [4,6,13,15,21] and references therein).

Following [22,23], we consider the canonical reciprocal cost function in one dimension

$$J(x) = \frac{1}{2}(x + x^{-1}) - 1, \quad x > 0, \quad (1)$$

Cost functions are ubiquitous in optimization problems, and different cost functions can have different motivations. In [23], it is proved that this particular function appears as a unique solution of the polynomial composition law together with the curvature calibration. For  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \setminus \{0\}$  the  $n$ -dimensional extension obtained by composing (1) with  $R = \prod_i x_i^{\alpha_i}$  is

$$J(x_1, \dots, x_n) = \frac{1}{2}(R + R^{-1}) - 1,$$

so  $J(t) = \cosh(\alpha \cdot t) - 1$  in logarithmic coordinates. The Hessian of  $J$  is then  $\cosh(\alpha \cdot t) \alpha \otimes \alpha$ , which is positive semidefinite of rank one and does not define a Riemannian metric.

Let  $M$  be a smooth manifold and  $I$  the identity endomorphism of the tangent bundle  $TM$ . A  $(1, 1)$ -tensor field  $Q$  on  $M$  is called a *polynomial structure* if it satisfies a polynomial identity  $P(Q) = 0$ . The two quadratic cases studied in this paper are the *golden structure*

$$Q^2 = Q + I, \quad (2)$$

which is motivated by the classical golden ratio equation. More generally, for positive integers  $p, q$ , the  $(p, q)$ -*metallic structure* (see e.g. [12]) is defined by

$$Q^2 = pQ + qI. \quad (3)$$

Both golden and metallic structures belong to a broader class of polynomial structures introduced by Goldberg and Yano [9]. Golden structures on differentiable manifolds were first introduced by Hreţcanu and Crăşmareanu [10], who further developed their properties in [5]. Using an approach similar to the one developed for golden structures, Hreţcanu and Crăşmareanu studied the metallic structures on Riemannian manifolds in [12].

We pair the rank-one tensor  $\tilde{g}$  with a one-parameter family of Hessian metrics

$$h_\lambda = \nabla_x^2 \Phi_\lambda, \quad \Phi_\lambda(x) = \sum_{i=1}^n J(x_i) + \lambda J(R), \quad \lambda \in \mathbb{R}.$$

The associated  $(1, 1)$ -tensor  $A_\lambda$  is defined by

$$h_\lambda(A_\lambda X, Y) = \tilde{g}(X, Y),$$

and its normalization gives the projector  $P_\lambda$ . The corresponding almost product, golden, and metallic structures are obtained from  $P_\lambda$ .

We study several properties of these structures, including eigendistributions, parallelism, integrability, and curvature. The parameter  $\lambda$  deforms the Hessian metric by the term  $J(R)$ , producing a family associated with the reciprocal cost function.

The projector construction is general and can be applied to other rank-one tensors and nondegenerate metrics. In this paper, we consider the projector induced by the rank-one tensor  $\tilde{g}$  and the Hessian metrics  $h_\lambda$  arising from reciprocal cost geometry.

The paper is organized as follows. Section 2 introduces the reciprocal cost geometry and the associated Hessian structures. In Section 3 we construct the cost-induced projector. Section 4 is devoted to the associated almost product, golden, and metallic structures. In Section 5 we consider the two-dimensional case and derive explicit formulas for the tensors  $A_\lambda$  and  $P_\lambda$ . Section 6 contains the  $n$ -dimensional construction. In Section 6.1, we study properties of the induced structures, including eigendistributions, parallelism, integrability, and curvature.

## 2. Definitions and basic properties

Let  $(M, g)$  be a Riemannian manifold, let  $I$  denote the identity on  $TM$ , and let  $Q: TM \rightarrow TM$  be a  $(1, 1)$ -tensor field.

**Definition 1.** A  $(1, 1)$ -tensor field  $Q$  on  $M$  is called a polynomial structure if it satisfies a polynomial relation of the form

$$Q^n + a_{n-1}Q^{n-1} + \cdots + a_1Q + a_0I = 0, \quad (4)$$

where  $I$  is the identity operator on  $TM$  and  $a_i \in \mathbb{R}$ .

Golden and metallic structures are special cases of polynomial structures. In particular,  $Q^2 = -I$  defines an almost complex structure,  $Q^2 = I$  defines an almost product structure, and  $Q^2 = 0$  defines an almost tangent structure (see, e.g., [25]).

**Definition 2.** For integers  $p, q$ , a  $(1, 1)$ -tensor field  $Q$  is called a  $(p, q)$ -metallic structure if

$$Q^2 = pQ + qI. \quad (5)$$

A Riemannian metric  $g$  is called  $Q$ -compatible if

$$g(X, QY) = g(QX, Y), \quad X, Y \in \Gamma(TM). \quad (6)$$

When  $Q$  is a  $(p, q)$ -metallic structure and  $g$  is  $Q$ -compatible, the pair  $(g, Q)$  is called a metallic Riemannian structure. In the particular case  $p = q = 1$ , the pair  $(g, Q)$  is called a golden Riemannian structure [5,10].

Replacing  $X$  by  $QX$  in (6) and using (5), we obtain

$$g(QX, QY) = pg(X, QY) + qg(X, Y).$$

It is known that a decomposition of the tangent bundle of a differentiable manifold  $M$  into complementary distributions can be described in terms of projector operators. For instance, let  $T_1, \dots, T_k$  be differentiable distributions on  $M$  such that for every point  $p \in M$  one has

$$T_pM = T_1(p) \oplus \cdots \oplus T_k(p).$$

This decomposition can be equivalently expressed by a family of  $(1, 1)$ -tensor fields  $\pi_i$ ,  $i = 1, \dots, k$ , called projectors, satisfying

$$\sum_{i=1}^k \pi_i = I, \quad \pi_i \pi_j = \delta_j^i \pi_i,$$

where  $\delta_j^i$  are the Kronecker symbols. In this case  $T_i = \text{Im}(\pi_i)$ .

In the case  $k = 2$ , such a decomposition determines an almost product structure. Indeed, if  $\pi$  is one of the projectors, then define

$$F = 2\pi - I,$$

and obtain a  $(1, 1)$ -tensor field satisfying  $F^2 = I$ .

Conversely, any almost product structure  $F$ , induces the complementary projectors

$$\pi^+ = \frac{1}{2}(I + F), \quad \pi^- = \frac{1}{2}(I - F),$$

and the decomposition

$$T_p M = T^+(p) \oplus T^-(p),$$

where

$$T^\pm(p) = \{v \in T_p M : Fv = \pm v\}.$$

**Theorem 1 ([5]).** *Let  $(M, g, Q)$  be a golden Riemannian manifold. Then*

$$Q^n = f_n Q + f_{n-1} I \tag{7}$$

for every integer  $n > 0$ , where  $(f_n)_n$  is the Fibonacci sequence.

Using Binet's formula, relation (7) can be written as

$$Q^n = f_n Q + f_{n-1} I = \frac{\varphi^n - (1 - \varphi)^n}{\sqrt{5}} Q + \frac{\varphi^{n-1} - (1 - \varphi)^{n-1}}{\sqrt{5}} I,$$

for every  $n > 1$ .

Some structures in this paper are related to generalized secondary Fibonacci sequences (GSFS) (see [19,20]) given by

$$G(n+1) = pG(n) + qG(n-1), \quad n \geq 1,$$

with  $G(0) = a \in \mathbb{R}$ ,  $G(1) = b \in \mathbb{R}$  and  $p, q \in \mathbb{R}$ .

The ratio  $G(n+1)/G(n)$  of two consecutive terms of GSFS converges to:

- the golden mean  $\varphi = \frac{1+\sqrt{5}}{2}$ , for  $p = q = 1$ , determined by the ratio of two consecutive classical Fibonacci numbers;
- the silver mean  $\sigma_{2,1} = 1 + \sqrt{2}$ , for  $p = 2$  and  $q = 1$ , determined by the ratio of two consecutive Pell numbers;
- the bronze mean  $\sigma_{3,1} = \frac{3+\sqrt{13}}{2}$ , for  $p = 3$  and  $q = 1$ ;
- the subtle mean  $\sigma_{4,1} = 2 + \sqrt{5} = \varphi^3$ , for  $p = 4$  and  $q = 1$ ;
- the copper mean  $\sigma_{1,2} = 2$ , for  $p = 1$  and  $q = 2$ ;
- the nickel mean  $\sigma_{1,3} = \frac{1+\sqrt{13}}{2}$ , for  $p = 1$  and  $q = 3$ .

In the case  $q = 1$  and  $p = k$ , one gets the  $k$ -Fibonacci sequence

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad F_{k,0} = 0, F_{k,1} = 1,$$

which generalizes the classical Fibonacci sequence.

### 2.1. Reciprocal cost geometry

The main point of our construction is related to the canonical reciprocal cost function  $J : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ ,

$$J(x) = \frac{1}{2}(x + x^{-1}) - 1, \quad (8)$$

which is the unique solution of the polynomial composition law together with the curvature calibration (for more details see [22]). The function  $J$  is reciprocal,  $J(x) = J(x^{-1})$ , nonnegative, with a minimum at  $x = 1$ . In logarithmic coordinates,

$$J(e^t) = \cosh(t) - 1.$$

Near  $t = 0$ , one has

$$J(e^t) = \frac{t^2}{2} + O(t^4).$$

Among many possible multidimensional extensions, the form considered here is motivated by the multiplicative structure of the one-dimensional reciprocal cost and by the logarithmic representation  $J(e^t) = \cosh(t) - 1$ .

We study the family of reciprocal cost functions (see, e.g., [24])

$$J(x_1, \dots, x_n) = \frac{1}{2}(R + R^{-1}) - 1, \quad R = \prod_{i=1}^n x_i^{\alpha_i}, \quad (9)$$

where  $x_i > 0$  and  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \setminus \{0\}$ . In logarithmic coordinates  $t_i = \log x_i$ , the function (9) takes the form

$$J(t) = \cosh(\alpha \cdot t) - 1.$$

Therefore, the function  $J$  depends only on the scalar  $S(t) = \alpha \cdot t = \sum_{i=1}^n \alpha_i t_i$ . Its Hessian is the rank-one tensor

$$\nabla^2 J = \cosh(\alpha \cdot t) \left( \sum_{i=1}^n \alpha_i dt_i \right) \otimes \left( \sum_{i=1}^n \alpha_i dt_i \right).$$

Since  $\nabla^2 J$  has rank one, the tensor  $\tilde{g} := \nabla^2 J$  is degenerate for  $n \geq 2$ .

The induced geometry is degenerate, with a distinguished direction generated by  $\alpha$  and an integrable  $(n - 1)$ -dimensional null distribution. In particular, the ambient space is  $n$ -dimensional and the associated Hessian structure in logarithmic coordinates reduces to a one-dimensional geometry. Hessian geometry and Hessian manifolds play an important role in affine differential geometry and information geometry (see, e.g., [1,16]).

The rank-one property of this Hessian tensor motivates the construction of additional geometric structures. To obtain a nondegenerate geometric structure, we combine the rank-one tensor associated with the reciprocal cost geometry with a family of Hessian metrics. This construction produces an associated  $(1, 1)$ -tensor field whose normalization defines a projector. The projector then induces an almost product structure and the corresponding golden and metallic structures.

More precisely, the rank-one tensor  $\tilde{g}$  is defined by (27). By combining  $\tilde{g}$  with the nondegenerate Hessian metric  $h_\lambda$  defined by (26), we obtain the associated  $(1, 1)$ -tensor  $A_\lambda$ , given by (28). Its normalization gives the projector  $P_\lambda$ , (29). The corresponding almost product, golden, and metallic structures are introduced in (31), (32) and (33), respectively.

### 3. The Cost Induced Projector

Let  $M$  be a smooth manifold, let  $g$  be a nondegenerate metric on  $M$ , and let  $\tilde{g}$  be a positive semidefinite symmetric  $(0, 2)$ -tensor field of rank one. On an open set  $U \subseteq M$  where  $\tilde{g} \neq 0$  there exists a vector field  $V$  such that

$$\tilde{g}(X, Y) = g(V, X)g(V, Y), \quad X, Y \in \Gamma(TU). \quad (10)$$

The associated  $(1, 1)$ -tensor field  $A$  is defined by

$$g(AX, Y) = \tilde{g}(X, Y) \quad X, Y \in \Gamma(TU). \quad (11)$$

On  $U$ , using (10), we obtain

$$AX = g(V, X)V, \quad X \in \Gamma(TU). \quad (12)$$

We have

$$g(AX, Y) = \tilde{g}(X, Y) = \tilde{g}(Y, X) = g(AY, X) = g(X, AY)$$

**Lemma 1.** *The tensor  $A$  defined by (11) satisfies*

$$A^2 = \mu A, \quad (13)$$

where  $\mu = g(V, V)$ . Moreover,  $\mu = \text{tr}(A)$ .

**Proof.** Using (12), we compute

$$A^2X = A(AX) = A(g(V, X)V) = g(V, X)AV = g(V, X)g(V, V)V.$$

Hence

$$A^2X = g(V, V)AX,$$

so  $A^2 = \mu A$  with  $\mu = g(V, V)$ . From  $AX = g(V, X)V$ , we obtain

$$\text{tr}(A) = g(V, V),$$

which completes the proof.  $\square$

**Corollary 1.** *On the open subset  $U \subseteq M$  where  $\mu = g(V, V) \neq 0$ , the tensor*

$$P := \frac{1}{\mu}A \quad (13)$$

is a projector. Moreover,

$$\text{im}(P) = \text{span}\{V\}, \quad \ker(P) = \{X \in TM|_U : g(V, X) = 0\}.$$

Hence

$$TM|_U = \text{im}(P) \oplus \ker(P).$$

**Proof.** Since  $A^2 = \mu A$  and  $\mu \neq 0$ , we have

$$P^2 = \frac{1}{\mu^2}A^2 = \frac{1}{\mu}A = P.$$

Thus  $P$  is a projector.

Using  $AX = g(V, X)V$ , we get

$$PX = \frac{g(V, X)}{g(V, V)} V.$$

Hence  $\text{im}(P) = \text{span}\{V\}$ . Also,

$$PX = 0 \iff g(V, X) = 0,$$

so

$$\ker(P) = \{X \in TM|_U : g(V, X) = 0\}.$$

Finally, every vector field  $X \in \Gamma(TM|_U)$  decomposes as

$$X = PX + (X - PX),$$

where  $PX \in \text{im}(P)$  and  $X - PX \in \ker(P)$ . Therefore

$$TM|_U = \text{im}(P) \oplus \ker(P).$$

□

#### 4. Golden and Metallic Operators

In this section, starting from the projector  $P$  given by (13) and induced splitting

$$TM|_U = \text{im}(P) \oplus \ker(P),$$

we construct the almost-product, golden, and metallic structures. The constructions below follow from the identity  $P^2 = P$  and hold for any projector. The reciprocal cost geometry provide a particular projector to which this construction is applied. Let us define

$$F := 2P - I.$$

**Proposition 1.** *The tensor  $F$  satisfies*

$$F^2 = I.$$

Moreover,  $F|_{\text{im}(P)} = I$  and  $F|_{\ker(P)} = -I$ .

**Proof.** Since  $F = 2P - I$  and  $P$  is a projector i.e.  $P^2 = P$ , we have

$$F^2 = (2P - I)^2 = 4P^2 - 4P + I = I.$$

Moreover, for  $X \in \text{im}(P)$  one has  $PX = X$ , so

$$FX = 2X - X = X.$$

For  $X \in \ker(P)$  one has  $PX = 0$ , so

$$FX = -X.$$

This completes the proof. □

Let us now consider an operator of the form

$$G = \alpha P + \beta(I - P), \quad \alpha, \beta \in \mathbb{R}.$$

Since  $P^2 = P$  and  $P(I - P) = 0$ , we have

$$G^2 = \alpha^2 P + \beta^2 (I - P).$$

We require that  $G$  satisfies the golden equation  $G^2 = G + I$ , then

$$\alpha^2 = \alpha + 1, \quad \beta^2 = \beta + 1.$$

Thus  $\alpha$  and  $\beta$  are roots of the equation

$$x^2 = x + 1,$$

which has two solutions

$$\varphi = \frac{1 + \sqrt{5}}{2}, \quad 1 - \varphi = \frac{1 - \sqrt{5}}{2}.$$

Taking the two roots, we obtain

$$\alpha = \varphi, \quad \beta = 1 - \varphi = -\varphi^{-1}.$$

Therefore

$$G = \varphi P + (1 - \varphi)(I - P).$$

or equivalently, since  $F = 2P - I$ ,

$$G = \frac{1}{2}(I + \sqrt{5}F). \quad (14)$$

A direct computation shows that  $G$  satisfies

$$G^2 = G + I.$$

Thus the golden structure is induced by the projector.

**Corollary 2.** *The operator  $G$  has eigenvalues  $\varphi$  on  $\text{im}(P)$  and  $-\varphi^{-1}$  on  $\text{ker}(P)$ .*

**Proof.** On  $\text{im}(P)$  one has  $F = I$ , hence  $G = \frac{1}{2}(1 + \sqrt{5})I = \varphi I$ . On  $\text{ker}(P)$  one has  $F = -I$ , hence  $G = \frac{1}{2}(1 - \sqrt{5})I = -\varphi^{-1}I$ .  $\square$

Let us now, for  $p, q \in \mathbb{N}$ , define

$$M_{p,q} := \frac{p}{2}I + \frac{1}{2}\sqrt{p^2 + 4q}F. \quad (15)$$

**Theorem 2.** *The operator  $M_{p,q}$  given by (15) satisfies*

$$M_{p,q}^2 = p M_{p,q} + q I.$$

**Proof.** Using  $F^2 = I$ , from (15) we obtain

$$M_{p,q}^2 = \left( \frac{p}{2}I + \frac{1}{2}\sqrt{p^2 + 4q}F \right)^2 = \left( \frac{p^2}{4} + \frac{p^2 + 4q}{4} \right) I + \frac{p}{2}\sqrt{p^2 + 4q}F.$$

Therefore

$$M_{p,q}^2 = \left( \frac{p^2}{2} + q \right) I + \frac{p}{2}\sqrt{p^2 + 4q}F.$$

On the other hand,

$$p M_{p,q} + qI = p \left( \frac{p}{2} I + \frac{1}{2} \sqrt{p^2 + 4q} F \right) + qI = \left( \frac{p^2}{2} + q \right) I + \frac{p}{2} \sqrt{p^2 + 4q} F.$$

which coincides with  $M_{p,q}^2$ .  $\square$

By construction, each operator in the family  $\{M_{p,q}\}_{p,q \in \mathbb{N}}$  is a polynomial expression in the projector  $P$ , and the golden structure  $G = M_{1,1}$  corresponds to the case  $p = q = 1$ .

The following proposition gives the main properties of a general metallic structure.

**Proposition 2** (Hreţcanu-Crăşmăreanu [12]). *Let  $M_{p,q}$  be defined by*

$$M_{p,q} = \frac{p}{2} I + \frac{1}{2} \sqrt{p^2 + 4q} F, \quad p, q \in \mathbb{N},$$

where  $F^2 = I$ . Then the following properties hold:

1. For every integer  $n \geq 1$ ,

$$M_{p,q}^n = G(n) M_{p,q} + q G(n-1) I,$$

where  $(G(n))_{n \geq 0}$  is the generalized secondary Fibonacci sequence defined by

$$G(n+1) = pG(n) + qG(n-1), \quad G(0) = 0, \quad G(1) = 1.$$

2. The operator  $M_{p,q}$  is an isomorphism on each tangent space  $T_x M$ , hence invertible. Its inverse is polynomial (of quadratic type, but not metallic) and is given by

$$\bar{M}_{p,q} = M_{p,q}^{-1} = \frac{1}{q} M_{p,q} - \frac{p}{q} I.$$

It satisfies

$$q \bar{M}_{p,q}^2 + p \bar{M}_{p,q} - I = 0.$$

3. The eigenvalues of  $M_{p,q}$  are

$$\frac{p + \sqrt{p^2 + 4q}}{2}, \quad \frac{p - \sqrt{p^2 + 4q}}{2}.$$

## 5. The two-dimensional case

We now illustrate the construction in the two-dimensional case. Consider the canonical reciprocal cost function (8). Following the  $n$ -dimensional case (9), define on  $\mathbb{R}_{>0}^2$ :

$$R(x, y) := \frac{x}{y}, \quad J(x, y) := \frac{1}{2} (R + R^{-1}) - 1. \quad (16)$$

This corresponds to the choice  $\alpha = (1, -1)$  in the  $n$ -dimensional model.

### 5.1. Logarithmic coordinates

Introduce logarithmic coordinates  $u = \log x$ ,  $v = \log y$ . Then

$$R = e^{u-v}, \quad J(u, v) = \cosh(u-v) - 1. \quad (17)$$

In this case, the function  $J(u, v)$  depends only on the quantity  $u - v$ . The Hessian tensor  $\tilde{g} = \nabla^2 J$  of  $J(u, v) = \cosh(u - v) - 1$  is

$$\tilde{g} = \cosh(u - v) (du - dv) \otimes (du - dv) = \cosh(u - v) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \quad (18)$$

Since  $\cosh(u - v) > 0$ , the tensor  $\tilde{g}$  is positive semidefinite of rank one, with distinguished direction  $V = \partial_u - \partial_v$ .

### 5.2. $(x, y)$ -coordinates

Passing to the original  $(x, y)$ -coordinates, and using  $u = \log x$ ,  $v = \log y$ , we obtain

$$V = x\partial_x - y\partial_y.$$

Using (18),  $du = \frac{dx}{x}$ ,  $dv = \frac{dy}{y}$ , and

$$\cosh\left(\log\left(\frac{x}{y}\right)\right) = \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right) = \frac{x^2 + y^2}{2xy}$$

we obtain

$$\tilde{g} = \frac{x^2 + y^2}{2xy} \left(\frac{dx}{x} - \frac{dy}{y}\right) \otimes \left(\frac{dx}{x} - \frac{dy}{y}\right).$$

Hence

$$\tilde{g} = \frac{x^2 + y^2}{2xy} \begin{pmatrix} x^{-2} & -\frac{1}{xy} \\ -\frac{1}{xy} & y^{-2} \end{pmatrix}. \quad (19)$$

So, the tensor  $\tilde{g}$  is a rank-one tensor.

On the other hand, we can also consider the full Hessian of the cost function directly in the original  $(x, y)$ -coordinates, i.e.

$$\tilde{g}_x = \nabla_x^2 J(x, y).$$

A direct computation gives

$$\det(\tilde{g}_x) = -\frac{(x^2 - y^2)^2}{4x^4y^4}.$$

Thus  $\tilde{g}_x$  is generically nondegenerate and indefinite, while it becomes singular on the locus  $x = y$ . The geometry determined by the Hessian metric  $\tilde{g}_x$  was studied in [24].

**Remark 1.** The rank-one tensor  $\tilde{g}$  obtained from the logarithmic representation of the cost function and the full Hessian metric  $\tilde{g}_x$  in the original  $(x, y)$ -coordinates carry different geometry. In particular,  $\tilde{g}$  is degenerate of constant rank one and determines the distinguished comparison direction generated by  $V = x\partial_x - y\partial_y$ , while  $\tilde{g}_x$  is generically nondegenerate and becomes singular on the locus  $x = y$ .

By Lemma 1 and Corollary 1, the rank-one property of  $\tilde{g}$ , when paired with a nondegenerate metric, leads to the construction of a projector. The tensor  $\tilde{g}_x$  does not have this property and does not produce the projector-type structures. For this reason, the projector construction developed in the following part is associated with  $\tilde{g}$ .

### 5.3. The induced projector

We introduce a one-parameter family of nondegenerate Hessian metrics on  $\mathbb{R}_{>0}^2$  by

$$\Phi_\lambda(x, y) = J(x) + J(y) + \lambda J\left(\frac{x}{y}\right), \quad \lambda \in \mathbb{R}. \quad (20)$$

For  $\lambda = 0$  this reduces to  $\Phi_0(x, y) = J(x) + J(y)$ , whose Hessian 324

$$h_0 = \nabla^2 \Phi_0 = \begin{pmatrix} x^{-3} & 0 \\ 0 & y^{-3} \end{pmatrix} \quad (21) \quad 325$$

is positive definite on  $\mathbb{R}_{>0}^2$  and it is used as a separable reference metric. 326

The associated  $(1, 1)$ -tensor field is defined by 327

$$A_0 = h_0^{-1} \tilde{g}. \quad (22) \quad 328$$

The metric  $h_0$  provides a nondegenerate reference metric for the distinguished comparison direction generated by  $V = x\partial_x - y\partial_y$ . 329

The term  $\lambda J(x/y)$  introduces mixed second derivatives along  $\omega$ , where 330

$$\omega = d \log(x/y) = \frac{dx}{x} - \frac{dy}{y}, \quad 331$$

so the Hessian metric  $h_\lambda = \nabla_x^2 \Phi_\lambda$  is nondiagonal for  $\lambda \neq 0$ . The associated  $(1, 1)$ -tensor is 332

$$A_\lambda = h_\lambda^{-1} \tilde{g}, \quad 333$$

and the projector is obtained by normalization, as given in Section 3. 334

**Remark 2.** *The construction involves two natural affine structures associated with the reciprocal cost geometry.* 335

*In the two-dimensional case the rank-one tensor  $\tilde{g} = \cosh(u - v) (du - dv) \otimes (du - dv)$  is derived with respect to the flat structure in the logarithmic coordinates  $(u, v)$ , while the Hessian metric  $h_\lambda = \nabla_x^2 \Phi_\lambda$  is used with respect to the flat structure in the original  $(x, y)$ -coordinates. The  $n$ -dimensional construction considered in Section 6 is based on the same choice of affine structures.* 336

Let 337

$$h_\lambda = \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \quad a = \frac{1 + \lambda y}{x^3}, \quad b = -\lambda \frac{x^2 + y^2}{2x^2 y^2}, \quad d = \frac{1 + \lambda x}{y^3}, \quad (23) \quad 338$$

Then  $A_\lambda$  has the form given in the next proposition. 339

**Proposition 3.** *Let  $h_\lambda$  is defined by* 340

$$h_\lambda = \nabla_x^2 \Phi_\lambda, \quad \Phi_\lambda(x, y) = J(x) + J(y) + \lambda J\left(\frac{x}{y}\right), \quad (24) \quad 341$$

and let  $\tilde{g}$  is given by (19). Assume that  $\det(h_\lambda) = ad - b^2 \neq 0$ , then 342

$$A_\lambda = \frac{x^2 + y^2}{2xy} \frac{1}{ad - b^2} \begin{pmatrix} \frac{d}{x^2} + \frac{b}{xy} & -\frac{d}{xy} - \frac{b}{y^2} \\ -\frac{b}{x^2} - \frac{a}{xy} & \frac{b}{xy} + \frac{a}{y^2} \end{pmatrix}. \quad (25) \quad 343$$

**Proof.** From the definition of  $\Phi_\lambda$ , we have 344

$$h_\lambda = \nabla_x^2 \Phi_\lambda = \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \quad 345$$

where

$$a = \frac{1 + \lambda y}{x^3}, \quad b = -\lambda \frac{x^2 + y^2}{2x^2y^2}, \quad d = \frac{1 + \lambda x}{y^3}.$$

Thus

$$h_\lambda^{-1} = \frac{1}{ad - b^2} \begin{pmatrix} d & -b \\ -b & a \end{pmatrix}.$$

Multiplying  $h_\lambda^{-1}$  with the matrix of  $\tilde{g}$  given by (19), we obtain (25).  $\square$

**Theorem 3.** *The tensor  $A_\lambda$  given by (25) satisfies  $A_\lambda^2 = \mu_\lambda A_\lambda$ , where  $\mu_\lambda = \text{tr}(A_\lambda)$ . The tensor*

$$P_\lambda := \frac{1}{\mu_\lambda} A_\lambda$$

is a projector on the open subset of  $\mathbb{R}^2$  where  $ad - b^2 \neq 0$  and  $\mu_\lambda \neq 0$ .

**Proof.** Since  $h_\lambda$  is nondegenerate and  $A_\lambda = h_\lambda^{-1} \tilde{g}$ , the tensors  $A_\lambda$  and  $\tilde{g}$  have same rank. Since  $\tilde{g}$  has rank one, then  $A_\lambda$  also has rank one. Therefore

$$A_\lambda^2 = \mu_\lambda A_\lambda,$$

where  $\mu_\lambda = \text{tr}(A_\lambda)$ . If  $\mu_\lambda \neq 0$ , then

$$P_\lambda := \frac{1}{\mu_\lambda} A_\lambda$$

is well defined and satisfies

$$P_\lambda^2 = P_\lambda.$$

$\square$

**Corollary 3.** *The corresponding projector is*

$$P_\lambda(X) = \frac{\omega(X)}{\omega(V_\lambda)} V_\lambda, \quad V_\lambda = h_\lambda^{-1} \omega,$$

where

$$\omega = \frac{dx}{x} - \frac{dy}{y}.$$

Moreover,

$$\text{im}(P_\lambda) = \text{span}\{V_\lambda\}, \quad \ker(P_\lambda) = \ker \omega.$$

In particular,  $\ker(P_\lambda)$  is generated by  $x\partial_x + y\partial_y$ .

**Proof.** Since

$$\tilde{g} = \frac{x^2 + y^2}{2xy} \omega \otimes \omega$$

and  $V_\lambda = h_\lambda^{-1} \omega$ , the tensor  $A_\lambda$  has the form

$$A_\lambda X = \frac{x^2 + y^2}{2xy} \omega(X) V_\lambda.$$

After normalization, we have

$$P_\lambda(X) = \frac{\omega(X)}{\omega(V_\lambda)} V_\lambda.$$

Hence  $\text{im}(P_\lambda) = \text{span}\{V_\lambda\}$ . Also,

$$P_\lambda X = 0 \iff \omega(X) = 0,$$

so  $\ker(P_\lambda) = \ker \omega$ . Since

$$\omega(x\partial_x + y\partial_y) = \frac{dx}{x}(x\partial_x + y\partial_y) - \frac{dy}{y}(x\partial_x + y\partial_y) = 1 - 1 = 0,$$

and  $\ker \omega$  is one-dimensional, we get

$$\ker(P_\lambda) = \text{span}\{x\partial_x + y\partial_y\}.$$

□

## 6. The $n$ -dimensional Hessian construction

In this part, we will describe the construction of a projector in the  $n$ -dimensional case. For  $\alpha = (\alpha_1, \dots, \alpha_n) \neq 0$ ,  $\lambda \in \mathbb{R}$ , and  $x \in \mathbb{R}_{>0}^n$ , we have

$$\Phi_\lambda(x_1, \dots, x_n) = \sum_{i=1}^n J(x_i) + \lambda J(R), \quad R = \prod_{i=1}^n x_i^{\alpha_i}.$$

The associated Hessian metric is

$$h_\lambda = \nabla_x^2 \Phi_\lambda \tag{26}$$

defines the associated Hessian metric. The parameter  $\lambda$  can be viewed as a deformation parameter. For  $\lambda = 0$ , we obtain

$$h_0 = \text{diag}(x_1^{-3}, \dots, x_n^{-3}),$$

which is positive definite on  $\mathbb{R}_{>0}^n$ .

For  $\lambda \neq 0$ , the term  $\lambda J(R)$  introduces mixed second derivatives, so  $h_\lambda$  is generally nondiagonal. Its positive definite locus depends on  $(x, \lambda)$ . Moreover, for every fixed point  $(x, y)$ , positive definiteness is preserved for sufficiently small values of  $|\lambda|$ . An analysis of the signature and singular loci of  $h_\lambda$  in  $n$ -dimensional lies beyond the scope of the present paper. We work on open subsets where  $h_\lambda$  is positive definite.

In the two-dimensional case, explicit conditions for positive definiteness are given in the following example.

**Example 1.** Let us consider the two-dimensional case, with  $h_\lambda$  given by (23). We have

$$\det(h_\lambda) = ad - b^2 = \frac{4xy(1 + \lambda(x + y)) - \lambda^2(x^2 - y^2)^2}{4x^4y^4}.$$

By Sylvester's criterion,  $h_\lambda$  is positive definite at  $(x, y) \in \mathbb{R}_{>0}^2$  if and only if  $a > 0$  and  $\det(h_\lambda) > 0$  or equivalently,

$$1 + \lambda y > 0 \quad \text{and} \quad 4xy(1 + \lambda(x + y)) - \lambda^2(x^2 - y^2)^2 > 0.$$

**Remark 3.** For  $\lambda \neq 0$ , the metric  $h_\lambda$  is not positive definite on  $\mathbb{R}_{>0}^2$ . For fixed  $x > 0$  and sufficiently large  $y$ ,  $\det(h_\lambda) < 0$  because the term  $-\lambda^2 y^4$  dominates the numerator. Therefore, we restrict the construction to open subsets of  $\mathbb{R}_{>0}^2$  where  $h_\lambda$  is positive definite.

**Remark 4.** For  $\lambda = 0$ , the metric  $h_0$  is positive definite on  $\mathbb{R}_{>0}^n$ . Since the coefficients of  $h_\lambda$  depend on  $\lambda$  and on variables  $x_i$ , this property is preserved for sufficiently small values of  $|\lambda|$  at every fixed point  $(x_1, \dots, x_n)$ . We consider open subsets where  $h_\lambda$  is positive definite.

Recall that, in logarithmic coordinates  $t_i = \log x_i$ , the reciprocal cost function (9) has the form

$$J(t) = \cosh(\alpha \cdot t) - 1, \quad \alpha \cdot t = \sum_{i=1}^n \alpha_i t_i.$$

with Hessian

$$\nabla^2 J = \cosh(\alpha \cdot t) \alpha \otimes \alpha.$$

In the original  $x$ -coordinates, let

$$\omega = \sum_{i=1}^n \alpha_i \frac{dx_i}{x_i}.$$

Then the rank-one tensor is

$$\tilde{g} = \cosh(\alpha \cdot t) \omega \otimes \omega. \quad (27)$$

The associated  $(1, 1)$ -tensor field  $A_\lambda := h_\lambda^{-1} \tilde{g}$  is defined by

$$h_\lambda(A_\lambda X, Y) = \tilde{g}(X, Y), \quad X, Y \in \Gamma(T\mathbb{R}_{>0}^n).$$

Let  $V_\lambda := h_\lambda^{-1}(\omega)$  defined by  $h_\lambda(V_\lambda, X) = \omega(X)$  for all  $X \in \Gamma(T\mathbb{R}_{>0}^n)$ . Then, by (27), we obtain

$$\begin{aligned} \tilde{g}(X, Y) &= \cosh(\alpha \cdot t) \omega(X) \omega(Y) \\ &= \cosh(\alpha \cdot t) \omega(X) h_\lambda(V_\lambda, Y). \end{aligned}$$

Applying Lemma 1 and Corollary 1 with  $g = h_\lambda$ ,  $V = V_\lambda$ , and  $\tilde{g}$  as in (27), we get

$$A_\lambda X = \cosh(\alpha \cdot t) \omega(X) V_\lambda, \quad A_\lambda = \cosh(\alpha \cdot t) V_\lambda \otimes \omega, \quad (28)$$

and

$$A_\lambda^2 = \mu_\lambda A_\lambda, \quad \mu_\lambda = \cosh(\alpha \cdot t) \omega(V_\lambda) = \text{tr}(A_\lambda).$$

We consider open subset of  $\mathbb{R}_{>0}^n$  where  $\mu_\lambda \neq 0$ . On this set the tensor

$$P_\lambda = \frac{1}{\mu_\lambda} A_\lambda \quad (29)$$

is a projector. More precisely,

$$P_\lambda X = \frac{\omega(X)}{\omega(V_\lambda)} V_\lambda. \quad (30)$$

Since  $\tilde{g}$  is symmetric,

$$h_\lambda(A_\lambda X, Y) = h_\lambda(X, A_\lambda Y),$$

hence  $A_\lambda$ , and therefore  $P_\lambda$ , are self-adjoint with respect to  $h_\lambda$ . Applying the construction given in Section 4, we obtain the induced almost product, golden, and metallic structures

$$F_\lambda = 2P_\lambda - I, \quad F_\lambda^2 = I, \quad (31)$$

$$G_\lambda = \frac{1}{2}(I + \sqrt{5} F_\lambda), \quad G_\lambda^2 = G_\lambda + I, \quad (32)$$

and, for  $p, q \in \mathbb{N}$ ,

$$M_{p,q}^\lambda = \frac{p}{2}I + \frac{1}{2}\sqrt{p^2 + 4q}F_\lambda, \quad (M_{p,q}^\lambda)^2 = pM_{p,q}^\lambda + qI. \quad (33)$$

Consequently,  $F_\lambda$ ,  $G_\lambda$ , and  $M_{p,q}^\lambda$  are also self-adjoint with respect to  $h_\lambda$ , since they are polynomial expressions in the self-adjoint operator  $P_\lambda$  with real coefficients.

### 6.1. Properties of the induced projector

We now study properties of the projector  $P_\lambda$  constructed in the previous sections. Let  $\nabla^\lambda$  denote the Levi-Civita connection of the Hessian metric  $h_\lambda = \nabla_x^2 \Phi_\lambda$  on the open subset of  $\mathbb{R}_{>0}^n$  where  $h_\lambda$  is positive definite. We denote by  $D$  the canonical flat affine connection associated with the chosen affine structure. In affine coordinates, its Christoffel symbols vanish identically.

We first check whether  $P_\lambda$  is parallel with respect to the canonical flat affine connection in logarithmic coordinates, using the two-dimensional case.

**Example 2.** Let  $n = 2$ ,  $\alpha = (1, -1)$ , and  $\lambda = 0$ . In logarithmic coordinates  $u = \log x$ ,  $v = \log y$ , we have

$$\omega = du - dv.$$

For the metric  $h_0$ , we have

$$V_0 = h_0^{-1}(\omega) = e^u \partial_u - e^v \partial_v,$$

and

$$\omega(V_0) = e^u + e^v.$$

Therefore

$$P_0 = \frac{1}{e^u + e^v} \begin{pmatrix} e^u & -e^u \\ -e^v & e^v \end{pmatrix}.$$

Hence

$$\partial_u(P_0)^1_1 = \frac{e^{u+v}}{(e^u + e^v)^2} \neq 0.$$

Since the canonical flat affine connection  $D$  satisfies

$$D_{\partial_u} \partial_u = D_{\partial_u} \partial_v = D_{\partial_v} \partial_u = D_{\partial_v} \partial_v = 0,$$

we obtain

$$(D_{\partial_u} P_0)^1_1 = \partial_u(P_0)^1_1 \neq 0.$$

Therefore  $DP_0 \neq 0$ , i.e. the projector  $P_0$  is not parallel with respect to the canonical flat affine connection in logarithmic coordinates.

**Remark 5.** The induced tensor  $G_\lambda$  satisfies

$$G_\lambda^2 = G_\lambda + I,$$

and is self-adjoint with respect to the Hessian metric  $h_\lambda$ . On every open subset of  $\mathbb{R}_{>0}^n$  where  $h_\lambda$  is positive definite,  $(\mathbb{R}_{>0}^n, h_\lambda, G_\lambda)$  defines a golden Riemannian manifold.

The following theorem describes the family of linear connections preserving the induced golden structure  $G_\lambda$ .

**Theorem 4** (Theorem 5.1, [5]). Let  $F_\lambda = 2P_\lambda - I$  be the induced almost product structure associated with the golden structure  $G_\lambda$ . Then the set of linear connections  $\nabla$  satisfying  $\nabla G_\lambda = 0$  is given by

$$\nabla_X Y = \frac{1}{5} \left[ 3\tilde{\nabla}_X Y + 2G_\lambda(\tilde{\nabla}_X G_\lambda Y) - G_\lambda(\tilde{\nabla}_X Y) - \tilde{\nabla}_X G_\lambda Y \right] + \mathcal{O}_{F_\lambda} Q(X, Y),$$

where  $\tilde{\nabla}$  is an arbitrary fixed linear connection and  $Q$  is a  $(1,2)$ -tensor field for which  $\mathcal{O}_{F_\lambda} Q$  is an associated Obata operator

$$\mathcal{O}_{F_\lambda} Q(X, Y) = \frac{1}{2} \left[ Q(X, Y) + F_\lambda Q(X, F_\lambda Y) \right].$$

We now study the parallelism properties of  $P_\lambda$  with respect to the Levi-Civita connection  $\nabla^\lambda$  of  $h_\lambda$ .

**Example 3.** In the two-dimensional case, the projector  $P_0$  is not parallel with respect to the Levi-Civita connection  $\nabla^0$  of  $h_0$ . We have

$$h_0 = \begin{pmatrix} x^{-3} & 0 \\ 0 & y^{-3} \end{pmatrix},$$

and

$$P_0 = \frac{1}{x+y} \begin{pmatrix} x & -x^2/y \\ -y^2/x & y \end{pmatrix}.$$

A direct computation gives

$$(\nabla_{\partial_x}^0 P_0)^x_x = \partial_x \left( \frac{x}{x+y} \right) = \frac{y}{(x+y)^2} \neq 0.$$

Therefore,  $\nabla^0 P_0 \neq 0$ .

**Example 4.** For  $\lambda \neq 0$ , the projector  $P_\lambda$  is not parallel with respect to  $\nabla^\lambda$ .

The coefficients of  $P_\lambda$  and the Christoffel symbols of  $\nabla^\lambda$  contain nontrivial mixed terms from  $J(x/y)$ . For example, at the point  $(x, y) = (1, 1)$ , we obtain

$$\left( \nabla_{\partial_x}^\lambda P_\lambda \right)^x_x = \frac{1+\lambda}{4(1+2\lambda)}.$$

At the point  $(1, 1)$ , the metric  $h_\lambda$  is positive definite only for  $1 + 2\lambda > 0$ , and consequently  $\left( \nabla_{\partial_x}^\lambda P_\lambda \right)^x_x \neq 0$ . Hence  $P_\lambda$  is not parallel with respect to the Levi-Civita connection  $\nabla^\lambda$ .

The two-dimensional case represents the simplest nontrivial realization of the general construction. The above examples show that the induced almost product, golden, and metallic structures are non-parallel in general.

## 6.2. Integrability of the eigendistributions

Motivated by Proposition 5.3 of [5], we study the integrability of the distributions induced by the projector  $P_\lambda$ .

Since the almost product, golden, and metallic structures  $F_\lambda = 2P_\lambda - I$ ,  $G_\lambda$ ,  $M_{p,q}^\lambda$  are obtained from  $P_\lambda$ , they have the same eigendistributions,  $\text{Im}(P_\lambda)$  and  $\text{ker}(P_\lambda)$ .

The distribution  $\text{Im}(P_\lambda) = \text{span}\{V_\lambda\}$  is one-dimensional and therefore integrable. For  $\text{ker}(P_\lambda)$ , recall from (30) that

$$\text{ker}(P_\lambda) = \text{ker}(\omega),$$

where

$$\omega = \sum_{i=1}^n \alpha_i \frac{dx_i}{x_i} = d(\log R).$$

Since  $\omega$  is exact, it is closed. Hence, by the Frobenius theorem, the distribution  $\ker(P_\lambda)$  is integrable.

Therefore, both eigendistributions of  $P_\lambda$  are integrable. Consequently, the induced almost product, golden, and metallic structures are integrable.

### 6.3. A rank-one representation of the cost-induced projector

Let  $P_\lambda$  be the cost induced projector. Since

$$P_\lambda X = \frac{h_\lambda(V_\lambda, X)}{h_\lambda(V_\lambda, V_\lambda)} V_\lambda,$$

on the open set where  $h_\lambda(V_\lambda, V_\lambda) \neq 0$ , we define

$$\xi_\lambda := V_\lambda, \quad \eta_\lambda(X) := \frac{h_\lambda(V_\lambda, X)}{h_\lambda(V_\lambda, V_\lambda)}.$$

Then  $\eta_\lambda(\xi_\lambda) = 1$  and  $P_\lambda X = \eta_\lambda(X)\xi_\lambda$ . Therefore,

$$P_\lambda = \eta_\lambda \otimes \xi_\lambda.$$

Consequently, the cost induced golden structure  $G_\lambda$  given by (32) and the metallic structures  $M_{p,q}^\lambda$  given by (33) can be written in the form

$$G_\lambda = (1 - \varphi)I + \sqrt{5}\eta_\lambda \otimes \xi_\lambda,$$

and

$$M_{p,q}^\lambda = \left( \frac{p}{2} - \frac{1}{2}\sqrt{p^2 + 4q} \right) I + \sqrt{p^2 + 4q}\eta_\lambda \otimes \xi_\lambda.$$

Thus the reciprocal cost geometry together with the Hessian metric  $h_\lambda$  determines the projector  $P_\lambda$  and the associated polynomial structures. The vector field  $V_\lambda$  generates  $\text{im}(P_\lambda)$ , while  $\ker(P_\lambda)$  gives the complementary distribution.

### 6.4. Curvature properties of the Hessian metric $h_\lambda$

We now consider curvature properties of the Hessian metric  $h_\lambda = Dd\Phi_\lambda$ , where  $D$  denotes the canonical flat affine connection on  $\mathbb{R}_{>0}^n$ .

Let  $\nabla^\lambda$  be the Levi-Civita connection of  $h_\lambda$ . The difference tensor  $K^\lambda$  of the Levi-Civita connection  $\nabla^\lambda$  of  $h_\lambda$  and  $D$  is defined by

$$K_X^\lambda Y := \nabla_X^\lambda Y - D_X Y. \quad (34)$$

Since both  $\nabla^\lambda$  and  $D$  are symmetric connections, the tensor  $K^\lambda$  is symmetric, i.e.  $K_X^\lambda Y = K_Y^\lambda X$ . Starting from

$$X(h_\lambda(Y, Z)) = h_\lambda(\nabla_X^\lambda Y, Z) + h_\lambda(Y, \nabla_X^\lambda Z).$$

and substituting

$$\nabla_X^\lambda Y = D_X Y + K_X^\lambda Y,$$

we have

$$\begin{aligned} X(h_\lambda(Y, Z)) &= h_\lambda(D_X Y, Z) + h_\lambda(K_X^\lambda Y, Z) \\ &\quad + h_\lambda(Y, D_X Z) + h_\lambda(Y, K_X^\lambda Z). \end{aligned}$$

Hence

$$(D_X h_\lambda)(Y, Z) = h_\lambda(K_X^\lambda Y, Z) + h_\lambda(Y, K_X^\lambda Z).$$

Since  $h_\lambda$  is Hessian metric, the cubic tensor

$$C_\lambda(X, Y, Z) := (D_X h_\lambda)(Y, Z) \quad (35)$$

is totally symmetric. Therefore,  $h_\lambda(K_X^\lambda Y, Z) = h_\lambda(Y, K_X^\lambda Z)$ , and consequently

$$h_\lambda(K_X^\lambda Y, Z) = \frac{1}{2}(D_X h_\lambda)(Y, Z). \quad (36)$$

Therefore  $K^\lambda$  is completely determined by the tensor  $C_\lambda(X, Y, Z)$ .

Since the connection  $D$  is flat, the curvature tensor of  $\nabla^\lambda$  is given by

$$R^\lambda(X, Y)Z = K_X^\lambda K_Y^\lambda Z - K_Y^\lambda K_X^\lambda Z + (D_X K^\lambda)(Y, Z) - (D_Y K^\lambda)(X, Z).$$

Using (35), (36), and the Codazzi-type identity  $(D_X K^\lambda)(Y, Z) = (D_Y K^\lambda)(X, Z)$ , we obtain

$$R^\lambda(X, Y, Z, W) = \frac{1}{4} \left( h_\lambda^{-1}(C_\lambda(X, Z, \cdot), C_\lambda(Y, W, \cdot)) - h_\lambda^{-1}(C_\lambda(Y, Z, \cdot), C_\lambda(X, W, \cdot)) \right),$$

where  $R^\lambda(X, Y, Z, W) := h_\lambda(R^\lambda(X, Y)Z, W)$ . Using  $C_\lambda(X, Y, Z) = 2h_\lambda(K_X^\lambda Y, Z)$ , the last formula becomes

$$R^\lambda(X, Y, Z, W) = \frac{1}{4} \left( h_\lambda(K_X^\lambda Z, K_Y^\lambda W) - h_\lambda(K_Y^\lambda Z, K_X^\lambda W) \right). \quad (37)$$

Thus the Riemannian curvature of  $h_\lambda$  is completely determined by the cubic tensor  $C_\lambda$ . If  $\{E_i\}_{i=1}^n$  is a local  $h_\lambda$ -orthonormal frame, then the corresponding Ricci tensor is

$$\text{Ric}_{h_\lambda}(Y, Z) = \sum_{i=1}^n R^\lambda(E_i, Y, Z, E_i).$$

Using (37), we obtain

$$\text{Ric}_{h_\lambda}(Y, Z) = \frac{1}{4} \sum_{i=1}^n \left( h_\lambda(K_Y^\lambda Z, K_{E_i}^\lambda E_i) - h_\lambda(K_{E_i}^\lambda Z, K_Y^\lambda E_i) \right).$$

Therefore the Ricci tensor is also completely determined by the cubic tensor  $C_\lambda$ . The scalar curvature of  $h_\lambda$  is defined by  $\text{Scal}_{h_\lambda} = \text{tr}_{h_\lambda}(\text{Ric}_{h_\lambda})$ .

In the two-dimensional case, the Ricci tensor is completely determined by the scalar curvature. For  $\lambda = 0$ , the Hessian metric is  $h_0 = \text{diag}(x^{-3}, y^{-3})$ , and the corresponding curvature vanishes, and  $\text{Scal}_{h_0} = 0$ .

For  $\lambda \neq 0$ , the scalar curvature is nonzero in general and depends on  $\lambda$ . By direct calculation, we have

$$\text{Scal}_{h_\lambda} = - \frac{4\lambda x^2 y^2 (\lambda(x+y)^3 + 3(x^2 + y^2))}{(\lambda^2(x^2 - y^2)^2 - 4\lambda xy(x+y) - 4xy)^2}.$$

Since the expression is not identically zero for  $\lambda \neq 0$ , the Hessian metric  $h_\lambda$  is non-flat in general. Thus the interaction term  $\lambda J(x/y)$  produces nontrivial curvature of  $h_\lambda$ .

## 7. Conclusion

In this paper, we studied a family of projector induced polynomial structures associated with the reciprocal cost geometry. Starting from the rank-one tensor  $\tilde{g}$  determined by the reciprocal cost function in logarithmic coordinates and a family of Hessian metrics  $h_\lambda$ , we constructed  $(1,1)$ -tensor field  $A_\lambda$  and the projector  $P_\lambda$ . The projector  $P_\lambda$  determines a splitting

$$TU = \text{im}(P_\lambda) \oplus \text{ker}(P_\lambda),$$

which induces the almost product structure  $F_\lambda = 2P_\lambda - I$ , and the corresponding golden and metallic structures. The Hessian structure is determined by the metric  $h_\lambda = \nabla_x^2 \Phi_\lambda$  with respect to the  $x$ -coordinates, while the rank-one tensor  $\tilde{g}$  is obtained from the logarithmic reciprocal cost geometry.

We studied several properties of the projector  $P_\lambda$  and the induced structures, including eigendistributions, parallelism, integrability, and curvature. In particular, we showed that  $P_\lambda$  is generally not parallel with respect to either the canonical flat affine connection in logarithmic coordinates or the Levi-Civita connection of the Hessian metric  $h_\lambda$ . The eigendistributions of  $P_\lambda$  are integrable and are determined by the vector field  $V_\lambda$  and the one-form  $\omega = d(\log R)$ . The polynomial structures are induced by the projector associated with the rank-one tensor  $\tilde{g}$  and the Hessian metric  $h_\lambda$ . The construction is developed in arbitrary dimension, while the two-dimensional case is considered in detail.

The obtained structures depend on the choice of the Hessian metric  $h_\lambda$ . In particular, the deformation parameter  $\lambda$  changes the corresponding projector, the induced polynomial structures, and their geometric properties.

The construction in the paper shows that a rank-one tensor obtained from reciprocal cost geometry, together with a nondegenerate Hessian metric, determines a projector and the associated almost product, golden, and metallic structures. In this way, reciprocal cost geometry leads to polynomial structures on Hessian manifolds.

Further directions will include the study of the positive-definite locus and curvature properties of  $h_\lambda$  and possible relations with Hessian [16] and information geometry [1].

**Author Contributions:** Conceptualization, J.W.; Methodology, J.W. and M.Z.; Software, J.W.; Validation, J.W. and M.Z.; Formal Analysis, M.Z. and J.W.; Investigation, J.W. and M.Z.; Resources, J.W.; Writing-Original Draft Preparation, J.W.; Writing-Review and Editing, M.Z. and J.W.; Funding Acquisition, J.W. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** No new data were created or analyzed in this study.

**Conflicts of Interest:** The authors declare no conflicts of interest.

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