Goldbach via a Mod-8 Kernel: Density-One and Short-Interval Positivity

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Abstract

We present a purely classical framework for Goldbach's conjecture based on a mod-8 periodic kernel K_8 and the circle method. On major arcs we obtain a positive main term equal to a 2-adic gate $c_8(2m) \in \{1, \frac{1}{2}\}$ times the Hardy–Littlewood singular series $\mathfrak{S}(2m)$. On minor arcs we prove unconditional density-one positivity via mean-square bounds, and we convert fourth-moment control into pointwise positivity in every short interval, giving a bounded gap between exceptional even integers. A quantified medium-arc dispersion lemma (with an explicit small saving $\delta_{\text{med}} > 0$) lowers the short-interval exponent from $(\log N)^8$ to $(\log N)^{8-\delta_{\text{med}}}$. We also include an unconditional Chen/Selberg variant (prime + almost-prime), explicit constants and parameter choices, a smoothed-to-sharp transfer with numerical bounds, and a reproducible computational protocol. An optional GRH template is recorded for comparison.

1 Introduction

Goldbach's conjecture asserts that every even integer $2m > 2$ can be expressed as a sum of two primes. We develop a classical circle-method framework using a mod-8 periodic kernel K_8 that preserves the natural residue structure and isolates the 2-adic local factor.

Contributions.

- **Mod-8 kernel and major arcs.** A periodic kernel K_8 yields a positive major-arc main term $(c_8(2m) + o(1))\,\mathfrak{S}(2m)\,N/\log^2 N$ with $c_8(2m) \in \{1, \frac{1}{2}\}$.
- **Minor arcs: density-one and short-interval positivity.** Mean-square bounds give unconditional density-one positivity. A fourth-moment argument gives bounded gaps between exceptions, initially $\ll (\log N)^8$.
- **Medium-arc dispersion (quantified).** A dispersion lemma on medium arcs provides a small saving $\delta_{\text{med}} > 0$, improving the short-interval exponent to $(\log N)^{8-\delta_{\text{med}}}$.
- **Chen/Selberg variant.** An unconditional prime + almost-prime result holds for all sufficiently large even integers, with computable threshold.
- **Explicit constants and protocol.** We record explicit parameter choices, constants, a smoothed-to-sharp transfer with numerical bounds, and a reproducible computational check.

2 Notation and setup

We write $e(x) := e^{2\pi i x}$. Denote by $\mathbb{P}$ the set of primes. Residues are taken modulo 8, with the odd classes $\{1,3,5,7\}$ and even classes $\{0,2,4,6\}$. Let $\pi(n)$ be the prime indicator (or a smoothed/weighted variant, as needed for analysis).

3 Classical mod-8 gate and density-one positivity

We record a purely arithmetic approach using a periodic kernel at modulus 8 and derive density-one positivity via the circle method, together with an unconditional "prime + almost-prime" variant à la Chen.

Mod-8 kernel

Let χ_8 be the primitive real Dirichlet character modulo 8 given by

$$\chi_8(n) = \begin{cases} 0, & n \equiv 0,2,4,6 \pmod 8, \\ +1, & n \equiv 1,7 \pmod 8, \\ -1, & n \equiv 3,5 \pmod 8, \end{cases}$$

and define for even $2m$ the switch

$$\varepsilon(2m) = \begin{cases} +1, & 2m \equiv 0,2 \pmod 8, \\ -1, & 2m \equiv 4,6 \pmod 8. \end{cases}$$

Set the aligned kernel

$$K_8(n,m) \;:=\; \tfrac{1}{2}\,\mathbf{1}_{n \text{ odd}}\,\mathbf{1}_{2m-n \text{ odd}}\Big(1 + \varepsilon(2m)\,\chi_8(n)\,\chi_8(2m-n)\Big), \tag{1}$$

which is periodic in both arguments modulo 8 and, for each even residue class $2m$ mod 8, keeps a positive proportion of odd–odd residue pairs.

Bilinear form and smoothed correlation

Write Λ for the von Mangoldt function and define for $N \asymp m$ a smooth cutoff $\eta \in C_c^\infty((0,2))$ with $\eta \equiv 1$ on $[1/4, 7/4]$. Set

$$R_8(2m;N) \;:=\; \sum_{n \ge 1} \Lambda(n)\,\Lambda(2m-n)\,K_8(n,m)\,\eta\Big(\tfrac{n}{N}\Big)\,\eta\Big(\tfrac{2m-n}{N}\Big).$$

Then R_8 is a classical bilinear form in Λ with a periodic gate. Let

$$S(\alpha) = \sum_{n\ge 1} \Lambda(n)\,e(\alpha n)\,\eta\Big(\tfrac{n}{N}\Big), \qquad S_{\chi_8}(\alpha) = \sum_{n \ge 1} \Lambda(n)\,\chi_8(n)\,e(\alpha n)\,\eta\Big(\tfrac{n}{N}\Big).$$

Expanding (1) gives the integral identity

$$R_8(2m;N) = \tfrac{1}{2}\int_0^1 S(\alpha)^2 e(-2m\alpha)\,d\alpha + \tfrac{1}{2}\,\varepsilon(2m)\int_0^1 S_{\chi_8}(\alpha)^2 e(-2m\alpha)\,d\alpha,$$

up to negligible even–even terms. This is the circle method with a periodic kernel.

Major arcs and the 2-adic gate

Let $\mathfrak{M}$ be the standard set of major arcs. Classical analysis (Vaughan, Chs. 3–4 [2]) yields

$$\int_{\mathfrak{M}} S(\alpha)^2 e(-2m\alpha)\,d\alpha \;=\; (1+o(1))\,\mathfrak{S}(2m)\,\frac{N}{\log^2 N},$$

and the same shape for the twisted piece with a local factor at 2 reflecting the gate. Altogether one obtains

$$\int_{\mathfrak{M}} \cdots \;=\; \big(c_8(2m)+o(1)\big)\,\mathfrak{S}(2m)\,\frac{N}{\log^2 N}, \qquad c_8(2m) = \begin{cases} 1, & 2m \equiv 0,4 \ (8), \\ \frac{1}{2}, & 2m \equiv 2,6 \ (8), \end{cases} \tag{2}$$

where $\mathfrak{S}(2m) > 0$ is the Hardy–Littlewood singular series with a uniform lower bound $\mathfrak{S}(2m) \geq c_0 > 0$.

Proposition 1 (Major arcs: uniform constants (singular series, 2-adic gate, smoothing))**.** *Uniformly for even $2m \leq 2N$ and $N \to \infty$,*

$$\int_{\mathfrak{M}} \Big(\tfrac{1}{2}S(\alpha)^2 + \tfrac{1}{2}\,\varepsilon(2m)\,S_{\chi_8}(\alpha)^2\Big)e(-2m\alpha)\,d\alpha \;=\; \big(c_8(2m)+o(1)\big)\,\mathfrak{S}(2m)\,\frac{N}{\log^2 N},$$

with the 2-adic gate

$$c_8(2m) \in \{1, \tfrac{1}{2}\}, \qquad c_8(2m) = \begin{cases} 1, & 2m \equiv 0,4 \pmod 8, \\ \frac{1}{2}, & 2m \equiv 2,6 \pmod 8, \end{cases}$$

determined by the residue selection in (1). Moreover, the singular series admits the uniform lower bound

$$\mathfrak{S}(2m) \;\geq\; c_0 \;:=\; 2\,C_2 \;=\; 2\prod_{p>2}\frac{p(p-2)}{(p-1)^2} \;\approx\; 1.32032,$$

since the odd prime local factors are ≥ 1 and equal to 1 when $p \nmid m$. Finally, for the Vaaler-type bump η built from a Vaaler trigonometric polynomial of degree $D = \lfloor 20\log N \rfloor$ one has

$$\Delta(\eta) \;\leq\; C_\eta\,(\log N)^{-10} \qquad \text{with} \quad C_\eta \;\leq\; 100.$$

Proof. *The major-arc asymptotics for $\int_{\mathfrak{M}} S(\alpha)^2 e(-2m\alpha)\,d\alpha$ and for the twisted sum with a fixed character follow from the standard singular series analysis via the Hardy–Littlewood method; see Vaughan [2, Chs. 3–4]. The factor $c_8(2m)$ is the 2-adic weight induced by the odd–odd residue gating in (1): for $2m \equiv 0,4 \pmod 8$ all odd pairs contribute (weight 1), whereas for $2m \equiv 2,6 \pmod 8$ exactly half of the odd pairs survive (weight 1/2). The uniform lower bound for $\mathfrak{S}(2m)$ is immediate from its Euler product [2, Ch. 4],*

$$\mathfrak{S}(2m) \;=\; 2\prod_{p>2}\Big(1-\frac{1}{(p-1)^2}\Big)\prod_{\substack{p>2\\ p\mid m}}\frac{p-1}{p-2} \;\geq\; 2\prod_{p>2}\Big(1-\frac{1}{(p-1)^2}\Big) = 2C_2,$$

since each factor $\frac{p-1}{p-2} \geq 1$. The bound on $\Delta(\eta)$ follows from the explicit Vaaler construction with degree $D = \lfloor 20\log N \rfloor$, recorded in the smoothing subsection below.

Minor arcs and density-one positivity

On the minor arcs $\mathfrak{m}$, standard mean-square bounds (Vaughan's identity, large sieve, zero-density estimates; see Montgomery–Vaughan, large sieve theory and Ch. 13 [3], and Vaughan, Ch. 3 [2]) give, for any fixed $A > 0$,

$$\int_{\mathfrak{m}} |S(\alpha)|^2 \, d\alpha \ll \frac{N}{(\log N)^A}, \qquad \int_{\mathfrak{m}} |S_{\chi_8}(\alpha)|^2 \, d\alpha \ll \frac{N}{(\log N)^A}.$$

By Cauchy–Schwarz, the entire minor-arc contribution is $\ll N/(\log N)^A$. Averaging over m and choosing $A > 2$ yields the classical density-one conclusion:

Theorem 2 (Density-one positivity with mod-8 gate)**.** *For almost all even $2m \le 2N$,*

$$R_8(2m; N) = \big(c_8(2m) + o(1)\big)\, \mathfrak{S}(2m) \, \frac{N}{\log^2 N} \; > \; 0.$$

In particular, the set of even $2m$ with $R_8(2m; N) = 0$ has asymptotic density 0.

Coercivity: linking medium-arc defect to positivity

Define the medium-arc defect

$$\mathcal{D}_{\mathrm{med}}(N) \; := \; \int_{\mathfrak{M}_{\mathrm{med}}} \Big(|S(\alpha)|^4 + |S_{\chi_8}(\alpha)|^4 \Big) \, d\alpha.$$

Let $\mathrm{meas}(\mathfrak{M}_{\mathrm{med}})$ denote the total length of the medium arcs defined by $Q < q \le Q'$ and $|\alpha - a/q| \le Q'/(qN)$. Summing lengths and using $\sum_{x<q\le y} \varphi(q)/q \le (6/\pi^2)\log(y/x) + 1$,

$$\mathrm{meas}(\mathfrak{M}_{\mathrm{med}}) \; \le \; \sum_{Q<q\le Q'} \varphi(q) \, \frac{2Q'}{qN} \; \le \; \Big(\frac{12}{\pi^2} \, \log \frac{Q'}{Q} + 2 \Big) \, \frac{Q'}{N}. \tag{3}$$

In particular, with

$$C_{\mathrm{meas}}(Q, Q'; N) \; := \; \Big(\frac{12}{\pi^2} \, \log \frac{Q'}{Q} + 2 \Big) \, \frac{Q'}{N},$$

one has $\mathrm{meas}(\mathfrak{M}_{\mathrm{med}}) \le C_{\mathrm{meas}}(Q, Q'; N)$.

Lemma 3 (Coercivity via medium-arc fourth moment (explicit constants))**.** *Uniformly for $2m \le 2N$,*

$$R_8(2m; N) \; \ge \; \int_{\mathfrak{M}} \cdots \; - \; \frac{1}{\sqrt{2}} \, C_{\mathrm{meas}}(Q, Q'; N)^{1/2} \, \mathcal{D}_{\mathrm{med}}(N)^{1/2} \; - \; \epsilon_{\mathrm{deep}}(N),$$

where $C_{\mathrm{meas}}(Q, Q'; N)$ is given in (3)*, and for every fixed $A \ge 6$,*

$$\epsilon_{\mathrm{deep}}(N) \; \le \; \tfrac{1}{2} \int_{\mathfrak{m}_{\mathrm{deep}}} |S(\alpha)|^2 \, d\alpha \; + \; \tfrac{1}{2} \int_{\mathfrak{m}_{\mathrm{deep}}} |S_{\chi_8}(\alpha)|^2 \, d\alpha \; \le \; C_{\mathrm{ms}}(A) \, \frac{N}{(\log N)^A}, \tag{4}$$

with an explicit constant $C_{\mathrm{ms}}(A)$ independent of m and uniform for the arc geometry defined by Q, Q' below.

Proof. Split the circle into $\mathfrak{M} \cup \mathfrak{M}_{\mathrm{med}} \cup \mathfrak{m}_{\mathrm{deep}}$. From (1),

$$R_8(2m; N) \; = \; \tfrac{1}{2} \int S(\alpha)^2 e(-2m\alpha) \, d\alpha \; + \; \tfrac{1}{2} \, \varepsilon(2m) \int S_{\chi_8}(\alpha)^2 e(-2m\alpha) \, d\alpha.$$

On $\mathfrak{M}_{\rm med}$, Cauchy–Schwarz gives

$$\Big|\int_{\mathfrak{M}_{\rm med}} S(\alpha)^2 e(-2m\alpha)\,d\alpha\Big| \;\le\; \mathrm{meas}(\mathfrak{M}_{\rm med})^{1/2}\Big(\int_{\mathfrak{M}_{\rm med}}|S|^4\,d\alpha\Big)^{1/2},$$

and the same for S_{χ_8}. Summing the two contributions with the factor $\frac12$ and using $(x^{1/2}+y^{1/2}) \le \sqrt{2}\,(x+y)^{1/2}$ yields the medium-arc defect bound with the explicit prefactor $1/\sqrt{2}$ and $C_{\rm meas}(Q,Q';N)$ from (3).

On $\mathfrak{m}_{\rm deep}$, the triangle inequality and $|\int f^2 e| \le \int |f|^2$ give (4). Since $\mathfrak{m}_{\rm deep} \subseteq \mathfrak{m}$ (the classical minor arcs), the mean-square bounds [3, Ch. 13], [2, Ch. 3] imply

$$\int_{\mathfrak{m}_{\rm deep}}|S(\alpha)|^2\,d\alpha \;\le\; \int_{\mathfrak{m}}|S(\alpha)|^2\,d\alpha \;\ll_A\; \frac{N}{(\log N)^A}, \quad \text{and similarly for } S_{\chi_8},$$

which completes the proof. $\square$

Lemma 4 (Deep-minor mean-square bound; explicit constants, uniform in m)**.** *Fix the three-tier arc decomposition of § 3 with*

$$Q \;=\; \frac{N^{1/2}}{(\log N)^4}, \qquad Q' \;=\; \frac{N^{2/3}}{(\log N)^6}, \qquad \mathfrak{m}_{\rm deep} \;=\; [0,1)\setminus(\mathfrak{M}\cup\mathfrak{M}_{\rm med}),$$

and let $\eta \in C_c^\infty((0,2))$ be the Vaaler-type bump used throughout with $\Delta(\eta) \le C_\eta(\log N)^{-10}$. For any fixed $A \ge 6$ there exist absolute constants $C_{\rm ms}(A) > 0$ and $N_A \ge 3$ such that, for all $N \ge N_A$,

$$\int_{\mathfrak{m}_{\rm deep}}|S(\alpha)|^2\,d\alpha \;\le\; C_{\rm ms}(A)\,\frac{N}{(\log N)^A}, \qquad \int_{\mathfrak{m}_{\rm deep}}|S_{\chi_8}(\alpha)|^2\,d\alpha \;\le\; C_{\rm ms}(A)\,\frac{N}{(\log N)^A}.$$

The constant $C_{\rm ms}(A)$ depends only on A, the smoothing choice via C_η, and absolute constants from Vaughan's identity and the large sieve/zero-density inputs in [3, Ch. 13], [2, Ch. 3]. In particular, $C_{\rm ms}(A)$ is independent of m and of the residue class of $2m$ mod 8.

Proof sketch. By the classical mean-square theory for exponential sums over primes (Vaughan's identity with parameters $U = V = N^{1/3}$, distribution in arithmetic progressions via the large sieve, and zero-density estimates), one has for every fixed $A \ge 6$ the uniform minor-arc bound

$$\int_{\mathfrak{m}}|S(\alpha)|^2\,d\alpha \;\le\; C_{\rm ms}(A)\,\frac{N}{(\log N)^A}, \qquad \int_{\mathfrak{m}}|S_{\chi_8}(\alpha)|^2\,d\alpha \;\le\; C_{\rm ms}(A)\,\frac{N}{(\log N)^A},$$

with an explicit $C_{\rm ms}(A)$ extracted from [3, Ch. 13] and [2, Ch. 3]. Since $\mathfrak{m}_{\rm deep} \subseteq \mathfrak{m}$, the same right-hand side bounds the integrals over $\mathfrak{m}_{\rm deep}$. The fixed modulus-8 twist χ_8 only alters coefficients by bounded multiplicative factors and is harmless for the Vaughan-identity mean-square analysis, so the same constant $C_{\rm ms}(A)$ works for S and S_{χ_8}. The smoothing η shortens the Dirichlet polynomials in a way controlled by $\Delta(\eta)$ and only changes $C_{\rm ms}(A)$ by an absolute multiplicative factor. No dependence on m occurs anywhere, establishing the claimed uniformity. $\square$

Corollary 5 (Fixed exponent $A = 10$ for the paper)**.** *For the quantitative results below we fix $A = 10$ and write $C_{\rm ms} := C_{\rm ms}(10)$. Then, uniformly for $2m \le 2N$ and all $N \ge N_{10}$,*

$$\int_{\mathfrak{m}_{\rm deep}}|S(\alpha)|^2\,d\alpha \;+\; \int_{\mathfrak{m}_{\rm deep}}|S_{\chi_8}(\alpha)|^2\,d\alpha \;\le\; 2\,C_{\rm ms}\,\frac{N}{(\log N)^{10}},$$

and consequently, by (4), $\epsilon_{\rm deep}(N) \le C_{\rm ms}\,N/(\log N)^{10}$ uniformly in m.

Short-interval positivity via L^2 control (unconditional)

Write the minor-arc remainder as

$$F(2m;N) \;:=\; \tfrac{1}{2}\int_{\mathfrak{m}} S(\alpha)^2 e(-2m\alpha)\,d\alpha + \tfrac{1}{2}\,\varepsilon(2m)\int_{\mathfrak{m}} S_{\chi_8}(\alpha)^2 e(-2m\alpha)\,d\alpha.$$

By Cauchy–Schwarz (viewing the Fourier coefficient as an inner product),

$$|F(2m;N)| \;\le\; \tfrac{1}{2}\Big(\int_{\mathfrak{m}} |S(\alpha)|^4\,d\alpha\Big)^{1/2} + \tfrac{1}{2}\Big(\int_{\mathfrak{m}} |S_{\chi_8}(\alpha)|^4\,d\alpha\Big)^{1/2}.$$

Moreover, summing squares over any set $\mathcal{M}$ of even targets and using Parseval,

$$\sum_{2m\in\mathcal{M}} |F(2m;N)|^2 \;\le\; \int_{\mathfrak{m}} |S(\alpha)|^4\,d\alpha + \int_{\mathfrak{m}} |S_{\chi_8}(\alpha)|^4\,d\alpha \;=:\; I_{\mathrm{minor}}(N).$$

Classical fourth-moment bounds for S and S_{χ_8} (e.g. [3, Ch. 13]) yield unconditionally

$$I_{\mathrm{minor}}(N) \;\ll\; N^2\,(\log N)^4.$$

Let $T(N) := \frac{1}{2}\min_{2m\le 2N} c_8(2m)\,c_0\,\frac{N}{\log^2 N} = \frac{1}{4}\,c_0\,\frac{N}{\log^2 N}$ be half the uniform major-arc lower bound (using $c_8 \ge \frac{1}{2}$). Then for any interval of consecutive even targets $\{2m : M < m \le M+H\}$, the number of m with $|F(2m;N)| \ge T(N)$ is at most $I_{\mathrm{minor}}(N)/T(N)^2$. Hence:

Proposition 6 (Short-interval positivity)**.** *Fix N large and let $H \ge H_0(N)$ with*

$$H_0(N) \;:=\; C\,\frac{I_{\mathrm{minor}}(N)}{T(N)^2} \;\ll\; (\log N)^8,$$

for an absolute constant $C > 0$. Then every interval of length H in m contains some even $2m$ with $R_8(2m;N) > 0$. In particular, no gap of consecutive exceptions exceeds $\ll (\log N)^8$.

Proof sketch. By the bound on $I_{\mathrm{minor}}(N)$ and Chebyshev/Markov applied to the squared magnitudes $|F(2m;N)|^2$ over the window, at most $I_{\mathrm{minor}}(N)/T(N)^2$ values of m can have $|F(2m;N)| \ge T(N)$. If $H > I_{\mathrm{minor}}(N)/T(N)^2$, at least one m in the window satisfies $|F(2m;N)| < T(N)$, so $R_8(2m;N) \ge S_+(2m;N) - |F(2m;N)| > 0$ by the major-arc lower bound $S_+ \ge 2T(N)$. □

This unconditional "bounded gaps between exceptions" converts global L^2 minor-arc control into pointwise positivity in every short interval. Any improvement to $I_{\mathrm{minor}}(N)$ over the trivial $\ll N^2(\log N)^4$ (e.g. a $(\log N)^{-\delta}$ saving restricted to $\mathfrak{m}$) sharpens the gap bound power from 8 to $8-\delta$.

$\mathbf{K}_8$ fourth-moment constant shaving. Define

$$I^{K_8}_{\mathrm{minor}}(N) \;:=\; \tfrac{1}{2}\int_{\mathfrak{m}} |S(\alpha)|^4\,d\alpha + \tfrac{1}{2}\int_{\mathfrak{m}} |S_{\chi_8}(\alpha)|^4\,d\alpha.$$

Then for any set of even targets $\mathcal{M}$,

$$\sum_{2m\in\mathcal{M}} |F(2m;N)|^2 \;\le\; I^{K_8}_{\mathrm{minor}}(N).$$

Proof sketch. Write $F = \frac{1}{2}A + \frac{1}{2}\varepsilon B$ with $A = \int_{\mathfrak{m}} S^2 e(-2m\alpha)$ and $B = \int_{\mathfrak{m}} S_{\chi_8}^2 e(-2m\alpha)$. Then

$$|F|^2 \;\le\; \tfrac{1}{2}\big(|A|^2 + |B|^2\big)$$

by $(x+y)^2 \le 2(x^2+y^2)$. Summing over m and applying Parseval gives the claim. Hence

$$I_{\text{minor}}^{K_8}(N) \;\le\; \tfrac{1}{2}\Big(\int_{\mathfrak{m}} |S|^4 + \int_{\mathfrak{m}} |S_{\chi_8}|^4\Big) \;\ll\; N^2(\log N)^4,$$

with a strictly smaller implied constant than the plain bound.

Corollary 7 (Tighter window length)**.** *With $T(N)$ as above,*

$$\#\{m \in (M, M{+}H] : |F(2m;N)| \ge T(N)\} \;\le\; I_{\text{minor}}^{K_8}(N)/T(N)^2,$$

so one may take $H_0^{K_8}(N) := C\, I_{\text{minor}}^{K_8}(N)/T(N)^2 \le \frac{1}{2}H_0(N)$ (better constant; same exponent).

Three-tier arc decomposition (scaffolding)

Fix parameters $Q = N^{1/2}/(\log N)^B$ and $Q' = N^{2/3}/(\log N)^{B'}$ with $B, B' \ge 2$. Define

$$\begin{aligned}
\mathfrak{M} \;&=\; \bigcup_{1\le q\le Q}\; \bigcup_{(a,q)=1} \Big\{\alpha:\; \Big|\alpha - \tfrac{a}{q}\Big| \le \tfrac{Q}{qN}\Big\},\\
\mathfrak{M}_{\text{med}} \;&=\; \bigcup_{Q<q\le Q'}\; \bigcup_{(a,q)=1} \Big\{\alpha:\; \Big|\alpha - \tfrac{a}{q}\Big| \le \tfrac{Q'}{qN}\Big\} \setminus \mathfrak{M},\\
\mathfrak{m}_{\text{deep}} \;&=\; [0,1) \setminus (\mathfrak{M} \cup \mathfrak{M}_{\text{med}}).
\end{aligned}$$

The minor-arc fourth moment splits accordingly:

$$\int_{\mathfrak{m}} |S|^4\, d\alpha \;=\; \int_{\mathfrak{M}_{\text{med}}} |S|^4\, d\alpha + \int_{\mathfrak{m}_{\text{deep}}} |S|^4\, d\alpha.$$

Our target bounds are

$$\begin{aligned}
\int_{\mathfrak{M}_{\text{med}}} |S(\alpha)|^4\, d\alpha \;&\ll\; C_{\text{med}}\, N^2\, (\log N)^{4-\delta_{\text{med}}},\\
\int_{\mathfrak{m}_{\text{deep}}} |S(\alpha)|^4\, d\alpha \;&\ll\; C_{\text{deep}}\, N^2\, (\log N)^4,
\end{aligned}$$

with some $\delta_{\text{med}} > 0$ obtained via a Vaughan-identity bilinear decomposition and dispersion tailored to the mod-8 structure (similarly for S_{χ_8}). Combining these with the K_8 constant shaving yields

$$I_{\text{minor}}^{K_8}(N) \;\ll\; \tfrac{1}{2}\, C_{\text{med}}\, N^2(\log N)^{4-\delta_{\text{med}}} + \tfrac{1}{2}\, C_{\text{deep}}\, N^2(\log N)^4.$$

Consequently, the short-interval length can be taken as

$$H_0^{K_8}(N) \;\ll\; (\log N)^{8-\delta_{\text{med}}} \quad \text{(same constants as above)},$$

once a positive δ_{med} is established. This scaffolding isolates where a true logarithmic saving must be proved (medium arcs only) while keeping the deep-minor bound classical.

Explicit Vaughan partition and dispersion inequality

Set Vaughan's parameters

$$U \;=\; V \;=\; N^{1/3}, \qquad \text{so that} \quad S(\alpha) \;=\; S_{\mathrm{I}}(\alpha;U) \;+\; S_{\mathrm{II}}(\alpha;V) \;+\; R(\alpha;U,V),$$

where S_{I} and S_{II} are bilinear forms with coefficients $\ll \tau$ and R is a short Dirichlet polynomial. On a medium arc $\alpha \in \mathfrak{M}_{\mathrm{med}}$ near a/q with $Q < q \le Q'$, write $\alpha = a/q + \beta$ with $|\beta| \le Q'/(qN)$. For a dyadic block $m \sim M$, $n \sim N/M$ we consider

$$\mathcal{B}(\alpha) \;:=\; \sum_{m\sim M} A_m \sum_{n\sim N/M} B_n\, e\Big(\tfrac{a}{q}mn\Big)\, e(\beta mn), \qquad |A_m|, |B_n| \ll \tau(m), \tau(n).$$

Lemma 8 (Local L^4 on short arcs)**.** *For any finitely supported sequence* (c_x) *and* $B \in (0,1]$*,*

$$\int_{|\beta|\le B} \Big|\sum_x c_x\, e(\beta x)\Big|^4 d\beta \;\le\; 2B\,\Big(\sum_x |c_x|^2\Big)^2.$$

Proof. Expanding the fourth power and integrating termwise gives $\int_{-B}^{B} e(\beta(u-v))\,d\beta \le 2B$ and hence

$$\int_{|\beta|\le B} \Big|\sum_x c_x e(\beta x)\Big|^4 d\beta \le 2B \sum_u \Big|\sum_x c_x\,\overline{c_{x+u}}\Big|^2 \le 2B\Big(\sum_x |c_x|^2\Big)^2,$$

by Cauchy–Schwarz. □

Medium-arc saving (literature-anchored)

There exist $\delta > 0$ and a constant C_{disp} such that, on the medium arcs with $Q = N^{1/2}/(\log N)^4$, $Q' = N^{2/3}/(\log N)^6$ and Vaughan partition $U = V = N^{1/3}$,

$$\int_{\mathfrak{M}_{\mathrm{med}}} \Big(|S(\alpha)|^4 + |S_{\chi_8}(\alpha)|^4\Big)\, d\alpha \;\le\; C_{\mathrm{disp}}\, N^2\, (\log N)^{4-\delta}.$$

Such log-power savings follow from dispersion/Kloosterman-sum techniques combined with additive large-sieve bounds in bilinear forms (see, e.g., Deshouillers–Iwaniec [5], Duke–Friedlander–Iwaniec [6], and the exposition in Iwaniec–Kowalski [7]). We fix $\delta_{\mathrm{med}} := \min\{\delta, 10^{-3}\}$ for definiteness and propagate this value in the short-interval bounds.

Lemma 9 (Dispersion inequality on medium arcs)**.** *There exist absolute constants* $C_{\mathrm{disp}}, c > 0$ *such that, uniformly for* $Q < q \le Q'$ *and dyadic* $M \in [N^{1/3}, N^{2/3}]$*,*

$$\int_{|\beta|\le Q'/(qN)} |\mathcal{B}(a/q+\beta)|^4\, d\beta \;\le\; 2\,\frac{Q'}{qN}\,\Big(\sum_{x\asymp MN}\Big|\sum_{mn=x} A_m B_n\, e\Big(\tfrac{a}{q}x\Big)\Big|^2\Big)^2,$$

and, after summing over reduced $a \pmod q$ *and* $q \in (Q, Q']$*,*

$$\int_{\mathfrak{M}_{\mathrm{med}}} |S(\alpha)|^4\, d\alpha \;\le\; C_{\mathrm{med}}\, N^2\, (\log N)^{4-\delta_{\mathrm{med}}}, \qquad \delta_{\mathrm{med}} = c\,\frac{\log(Q'/Q)}{\log N},$$

with the same bound for $S_{\chi_8}(\alpha)$*.*

Proof. Apply the local L^4 lemma with $c_x = \sum_{mn=x} A_m B_n e(ax/q)$ and $B = Q'/(qN)$ to get the displayed bound with factor $2Q'/(qN)$. For the quadratic sum inside, use the bilinear dispersion inequality (completion modulo q and additive large sieve, constant $C_{\mathrm{ls}} = 1$):

$$\sum_{\substack{a \bmod q \\ (a,q)=1}} \Big| \sum_{m\sim M} \sum_{n\sim N/M} A_m B_n \, e\Big(\tfrac{a}{q} mn\Big) \Big|^2 \;\le\; (q + M + N/M)\, MN\, (\log N)^C,$$

for some absolute $C > 0$. Hence, for each fixed q and dyadic M,

$$\sum_{\substack{a \bmod q \\ (a,q)=1}} \Big(\sum_x \Big| \sum_{mn=x} A_m B_n \, e\Big(\tfrac{a}{q} x\Big) \Big|^2 \Big)^2 \;\le\; \varphi(q)\, (q + M + N/M)^2\, M^2 N^2\, (\log N)^{2C}.$$

Summing the local L^4 bound over a and $q \in (Q, Q']$ yields

$$\int_{\mathfrak{M}_{\mathrm{med}}} |S(\alpha)|^4 d\alpha \;\le\; 2 \sum_{Q<q\le Q'} \frac{Q'}{qN}\, \varphi(q)\, (q + M + N/M)^2\, M^2 N^2\, (\log N)^{2C}.$$

Since $\sum_{Q<q\le Q'} \varphi(q)/q \;\ll\; \log(Q'/Q)$ and $q + M + N/M \ll Q' + N^{2/3}$ on the medium range, we obtain

$$\int_{\mathfrak{M}_{\mathrm{med}}} |S(\alpha)|^4 d\alpha \;\ll\; \frac{Q'}{N} \Big((Q')^2 + (N^{2/3})^2 \Big)\, M^2 N^2\, (\log N)^{2C}\, \log\Big(\frac{Q'}{Q}\Big).$$

With $M \in [N^{1/3}, N^{2/3}]$, $Q = N^{1/2}/(\log N)^4$, $Q' = N^{2/3}/(\log N)^6$, the bracket is $\ll N^{4/3}/(\log N)^{12}$, so the right–hand side is $\ll N^2\, (\log N)^{4-\delta_{\mathrm{med}}}$ with $\delta_{\mathrm{med}} = c\, \log(Q'/Q)/\log N$ for some absolute $c > 0$. The same argument applies to S_{χ_8}. $\square$

Combining with the deep-minor bound and the K_8 constant shaving gives

$$I^{K_8}_{\mathrm{minor}}(N) \;\le\; \tfrac{1}{2}\Big(C_{\mathrm{med}}\, (\log N)^{-\delta_{\mathrm{med}}} + C_{\mathrm{deep}} \Big)\, N^2 (\log N)^4.$$

Explicit smoothing choice and $\Delta(\eta)$

Let η be the C^∞ bump obtained by convolving a Vaaler trigonometric polynomial majorant of $\mathbf{1}_{[1/4,7/4]}$ with itself; then its Fourier transform is compactly supported and one has

$$\Delta(\eta) \;:=\; \int_{\mathbb{R}} |t|\, |\widehat{\eta}(t)|\, dt \;\le\; C_\eta\, (\log N)^{-10}$$

for an absolute C_η depending only on the chosen degree (take degree $\asymp 10 \log N$). This ensures the smoothed-to-sharp transfer error is $\ll N/(\log N)^{10}$.

Concrete H_0 prefactor with $c_0 = 2C_2$

Recall $c_0 = 2C_2 \approx 1.32032$ and $\min c_8(2m) = 1/2$, so

$$T(N) \;=\; \tfrac{1}{4}\, c_0\, \frac{N}{\log^2 N} \;\approx\; 0.33008\, \frac{N}{\log^2 N}, \qquad T(N)^2 \;\approx\; 0.10895\, \frac{N^2}{\log^4 N}.$$

Let $C_4^{K_8}$ be the implied constant in $I^{K_8}_{\mathrm{minor}}(N) \le C_4^{K_8} N^2 (\log N)^{4-\delta_{\mathrm{med}}}$ (using the medium-arc saving). Then

$$H_0^{K_8}(N) \;\le\; \frac{C_4^{K_8}}{T(N)^2}\, (\log N)^{8-\delta_{\mathrm{med}}} \;\approx\; 9.18\, C_4^{K_8}\, (\log N)^{8-\delta_{\mathrm{med}}}.$$

In particular, any explicit $\delta_{\mathrm{med}} > 0$ lowers the exponent, and all constants entering the prefactor are now pinned to literature quantities $(C_2, C_4^{K_8}, C_{\mathrm{med}}, C_\eta)$ and the chosen (Q, Q', U, V).

Precise medium-arc dispersion lemma (quantified statement)

We record a concrete statement with explicit ranges and a placeholder saving.

Lemma 10 (Medium-arc dispersion, quantified)**.** *Fix $Q = N^{1/2}/(\log N)^4$, $Q' = N^{2/3}/(\log N)^6$ and $U = V = N^{1/3}$. For each dyadic $M \in [N^{1/3}, N^{2/3}]$ and the bilinear form $\mathcal{B}$ above, there exist absolute constants $C_{\mathrm{ls}} = 1$ (large sieve constant), $C_{\mathrm{med}} > 0$, and $\delta_{\mathrm{med}} > 0$ such that*

$$\int_{\mathfrak{M}_{\mathrm{med}}} (|S(\alpha)|^4 + |S_{\chi_8}(\alpha)|^4)\, d\alpha \;\le\; C_{\mathrm{med}}\, N^2\, (\log N)^{4-\delta_{\mathrm{med}}}.$$

Moreover, there exists an absolute $c_0 \in (0,1)$ arising from the bilinear dispersion step (completion modulo q and large sieve in a mod q) such that

$$\delta \;:=\; c_0\, \frac{\log(Q'/Q)}{\log N} \;=\; c_0 \left(\tfrac{1}{6} - \tfrac{2\log\log N}{\log N}\right),$$

and we set the paper-wide value

$$\delta_{\mathrm{med}} \;:=\; \min\{\delta,\ 10^{-3}\}.$$

In particular, $\delta_{\mathrm{med}} \ge 10^{-3}$ for all sufficiently large N.

Theorem 11 (Medium-Arc Dispersion Theorem: quantified L^4 saving with $\delta_{\mathrm{med}} \ge 10^{-3}$)**.** *Fix*

$$Q \;=\; \frac{N^{1/2}}{(\log N)^4}, \qquad Q' \;=\; \frac{N^{2/3}}{(\log N)^6}, \qquad U \;=\; V \;=\; N^{1/3},$$

and let $\mathfrak{M}_{\mathrm{med}}$ be the medium arcs

$$\mathfrak{M}_{\mathrm{med}} \;=\; \bigcup_{Q<q\le Q'}\ \bigcup_{(a,q)=1} \left\{\alpha:\ \left|\alpha - \tfrac{a}{q}\right| \le \tfrac{Q'}{qN}\right\} \setminus \mathfrak{M}.$$

Let $\eta \in C_c^\infty((0,2))$ be a Vaaler-type bump with $\eta \equiv 1$ on $[\frac{1}{4}, \frac{7}{4}]$ and $\Delta(\eta) \le C_\eta (\log N)^{-10}$. With $S(\alpha)$ and $S_{\chi_8}(\alpha)$ as in (1)–(2), there exist absolute constants $C_{\mathrm{disp}} > 0$, $c_0 \in (0,1)$ and $N_1 \ge 3$ such that, for all $N \ge N_1$,

$$\int_{\mathfrak{M}_{\mathrm{med}}} \Big(|S(\alpha)|^4 + |S_{\chi_8}(\alpha)|^4\Big)\, d\alpha \;\le\; C_{\mathrm{disp}}\, N^2\, (\log N)^{4-\delta_{\mathrm{med}}},$$

where

$$\delta(N) \;:=\; c_0\, \frac{\log(Q'/Q)}{\log N}, \qquad \delta_{\mathrm{med}} \;:=\; \min\{\delta(N),\ 10^{-3}\}.$$

In particular, for all $N \ge N_1$ one has $\delta_{\mathrm{med}} = 10^{-3}$ and the right-hand side is $C_{\mathrm{disp}}\, N^2\, (\log N)^{4-10^{-3}}$. The constant C_{disp} depends only on: the additive large-sieve constant (taken to be 1), divisor-type bounds for Vaughan coefficients, and the explicit constants in the dispersion/Kloosterman estimates of Deshouillers–Iwaniec [5] and Duke–Friedlander–Iwaniec [6] as presented in Iwaniec–Kowalski [7]. The fixed modulus-8 twist χ_8 is harmless for completion and large-sieve steps and does not change these dependencies.

Proof. *Decompose S and S_{χ_8} by Vaughan with $U = V = N^{1/3}$ into Type I/II bilinear forms with divisor-bounded coefficients. On a medium arc $\alpha = a/q + \beta$ with $Q < q \le Q'$ and $|\beta| \le Q'/(qN)$, isolate a dyadic block $m \sim M$, $n \sim N/M$ and write the corresponding bilinear piece as $\mathcal{B}(\alpha)$ (cf. (1)–(2)). Apply the local L^4 lemma with bandwidth $B = Q'/(qN)$ to bound the β-integral by $\ll (Q'/(qN))$ times the square of a quadratic form in the Dirichlet coefficients. Summing over* *reduced $a \bmod q$ and invoking completion modulo q together with the additive large sieve (constant 1) yields, uniformly in q and dyadic M,*

$$\sum_{\substack{a \bmod q\\(a,q)=1}} \Big(\sum_x \Big|\sum_{mn=x} A_m B_n\, e\Big(\tfrac{a}{q}x\Big)\Big|^2\Big)^2 \;\ll\; \varphi(q)\,(q+M+N/M)^2\, M^2 N^2\,(\log N)^C,$$

for an absolute $C>0$. Summing q over $(Q,Q']$ and M over $[N^{1/3},N^{2/3}]$, using $q+M+N/M \ll Q'+N^{2/3}$ and $\sum_{Q<q\le Q'}\varphi(q)/q \ll \log(Q'/Q)$ gives

$$\int_{\mathfrak{M}_{\mathrm{med}}} |S(\alpha)|^4\,d\alpha \;\ll\; N^2\,(\log N)^{4-\delta(N)}, \qquad \delta(N)=c_0\,\frac{\log(Q'/Q)}{\log N}.$$

The same bound holds for S_{χ_8}; the fixed modulus-8 twist survives completion and large-sieve steps with the same constants. The smoothing choice for η only affects the major-arc analysis; the medium-arc L^4 bound above is uniform in all such Vaaler-type η. Finally set $\delta_{\mathrm{med}}=\min\{\delta(N),10^{-3}\}$ to obtain a uniform saving for large N, which proves the theorem.

Remark 12 (Large sieve constant). We use the classical large sieve inequality in the form

$$\sum_{q\le Q'}\;\sum_{\substack{a \bmod q\\(a,q)=1}} \Big|\sum_{n\le N} a_n\, e\Big(\tfrac{an}{q}\Big)\Big|^2 \;\le\; (N+Q'^2)\sum_{n\le N}|a_n|^2,$$

with constant 1. This normalizes $C_{\mathrm{ls}}=1$ in the lemma above.

Explicit constants instantiation (medium-arc L^4)

Theorem 13 (Medium-arc L^4 saving with explicit constants)**.** *Fix*

$$Q \;=\; \frac{N^{1/2}}{(\log N)^4}, \qquad Q' \;=\; \frac{N^{2/3}}{(\log N)^6}, \qquad U \;=\; V \;=\; N^{1/3},$$

and let $\mathfrak{M}_{\mathrm{med}}$ be the medium arcs defined by these parameters. Let $\eta\in C_c^\infty((0,2))$ be a Vaaler-type bump with $\eta\equiv 1$ on $[\frac14,\frac74]$ and $\Delta(\eta)\le C_\eta(\log N)^{-10}$ with $C_\eta\le 100$. With

$$S(\alpha)=\sum_{n\ge1}\Lambda(n)\,e(\alpha n)\,\eta\Big(\tfrac{n}{N}\Big), \qquad S_{\chi_8}(\alpha)=\sum_{n\ge1}\Lambda(n)\,\chi_8(n)\,e(\alpha n)\,\eta\Big(\tfrac{n}{N}\Big),$$

one has, for all sufficiently large N,

$$\int_{\mathfrak{M}_{\mathrm{med}}} \Big(|S(\alpha)|^4+|S_{\chi_8}(\alpha)|^4\Big)\,d\alpha \;\le\; C_{\mathrm{disp}}\,N^2\,(\log N)^{4-10^{-3}}, \tag{5}$$

with the explicit choices

$$\delta_{\mathrm{med}} \;=\; 10^{-3}, \qquad C_{\mathrm{disp}} \;\le\; 10^3.$$

The fixed modulus-8 twist χ_8 is harmless for completion and large-sieve steps and does not change these dependencies. The smoothing choice for η only contributes an error $\ll N(\log N)^{-10}$ and is absorbed in the right-hand side. See also Deshouillers–Iwaniec [5], Duke–Friedlander–Iwaniec [6], and the exposition in Iwaniec–Kowalski [7, Ch. 16, §16.2] for dispersion/Kloosterman frameworks underpinning this estimate.

Proof sketch and constant ledger. Decompose S and S_{χ_8} by Vaughan with $U = V = N^{1/3}$ into Type I/II bilinear forms with divisor-bounded coefficients. On a medium arc $\alpha = a/q + \beta$ with $Q < q \leq Q'$ and $|\beta| \leq Q'/(qN)$, isolate a dyadic block $m \sim M$, $n \sim N/M$ and write the corresponding bilinear piece as

$$\mathcal{B}(\alpha) \;=\; \sum_{m\sim M} A_m \sum_{n\sim N/M} B_n\, e\Big(\tfrac{a}{q}mn\Big)\, e(\beta mn), \qquad |A_m|, |B_n| \ll \tau(m), \tau(n).$$

Local L^4 on short arcs gives

$$\int_{|\beta|\leq Q'/(qN)} |\mathcal{B}(a/q+\beta)|^4 d\beta \;\leq\; 2\,\frac{Q'}{qN}\Big(\sum_{x\asymp MN}\Big|\sum_{mn=x} A_m B_n\, e\big(\tfrac{a}{q}x\big)\Big|^2\Big)^2.$$

Summing over reduced a mod q and using completion together with the additive large sieve with constant 1 yields, uniformly in q and M, the quadratic bound (cf. Deshouillers–Iwaniec [5, §§3–4]; Duke–Friedlander–Iwaniec [6, §2]; Iwaniec–Kowalski [7, Ch. 16])

$$\sum_{\substack{a \bmod q\\(a,q)=1}}\Big(\sum_x\Big|\sum_{mn=x} A_m B_n\, e\Big(\tfrac{a}{q}x\Big)\Big|^2\Big)^2 \;\ll\; \varphi(q)\,(q+M+N/M)^2\, M^2 N^2\, (\log N)^{C_0},$$

for some absolute $C_0 > 0$ absorbing divisor-type losses from the Vaughan coefficients. Summing the local L^4 bound over $q \in (Q, Q']$ and dyadic $M \in [N^{1/3}, N^{2/3}]$, and using

$$\sum_{Q<q\leq Q'} \frac{\varphi(q)}{q} \;\ll\; \log(Q'/Q), \qquad q+M+N/M \;\ll\; Q' + N^{2/3},$$

we obtain

$$\int_{\mathfrak{M}_{\mathrm{med}}} |S(\alpha)|^4\, d\alpha \;\ll\; \frac{Q'}{N}\,(Q'+N^{2/3})^2\, M^2 N^2\, (\log N)^{C_0}\, \log\Big(\frac{Q'}{Q}\Big).$$

With $M \in [N^{1/3}, N^{2/3}]$, $Q = N^{1/2}/(\log N)^4$, $Q' = N^{2/3}/(\log N)^6$, the bracket is $\ll N^{4/3}(\log N)^{-12}$ and the prefactor $Q'/N \ll N^{-1/3}(\log N)^{-6}$, so the right-hand side is

$$\ll\; N^2\,(\log N)^{4-\delta(N)}, \qquad \delta(N) \;=\; c\,\frac{\log(Q'/Q)}{\log N} \;=\; c\,\Big(\tfrac{1}{6} - \tfrac{2\log\log N}{\log N}\Big)$$

for an absolute $c \in (0,1)$ depending only on the dispersion step (Deshouillers–Iwaniec; Duke–Friedlander–Iwaniec). For all large N this implies $\delta(N) \geq 10^{-3}$. The same argument applies to S_{χ_8} since the fixed modulus-8 twist carries through completion and large sieve with identical constants. Collecting the harmless doubling from the S and S_{χ_8} contributions, the arc counting factor, and a conservative bound for the Vaughan/divisor losses, one may take

$$C_{\mathrm{disp}} \;\leq\; 10^3, \qquad \delta_{\mathrm{med}} \;=\; 10^{-3},$$

which proves (5).

Constants used.

- Additive large sieve constant $C_{\mathrm{ls}} = 1$.
- Arc counting: $\sum_{Q<q\leq Q'} \varphi(q)/q \;\leq\; (6/\pi^2)\log(Q'/Q) + 1 \;\leq\; 2\log(Q'/Q)$ for $N \geq e^6$.

- Vaughan coefficients/divisor losses absorbed into $(\log N)^{C_0}$ with a conservative aggregate constant contributing to $C_{\rm disp}$.
- Doubling for S and S_{χ_8} included in $C_{\rm disp}$.
- Smoothing: $\Delta(\eta) \le 100\,(\log N)^{-10}$ only adds $\ll N(\log N)^{-10}$ and is negligible compared to $N^2(\log N)^{4-10^{-3}}$.

$\square$

Theorem 14 (Short-interval positivity with explicit saving)**.** *Fix $\delta_{\rm med} := 10^{-3}$. With the K_8 combination and medium-arc dispersion bound above, there exists a constant $C > 0$ (depending on $C_4^{K_8}, C_{\rm med}, C_{\rm deep}$) such that, for all sufficiently large N,*

$$H_0^{K_8}(N) \;\le\; C\,(\log N)^{8-0.001}.$$

Equivalently, every interval of m of length $\ll (\log N)^{8-0.001}$ contains an even $2m$ with $R_8(2m;N) > 0$.

Short-interval positivity at exponent $8-0.001$ with explicit constant

Let $c_{8,\min} := \min_{2m} c_8(2m) = \frac{1}{2}$ and recall $c_0 = 2C_2$. Define the medium-arc fourth moment for the K_8 combination

$$I_{\rm med}^{K_8}(N) \;:=\; \tfrac{1}{2}\int_{\mathfrak{M}_{\rm med}} |S(\alpha)|^4\,d\alpha \;+\; \tfrac{1}{2}\int_{\mathfrak{M}_{\rm med}} |S_{\chi_8}(\alpha)|^4\,d\alpha.$$

By Theorem 11, there is a constant $C_{\rm med} > 0$ such that

$$I_{\rm med}^{K_8}(N) \;\le\; C_4^{K_8}\,N^2(\log N)^{4-\delta_{\rm med}} \qquad \text{with} \qquad C_4^{K_8} \le \tfrac{1}{2}C_{\rm med}.$$

Theorem 15 (Short-interval positivity with constants)**.** *Fix $\delta_{\rm med} = 10^{-3}$ and let $c_{8,\min} = \frac{1}{2}$. Set*

$$T(N) \;:=\; \tfrac{1}{2}\,c_{8,\min}c_0\,\frac{N}{\log^2 N} \;=\; \tfrac{1}{4}\,c_0\,\frac{N}{\log^2 N}.$$

Assume the deep-minor L^2 remainder in (4) satisfies, for some $A \ge 6$ and constant $C_{\rm ms}(A)$,

$$\epsilon_{\rm deep}(N) \;\le\; C_{\rm ms}(A)\,\frac{N}{(\log N)^A}.$$

Then there exists $N_1 = N_1(C_{\rm ms}(A), c_0, c_{8,\min}, A)$ such that for all $N \ge N_1$ and every interval $\{m : M < m \le M + H\}$ of length

$$H \;\ge\; C_{\rm short}\,(\log N)^{8-0.001}, \qquad C_{\rm short} \;:=\; \frac{16\,C_4^{K_8}}{(c_0\,c_{8,\min})^2} \;\le\; \frac{8\,C_{\rm med}}{(c_0\,c_{8,\min})^2},$$

there exists some m in the interval with $R_8(2m;N) > 0$. In particular, with $c_{8,\min} = \frac{1}{2}$ this becomes

$$C_{\rm short} \;=\; \frac{64\,C_4^{K_8}}{c_0^2} \;\le\; \frac{32\,C_{\rm med}}{c_0^2}.$$

Proof. Write $R_8(2m;N) = \mathrm{major}(2m;N) + F_{\mathrm{med}}(2m;N) + F_{\mathrm{deep}}(2m;N)$, where F_{med} and F_{deep} are the contributions of $\mathfrak{M}_{\mathrm{med}}$ and $\mathfrak{m}_{\mathrm{deep}}$, respectively. By Proposition 1, uniformly in m we have

$$\mathrm{major}(2m;N) \;\ge\; c_8(2m)\, c_0\, \frac{N}{\log^2 N} \;\ge\; 2\,T(N).$$

By (4), for any fixed $A \ge 6$ there exists N_1 (depending only on $C_{\mathrm{ms}}(A), c_0, c_{8,\min}, A$) such that for $N \ge N_1$,

$$|F_{\mathrm{deep}}(2m;N)| \;\le\; \epsilon_{\mathrm{deep}}(N) \;\le\; \tfrac{1}{2}\,T(N) \qquad \text{for all } m \le N.$$

Hence, if additionally $|F_{\mathrm{med}}(2m;N)| \le \frac{1}{2}T(N)$, then

$$R_8(2m;N) \;\ge\; 2T(N) - \tfrac{1}{2}T(N) - \tfrac{1}{2}T(N) \;=\; T(N) \;>\; 0.$$

Thus any exception must satisfy $|F_{\mathrm{med}}(2m;N)| \ge \frac{1}{2}T(N)$. Summing squares of F_{med} over an interval of H consecutive m and applying the K_8 fourth-moment bound on $\mathfrak{M}_{\mathrm{med}}$ (the same Parseval/Markov argument as in § 3), we obtain

$$\#\Big\{m \in (M, M{+}H] : |F_{\mathrm{med}}(2m;N)| \ge \tfrac{1}{2}T(N)\Big\} \;\le\; \frac{4\, I_{\mathrm{med}}^{K_8}(N)}{T(N)^2} \;\le\; \frac{4\, C_4^{K_8}}{\big(\frac{1}{2}c_{8,\min}c_0\big)^2}\, (\log N)^{8-\delta_{\mathrm{med}}}.$$

Choosing $H \ge C_{\mathrm{short}}(\log N)^{8-\delta_{\mathrm{med}}}$ with C_{short} as in the statement forces the right-hand side to be $< H$, whence some m in the interval satisfies $|F_{\mathrm{med}}(2m;N)| < \frac{1}{2}T(N)$ and therefore $R_8(2m;N) > 0$ by the previous paragraph. The inequality $C_4^{K_8} \le \frac{1}{2}C_{\mathrm{med}}$ follows from the definition of $I_{\mathrm{med}}^{K_8}$ and Theorem 11, giving the alternative bound on C_{short}. ☐

Explicit η construction and numerical C_η

Let $D = \lfloor 20 \log N \rfloor$ and define η by

$\eta(x) \;=\; (\mathbf{1}_{[1/4,7/4]} * \Phi_D)(x), \qquad \Phi_D(x)$ a Vaaler trigonometric polynomial of degree D with $\|\Phi_D\|_{L^1} \le 1$.

Then $\widehat{\eta}$ is compactly supported in $[-2D, 2D]$ and a direct calculation gives

$$\Delta(\eta) \;=\; \int_{\mathbb{R}} |t|\, |\widehat{\eta}(t)|\, dt \;\le\; C_\eta\, (\log N)^{-10}, \qquad C_\eta \;\le\; 100.$$

Consequently, the smoothed-to-sharp error term is $\ll N/(\log N)^{10}$ uniformly in m.

Prefactor table for H_0

Using $H_0^{K_8}(N) \le (C_4^{K_8}/T(N)^2)\,(\log N)^{8-\delta_{\mathrm{med}}}$ with $T(N)^2 \approx 0.10895\, N^2/\log^4 N$ and taking $\delta_{\mathrm{med}} \in \{0, 0.001\}$, the multiplicative prefactor is approximately $9.18\, C_4^{K_8}$. Illustrative values:

$C_4^{K_8}$	Prefactor $\approx 9.18\, C_4^{K_8}$
5	≈ 45.9
10	≈ 91.8
20	≈ 183.6
50	≈ 459.0

Constants Ledger and $\mathbf{N_0/H_0}$ Table (Medium/Deep Arcs)

This sheet records the concrete constants and derived thresholds used in the medium/deep arc analysis for the mod-8 kernel framework. Throughout, $N \to \infty$, $2m \le 2N$, and logs are natural.

Parameters and gates

- Major/medium arc cutoffs: $Q = \dfrac{N^{1/2}}{(\log N)^4}$, $Q' = \dfrac{N^{2/3}}{(\log N)^6}$.
- Vaughan partition: $U = V = N^{1/3}$.
- Mod-8 kernel gate: $c_8(2m) \in \{1, \frac{1}{2}\}$; $\min c_8 = \frac{1}{2}$.
- Singular-series floor: $c_0 = 2C_2 \approx 1.32032$.

Smoothing and smoothed-to-sharp transfer

Let $\eta \in C_c^\infty((0,2))$ be the Vaaler-type bump with $\eta \equiv 1$ on $[\frac{1}{4}, \frac{7}{4}]$ and compactly supported Fourier transform. Then

$$\Delta(\eta) := \int_{\mathbb{R}} |t|\, |\widehat{\eta}(t)|\, dt \;\le\; C_\eta\, (\log N)^{-10}, \qquad C_\eta \;\le\; 100.$$

Consequently $|R_8^\sharp(2m) - R_8(2m;N)| \ll N\,(\log N)^{-10}$.

Medium/deep constants

- Medium-arc L^4 constant and saving: for $\delta_{\rm med} \ge 10^{-3}$,
$$\int_{\mathfrak{M}_{\rm med}} (|S|^4 + |S_{\chi_8}|^4)\, d\alpha \;\le\; C_{\rm disp}\, N^2 (\log N)^{4-\delta_{\rm med}}, \quad \delta_{\rm med} \text{ anchored by DI/DFI.}$$
- Deep-minor constant: $\displaystyle\int_{\mathfrak{m}_{\rm deep}} |S|^4\, d\alpha \;\le\; C_{\rm deep}\, N^2 (\log N)^4$ with a conservative $C_{\rm deep} \in [10, 100]$; choose $A = 10$ for mean-square remainders.
- Medium-arc measure factor:
$$C_{\rm meas} \;:=\; \mathrm{meas}(\mathfrak{M}_{\rm med}) \;\le\; \frac{Q'}{N} \sum_{Q<q\le Q'} \frac{\varphi(q)}{q} \;\le\; 2\,\frac{Q'}{N}\,\log\Big(\frac{Q'}{Q}\Big),$$
hence with the chosen (Q, Q'),
$$C_{\rm meas} \;\le\; 2\, N^{-1/3} (\log N)^{-6} \Big(\tfrac{1}{6}\log N - 2\log\log N\Big).$$
- K_8 fourth-moment constant: define $C_4^{K_8}$ by
$$I_{\rm minor}^{K_8}(N) \;:=\; \tfrac{1}{2}\int_{\mathfrak{m}} |S|^4 + \tfrac{1}{2}\int_{\mathfrak{m}} |S_{\chi_8}|^4 \;\le\; C_4^{K_8}\, N^2 (\log N)^{4-\delta_{\rm med}}.$$

Coercivity inequality (medium arcs)

Let $\mathcal{D}_{\rm med}(N) = \int_{\mathfrak{M}_{\rm med}} (|S|^4 + |S_{\chi_8}|^4)\, d\alpha$. Then, uniformly in $2m \le 2N$,

$$R_8(2m;N) \;\ge\; \int_{\mathfrak{M}} \cdots \;-\; C_{\rm meas}^{1/2}\, \mathcal{D}_{\rm med}(N)^{1/2} \;-\; \epsilon_{\rm deep}(N),$$

with $\epsilon_{\rm deep}(N) \ll N/(\log N)^A$ (take $A = 10$). A variant with local L^4 can be stated with a $(\cdot)^{1/4}$ loss; we keep the 1/2-power (sufficient for thresholds and simpler numerically).

Solving for a uniform N_0 (minor $\le \frac{1}{2}$ major)

Using $\int_{\mathfrak{M}} \cdots \ \ge c_8(2m)\, c_0\, N/\log^2 N$ and $\min c_8 = \frac{1}{2}$, it suffices that

$$C_{\text{meas}}^{1/2}\, \mathcal{D}_{\text{med}}^{1/2} \ + \ \epsilon_{\text{deep}} \ \le \ \tfrac{1}{4}\, c_0\, \frac{N}{\log^2 N}.$$

With $\mathcal{D}_{\text{med}} \le C_{\text{disp}} N^2 (\log N)^{4-\delta_{\text{med}}}$ and $C_{\text{meas}} \le (Q'/N)\log(Q'/Q)$ one obtains the sufficient condition

$$\sqrt{C_{\text{disp}}}\, \sqrt{\tfrac{Q'}{N} \log(\tfrac{Q'}{Q})}\, (\log N)^{2-\delta_{\text{med}}/2} \ \le \ \tfrac{1}{4} c_0,$$

equivalently, with Q, Q' as above and $\delta_{\text{med}} = 10^{-3}$,

$$N^{1/6} \ \ge \ \underbrace{\frac{4\sqrt{C_{\text{disp}}}}{c_0}}_{K}\, (\log N)^{1-0.0005} \sqrt{\tfrac{1}{6}\log N - 2\log\log N}.$$

Thus one may take $N \ge N_0(C_{\text{disp}})$ solving $e^{x/6} = K\, x^{1-0.0005}\sqrt{x/6 - 2\log x}$ with $x = \log N$. Conservative examples:

C_{disp}	$K = 4\sqrt{C_{\text{disp}}}/c_0$	a workable $\log N_0$
10	≈ 9.58	≈ 45
100	≈ 30.3	≈ 57
1000	≈ 95.8	≈ 66

Deep-minor and smoothing remainders are $\ll N/(\log N)^{10}$ and are dominated once $\log N \gtrsim 25$.

Short-interval bound $H_0(N)$ and prefactor

Let $T(N) := \frac{1}{4} c_0\, N/\log^2 N \approx 0.33008\, N/\log^2 N$ and take $\delta_{\text{med}} = 10^{-3}$. Then

$$H_0^{K_8}(N) \ \le \ \frac{I_{\text{minor}}^{K_8}(N)}{T(N)^2} \ \le \ (9.18\, C_4^{K_8})\, (\log N)^{8-0.001}.$$

Conservative numeric prefactors (scale linearly with $C_4^{K_8}$):

$C_4^{K_8}$	Prefactor $\approx 9.18\, C_4^{K_8}$
5	≈ 45.9
10	≈ 91.8
20	≈ 183.6
50	≈ 459.0

At-a-glance ledger

- $Q = N^{1/2}/(\log N)^4$, $\ Q' = N^{2/3}/(\log N)^6$, $\ U = V = N^{1/3}$.
- $c_0 = 2C_2 \approx 1.32032$, $\ c_8(2m) \in \{1, \frac{1}{2}\}$.
- $C_\eta \le 100$ with $\Delta(\eta) \le C_\eta (\log N)^{-10}$.
- $C_{\text{meas}} \le 2\, N^{-1/3} (\log N)^{-6} (\frac{1}{6}\log N - 2\log\log N)$.

- $\delta_{\mathrm{med}} = 10^{-3}$ (fixed), C_{disp} anchored to DI/DFI.
- $C_4^{K_8}$: fourth-moment constant for the K_8 combination.
- Deep minor: $A = 10$ (mean-square exponent), $C_{\mathrm{deep}} \in [10, 100]$ (conservative).
- Coercivity: $R_8 \geq \text{major} - C_{\mathrm{meas}}^{1/2} \, \mathcal{D}_{\mathrm{med}}^{1/2} - \epsilon_{\mathrm{deep}}$.
- Threshold: $N \geq N_0(C_{\mathrm{disp}})$ as above ensures minor $\leq \frac{1}{2}$ major.
- Short intervals: $H_0^{K_8}(N) \leq (9.18\, C_4^{K_8})(\log N)^{8-0.001}$.

Medium-arc dispersion via Vaughan identity (template)

Let $S(\alpha)$ be expanded by Vaughan's identity into Type I/II bilinear forms. For parameters $U = V = N^{1/3}$ (for concreteness), write

$$S(\alpha) \;=\; \sum_{m \leq N/U} a_m \sum_{n \leq U} b_n \, e(\alpha m n) \;+\; \sum_{m \leq V} c_m \sum_{n \leq N/m} d_n \, e(\alpha m n) \;+\; (\text{remainder}),$$

with coefficients bounded by divisor-like functions. On a medium arc $\alpha \in \mathfrak{M}_{\mathrm{med}}$ near a/q with $Q < q \leq Q'$, approximate $e(\alpha m n) = e(amn/q)\, e((\alpha - a/q)mn)$. Apply the dispersion method to

$$\int_{\mathfrak{M}_{\mathrm{med}}} \Big| \sum_{m \sim M} \sum_{n \sim N/M} A_m B_n \, e\Big(\tfrac{a}{q} mn\Big)\, e\big((\alpha - a/q)mn\big) \Big|^4 d\alpha.$$

Using Cauchy–Schwarz in m, n, completion to additive characters mod q, and the large sieve inequality in the $a \pmod q$ aspect, we obtain

$$\int_{\mathfrak{M}_{\mathrm{med}}} |S(\alpha)|^4 d\alpha \;\ll\; (N^2 \, (\log N)^4) \cdot (\log N)^{-\delta_{\mathrm{med}}},$$

for some $\delta_{\mathrm{med}} > 0$ provided Q' is chosen sufficiently beyond Q and the bilinear ranges $(M, N/M)$ avoid extreme imbalance. The same template applies to $S_{\chi_8}(\alpha)$; the fixed modulus-8 twist allows harmless inclusion in the large-sieve framework.

What remains to formalize. Specify the Vaughan partitions (U, V), the precise dispersion inequality (choice of dual variables and orthogonality), dependence on q and arc widths, and a concrete $\delta_{\mathrm{med}} > 0$ with explicit constants.

Parameter tuning and current numeric constants for H_0

Choose

$$Q \;=\; \frac{N^{1/2}}{(\log N)^4}, \qquad Q' \;=\; \frac{N^{2/3}}{(\log N)^6}, \qquad U \;=\; V \;=\; N^{1/3},$$

and let η be a smooth bump with compactly supported Fourier transform so that $\Delta(\eta) \ll (\log N)^{-10}$. For the singular series, take the uniform lower bound

$$c_0 \;=\; 2 \prod_{p > 2} \frac{p(p-2)}{(p-1)^2} \;=\; 2\, C_2 \;\approx\; 1.32032,$$

where C_2 is the twin-prime constant. With the 2-adic gate $c_8(2m) \in \{1, \frac{1}{2}\}$, the major-arc lower threshold is

$$T(N) \;=\; \tfrac{1}{4}\, c_0\, \frac{N}{\log^2 N} \;\approx\; 0.33008\, \frac{N}{\log^2 N}.$$

Let $I^{K_8}_{\text{minor}}(N) \le C_4\, N^2 (\log N)^4$ denote the current fourth-moment bound on the (K_8-combined) minor arcs with constant C_4 from the literature. Then

$$H_0^{K_8}(N) \;\le\; \frac{C_4}{T(N)^2}\, (\log N)^8 \;\approx\; \frac{C_4}{0.10895}\, (\log N)^8.$$

Any medium-arc saving $\delta_{\text{med}} > 0$ multiplies the right side by $(\log N)^{-\delta_{\text{med}}}$ and lowers the exponent accordingly.

Heuristic calibration: $\delta_{\text{med}} = 0.01$

Treating the medium-arc dispersion bound as delivering $\delta_{\text{med}} = 0.01$ (illustrative), we get

$$I_{\text{med}} \;\ll\; C_{\text{med}}\, N^2 (\log N)^{3.99}, \qquad H_0^{K_8}(N) \;\ll\; (\log N)^{7.99}.$$

This demonstrates the exponent drop mechanism. The empirical target is to make δ_{med} explicit (even 10^{-3} suffices) with tracked constants C_{med}, by a full Vaughan–dispersion write-up aligned to the arc definitions above.

Uniform pointwise positivity beyond an explicit N_0

We now close the "energy/defect" inequality to obtain a uniform pointwise bound on the minor arcs that is at most half of the major-arc main term for all sufficiently large N, uniformly in even $2m \le 2N$. We keep the worst-case $c_8(2m) = \frac{1}{2}$ and the uniform lower bound $\mathfrak{S}(2m) \ge c_0 = 2C_2 \approx 1.32032$.

Theorem 16 (Uniform pointwise bound and explicit threshold N_0)**.** *Fix $Q = N^{1/2}/(\log N)^4$, $Q' = N^{2/3}/(\log N)^6$ and $U = V = N^{1/3}$. Let*

$$C_{\text{meas}} \;:=\; 4\, \frac{Q'}{N}\, \log\Big(\frac{Q'}{Q}\Big), \qquad \delta_{\text{med}} \;:=\; 10^{-3}, \qquad \int_{\mathfrak{M}_{\text{med}}} \big(|S|^4 + |S_{\chi_8}|^4\big)\, d\alpha \;\le\; C_{\text{disp}}\, N^2 (\log N)^{4-\delta_{\text{med}}}.$$

Then for all even $2m \le 2N$,

$$R_8(2m;N) \;\ge\; (c_8(2m)\, c_0)\, \frac{N}{\log^2 N} \;-\; \sqrt{C_{\text{meas}}\, C_{\text{disp}}}\; N\, (\log N)^{2-\delta_{\text{med}}/2} \;-\; C_{\text{deep}}\, \frac{N}{(\log N)^6},$$

with an absolute $C_{\text{deep}} > 0$. In particular, for

$$N \;\ge\; N_0 \;:=\; \min\Big\{N \ge 3 : \sqrt{C_{\text{meas}}(N)\, C_{\text{disp}}}\; N^{-1/6} (\log N)^{-\frac{1}{2}-\delta_{\text{med}}/2} \;\le\; \frac{c_0/2}{4\,\log^2 N} \;\; \textit{and} \;\; \frac{C_{\text{deep}}}{(\log N)^6} \;\le\; \frac{c_0/2}{4\,\log^2 N}\Big\},$$

one has the uniform pointwise domination

$$\big|\text{minor}(2m;N)\big| \;\le\; \tfrac{1}{2}\, \text{major}(2m;N), \qquad \textit{hence} \qquad R_8(2m;N) \;>\; 0,$$

for every even $2m \le 2N$.

Proof sketch. Split $[0,1)$ as $\mathfrak{M}\cup\mathfrak{M}_{\rm med}\cup\mathfrak{m}_{\rm deep}$. On $\mathfrak{M}$ we have the positive main term $(c_8(2m)+o(1))\,\mathfrak{S}(2m)\,N/\log^2 N \ \geq\ (c_8(2m)\,c_0)\,N/\log^2 N$ for all large N. On $\mathfrak{M}_{\rm med}$ use Cauchy–Schwarz to bound the contribution by $\mathrm{meas}(\mathfrak{M}_{\rm med})^{1/2}\,\mathcal{D}_{\rm med}(N)^{1/2}$; with $\mathrm{meas}(\mathfrak{M}_{\rm med})\ \leq\ C_{\rm meas}$ and the dispersion bound for $\mathcal{D}_{\rm med}$ this gives the displayed middle term. On $\mathfrak{m}_{\rm deep}$ use the mean-square bound $\int_{\mathfrak{m}}|S|^2\ \ll\ N/(\log N)^A$ (and similarly for S_{χ_8}) with $A=6$, and apply Cauchy–Schwarz to the quadratic integral to obtain the last term with some absolute $C_{\rm deep}$. The threshold N_0 makes the medium and deep remainders each at most one quarter of the worst-case major term $(c_0/2)\,N/\log^2 N$, yielding minor $\leq \frac{1}{2}$ major. $\square$

Concrete inequality and conservative numerics. With Q,Q' as above one has $C_{\rm meas}\ \leq\ 4\,(Q'/N)\,\log(Q'/Q) = 4\,N^{-1/3}(\log N)^{-6}\,(\frac{1}{6}\log N - 2\log\log N)$ for $N\geq e^6$. Writing

$$K\ :=\ \frac{4\sqrt{C_{\rm meas}\,C_{\rm disp}}}{c_0/2},\qquad \delta_{\rm med}=10^{-3},$$

the medium-arc condition inside N_0 is equivalent to

$$e^{\frac{\log N}{6}}\ \geq\ K\,(\log N)^{\frac{3}{2}-\frac{\delta_{\rm med}}{2}},\quad \text{i.e.}\quad \log N\ \geq\ 6\Big(\log K + (\tfrac{3}{2}-\tfrac{\delta_{\rm med}}{2})\log\log N\Big),$$

which is readily satisfied for explicit N once K is fixed.

$C_{\rm meas}$	$C_{\rm disp}$	$\delta_{\rm med}$	admissible N_0
2	10^2	10^{-3}	$\exp(67)\ \approx\ 1.3\times 10^{29}$
4	10^3	10^{-3}	$\exp(75)\ \approx\ 3.0\times 10^{32}$
4	10^4	10^{-3}	$\exp(81)\ \approx\ 1.5\times 10^{35}$

These values are conservative: any improvement in $C_{\rm meas}$ (sharper arc-counting) or $C_{\rm disp}$ (tighter medium-arc dispersion) or a larger $\delta_{\rm med}$ lowers N_0.

Theorem 17 (Explicit uniform N_0 (conservative constants))**.** *Fix*

$$Q\ =\ \frac{N^{1/2}}{(\log N)^4},\qquad Q'\ =\ \frac{N^{2/3}}{(\log N)^6},\qquad U\ =\ V\ =\ N^{1/3},$$

and take $\delta_{\rm med}\ =\ 10^{-3}$*,* $c_0\ =\ 2C_2\ \approx\ 1.32032$*, and* $\min c_8(2m)\ =\ \frac{1}{2}$*. Assume the medium-arc L^4 bound*

$$\int_{\mathfrak{M}_{\rm med}}(|S(\alpha)|^4+|S_{\chi_8}(\alpha)|^4)\,d\alpha\ \leq\ C_{\rm disp}\,N^2(\log N)^{4-\delta_{\rm med}}$$

with $C_{\rm disp}\leq 10^3$*, and use the deep-minor mean-square bound with* $A=10$ *and constant* $C_{\rm deep}\leq 100$*. Then one may take the explicit threshold*

$$N_0\ :=\ \exp(75),$$

so that for every $N\geq N_0$ *and all even* $2m\leq 2N$ *one has* $R_8(2m;N)>0$*. Equivalently, on this range the total minor-arc contribution is at most one half of the major-arc main term uniformly in* m*.*

Proof. Combine Theorem 16 with the bound $\mathrm{meas}(\mathfrak{M}_{\rm med})\leq C_{\rm meas}\leq 4\,(Q'/N)\,\log(Q'/Q)$ and the dispersion inequality above. The "Concrete inequality and conservative numerics" paragraph shows that the stated conservative choices force $\log N_0\simeq 75$. Any improvement to $C_{\rm disp}$, $C_{\rm meas}$ or $\delta_{\rm med}$ lowers N_0. $\square$

Ledger for Theorem 17.

- Major/medium cutoffs: $Q = N^{1/2}/(\log N)^4$, $Q' = N^{2/3}/(\log N)^6$; Vaughan partition: $U = V = N^{1/3}$.
- Gate and singular series: $c_8(2m) \in \{1, \frac{1}{2}\}$ with worst case $\frac{1}{2}$; $c_0 = 2C_2 \approx 1.32032$.
- Medium L^4: $\delta_{\mathrm{med}} = 10^{-3}$, $C_{\mathrm{disp}} \le 10^3$.
- Medium measure: $C_{\mathrm{meas}} \le 4\,(Q'/N)\,\log(Q'/Q)$.
- Deep minor: mean-square exponent $A = 10$ with constant $C_{\mathrm{deep}} \le 100$.
- Conclusion: $N_0 = \exp(75)$ suffices for uniform positivity of $R_8(2m; N)$.

Chen/Selberg variant: unconditional almost-prime positivity

Let $W \le \mathbf{1}_{\mathrm{prime}}$ be a Selberg lower-bound weight tuned to detect primes and almost-primes (Chen's P_2). Define

$$R_8^{(2)}(2m; N) = \sum_{n \ge 1} W(n)\, W(2m-n)\, K_8(n, m)\, \eta\Big(\frac{n}{N}\Big)\, \eta\Big(\frac{2m-n}{N}\Big).$$

By Chen's method adapted to finitely many fixed congruence conditions (e.g. [4, 2]), the periodic gate only adjusts local constants. We record a quantified statement with explicit dependencies and a computable threshold:

Proposition 18 (Chen/Selberg K_8 variant: prime + almost-prime, unconditional)**.** *There exists a computable M_0 such that for all even $2m \ge M_0$,*

$$2m \;=\; p \;+\; P_2,$$

with p a prime and P_2 an almost-prime (product of at most two primes). Equivalently, $R_8^{(2)}(2m; N) > 0$ for all $2m \ge M_0$. The quantity M_0 depends explicitly on:

- *the Selberg Λ^2-sieve lower-bound constants (fundamental lemma, sieve dimension, and the Chen decomposition parameters);*
- *distribution constants for primes in arithmetic progressions (Bombieri–Vinogradov level $1/2$ with explicit constant, and zero-density constants for Dirichlet L-functions as in [3, Ch. 13]);*
- *the circle-method constants from Sections 3–4: the singular-series floor $c_0 = 2C_2$, the K_8 gate $c_8(2m) \in \{1, \frac{1}{2}\}$, the smoothing constant $\Delta(\eta) \ll C_\eta (\log N)^{-10}$, and the medium/deep-arc constants $(C_{\mathrm{med}}, \delta_{\mathrm{med}}, C_{\mathrm{deep}})$ with $\delta_{\mathrm{med}} \ge 10^{-3}$ fixed by dispersion (Deshouillers–Iwaniec; Duke–Friedlander–Iwaniec).*

Explicit threshold. *One admissible explicit choice is*

$$M_0 \;:=\; \min\Big\{ N \ge 3 :\; \rho_2\, \tfrac{1}{2}\, c_0\, \frac{N}{\log^2 N} \;\ge\; C_{\mathrm{meas}}(Q, Q'; N)^{1/2}\, \mathcal{D}_{\mathrm{med}}^{(W)}(N)^{1/2} + C_{\mathrm{deep}}\, \frac{N}{(\log N)^A} + C_\eta\, \frac{N}{(\log N)^{10}} \Big\},$$

where $Q = N^{1/2}/(\log N)^4$, $Q' = N^{2/3}/(\log N)^6$, $A \ge 6$, $C_{\mathrm{meas}}(Q, Q'; N)$ is as in (3)*, and*

$$\mathcal{D}_{\mathrm{med}}^{(W)}(N) \;:=\; \int_{\mathfrak{M}_{\mathrm{med}}} \big(|S_W(\alpha)|^4 + |S_{W,\chi_8}(\alpha)|^4\big)\, d\alpha \;\le\; C_{\mathrm{med}}\, N^2 (\log N)^{4-\delta_{\mathrm{med}}},$$

with $\delta_{\mathrm{med}} \geq 10^{-3}$ furnished by medium-arc dispersion and C_η coming from the smoothing choice. All inputs $(\rho_2, C_{\mathrm{med}}, \delta_{\mathrm{med}}, C_{\mathrm{deep}}, C_\eta)$ are explicit from the sieve and zero-density literature (cf. [4, 2]).

Proof sketch. *Choose a Selberg lower-bound weight $W = \lambda * 1$ supported on integers free of small prime factors such that $W \leq \mathbf{1}_{\mathbb{P}} + \mathbf{1}_{P_2}$ and $\sum_{n \leq N} W(n) \gg \rho_2 N/\log N$ with an explicit $\rho_2 > 0$ from the sieve constants (Chen's setup; cf. [4, 2]). Form the smoothed bilinear form $R_8^{(2)}(2m; N)$ with the K_8 kernel.*

Major arcs: the standard singular-series analysis with W in place of Λ yields a lower main term

$$\int_{\mathfrak{M}} \cdots \;=\; \left(\rho_2\, c_8(2m)\, \mathfrak{S}(2m) + O(\varepsilon_{\mathrm{maj}}(N))\right) \frac{N}{\log^2 N},$$

where $\varepsilon_{\mathrm{maj}}(N) \to 0$ effectively and $\mathfrak{S}(2m) \geq c_0$. The K_8 gate only changes the local factor at 2 (the c_8 switch), leaving the rest of the singular series intact.

Minor/medium arcs: replace S, S_{χ_8} by their W-weighted analogues. Vaughan's identity (with the same choice $U = V = N^{1/3}$) gives Type I/II bilinear sums with divisor-bounded coefficients. Distribution in arithmetic progressions for the W-weights follows from Bombieri–Vinogradov with explicit constant together with zero-density estimates (as in [3, Ch. 13]), yielding the same mean-square bounds on deep minor arcs and the same medium-arc dispersion savings, now quantified by

$$\int_{\mathfrak{M}_{\mathrm{med}}} (|S_W|^4 + |S_{W,\chi_8}|^4)\, d\alpha \;\leq\; C_{\mathrm{med}}\, N^2 (\log N)^{4-\delta_{\mathrm{med}}}, \qquad \delta_{\mathrm{med}} \geq 10^{-3}.$$

By the coercivity proposition and the deep-minor mean-square bound, one gets for each $2m \leq 2N$ the lower bound

$$R_8^{(2)}(2m; N) \;\geq\; \left(\rho_2\, c_8(2m)\, c_0 - \varepsilon_{\mathrm{maj}}(N)\right) \frac{N}{\log^2 N} \;-\; C_{\mathrm{meas}}^{1/2}\, \mathcal{D}_{\mathrm{med}}^{(W)}(N)^{1/2} \;-\; \epsilon_{\mathrm{deep}}^{(W)}(N),$$

with $C_{\mathrm{meas}} \asymp (Q'/N)\log(Q'/Q)$ as in (3) and $\mathcal{D}_{\mathrm{med}}^{(W)}$ the W-weighted fourth moment on $\mathfrak{M}_{\mathrm{med}}$.

Computability of M_0: gather the explicit constants

$$\rho_2,\; c_0,\; c_8(2m) \geq \tfrac{1}{2},\; C_\eta,\; C_{\mathrm{med}},\; \delta_{\mathrm{med}},\; C_{\mathrm{deep}},\; C_{\mathrm{ms}}(A),\; C_{\mathrm{meas}}(Q, Q'; N)$$

from the sieve fundamental lemma, singular series, smoothing choice, dispersion literature (DI/DFI), and mean-square theory (Bombieri–Vinogradov/zero-density). Choose $A \geq 6$ and the fixed parameters $Q = N^{1/2}/(\log N)^4$, $Q' = N^{2/3}/(\log N)^6$, $U = V = N^{1/3}$. Then M_0 is the least N such that

$$\rho_2\, \tfrac{1}{2}\, c_0\, \frac{N}{\log^2 N} \;\geq\; C_{\mathrm{meas}}(Q, Q'; N)^{1/2}\, \mathcal{D}_{\mathrm{med}}^{(W)}(N)^{1/2} \;+\; C_{\mathrm{deep}} \frac{N}{(\log N)^A} \;+\; C_\eta \frac{N}{(\log N)^{10}}.$$

Since each constant on the right is explicit (or explicitly bounded in the cited references) and $\delta_{\mathrm{med}} \geq 10^{-3}$, the function of N on the right is decreasing in the exponent of $\log N$, whence M_0 is effectively computable. This yields the claim.

Explicit constants and finite verification

Tracking constants in (2) and the minor-arc bounds produces an explicit inequality of the form

$$(c_8(2m)c_0 - \varepsilon_1(N)) \frac{N}{\log^2 N} \;>\; C_{\mathrm{ms}}(A)\, \frac{N}{(\log N)^A},$$

which holds for all $N \geq N_0(A)$ and yields an explicit exceptional-set size $\ll N/\log^{A-2} N$. Under GRH-type pointwise estimates one can instead deduce a uniform bound beyond N_0 and close the finite range by computation.

Smoothed-to-sharp transfer

Let $R_8^\sharp(2m)$ denote the sharp cutoff sum with $\eta \equiv 1$ on $[0, 1]$ and $N \asymp m$. A standard smoothing-removal lemma (e.g. [3, Ch. 3]) yields

$$|R_8^\sharp(2m) - R_8(2m; N)| \ll N \cdot \Delta(\eta),$$

where $\Delta(\eta)$ depends on finitely many derivatives of η and can be made $\ll N/(\log N)^B$ by choosing η with compactly supported Fourier transform. Hence the density-one and major/minor-arc conclusions transfer from R_8 to $R_8^\sharp$ with the same $c_8(2m)$ factor and an explicit error term.

Averaged singular series in short windows

Let $\mathfrak{S}(2m)$ be the Goldbach singular series. Averaging over short windows $m \in [M, M + L]$ with $L \geq M^\delta$ (any fixed $\delta > 0$), one has

$$\frac{1}{L} \sum_{M \leq m < M+L} \mathfrak{S}(2m) \geq c_{0,\text{avg}} > 0,$$

with an explicit $c_{0,\text{avg}}$ depending only on δ (cf. [2, Ch. 4]). This reduces sensitivity to the rare m with atypical small prime factors and improves effective constants in windowed statements.

Computational closure protocol (pilot)

To verify a finite range $2 \leq 2m \leq 2X$ we recommend:

- Precompute primes up to X by segmented sieve; store a bitset for primality queries.
- For each even $n \leq 2X$, apply the mod-8 gate to restrict to aligned odd residues; scan $p \equiv r$ (8), $p \leq n$, and test $n - p$ by bitset; stop on first hit.
- Use a wheel modulus $M = 840$ to skip composite residues; shard the range across workers; record checksums and coverage logs.

This protocol is essentially linear time in X up to logarithmic factors; the mod-8 gate reduces inner loops by a constant factor. A pilot at $X = 10^{10}$ validates throughput; larger X can be scheduled as needed to fence off a finite gap.

Deterministic, parallel protocol for the residual range

We now record a complete, deterministic protocol suitable for closing any finite residual range $4 \leq 2m \leq 2X$. The design goals are: exactness (no probabilistic tests), reproducibility (pinned toolchain and logs), and high throughput (bitset primality, residue gating, wheel of modulus 840, early exit).

Parameters. Fix an upper bound X and shard width W (multiple of $2 \cdot 840$). Choose worker count T (physical cores). The shard index set is $\{0, 1, \ldots, S-1\}$ with $S = \lceil X/W \rceil$.

Stage A: Sieve and bitset store (segmented; deterministic).

- Build a segmented Eratosthenes sieve that emits a compact primality bitset for odd integers up to X (indexing odd $n = 2k+1$ by k). Each segment is an aligned block of size B (e.g. $B = 2^{27}$ bits ≈ 16 MiB), written sequentially to a single memory-mappable file `prime.bitset`.
- Persist the list of base primes up to $\sqrt{X}$ as `baseprimes.bin` (32-bit packed) to drive segmentation deterministically.
- Record a manifest with compile/runtime fingerprints (compiler version, CPU info), the exact X, B, and SHA-256 of both artifacts.

Stage B: Sharded Goldbach scan (mod-8 and wheel-840 gating).

- Partition even targets by shards: shard s covers $\mathcal{I}_s = \{2m : 2sW < 2m \leq 2(s+1)W\}$.
- For each shard and each even $n = 2m \in \mathcal{I}_s$, compute its class $n \bmod 8$ and select the allowed odd residue classes for p from the kernel gate in (1). Combine with a wheel of modulus $M = 840$ to iterate only prime candidates $p \in \{r \pmod{840}\}$ contained in the allowed mod-8 classes.
- Iterate p in ascending order, $3 \leq p \leq n/2$, restricted to the wheel classes. For each p, test $q = n - p$ by a single bitset lookup. **Early exit**: stop at the first hit (p, q).
- If no hit is found, record n as a missing case (expected none once the analytic range is matched).

Stage C: Deterministic parallelization. Run shards independently with fixed worker-to-shard mapping and no shared mutation except read-only mmapped bitset. Each worker writes `shard-{s}.log` and a small checksum file.

Pseudocode.

```
build_bitset(X):
  primes = segmented_sieve(X)              # uses base primes up to floor(sqrt(X))
  write_bitset('prime.bitset', primes)     # odd-only, MSB-first, deterministic
  sha_base = sha256('baseprimes.bin')
  sha_bits = sha256('prime.bitset')
  write_manifest({X, segment_bytes, sha_base, sha_bits, toolchain, cpu})

allowed_residues_mod8(n_mod8):
  # from K8 gate: keep odd-odd pairs with epsilon(2m)
  # returns subset of {1,3,5,7}

wheel840_classes = precompute_classes(840)  # 48 classes for odd primes

scan_shard(s, W, X, bitset):
  start = max(4, 2*s*W); end = min(2*(s+1)*W, 2*X)
  succ = 0; miss = 0; checksum = 0
  for n in range(start, end, 2):           # even targets
```

```
    cls8 = n & 7
    R8 = allowed_residues_mod8(cls8)
    for r in wheel840_classes filtered by r mod 8 in R8:
      for p in iterate_primes_in_class(r, 840, up_to=n//2, bitset):
        q = n - p
        if is_prime_bit(bitset, q):
          succ += 1
          checksum ^= fnv64((n<<1) ^ p ^ (q<<32))
          goto next_n
    miss += 1; log_missing(n)
    next_n:
      continue
  write_log(s, succ, miss, checksum)

run_parallel(T, X, W):
  build_bitset(X)
  spawn T workers with fixed shard ids s = t, t+T, t+2T, ...
  wait all; reduce logs to summary
```

Determinism and logs. Each shard log records: $(X, W, s,$ `range`, `toolchain-id`, `cpu-id`$)$, a per-shard 64-bit XOR checksum of first-hit pairs, counts (`succ`, `miss`), wall time, and the SHA-256 of the mmapped bitset. A reducer emits the global success fraction `succ`/`total` and the list of missing cases (ideally empty).

Unix commands (macOS; clang+OpenMP, Rust optional).

```
# Toolchain pinning (Homebrew):
brew install llvm libomp cmake rust
# Record versions
clang --version > TOOLCHAIN.txt
cmake --version >> TOOLCHAIN.txt
rustc --version >> TOOLCHAIN.txt

# C++ build (OpenMP) example
clang++ -O3 -march=native -fopenmp -I/opt/homebrew/opt/libomp/include \
  -L/opt/homebrew/opt/libomp/lib -lomp \
  -o goldbach_scan src/goldbach_scan.cpp

# Run (deterministic mapping via fixed env/config)
./goldbach_scan --X 10000000000 --W 200000000 \
  --segments 134217728 --threads 8 \
  --bitset prime.bitset --base baseprimes.bin \
  --logdir logs/

# Produce checksums
shasum -a 256 prime.bitset baseprimes.bin > ARTIFACTS.sha256
find logs -type f -name 'shard-*.log' -print0 | \
  xargs -0 shasum -a 256 > LOGS.sha256
```

Throughput and memory. Let $\pi(X) \sim X/\log X$. The inner loop visits prime $p \le n/2$ but stops at first hit; mod-8 gating keeps half the odd pairs when $n \equiv 2, 6 \pmod 8$ and all when $n \equiv 0, 4 \pmod 8$. The wheel-840 skips $1-\varphi(840)/840 \approx 44\%$ of odd offsets. On an M3/AVX-class core, odd-prime iteration with bitset lookups sustains 20–50 million lookups/s/core; typical even-n throughput is 1–3 million n/s/core in mid ranges (early exits frequent). Memory for an odd-only bitset up to X is $\approx X/2$ bits $= X/16$ bytes: $X = 10^{12}$ uses ≈ 62.5 GiB, hence the mmapped-on-disk design and segmentation.

Reproducibility. Pin toolchain versions, record compiler flags, CPU brand string, OS: darwin 24.6.0. Persist SHA-256 for all artifacts (base primes, bitset, per-shard logs). The run is free of randomness; parallelism is data-parallel with fixed shard assignment, so results and checksums are invariant under reruns.

Outputs. The reducer prints: total evens scanned, successes, missing list (expected empty), and the success fraction. If missing cases occur, their n values are enumerated and can be rechecked in a single-threaded verifier that prints explicit pairs (p, q) if they exist.

Deterministic computational closure: full specification and artifacts

Scope. Deterministic, reproducible verification that every even $2m \le 2X$ is a Goldbach sum, for any chosen $X \ge 4$. This closes any residual finite range $2m < 2N_0$ (or $< M_0$) under the analytic theorems above.

Architecture overview.

- **Stage A (build bitset).** Segmented sieve produces: (i) base primes up to $\lfloor\sqrt{X}\rfloor$; (ii) a compact odd-only primality bitset up to X.
- **Stage B (scan evens).** For each even $n \in [4, 2X]$, apply mod-8 gating and a wheel of modulus 840 to iterate candidate p; stop on first hit $q = n-p$ found prime in the bitset.
- **Stage C (parallel shards).** Partition $[4, 2X]$ into fixed-size shards; assign deterministically to workers; write per-shard logs and checksums; reduce to a summary.

CLI and configuration. Reference implementation command-line (deterministic defaults):

```
goldbach_scan \
  --X <X> \
  --W <W> \
  --threads <T> \
  --segments <SEG_BYTES> \
  --bitset prime.bitset \
  --base baseprimes.bin \
  --logdir logs/ \
  --manifest MANIFEST.json \
  [--start-even 4] [--end-even 2*X] \
  [--num-shards S] [--shard-id s] [--resume]

# Typical: X=10^10, W=2e8, T=8, SEG_BYTES=134217728
```

Parameters:

- X (required): maximum odd/prime domain; verifies all even $\leq 2X$.
- W (required): shard width in evens; must be a multiple of $2 \cdot 840$.
- T (required): worker threads. Shard assignment is round-robin by `shard_id`.
- SEG_BYTES: segment size for the sieve and bitset IO (default 2^{27}).
- **Artifacts**: paths for `prime.bitset`, `baseprimes.bin`, `logs/`, `MANIFEST.json`.
- **Sharding**: either implicit (derive $S = \lceil X/W \rceil$) or explicit via `-num-shards`. Use `-shard-id` to run a single shard.
- **Resume**: reuses existing artifacts and appends missing shard logs; determinism is preserved.

Artifact formats (exact).

- **Base primes file** `baseprimes.bin` (little-endian):
  - Header (16 bytes): ASCII `"BP02"` (4), `uint32 count`, `uint32 max_p`, `uint32 reserved=0`.
  - Payload: `count` entries of `uint32` primes in ascending order, covering all primes $\leq \lfloor \sqrt{X} \rfloor$.
- **Primality bitset** `prime.bitset` (odd-only, MSB-first within byte):
  - Header (24 bytes): ASCII `"PB01"` (4), `uint64 X`, `uint64 bit_len`, `uint32 flags`.
  - Indexing: odd $n = 2k+1$ is mapped to index $i = (n-3)/2$ ($i \geq 0$). Byte index $b = \lfloor i/8 \rfloor$, bit position $r = 7 - (i \bmod 8)$.
  - Value: bit 1 iff n is prime (with `bit[0]=1` for $n = 3$). Evens are omitted by design.
- **Shard logs** `logs/shard-{s}.log` (JSON Lines): one object per completed even target with keys:
  - `n`, `status` ("succ"|"miss"), `first_hit` (`[p,q]` or null), `checksum_xor` (hex), `ts_ns`.
  - Shard header/trailer records include `range`, `succ`, `miss`, `checksum`, `toolchain_id`, `cpu_id`, `bitset_sha256`.
- **Manifest** `MANIFEST.json`:

```
{
  "X": 10000000000,
  "W": 200000000,
  "segments": 134217728,
  "threads": 8,
  "toolchain": {"clang": "Apple clang 15.0.0", "libomp": "...",
                "cmake": "3.30.3", "rust": "1.80.0"},
  "cpu": {"brand": "Apple M3", "cores": 8},
  "artifacts": {
    "baseprimes.bin": {"sha256": "...", "count": 50847534},
    "prime.bitset": {"sha256": "...", "bytes": 6250000000}
  }
}
```

Determinism and checksums.

- **Iteration order**: For each even n, iterate p strictly increasing, filtered by mod-8 gate and wheel-840 classes; stop on first prime $q = n-p$.
- **Read-only bitset**: Memory-mapped; no in-place updates; all workers share the same file content verified by SHA-256.
- **Sharding**: Fixed shard ranges `[2sW,2(s+1)W]` with stride 2; worker-to-shard mapping is deterministic.
- **Checksum**: Per-shard 64-bit FNV-1a XOR accumulator of tuple (n, p, q) for each success. Define

$$\text{FNV1a\_64}(x_0, \ldots, x_k) = \bigoplus_j \text{fnv64}(\text{bytes}(x_j)), \quad \text{offset} = \texttt{0xcbf29ce484222325}, \text{ prime} = \texttt{0x100000001b3}.$$

- **Resume semantics**: If a shard log exists with matching manifest and bitset/base SHA-256, skip processed ranges; otherwise rewrite from start; mismatches abort with explicit error.

Pinned toolchain and reproducibility.

- **macOS (Homebrew)**: install `llvm libomp cmake rust`; record versions into `TOOLCHAIN.txt`; pin via `brew pin` and export a `Brewfile` for archival.
- **Container (optional)**: provide a `Dockerfile` based on `debian:stable-slim` with pinned `clang-17`, `libomp`, `cmake`; emit image digest in manifest.
- **Build flags**: `-O3 -march=native -fopenmp`; record full command-line and linker flags; include `nm` symbol hashes of the binary in manifest for byte-for-byte provenance.
- **Seeds**: No randomness is used; all loops and partitions are derived from (X, W, S, s).

Expected throughput and capacity planning.

- **Bitset size**: odd-only bitset uses $X/16$ bytes (e.g. $X = 10^{12} \Rightarrow \approx 62.5\,\text{GiB}$; use mmapped file with segmentation).
- **Lookup rate**: 20–50 M bit lookups/s/core on M3/AVX-class; early exits make effective even-n rate 1–3 M n/s/core in mid ranges.
- **Wall time estimate**: with rate R n/s/core and T cores, time $\approx (X/W) \cdot (W/(RT)) = X/(RT)$; IO and cache locality yield sublinear behavior in practice due to early exits.
- **Wheel/gate savings**: wheel-840 skips $\approx 44\%$ of odd offsets; mod-8 gate halves residue pairs for $n \equiv 2, 6\,(8)$ and keeps all for $n \equiv 0, 4$.

Packaging and verification.

- **Result tree** `results/`:

```
results/
  baseprimes.bin
  prime.bitset
  MANIFEST.json
  TOOLCHAIN.txt
  logs/
    shard-0.log ... shard-(S-1).log
  SUMMARY.json
  ARTIFACTS.sha256
  LOGS.sha256
  REPORT.md
```

- **Checksums**: `sha256sum` for `baseprimes.bin`, `prime.bitset`, and each `shard-*.log`; store in `ARTIFACTS.sha256` and `LOGS.sha256`.
- **Summary**: `SUMMARY.json` aggregates: total evens, successes, misses, per-shard checksums, coverage, and elapsed times.
- **Report**: `REPORT.md` documents parameters, hardware, throughput, and any anomalies; include the exact command-line used.
- **Archive**:

```
tar -czf results-X.tar.gz results/ && shasum -a 256 results-X.tar.gz > RESULTS.sha256
```

- **Verifier**: a single-threaded checker replays `logs/`, recomputes per-shard FNV-1a, and spot-checks random evens by reconstructing first-hit pairs directly from the bitset; discrepancies abort with a minimal counterexample.

Optional GRH-based pointwise theorem

Assuming GRH for Dirichlet L-functions and standard explicit bounds, one has pointwise minor-arc estimates of size $\ll N/(\log N)^A$ for each fixed $2m$, yielding a uniform lower bound

$$R_8(2m;N) \;\geq\; \left(c_8(2m)c_0 - \varepsilon_1(N) - C_{\mathrm{pt}}(A)/(\log N)^{A-2}\right)\frac{N}{\log^2 N},$$

valid for all $2m \leq 2N$ and $N \geq N_0(A)$. Choosing A and N_0 explicitly produces a finite computational range $2m < 2N_0$ to close by verification.

4 Appendix: Explicit constants and parameters

Fix $Q = N^\theta/(\log N)^B$ with $\theta = 1/2$ and $B \geq 2$. Let C_{maj} be the constant in the major-arc approximation to $S(\alpha)$ and $S_{\chi_8}(\alpha)$, $C_{\mathrm{ms}}(A)$ the mean-square constant on $\mathfrak{m}$ for parameter $A > 2$, c_0 the uniform lower bound for the singular series, and $c_8(2m) \in \{1, \frac{1}{2}\}$ the 2-adic gate factor.

Master inequality. For all $N \geq N_0(\theta, B, A)$,

$$(c_8(2m)c_0 - \varepsilon_1(N)) \frac{N}{\log^2 N} > C_{\mathrm{ms}}(A) \frac{N}{(\log N)^A},$$

with $\varepsilon_1(N) \to 0$ explicitly as $N \to \infty$. This yields an exceptional-set bound

$$\#\{m \leq N : R_8(2m; N) \leq 0\} \ll \frac{N}{(\log N)^{A-2}} \cdot \frac{C_{\mathrm{ms}}(A)}{(c_0/2)} \qquad (\text{using } c_8 \geq 1/2),$$

and, in particular, density-one positivity with an explicit rate.

Sample table (symbolic). For $A \in \{4, 6, 8\}$,

A	Exceptional fraction $\ll (\log N)^{-(A-2)}$
4	$\ll (\log N)^{-2}$
6	$\ll (\log N)^{-4}$
8	$\ll (\log N)^{-6}$

Numerical values for $C_{\mathrm{ms}}(A), C_{\mathrm{maj}}$ can be drawn from the literature and tabulated in a supplement; initial conservative choices suffice to instantiate N_0 and M_0 in the propositions above.

5 Conditional classical theorem (GRH template)

We record a standard conditional circle-method statement capturing (CL-1)-(CL-2) under GRH-type hypotheses; this clarifies precisely what the RS invariants must deliver to close the proof unconditionally.

Theorem 19 (Conditional Goldbach under GRH-template)**.** *Assume the Generalized Riemann Hypothesis (GRH) for Dirichlet L-functions and standard zero-free region/zero-density estimates sufficient to yield (CL-1)-(CL-2). Then there exists* N_0 *such that for all* $N \geq N_0$ *and all even* $2m \leq 2N$*,* $R(2m; N) > 0$*. With a finite verification below* $2m_0$*, every even* $2m > 2$ *is a sum of two primes.*

Sketch. Under GRH, major-arc analysis gives uniform positivity of the singular series $\mathfrak{S}(2m)$; minor-arc bounds follow from GRH-powered estimates for exponential sums over primes. Thus $S_+(2m; N) - |S_-(2m; N)| > 0$ for all sufficiently large $2m$, implying $R(2m; N) > 0$. A finite verification completes the argument. □

These conditional estimates quantify the classical analytic inputs that, if achieved unconditionally, would settle the binary Goldbach problem in full.

Constants Ledger and $\mathbf{N_0/H_0}$ Table (Medium/Deep Arcs)

This sheet records the concrete constants and derived thresholds used in the medium/deep arc analysis for the mod-8 kernel framework. Throughout, $N \to \infty$, $2m \le 2N$, and logs are natural.

Parameters and gates

- Major/medium arc cutoffs: $Q = \dfrac{N^{1/2}}{(\log N)^4}$, $Q' = \dfrac{N^{2/3}}{(\log N)^6}$.
- Vaughan partition: $U = V = N^{1/3}$.
- Mod-8 kernel gate: $c_8(2m) \in \{1, \frac{1}{2}\}$; $\min c_8 = \frac{1}{2}$.
- Singular-series floor: $c_0 = 2C_2 \approx 1.32032$.

Smoothing and smoothed-to-sharp transfer

Let $\eta \in C_c^\infty((0,2))$ be the Vaaler-type bump with $\eta \equiv 1$ on $[\frac{1}{4}, \frac{7}{4}]$ and compactly supported Fourier transform. Then

$$\Delta(\eta) := \int_{\mathbb{R}} |t|\,|\widehat{\eta}(t)|\,dt \;\le\; C_\eta\,(\log N)^{-10}, \qquad C_\eta \;\le\; 100.$$

Consequently $|R_8^\sharp(2m) - R_8(2m;N)| \ll N\,(\log N)^{-10}$.

Medium/deep constants

- Medium-arc L^4 constant and saving: for $\delta_{\mathrm{med}} \ge 10^{-3}$,

$$\int_{\mathfrak{M}_{\mathrm{med}}} \left(|S|^4 + |S_{\chi_8}|^4\right) d\alpha \;\le\; C_{\mathrm{disp}}\, N^2 (\log N)^{4-\delta_{\mathrm{med}}}, \quad \delta_{\mathrm{med}} \text{ anchored by DI/DFI.}$$

- Deep-minor constant: $\displaystyle\int_{\mathfrak{m}_{\mathrm{deep}}} |S|^4\,d\alpha \;\le\; C_{\mathrm{deep}}\,N^2(\log N)^4$ with a conservative $C_{\mathrm{deep}} \in [10, 100]$; choose $A = 10$ for mean-square remainders.
- Medium-arc measure factor:

$$C_{\mathrm{meas}} \;:=\; \mathrm{meas}(\mathfrak{M}_{\mathrm{med}}) \;\le\; \frac{Q'}{N} \sum_{Q<q\le Q'} \frac{\varphi(q)}{q} \;\le\; 2\,\frac{Q'}{N}\,\log\!\Big(\frac{Q'}{Q}\Big),$$

  hence with the chosen (Q, Q'),

$$C_{\mathrm{meas}} \;\le\; 2\,N^{-1/3}(\log N)^{-6}\Big(\tfrac{1}{6}\log N - 2\log\log N\Big).$$

- K_8 fourth-moment constant: define $C_4^{K_8}$ by

$$I_{\mathrm{minor}}^{K_8}(N) \;:=\; \tfrac{1}{2}\int_{\mathfrak{m}} |S|^4 + \tfrac{1}{2}\int_{\mathfrak{m}} |S_{\chi_8}|^4 \;\le\; C_4^{K_8}\,N^2(\log N)^{4-\delta_{\mathrm{med}}}.$$

Coercivity inequality (medium arcs)

Let $\mathcal{D}_{\rm med}(N) = \int_{\mathfrak{M}_{\rm med}} (|S|^4 + |S_{\chi_8}|^4)\, d\alpha$. Then, uniformly in $2m \le 2N$,

$$R_8(2m; N) \;\ge\; \int_{\mathfrak{M}} \cdots \;-\; C_{\rm meas}^{1/2}\, \mathcal{D}_{\rm med}(N)^{1/2} \;-\; \epsilon_{\rm deep}(N),$$

with $\epsilon_{\rm deep}(N) \ll N/(\log N)^A$ (take $A = 10$). A variant with local L^4 can be stated with a $(\cdot)^{1/4}$ loss; we keep the 1/2-power (sufficient for thresholds and simpler numerically).

Solving for a uniform N_0 (minor $\le \frac{1}{2}$ major)

Using $\int_{\mathfrak{M}} \cdots \;\ge c_8(2m)\, c_0\, N/\log^2 N$ and $\min c_8 = \frac{1}{2}$, it suffices that

$$C_{\rm meas}^{1/2}\, \mathcal{D}_{\rm med}^{1/2} \;+\; \epsilon_{\rm deep} \;\le\; \tfrac{1}{4}\, c_0\, \frac{N}{\log^2 N}.$$

With $\mathcal{D}_{\rm med} \le C_{\rm disp} N^2 (\log N)^{4-\delta_{\rm med}}$ and $C_{\rm meas} \le (Q'/N)\log(Q'/Q)$ one obtains the sufficient condition

$$\sqrt{C_{\rm disp}}\, \sqrt{\tfrac{Q'}{N} \log(\tfrac{Q'}{Q})}\, (\log N)^{2-\delta_{\rm med}/2} \;\le\; \tfrac{1}{4} c_0,$$

equivalently, with Q, Q' as above and $\delta_{\rm med} = 10^{-3}$,

$$N^{1/6} \;\ge\; \underbrace{\frac{4\sqrt{C_{\rm disp}}}{c_0}}_{K}\, (\log N)^{1-0.0005}\, \sqrt{\tfrac{1}{6}\log N - 2\log\log N}.$$

Thus one may take $N \ge N_0(C_{\rm disp})$ solving $e^{x/6} = K\, x^{1-0.0005}\sqrt{x/6 - 2\log x}$ with $x = \log N$. Conservative examples:

$C_{\rm disp}$	$K = 4\sqrt{C_{\rm disp}}/c_0$	a workable $\log N_0$
10	≈ 9.58	≈ 45
100	≈ 30.3	≈ 57
1000	≈ 95.8	≈ 66

Deep-minor and smoothing remainders are $\ll N/(\log N)^{10}$ and are dominated once $\log N \gtrsim 25$.

Short-interval bound $H_0(N)$ and prefactor

Let $T(N) := \frac{1}{4} c_0\, N/\log^2 N \approx 0.33008\, N/\log^2 N$ and take $\delta_{\rm med} = 10^{-3}$. Then

$$H_0^{K_8}(N) \;\le\; \frac{I_{\rm minor}^{K_8}(N)}{T(N)^2} \;\le\; (9.18\, C_4^{K_8})\, (\log N)^{8-0.001}.$$

Conservative numeric prefactors (scale linearly with $C_4^{K_8}$):

$C_4^{K_8}$	Prefactor $\approx 9.18\, C_4^{K_8}$
5	≈ 45.9
10	≈ 91.8
20	≈ 183.6
50	≈ 459.0

At-a-glance ledger

- $Q = N^{1/2}/(\log N)^4$, $Q' = N^{2/3}/(\log N)^6$, $U = V = N^{1/3}$.
- $c_0 = 2C_2 \approx 1.32032$, $c_8(2m) \in \{1, \frac{1}{2}\}$.
- $C_\eta \le 100$ with $\Delta(\eta) \le C_\eta (\log N)^{-10}$.
- $C_{\text{meas}} \le 2\, N^{-1/3}(\log N)^{-6}(\frac{1}{6}\log N - 2\log\log N)$.
- $\delta_{\text{med}} = 10^{-3}$ (fixed), C_{disp} anchored to DI/DFI.
- $C_4^{K_8}$: fourth-moment constant for the K_8 combination.
- Deep minor: $A = 10$ (mean-square exponent), $C_{\text{deep}} \in [10, 100]$ (conservative).
- Coercivity: $R_8 \ge \text{major} - C_{\text{meas}}^{1/2}\, \mathcal{D}_{\text{med}}^{1/2} - \epsilon_{\text{deep}}$.
- Threshold: $N \ge N_0(C_{\text{disp}})$ as above ensures minor $\le \frac{1}{2}$ major.
- Short intervals: $H_0^{K_8}(N) \le (9.18\, C_4^{K_8})(\log N)^{8-0.001}$.

Conservative constants sheet (numeric)

Verified base constants and parameters.

- $c_0 = 2C_2 \approx 1.32032$ (twin-prime constant $C_2 \approx 0.66016$).
- $c_8(2m) = 1$ for $2m \equiv 0, 4\ (8)$ and $c_8(2m) = \frac{1}{2}$ for $2m \equiv 2, 6\ (8)$.
- Arc/partition parameters: $Q = \dfrac{N^{1/2}}{(\log N)^4}$, $Q' = \dfrac{N^{2/3}}{(\log N)^6}$, $U = V = N^{1/3}$.
- Medium-arc saving: fix $\delta_{\text{med}} = 10^{-3}$.
- Smoothing: $\Delta(\eta) \le C_\eta (\log N)^{-10}$ with $C_\eta \le 100$.

C_{meas} **size (tight vs conservative).** With the definitions above,

$$C_{\text{meas}}(N) \;\le\; \underbrace{2\,\frac{Q'}{N}\,\log\Big(\frac{Q'}{Q}\Big)}_{\text{tight}} \;=\; 2\,N^{-1/3}(\log N)^{-6}\,\Big(\tfrac{1}{6}\log N - 2\log\log N\Big),$$

and we also record a conservative variant (used in pointwise bounds)

$$C_{\text{meas}}^{\text{cons}}(N) \;:=\; 4\,N^{-1/3}(\log N)^{-6}\,\Big(\tfrac{1}{6}\log N - 2\log\log N\Big).$$

Illustrative numerical values:

$\log N$	N (approx.)	C_{meas} (tight)	$C_{\text{meas}}^{\text{cons}}$
48	$\approx 4.1 \times 10^{20}$	$\approx 4.75 \times 10^{-18}$	$\approx 9.50 \times 10^{-18}$
57	$\approx 5.7 \times 10^{24}$	$\approx 4.63 \times 10^{-19}$	$\approx 9.26 \times 10^{-19}$
66	$\approx 4.6 \times 10^{28}$	$\approx 1.77 \times 10^{-20}$	$\approx 3.54 \times 10^{-20}$

C_{disp} and $C_4^{K_8}$ (conservative choices). The dispersion constant C_{disp} is anchored in DI/DFI; we use order-of-magnitude placeholders

$$C_{\text{disp}} \in \{10,\ 10^2,\ 10^3\}, \qquad C_4^{K_8} \in \{5, 10, 20, 50\}.$$

Explicit N_0 for uniform pointwise positivity

We use the ledger inequality

$$N^{1/6} \ \geq \ \frac{4\sqrt{C_{\text{disp}}}}{c_0} (\log N)^{1-\delta_{\text{med}}/2} \sqrt{\tfrac{1}{6}\log N - 2\log\log N} \qquad (\delta_{\text{med}} = 10^{-3}).$$

Solving conservatively (rounding $\log N$ up to ensure the bracket is positive) gives the following thresholds:

C_{disp}	$\log N_0$ (adopted)	N_0 (approx.)
10	48	$\approx 4.1 \times 10^{20}$
10^2	57	$\approx 5.7 \times 10^{24}$
10^3	66	$\approx 4.6 \times 10^{28}$

These values are conservative; larger δ_{med} or smaller C_{disp} reduce N_0.

Short-interval H_0 prefactors (with $\delta_{\text{med}} = 10^{-3}$)

The exponent is $8 - \delta_{\text{med}}$ and the prefactor is $\approx 9.18\, C_4^{K_8}$:

$C_4^{K_8}$	Prefactor $\approx 9.18\, C_4^{K_8}$
5	≈ 45.9
10	≈ 91.8
20	≈ 183.6
50	≈ 459.0

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