Finite-Volume Rigidity and Flux Sectors in a Reciprocal Ratio Gradient Model on Graphs

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We study the finite-volume nearest-neighbor energy generated by the symmetric reciprocal-ratio penalty, or equivalently the gradient potential $V(t) = \cosh t - 1$ in logarithmic variables. Uniform convexity places the model in the standard class of noncompact height theories and yields weighted-Laplacian Hessians, strict convexity on fixed-mean slices, ground-state characterization, and coercive finite-graph bounds. The specific cosh form gives several closed-form results. In each cohomology class of oriented edge fields there is a unique minimizing representative, characterized by the nonlinear coclosed equation $\delta \sinh \omega = 0$, and the sector energy admits a quadratic gap. On twisted discrete tori this representative is explicit, so affine fields are the unique minimizers modulo constants and the finite-volume energy density equals $\sum_{i=1}^{d}(\cosh a_i - 1)$, independent of torus size. For boxes and discrete tori in $\mathbb{Z}^d$, the spectral gaps are explicit, giving in $d = 3$ an $o(L)$ sufficient condition under which the normalized logarithmic field vanishes in averaged L^2.

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I. INTRODUCTION

Local edge energies built from disagreement variables are basic objects in variational statistical mechanics, lattice interface theory, and network dynamics. In additive models one penalizes differences $r_v - r_w$. In multiplicative models the natural observable is instead a ratio x_v/x_w. Variational, convex-analytic, and equilibrium formulations of such lattice systems are classical in rigorous statistical mechanics and large-deviation theory.[1–6]

In this paper, we study the reciprocal ratio cost

$$J(x) = \frac{1}{2}\left(x + x^{-1}\right) - 1, \qquad x > 0, \tag{1}$$

and the associated graph energy

$$C_G[x] = \sum_{\langle v,w\rangle \in E} J\left(\frac{x_v}{x_w}\right). \tag{2}$$

The energy is invariant under global rescaling $x \mapsto cx$, and in logarithmic variables $r_v = \ln x_v$ it becomes

$$F_G[r] := C_G[e^r] = \sum_{\langle v,w\rangle \in E} \big(\cosh(r_v - r_w) - 1\big). \tag{3}$$

Thus the model is a nearest-neighbor gradient system with edge potential $V(t) = \cosh t - 1$. Since $V''(t) = \cosh t \geq 1$, it lies in the standard class of uniformly convex gradient interactions familiar from Brascamp–Lieb inequalities and the theory of gradient Gibbs or interface models.[7–12] Near consensus one has $V(t) = \frac{1}{2}t^2 + O(t^4)$, so the small-fluctuation regime is governed by the usual graph Dirichlet energy and its spectral-gap estimates.[13–16]

The logarithmic formulation also gives the model a direct physical reading. The original variables $x_v > 0$ may be viewed as positive local amplitudes or scale variables attached to the vertices, and the reciprocal-ratio penalty measures relative mismatch across an edge rather than additive discrepancy. Passing to $r_v = \ln x_v$ turns this multiplicative disagreement into an additive height difference, with the global rescaling symmetry $x \mapsto cx$ becoming the global height-shift symmetry $r \mapsto r + (\ln c)\mathbf{1}$. In this sense the model is a zero-temperature noncompact interface model with explicit strictly convex nearest-neighbor potential $V(t) = \cosh t - 1$. The main point of the paper is that this multiplicative-to-additive change of variables makes the finite-volume sector structure and coercive estimates completely explicit.

The exact logarithmic representation (3) has four concrete finite-volume consequences. First, on arbitrary finite connected graphs, the Hessian of F_G is a weighted graph Laplacian with conductances $\cosh(r_v - r_w)$, so after removing the one-dimensional shift symmetry the energy is strongly convex and quantitatively controls deviations from the constant manifold. Second, the model admits a natural finite-graph flux-sector decomposition: if one fixes an oriented edge field h modulo exact shifts $h \sim h + d\phi$, then each class $[h] \in H^1(G;\mathbb{R})$ contains a unique minimizing representative, characterized by the nonlinear coclosed (discrete divergence-free) equation $\delta \sinh \omega = 0$, and the sector gap is again quadratic. Third, on boxes and discrete tori in $\mathbb{Z}^d$, the relevant spectral gaps are explicit, yielding coercive bounds at the natural Laplacian scale L^{-2} and an $o(L^{d-2})$ sufficient condition under which the normalized logarithmic field vanishes in averaged L^2. Fourth, on twisted discrete tori with prescribed logarithmic slope, the coordinate-constant sector representative is already coclosed, so the finite-volume minimum is exact: affine fields are the unique minimizers modulo constants and the associated zero-temperature surface tension is $\sum_{i=1}^{d}(\cosh a_i - 1)$.

Not all of these consequences are equally model-specific, and it is useful to separate the general mechanism from the explicit solvable features. The coercive estimates in Secs. II–V are driven by uniform convexity: whenever a gradient potential satisfies $V'' \geq c > 0$, its Hessian dominates c times the graph Laplacian, yielding slice-wise strong convexity, existence and uniqueness of sector minimizers, and spectral-gap bounds. What is special here is that $V(t) = \cosh t - 1$ remains explicit at every step: the Euler–Lagrange equation becomes $\delta \sinh \omega = 0$, the bounded-gradient comparison factor $A(M)$ is closed form, the twisted-torus energy density is exactly $\sum_{i=1}^{d}(\cosh a_i - 1)$, and the compactified one-edge potential can be written explicitly.

From the viewpoint of the gradient-Gibbs literature, the flux-sector discussion below should be read as a deterministic finite-graph analogue of working at fixed slope or tilt in uniformly convex interface models. Positive-temperature fluctuation, homogenization, and infinite-volume questions are treated in the Brascamp–Lieb, Naddaf–Spencer, Funaki, Biskup–Kotecký, Sheffield, and Velenik literature; the present paper stays at zero temperature and finite volume, where the specific cosh potential allows several sector quantities to be computed exactly. Throughout, "rigidity" is used only in this quantitative stability sense: the total energy controls distance to the constant manifold, rather than providing an infinite-volume classification theorem; compare the quantitative use of the term in geometric rigidity.[7–12,17]

A separate but related construction concerns compact phase reduction. Reducing the logarithmic target space modulo $\ln b$ at each site yields a circle-valued field and a one-edge periodic interaction that can be written in closed form. We record this in Appendix B because it isolates the local compact phase model naturally associated with the logarithmic variable and clarifies how that local reduction compares with the genuine graph-level quotient over sitewise lifts: the two coincide on trees and differ on general cyclic graphs.

We work throughout in finite volume and at zero temperature. Infinite-volume Gibbs states, DLR ground states, and positive-temperature questions are left aside. Section II records the basic identities and comparison estimates. Section III proves strong convexity and quantitative stability on finite connected graphs. Section IV formulates flux sectors on arbitrary finite graphs and proves existence, uniqueness, and a quantitative gap in each cohomology class. Section V specializes the zero-sector bounds to boxes and tori in $\mathbb{Z}^d$. Section VI extracts the exact twisted torus formula as an explicit flux-sector specialization. Section VII closes the main text. Appendix A records the product-spectrum calculation used for boxes and tori, and Appendix B records the local phase reduction.

II. THE LOGARITHMIC FORMULATION AND BASIC COMPARISON ESTIMATES

Let $G = (V, E)$ be a simple graph, finite or countably infinite. A configuration is a positive field $x\colon V \to \mathbb{R}_{>0}$. For finite graphs the energy (2) is well defined. On infinite graphs we work only through finite-volume restrictions. The ratio form makes the global scale symmetry immediate:

$$C_G[cx] = C_G[x], \qquad c > 0. \tag{4}$$

In logarithmic variables this becomes the additive symmetry $r \mapsto r + (\ln c)\mathbf{1}$.

For a finite graph and a vertex function $f \in \mathbb{R}^V$, the combinatorial Laplacian is

$$(L_G f)(v) := \sum_{w\,:\,\langle v,w\rangle \in E} \big(f(v) - f(w)\big), \tag{5}$$

and the associated Dirichlet form is

$$\mathcal{E}_G[r] := \frac{1}{2} \sum_{\langle v,w\rangle \in E} (r_v - r_w)^2. \tag{6}$$

We write $d_G(v,w)$ for graph distance and, when G is finite, $\mathrm{diam}(G) = \max_{v,w\in V} d_G(v,w)$ for the graph diameter. Standard references on graph Laplacians and discrete Dirichlet forms include Refs.[13–15]. The purpose of this section is to record the elementary comparison estimates that later drive the coercive bounds and sector analysis.

Lemma II.1 (Elementary identities for the reciprocal cost)**.** *For $x > 0$ and $t \in \mathbb{R}$, the function J satisfies:*

1.
$$J(x) = \frac{(x-1)^2}{2x} = \frac{1}{2}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \geq 0, \tag{7}$$
with equality if and only if $x = 1$;

2. $J(x) = J(x^{-1})$;

3. $J(e^t) = \cosh t - 1$;

4.
$$\cosh t - 1 \geq \frac{1}{2}t^2 \qquad (t \in \mathbb{R}). \tag{8}$$

Proof. The algebraic identities are immediate. For (8), set $f(t) = \cosh t - 1 - \frac{1}{2}t^2$. Then $f(0) = f'(0) = 0$ and $f''(t) = \cosh t - 1 \geq 0$, so f attains its minimum at $t = 0$. □

The lower bound (8) is the basic bridge from the nonquadratic ratio model to the graph Dirichlet form.

Proposition II.2 (Dirichlet lower bound)**.** *For every finite graph G and every configuration $x\colon V \to \mathbb{R}_{>0}$ with logarithmic field $r_v = \ln x_v$,*

$$C_G[x] = F_G[r] \geq \mathcal{E}_G[r]. \tag{9}$$

If G is connected, equality holds if and only if r is constant.

Proof. Apply Lemma II.1(iv) to each edge term in (3). Equality in (8) holds only at $t = 0$, so equality in (9) forces $r_v - r_w = 0$ on every edge. Connectedness then implies that r is constant. □

For a variation direction $\eta \in \mathbb{R}^V$, we use the directional notation $DF_G[r](\eta) := \left.\frac{d}{ds}F_G[r+s\eta]\right|_{s=0}$ and $D^2F_G[r](\eta,\eta) := \left.\frac{d^2}{ds^2}F_G[r+s\eta]\right|_{s=0}$. In particular, the symbol η always denotes a test direction in vertex-function space.

Lemma II.3 (Gradient and Hessian). *Let G be finite. For $r, \eta \in \mathbb{R}^V$,*

$$\big(\nabla F_G[r]\big)_v = \sum_{w\,:\,\langle v,w\rangle \in E} \sinh(r_v - r_w), \tag{10}$$

and

$$D^2 F_G[r](\eta,\eta) = \sum_{\langle v,w\rangle \in E} \cosh(r_v - r_w)(\eta_v - \eta_w)^2. \tag{11}$$

Equivalently, the Hessian is the weighted Laplacian with conductances

$$a_{vw}(r) := \cosh(r_v - r_w) \geq 1. \tag{12}$$

In particular,

$$D^2 F_G[r](\eta,\eta) \geq 2\mathcal{E}_G[\eta] \qquad (r, \eta \in \mathbb{R}^V). \tag{13}$$

Proof. Differentiate the curve $s \mapsto F_G[r + s\eta]$ once and twice at $s = 0$. Formula (13) follows from (11) and $\cosh \geq 1$. $\square$

Proposition II.4 (Bounded-gradient comparison). *For $M \geq 0$, define*

$$A(M) := \begin{cases} 1, & M = 0, \\ \dfrac{2(\cosh M - 1)}{M^2}, & M > 0. \end{cases} \tag{14}$$

If a logarithmic field satisfies $|r_v - r_w| \leq M$ on every edge, then

$$\mathcal{E}_G[r] \leq F_G[r] \leq A(M)\, \mathcal{E}_G[r]. \tag{15}$$

Moreover,

$$A(M) = 1 + \frac{M^2}{12} + O(M^4) \qquad (M \to 0). \tag{16}$$

Proof. The lower bound is Proposition II.2. For $|t| \leq M$ and $M > 0$,

$$\cosh t - 1 = \sum_{k\geq 1} \frac{t^{2k}}{(2k)!} \leq \frac{t^2}{M^2} \sum_{k\geq 1} \frac{M^{2k}}{(2k)!} = \frac{\cosh M - 1}{M^2}\, t^2.$$

Summing over edges yields the upper bound. The expansion (16) is the Taylor series of $\cosh M$. $\square$

Lemma II.5 (Tangent-plus-quadratic lower bound). *For all $a, s \in \mathbb{R}$,*

$$\cosh(a + s) - 1 \geq \cosh a - 1 + (\sinh a)s + \frac{1}{2}s^2. \tag{17}$$

Equality holds if and only if $s = 0$.

Proof. Set

$$g_a(s) := \cosh(a+s) - 1 - \big(\cosh a - 1\big) - (\sinh a)s - \frac{1}{2}s^2.$$

Then $g_a(0) = g_a'(0) = 0$ and $g_a''(s) = \cosh(a+s) - 1 \geq 0$. Taylor's theorem with integral remainder gives

$$g_a(s) = s^2 \int_0^1 (1-\tau)\big(\cosh(a+\tau s) - 1\big)\, d\tau \geq 0,$$

which proves (17). If $s \neq 0$, then $\tau \mapsto a + \tau s$ ranges over a nondegenerate interval, while $\cosh t - 1$ vanishes only at $t = 0$; hence the integrand is positive on a set of positive measure, so the inequality is strict. Therefore equality holds if and only if $s = 0$. □

Propositions II.2 and II.4, together with Lemma II.5, provide the basic comparison estimates used in the finite-volume analysis below.

III. FINITE CONNECTED GRAPHS: STRONG CONVEXITY, COERCIVITY, AND QUANTITATIVE STABILITY

In this section $G = (V, E)$ is finite and connected. Write $|V| = N$ and

$$\bar{r} := \frac{1}{N} \sum_{v \in V} r_v. \tag{18}$$

For $m \in \mathbb{R}$, define the affine slice

$$\mathcal{H}_m := \left\{ r \in \mathbb{R}^V : \frac{1}{N} \sum_{v \in V} r_v = m \right\}. \tag{19}$$

The decomposition $\mathbb{R}^V = \mathbb{R}\mathbf{1} \oplus \mathbf{1}^\perp$ shows that fixing the mean removes exactly the global shift mode. Let $\lambda_2(G) > 0$ denote the first nonzero eigenvalue of L_G.

Proposition III.1 (Strong convexity on fixed-mean slices)**.** *For every $m \in \mathbb{R}$, the restriction of F_G to $\mathcal{H}_m$ is $\lambda_2(G)$-strongly convex. Equivalently, for every $r \in \mathcal{H}_m$, every $\eta \in \mathbf{1}^\perp$, and every $s \in \mathbb{R}$,*

$$\frac{d^2}{ds^2} F_G[r + s\eta] \geq \lambda_2(G) \sum_{v \in V} \eta_v^2. \tag{20}$$

Consequently, the unique minimizer of F_G on $\mathcal{H}_m$ is the constant field $r_v \equiv m$.

Proof. By Lemma II.3,

$$\frac{d^2}{ds^2} F_G[r + s\eta] = D^2 F_G[r + s\eta](\eta, \eta) \geq 2\mathcal{E}_G[\eta].$$

Since $\eta \in \mathbf{1}^\perp$, the graph Poincaré inequality gives

$$2\mathcal{E}_G[\eta] \geq \lambda_2(G) \sum_{v \in V} \eta_v^2.$$

This proves (20). The constant field $r_v \equiv m$ belongs to $\mathcal{H}_m$ and has zero energy, hence it is the unique minimizer. □

Proposition III.2 (Global coercive bound after gauge fixing)**.** *For every $r \in \mathbb{R}^V$,*

$$F_G[r] \geq \mathcal{E}_G[r] \geq \frac{\lambda_2(G)}{2} \sum_{v \in V} (r_v - \bar{r})^2. \tag{21}$$

Equivalently, for every $x\colon V \to \mathbb{R}_{>0}$,

$$C_G[x] \geq \frac{\lambda_2(G)}{2} \sum_{v \in V} (\ln x_v - \bar{r})^2. \tag{22}$$

Proof. The first inequality is Proposition II.2. The second is the spectral-gap inequality applied to $r - \bar{r}\,\mathbf{1} \in \mathbf{1}^\perp$. □

Remark III.3. At positive temperature, Brascamp–Lieb comparison gives corresponding variance bounds for Gibbs measures whose Hessian is bounded below by the graph Laplacian. Proposition III.2 is the zero-temperature deterministic counterpart used in the present finite-volume analysis.[7,8]

Corollary III.4 (Finite-volume ground states)**.** *The finite-volume ground states of the ratio energy are exactly the constant configurations:*

$$C_G[x] = 0 \quad \Longleftrightarrow \quad x_v \equiv c \text{ on } V \text{ for some } c > 0. \tag{23}$$

Equivalently, the ground-state manifold is the global scale orbit of a constant configuration.

Proof. Constant configurations have zero energy. Conversely, if $C_G[x] = 0$, then (7) forces $x_v = x_w$ on every edge, hence on all of V by connectedness. □

Theorem III.5 (Quantitative stability from total energy)**.** *Let $x\colon V \to \mathbb{R}_{>0}$ be a configuration and let $r_v = \ln x_v$. Define*

$$\rho_x(v,w) := \sqrt{2\, d_G(v,w)\, C_G[x]}, \qquad R_x := \sqrt{2\ \mathrm{diam}(G)\, C_G[x]}. \tag{24}$$

Then

$$C_G[x] \geq \frac{\lambda_2(G)}{2} \sum_{v \in V} (r_v - \bar{r})^2, \tag{25}$$

$$|r_v - r_w| \leq \rho_x(v, w) \qquad (v, w \in V), \tag{26}$$

$$e^{-\rho_x(v,w)} \leq \frac{x_v}{x_w} \leq e^{\rho_x(v,w)} \qquad (v, w \in V), \tag{27}$$

$$\max_{v,w \in V} |r_v - r_w| \leq R_x, \tag{28}$$

$$\max_{v \in V} |r_v - \bar{r}| \leq R_x, \tag{29}$$

$$e^{-R_x} \leq e^{-\bar{r}} x_v \leq e^{R_x} \qquad (v \in V). \tag{30}$$

Proof. Equation (25) is Proposition III.2. For (26), let $v = v_0, v_1, \ldots, v_m = w$ be a shortest path, so $m = d_G(v, w)$. By Cauchy–Schwarz,

$$\begin{aligned} |r_v - r_w|^2 &= \left| \sum_{i=0}^{m-1} (r_{v_i} - r_{v_{i+1}}) \right|^2 \\ &\leq m \sum_{i=0}^{m-1} (r_{v_i} - r_{v_{i+1}})^2 \\ &\leq m \sum_{\langle u,u' \rangle \in E} (r_u - r_{u'})^2 = 2m\, \mathcal{E}_G[r] \\ &\leq 2m\, C_G[x], \end{aligned}$$

where the last step is Proposition II.2. This proves (26), and (27) follows by exponentiation. Maximizing (26) over pairs gives (28). For (29),

$$|r_v - \bar{r}| = \left| \frac{1}{N} \sum_{w \in V} (r_v - r_w) \right| \leq \frac{1}{N} \sum_{w \in V} |r_v - r_w| \leq \max_{w \in V} |r_v - r_w|.$$

Applying (28) proves (29), and (30) is again the exponential form. □

Remark III.6. The quantities $\rho_x(v, w)$ and R_x package the pathwise and diameter-scale pointwise consequences of the total-energy bound. They are not needed later in the paper, but they record the direct ratio and sup-norm control that follows from the same energy estimate without additional spectral input.

The ground-state characterization and the coercive estimates of this section are not special to the cosh potential: any nonnegative edge cost with a unique zero and a uniformly convex logarithmic representation gives analogous finite-volume stability statements. What is special here is that the exact logarithmic form (3) keeps the nonlinear stationarity equation and the sector energies explicit. The next sections exploit that extra explicitness to identify the coclosed representative $\delta \sinh \omega = 0$, compute twisted-torus surface tensions exactly, and compare the local compact reduction with the full graph-level quotient.

IV. FLUX SECTORS ON ARBITRARY FINITE GRAPHS

The flux-sector framework introduced here later specializes to the twisted discrete tori treated explicitly in Sec. VI. From the physical point of view, fixing a cohomology class is the finite-graph analogue of fixing a background flux or macroscopic tilt carried by the cycles of the graph; the sector minimizer identifies the energetically preferred representative within that constrained class. The point of the flux-sector formalism is to separate the additive gauge freedom from the cycle data carried by the graph. We use u rather than r for the vertex variable in this section to emphasize that we are now minimizing inside a fixed cohomology class of edge fields. Choose once and for all an orientation $E^{\to}$ of the edges of a finite connected graph $G = (V, E)$, and write

$$C^0(G) := \mathbb{R}^V, \qquad C^1(G) := \mathbb{R}^{E^{\to}}. \tag{31}$$

We retain the notation

$$\mathcal{H}_m := \left\{ u \in C^0(G) : \frac{1}{|V|} \sum_{v \in V} u_v = m \right\} \qquad (m \in \mathbb{R}) \tag{32}$$

for the fixed-mean slices. For $u \in C^0(G)$ and $\omega \in C^1(G)$, define the coboundary and divergence by

$$(du)(e) := u(e_+) - u(e_-), \tag{33}$$

$$(\delta\omega)(v) := \sum_{e_+ = v} \omega(e) - \sum_{e_- = v} \omega(e), \tag{34}$$

where e_- and e_+ denote the tail and head of the oriented edge e. We use both $u(v)$ and u_v for the value of a vertex field at v; in particular, $u(e_\pm)$ simply means the value of u at the head or tail of the oriented edge e. Then $\delta d = L_G$, and the discrete integration-by-parts identity

$$\sum_{e \in E^{\to}} \omega(e)(du)(e) = \sum_{v \in V} (\delta\omega)(v) u_v \tag{35}$$

holds for all $u \in C^0(G)$ and $\omega \in C^1(G)$. We use only this elementary cochain language of finite graphs; see, for example, Refs.[13,15,18].

Fix a sector representative $h \in C^1(G)$ and define the corresponding flux-sector energy by

$$F_{G,h}[u] := \sum_{e \in E^{\to}} \big(\cosh(h_e + (du)(e)) - 1\big), \qquad u \in C^0(G). \tag{36}$$

Changing h by an exact 1-form leaves the sector unchanged.

Proposition IV.1 (Exact-shift equivalence)**.** *If $\phi \in C^0(G)$ and $h' := h + d\phi$, then*

$$F_{G,h'}[u] = F_{G,h}[u + \phi] \qquad (u \in C^0(G)). \tag{37}$$

Consequently, the minimum of (36) *depends only on the cohomology class*

$$[h] \in H^1(G;\mathbb{R}) := C^1(G)/dC^0(G). \tag{38}$$

Proof. Since $d(u + \phi) = du + d\phi$, one has $h' + du = h + d(u + \phi)$ edgewise, which is exactly (37). The class dependence follows immediately. □

Theorem IV.2 (Unique minimizing representative in each flux sector)**.** *Let G be finite and connected, and let $h \in C^1(G)$. Define the sector minimum by*

$$\sigma_G([h]) := \inf_{u \in C^0(G)} F_{G,h}[u]. \tag{39}$$

Then the following hold.

1. *For every $m \in \mathbb{R}$, the restriction of $F_{G,h}$ to the slice $\mathcal{H}_m$ is $\lambda_2(G)$-strongly convex. In particular, on each slice there is a unique minimizer, and the minimizers on different slices differ only by additive constants.*

2. *If $u_h \in \mathcal{H}_0$ is the unique mean-zero minimizer and*

$$\omega_h := h + du_h \in [h], \tag{40}$$

then

$$\delta \sinh \omega_h = 0. \tag{41}$$

Moreover, ω_h is the unique representative of the class $[h]$ satisfying (41).

3. For every $u \in C^0(G)$, with $\eta := u - u_h$ and $\bar\eta := |V|^{-1}\sum_{v\in V}\eta_v$,

$$F_{G,h}[u] - \sigma_G([h]) \geq \frac{1}{2}\sum_{e\in E^{\to}} (d\eta)(e)^2 \tag{42}$$

$$\geq \frac{\lambda_2(G)}{2}\sum_{v\in V}(\eta_v - \bar\eta)^2. \tag{43}$$

Hence each cohomology class contains a unique energy-minimizing representative.

Proof. For $u, \zeta \in C^0(G)$,

$$D^2F_{G,h}[u](\zeta,\zeta) = \sum_{e\in E^{\to}} \cosh\big(h_e + (du)(e)\big)\,(d\zeta)(e)^2. \tag{44}$$

Since $\cosh \geq 1$, one has

$$D^2F_{G,h}[u](\zeta,\zeta) \geq \sum_{e\in E^{\to}} (d\zeta)(e)^2 = 2\mathcal{E}_G[\zeta]. \tag{45}$$

If $\zeta \in \mathbf{1}^{\perp}$, the graph Poincaré inequality gives $2\mathcal{E}_G[\zeta] \geq \lambda_2(G)\sum_v \zeta_v^2$, so $F_{G,h}|_{\mathcal{H}_m}$ is $\lambda_2(G)$-strongly convex. In finite dimension, strong convexity implies coercivity on each affine slice and therefore existence and uniqueness of a minimizer. Since adding a constant to u leaves du unchanged, the minimizers on different slices differ only by constants.

Let $u_h \in \mathcal{H}_0$ be the mean-zero minimizer and set $\omega_h = h + du_h$. Because $F_{G,h}[u + c\mathbf{1}] = F_{G,h}[u]$ for every $c \in \mathbb{R}$, the slice minimizer is stationary under arbitrary variations: any $\zeta \in C^0(G)$ decomposes as $(\zeta - \bar\zeta\,\mathbf{1}) + \bar\zeta\,\mathbf{1}$, and the constant part does not change the energy. Differentiating (36) at u_h in the direction $\zeta \in C^0(G)$ gives

$$0 = \frac{d}{ds}F_{G,h}[u_h + s\zeta]\bigg|_{s=0} = \sum_{e\in E^{\to}} \sinh\big(\omega_h(e)\big)(d\zeta)(e) = \sum_{v\in V}(\delta\sinh\omega_h)(v)\zeta_v,$$

where the last step is (35). Since this holds for every ζ, (41) follows.

Now let $\omega' \in [h]$ also satisfy $\delta\sinh\omega' = 0$. Then $\omega' - \omega_h = d\phi$ for some $\phi \in C^0(G)$. Using (35) again,

$$\begin{aligned}
0 &= \sum_{v\in V}\phi_v\,\big(\delta(\sinh\omega' - \sinh\omega_h)\big)(v) \\
&= \sum_{e\in E^{\to}} (d\phi)(e)\,\big(\sinh\omega'(e) - \sinh\omega_h(e)\big) \\
&= \sum_{e\in E^{\to}} \big(\omega'(e) - \omega_h(e)\big)\,\big(\sinh\omega'(e) - \sinh\omega_h(e)\big).
\end{aligned}$$

Because sinh is strictly increasing, every summand is nonnegative and vanishes only when $\omega'(e) = \omega_h(e)$. Hence $\omega' = \omega_h$, proving uniqueness of the nonlinear coclosed representative.

Finally, let $u \in C^0(G)$ and set $\eta = u - u_h$. Apply Lemma II.5 edgewise with $a = \omega_h(e)$ and $s = (d\eta)(e)$:

$$\cosh\bigl(\omega_h(e) + (d\eta)(e)\bigr) - 1 \geq \cosh\bigl(\omega_h(e)\bigr) - 1 + \sinh\bigl(\omega_h(e)\bigr)(d\eta)(e) + \frac{1}{2}(d\eta)(e)^2.$$

Summing over $e \in E^{\to}$ and using (35) together with $\delta \sinh \omega_h = 0$ removes the linear term, which yields (42). Since $d\eta = d(\eta - \bar{\eta}\,\mathbf{1})$, the graph Poincaré inequality gives (43). □

Remark IV.3. If G is a tree, every 1-cochain is exact, so $H^1(G;\mathbb{R}) = 0$ and there is only one flux sector. Nontrivial sectors appear precisely when G has cycles. In that sense, Theorem IV.2 is the finite-graph, zero-temperature analogue of working at fixed slope or tilt in gradient Gibbs and interface models.[9,11,12]

V. BOXES AND TORI IN $\mathbb{Z}^d$

We now specialize the finite-volume bounds to the standard cubic boxes in $\mathbb{Z}^d$. Let

$$\Lambda_{L,d} := [-L, L]^d \cap \mathbb{Z}^d, \qquad N_L := 2L + 1, \tag{46}$$

and consider either the free box graph on $\Lambda_{L,d}$ or the periodic torus $(\mathbb{Z}/N_L\mathbb{Z})^d$. For a configuration x on either graph, write $r_v := \ln x_v$ and

$$\bar{r}_{L,d} := \frac{1}{N_L^d} \sum_v r_v, \tag{47}$$

with the sum over the relevant vertex set. The corresponding energies are denoted by $C^{\mathrm{f}}_{L,d}[x]$ and $C^{\mathrm{p}}_{L,d}[x]$.

Lemma V.1 (Spectral gaps of boxes and tori). *For every $d \geq 1$ and every $L \geq 1$, the Laplacian spectral gaps of the free box and periodic torus are*

$$\lambda^{\mathrm{f}}_{2,d}(L) = 2\left(1 - \cos\frac{\pi}{N_L}\right), \tag{48}$$

$$\lambda^{\mathrm{p}}_{2,d}(L) = 2\left(1 - \cos\frac{2\pi}{N_L}\right). \tag{49}$$

In particular,

$$\lambda^{\mathrm{f}}_{2,d}(L) \geq \frac{4}{N_L^2}, \qquad \lambda^{\mathrm{p}}_{2,d}(L) \geq \frac{16}{N_L^2}. \tag{50}$$

Proof. The box and torus are Cartesian products of d copies of the path graph P_{N_L} and the cycle graph C_{N_L}, respectively. Their Laplacian eigenvalues therefore add. The product formulas are recorded in Appendix A. The smallest positive eigenvalue is obtained by taking one factor in its first nontrivial mode and the remaining factors in the zero mode, which yields (48) and (49). The bounds (50) follow from $1 - \cos t = 2\sin^2(t/2) \geq 2(t/\pi)^2$ for $0 \leq t \leq \pi$. □

Theorem V.2 (Averaged coercive bounds on boxes and tori)**.** *For every $d \geq 1$, every $L \geq 1$, and every configuration x on the free box, with $r_v := \ln x_v$,*

$$\frac{1}{N_L^d} \sum_{v \in \Lambda_{L,d}} (r_v - \bar{r}_{L,d})^2 \leq \frac{1}{2N_L^{d-2}}\, C^{\mathrm{f}}_{L,d}[x]. \tag{51}$$

For every configuration x on the periodic torus, with $r_v := \ln x_v$,

$$\frac{1}{N_L^d} \sum_{v \in (\mathbb{Z}/N_L\mathbb{Z})^d} (r_v - \bar{r}_{L,d})^2 \leq \frac{1}{8N_L^{d-2}}\, C^{\mathrm{p}}_{L,d}[x]. \tag{52}$$

Consequently, if $\{x^{(L)}\}_{L\geq 1}$ is a sequence of free-boundary configurations such that

$$C^{\mathrm{f}}_{L,d}[x^{(L)}] = o(L^{d-2}) \qquad (L \to \infty), \tag{53}$$

or a sequence of periodic configurations such that

$$C^{\mathrm{p}}_{L,d}[x^{(L)}] = o(L^{d-2}) \qquad (L \to \infty), \tag{54}$$

then, writing

$$\bar{r}^{(L)}_{L,d} := \frac{1}{N_L^d} \sum_v \ln x_v^{(L)}, \tag{55}$$

with the sum over the relevant vertex set, the normalized fields

$$y_v^{(L)} := e^{-\bar{r}^{(L)}_{L,d}} x_v^{(L)} \tag{56}$$

satisfy

$$\frac{1}{N_L^d} \sum_v \big(\ln y_v^{(L)}\big)^2 \longrightarrow 0. \tag{57}$$

Proof. For free boundary conditions, Proposition III.2 and Lemma V.1 give

$$\sum_{v \in \Lambda_{L,d}} (r_v - \bar{r}_{L,d})^2 \leq \frac{2}{\lambda^{\mathrm{f}}_{2,d}(L)}\, C^{\mathrm{f}}_{L,d}[x] \leq \frac{N_L^2}{2}\, C^{\mathrm{f}}_{L,d}[x].$$

Dividing by N_L^d yields (51). The periodic estimate (52) is identical, using $\lambda^{\mathrm{p}}_{2,d}(L) \geq 16/N_L^2$. Finally, $\ln y_v^{(L)} = \ln x_v^{(L)} - \bar{r}^{(L)}_{L,d}$, so (57) says precisely that the mean-zero logarithmic field converges to zero in averaged L^2. Since $N_L \sim 2L$, either (53) or (54) implies (57). □

Remark V.3 (Three-dimensional scaling). In dimension $d = 3$, (51) and (52) reduce to

$$\frac{1}{N_L^3} \sum_{v \in \Lambda_{L,3}} (r_v - \bar{r}_{L,3})^2 \le \frac{1}{2N_L} C^{\mathrm{f}}_{L,3}[x], \tag{58}$$

and

$$\frac{1}{N_L^3} \sum_{v \in (\mathbb{Z}/N_L\mathbb{Z})^3} (r_v - \bar{r}_{L,3})^2 \le \frac{1}{8N_L} C^{\mathrm{p}}_{L,3}[x]. \tag{59}$$

Thus $o(L)$ total energy is a sufficient condition for the mean-zero logarithmic field to converge to zero in box-averaged L^2 after normalization.

Remark V.4. Theorem V.2 is a finite-size coercive statement, not an infinite-volume classification theorem. It identifies the scale at which the mean-zero logarithmic field $\ln y^{(L)}$ is forced to converge to zero in averaged L^2, but it does not classify infinite-volume local minimizers or Gibbs states.

VI. TWISTED TORUS SECTORS AS AN EXPLICIT FLUX-SECTOR SPECIALIZATION

Theorem IV.2 becomes completely explicit on the discrete torus. Prescribing a logarithmic slope is equivalent to fixing a cohomology class represented by a coordinate-constant oriented edge field, and in this geometry the minimizing representative can be written down in closed form.

Let

$$T_{N,d} := (\mathbb{Z}/N\mathbb{Z})^d, \qquad N \ge 2, \tag{60}$$

and for $u \colon T_{N,d} \to \mathbb{R}$ write

$$\nabla_i u(n) := u(n + e_i) - u(n), \qquad i = 1, \dots, d, \tag{61}$$

where e_i is the ith coordinate vector and addition is modulo N. Fix a slope vector $a = (a_1, \dots, a_d) \in \mathbb{R}^d$ and define the twisted torus energy

$$F^{(a)}_{N,d}[u] := \sum_{n \in T_{N,d}} \sum_{i=1}^{d} \big(\cosh(a_i + \nabla_i u(n)) - 1\big). \tag{62}$$

Equivalently, if $r \colon \mathbb{Z}^d \to \mathbb{R}$ satisfies the quasi-periodicity condition

$$r(n + Ne_i) = r(n) + Na_i \qquad (i = 1, \dots, d), \tag{63}$$

then $r(n) = a \cdot n + u(n)$ with u N-periodic, and the energy on one fundamental cell is exactly (62).

Theorem VI.1 (Exact affine minimizers in every twisted torus sector)**.** *For every $N \geq 2$, every $d \geq 1$, every slope $a \in \mathbb{R}^d$, and every periodic field $u \colon T_{N,d} \to \mathbb{R}$,*

$$F^{(a)}_{N,d}[u] \geq N^d \sum_{i=1}^{d} \bigl(\cosh a_i - 1\bigr), \tag{64}$$

$$F^{(a)}_{N,d}[u] - N^d \sum_{i=1}^{d} \bigl(\cosh a_i - 1\bigr) \geq \frac{1}{2} \sum_{n \in T_{N,d}} \sum_{i=1}^{d} \bigl(\nabla_i u(n)\bigr)^2 \tag{65}$$

$$\geq \left(1 - \cos \frac{2\pi}{N}\right) \sum_{n \in T_{N,d}} \bigl(u(n) - \bar{u}\bigr)^2, \tag{66}$$

where

$$\bar{u} := \frac{1}{N^d} \sum_{n \in T_{N,d}} u(n). \tag{67}$$

Equality in (64) *holds if and only if u is constant. Equivalently, among all quasi-periodic lifts satisfying* (63)*, the unique minimizers modulo additive constants are the affine fields*

$$r(n) = a \cdot n + c. \tag{68}$$

Proof. Orient every torus edge in the positive coordinate direction and define the sector representative $h^a \in C^1(T_{N,d})$ by

$$h^a_{(n,n+e_i)} := a_i, \qquad n \in T_{N,d}, \quad i = 1, \ldots, d. \tag{69}$$

Then $F^{(a)}_{N,d}[u] = F_{T_{N,d},h^a}[u]$. At every vertex $n \in T_{N,d}$,

$$(\delta \sinh h^a)(n) = \sum_{i=1}^{d} \sinh(a_i) - \sum_{i=1}^{d} \sinh(a_i) = 0,$$

so h^a already satisfies the Euler–Lagrange condition (41). By Theorem IV.2, it is therefore the unique energy-minimizing representative of its cohomology class, and the unique mean-zero minimizing vertex field is $u_h \equiv 0$. Evaluating the energy at a constant field gives

$$\sigma_{T_{N,d}}([h^a]) = N^d \sum_{i=1}^{d} \bigl(\cosh a_i - 1\bigr),$$

which is exactly (64). The uniqueness statement in Theorem IV.2 shows that equality in (64) holds precisely for constant u, equivalently for affine quasi-periodic lifts (68) modulo additive constants.

For the quantitative gap, apply (42) from Theorem IV.2 with $u_h \equiv 0$:

$$F^{(a)}_{N,d}[u] - N^d \sum_{i=1}^{d} (\cosh a_i - 1) \ge \frac{1}{2} \sum_{n \in T_{N,d}} \sum_{i=1}^{d} \big(\nabla_i u(n)\big)^2,$$

which is (65). The bound (66) then follows from the torus spectral-gap estimate, whose first nonzero Laplacian eigenvalue is $2(1 - \cos(2\pi/N))$ by (A5) and the Cartesian-product formula (A8). $\square$

Corollary VI.2 (Exact zero-temperature surface tension)**.** *Define the finite-size twisted energy density by*

$$\sigma_{N,d}(a) := \frac{1}{N^d} \inf_{u\colon T_{N,d} \to \mathbb{R}} F^{(a)}_{N,d}[u]. \tag{70}$$

Then

$$\sigma_{N,d}(a) = \sum_{i=1}^{d} \big(\cosh a_i - 1\big), \tag{71}$$

so the zero-temperature surface tension is already exact at finite size and does not depend on N*. In particular,*

$$\sigma_{N,d}(a) = \frac{1}{2}|a|^2 + O(|a|^4) \qquad (a \to 0). \tag{72}$$

Proof. Equation (71) is (64) divided by N^d, with equality achieved by any constant u, equivalently by the affine quasi-periodic lift $r(n) = a \cdot n + c$. The expansion (72) follows from the Taylor series of cosh. $\square$

VII. CONCLUSION

For the reciprocal ratio cost (1), the logarithmic change of variables converts the multiplicative graph energy into the uniformly convex noncompact height functional (3). At that general level, the paper belongs to the standard theory of uniformly convex gradient models: the Hessian is a weighted Laplacian, strong convexity holds after removing the shift mode, the ground states are exactly the constant configurations, and the total energy controls both averaged and pointwise deviations from the constant manifold.

What is specific to the present model is the persistence of closed form after the logarithmic transform. Fixing an oriented edge field modulo exact shifts leads to a finite-graph flux-sector theorem in which each cohomology class contains a unique minimizing representative satisfying the explicit nonlinear coclosed equation $\delta \sinh \omega = 0$, and the sector gap remains quadratic. On d-dimensional boxes and tori the spectral gaps can be written explicitly, so the zero-sector coercive estimates become concrete finite-size bounds. In dimension three, the resulting L^{-1} scale gives an $o(L)$ sufficient condition under which the normalized logarithmic field vanishes in averaged L^2.

Twisted discrete tori are an explicit special case of this sector theorem: the coordinate-constant representative is already coclosed, so affine quasi-periodic fields are the unique minimizers modulo constants and the zero-temperature surface tension is $\sum_{i=1}^{d}(\cosh a_i - 1)$, independent of torus size. Appendix B records a complementary local phase reduction obtained by reducing the logarithmic variable modulo $\ln b$ sitewise. The resulting periodic interaction is explicit, agrees with the graph-level quotient on trees, and differs from it on general cyclic graphs because local edgewise reduction does not retain the global cycle constraints. From the physical side, the model should therefore be viewed not as a full positive-temperature interface theory, but as an exactly analyzable zero-temperature representative of a multiplicative nearest-neighbor interaction: relative amplitudes become a noncompact height field after the logarithmic change of variables, and imposed flux or tilt becomes a finite-graph sector constraint. This combination of multiplicative origin, explicit convex-gradient structure, and exact sector energies is the main feature isolated here.

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Conceptualization, J.W.; methodology, A.T. and J.W.; formal analysis, A.T. and J.W.; writing—original draft preparation, J.W.; writing—review and editing, A.T. and J.W. All authors have read and agreed to the published version of the manuscript.

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DATA AVAILABILITY

No new data were generated or analyzed in support of this research.

Appendix A: Product-spectrum formulas for boxes and tori

This appendix records the spectral calculation used in Lemma V.1. Fix $d \geq 1$ and set $N = N_L = 2L + 1$.

For the path graph P_N on vertices $\{1, \ldots, N\}$, define

$$u_m(j) := \cos\left(\frac{\pi m}{N}\left(j - \frac{1}{2}\right)\right), \qquad m = 0, 1, \ldots, N-1. \tag{A1}$$

A direct computation shows that these are Laplacian eigenvectors with eigenvalues

$$\lambda_m(P_N) = 2\left(1 - \cos\frac{\pi m}{N}\right), \qquad m = 0, 1, \ldots, N-1. \tag{A2}$$

In particular, the first nonzero eigenvalue is

$$\lambda_2(P_N) = 2\left(1 - \cos\frac{\pi}{N}\right). \tag{A3}$$

For the cycle graph C_N on $\mathbb{Z}/N\mathbb{Z}$, the Fourier modes

$$\psi_k(j) := e^{2\pi i k j/N}, \qquad k = 0, 1, \ldots, N-1, \tag{A4}$$

diagonalize the Laplacian with eigenvalues

$$\lambda_k(C_N) = 2\left(1 - \cos\frac{2\pi k}{N}\right). \tag{A5}$$

Hence the first nonzero eigenvalue is

$$\lambda_2(C_N) = 2\left(1 - \cos\frac{2\pi}{N}\right). \tag{A6}$$

The free box $\Lambda_{L,d}$ is the Cartesian product of d copies of P_N, while the periodic torus $(\mathbb{Z}/N\mathbb{Z})^d$ is the Cartesian product of d copies of C_N. For Cartesian products, Laplacian eigenvalues add. Hence the free-box eigenvalues are

$$\lambda^{\mathrm{f}}_{m_1,\ldots,m_d} = \sum_{i=1}^{d} 2\left(1 - \cos\frac{\pi m_i}{N}\right), \qquad m_i \in \{0, \ldots, N-1\}, \tag{A7}$$

and the periodic-torus eigenvalues are

$$\lambda^{\mathrm{p}}_{k_1,\ldots,k_d} = \sum_{i=1}^{d} 2\left(1 - \cos\frac{2\pi k_i}{N}\right), \qquad k_i \in \{0, \ldots, N-1\}. \tag{A8}$$

The smallest positive eigenvalue is obtained by taking one coordinate equal to 1 and the remaining coordinates equal to 0, which yields (48) and (49).

Appendix B: Local phase reduction: reduced edge potential and graph-level quotient

This appendix is included for a specific reason. The main body studies a noncompact gradient model, but the logarithmic variable also admits a natural compact phase obtained by reducing it modulo $\ln b$ sitewise. That local reduction produces an explicit one-edge periodic interaction, so one can build a compact phase energy directly and then compare it with the genuine graph-level quotient over sitewise integer lifts. This operation should be distinguished from quotienting only the global rescaling symmetry, and it is also different from the twisted torus sectors of Sec. VI, where the logarithmic variable remains noncompact but carries a prescribed quasi-periodic slope. On graphs with cycles, the local compact reduction need not coincide with the full graph-level quotient: optimizing edge by edge forgets the global compatibility constraints, or equivalently the cycle data, retained by the graph-level quotient.

Proposition B.1 (Global quotient versus sitewise phase reduction)**.** *Assume G is finite with $|V| = N$, and fix $b > 1$. Quotienting the logarithmic configuration space $\mathbb{R}^V$ by the global subgroup $(\ln b)\mathbb{Z}\,\mathbf{1}$ gives*

$$\mathbb{R}^V/\big((\ln b)\mathbb{Z}\,\mathbf{1}\big) \cong \mathbf{1}^\perp \times \big(\mathbb{R}/(\ln b)\mathbb{Z}\big). \tag{B1}$$

In particular, the global quotient compactifies only the zero mode and leaves the $N-1$ relative directions noncompact. By contrast, reducing each site modulo $\ln b$ gives the phase space

$$\big(\mathbb{R}/(\ln b)\mathbb{Z}\big)^V \cong (\mathbb{R}/\mathbb{Z})^V. \tag{B2}$$

Proof. Every $r \in \mathbb{R}^V$ decomposes uniquely as $r = \bar{r}\,\mathbf{1} + r^\perp$ with $r^\perp \in \mathbf{1}^\perp$. Translation by $(\ln b)\mathbb{Z}\,\mathbf{1}$ acts only on the scalar coordinate $\bar{r}$ and leaves $r^\perp$ unchanged, giving (B1). Reducing each coordinate modulo $\ln b$ separately yields (B2). $\square$

We therefore introduce the phase variable as the class of $r_v/\ln b$ in $\mathbb{R}/\mathbb{Z}$:

$$\Theta_v := \frac{r_v}{\ln b} + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}, \qquad b > 1, \tag{B3}$$

and for a phase difference $\delta \in \mathbb{R}$ define the reduced edge potential by minimizing over integer lifts,

$$\widetilde{J}_b(\delta) := \min_{n\in\mathbb{Z}} J\big(b^{n+\delta}\big) = \min_{n\in\mathbb{Z}}\big(\cosh((\ln b)(n+\delta)) - 1\big). \tag{B4}$$

Let

$$d_{\mathbb{Z}}(\delta) := \min_{n\in\mathbb{Z}} |\delta - n| \in \left[0, \frac{1}{2}\right] \tag{B5}$$

denote the distance to the nearest integer.

Proposition B.2 (Closed form of the reduced periodic interaction)**.** *For every $b > 1$ and every $\delta \in \mathbb{R}$,*

$$\widetilde{J}_b(\delta) = \cosh\!\big((\ln b)\, d_{\mathbb{Z}}(\delta)\big) - 1. \tag{B6}$$

Hence $\widetilde{J}_b$ is even and 1-periodic, satisfies $\widetilde{J}_b \geq 0$, and vanishes if and only if $\delta \in \mathbb{Z}$. It is continuous and piecewise analytic, with nondifferentiable points at the half-integers.

Proof. The map $u \mapsto \cosh((\ln b)u) - 1$ is even and strictly increasing for $u \geq 0$. Therefore the minimizing integer in (B4) is the one that makes $|n + \delta|$ minimal, namely $|n + \delta| = d_{\mathbb{Z}}(\delta)$. □

Proposition B.3 (Quadratic bounds for the reduced interaction)**.** *For every $b > 1$ and every $\delta \in \mathbb{R}$,*

$$\frac{(\ln b)^2}{2}\, d_{\mathbb{Z}}(\delta)^2 \leq \widetilde{J}_b(\delta) \leq 4\big(\cosh((\ln b)/2) - 1\big)\, d_{\mathbb{Z}}(\delta)^2. \tag{B7}$$

Proof. Set $t = (\ln b)\, d_{\mathbb{Z}}(\delta)$, so $|t| \leq (\ln b)/2$. The lower bound is (8) applied to (B6). For the upper bound, apply Proposition II.4 to the one-edge logarithmic variable t with $M = (\ln b)/2$. □

The quantity $\widetilde{J}_b$ is the one-edge building block of the compactified phase model: once it is known explicitly, the corresponding finite-volume energy is obtained by summing it over edges. Figure 1 displays the periodic well structure, the half-integer cusps, and the dependence on the base b. The harmonic stiffness extracted in Proposition B.6 increases with b through the factor $(\ln b)^2/2$.

The associated local reduced phase energy on a finite graph is

$$\widetilde{C}_{G,b}[\Theta] := \sum_{\langle v,w\rangle \in E} \widetilde{J}_b(\vartheta_v - \vartheta_w), \tag{B8}$$

where $\vartheta_v \in \mathbb{R}$ is any lift of $\Theta_v \in \mathbb{R}/\mathbb{Z}$. Because $\widetilde{J}_b$ is 1-periodic, the right-hand side does not depend on the chosen lifts.

For comparison, the genuine graph-level quotient over sitewise integer lifts is

$$Q_{G,b}[\Theta] := \inf_{n\in\mathbb{Z}^V} \sum_{\langle v,w\rangle\in E} \big(\cosh((\ln b)(\vartheta_v - \vartheta_w + n_v - n_w)) - 1\big), \tag{B9}$$

which is likewise independent of the chosen lift ϑ.

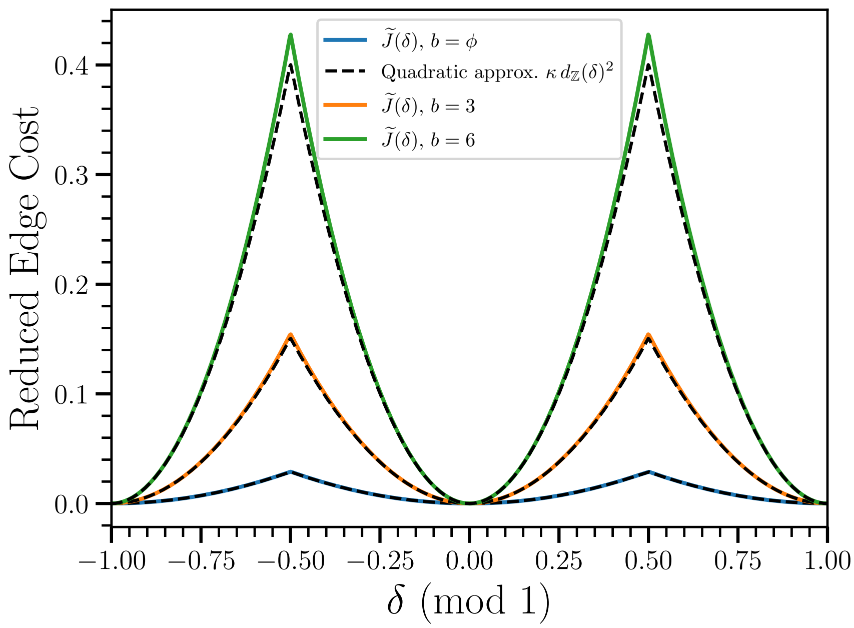

FIG. 1. The reduced periodic edge potential $\widetilde{J}_b(\delta)$ for three representative bases $b = 6$, 3, and $\phi = (1+\sqrt{5})/2$. Larger b produces steeper wells. All three curves are 1-periodic, have minima at the integers, and develop cusps at the half-integers because the minimizing lift in (B4) changes there.

Proposition B.4 (Local reduction versus graph-level quotient). *For every finite graph G and every phase field $\Theta \in (\mathbb{R}/\mathbb{Z})^V$,*

$$\widetilde{C}_{G,b}[\Theta] \le Q_{G,b}[\Theta]. \tag{B10}$$

If G is a tree, equality holds for every Θ. If G contains a cycle, equality need not hold.

Proof. For any $n \in \mathbb{Z}^V$ and any edge $\langle v, w\rangle$,

$$\cosh((\ln b)(\vartheta_v - \vartheta_w + n_v - n_w)) - 1 \ge \widetilde{J}_b(\vartheta_v - \vartheta_w)$$

by the definition (B4). Summing over edges and taking the infimum in n gives (B10).

Assume now that G is a tree. Choose an orientation of the edges away from a root v_0. For each oriented edge $e = (v, w)$, choose an integer m_e attaining the minimum in $\widetilde{J}_b(\vartheta_v - \vartheta_w)$. Set $n_{v_0} = 0$ and define n_w recursively by $n_v - n_w = m_e$ along every oriented edge. Since G has no cycles, this prescription is consistent and yields an integer field $n \in \mathbb{Z}^V$ realizing every edgewise minimum simultaneously. Hence $Q_{G,b}[\Theta] = \widetilde{C}_{G,b}[\Theta]$.

If G contains a cycle, equality can fail. On the triangle with phase lifts $(\vartheta_1, \vartheta_2, \vartheta_3) = (0, 1/3, 2/3)$, one has

$$\widetilde{C}_{G,b}[\Theta] = 3\big(\cosh((\ln b)/3) - 1\big). \tag{B11}$$

For the graph-level quotient, write the three lifted edge differences as

$$\Delta_{12} = -\frac{1}{3} + n_1 - n_2, \qquad \Delta_{23} = -\frac{1}{3} + n_2 - n_3, \qquad \Delta_{31} = \frac{2}{3} + n_3 - n_1,$$

so that $\Delta_{12} + \Delta_{23} + \Delta_{31} = 0$. Each Δ_{ij} lies in $\mathbb{Z} \pm 1/3$, so its absolute value is at least $1/3$. If all three absolute values were $1/3$, then each Δ_{ij} would equal $\pm 1/3$, but no choice of signs on three numbers of magnitude $1/3$ sums to zero. Hence at least one of the three absolute values is at least $2/3$. Since cosh is even and strictly increasing on $[0, \infty)$, the minimum is therefore attained when two edges have absolute value $1/3$ and the third has absolute value $2/3$, namely

$$Q_{G,b}[\Theta] = 2\big(\cosh((\ln b)/3) - 1\big) + \big(\cosh(2(\ln b)/3) - 1\big), \tag{B12}$$

which is strictly larger than (B11). □

Corollary B.5 (Finite-volume ground states of the reduced compact-phase model)**.** *If G is finite and connected, the minimizers of $\widetilde{C}_{G,b}$ are exactly the constant phase fields $\Theta_v \equiv \Theta_0$.*

Proof. By Proposition B.2, every edge term in (B8) is nonnegative and vanishes exactly when adjacent phases agree in $\mathbb{R}/\mathbb{Z}$. Connectedness propagates that agreement across the graph. □

Proposition B.6 (Small-gradient stiffness of the reduced interaction)**.** *For $|\delta| < 1/2$,*

$$\widetilde{J}_b(\delta) = \frac{(\ln b)^2}{2}\,\delta^2 + O(\delta^4) \qquad (\delta \to 0). \tag{B13}$$

Thus the harmonic stiffness is

$$\kappa_b = \frac{(\ln b)^2}{2}. \tag{B14}$$

Proof. If $|\delta| < 1/2$, then $d_{\mathbb{Z}}(\delta) = |\delta|$, so (B6) reduces to $\widetilde{J}_b(\delta) = \cosh((\ln b)\delta) - 1$. The expansion follows from the Taylor series of cosh. □

REFERENCES

[1]D. Ruelle, *Statistical Mechanics: Rigorous Results* (World Scientific, Singapore, 1999).

[2]B. Simon, *The Statistical Mechanics of Lattice Gases, Volume I* (Princeton University Press, Princeton, NJ, 1993).

[3]R. S. Ellis, *Entropy, Large Deviations, and Statistical Mechanics*, Grundlehren der mathematischen Wissenschaften, Vol. 271 (Springer, New York, 1985).

[4]R. T. Rockafellar, *Convex Analysis* (Princeton University Press, Princeton, NJ, 1970).

[5]S. Friedli and Y. Velenik, *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction* (Cambridge University Press, Cambridge, 2017).

[6]H.-O. Georgii, *Gibbs Measures and Phase Transitions*, 2nd ed. (De Gruyter, Berlin, 2011).

[7]H. J. Brascamp and E. H. Lieb, Journal of Functional Analysis **22**, 366 (1976).

[8]A. Naddaf and T. Spencer, Communications in Mathematical Physics **183**, 55 (1997).

[9]T. Funaki, in *Lectures on Probability Theory and Statistics*, Lecture Notes in Mathematics, Vol. 1869, edited by J. Picard (Springer, Berlin, 2005) pp. 1–136.

[10]Y. Velenik, Probability Surveys **3**, 112 (2006).

[11]M. Biskup and R. Kotecký, Probability Theory and Related Fields **139**, 1 (2007).

[12]S. Sheffield, Probability Theory and Related Fields **139**, 521 (2007).

[13]F. R. K. Chung, *Spectral Graph Theory*, CBMS Regional Conference Series in Mathematics, Vol. 92 (American Mathematical Society, Providence, RI, 1997).

[14]M. Fiedler, Czechoslovak Mathematical Journal **23**, 298 (1973).

[15]M. Keller, D. Lenz, and R. K. Wojciechowski, *Graphs and Discrete Dirichlet Spaces* (Springer, Cham, 2021).

[16]R. Olfati-Saber, J. A. Fax, and R. M. Murray, Proceedings of the IEEE **95**, 215 (2007).

[17]G. Friesecke, R. D. James, and S. Müller, Communications on Pure and Applied Mathematics **55**, 1461 (2002).

[18]N. Biggs, *Algebraic Graph Theory*, 2nd ed. (Cambridge University Press, Cambridge, 1993).