Article

d'Alembert's Functional Equation and a Globally Convex Free-Action Principle on Positive Paths

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Abstract

We study the kinetic action that d'Alembert's functional equation induces on positive paths in $\mathbb{R}_{>0}$, and prove that it is strongly convex. Calibrated d'Alembert forces the cosh cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$, equivalently $\widetilde{J}(\xi) = \cosh \xi - 1$ in the log coordinate $\xi = \log x$. Evaluating this log-cost at the log-*velocity* $\dot{\xi}$ rather than the log-position – a single named modeling postulate (Postulate 2.10) – yields the kinetic action $\mathcal{A}[\gamma] = \int_a^b (\cosh \dot{\xi} - 1)\, dt$, which is strongly convex under geometric (log-space) interpolation of paths. This convexity has three consequences, none of which requires an Euler–Lagrange equation, a Fréchet derivative, or a second variation. First, a one-sided chord condition characterizes global minimality. Second, the unique fixed-endpoint minimizer is the uniform-log-velocity path. Third, the action gap obeys an exact Bregman / Pythagorean identity $\mathcal{A}[\gamma] - \mathcal{A}[\gamma_*] = \int D_K(\dot{\xi} \,\|\, \dot{\xi}_*)\, dt$, sharpened by a quantitative Friedrichs–Poincaré bound on $\log(\gamma/\gamma_*)$. The whole package carries a dually-flat / Hessian-manifold reading in the additive coordinate ξ.
This convexity theorem is purely mathematical, and we are careful to delimit it. The bridge to Newtonian and rapidity mechanics is *conditional*, requiring structure beyond Postulate 2.10: a kinematic embedding, a mass coupling, a time calibration, and a Hamiltonian-primary Legendre structure. Granted these, the cosh action recovers the Newtonian small-step limit and the relativistic rapidity profile $K_m(\phi) = m(\gamma_L - 1)$; even so, the cosh-dual Hamiltonian is *not* the special-relativistic free-particle Hamiltonian (Proposition 6.9), the agreement being one of profile in rapidity rather than an identity of Hamiltonians. Global minimality, finally, is a free-sector phenomenon: once a non-affine strictly convex potential is added, joint convexity is lost and the classical local-minimum / stationary-action picture returns.

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1. Introduction

The principle of least action is normally introduced as a variational postulate [1]. In this work, we isolate a narrower mathematical question: can a functional equation, once paired with a single explicit step-evaluation postulate, select a kinetic action whose fixed-endpoint minimizer is global rather than merely stationary? We answer this question for positive paths. Throughout, we take the choice manifold to be the positive half-line $\mathbb{R}_{>0}$, interpreted as the space of comparison ratios in the cost-first ledger framework of [2,3], and written in logarithmic coordinate $\xi = \log x$. We do not re-derive this choice here. The geometric and information-theoretic content of $\mathbb{R}_{>0}$ as a Hessian manifold is developed in the multidimensional cost-geometry paper [4]; the cost-first ledger interpretation of $x \in \mathbb{R}_{>0}$ as a ratio of ledger entries (with $x = 1$ the equilibrium "no comparison required" state) is the subject of [2]. We take both upstream choices as inputs to our free-sector analysis, not as consequences of it.

Our input is d'Alembert's functional equation

$$H(t+u) + H(t-u) = 2\,H(t)\,H(u), \qquad t, u \in \mathbb{R}. \tag{1}$$

With the calibration $H(0) = 1$ and $H''(0) = 1$, the continuous classification gives $H(t) = \cosh t$ [5,6]. Thus the induced cost on $\mathbb{R}_{>0}$ is

$$J(x) = \cosh(\log x) - 1 = \tfrac{1}{2}(x + x^{-1}) - 1.$$

The dynamical content of the work then rests on a single modeling choice, formalized below as Postulate 2.10: the d'Alembert log-cost $\widetilde{J}$ is evaluated at the log-*velocity* $\dot{\xi}$, not at the log-position ξ. This is a modeling postulate, not a theorem of d'Alembert; (1) fixes the *function* $\widetilde{J}$ but says nothing about its argument. The motivation for the postulate is structural: d'Alembert's equation is a composition law for additive steps (rapidity addition, Remark 6.3), so the natural object the cost applies to is the infinitesimal log-step $\dot{\xi} = d(\log \gamma)/dt$. Applying $\widetilde{J}$ to that step gives the kinetic integrand

$$K(\dot{\xi}) = \widetilde{J}(\dot{\xi}) = \cosh(\dot{\xi}) - 1.$$

Our main result is then a convexity theorem. On the class of kinetically admissible positive paths, the kinetic action

$$\mathcal{A}[\gamma] = \int_a^b K(\dot{\xi}(t))\,dt, \qquad \xi = \log \gamma,$$

is *strongly* convex under geometric, equivalently log-space, interpolation of paths, with explicit L^2 slack $\frac{s(1-s)}{2}\|\dot{\xi}_1 - \dot{\xi}_2\|_{L^2}^2$. Consequently, a one-sided chord condition implies global minimality without invoking Euler–Lagrange equations, Fréchet derivatives, or second variations. More concretely, for fixed endpoints $x_a, x_b > 0$ the unique global minimizer is

$$\gamma_*(t) = \exp\!\Big(\log x_a + \frac{t-a}{b-a}(\log x_b - \log x_a)\Big),$$

the uniform-log-velocity path. The action gap admits an exact Bregman / Pythagorean decomposition [7]

$$\mathcal{A}[\gamma] - \mathcal{A}[\gamma_*] \;=\; \int_a^b D_K\big(\dot{\xi}(t)\,\|\,\dot{\xi}_*\big)\,dt \;\geq\; \tfrac{1}{2}\|\dot{\xi} - \dot{\xi}_*\|_{L^2}^2 \;\geq\; \frac{\pi^2}{2(b-a)^2}\,\|\log(\gamma/\gamma_*)\|_{L^2}^2,$$

where $D_K(v\|w) = \cosh v - \cosh w - \sinh w\,(v - w)$ is the cosh Bregman divergence. The minimum-action profile $\mathcal{A}_*(T, \Delta) = T(\cosh(\Delta/T) - 1)$ is jointly convex and positively 1-homogeneous, hence subadditive in time. These quantitative refinements admit a clean dually-flat / Hessian-manifold reading [8,9] in the additive coordinate $\xi = \log x$: the log-cost $\widetilde{J}(\xi) = \cosh \xi - 1$ is the Hessian potential of a 1D dually-flat structure with metric $\cosh \xi\, d\xi^2$, whose e-affine (log-affine) geodesics are exactly the geometric interpolations, and $\mathcal{A}$ is the corresponding Bregman (cosh-)energy. This log-coordinate structure is distinct from the Hessian metric $g_J = x^{-3}\,dx^2$ of J in the coordinate x (Appendix A).

We confine this global-minimizer statement to the free sector. Once a non-affine strictly convex potential is added, joint convexity of the Lagrangian is lost and the classical local-minimum / stationary-action picture reappears. We therefore make the later bridge to Newtonian mechanics explicitly conditional: it requires a kinematic embedding, a mass coupling, a time calibration, and a Hamiltonian-primary Legendre structure. These choices explain how the cosh kinetic cost has the Newtonian small-step limit and the relativistic rapidity profile, but they are not part of the free convexity theorem itself.

We retain the velocity-free integral $\mathcal{J}[\gamma] = \int J(\gamma(t))\,dt$ only as a static cost (Section 3); its integrand has no velocity dependence and its Euler–Lagrange equation is algebraic, selecting the ground state $\gamma \equiv 1$ (Section 5). We defer the comparison of $\mathcal{A}$ with the Hessian Riemannian path-energy on $(\mathbb{R}_{>0}, g_J)$ to Appendix A.

Nature of the contribution. The free-sector arguments are deliberately elementary: once d'Alembert fixes the cosh cost and Postulate 2.10 places it at the log-velocity, the convexity results follow from pointwise convexity of cosh after the log change of coordinates, Jensen's inequality, a perspective construction [10,11], the one-dimensional Friedrichs inequality [12], and textbook Bregman / dually-flat geometry [7–9]. The novelty we claim is therefore one of *provenance and organization*, not of analytic depth: a functional equation together with a single named postulate forces a globally (not merely locally) minimizing free action with an exact Bregman gap and a closed-form geodesic minimizer. The fuller scope statement, including the conditional status of the mechanics bridge, is given in Section 9.

1.1. Structure of the paper

The paper has two parts. The free-sector mathematical core occupies Sections 2–5, and the conditional physical bridge occupies Sections 6–8.

- §2 recalls the calibrated d'Alembert classification and introduces the costs $J, \widetilde{J}, K$.
- §3 sets up positive path spaces, $\mathcal{A}$, and the geometric and arithmetic interpolations.
- §4 proves the strong convexity theorem, the fixed-endpoint minimizer, the Pythagorean / Bregman identity with its Friedrichs–Poincaré bound, the perspective convexity of the minimum-action profile, and the dually flat reformulation.
- §5 compares the static, kinetic, and Hessian Euler–Lagrange pictures.
- §6 derives Newton's law from the cosh Lagrangian.
- §7 gives the Hamiltonian formulation and the Noether conservation laws.
- §8 records the classical local-min / stationary-action picture for general potentials.

Section 9 summarizes the results and lists open directions, and Appendix A collects the comparison material on the Hessian Riemannian path-energy $\mathcal{E}_{\mathrm{Hess}}$.

Remark 1.1 (Scope of the framework: what is and is not claimed)**.** Because the framework draws on several traditions (d'Alembert's functional equation, Bregman / dually flat geometry, Newtonian mechanics, special relativity), it is worth stating up front what each piece does and does not assert.

- *Two velocity notions.* Under the kinematic embedding of Definition 6.1, the coordinate velocity $\dot{q}$ equals the boost rapidity ϕ (in natural units), *not* the SR 3-velocity $v_{\mathrm{SR}} = c\tanh\phi$. The two agree only at leading order, $\phi = v_{\mathrm{SR}}/c + O((v_{\mathrm{SR}}/c)^3)$; for large rapidity v_{SR} saturates at $\pm c$ while $\dot{q} = \phi$ remains unbounded.
- *Cosh-dual Hamiltonian is not the SR Hamiltonian.* The Hamiltonian $T_H(p) = p\,\mathrm{arsinh}(p/m) - \sqrt{m^2+p^2} + m$ of §7 is the Legendre dual of the cosh kinetic cost. It is *not* the SR free-particle Hamiltonian $\sqrt{m^2+p^2} - m$; the two agree only at $O(p^2)$ and differ strictly at $O(p^4)$, with $T_H(p) > \sqrt{m^2+p^2} - m$ for all $p \neq 0$ (Proposition 6.9). The Proposition 6.5 match is one of *profile* in rapidity, not Hamiltonian identity.
- *General potentials are external.* The native cosh-sinh potential $V_{\mathrm{nat}} = k\widetilde{J}$ is the only potential forced by the same d'Alembert uniqueness that forces the kinetic term. All

other potentials (harmonic, Coulomb, gravitational, polynomial, lattice, periodic, etc.) are external inputs from the cost-field environment; we do not derive them from (1).

- *Global minimality is a free-sector claim.* The chord-form free-action principle (Theorem 4.7) gives global minimality of $\mathcal{A}$ on kinetically admissible positive paths sharing endpoints. Once a non-affine strictly convex potential is added, the Lagrangian becomes indefinite in $(\xi, \dot{\xi})$ and the classical short-time local-minimum / long-time stationary-action / conjugate-time obstruction picture replaces global minimality (§8).

2. The Cost Functional, Log Coordinates, and the Kinetic Cost

This section assembles the cost functions on which the rest of the paper is built. We first recall the calibrated d'Alembert classification, then introduce the static cost J on $\mathbb{R}_{>0}$, its logarithmic form $\widetilde{J}$, and the kinetic cost K obtained by evaluating $\widetilde{J}$ at the log-velocity (Postulate 2.10). We begin by fixing notation.

Remark 2.1 (Notation glossary)**.** For convenience we list here the principal symbols used in the cost, action, Lagrangian, and Hamiltonian layers of the paper, with a pointer to where each is first introduced. The free-sector symbols (J through $\mathcal{A}_*$) are dimensionless throughout; the mechanics-layer symbols (K_m onward) carry the dimensions recorded in the dimensional dictionary (Remark 6.2).

Symbol	Role	First introduced
$J(x)$	static cost on $\mathbb{R}_{>0}$, $\frac{1}{2}(x + x^{-1}) - 1$	§2
$\widetilde{J}(\xi)$	log-cost, $\cosh \xi - 1$	§2
$K(v)$	kinetic cost, $\widetilde{J}(v) = \cosh v - 1$	§2, Postulate 2.10
$\mathcal{J}[\gamma]$	static integral $\int_a^b J(\gamma)\, dt$	§3
$\mathcal{A}[\gamma]$	kinetic action $\int_a^b K(\dot{\xi})\, dt$	§3
$\mathcal{A}_*(T, \Delta)$	minimum-action profile, $T(\cosh(\Delta/T) - 1)$	§4.6
$\mathcal{E}_{\mathrm{Hess}}[\gamma]$	Hessian Riemannian path-energy (distinct from $\mathcal{A}$)	§5.2, App. A
$K_m(\phi)$	physical kinetic cost, $m(\cosh \phi - 1)$	§6, Def. 6.4
$\mathcal{L}(\xi, \dot{\xi})$	cosh Lagrangian, $K_m(\dot{\xi}) - V(\xi)$	§6.4
L_{nat}	native d'Alembert Lagrangian	Def. 6.11
$T_H(p)$	cosh-dual Hamiltonian ($\neq \sqrt{m^2 + p^2} - m$, Prop. 6.9)	Prop. 6.8
$H(\xi, p)$	Hamiltonian, $T_H(p) + V(\xi)$	Def. 6.7

The kinematic symbols (ξ, ϕ, v_{SR}, q, τ, t_0) used in the mechanics layer are catalogued separately in Remark 6.2. Two symbols carry conventional double duty within standard physics usage but are kept distinct here by notation: the Hessian path-energy is written $\mathcal{E}_{\mathrm{Hess}}$, never E (reserved for the conserved energy along a trajectory in §7); and the d'Alembert solution function H of (1) (Theorem 2.2) appears only inside §2, never in the same equation as the Hamiltonian $H(\xi, p)$ of §7.

2.1. Calibrated d'Alembert classification

Theorem 2.2 (Calibrated d'Alembert classification, recalled)**.** *Let $H : \mathbb{R} \to \mathbb{R}$ be a continuous solution of the d'Alembert functional equation* (1)*.*

(i) ***Smoothness (Aczél).*** *By Aczél's smoothness theorem for d'Alembert's equation [5, Ch. 3, §3.1.3], continuity of H on $\mathbb{R}$ implies $H \in C^\infty(\mathbb{R})$. This is the only external bridge result used in the classification.*

(ii) ***Classification.*** *Every continuous solution of* (1) *is exactly one of the following disjoint alternatives:*

$$H(t) \equiv 0, \qquad H(t) = \cos(\alpha t)\ (\alpha > 0), \qquad H(t) = \cosh(\alpha t)\ (\alpha > 0), \qquad H(t) \equiv 1.$$

The zero branch occurs when $H(0) = 0$*. If* $H(0) = 1$*, then* H *is even,* $H'(0) = 0$*, and differentiating* (1) *twice in the second variable at* 0 *yields* $H''(t) = H''(0)H(t)$*. The sign of* $H''(0)$ *gives the cosine, cosh, or constant branch.*

(iii) ***Calibration to*** **cosh.** *The condition* $H(0) = 1$ *excludes the zero branch, and the condition* $H''(0) > 0$ *excludes the cosine and constant branches, forcing* $H(t) = \cosh(\alpha t)$ *for some* $\alpha > 0$*. Fixing the unit of cost by the further normalization* $H''(0) = 1$ *then selects* $\alpha = 1$*, hence* $H(t) = \cosh t$*.*

Remark 2.3 ($H''(0) = 1$ is a unit choice, not forced by d'Alembert)**.** d'Alembert's equation (1) together with $H(0) = 1$ and $H''(0) > 0$ forces $H(t) = \cosh(\alpha t)$ only *up to* the scale parameter $\alpha = \sqrt{H''(0)} > 0$. The choice $\alpha = 1$ we adopt throughout is a normalization of the cost scale, equivalent to choosing the unit in which the kinetic coefficient $K''(0) = \cosh 0 = 1$ at the ground state. Any other choice $\alpha > 0$ produces the same theory under the rescaling $\xi \mapsto \alpha\xi$, $K \mapsto \cosh(\alpha\,\cdot) - 1$. No qualitative result here depends on the value of α; only the numerical action constants do.

Remark 2.4 (Aczél's smoothness theorem, restated for self-containedness)**.** For ease of reference and completeness, we restate the precise statement of the single external bridge result on which Theorem 2.2(i) relies.

(Aczél) [5, Ch. 3, §3.1.3]. *Let* $H : \mathbb{R} \to \mathbb{R}$ *be a continuous, not identically zero solution of d'Alembert's equation* (1)*. Then* H *is infinitely differentiable on* $\mathbb{R}$*.*

The proof in Aczél proceeds by a translation-and-averaging argument. Setting $u = t$ in (1) gives the algebraic identity

$$H(2t) + H(0) \;=\; 2H(t)^2,$$

which expresses H on a dilated argument as a polynomial in $H(t)$ and so propagates information about H to all dyadic scales; this identity by itself does not raise regularity. The regularity bootstrap itself comes from integrating (1) in u over $[0, h]$. Choosing $h > 0$ small enough that $C_h := \int_0^h H(u)\,du \neq 0$ (possible on the nontrivial branch, where $H(0) = 1$ and H is continuous), the substitutions $s = t \pm u$ give

$$2\,C_h\,H(t) \;=\; \int_t^{t+h} H(s)\,ds + \int_{t-h}^{t} H(s)\,ds.$$

The right-hand side is C^1 in t by the fundamental theorem of calculus, hence so is H; iterating, if $H \in C^k$ then the right-hand side is C^{k+1}, hence $H \in C^{k+1}$, so $H \in C^\infty$. The upshot is that $H'(0)$ and $H''(0)$ both exist as ordinary derivatives, which is exactly what the calibration in Theorem 2.2(iii) requires. This is the only place in our development where we invoke a continuity-to-smoothness upgrade from outside; everything downstream stays inside C^∞.

2.2. The static cost functional

Definition 2.5 (Cost functional)**.** The *cost functional* is

$$J(x) \;=\; \tfrac{1}{2}\Big(x + x^{-1}\Big) - 1, \qquad x > 0.$$

J is non-negative, vanishes iff $x = 1$, is symmetric under inversion ($J(x) = J(1/x)$), and diverges at 0 and ∞.

Theorem 2.6 (Strict convexity of J)**.** *J is strictly convex on* $(0, \infty)$*.*

Proof. $J''(x) = x^{-3} > 0$ for $x > 0$. □

2.3. Logarithmic coordinates

Set $\xi := \log x$, so that $x = e^{\xi}$ and ξ ranges over $\mathbb{R}$.

Definition 2.7 (Log-cost functional)**.** The *log-cost functional* is

$$\widetilde{J}(\xi) \;:=\; J(e^{\xi}) \;=\; \tfrac{1}{2}(e^{\xi}+e^{-\xi})-1 \;=\; \cosh(\xi)-1.$$

Proposition 2.8 (Properties of $\widetilde{J}$)**.** *(i)* *$\widetilde{J}(\xi) \geq 0$ with equality iff $\xi = 0$.*

(ii) *$\widetilde{J}$ is even: $\widetilde{J}(-\xi) = \widetilde{J}(\xi)$, which is the log-form of $J(x) = J(1/x)$.*

(iii) *$\widetilde{J}$ is strictly convex on $\mathbb{R}$, with $\widetilde{J}''(\xi) = \cosh(\xi) \geq 1 > 0$.*

(iv) *Small-ξ Taylor: $\widetilde{J}(\xi) = \frac{1}{2}\xi^2 + \frac{1}{24}\xi^4 + O(\xi^6)$.*

Proof. All from $\widetilde{J}(\xi) = \cosh(\xi) - 1$ and the standard Taylor series of cosh. □

Remark 2.9. The passage $J \leftrightarrow \widetilde{J}$ via $\xi = \log x$ is the natural one for d'Alembert. In the additive variable ξ, (1) is precisely the cosh-composition identity $\cosh(\xi+\eta)+\cosh(\xi-\eta) = 2\cosh\xi\cosh\eta$, so $\widetilde{J}$ is the d'Alembert-calibrated cost *evaluated at an additive argument*. The remainder of our analysis lives naturally in these coordinates.

2.4. The kinetic cost

The classification of d'Alembert solutions (Theorem 2.2) fixes the *function* $\widetilde{J}(\cdot) = \cosh(\cdot) - 1$, but it does not specify which argument that function takes on. Two natural choices arise on a positive path $\gamma : [a,b] \to \mathbb{R}_{>0}$ with $\xi = \log\gamma$: the log-*position* $\xi(t)$ and the log-*velocity* $\dot{\xi}(t) = d(\log\gamma)/dt$. We make the second choice explicit as a postulate, since it is what generates the dynamics of this paper.

Postulate 2.10 (Step-evaluation of the d'Alembert log-cost)**.** Along a positive path $\gamma : [a,b] \to \mathbb{R}_{>0}$ with $\xi = \log\gamma$, the d'Alembert log-cost $\widetilde{J}$ is evaluated at the log-*velocity* $\dot{\xi}$, not at the log-position ξ. The cost per unit parameter time of motion with log-velocity $\dot{\xi}$ is therefore

$$\widetilde{J}(\dot{\xi}) \;=\; \cosh(\dot{\xi}) - 1.$$

Remark 2.11 (Status and consequences of Postulate 2.10)**.** Postulate 2.10 is a *modeling choice*, not a consequence of d'Alembert's equation: (1) fixes the function $\widetilde{J}$ but says nothing about its argument. Its motivation is structural – d'Alembert's equation is the composition law for additive *steps* (rapidity addition, in the kinematic reading of §6, Remark 6.3), and the canonical additive object on a positive path is the infinitesimal log-step $\dot{\xi}$. We do not claim this motivation forces the postulate.

Evaluating $\widetilde{J}$ at the log-*position* ξ instead gives the velocity-free static cost integrand of §3, whose Euler–Lagrange equation is algebraic (Remark 5.2) and produces no dynamics. The strongly convex kinetic action of §4, and with it the entire free-sector global-minimum result of this paper, is downstream of Postulate 2.10. The same $\widetilde{J}$ also reappears at log-position in §6 as the native potential V_{nat} once a Hamiltonian-primary additive cost postulate is added; until then, only the velocity reading is in force.

Definition 2.12 (Kinetic cost)**.** Under Postulate 2.10, the *kinetic cost* is the real-valued function $K : \mathbb{R} \to \mathbb{R}$,

$$K(v) \;:=\; \widetilde{J}(v) \;=\; \cosh(v) - 1.$$

The value $K(\dot{\xi})$ is the d'Alembert-calibrated cost per unit parameter time of motion with log-velocity $\dot{\xi}$. In the physical bridge of Section 6, this parameter is calibrated to the dimensionless time $\tau = t_{\mathrm{phys}}/t_0$; before that bridge, t is only the variational parameter on the path.

Proposition 2.13 (Properties of K)**.** *(i)* *$K(v) \geq 0$ with equality iff $v = 0$ (zero log-velocity).*

(ii) *K is even.*

(iii) *K is strictly convex on $\mathbb{R}$, with $K''(v) = \cosh(v) \geq 1$.*

(iv) ***Sharp quartic remainder.** For all $|v| < 1$,*

$$\left|K(v) - \tfrac{1}{2}v^2\right| \;\leq\; \frac{v^4}{24\,(1-v^2)}. \tag{2}$$

In particular, $K(v) - \frac{1}{2}v^2 = \frac{v^4}{24} + O(v^6)$ as $v \to 0$, and the leading constant $1/24$ is sharp.

Proof. (i)–(iii) are immediate from $K(v) = \cosh(v) - 1$ and $K''(v) = \cosh(v) \geq 1$.

For (iv), the cosh Taylor series gives

$$K(v) - \tfrac{1}{2}v^2 \;=\; \sum_{k \geq 2} \frac{v^{2k}}{(2k)!}.$$

All terms are non-negative, so the difference is non-negative (the absolute value can be dropped). For each $k \geq 2$, $(2k)! \geq 4! = 24$ and $v^{2k} = v^4 \cdot (v^2)^{k-2}$, so $v^{2k}/(2k)! \leq v^4(v^2)^{k-2}/24$. Summing the resulting geometric series in v^2 (valid for $|v| < 1$),

$$0 \;\leq\; \sum_{k \geq 2} \frac{v^{2k}}{(2k)!} \;\leq\; \frac{v^4}{24} \sum_{j \geq 0} (v^2)^j \;=\; \frac{v^4}{24\,(1-v^2)},$$

which is (2). Sharpness of the leading $1/24$ follows from the $k = 2$ term: as $v \to 0$, $K(v) - \frac{1}{2}v^2 \sim v^4/24$. □

Remark 2.14 (Rapidity interpretation)**.** If we read the dimensionless step variable $\phi := d\xi/d\tau$ as the rapidity of a one-parameter boost, then $K(\phi) = \cosh(\phi) - 1$ is the excess cosh of rapidity – the relativistic kinetic-energy profile. d'Alembert's equation (1) is the rapidity-addition law, so this profile is internal to the d'Alembert framework. The dynamical identification of ϕ with physical rapidity is the kinematic calibration we make in Definition 6.1.

3. Path Space, Static and Kinetic Actions, and Interpolation

With the cost functions J, $\widetilde{J}$, and K now in hand, we turn to the spaces of paths on which they act. This section fixes the admissibility classes used throughout the paper, defines the static cost integral $\mathcal{J}$ and the kinetic action $\mathcal{A}$, and introduces the two interpolations – arithmetic and geometric – whose contrast drives the convexity theory of Section 4.

3.1. Admissible paths

Definition 3.1 (Admissible path)**.** For $a, b \in \mathbb{R}$ with $a < b$, an *admissible path on* $[a, b]$ is a function $\gamma : [a, b] \to \mathbb{R}_{>0}$ that is absolutely continuous on $[a, b]$, satisfies $\gamma(t) > 0$ for every $t \in [a, b]$, and has $\dot{\gamma} \in L^2([a, b])$.

Remark 3.2 (Automatic uniform positivity)**.** Absolute continuity on the compact interval $[a, b]$ implies continuity, so the strict positivity condition $\gamma(t) > 0$ on the compact $[a, b]$ together with continuity yields a positive lower bound: there exists $c_\gamma > 0$ such that $\gamma(t) \geq c_\gamma$ for all $t \in [a, b]$. We therefore use "strictly positive on $[a, b]$" and "bounded away from 0" interchangeably for admissible paths.

Strict positivity is essential: J is defined only on $(0, \infty)$, and we will need $\log \gamma$ to be absolutely continuous with L^2 derivative. The log-coordinate of an admissible path, $\xi(t) := \log \gamma(t)$, is absolutely continuous on $[a, b]$ with $\dot{\xi} = \dot{\gamma}/\gamma \in L^2$ (using Remark 3.2 to bound $1/\gamma$ above by $1/c_\gamma < \infty$).

Definition 3.3 (Finite-action (kinetically admissible) path)**.** An admissible path γ on $[a, b]$ is *kinetically admissible* (or *finite-action*) if, writing $\xi := \log \gamma$, one has $K(\dot{\xi}) \in L^1([a, b])$, equivalently

$$\int_a^b \big[\cosh(\dot{\xi}(t)) - 1\big]\, dt < \infty.$$

Remark 3.4. The condition $\dot{\xi} \in L^2([a, b])$ does *not* by itself imply $K(\dot{\xi}) \in L^1([a, b])$, since cosh has super-quadratic growth. We therefore treat the kinetic action $\mathcal{A}$ below as a real-valued functional on the class of kinetically admissible paths.

3.2. The static cost integral

Definition 3.5 (Static cost integral)**.** For an admissible path γ on $[a, b]$, the *static cost integral* is

$$\mathcal{J}[\gamma] \;:=\; \int_a^b J(\gamma(t))\, dt.$$

Proposition 3.6 (Basic properties of $\mathcal{J}$)**.** *$\mathcal{J}[\gamma] \geq 0$, and $\mathcal{J}[\gamma_{\text{const}=1}] = 0$.*

Proof. An admissible path is continuous on the compact interval $[a, b]$ and takes values in $(0, \infty)$, hence $J \circ \gamma$ is continuous and therefore integrable. Since $J(x) \geq 0$ for all $x > 0$, its integral is non-negative. If $\gamma \equiv 1$, then $J(\gamma(t)) = J(1) = 0$ for all t, so the integral is zero. □

$\mathcal{J}$ measures the time-integrated cost of *being in* a non-ground state and plays a role analogous to a potential term in the multiplicative coordinate. It is not an action in the dynamical sense: its integrand has no velocity dependence and its Euler–Lagrange equation is algebraic (Section 5).

3.3. The kinetic action

Definition 3.7 (Kinetic action)**.** For a kinetically admissible path γ on $[a, b]$, with $\xi := \log \gamma$, the *kinetic action* is

$$\mathcal{A}[\gamma] \;:=\; \int_a^b K(\dot{\xi}(t))\, dt \;=\; \int_a^b \big[\cosh(\dot{\xi}(t)) - 1\big]\, dt.$$

Proposition 3.8 (Basic properties of $\mathcal{A}$)**.** *(i)* $\mathcal{A}[\gamma] \geq 0$.

(ii) *$\mathcal{A}[\gamma] = 0$ iff γ is constant on $[a, b]$ (any positive constant; not only $\gamma \equiv 1$).*

(iii) *If $\|\dot{\xi}\|_\infty \leq 1/10$, then the kinetic action approximates the Newtonian L^2 integral with relative error at most $1/1188$:*

$$\Big|\mathcal{A}[\gamma] - \tfrac{1}{2}\int_a^b \dot{\xi}(t)^2\, dt\Big| \;\leq\; \tfrac{100}{99\cdot 24}\int_a^b \dot{\xi}(t)^4\, dt \;\leq\; \tfrac{1}{2376}\int_a^b \dot{\xi}(t)^2\, dt.$$

Proof. (i) $K \geq 0$ pointwise. (ii) $K(\dot{\xi}) = 0$ a.e. $\iff$ $\dot{\xi} = 0$ a.e., and absolute continuity of ξ gives ξ constant, equivalently γ constant. (iii) Apply the sharp quartic remainder (2) pointwise at $v = \dot{\xi}(t)$ with $|v| \leq 1/10$, so that $(1 - v^2)^{-1} \leq 100/99$ and $v^4 \leq v^2/100$; integration yields both bounds. □

3.4. Two interpolations

The variational theorems below require an interpolation between two admissible paths. Two natural candidates arise: *arithmetic* interpolation (appropriate for $\mathcal{J}$) and *geometric* (log- space) interpolation (appropriate for $\mathcal{A}$).

Definition 3.9 (Arithmetic interpolation)**.** For admissible γ_1, γ_2 on $[a, b]$ and $s \in [0, 1]$,

$$\operatorname{interp}^{+}(\gamma_1, \gamma_2, s)(t) \;:=\; (1-s)\,\gamma_1(t) + s\,\gamma_2(t).$$

Definition 3.10 (Geometric (log-space) interpolation)**.** For admissible γ_1, γ_2 on $[a, b]$ and $s \in [0, 1]$,

$$\operatorname{interp}^{\times}(\gamma_1, \gamma_2, s)(t) \;:=\; \gamma_1(t)^{1-s}\,\gamma_2(t)^{s} \;=\; \exp\bigl((1-s)\log\gamma_1(t) + s\log\gamma_2(t)\bigr).$$

Equivalently, in log coordinates, $\xi_s(t) = (1-s)\xi_1(t) + s\xi_2(t)$.

Lemma 3.11 (Both interpolations preserve admissibility)**.** *If γ_1, γ_2 are admissible on $[a, b]$ and $s \in [0, 1]$, then $\operatorname{interp}^{+}(\gamma_1, \gamma_2, s)$ and $\operatorname{interp}^{\times}(\gamma_1, \gamma_2, s)$ are admissible on $[a, b]$.*

Proof. For $\operatorname{interp}^{+}$, absolute continuity and the L^2 derivative condition are preserved under linear combinations, and positivity holds since $(1-s)\gamma_1 + s\gamma_2 \geq \min(\gamma_1, \gamma_2) > 0$.

For $\operatorname{interp}^{\times}$, write $\xi_i := \log\gamma_i$. Since each γ_i is bounded away from 0 and absolutely continuous, ξ_i is absolutely continuous with $\dot{\xi}_i = \dot{\gamma}_i/\gamma_i \in L^2([a, b])$. Then $\xi_s := (1-s)\xi_1 + s\xi_2$ is absolutely continuous with $\dot{\xi}_s \in L^2$, and $\gamma_s := \operatorname{interp}^{\times}(\gamma_1, \gamma_2, s) = e^{\xi_s}$ is absolutely continuous with $\dot{\gamma}_s = \gamma_s\,\dot{\xi}_s$ a.e. Since γ_1, γ_2 are bounded away from 0, there exist $c_1, c_2 > 0$ with $\gamma_i(t) \geq c_i$ on $[a, b]$, hence $\gamma_s(t) = \gamma_1(t)^{1-s}\gamma_2(t)^s \geq c_1^{1-s}c_2^s > 0$, so γ_s is bounded away from 0. On the compact interval $[a, b]$, γ_s is continuous and strictly positive, hence also bounded above, so $\dot{\gamma}_s = \gamma_s\dot{\xi}_s \in L^2([a, b])$. □

Lemma 3.12 (Geometric interpolation preserves finite kinetic action)**.** *If γ_1, γ_2 are kinetically admissible on $[a, b]$ and $s \in [0, 1]$, then $\operatorname{interp}^{\times}(\gamma_1, \gamma_2, s)$ is kinetically admissible on $[a, b]$.*

Proof. Let $\xi_i = \log\gamma_i$ and $\xi_s = (1-s)\xi_1 + s\xi_2$, so $\dot{\xi}_s = (1-s)\dot{\xi}_1 + s\dot{\xi}_2$ a.e. By convexity of K,

$$K(\dot{\xi}_s(t)) \leq (1-s)K(\dot{\xi}_1(t)) + sK(\dot{\xi}_2(t)) \qquad \text{for a.e. } t.$$

The right-hand side lies in $L^1([a, b])$ by kinetic admissibility of γ_1, γ_2, hence so does $K(\dot{\xi}_s)$. □

4. Convexity and the Free-Action Principle

We now reach the mathematical core of the paper. Having assembled the cost functions and the path spaces, we establish the convexity properties of the static and kinetic actions, derive the free-action principle in chord form, and extract its quantitative consequences: the closed-form fixed-endpoint minimizer, the Bregman–Pythagorean identity with its Friedrichs–Poincaré bound, the perspective convexity of the minimum-action profile, and the dually flat reformulation.

4.1. Static convexity

Theorem 4.1 (Convexity of $\mathcal{J}$ under arithmetic interpolation)**.** *For admissible γ_1, γ_2 on $[a, b]$ and $s \in [0, 1]$,*

$$\mathcal{J}[\operatorname{interp}^{+}(\gamma_1, \gamma_2, s)] \;\leq\; (1-s)\,\mathcal{J}[\gamma_1] + s\,\mathcal{J}[\gamma_2].$$

Proof. Pointwise convexity of J (Theorem 2.6): at every t, $J((1-s)\gamma_1(t)+s\gamma_2(t)) \leq (1-s)J(\gamma_1(t))+sJ(\gamma_2(t))$. Integrate. □

4.2. Kinetic convexity (the central theorem)

The kinetic action $\mathcal{A}$ is *not* in general convex under arithmetic interpolation in γ. The correct interpolation is geometric: in log coordinates, this is affine, and convexity of the integrand $K(\dot{\xi})$ in $\dot{\xi}$ propagates immediately.

Theorem 4.2 (Strong convexity of $\mathcal{A}$ under geometric interpolation)**.** *For kinetically admissible γ_1, γ_2 on $[a,b]$, write $\xi_i := \log \gamma_i$. For every $s \in [0,1]$,*

$$\mathcal{A}[\mathrm{interp}^{\times}(\gamma_1,\gamma_2,s)] \;\leq\; (1-s)\,\mathcal{A}[\gamma_1] + s\,\mathcal{A}[\gamma_2] \;-\; \frac{s(1-s)}{2}\int_a^b \left(\dot{\xi}_1(t)-\dot{\xi}_2(t)\right)^2 dt. \quad (3)$$

In particular, $\mathcal{A}$ is convex along geometric interpolation, and inequality (3) is strict for $s \in (0,1)$ unless $\dot{\xi}_1 = \dot{\xi}_2$ a.e. on $[a,b]$; equivalently, γ_2/γ_1 is constant on $[a,b]$.

Proof. Set $\xi_s := (1-s)\xi_1 + s\xi_2$, so ξ_s is the log of $\mathrm{interp}^{\times}(\gamma_1,\gamma_2,s)$ and $\dot{\xi}_s = (1-s)\dot{\xi}_1 + s\dot{\xi}_2$ a.e. The kinetic integrand $K(v) = \cosh(v) - 1$ satisfies $K''(v) = \cosh(v) \geq 1$ on $\mathbb{R}$, hence is 1-strongly convex: for all $v_1, v_2 \in \mathbb{R}$ and $s \in [0,1]$, with $v_s := (1-s)v_1 + sv_2$,

$$K(v_s) \;\leq\; (1-s)\,K(v_1) + s\,K(v_2) \;-\; \tfrac{s(1-s)}{2}\,(v_1-v_2)^2. \quad (4)$$

Indeed, Taylor's theorem at v_s applied separately to v_1 and v_2 gives, for some ζ_i between v_s and v_i, $K(v_i) = K(v_s) + K'(v_s)(v_i - v_s) + \frac{1}{2}\,K''(\zeta_i)(v_i - v_s)^2$. Combining with weights $(1-s)$ and s, the linear $K'(v_s)$ contribution vanishes (because v_s is the weighted mean), and the remainders are bounded below by $\frac{1}{2}(v_i - v_s)^2$ via $K'' \geq 1$. Using $v_1 - v_s = s(v_1 - v_2)$ and $v_2 - v_s = (1-s)(v_2 - v_1)$, the quadratic floor sums to $\frac{1}{2}s(1-s)(v_1-v_2)^2$, which yields (4).

Apply (4) pointwise at $v_i = \dot{\xi}_i(t)$ for a.e. t. The right-hand side is integrable: the kinetic terms by kinetic admissibility of γ_1, γ_2 (Definition 3.3), and the quadratic term because $\dot{\xi}_i \in L^2([a,b])$ (Definition 3.1). Lemma 3.12 gives $K(\dot{\xi}_s) \in L^1$. Integrating yields (3).

For the equality clause, suppose equality holds in (3) for some $s_0 \in (0,1)$. Then the pointwise inequality (4), applied at $v_i = \dot{\xi}_i(t)$, must hold with equality for a.e. $t \in [a,b]$: if the pointwise gap were strictly positive on a set of positive measure, integration would give a strict integral gap, contradicting integral equality. We now show that pointwise equality in (4) at $s_0 \in (0,1)$ forces $v_1 = v_2$.

Define, for $v, w \in \mathbb{R}$, the non-negative quantity

$$\Delta(v,w) \;:=\; K(v) - K(w) - K'(w)(v-w) \;=\; \int_w^v (v-u)\cosh(u)\,du,$$

by Taylor's theorem with integral remainder for K at w. The identity $(1-s_0)(v_1 - v_s) + s_0(v_2 - v_s) = 0$ kills the linear term in the weighted combination, giving the exact identity

$$(1-s_0)K(v_1) + s_0K(v_2) - K(v_s) \;=\; (1-s_0)\,\Delta(v_1,v_s) + s_0\,\Delta(v_2,v_s).$$

Combining with $(v_1 - v_s)^2 = s_0^2(v_1-v_2)^2$ and $(v_2 - v_s)^2 = (1-s_0)^2(v_1-v_2)^2$, the strong-convex bound (4) rearranges to the manifestly non-negative expression

$$(1-s_0)\left[\Delta(v_1,v_s) - \tfrac{1}{2}(v_1-v_s)^2\right] + s_0\left[\Delta(v_2,v_s) - \tfrac{1}{2}(v_2-v_s)^2\right] \;\geq\; 0,$$

where each bracket is ≥ 0 since cosh ≥ 1 in the integrand. Since $s_0, (1-s_0) > 0$, equality forces each bracket to vanish:

$$\Delta(v_i, v_s) \;=\; \tfrac{1}{2}(v_i - v_s)^2, \qquad i = 1, 2.$$

We now show that $\Delta(v,w) = \frac{1}{2}(v-w)^2$ implies $v = w$. Subtracting,

$$\Delta(v,w) - \tfrac{1}{2}(v-w)^2 = \int_w^v (v-u)\,[\cosh(u) - 1]\,du.$$

We claim this is *strictly positive* whenever $v \neq w$, irrespective of the ordering of v and w. Indeed, on the open interval with endpoints w and v the factor $(v-u)$ has constant sign, and $\cosh u - 1 \geq 0$ vanishes only at the single point $u = 0$; hence the product $(v-u)[\cosh u - 1]$ has constant sign and is nonzero off the null set $\{u = 0\}$. If $w < v$ then $v - u > 0$ on (w,v), so the integrand is ≥ 0 and positive a.e., giving $\int_w^v (\cdots)\,du > 0$. If $w > v$ then $v - u < 0$ on (v,w), so $\int_v^w (\cdots)\,du < 0$ and therefore $\int_w^v (\cdots)\,du = -\int_v^w (\cdots)\,du > 0$. In both cases the quantity is strictly positive, so $\Delta(v,w) > \frac{1}{2}(v-w)^2$ for $v \neq w$, and the only solution of $\Delta(v,w) = \frac{1}{2}(v-w)^2$ is $v = w$.

Therefore $v_1 = v_s = v_2$, i.e. $v_1 = v_2$. Applying this pointwise, $\dot{\xi}_1(t) = \dot{\xi}_2(t)$ for a.e. $t \in [a,b]$. By absolute continuity, $\xi_2 - \xi_1$ is constant on $[a,b]$, i.e. γ_2/γ_1 is constant. □

Remark 4.3 (The quantity $\Delta(v,w)$ is the cosh Bregman divergence)**.** The quantity $\Delta(v,w)$ introduced in the equality clause of the preceding proof is, up to notation, the *cosh Bregman divergence* $D_K(v\|w)$ formally defined in Section 4.5 (Definition 4.11). The properties exploited above – non-negativity, integral representation, and the strict quadratic lower bound – are collected later in Proposition 4.12; we used them inline here to keep the proof of Theorem 4.2 self-contained and free of forward references.

Remark 4.4 (Mechanism of the convexity theorem)**.** The proof uses only two ingredients: (a) the log-change of coordinates $\gamma \mapsto \xi := \log\gamma$, which converts the multiplicative interpolation on $\mathbb{R}_{>0}$ into the affine interpolation $\xi_s = (1-s)\xi_1 + s\xi_2$ on $\mathbb{R}$; and (b) pointwise convexity of K, propagated by integration. The argument is structurally a Jensen-type estimate *after* the log-change of coordinates – no Fréchet differentiation, no calculus of variations, no analytic estimate beyond pointwise convexity. The strength comes from matching the interpolation to the d'Alembert log-cost; Theorem 4.7 then converts this strong convexity into global minimality via a single chord check.

4.3. The convex free-action principle

The next two theorems give the convex form of the principle of least action in the *free sector*, i.e., for the kinetic action $\mathcal{A}$ with no potential term. Their hypothesis is the weakest possible local-minimality condition: the action does not strictly decrease along a single positive geometric-interpolation step toward any competitor.

Theorem 4.5 (Local-min implies global-min, kinetic form)**.** *Let $\gamma_{\rm geo}$ and $\gamma_{\rm other}$ be kinetically admissible paths on $[a,b]$ sharing endpoints. If there exists $s_0 \in (0,1]$ such that*

$$\mathcal{A}[\gamma_{\rm geo}] \;\leq\; \mathcal{A}[\mathrm{interp}^{\times}(\gamma_{\rm geo}, \gamma_{\rm other}, s_0)],$$

then

$$\mathcal{A}[\gamma_{\rm geo}] \;\leq\; \mathcal{A}[\gamma_{\rm other}] \;-\; \frac{1-s_0}{2}\int_a^b \big(\dot{\xi}_{\rm geo}(t) - \dot{\xi}_{\rm other}(t)\big)^2\,dt, \tag{5}$$

where $\xi_\bullet := \log\gamma_\bullet$. In particular, $\mathcal{A}[\gamma_{\rm geo}] \leq \mathcal{A}[\gamma_{\rm other}]$. Moreover, if $s_0 \in (0,1)$, then the inequality is strict unless $\gamma_{\rm other} \equiv \gamma_{\rm geo}$ on $[a,b]$.

Remark 4.6 (Vacuity of the boundary case $s_0 = 1$)**.** At $s_0 = 1$, the geometric interpolant $\mathrm{interp}^{\times}(\gamma_{\mathrm{geo}}, \gamma_{\mathrm{other}}, 1)$ collapses to γ_{other}, so the hypothesis of Theorem 4.5 reduces to $\mathcal{A}[\gamma_{\mathrm{geo}}] \le \mathcal{A}[\gamma_{\mathrm{other}}]$, which is identical to its conclusion. The slack $\frac{1-s_0}{2}\int(\dot{\xi}_{\mathrm{geo}} - \dot{\xi}_{\mathrm{other}})^2 dt$ in (5) also vanishes, so the strictness clause is vacuous at $s_0 = 1$. The boundary value $s_0 = 1$ is retained in Theorem 4.5 only because the quantitative bound (5) remains a (trivially true) identity there; the nontrivial mathematical content lives entirely in $s_0 \in (0,1)$. For exactly this reason the global-minimality principle Theorem 4.7 and its converse (Remark 4.8) are stated with the *interior* range $s_0 \in (0,1)$, which avoids the circularity in which the $s_0 = 1$ chord inequality would coincide with the conclusion itself (Remark 4.9).

Proof. By Theorem 4.2 (the strong-convexity bound (3)) with $s = s_0$,

$$\mathcal{A}[\mathrm{interp}^{\times}(\gamma_{\mathrm{geo}}, \gamma_{\mathrm{other}}, s_0)] \;\le\; (1-s_0)\,\mathcal{A}[\gamma_{\mathrm{geo}}] + s_0\,\mathcal{A}[\gamma_{\mathrm{other}}] - \frac{s_0(1-s_0)}{2}\int_a^b (\dot{\xi}_{\mathrm{geo}} - \dot{\xi}_{\mathrm{other}})^2\, dt.$$

Chain with the hypothesis, subtract $(1-s_0)\mathcal{A}[\gamma_{\mathrm{geo}}]$, and divide by $s_0 > 0$ to obtain (5).

For the strictness statement, assume $s_0 \in (0,1)$ and $\gamma_{\mathrm{other}} \not\equiv \gamma_{\mathrm{geo}}$. The shared-endpoint hypothesis forces $\xi_{\mathrm{other}} - \xi_{\mathrm{geo}} \in H_0^1([a,b])$, so by absolute continuity its derivative cannot be identically zero a.e. (otherwise the difference would be constant, hence zero by the endpoint condition). Therefore $\int_a^b (\dot{\xi}_{\mathrm{geo}} - \dot{\xi}_{\mathrm{other}})^2\, dt > 0$, and the slack $\frac{1-s_0}{2}\int(\dot{\xi}_{\mathrm{geo}} - \dot{\xi}_{\mathrm{other}})^2\, dt$ in (5) is strictly positive, yielding strict inequality. At $s_0 = 1$ the slack vanishes, so the conclusion reduces to the hypothesis. ☐

Theorem 4.7 (Free-action principle, chord form)**.** *Let γ_{geo} be a kinetically admissible path on $[a,b]$. Suppose that for every kinetically admissible competitor γ_{other} sharing endpoints with γ_{geo} there exists some* interior *chord parameter $s_0 \in (0,1)$ with*

$$\mathcal{A}[\gamma_{\mathrm{geo}}] \;\le\; \mathcal{A}[\mathrm{interp}^{\times}(\gamma_{\mathrm{geo}}, \gamma_{\mathrm{other}}, s_0)].$$

Then γ_{geo} is a global minimizer of $\mathcal{A}$ among kinetically admissible competitors sharing its endpoints.

Proof. Apply Theorem 4.5 for each such γ_{other}. ☐

Remark 4.8 (Chord condition is a characterization of global minimality)**.** The chord condition of Theorem 4.7 is in fact equivalent to global minimality, not merely sufficient. The non-trivial direction (sufficiency) is Theorem 4.7 itself. The converse is elementary: if γ_{geo} is a global minimizer of $\mathcal{A}$ on the kinetically admissible class with fixed endpoints, then for every kinetically admissible competitor γ_{other} sharing the endpoints and every $s \in (0,1)$, the interpolant $\mathrm{interp}^{\times}(\gamma_{\mathrm{geo}}, \gamma_{\mathrm{other}}, s)$ is itself a kinetically admissible competitor sharing the endpoints (Lemma 3.11 together with Lemma 3.12), so global minimality of γ_{geo} gives $\mathcal{A}[\gamma_{\mathrm{geo}}] \le \mathcal{A}[\mathrm{interp}^{\times}(\cdots, s)]$ trivially. Theorem 4.7 therefore gives a one-line characterization of global minimality of $\mathcal{A}$ via a directional inequality, with no calculus involved on either side.

Remark 4.9 (On the meaning of the free-sector claim)**.** The hypothesis of Theorem 4.7 is a one-sided *chord* condition: for each competitor, the action does not strictly *decrease* along at least one *interior* positive geometric-interpolation step toward that competitor. This is exactly the classical first-order optimality criterion for a convex functional – a point is a global minimizer iff every admissible one-sided directional variation is non-negative – specialized to geometric interpolation. The mathematical content of Theorem 4.7 is therefore the strong convexity of $\mathcal{A}$ along $\mathrm{interp}^{\times}$ (Theorem 4.2), not a new variational mechanism; no Euler–Lagrange equation, Fréchet derivative, or second-variation condition is invoked, and the genuine quantitative output is the explicit Jensen minimizer of Corollary 4.10.

We require the chord parameter to be *interior*, $s_0 \in (0,1)$, for a substantive reason. At the boundary $s_0 = 1$ the interpolant $\mathrm{interp}^{\times}(\gamma_{\mathrm{geo}}, \gamma_{\mathrm{other}}, 1)$ collapses to γ_{other}, so the chord inequality $\mathcal{A}[\gamma_{\mathrm{geo}}] \le \mathcal{A}[\mathrm{interp}^{\times}(\cdots, 1)]$ degenerates into the conclusion $\mathcal{A}[\gamma_{\mathrm{geo}}] \le \mathcal{A}[\gamma_{\mathrm{other}}]$ itself and carries no deductive content (cf. the vacuity of $s_0 = 1$ in Remark 4.6). Admitting $s_0 = 1$ would make the hypothesis formally equivalent to the conclusion; restricting to $s_0 \in (0,1)$ removes this circularity while leaving the converse of Remark 4.8 intact. The extension to Lagrangians with potential is necessarily weaker (Section 8).

4.4. Closed-form minimizer and uniqueness

Corollary 4.10 (Uniform-log-velocity minimizer)**.** *Given endpoints $\gamma(a) = x_a > 0$ and $\gamma(b) = x_b > 0$ on $[a,b]$ with $a < b$, the unique minimizer of $\mathcal{A}$ among kinetically admissible competitors is the* uniform-log-velocity path

$$\gamma_*(t) \;=\; \exp\!\Big(\log x_a + \frac{t-a}{b-a}\,(\log x_b - \log x_a)\Big),$$

whose log-velocity is the constant $\dot{\xi}_* \equiv (\log x_b - \log x_a)/(b-a)$. Its kinetic action is

$$\mathcal{A}[\gamma_*] \;=\; (b-a)\Big[\cosh\!\Big(\tfrac{\log x_b - \log x_a}{b-a}\Big) - 1\Big].$$

Proof. The path γ_* has constant $\dot{\xi}_*$, hence is kinetically admissible and has finite action. For any *kinetically admissible* γ with these endpoints, $\int_a^b \dot{\xi}\,dt = \xi(b) - \xi(a) = \log x_b - \log x_a$ is fixed. Here $\dot{\xi} \in L^2([a,b]) \subset L^1([a,b])$ because $a < b < \infty$, and kinetic admissibility gives $K(\dot{\xi}) \in L^1([a,b])$. Thus Jensen's inequality applies to the probability measure $(b-a)^{-1}dt$ and the strictly convex function K:

$$\int_a^b K(\dot{\xi}(t))\,dt \;\ge\; (b-a)\,K\!\Big(\tfrac{1}{b-a}\int_a^b \dot{\xi}\,dt\Big) \;=\; (b-a)\,K(\dot{\xi}_*),$$

with equality iff $\dot{\xi}$ is a.e. constant, in which case absolute continuity and the endpoint condition force $\gamma = \gamma_*$. □

4.5. Pythagorean identity and the action gap

Strong convexity of $\mathcal{A}$ (Theorem 4.2) is reflected, at the level of the fixed-endpoint minimizer, by an exact *Bregman / Pythagorean identity* that decomposes the action gap $\mathcal{A}[\gamma] - \mathcal{A}[\gamma_*]$ into a non-negative divergence integral. This refines the qualitative uniqueness of Corollary 4.10 into a quantitative L^2 / Sobolev bound.

Definition 4.11 (Cosh Bregman divergence)**.** The *cosh Bregman divergence* is the non-negative function $D_K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,

$$D_K(v \,\|\, w) \;:=\; K(v) - K(w) - K'(w)(v-w) \;=\; \cosh(v) - \cosh(w) - \sinh(w)(v-w).$$

Proposition 4.12 (Properties of D_K)**.** *(i)* ***Integral form.*** $D_K(v \,\|\, w) = \int_w^v (v-u)\cosh(u)\,du.$

(ii) ***Non-negativity.*** *$D_K(v \,\|\, w) \ge 0$, with equality iff $v = w$.*

(iii) ***Quadratic lower bound.*** *$D_K(v \,\|\, w) \ge \frac{1}{2}(v-w)^2$ for all $v, w \in \mathbb{R}$.*

(iv) ***Asymmetry near the ground state.*** *If $w = 0$, $D_K(v \,\|\, 0) = K(v) = \cosh(v) - 1$, so the* Bregman divergence at the ground velocity *reduces to the kinetic cost itself.*

Proof. Part (i) is Taylor's theorem with integral remainder for K at w: $K(v) - K(w) - K'(w)(v-w) = \int_w^v (v-u)K''(u)\,du = \int_w^v (v-u)\cosh(u)\,du$. Parts (ii) and (iii) follow from $\cosh \ge 1 > 0$ in (i): the integrand $(v-u)\cosh(u)$ has a constant sign on

$[\min(v,w),\max(v,w)]$, and replacing $\cosh(u)$ by its lower bound 1 gives $\int_w^v(v-u)\,du = \frac{1}{2}(v-w)^2$. Part (iv) is direct evaluation. □

Theorem 4.13 (Pythagorean identity for the geodesic minimizer)**.** *Let γ be a kinetically admissible path on $[a,b]$ with $\gamma(a)=x_a$ and $\gamma(b)=x_b$, and let γ_* be the uniform-log-velocity minimizer of Corollary 4.10, with constant log-velocity $\dot{\xi}_* = (\log x_b - \log x_a)/(b-a)$. Writing $\xi := \log\gamma$,*

$$\mathcal{A}[\gamma] \;=\; \mathcal{A}[\gamma_*] \;+\; \int_a^b D_K\big(\dot{\xi}(t)\,\|\,\dot{\xi}_*\big)\,dt. \tag{6}$$

In particular, by Proposition 4.12(iii),

$$\mathcal{A}[\gamma]-\mathcal{A}[\gamma_*] \;\ge\; \tfrac{1}{2}\int_a^b\big(\dot{\xi}(t)-\dot{\xi}_*\big)^2\,dt \;=\; \tfrac{1}{2}\,\|\dot{\xi}-\dot{\xi}_*\|^2_{L^2([a,b])}, \tag{7}$$

with equality iff $\gamma\equiv\gamma_*$ on $[a,b]$.

Proof. Define $\xi_*(t) := \log x_a + \dot{\xi}_*\,(t-a)$, so ξ_* is affine with constant derivative $\dot{\xi}_*$. By Definition 4.11,

$$K(\dot{\xi}(t)) \;=\; K(\dot{\xi}_*) + K'(\dot{\xi}_*)\big(\dot{\xi}(t)-\dot{\xi}_*\big) \;+\; D_K\big(\dot{\xi}(t)\,\|\,\dot{\xi}_*\big). \tag{8}$$

Before integrating, note that each of the three terms lies in $L^1([a,b])$: the constant $K(\dot{\xi}_*)$ trivially; the linear term because $\dot{\xi}-\dot{\xi}_* \in L^2([a,b]) \subset L^1([a,b])$ and $K'(\dot{\xi}_*) = \sinh(\dot{\xi}_*)$ is a finite constant; and the Bregman term because, by (8), it equals the L^1 function $K(\dot{\xi})$ (integrable by kinetic admissibility of γ, Definition 3.3) minus the two L^1 terms just named. In particular $D_K(\dot{\xi}\,\|\,\dot{\xi}_*) \in L^1([a,b])$, so the identity (6) is an equality between finite quantities. The first term integrates to $(b-a)K(\dot{\xi}_*) = \mathcal{A}[\gamma_*]$ (Corollary 4.10). The second term integrates to $K'(\dot{\xi}_*)\int_a^b(\dot{\xi}-\dot{\xi}_*)\,dt = K'(\dot{\xi}_*)\big[(\xi(b)-\xi(a))-(\xi_*(b)-\xi_*(a))\big]=0$, because γ and γ_* share endpoints, so $\int_a^b\dot{\xi}\,dt = \log x_b - \log x_a = \int_a^b \dot{\xi}_*\,dt$ and the prefactor $K'(\dot{\xi}_*) = \sinh(\dot{\xi}_*)$ is a finite constant. The third term integrates to the right-hand side of (6). The bound (7) is Proposition 4.12(iii) integrated, with equality iff $\dot{\xi}=\dot{\xi}_*$ a.e., equivalently $\gamma=\gamma_*$ by absolute continuity and the endpoint condition. □

Remark 4.14 (Pythagorean reading)**.** Equation (6) is the Pythagorean theorem of Bregman geometry in 1D: the geodesic γ_* is the *Bregman projection* of any kinetically admissible competitor γ onto the constant-velocity submanifold subject to the endpoint constraint, and the action gap is the Bregman "squared distance" to that projection. In this sense Corollary 4.10 is not only an existence/uniqueness statement but a projection identity, and Theorem 4.7 can be read as a sub-gradient characterization of the projection.

Corollary 4.15 (Sobolev / Friedrichs–Poincaré gap)**.** *With the notation of Theorem 4.13, set $\eta := \xi-\xi_* \in H_0^1([a,b])$ (so $\eta(a)=\eta(b)=0$). Then*

$$\mathcal{A}[\gamma]-\mathcal{A}[\gamma_*] \;\ge\; \tfrac{1}{2}\int_a^b\dot{\eta}(t)^2\,dt \;\ge\; \frac{\pi^2}{2(b-a)^2}\int_a^b\eta(t)^2\,dt \;=\; \frac{\pi^2}{2(b-a)^2}\,\big\|\log(\gamma/\gamma_*)\big\|^2_{L^2([a,b])}. \tag{9}$$

The constant $\pi^2/(b-a)^2$ is the optimal Friedrichs constant on $H_0^1([a,b])$, attained by $\eta(t) = \sin(\pi(t-a)/(b-a))$.

Proof. The first inequality is (7), written in η-coordinates: $\dot{\xi}-\dot{\xi}_* = \dot{\eta}$. The second inequality is the Friedrichs (one-dimensional Wirtinger) inequality on $H_0^1([a,b])$, $\int\dot{\eta}^2 \ge \frac{\pi^2}{(b-a)^2}\int\eta^2$. □

4.6. Joint convexity of the minimum-action profile

Proposition 4.16 (Perspective convexity of $\mathcal{A}_*$)**.** *Define the* minimum-action profile

$$\mathcal{A}_*(T,\Delta) \;:=\; T\big[\cosh(\Delta/T)-1\big], \qquad T>0,\ \Delta\in\mathbb{R}.$$

By Corollary 4.10, $\mathcal{A}_*(b-a,\,\log x_b-\log x_a)=\mathcal{A}[\gamma_*]$ on $[a,b]$. Then $\mathcal{A}_*$ is:

(i) the perspective transform of $u\mapsto \cosh(u)-1$, hence jointly convex *on $\mathbb{R}_{>0}\times\mathbb{R}$;*
(ii) strictly convex in Δ for each fixed $T>0$;
(iii) positively homogeneous of degree 1: $\mathcal{A}_*(\lambda T,\lambda\Delta)=\lambda\,\mathcal{A}_*(T,\Delta)$ for every $\lambda>0$;
(iv) monotone non-increasing in T for each fixed Δ, with $\partial_T\mathcal{A}_*\le 0$ and $\mathcal{A}_*(T,\Delta)\to 0$ as $T\to\infty$.

Proof. Let $f(u):=\cosh(u)-1$ and write $u:=\Delta/T$, so $\mathcal{A}_*(T,\Delta)=Tf(u)$. From $\partial_T u=-u/T$ and $\partial_\Delta u=1/T$ one computes the partial derivatives

$$\partial_T\mathcal{A}_*=f(u)-uf'(u),\qquad \partial_\Delta\mathcal{A}_*=f'(u),$$

$$\partial_T^2\mathcal{A}_*=\frac{u^2f''(u)}{T},\qquad \partial_\Delta^2\mathcal{A}_*=\frac{f''(u)}{T},\qquad \partial_T\partial_\Delta\mathcal{A}_*=-\frac{uf''(u)}{T}.$$

With ordering (T,Δ), the Hessian factors as a rank-one outer product:

$$\nabla^2\mathcal{A}_*(T,\Delta)\;=\;\frac{f''(u)}{T}\begin{pmatrix}u\\-1\end{pmatrix}\begin{pmatrix}u & -1\end{pmatrix}\;=\;\frac{\cosh(\Delta/T)}{T}\begin{pmatrix}\Delta/T\\-1\end{pmatrix}\begin{pmatrix}\Delta/T & -1\end{pmatrix}.$$

Since $\cosh>0$ and $T>0$, the Hessian is positive semidefinite, giving joint convexity (i). For fixed T, $\partial_\Delta^2\mathcal{A}_*=T^{-1}\cosh(\Delta/T)>0$, giving strict convexity in Δ, hence (ii). Positive homogeneity (iii) is immediate from $\lambda T\,f(\lambda\Delta/(\lambda T))=\lambda Tf(\Delta/T)$.

For (iv), use the supporting-hyperplane inequality for the convex function f at the point u, evaluated at 0:

$$f(0)\;\ge\;f(u)+f'(u)(0-u)\;=\;f(u)-uf'(u),$$

i.e. $f(u)-uf'(u)\le f(0)=0$, with strict inequality when $u\neq 0$ by strict convexity (the supporting line at u touches f only at u). Hence $\partial_T\mathcal{A}_*=f(\Delta/T)-(\Delta/T)f'(\Delta/T)\le 0$, with equality iff $\Delta/T=0$, i.e. iff $\Delta=0$. The asymptotic $\mathcal{A}_*(T,\Delta)=\Delta^2/(2T)+O(\Delta^4/T^3)\to 0$ as $T\to\infty$ follows from the Taylor expansion $\cosh(u)-1=u^2/2+O(u^4)$ at $u=\Delta/T\to 0$. □

Corollary 4.17 (Geodesic concatenation: subadditivity in time)**.** *For $T_1,T_2>0$ and $\Delta_1,\Delta_2\in\mathbb{R}$,*

$$\mathcal{A}_*(T_1+T_2,\,\Delta_1+\Delta_2)\;\le\;\mathcal{A}_*(T_1,\Delta_1)+\mathcal{A}_*(T_2,\Delta_2),\tag{10}$$

with equality iff $\Delta_1/T_1=\Delta_2/T_2$. Equivalently, the unique global minimizer between fixed positive endpoints on $[a,c]$ is the single uniform-log-velocity geodesic; splitting the interval at any intermediate $b\in(a,c)$ and relaxing the value at b never beats the global geodesic, and ties it only when the two halves continue at the same log-velocity.

Proof. By Proposition 4.16(i),(iii), $\mathcal{A}_*$ is convex and positively homogeneous of degree 1 on the open half-plane $\mathbb{R}_{>0}\times\mathbb{R}$. A convex positively 1-homogeneous function is subadditive: applying convexity at $\frac{1}{2}((2T_1,2\Delta_1)+(2T_2,2\Delta_2))$ and using 1-homogeneity to absorb the factor 2 yields (10).

For the equality clause, write $X_i = (T_i, \Delta_i) \in \mathbb{R}_{>0} \times \mathbb{R}$. Equality in (10) is exactly equality in the midpoint convexity step $\mathcal{A}_*\big(\tfrac{1}{2}(2X_1) + \tfrac{1}{2}(2X_2)\big) = \tfrac{1}{2}\mathcal{A}_*(2X_1) + \tfrac{1}{2}\mathcal{A}_*(2X_2)$, which holds iff $\mathcal{A}_*$ is affine on the segment joining $2X_1$ and $2X_2$. By Proposition 4.16, the Hessian $\nabla^2\mathcal{A}_*(T,\Delta) = \frac{\cosh(\Delta/T)}{T}(\Delta/T,-1)^\top(\Delta/T,-1)$ is rank one and positive semidefinite, with its single null direction the *radial* one (T,Δ) (along which 1-homogeneity makes $\mathcal{A}_*$ linear); in every transverse direction it is strictly positive. Hence $\mathcal{A}_*$ is strictly convex along any segment that does not lie on a ray through the origin, and the midpoint identity forces $2X_1$ and $2X_2$ – equivalently X_1 and X_2 – to be positively proportional, i.e. $\Delta_1/T_1 = \Delta_2/T_2$. Conversely, on a common ray 1-homogeneity gives equality. (This sharpening also uses the strict convexity in Δ of Proposition 4.16(ii), which guarantees the transverse Hessian does not degenerate.) ☐

Remark 4.18 (Causal / adiabatic asymptotics)**.** Two limits of $\mathcal{A}_*$ are physically significant.

- **Short-time / ultra-relativistic.** As $T \to 0^+$ with $\Delta \neq 0$ fixed, $\mathcal{A}_*(T,\Delta) \sim \frac{1}{2}e^{|\Delta|/T}\,T \to \infty$ exponentially. The cosh cost imposes a natural *exponential barrier in log-displacement*: a log-displacement Δ over time T is exponentially expensive when $|\Delta| \gg T$.
- **Long-time / adiabatic.** As $T \to \infty$ with Δ fixed, $\mathcal{A}_*(T,\Delta) = \frac{\Delta^2}{2T} + O(\Delta^4/T^3) \to 0$. The minimum-action profile is asymptotically Newtonian on the minimizer alone, even though no individual integrand has been linearized.

4.7. Structural reformulation: dually flat / Hessian geometry

Remark 4.19 (Dually flat / Hessian-manifold reading)**.** The convexity results admit a clean reading in the language of Hessian / dually flat manifolds (Amari–Nagaoka [8], Shima [9]), provided one is careful about *which* Hessian structure carries the geometric interpolation. The relevant structure lives in the *additive* coordinate $\xi = \log x$: the log-cost $\widetilde{J}(\xi) = \cosh\xi - 1$ is a strictly convex Hessian potential on $\mathbb{R}$, and the pair $(\mathbb{R}, \widetilde{J})$ is a one-dimensional dually flat manifold with metric

$$g_{\widetilde{J}}(\xi) = \widetilde{J}''(\xi)\,d\xi^2 = \cosh\xi\,d\xi^2.$$

In this structure:

- geometric interpolation $\mathrm{interp}^\times$, being affine in ξ, is precisely the e-affine (log-affine) geodesic;
- the cosh Bregman divergence (Definition 4.11) is the Bregman divergence *generated by* $\widetilde{J}$, entering the Pythagorean identity (Theorem 4.13) evaluated on the log-*velocity* $\dot{\xi}$;
- the kinetic action $\mathcal{A}$ is the corresponding Bregman / cosh-energy of paths;
- Theorems 4.2, 4.13 and Corollary 4.10 are the 1D instances of the standard “geodesic convexity + projection theorem” for Bregman energies on dually flat manifolds.

This is not the Hessian metric of J in the coordinate x. The pair $(\mathbb{R}_{>0}, J)$ is *also* a Hessian manifold, with affine coordinate x and metric $g_J(x) = J''(x)\,dx^2 = x^{-3}\,dx^2$; its primal-affine geodesics are parameterized linearly in x (arithmetic interpolation interp^+), and its Levi–Civita geodesics are the power-law family $\gamma(t) = (at+b)^{-2}$ of Appendix A (Theorem A1, Remark A3). On the 1-manifold $\mathbb{R}_{>0}$ these are not distinct *curves* – a fixed-endpoint geodesic of any connection traces the same arc of $\mathbb{R}_{>0}$ – but distinct *parameterizations* of that common arc, each connection fixing its own affine time-law. Because $\xi = \log x$ is a *nonlinear* reparametrization, it does not transport the affine/geodesic structure of x: indeed g_J pulls back to $e^{-\xi}\,d\xi^2$, which differs from $g_{\widetilde{J}} = \cosh\xi\,d\xi^2$. The two Hessian structures share the carrier but use different potentials, metrics, and connections, and only $(\mathbb{R}, \widetilde{J})$ carries the geometric-interpolation convexity of $\mathcal{A}$. The Otto-type geometry of $\mathcal{E}_{\mathrm{Hess}}$ on $(\mathbb{R}_{>0}, g_J)$ is left to future work.

5. Euler–Lagrange Equations and the Geodesic Picture

Three natural Euler–Lagrange equations appear in the cost-functional setting: one for $\mathcal{J}$, one for $\mathcal{A}$, and one for the Hessian path-energy $\mathcal{E}_{\mathrm{Hess}}$. All three are satisfied at the ground state $\gamma \equiv 1$; away from it they differ in content.

Definition 5.1 (Critical points used in this section)**.** Let $a < b$.

(i) An admissible path γ is a *static critical point* of $\mathcal{J}$ if, for every $\eta \in C^1([a,b])$ with $\eta(a) = \eta(b) = 0$, the admissibility condition $\gamma + \varepsilon\eta > 0$ holds for all sufficiently small $|\varepsilon|$, and

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_a^b J(\gamma(t) + \varepsilon\eta(t))\, dt = 0.$$

(ii) A C^2 log-path ξ is a *kinetic critical point* of $\mathcal{A}$ if, for every $\eta \in C^1([a,b])$ with $\eta(a) = \eta(b) = 0$,

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_a^b K(\dot{\xi}(t) + \varepsilon\dot{\eta}(t))\, dt = 0.$$

For an admissible path γ with $\xi = \log\gamma$, this is the critical-point notion used for $\mathcal{A}[\gamma]$.

Remark 5.2 (Cost-rate Euler–Lagrange equation for $\mathcal{J}$)**.** Because the integrand of $\mathcal{J}$ depends only on γ and not on $\dot{\gamma}$, the variational equation for $\mathcal{J}$ is purely algebraic. For admissible γ and $\eta \in C^1([a,b])$ with $\eta(a) = \eta(b) = 0$, differentiating under the integral (justified because γ is continuous and strictly positive on the compact $[a,b]$, so $\gamma + \varepsilon\eta$ stays in a compact subinterval of $(0,\infty)$ where J' is bounded, for $|\varepsilon|$ small) gives the first variation

$$\delta\mathcal{J}[\gamma](\eta) \;=\; \int_a^b J'(\gamma(t))\, \eta(t)\, dt.$$

The du Bois-Reymond lemma forces $J'(\gamma(t)) = 0$ a.e.; with $J' \circ \gamma$ continuous on $[a,b]$, this upgrades to equality for every t. Since $J'(x) = \frac{1}{2}(1 - x^{-2})$ vanishes on $(0,\infty)$ only at $x = 1$, the unique admissible static critical point of $\mathcal{J}$ (in the sense of Definition 5.1) is the ground state $\gamma \equiv 1$. We retain this fact only for the ground-state coexistence observation of Remark 5.5 below; no theorem downstream of this remark uses the cost-rate Euler–Lagrange equation.

5.1. The kinetic EL for $\mathcal{A}$

The integrand of $\mathcal{A}$ is $K(\dot{\xi})$ – a function of $\dot{\xi}$ only, independent of ξ. The Euler–Lagrange equation is therefore that $\partial_{\dot{\xi}} K$ is conserved in time:

$$\frac{d}{dt}\sinh(\dot{\xi}(t)) \;=\; 0 \quad\Longleftrightarrow\quad \cosh(\dot{\xi}(t))\,\ddot{\xi}(t) \;=\; 0 \quad\Longleftrightarrow\quad \ddot{\xi}(t) \;=\; 0,$$

the last equivalence because $\cosh > 0$.

Theorem 5.3 (Kinetic EL selects uniform log-velocity)**.** *Let γ be an admissible path on $[a,b]$ with $\xi := \log\gamma \in C^2$. Then γ is automatically kinetically admissible (Definition 3.3), and γ is a kinetic critical point of $\mathcal{A}$ in the sense of Definition 5.1 iff $\ddot{\xi} \equiv 0$, i.e., ξ is an affine function of t and $\gamma(t) = x_a\, e^{\dot{\xi}_0(t-a)}$ for some $x_a > 0$ and $\dot{\xi}_0 \in \mathbb{R}$. With fixed endpoints $\gamma(a) = x_a$, $\gamma(b) = x_b$, the unique critical point is the uniform-log-velocity path of Corollary 4.10, and it is simultaneously the global minimizer.*

Proof. Since $\xi \in C^2([a,b])$, the derivative $\dot{\xi}$ is continuous and bounded. Hence $K(\dot{\xi})$ is continuous and integrable, so γ is kinetically admissible.

For $\eta \in C^1([a,b])$ with $\eta(a) = \eta(b) = 0$, differentiating under the integral is justified because $\dot{\xi}$ and $\dot{\eta}$ are bounded and K' is continuous on the compact range swept out by $\dot{\xi} + \varepsilon\dot{\eta}$ for $|\varepsilon|$ small. The first variation is

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_a^b K(\dot{\xi} + \varepsilon\dot{\eta})\,dt = \int_a^b \sinh(\dot{\xi}(t))\,\dot{\eta}(t)\,dt.$$

Because $\xi \in C^2$, the function $\sinh(\dot{\xi})$ is C^1, and integration by parts gives

$$\int_a^b \sinh(\dot{\xi})\,\dot{\eta}\,dt = \left[\sinh(\dot{\xi})\eta\right]_a^b - \int_a^b \cosh(\dot{\xi})\ddot{\xi}\,\eta\,dt = -\int_a^b \cosh(\dot{\xi})\ddot{\xi}\,\eta\,dt.$$

Thus the first variation vanishes for all endpoint-fixed $\eta \in C^1([a,b])$ iff $\cosh(\dot{\xi})\ddot{\xi} = 0$ on $[a,b]$ by the fundamental lemma of the calculus of variations. Since $\cosh(\dot{\xi}) > 0$, this is equivalent to $\ddot{\xi} = 0$. Therefore ξ is affine, and conversely any affine ξ makes the displayed first variation vanish.

With fixed endpoints, the affine function ξ is uniquely determined, giving the path of Corollary 4.10. Its global minimality is exactly Corollary 4.10. □

Remark 5.4 (Regularity classes)**.** Theorem 5.3 uses the strong pointwise Euler–Lagrange notion of "critical point," which requires $\xi \in C^2$ (a proper subclass of the kinetically admissible paths of Definition 3.3). The chord condition in Theorem 4.7 is instead a convexity-based sufficient and necessary condition for global minimality on the larger kinetically admissible class, where ξ need only be absolutely continuous with L^2 log-derivative and finite kinetic action. For C^2 paths with fixed endpoints, the uniform-log-velocity path is simultaneously the strong-EL critical point of this theorem and the global minimizer furnished by Corollary 4.10.

5.2. Common ground state and the Hessian-energy comparison

So far we have compared the static $\mathcal{J}$ and kinetic $\mathcal{A}$ Euler–Lagrange pictures. A third path-level functional also lives on $\mathbb{R}_{>0}$ – the Hessian Riemannian path-energy

$$\mathcal{E}_{\text{Hess}}[\gamma] \;:=\; \int_a^b \tfrac{1}{2}\,g(\gamma)\,\dot{\gamma}^2\,dt \;=\; \int_a^b \tfrac{1}{2}\,\frac{\dot{\gamma}(t)^2}{\gamma(t)^3}\,dt$$

of the Hessian metric $g(x) = J''(x) = x^{-3}$, with geodesic Euler–Lagrange equation

$$\ddot{\gamma} + \Gamma(\gamma)\,\dot{\gamma}^2 \;=\; 0, \qquad \Gamma(x) = -\frac{3}{2x}. \tag{11}$$

$\mathcal{E}_{\text{Hess}}$ uses the Riemannian connection of g, whereas $\mathcal{A}$ uses the log-affine (e-)connection of §4.7, and the two have different nonconstant critical families – different *parameterizations* of the same arcs of $\mathbb{R}_{>0}$, not different trajectories, since the carrier is 1-dimensional (Appendix A, Remark A3). We include $\mathcal{E}_{\text{Hess}}$ here only for the comparison below; it does not enter any subsequent theorem of this work. (The symbol $\mathcal{E}_{\text{Hess}}$ is reserved for this Hessian path-energy throughout; the unadorned E later used in §7 for the conserved energy along a trajectory is a distinct object.)

Remark 5.5 (Coexistence at the ground state)**.** The constant path $\gamma \equiv 1$ on $[a,b]$ satisfies all three a priori distinct critical-point conditions simultaneously: the cost-rate Euler–Lagrange equation for $\mathcal{J}$ (Remark 5.2); the kinetic Euler–Lagrange equation $\ddot{\xi} = 0$ for $\mathcal{A}$ (Theorem 5.3); and the Hessian-metric geodesic equation (11). Verification is immediate: $J'(1) = \frac{1}{2}(1-1) = 0$ gives (i); $\dot{\xi} \equiv 0$ gives $\ddot{\xi} \equiv 0$ and hence (ii); $\dot{\gamma} \equiv 0$ together with $\ddot{\gamma} \equiv 0$ gives (iii). Although the dynamical functionals $\mathcal{A}$ and $\mathcal{E}_{\text{Hess}}$ have different nonconstant critical families – differing in parameterization rather than trajectory, the carrier being 1-dimensional – the additional static cost-rate condition (i) singles out $\gamma \equiv 1$ as the common normalized reference state shared by all three critical sets. The pictures remain distinct away from the ground state; we make no further bridging claim.

6. Conditional Bridge to Newtonian Mechanics

Having established in Sections 2–5 that the d'Alembert calibration alone forces the convex free-sector structure of $\mathcal{A}$, we now turn to the physical bridge. We shall see that the cosh kinetic profile recovers the Newtonian small-step limit and the relativistic rapidity dependence only after four explicit additional postulates, which we name and isolate at the outset:

- **Kinematic embedding.** The mechanical position q is identified with the log-coordinate $\xi = \log \gamma$ of a positive degree of freedom.
- **Mass coupling.** A species carries a coupling constant $m > 0$ that multiplies the dimensionless K; this gives the Newtonian small-step limit and matches the relativistic rapidity profile of the same cosh form.
- **Hamiltonian primacy.** The Hamiltonian $H = T_H(p) + V(\xi)$ is the primary, additively combined cost; the Lagrangian $L = K_m - V$ is its Legendre dual, and the minus sign on V is forced by the transform, not posited.
- **Native state cost.** The same d'Alembert uniqueness that forces K also forces a native static log-cost $\widetilde{J}(\xi) = \cosh \xi - 1$; the native Lagrangian is $L_{\text{nat}} = K_m(\dot{\xi}) - k\widetilde{J}(\xi)$, a cosh-sinh oscillator. General potentials V are external inputs from the cost-field environment, outside the scope of our analysis.

These four ingredients make the Newtonian interpretation a derivation of clearly bounded scope rather than an identification asserted at the outset.

6.1. Foundations of the bridge

Definition 6.1 (Kinematic embedding axiom)**.** *Status.* The identification below is an *axiom* of the physical bridge, not a theorem of d'Alembert. We postulate that the mechanical position coordinate q of classical mechanics is the log-coordinate $\xi = \log \gamma$ of a positive degree of freedom, and that the dimensionless log-step $\phi := d\xi/d\tau$ is calibrated as a boost rapidity. Neither identification is forced by the convexity theorems of Sections 2–5; both are physical postulates whose role is exactly to supply the kinematic content that d'Alembert's algebra alone cannot.

Statement.

Fix the species-relevant invariant speed $c > 0$ and a reference time scale $t_0 > 0$. We write

$$\tau := t/t_0$$

for the associated dimensionless evolution parameter. Throughout Sections 6–7 we work in *natural units*, in which $c = 1$ and $t_0 = 1$, and restore standard units explicitly at each appearance of a dimensional quantity.

For the physical bridge in Sections 6–7 we identify the variational parameter used in the preceding sections with τ. Thus a dot in these sections denotes differentiation with respect to the dimensionless parameter τ unless a physical-time derivative such as dq/dt is written explicitly. In natural units $t = \tau$, so the dot notation agrees with the usual mechanical notation.

We identify a positive degree of freedom $\gamma \in \mathbb{R}_{>0}$ with classical kinematic data via

$$\xi \;:=\; \log \gamma \;\in\; \mathbb{R}, \qquad q \;:=\; \xi \quad \text{(natural units)} \quad \Big[\text{equivalently } q := c\, t_0\, \xi \text{ in standard units}\Big],$$

and we identify the dimensionless log-step per unit dimensionless time with the boost rapidity:

$$\phi := \frac{d\xi}{d\tau}, \qquad v_{\mathrm{SR}}/c := \tanh(\phi), \qquad \frac{dq}{dt} = c\,\frac{d\xi}{d\tau} \quad \Big[= \dot{\xi} = \phi \text{ in natural units } c = t_0 = 1\Big]. \tag{12}$$

The mechanical ground state $q = 0$ corresponds to the ground state $\gamma = 1$.

Remark 6.2 (Dimensional dictionary)**.** Sections 6–7 are written in natural units ($c = t_0 = 1$), so every symbol is either dimensionless or carries some power of mass. The following table lists each symbol's role together with its standard-unit (i.e., c, t_0 restored) dimensions, so that the natural-unit expressions in the body can be read in either convention.

Symbol	Description	Standard-unit dimensions
ξ	log-coordinate, $\xi = \log\gamma$	dimensionless
$\tau = t/t_0$	dimensionless time	dimensionless
t	physical time	T
t_0	reference time scale	T
c	invariant speed	LT^{-1}
$\phi = d\xi/d\tau$	rapidity (boost parameter)	dimensionless
$v_{\mathrm{SR}} = c\tanh\phi$	SR 3-velocity	LT^{-1}
$q = c\,t_0\,\xi$	mechanical position	L
$dq/dt = c\,\phi$	coordinate velocity	LT^{-1}
m	mass coupling (Def. 6.4)	M
k	binding coupling (§6.3)	ML^2T^{-2} (energy)
$K_m(\phi) = mc^2(\cosh\phi - 1)$	physical kinetic cost	ML^2T^{-2} (energy)
$p = \partial L/\partial\dot{\xi}$	conjugate to ξ	ML^2T^{-2} (energy)
L, H	Lagrangian, Hamiltonian	ML^2T^{-2} (energy)
$\mathcal{A}[\gamma]$	free-sector kinetic action	dimensionless
$mc^2t_0\,\mathcal{A}[\gamma]$, $\mathcal{L}_V[\gamma] = \int L\,dt$	physical action	ML^2T^{-1} (action)

Notes. (i) Because ξ is dimensionless, the conjugate $p := \partial L/\partial\dot{\xi}$ has dimensions of *energy*, not of mechanical momentum MLT^{-1}; the conventional SR-mechanical momentum is p/c, with units MLT^{-1}. (ii) The position identification $q = c\,t_0\,\xi$ in standard units is the dimensionally consistent reading that makes $dq/dt = c\,\phi$ a proper velocity (LT^{-1}); in natural units it collapses to $q = \xi$ and $\dot{q} = \phi$ as written. (iii) In natural units the physical kinetic cost $K_m(\phi)$ and the binding cost $k(\cosh\xi - 1)$ both reduce to mass times a dimensionless cosh-bump; the explicit c^2 factors are restored only in expressions written outside of natural units. (iv) The free-sector functionals $\mathcal{A}[\gamma]$ and $\mathcal{A}_*(T, \Delta)$ of Sections 3–4 are *dimensionless*, consistent with Remark 2.1; the symbol $\mathcal{A}$ acquires action dimensions only through the mass–time prefactor, as the physical kinetic action $mc^2t_0\,\mathcal{A}[\gamma]$, which in the free case $V \equiv 0$ coincides with $\mathcal{L}_V[\gamma] = \int L\,dt$. The two rows above list these two distinct objects separately.

Remark 6.3 (Two velocity notions and why rapidity is primary)**.** Two observations force the rapidity identification of Definition 6.1 rather than a direct "$\dot{\xi}$ = Newtonian (SR 3-)velocity" reading, and Equation (12) carries two distinct "velocity-like" quantities that must not be confused.

(a) Algebra. d'Alembert's equation (1) on the additive variable ξ is precisely the *composition law for boosts*: $\cosh(\phi_1 + \phi_2) + \cosh(\phi_1 - \phi_2) = 2\cosh(\phi_1)\cosh(\phi_2)$. The canonical additive parameter on which this composition is linear is the rapidity ϕ, not the SR 3-velocity v_{SR} (which composes nonlinearly via the Einstein addition formula).

(b) Dimensions. If γ is dimensionless, then so is ξ. Differentiating ξ with respect to physical time gives a quantity with dimensions of inverse time, so we compare instead with the dimensionless step $\phi = d\xi/d\tau$, where $\tau = t/t_0$. The Newtonian identification "$q = \xi$" is dimensionally consistent only after choosing natural units; in standard units it becomes $q = c\, t_0\, \xi$ and $dq/dt = c\, d\xi/d\tau = c\,\phi$ (Remark 6.2). The rapidity identification (12) is dimension-free because it uses ϕ, not the unscaled derivative with respect to physical time.

(c) Two velocity notions: v_{SR} versus $\dot{q}$. Under the kinematic axiom (12), the *coordinate (log-) velocity* $\dot{q} = dq/dt = c\,\phi \overset{\text{nat. units}}{=} \phi$ is the rate of change of the log-coordinate $q = \xi$ with respect to lab time; under the kinematic axiom this *is the rapidity itself* (in natural units). The *special-relativistic* 3*-velocity* $v_{\mathrm{SR}} := c\tanh(\phi)$ is defined by the rapidity-to-velocity formula of special relativity. The two agree only to leading order. In natural units, where $\dot{q} = \phi$ and $c = 1$,

$$v_{\mathrm{SR}} = \tanh(\dot{q}) = \dot{q} - \frac{\dot{q}^3}{3} + O(\dot{q}^5),$$

while in standard units, using $\phi = \dot{q}/c$,

$$v_{\mathrm{SR}} = c\tanh(\dot{q}/c) = \dot{q} - \frac{\dot{q}^3}{3c^2} + O(\dot{q}^5/c^4).$$

For $|\dot{q}| \ll 1$ they coincide; for large $|\dot{q}|$, v_{SR} saturates at $\pm c$ while $\dot{q} = \phi$ is unbounded. The Newtonian small-step limit (Proposition 6.5(ii) and Theorem 6.14) lives where the distinction is invisible at leading order; the relativistic profile match (Proposition 6.5(i)) lives where the distinction is sharp. We always use v_{SR} for the SR 3-velocity $c\tanh\phi$, distinct from $\dot{q}$ except at $\phi = 0$.

The Newtonian-limit identification of the log-step variable with velocity in natural units is therefore not an independent embedding; it is the small-rapidity consequence $\phi \approx v_{\mathrm{SR}}/c$ for $|v_{\mathrm{SR}}| \ll c$ (Proposition 6.5(ii)). Under the kinematic axiom, $\dot{q} = \phi$ *exactly* in natural units.

With Definition 6.1 in place, the results of Sections 4–5 are re-readable as statements about a genuine scalar mechanical coordinate q.

Definition 6.4 (Species mass coupling)**.** Each particle species carries a *mass coupling* $m > 0$ (with dimensions of mass), a species-specific prefactor on the dimensionless d'Alembert-forced kinetic cost. The *physical kinetic cost* is, in natural units ($c = t_0 = 1$),

$$K_m(\phi) \;:=\; m\,K(\phi) \;=\; m\big[\cosh(\phi) - 1\big],$$

and in standard units the same object reads $K_m(\phi) = mc^2[\cosh(\phi) - 1]$, with dimensions of energy. When the dimensionless parameter is chosen as $\tau = t/t_0$, the physical action includes the corresponding time factor $dt = t_0\, d\tau$. We write K_m throughout for the natural-unit form, where $c = t_0 = 1$ and $\phi = \dot{\xi}$.

Proposition 6.5 (Single cosh profile covers Newtonian and rapidity regimes)**.** *Let $m > 0$. With the rapidity identification (12) of Definition 6.1, the notational distinction $\dot{q} = \phi$ versus $v_{\mathrm{SR}} = c\tanh\phi$ of Remark 6.3 in force:*

(i) ***Relativistic rapidity profile (as a function of rapidity, or equivalently of v_{SR}).*** *For all rapidities $\phi \in \mathbb{R}$,*

$$K_m(\phi) \;=\; m\big[\cosh(\phi) - 1\big] \;=\; m(\gamma_L - 1),$$

where $\gamma_L := \cosh\phi = 1/\sqrt{1 - (v_{\mathrm{SR}}/c)^2}$ is the Lorentz factor associated with $v_{\mathrm{SR}}/c = \tanh\phi$. This is the exact relativistic kinetic-energy dependence on rapidity; in standard units the right-hand side reads $mc^2(\gamma_L - 1)$. This is a profile match for the d'Alembert step *cost, not a claim that the Legendre Hamiltonian below is the standard special-relativistic Hamiltonian.*

(ii) ***Newtonian limit (consequence).*** *For $|\phi| \ll 1$ (equivalently $|v_{\mathrm{SR}}| \ll c$, equivalently $|\dot q| \ll 1$ in natural units), the Taylor expansion of* cosh *gives*

$$K_m(\phi) \;=\; \tfrac{1}{2}m\phi^2 + \tfrac{m}{24}\phi^4 + O(\phi^6).$$

Substituting either of the two leading-order equivalences $\phi = \dot q$ (exact, by ([12])) or $\phi = v_{\mathrm{SR}}/c + (v_{\mathrm{SR}}/c)^3/3 + O((v_{\mathrm{SR}}/c)^5)$ gives, in natural units $c = t_0 = 1$,

$$K_m \;=\; \tfrac{1}{2}m\,\dot q^2 + \tfrac{m}{24}\dot q^4 + O(\dot q^6) \;=\; \tfrac{1}{2}m\,v_{\mathrm{SR}}^2 + O(v_{\mathrm{SR}}^4),$$

the standard Newtonian kinetic energy. The two readings "$\dot q$" and "v_{SR}" coincide to leading order v^2 and differ only at $O(v^4)$.

Proof. (i) is the identity $\cosh(\operatorname{artanh}\beta) = (1-\beta^2)^{-1/2}$ applied to $\beta = v_{\mathrm{SR}}/c$ and $\phi = \operatorname{artanh}(v_{\mathrm{SR}}/c)$, multiplied by m; equivalently, $\cosh\phi = \gamma_L$ by definition of rapidity.

(ii) The first expansion is the standard cosh Taylor series $\cosh(\phi) - 1 = \frac{1}{2}\phi^2 + \frac{1}{24}\phi^4 + O(\phi^6)$ multiplied by m, with the sharp explicit error bound $|K_m(\phi) - \frac{1}{2}m\phi^2| \le \frac{m\phi^4}{24(1-\phi^2)}$ for $|\phi| < 1$ (Proposition 2.13(iv) multiplied by m).

Substituting $\phi = \dot q$ (an *exact* identification under the kinematic axiom ([12]) in natural units) directly yields $K_m = \frac{1}{2}m\dot q^2 + \frac{m}{24}\dot q^4 + O(\dot q^6)$.

Substituting instead the SR 3-velocity expansion $\phi = \operatorname{artanh}(v_{\mathrm{SR}}/c) = v_{\mathrm{SR}}/c + (v_{\mathrm{SR}}/c)^3/3 + O((v_{\mathrm{SR}}/c)^5)$ gives, in natural units, $\phi^2 = v_{\mathrm{SR}}^2 + 2v_{\mathrm{SR}} \cdot v_{\mathrm{SR}}^3/3 + O(v_{\mathrm{SR}}^6) = v_{\mathrm{SR}}^2 + O(v_{\mathrm{SR}}^4)$, so $\frac{1}{2}m\phi^2 = \frac{1}{2}mv_{\mathrm{SR}}^2 + O(v_{\mathrm{SR}}^4)$, and the quartic-and-higher tail $\frac{m}{24}\phi^4 + O(\phi^6) = O(v_{\mathrm{SR}}^4)$ is absorbed in the same remainder. The agreement of the two readings at the v_{SR}^2 level is then $\dot q^2 = v_{\mathrm{SR}}^2 + O(v_{\mathrm{SR}}^4)$, which follows from $\dot q = \phi = v_{\mathrm{SR}}/c + O((v_{\mathrm{SR}}/c)^3)$. ☐

Remark 6.6 (Provenance of the mass value)**.** The mass coupling m is the only particle-species parameter in the kinetic sector. We do not produce its provenance – why a given species carries a given numerical m – here; it is determined by deeper structural properties of the species that lie outside the scope of this paper. Within our scope, m is an input. What we derive is the *form* $mK(\phi)$ of the kinetic Lagrangian, with the same $\cosh - 1$ function covering the Newtonian small-step limit and the rapidity profile.

Definition 6.7 (Additive cost postulate)**.** The *total instantaneous cost* of a trajectory in a degree of freedom ξ with conjugate momentum p decomposes additively into a velocity cost and a state cost:

$$H(\xi, p) \;=\; T_H(p) \;+\; V(\xi),$$

where T_H is the Legendre transform of the physical kinetic cost K_m and $V : \mathbb{R} \to \mathbb{R}$ is the state cost. H is the *Hamiltonian* and is the primary dynamical object of the framework.

Proposition 6.8 (The cosh kinetic Hamiltonian)**.** *Let $m > 0$. For the physical kinetic cost $K_m(\dot\xi) = m[\cosh\dot\xi - 1]$,*

$$p \;=\; \frac{\partial K_m}{\partial \dot\xi} \;=\; m\sinh\dot\xi, \qquad \dot\xi \;=\; \operatorname{arsinh}\!\Big(\frac{p}{m}\Big),$$

and the Legendre transform is

$$T_H(p) \;=\; p\dot\xi - K_m(\dot\xi) \;=\; p\,\operatorname{arsinh}\!\Big(\frac{p}{m}\Big) - \sqrt{m^2 + p^2} + m.$$

Small-p Taylor:

$$T_H(p) \;=\; \frac{p^2}{2m} \;-\; \frac{p^4}{24\,m^3} \;+\; O(p^6/m^5).$$

Proof. The identities $p = m\sinh\dot{\xi}$ and $\dot{\xi} = \operatorname{arsinh}(p/m)$ are immediate from the definition of p and the inverse-hyperbolic-sine. Using $\cosh(\operatorname{arsinh}(u)) = \sqrt{1+u^2}$ with $u = p/m$, $K_m(\dot{\xi}) = m[\sqrt{1+p^2/m^2}-1] = \sqrt{m^2+p^2} - m$, and therefore $T_H(p) = p\operatorname{arsinh}(p/m) - \sqrt{m^2+p^2} + m$. For the Taylor expansion: $\operatorname{arsinh}(p/m) = p/m - p^3/(6m^3) + O(p^5/m^5)$, so $p\operatorname{arsinh}(p/m) = p^2/m - p^4/(6m^3) + O(p^6/m^5)$; and $\sqrt{m^2+p^2} = m + p^2/(2m) - p^4/(8m^3) + O(p^6/m^5)$. Subtracting,

$$T_H(p) = \tfrac{p^2}{m} - \tfrac{p^4}{6m^3} - m - \tfrac{p^2}{2m} + \tfrac{p^4}{8m^3} + m + O(p^6/m^5) = \tfrac{p^2}{2m} + \left(\tfrac{1}{8} - \tfrac{1}{6}\right)\tfrac{p^4}{m^3} + O(p^6/m^5) = \tfrac{p^2}{2m} - \tfrac{p^4}{24\,m^3} +$$

□

Proposition 6.9 (Cosh-dual Hamiltonian is distinct from the SR free-particle Hamiltonian). *Let $m > 0$. The cosh-dual Hamiltonian*

$$T_H(p) \;=\; p\ \operatorname{arsinh}\!\left(\frac{p}{m}\right) \;-\; \sqrt{m^2+p^2} \;+\; m$$

of Proposition 6.8 and the special-relativistic free-particle Hamiltonian (with rest energy subtracted)

$$H_{\mathrm{SR}}(p) \;:=\; \sqrt{m^2+p^2} \;-\; m$$

are distinct functions on $\mathbb{R}$. More precisely:

(i) ***Strict separation away from zero.*** *Let $f(p) := T_H(p) - H_{\mathrm{SR}}(p) = p\ \operatorname{arsinh}(p/m) - 2\sqrt{m^2+p^2} + 2m$. Then*

$$f''(p) \;=\; \frac{p^2}{(m^2+p^2)^{3/2}} \;\geq\; 0,$$

with equality only at $p = 0$, so f is convex on $\mathbb{R}$ and strictly convex off the single point $p = 0$. Together with $f(0) = 0$ and $f'(0) = 0$, this gives

$$T_H(p) \;>\; H_{\mathrm{SR}}(p) \qquad \text{for every } p \neq 0.$$

(ii) ***Newtonian agreement, quartic divergence.*** *The two Hamiltonians share the Newtonian small-p limit at order p^2 but differ at order p^4:*

$$\begin{aligned} T_H(p) \;&=\; \frac{p^2}{2m} - \frac{p^4}{24\,m^3} + O(p^6/m^5), \\ H_{\mathrm{SR}}(p) \;&=\; \frac{p^2}{2m} - \frac{p^4}{8\,m^3} + O(p^6/m^5), \end{aligned}$$

so

$$T_H(p) - H_{\mathrm{SR}}(p) \;=\; \frac{p^4}{12\,m^3} + O(p^6/m^5).$$

The match of Proposition 6.5 is therefore one of profile *in rapidity – kinetic energy as a function of rapidity equals $m(\cosh\phi - 1) = m(\gamma_L - 1)$ in both theories – not an identity of Hamiltonians. We use the cosh-dual T_H throughout.*

Proof. *(i)* Set $f(p) := T_H(p) - H_{\mathrm{SR}}(p) = p\,\operatorname{arsinh}(p/m) - 2\sqrt{m^2+p^2} + 2m$. Using $(d/dp)\operatorname{arsinh}(p/m) = 1/\sqrt{m^2+p^2}$, direct differentiation gives

$$f'(p) = \operatorname{arsinh}\Big(\frac{p}{m}\Big) + \frac{p}{\sqrt{m^2+p^2}} - \frac{2p}{\sqrt{m^2+p^2}} = \operatorname{arsinh}\Big(\frac{p}{m}\Big) - \frac{p}{\sqrt{m^2+p^2}},$$

and differentiating once more, using $(d/dp)\big[p/\sqrt{m^2+p^2}\big] = m^2/(m^2+p^2)^{3/2}$,

$$f''(p) = \frac{1}{\sqrt{m^2+p^2}} - \frac{m^2}{(m^2+p^2)^{3/2}} = \frac{(m^2+p^2)-m^2}{(m^2+p^2)^{3/2}} = \frac{p^2}{(m^2+p^2)^{3/2}}.$$

Hence $f''(p) \geq 0$ on $\mathbb{R}$, with equality iff $p = 0$. In particular f is convex on $\mathbb{R}$ and strictly convex on any open interval not containing 0. The critical-point equations $f(0) = 0 \cdot 0 - 2m + 2m = 0$ and $f'(0) = 0 - 0 = 0$ identify $p = 0$ as the unique critical point and, by convexity, the global minimum of f. Suppose, toward a contradiction, that $f(p_0) = 0$ for some $p_0 \neq 0$. By convexity of f on $[0, p_0]$ (or $[p_0, 0]$ if $p_0 < 0$) with $f(0) = f(p_0) = 0$, $f \leq 0$ on this interval; combined with $f \geq 0$ from minimality at 0, this forces $f \equiv 0$ on the interval, hence $f'' \equiv 0$ there. But $f''(p) > 0$ at every interior point $p \neq 0$, contradicting f'' vanishing on an open subinterval. Therefore $f(p) > 0$ for every $p \neq 0$, i.e., $T_H(p) > H_{\mathrm{SR}}(p)$ for every $p \neq 0$.

(ii) The T_H expansion is Proposition 6.8. For H_{SR}, apply the binomial series $\sqrt{1+u} = 1 + u/2 - u^2/8 + O(u^3)$ at $u = p^2/m^2 \geq 0$ to obtain

$$\sqrt{m^2+p^2} = m\sqrt{1+p^2/m^2} = m + \frac{p^2}{2m} - \frac{p^4}{8\,m^3} + O(p^6/m^5),$$

and subtract m to get the displayed H_{SR} expansion. Subtracting the two expansions term by term, the p^2 terms cancel and the p^4 coefficient of $T_H - H_{\mathrm{SR}}$ is

$$-\frac{1}{24\,m^3} - \Big(-\frac{1}{8\,m^3}\Big) = \frac{-1+3}{24\,m^3} = \frac{1}{12\,m^3},$$

giving $T_H(p) - H_{\mathrm{SR}}(p) = p^4/(12\,m^3) + O(p^6/m^5)$ as claimed. The positivity of this leading p^4 coefficient is also implied by part (i): $f(p) > 0$ for small $p \neq 0$ together with $f(p) = c\,p^4 + O(p^6)$ from the Taylor expansion forces $c \geq 0$, and the explicit value $c = 1/(12\,m^3)$ confirms $c > 0$. □

Remark 6.10 (The Legendre sign is derived, not posited)**.** From the additive postulate $H = T_H(p) + V(\xi)$, the Lagrangian is derived via the inverse Legendre transform:

$$L(\xi, \dot{\xi}) = p\dot{\xi} - H = \big[p\dot{\xi} - T_H(p)\big] - V(\xi) = K_m(\dot{\xi}) - V(\xi).$$

The minus sign on V in the Lagrangian is *forced* by the Legendre transform of an additive Hamiltonian; it is not an independent postulate. In a Hamiltonian-primary formulation, where the primary object is the additive cost H, the sign structure $L = T - V$ is a theorem, not an axiom.

In the free-sector analysis, the d'Alembert log-cost $\widetilde{J}(\xi) = \cosh\xi - 1$ of §2 played two distinct roles. Evaluated at the log-velocity $\dot{\xi}$, it produced the kinetic integrand $K(v) = \widetilde{J}(v) = \cosh v - 1$ that drives the convexity theorem of §4. Evaluated at the log-position ξ itself, the same log-cost is the integrand of the velocity-free static cost $\mathcal{J}[\gamma] = \int \widetilde{J}(\xi)\,dt$ of §3, whose algebraic Euler–Lagrange equation (Remark 5.2) selects the ground state $\gamma \equiv 1$. The d'Alembert calibration therefore fixes $\widetilde{J}$ on *both* arguments. Assigning the static cost its own species prefactor $k > 0$ – the *binding coupling* – gives the *native d'Alembert state cost* $V_{\mathrm{nat}}(\xi) = k\widetilde{J}(\xi) = k(\cosh\xi - 1)$. Thus $\widetilde{J}$ enters the native Lagrangian below twice under the same calibration: once at log-velocity (kinetic) and once at log-position (potential), with independent couplings m and k.

Definition 6.11 (Native d'Alembert Lagrangian)**.** With kinetic mass coupling m and binding coupling k, the *native d'Alembert Lagrangian* is

$$L_{\mathrm{nat}}(\xi,\dot{\xi}) \;:=\; K_m(\dot{\xi}) - k\,\widetilde{J}(\xi) \;=\; m\big[\cosh(\dot{\xi}) - 1\big] - k\big[\cosh(\xi) - 1\big].$$

In the *pure d'Alembert case* $k = m$, the constants cancel and $L_{\mathrm{nat}}(\xi,\dot{\xi}) = m\big[\cosh(\dot{\xi}) - \cosh(\xi)\big]$.

Remark 6.12 (Provenance of the binding coupling)**.** Like the mass coupling m of Definition 6.4, the binding coupling $k > 0$ is treated here as an external species parameter, subject to the same provenance caveat recorded for m in the remark above: its numerical value is governed by the same upstream structural mechanisms that fix m and lies outside the scope of this paper. What we derive is the *form* of the native Lagrangian, with the d'Alembert log-cost $\widetilde{J}$ appearing under the same calibration in both the kinetic slot (evaluated at $\dot{\xi}$) and the potential slot (evaluated at ξ); the two couplings (m,k) then enter as independent prefactors of those two slots.

Remark 6.13 (Native vs. general potentials)**.** L_{nat} is the minimal dynamical object produced by d'Alembert alone: kinetic cost plus native static cost, combined via the Hamiltonian-primary Legendre transform (Remark 6.10). A *general* potential $V(\xi)$ in Section 6.4 is the replacement of the native $k\widetilde{J}(\xi)$ by whatever interaction model supplies the state cost. Here, any $V \neq k\widetilde{J}$ is an external input that we do not derive from d'Alembert.

6.2. Small-velocity reduction of the kinetic action

Theorem 6.14 (Small-velocity reduction, mass-coupled)**.** *Let γ be an admissible path on $[a,b]$ with $\xi := \log\gamma$ and $\|\dot{\xi}\|_\infty \le 1/10$ (hence in particular kinetically admissible, since $\cosh(\dot{\xi}) \le \cosh(1/10) < \infty$ a.e.), and let $m > 0$. Then*

$$\left| m\,\mathcal{A}[\gamma] \;-\; \tfrac{m}{2}\int_a^b \dot{\xi}(t)^2\,dt \right| \;\le\; \frac{m}{24}\cdot\frac{100}{99}\int_a^b \dot{\xi}(t)^4\,dt \;\le\; \frac{m}{2376}\int_a^b \dot{\xi}(t)^2\,dt. \tag{13}$$

Under the kinematic embedding $q := \xi$ and the natural-unit calibration of the variational parameter with τ, the mass-weighted kinetic action $m\,\mathcal{A}[\gamma]$ reduces to the standard kinetic integral $T[q] = \frac{1}{2}\int_a^b m\,\dot{q}(t)^2\,dt$ with relative error at most $1/1188 \approx 8.4\times 10^{-4}$.

Proof. The sharp quartic remainder (2) applied pointwise at $v = \dot{\xi}(t)$ with $|\dot{\xi}(t)| \le 1/10$ gives $|K(\dot{\xi}) - \frac{1}{2}\dot{\xi}^2| \le \dot{\xi}^4/24 \cdot 100/99$, which integrated against $m\,dt$ yields the first inequality of (13). The second uses $\dot{\xi}^4 \le \dot{\xi}^2/100$ when $|\dot{\xi}| \le 1/10$, giving a relative-error bound of $2/2376 = 1/1188$ on the kinetic approximation. ☐

Remark 6.15. The kinetic term emerges from the Taylor of $K(\dot{\xi})$ – *velocity*, not of $J(1+\varepsilon)$ – state displacement. The mass coupling m enters as a uniform prefactor consistent with the Newtonian small-step limit and the rapidity profile of Proposition 6.5.

6.3. The native cosh-sinh oscillator

Theorem 6.16 (EL equation and small-amplitude limit of L_{nat})**.** *For the native d'Alembert Lagrangian (Definition 6.11) with couplings $m, k > 0$, the Euler–Lagrange equation is*

$$m\cosh(\dot{\xi})\,\ddot{\xi} + k\sinh(\xi) \;=\; 0. \tag{14}$$

In the small-velocity, small-amplitude limit ($|\dot{\xi}|, |\xi| \ll 1$), (14) *reduces to the linear harmonic oscillator* $m\,\ddot{\xi} + k\,\xi = 0$, *with angular frequency* $\omega = \sqrt{k/m}$.

Proof. $\partial_{\dot{\xi}} L_{\mathrm{nat}} = m \sinh \dot{\xi}$, so $(d/dt)\partial_{\dot{\xi}} L_{\mathrm{nat}} = m \cosh \dot{\xi}\, \ddot{\xi}$. $\partial_{\xi} L_{\mathrm{nat}} = -k \sinh \xi$. The Euler–Lagrange equation $(d/dt)\partial_{\dot{\xi}} L - \partial_{\xi} L = 0$ gives (14). Taylor-expanding $\cosh \dot{\xi} = 1 + O(\dot{\xi}^2)$ and $\sinh \xi = \xi + O(\xi^3)$ recovers the harmonic oscillator. ☐

Corollary 6.17 (Conserved energy of the native oscillator). *For couplings* $m, k > 0$, *the native Hamiltonian (Proposition 6.8 with* $V = k\widetilde{J}$*),*

$$H_{\mathrm{nat}}(\xi, p) \;=\; p\ \mathrm{arsinh}\big(\tfrac{p}{m}\big) - \sqrt{m^2 + p^2} + m + k\big[\cosh(\xi) - 1\big],$$

is constant along any C^2 *solution of* (14). *Its small-*(ξ, p) *expansion is* $H_{\mathrm{nat}} = \frac{p^2}{2m} + \frac{k}{2}\xi^2 - \frac{p^4}{24\,m^3} + \frac{k}{24}\xi^4 + O(p^6, \xi^6)$, *the harmonic oscillator Hamiltonian plus exactly determined quartic corrections.*

Proof. *Conservation.* Let $\xi \in C^2([a, b])$ satisfy (14), and set $p := m \sinh(\dot{\xi})$. Using $\cosh(\mathrm{arsinh}(p/m)) = \sqrt{1 + p^2/m^2}$, the Hamiltonian H_{nat} rewritten in $(\xi, \dot{\xi})$ coordinates along the trajectory is

$$E(t) \;:=\; H_{\mathrm{nat}}(\xi(t), p(t)) \;=\; m\big[\dot{\xi}(t) \sinh(\dot{\xi}(t)) - \cosh(\dot{\xi}(t)) + 1\big] + k\big[\cosh(\xi(t)) - 1\big].$$

Direct differentiation gives

$$\begin{aligned} \dot{E}(t) \;&=\; m\big[\ddot{\xi} \sinh(\dot{\xi}) + \dot{\xi} \cosh(\dot{\xi})\, \ddot{\xi} - \sinh(\dot{\xi})\, \ddot{\xi}\big] + k \sinh(\xi)\, \dot{\xi} \\ &=\; \dot{\xi}\big[m \cosh(\dot{\xi})\, \ddot{\xi} + k \sinh(\xi)\big] \;=\; 0 \end{aligned}$$

by (14); the $\ddot{\xi} \sinh(\dot{\xi})$ and $-\sinh(\dot{\xi})\ddot{\xi}$ terms cancel exactly. Hence E is constant on $[a, b]$.

Expansion. Combine Proposition 6.8 (giving $T_H(p) = p^2/(2m) - p^4/(24m^3) + O(p^6/m^5)$) with the small-$\xi$ Taylor of cosh: $k[\cosh \xi - 1] = k\xi^2/2 + k\xi^4/24 + O(\xi^6)$. ☐

Remark 6.18 (Joint saddle structure of L_{nat} and explicit dropout time)**.** We anticipate here the conjugate-time machinery of §8 (Definition 8.2 and Theorems 8.3/8.5) to compute the dropout time for the native oscillator; only the small-amplitude Jacobi computation is performed in this remark, and the statements about strict weak local minimality below are direct specializations of those theorems to $V = k\widetilde{J}$.

L_{nat} is *convex* in $\dot{\xi}$ (from $\cosh \dot{\xi}$) but *concave* in ξ (from $-\cosh \xi$). It is therefore a saddle jointly in $(\xi, \dot{\xi})$, and the free global-minimum theorem of Section 4 does *not* extend to it.

The dropout is quantitative. Linearize (14) along the trivial extremal $\xi_* \equiv 0$: with $L_{\dot{\xi}\dot{\xi}}(0, 0) = m$ and $L_{\xi\xi}(0, 0) = -k$, the Jacobi equation (18) reduces to the harmonic-oscillator equation

$$m\,\ddot{\eta} + k\,\eta = 0,$$

whose solution with $\eta(a) = 0$ is $\eta(t) = C \sin\!\big(\sqrt{k/m}\,(t - a)\big)$. The first conjugate time of Definition 8.2 along $\xi_* \equiv 0$ is therefore

$$t_c \;=\; a + \pi\sqrt{m/k} \;=\; a + \pi/\omega,$$

exactly half the harmonic period. Specializing Theorems 8.3/8.5 to $V = k\widetilde{J}$ gives three regimes:

- *Short-time regime,* $b - a < \pi/\omega$. Then $b < t_c$, so by Theorem 8.3, $\xi_* \equiv 0$ is a strict weak local minimizer of $\mathcal{L}_V$ on $\mathcal{V}$.

- *Long-time regime, $b - a > \pi/\omega$.* Then $t_c = a + \pi/\omega \in (a, b)$ is a conjugate point in the open interval, so by Theorem 8.5, $\xi_* \equiv 0$ is not a strict weak local minimizer.
- *Boundary case, $b - a = \pi/\omega$.* Then the first conjugate time lies exactly at $t_c = b$. This degenerate case falls outside Theorems 8.3/8.5 as stated, since Theorem 8.3 requires the strict inequality $b < t_c$ and Theorem 8.5 requires c in the open interval (a, b). The behaviour at this boundary is governed by the leading nonzero higher-order term of $\mathcal{L}_V$ along the Jacobi field $\eta(t) = \sin(\omega(t - a))$, and is resolved by Proposition 6.19 below for $k \neq m$ (according to the sign of $k - m$); in the pure case $k = m$ the action is exactly flat along the null field, so strict local minimality fails and non-strict minimality is left open.

Thus the native oscillator transitions from local-minimizer status to non-minimizer at exactly half the harmonic period, recovering the classical conjugate-point picture for the small-amplitude limit of (14); the degenerate boundary $b - a = \pi/\omega$ is settled by Proposition 6.19.

Proposition 6.19 (Resolution of the boundary conjugate case)**.** *Let $m, k > 0$, $\omega = \sqrt{k/m}$, and let $\xi_* \equiv 0$ be the trivial extremal of the native Lagrangian L_{nat} (Definition 6.11) on $[a, b]$ at the boundary length $b - a = \pi/\omega$, so that the first conjugate time lies exactly at $t_c = b$. Let $\mathcal{V}$ be the class of C^1 endpoint-fixed variations of Theorem 8.3. Then:*

(i) *The second variation $\delta^2\mathcal{L}_V[\eta] = \int_a^b (m\dot{\eta}^2 - k\eta^2)\, dt$ is positive semidefinite on $H_0^1([a, b])$, with one-dimensional null space* $\mathrm{span}\{\eta_0\}$, *$\eta_0(t) = \sin(\omega(t - a))$, and is strictly positive on the L^2-orthogonal complement of η_0 with spectral gap $\lambda_2 = 3k > 0$.*

(ii) *Along the null direction the leading nonvanishing term of the action is quartic,*

$$\mathcal{L}_V[\varepsilon\eta_0] \;=\; \frac{\pi\omega}{64}\,(k - m)\,\varepsilon^4 \;+\; O(\varepsilon^6).$$

(iii) *Hence, at $b - a = \pi/\omega$: if $k > m$ then $\xi_* \equiv 0$ is a strict weak local minimizer of $\mathcal{L}_V$ on $\mathcal{V}$; if $k < m$ it is* not *a weak local minimizer (the explicit family $\varepsilon\eta_0$ strictly lowers the action); and if $k = m$ (the pure d'Alembert case of Definition 6.11) the action is* exactly flat *along the entire family $\varepsilon\eta_0$, so $\xi_* \equiv 0$ is* not a strict *weak local minimizer; whether it is a (non-strict) weak local minimizer requires a mixed-direction analysis transverse to η_0 and is not decided by the one-parameter expansion along the null field alone.*

Proof. *(i)* Since $L_{\xi\dot{\xi}} = 0$, $L_{\dot{\xi}\dot{\xi}}(0,0) = m$ and $L_{\xi\xi}(0,0) = -V''(0) = -k$, the accessory quadratic form is $\delta^2\mathcal{L}_V[\eta] = \int_a^b (m\dot{\eta}^2 - k\eta^2)\, dt$. The Dirichlet Sturm–Liouville operator $-m\, d^2/dt^2 - k$ on $[a, b]$ has eigenpairs $\eta_n(t) = \sin\big(n\pi(t - a)/(b - a)\big)$, $\lambda_n = mn^2\pi^2/(b - a)^2 - k$. At $b - a = \pi/\omega$, i.e. $(b - a)^2 = \pi^2 m/k$, these become $\lambda_n = (n^2 - 1)k$, so $\lambda_1 = 0$ (eigenfunction $\eta_0 = \sin(\omega(t - a))$) and $\lambda_n = (n^2 - 1)k \geq 3k > 0$ for $n \geq 2$. Thus $\delta^2\mathcal{L}_V \geq 0$, vanishes exactly on $\mathrm{span}\{\eta_0\}$, and for $\zeta = \sum_{n\geq 2} c_n\hat{\eta}_n$ in the L^2-orthogonal complement (with $\hat{\eta}_n$ L^2-orthonormal) satisfies $\delta^2\mathcal{L}_V[\zeta] = \sum_{n\geq 2}\lambda_n c_n^2 \geq 3k\|\zeta\|_{L^2}^2$. Moreover, since $\|\dot{\zeta}\|_{L^2}^2 = \sum_{n\geq 2}\big(n\pi/(b - a)\big)^2 c_n^2 = \sum_{n\geq 2}(n^2 k/m)\, c_n^2$ at this length and $(n^2 - 1)/n^2 \geq \frac{3}{4}$ for $n \geq 2$, one has the H^1-coercivity

$$\delta^2\mathcal{L}_V[\zeta] = \sum_{n\geq 2}(n^2 - 1)k\, c_n^2 \;\geq\; \tfrac{3}{4}\, m\, \|\dot{\zeta}\|_{L^2}^2, \qquad \zeta \perp_{L^2} \eta_0. \tag{15}$$

Equivalently, the semidefiniteness is the equality case of the sharp Friedrichs inequality (Corollary 4.15) at $(b - a)^2 = \pi^2 m/k$.

(ii) With $\xi = \varepsilon\eta_0$, $\dot{\xi} = \varepsilon\dot{\eta}_0$ and $\cosh u - 1 = \frac{1}{2}u^2 + \frac{1}{24}u^4 + O(u^6)$,

$$\mathcal{L}_V[\varepsilon\eta_0] = \frac{\varepsilon^2}{2}\int_a^b (m\dot{\eta}_0^2 - k\eta_0^2)\,dt + \frac{\varepsilon^4}{24}\int_a^b (m\dot{\eta}_0^4 - k\eta_0^4)\,dt + O(\varepsilon^6).$$

The ε^2 coefficient is $\frac{1}{2}\delta^2\mathcal{L}_V[\eta_0] = 0$ by (i), and the cubic term vanishes since cosh is even. With $\dot{\eta}_0 = \omega\cos(\omega(t-a))$, the substitution $s = \omega(t-a) \in [0,\pi]$ and $\int_0^\pi \sin^4 = \int_0^\pi \cos^4 = \frac{3\pi}{8}$ give $\int_a^b \eta_0^4\,dt = \frac{3\pi}{8\omega}$ and $\int_a^b \dot{\eta}_0^4\,dt = \frac{3\pi\omega^3}{8}$, whence

$$\frac{1}{24}\int_a^b (m\dot{\eta}_0^4 - k\eta_0^4)\,dt = \frac{\pi}{64}\cdot\frac{m\omega^4 - k}{\omega} = \frac{\pi\omega}{64}\,(k-m),$$

using $\omega^2 = k/m$ so that $m\omega^4 - k = k(\omega^2 - 1) = k(k-m)/m$ and $k/(m\omega) = \omega$.

(iii), case $k < m$. The quartic coefficient $\frac{\pi\omega}{64}(k-m)$ is negative, so $\mathcal{L}_V[\varepsilon\eta_0] < 0 = \mathcal{L}_V[\xi_*]$ for all small $\varepsilon \neq 0$; the admissible one-parameter family $\varepsilon\eta_0 \in \mathcal{V}$ strictly lowers the action, hence $\xi_* \equiv 0$ is not a weak local minimizer.

Case $k > m$. We prove directly that $\mathcal{L}_V[\eta] > 0$ for every $\eta \in C_0^1([a,b])$ with $0 < \|\eta\|_{C^1} \leq \epsilon$ and a suitable $\epsilon > 0$, which is strict weak local minimality (the competitor path is $\xi_* + \eta = \eta$). Fix $\rho \in (0,1)$ and assume $\|\eta\|_\infty \leq \rho$. Two pointwise all-order cosh estimates hold: from $\cosh u - 1 - \frac{1}{2}u^2 - \frac{1}{24}u^4 = \sum_{j\geq 3} u^{2j}/(2j)! \geq 0$,

$$\cosh\dot{\eta} - 1 \;\geq\; \tfrac{1}{2}\dot{\eta}^2 + \tfrac{1}{24}\dot{\eta}^4,$$

while Proposition 2.13(iv) gives, for $|\eta| \leq \rho < 1$,

$$\cosh\eta - 1 \;\leq\; \tfrac{1}{2}\eta^2 + \frac{\eta^4}{24(1-\rho^2)}.$$

Substituting both into $\mathcal{L}_V$, with $Q[\eta] := \delta^2\mathcal{L}_V[\eta] = \int_a^b (m\dot{\eta}^2 - k\eta^2)\,dt$,

$$\mathcal{L}_V[\eta] \;\geq\; \tfrac{1}{2}Q[\eta] + \frac{m}{24}\int_a^b \dot{\eta}^4\,dt - \frac{k}{24(1-\rho^2)}\int_a^b \eta^4\,dt. \tag{16}$$

Decompose $\eta = c\,\eta_0 + \zeta$ with $\eta_0(t) = \sin(\omega(t-a))$ and $\langle\zeta,\eta_0\rangle_{L^2} = 0$; both $\eta_0, \zeta \in C_0^1([a,b])$. Integrating by parts and using $m\ddot{\eta}_0 = -m\omega^2\eta_0 = -k\eta_0$, the Q-cross term vanishes, $\int_a^b (m\dot{\eta}_0\dot{\zeta} - k\eta_0\zeta)\,dt = \int_a^b (k\eta_0\zeta - k\eta_0\zeta)\,dt = 0$, so $Q[\eta] = Q[\zeta] \geq \frac{3}{4}m\|\dot{\zeta}\|_{L^2}^2$ by (15). The splitting is bounded in C^1: as $\|\eta_0\|_{L^2}^2 = (b-a)/2$, the coefficient $c = \langle\eta,\eta_0\rangle_{L^2}/\|\eta_0\|_{L^2}^2$ obeys $|c| \leq \sqrt{2}\,\|\eta\|_\infty$, whence $\|\zeta\|_\infty \leq (1+\sqrt{2})\|\eta\|_\infty$ and $\|\dot{\zeta}\|_\infty \leq (1+\sqrt{2}\,\omega)\|\eta\|_{C^1}$.

For the quartics use, for any fixed $\delta \in (0,1)$, the elementary Young inequalities $(x+y)^4 \geq (1-\delta)x^4 - C_\delta y^4$ and $(x+y)^4 \leq (1+\delta)x^4 + C_\delta y^4$ (with C_δ depending only on δ), applied pointwise with $(x,y) = (c\dot{\eta}_0, \dot{\zeta})$ and $(x,y) = (c\eta_0, \zeta)$ and integrated:

$$\int_a^b \dot{\eta}^4\,dt \geq (1-\delta)c^4\int_a^b \dot{\eta}_0^4\,dt - C_\delta\int_a^b \dot{\zeta}^4\,dt, \qquad \int_a^b \eta^4\,dt \leq (1+\delta)c^4\int_a^b \eta_0^4\,dt + C_\delta\int_a^b \zeta^4\,dt.$$

Inserting these into (16) and using the values $\frac{m}{24}\int_a^b \dot{\eta}_0^4\,dt = \frac{k\pi\omega}{64}$ and $\frac{k}{24}\int_a^b \eta_0^4\,dt = \frac{k\pi}{64\omega}$ from (ii) (recall $m\omega^3 = k\omega$), the c^4 contribution is $c^4\,\mu(\delta,\rho)$ with

$$\mu(\delta,\rho) := (1-\delta)\frac{k\pi\omega}{64} - \frac{1+\delta}{1-\rho^2}\,\frac{k\pi}{64\omega} \xrightarrow[(\delta,\rho)\to(0,0)]{} \frac{k\pi}{64}\left(\omega - \frac{1}{\omega}\right) = \frac{\pi\omega}{64}(k-m) > 0.$$

Fix δ, ρ small enough that $\mu(\delta,\rho) \geq \frac{\pi\omega}{128}(k-m) =: \mu_0 > 0$. The remaining ζ-dependent terms are

$$\tfrac{1}{2}Q[\zeta] - \frac{mC_\delta}{24}\int_a^b \dot{\zeta}^4\,dt - \frac{kC_\delta}{24(1-\rho^2)}\int_a^b \zeta^4\,dt.$$

By $\int \dot{\zeta}^4 \leq \|\dot{\zeta}\|_\infty^2 \|\dot{\zeta}\|_{L^2}^2$, $\int \zeta^4 \leq \|\zeta\|_\infty^2 C_P^2 \|\dot{\zeta}\|_{L^2}^2$ (Poincaré $\|\zeta\|_{L^2} \leq C_P \|\dot{\zeta}\|_{L^2}$), and the C^1 bounds on ζ above, both negative terms are $\leq C_* \|\eta\|_{C^1}^2 \|\dot{\zeta}\|_{L^2}^2$ for a constant $C_* = C_*(\delta, \rho, b-a, \omega, m, k)$. With $\frac{1}{2}Q[\zeta] \geq \frac{3}{8}m\|\dot{\zeta}\|_{L^2}^2$, the ζ-terms are $\geq (\frac{3}{8}m - C_*\|\eta\|_{C^1}^2)\|\dot{\zeta}\|_{L^2}^2$. Choosing $\epsilon \leq \rho$ with $C_*\epsilon^2 \leq \frac{3}{16}m$ gives, for all $\|\eta\|_{C^1} \leq \epsilon$,

$$\mathcal{L}_V[\eta] \;\geq\; \mu_0\, c^4 + \tfrac{3}{16}m\,\|\dot{\zeta}\|_{L^2}^2 \;\geq\; 0,$$

with equality only if $c = 0$ and $\dot{\zeta} \equiv 0$, i.e. (since $\zeta \in H_0^1$) $\zeta = 0$ and so $\eta = 0$. Hence $\mathcal{L}_V[\eta] > 0 = \mathcal{L}_V[\xi_*]$ for every $\eta \in \mathcal{V} \setminus \{0\}$ with $\|\eta\|_{C^1} \leq \epsilon$, and $\xi_* \equiv 0$ is a strict weak local minimizer.

Case $k = m$. Here $\omega = 1$, the interval has length π, and $\eta_0(t) = \sin(t-a)$. Along the one-parameter family $\xi = \varepsilon\eta_0$ the action is *exactly flat for every* ε, not merely quartically degenerate: with $L_{\mathrm{nat}} = m[\cosh\dot{\xi} - \cosh\xi]$ and the substitution $s = t - a \in [0, \pi]$,

$$\mathcal{L}_V[\varepsilon\eta_0] = m\int_0^\pi \big[\cosh(\varepsilon\cos s) - \cosh(\varepsilon\sin s)\big]\,ds = 0,$$

because $\int_0^\pi f(\cos s)\,ds = \int_0^\pi f(\sin s)\,ds$ for every even f (both equal $2\int_0^{\pi/2} f(\sin s)\,ds$), applied to $f = \cosh(\varepsilon\,\cdot)$. Since $\varepsilon\eta_0 \in \mathcal{V}$ is admissible and arbitrarily small in C^1 with $\mathcal{L}_V[\varepsilon\eta_0] = 0 = \mathcal{L}_V[\xi_*]$, the strict inequality required for strict weak local minimality fails. The exact flatness also shows that no finite-order expansion along η_0 can decide the question: whether $\xi_* \equiv 0$ is a non-strict weak local minimizer depends on the behaviour of $\mathcal{L}_V$ in directions transverse to η_0 and is left open here. □

6.4. *Newton's law from the general cosh Lagrangian*

Definition 6.20 (Cosh Lagrangian with general potential)**.** Given a potential $V : \mathbb{R} \to \mathbb{R}$ of class C^1 (either the native $V = k\widetilde{J}$ or an external input for general interactions), and mass coupling $m > 0$, the *cosh Lagrangian* is

$$\mathcal{L}(\xi, \dot{\xi}) \;:=\; K_m(\dot{\xi}) - V(\xi) \;=\; m\big[\cosh(\dot{\xi}) - 1\big] - V(\xi),$$

with associated action $\mathcal{L}_V[\gamma] := \int_a^b \mathcal{L}(\xi(t), \dot{\xi}(t))\,dt$ for admissible paths γ with $\xi = \log\gamma$ for which this integral is defined.

Theorem 6.21 (Newton's second law from the cosh Lagrangian)**.** *Let $a < b$, $m > 0$, and $V \in C^1(\mathbb{R})$. For C^2 curves $\xi : [a,b] \to \mathbb{R}$, the Euler–Lagrange equation of $\mathcal{L}$ is*

$$m\cosh(\dot{\xi}(t))\,\ddot{\xi}(t) + V'(\xi(t)) \;=\; 0. \tag{17}$$

In the small-velocity limit $\cosh(\dot{\xi}) \to 1$, *this reduces to*

$$m\,\ddot{\xi}(t) \;=\; -V'(\xi(t)),$$

Newton's second law with mass m and force $F = -V'$. Under the kinematic embedding $q := \xi$, this is the standard form $m\ddot{q} = -V'(q)$.

Proof. $\partial_{\dot\xi}\mathcal{L} = m\sinh(\dot\xi)$, so $(d/dt)\partial_{\dot\xi}\mathcal{L} = m\cosh(\dot\xi)\ddot\xi$. $\partial_\xi\mathcal{L} = -V'(\xi)$. The Euler–Lagrange equation $(d/dt)\partial_{\dot\xi}\mathcal{L} - \partial_\xi\mathcal{L} = 0$ gives $m\cosh(\dot\xi)\ddot\xi - (-V'(\xi)) = 0$, i.e., (17). The small-velocity limit is immediate. □

Remark 6.22 (Newton's first law: inertia)**.** For $V' \equiv 0$, equation (17) reduces to $\ddot\xi = 0$, i.e., uniform log-velocity motion. This is the mechanical reading, under the kinematic embedding $q = \xi$, of the variational statement Theorem 5.3; it is also equivalent to the conservation of the cosh momentum $p = m\sinh\dot\xi$ in Corollary 7.6 below, via $\dot p = m\cosh(\dot\xi)\ddot\xi$ and $\cosh > 0$. We single it out by name only because the bridge to Newtonian mechanics motivates labeling, not because it is a separate fact.

7. Hamiltonian Formulation and Conservation Laws

Having established the cosh Lagrangian in Section 6, we now record its Hamiltonian dual and the associated conservation laws. The Hamiltonian $H = T_H(p) + V(\xi)$ is the foundationally primary object in the bridge construction (Definition 6.7): the additively combined total cost from which the Lagrangian is Legendre-derived. We collect below the mass-coupled Hamilton equations, the small-momentum expansion, and energy conservation.

Definition 7.1 (Cosh momentum and Hamiltonian, mass-coupled)**.** Let $m > 0$ and let $V : \mathbb{R} \to \mathbb{R}$. For $\mathcal{L}(\xi, \dot\xi) = K_m(\dot\xi) - V(\xi) = m[\cosh(\dot\xi) - 1] - V(\xi)$, the *conjugate momentum* is

$$p := \frac{\partial\mathcal{L}}{\partial\dot\xi} = m\sinh(\dot\xi), \qquad \dot\xi = \operatorname{arsinh}\Big(\frac{p}{m}\Big),$$

and the *Hamiltonian* is

$$H(\xi, p) := p\,\dot\xi - \mathcal{L} = T_H(p) + V(\xi) = p\ \operatorname{arsinh}\Big(\frac{p}{m}\Big) - \sqrt{m^2 + p^2} + m + V(\xi),$$

with T_H as in Proposition 6.8 and using the identity $\cosh(\operatorname{arsinh}(u)) = \sqrt{1 + u^2}$.

Proposition 7.2 (Small-momentum limit of H)**.** *Let $m > 0$ and $V : \mathbb{R} \to \mathbb{R}$. As $|p|/m \to 0$,*

$$H(\xi, p) = \frac{p^2}{2m} + V(\xi) - \frac{p^4}{24\,m^3} + O(p^6/m^5).$$

Under the kinematic embedding $q := \xi$, this is the standard non-relativistic Hamiltonian $H = p^2/(2m) + V(q)$ plus a species-scale-suppressed quartic correction.

Proof. Specialize Proposition 6.8 and add $V(\xi)$. □

Theorem 7.3 (Hamilton's equations from EL)**.** *Let $a < b$, $m > 0$, and let $V \in C^1(\mathbb{R})$. Let*

$$H(\xi, p) = p\ \operatorname{arsinh}(p/m) - \sqrt{m^2 + p^2} + m + V(\xi).$$

If $\xi \in C^2([a, b])$ satisfies (17) and $p(t) = m\sinh(\dot\xi(t))$, then Hamilton's equations hold:

$$\dot\xi = \partial_p H = \operatorname{arsinh}\Big(\frac{p}{m}\Big), \qquad \dot p = -\partial_\xi H = -V'(\xi).$$

Proof. The first equation is the inverse Legendre transform ($p = m\sinh(\dot\xi) \iff \dot\xi = \operatorname{arsinh}(p/m)$). The second follows from (17): $\dot p = (d/dt)[m\sinh(\dot\xi)] = m\cosh(\dot\xi)\ddot\xi = -V'(\xi)$. □

7.1. Conservation laws from one-parameter symmetries

Two standard conservation laws we use elsewhere in this work – energy and the cosh momentum – arise as Noether charges of explicit one-parameter symmetries of the action. We take the Noether derivation as primary; energy and momentum conservation are then immediate corollaries. This presentation also makes visible why the cosh momentum is *not* conserved for the native cosh–sinh oscillator.

Theorem 7.4 (Noether charges of the cosh Lagrangian)**.** *Let* $L(\xi,\dot\xi) = m\big[\cosh(\dot\xi) - 1\big] - V(\xi)$ *with* $m > 0$ *and* $V \in C^1(\mathbb{R})$*.*

(a) Time-translation symmetry. *The one-parameter group* $\Phi_t^\alpha : (t,\xi) \mapsto (t+\alpha,\xi)$ *leaves L invariant for every* $\alpha \in \mathbb{R}$*, since L has no explicit t-dependence. The associated Noether charge is the energy*

$$E \;=\; \dot\xi\,\partial_{\dot\xi}L \;-\; L \;=\; m\big[\dot\xi\sinh(\dot\xi) - \cosh(\dot\xi) + 1\big] + V(\xi),$$

and it is conserved on every C^2 *solution of* (17)*.*

(b) ξ***-translation symmetry, free case.*** *If V is constant on* $\mathbb{R}$*, the one-parameter group* $\Phi_\xi^\alpha : (t,\xi) \mapsto (t,\xi+\alpha)$ *leaves L invariant. The associated Noether charge is the cosh momentum* $p \;=\; \partial_{\dot\xi}L \;=\; m\sinh(\dot\xi)$*, conserved on every* C^2 *solution of* (17)*.*

(c) Native oscillator: no ξ***-translation symmetry.*** *For* $V = k\widetilde{J}$ *with* $k > 0$*, the native potential satisfies* $\widetilde{J}(\xi+\alpha) \;=\; \cosh(\xi+\alpha) - 1 \;\not\equiv\; \widetilde{J}(\xi)$ *whenever* $\alpha \neq 0$*. Hence* Φ_ξ^α *is* not *a Noether symmetry of* L_{nat}*, and the cosh momentum is not conserved along solutions of* (14)*; only the energy of Corollary 6.17 is.*

Proof. *(a)* For an autonomous first-order Lagrangian, the Noether identity for time-translation gives $dE/dt = \partial_t L = 0$ along solutions. Explicitly, with $E = m[\dot\xi\sinh(\dot\xi) - \cosh(\dot\xi) + 1] + V(\xi)$,

$$\dot E = m\big[\ddot\xi\sinh(\dot\xi) + \dot\xi\cosh(\dot\xi)\ddot\xi - \sinh(\dot\xi)\ddot\xi\big] + V'(\xi)\dot\xi = \dot\xi\big[m\cosh(\dot\xi)\ddot\xi + V'(\xi)\big] = 0$$

by (17).

(b) The infinitesimal generator is $(\delta t,\delta\xi) = (0,1)$. Under this generator, $\delta L = \partial_\xi L\cdot 1 = -V'(\xi)$, which vanishes identically iff V is constant. The Noether charge is $\partial_{\dot\xi}L\cdot\delta\xi = m\sinh(\dot\xi)$, and direct differentiation gives

$$\dot p \;=\; m\cosh(\dot\xi)\ddot\xi \;=\; -V'(\xi) \;=\; 0$$

by (17) and $V' \equiv 0$.

(c) We use the *quasi-invariance* (Bessel-Hagen) form of Noether's theorem [13–15]: Φ_ξ^α is a Noether symmetry of L_{nat} iff there exists a function $F = F(t,\xi,\dot\xi)$ of class C^1 such that $\delta L_{\text{nat}} = dF/dt$ along arbitrary C^1 curves, not merely along solutions. Here $\delta L_{\text{nat}} = \partial_\xi L_{\text{nat}}\cdot 1 = -k\sinh(\xi)$, a function of ξ alone. We show that $-k\sinh(\xi)$ is not a total t-derivative of any $F(t,\xi,\dot\xi)$.

Suppose for contradiction that $-k\sinh(\xi) = dF/dt = \partial_t F + F_\xi\,\dot\xi + F_{\dot\xi}\,\ddot\xi$ identically in $(t,\xi,\dot\xi,\ddot\xi)$. The left-hand side is independent of $\dot\xi$ and $\ddot\xi$, so equating coefficients of $\ddot\xi$ and of $\dot\xi$ gives $F_{\dot\xi} \equiv 0$ and $F_\xi \equiv 0$, hence $F = F(t)$. Then $dF/dt = F'(t)$ depends only on t, but $-k\sinh(\xi)$ depends nontrivially on ξ (since $k > 0$), contradiction. Hence no such F exists, Φ_ξ^α is not a quasi-invariance of L_{nat}, and Noether's theorem produces no conserved charge from ξ-translation for the native potential.

Direct verification of non-conservation. Along any solution of (14), $\dot p = m\cosh(\dot\xi)\ddot\xi = -k\sinh(\xi)$, which vanishes identically only at the trivial extremal $\xi \equiv 0$. So even without the symmetry analysis, p is manifestly not conserved. □

Corollary 7.5 (Energy conservation)**.** *Let $a < b$, $m > 0$, $V \in C^1(\mathbb{R})$, and let H be as in Theorem 7.3. If $\xi \in C^2([a,b])$ satisfies (17) and $p(t) = m\sinh(\dot{\xi}(t))$, then the total energy $E(t) := H(\xi(t), p(t))$ is constant on $[a,b]$.*

Proof. Theorem 7.4(a). ☐

Corollary 7.6 (Momentum conservation in the free case)**.** *Let $a < b$, $m > 0$, and let $V' \equiv 0$ on $\mathbb{R}$. If $\xi \in C^2([a,b])$ satisfies (17), then the cosh momentum $p(t) := m\sinh(\dot{\xi}(t))$ is constant on $[a,b]$. Equivalently, $\ddot{\xi} \equiv 0$ (Remark 6.22 / Theorem 5.3), since $\dot{p} = m\cosh(\dot{\xi})\ddot{\xi}$ and $\cosh > 0$.*

Proof. Theorem 7.4(b); the equivalent "$\ddot{\xi} = 0$" form is immediate from $\dot{p} = m\cosh(\dot{\xi})\ddot{\xi} = 0$ and $\cosh > 0$. ☐

Remark 7.7 (Symmetry inventory of the d'Alembert framework)**.** Within the cosh Lagrangian $L = m[\cosh\dot{\xi} - 1] - V(\xi)$, the maximal one-parameter symmetry group is generated by:

- time translation, always present, conserving energy;
- ξ-translation, present iff V is constant on $\mathbb{R}$, conserving the cosh momentum $m\sinh\dot{\xi}$;
- the discrete reflection $\xi \mapsto -\xi$ (with $\dot{\xi} \mapsto -\dot{\xi}$), present iff V is even, which is a $\mathbb{Z}/2\mathbb{Z}$ symmetry rather than a one-parameter group and so yields no continuous Noether charge.

The native potential $V = k\widetilde{J}$ is even but not translation invariant, so it admits the reflection symmetry but not a continuous ξ-translation symmetry; consequently the only continuous conservation law of the native oscillator is energy. This explains both *why* energy survives the addition of the native potential and *why* the cosh momentum does not.

8. Potentials and Classical Variational Scope

Having shown in Section 6 how Newton's law arises in the small-velocity limit of the cosh Lagrangian, we now ask what variational status the resulting extremals carry when a non-affine convex potential is added. Our convexity argument in Section 4 is *free* of any potential; adding $-V(\xi)$ breaks joint convexity of $\mathcal{L}$ in $(\xi, \dot{\xi})$ whenever V is strictly convex (the native case $V = k\widetilde{J}$ is the basic example). The Lagrangian is then a saddle: convex in velocity and concave in position. In this setting the convex global-minimizer theorem of Theorem 4.7 no longer applies, and we adopt the classical local-minimum / stationary-action picture as the correct replacement.

Remark 8.1 (Convention on regularity classes for this section)**.** *This convention is in force throughout Section 8.* The free-sector results of Sections 3–4 are stated for *kinetically admissible* paths in the sense of Definition 3.3: absolutely continuous positive paths $\gamma : [a,b] \to \mathbb{R}_{>0}$ with log-derivative in $L^2([a,b])$ and finite kinetic action $\int(\cosh\dot{\xi} - 1)\,dt < \infty$.

All theorems below (Theorems 8.3, 8.4, 8.5) instead require:

(R1) the potential satisfies $V \in C^2(\mathbb{R})$ (or $V \in C^1(\mathbb{R})$ for the bare stationarity theorem Theorem 8.4);

(R2) the extremal $\xi_* : [a,b] \to \mathbb{R}$ is of class $C^2([a,b])$ and satisfies the strong Euler–Lagrange equation (17) pointwise;

(R3) variations $\eta : [a,b] \to \mathbb{R}$ are of class $C^1([a,b])$ with $\eta(a) = \eta(b) = 0$.

This is a deliberate shift to stronger regularity than the free-sector path space of Section 3, made because the Jacobi sufficient/necessary conditions of Gelfand–Fomin [16] are formulated at this regularity. The free-sector theorems of Section 4 are *not* re-stated under (R1)–(R3); they remain valid in the larger kinetically admissible class. The two regularity worlds meet at C^2 uniform-log-velocity paths, which simultaneously satisfy the convex chord characterization of Theorem 4.7 and the pointwise Euler–Lagrange equation of Theorem 5.3 (see Remark 5.4).

Second variation, Jacobi fields, and conjugate times

Fix $a < b$, $m > 0$, and $V \in C^2(\mathbb{R})$. For a C^2 curve $\xi : [a, b] \to \mathbb{R}$, write

$$\mathcal{L}_V[\xi] \;:=\; \int_a^b \big[m(\cosh(\dot{\xi}(t)) - 1) - V(\xi(t))\big]\,dt.$$

For an admissible path γ with $\xi = \log\gamma$, this agrees with the earlier notation $\mathcal{L}_V[\gamma] = \mathcal{L}_V[\xi]$. Let $\xi_* \in C^2([a, b])$ be a solution of the Euler–Lagrange equation (17). The classical second-variation theory for first-order C^2 Lagrangians associates to ξ_* a Jacobi equation. In the present case $L_{\xi\dot{\xi}} = 0$, so the Jacobi equation is

$$\frac{d}{dt}\Big(m\cosh(\dot{\xi}_*(t))\,\dot{\eta}(t)\Big) + V''(\xi_*(t))\,\eta(t) \;=\; 0. \tag{18}$$

Definition 8.2 (Conjugate time)**.** A time $c \in (a, b]$ is *conjugate to a along ξ_** if there exists a nonzero C^2 solution η of (18) with $\eta(a) = 0$ and $\eta(c) = 0$. The *first conjugate time on $[a, b]$* is the infimum of such c in $(a, b]$, if any exists, and is defined to be ∞ if no conjugate time occurs in $(a, b]$.

Theorem 8.3 (Short-time local minimality)**.** *Let $a < b$, $m > 0$, $V \in C^2(\mathbb{R})$, and let $\mathcal{L}(\xi, \dot{\xi}) = K_m(\dot{\xi}) - V(\xi)$. Let $\xi_* \in C^2([a, b])$ be a solution of (17) with fixed endpoints, and let t_c be the first conjugate time of Definition 8.2. Denote by $\mathcal{V}$ the class of C^1 endpoint-fixed variations*

$$\mathcal{V} := \{\xi_* + \eta : \eta \in C^1([a, b]),\ \eta(a) = \eta(b) = 0\}.$$

If $b < t_c$ (equivalently, there is no conjugate point in $(a, b]$), then ξ_* is a strict local minimizer of $\mathcal{L}_V$ within $\mathcal{V}$, in a sufficiently small C^1 neighborhood of ξ_*.

Proof. We invoke the Jacobi sufficient-condition theorem in the form of Gelfand–Fomin [16, Ch. 5, Sufficient Conditions for a Weak Extremum]. Its hypotheses hold here: the Lagrangian $L(\xi, \dot{\xi}) = m(\cosh\dot{\xi} - 1) - V(\xi)$ is C^2, the strong Legendre condition $L_{\dot{\xi}\dot{\xi}} = m\cosh(\dot{\xi}) \geq m > 0$ holds, ξ_* is a C^2 extremal, and $b < t_c$ says that no nontrivial Jacobi field vanishing at a has a second zero in $(a, b]$. The classical theorem therefore gives strict weak local minimality in a sufficiently small C^1 neighborhood. □

Theorem 8.4 (Long-time stationarity)**.** *Let $a < b$, $m > 0$, $V \in C^1(\mathbb{R})$, and let $\xi_* \in C^2([a, b])$ be a solution of (17) with fixed endpoints. Then the first variation of $\mathcal{L}_V$ at ξ_* vanishes on every C^1 endpoint-fixed variation η, without any claim of local or global minimality.*

Proof. For $\eta \in C^1([a, b])$ with $\eta(a) = \eta(b) = 0$, the standard first-variation formula applies: the kinetic term is smooth in $\dot{\xi}$, $V \in C^1$, and $\xi_* \in C^2$, so differentiating under the integral and integrating the kinetic term by parts gives

$$\delta\mathcal{L}_V[\xi_*](\eta) = \int_a^b \Big(\frac{d}{dt}\partial_{\dot{\xi}} L(\xi_*, \dot{\xi}_*) - \partial_{\xi} L(\xi_*, \dot{\xi}_*)\Big)\eta(t)\,dt.$$

Since ξ_* satisfies the Euler–Lagrange equation, this integral vanishes for all endpoint-fixed η. □

Theorem 8.5 (Conjugate-time obstruction to local minimality)**.** *Let $a < b$, $m > 0$, $V \in C^2(\mathbb{R})$, and let $\xi_* \in C^2([a, b])$ be a solution of (17) with fixed endpoints. If there exists a time $c \in (a, b)$*

conjugate to a along ξ_* (Definition 8.2), then ξ_* is not a strict local minimizer of $\mathcal{L}_V$ on the class $\mathcal{V}$ of C^1 endpoint-fixed variations.

Proof. The Lagrangian is C^2 and satisfies the strong Legendre condition $L_{\dot{\xi}\dot{\xi}} = m\cosh(\dot{\xi}) \geq m > 0$. The classical Jacobi necessary condition for a strict weak local minimum therefore applies to the extremal ξ_*. That theorem states that a strict weak local minimum cannot have a conjugate point to the initial endpoint in the open interval (a, b); see Gelfand–Fomin [16, Ch. 5, Necessary Conditions; Ch. 6]. The existence of such a c contradicts the necessary condition, so ξ_* is not a strict local minimizer on $\mathcal{V}$. □

Thus the variational principle changes form when a non-affine strictly convex potential is introduced: the free convex global minimum is replaced by short-time local minimality up to conjugate points and long-time stationarity only.

9. Discussion

9.1. Summary and scope

Our main result is a free-sector variational theorem on positive paths. The calibrated d'Alembert equation forces the cosh cost, and applying that cost to the log-velocity yields the kinetic action $\mathcal{A}[\gamma] = \int_a^b [\cosh(\dot{\xi}(t)) - 1]\,dt$ with $\xi = \log\gamma$. On the kinetically admissible class, $\mathcal{A}$ is *strongly* convex under geometric interpolation (Theorem 4.2), with an explicit L^2 slack of 1-strong convexity. The chord condition of Theorem 4.7 therefore implies global minimality with a quantitative gap (Theorem 4.5), and Corollary 4.10 identifies the unique fixed-endpoint minimizer explicitly as the uniform-log-velocity path. The action gap admits an exact Pythagorean / Bregman identity (Theorem 4.13) together with the Friedrichs–Poincaré bound $\mathcal{A}[\gamma] - \mathcal{A}[\gamma_*] \geq \frac{\pi^2}{2(b-a)^2}\|\log(\gamma/\gamma_*)\|_{L^2}^2$ (Corollary 4.15), and the minimum-action profile $\mathcal{A}_*(T, \Delta) = T(\cosh(\Delta/T) - 1)$ is the perspective transform of $\cosh - 1$, jointly convex and positively 1-homogeneous, giving the geodesic-concatenation inequality of Corollary 4.17. These results form a Bregman / dually-flat interpretation of the d'Alembert log-cost $\widetilde{J}$ in the additive coordinate $\xi = \log x$ (metric $\cosh\xi\, d\xi^2$), which is distinct from the Hessian metric $g_J = x^{-3}\,dx^2$ of J in the coordinate x (Remark 4.19).

On the nature of the contribution. The free-sector arguments are deliberately elementary. Once d'Alembert fixes the cosh cost (Theorem 2.2) and Postulate 2.10 places it at the log-velocity, the entire convexity package follows from pointwise convexity of cosh after the log change of coordinates (Remark 4.4), Jensen's inequality (Corollary 4.10), the perspective construction (Proposition 4.16), the one-dimensional Friedrichs inequality (Corollary 4.15), and textbook Bregman / dually-flat geometry (Remark 4.19); Theorem 4.7 is the specialization of the standard first-order optimality criterion for convex functionals (Remark 4.9). The novelty we claim is therefore one of *provenance and organization* – that a functional equation together with a single named modeling postulate forces a globally (not merely locally) minimizing free action with an exact Bregman gap and a closed-form geodesic minimizer – rather than of analytic depth. Correspondingly, the dynamical content actually forced by d'Alembert is narrow: free motion is the uniform-log-velocity geodesic, and the interacting system selected within this framework is the native cosh–sinh oscillator (the sharper scope statement below); all other potentials enter as external inputs.

This conclusion is a free-sector result; it does not assert global minimality for Lagrangians with non-affine strictly convex potentials. In that case the Lagrangian becomes a saddle in $(\xi, \dot{\xi})$, and Section 8 records the classical replacement: short-time local minimality, long-time stationarity, and conjugate-time obstructions. The bridge to Newtonian mechanics is also conditional. It requires the kinematic identification $q = \xi$, a mass coupling, a dimensionless time calibration, and the Hamiltonian-primary Legendre structure

(Section 6). With those choices the cosh action has the Newtonian small-step limit and the cosh-dual Hamiltonian reduces to $p^2/(2m) + V(q)$ at small momentum. These bridge statements are interpretive consequences of additional structure, not part of the free convexity theorem itself.

A sharper version of the scope statement is the following. Among all Lagrangians of the form $\mathcal{L}(\xi, \dot{\xi}) = K_m(\dot{\xi}) - V(\xi)$ with $m > 0$, the *only one forced by the d'Alembert calibration* is the native cosh–sinh Lagrangian $L_{\text{nat}}(\xi, \dot{\xi}) = m[\cosh \dot{\xi} - 1] - k[\cosh \xi - 1]$ of Definition 6.11, in which the same cosh -1 function appears in both the kinetic and the static slot under the same d'Alembert uniqueness. Every other potential – harmonic, Coulomb, gravitational, polynomial, lattice, periodic, … – is an *external input* from the cost-field environment that we do not derive from d'Alembert's equation here. The mathematically privileged dynamical system is therefore the cosh–sinh oscillator carrying two species couplings (m, k); even there, the binding coupling $k > 0$ is left as an unconstrained input, with its provenance (like that of m) lying outside the scope of this paper. For any external V, the results of Sections 6.4–8 (Newton's law in the small-velocity limit, Hamilton's equations, energy conservation, short-time local min / long-time stationarity / conjugate-time obstruction) apply, but they describe a *compatible* embedding of an externally specified mechanical system into the cosh kinetic framework, not a derivation of that mechanical system from d'Alembert.

9.2. Open directions

The most direct extensions are mathematical, and the Bregman / dually flat reading of §4.7 makes several of them concrete.

- **Multi-component / matrix lift.** On the cone $\mathbb{R}^n_{>0}$ the cost $J_n(x) = \sum_i J(x_i)$ inherits the Hessian metric $\sum_i x_i^{-3} dx_i^2$, geometric interpolation acts componentwise, and Theorems 4.2–4.13 extend verbatim. The non-trivial lift is to $\mathrm{Sym}^n_{>0}$ via $J(X) = \frac{1}{2}\mathrm{tr}(X + X^{-1}) - n$, where the affine-invariant geodesic structure suggests comparisons with log-Euclidean / Bures–Wasserstein geometry [17,18].
- **Discrete-time actions.** The discrete kinetic action $\sum_k (\cosh(\xi_{k+1} - \xi_k) - 1)$ inherits geometric- interpolation convexity and the Pythagorean identity at the discrete level, giving a discrete optimal-control statement parallel to Corollary 4.10.
- **Probabilistic identification.** The function $K(v) = \cosh(v) - 1$ is, up to normalization, the cumulant generating function of the Skellam$(\frac{1}{2}, \frac{1}{2})$ distribution [19]. This suggests that $\mathcal{A}$ is the rate functional of a continuous-time Skellam-type random walk on $\mathbb{R}$, with the geodesic of Corollary 4.10 appearing as the corresponding Schrödinger bridge [20].
- **Field-theoretic cosh-Dirichlet energy.** For $\gamma : M \to \mathbb{R}_{>0}$ on a Riemannian manifold (M, g), $\mathcal{E}_M[\gamma] = \int_M (\cosh |\nabla \log \gamma|_g - 1)\, d\mathrm{vol}_g$ is convex along log-affine homotopies; its Euler–Lagrange equation is a degenerate-elliptic cosh-Laplacian with growth interpolating between $|\nabla|^2$ and $e^{|\nabla|}$.
- **Comparison with the Hessian energy $\mathcal{E}_{\text{Hess}}$.** The relation between the Bregman geometry of $\mathcal{A}$ (§4.7) and the Otto-type geometry of $\mathcal{E}_{\text{Hess}}$ on the same base manifold $(\mathbb{R}_{>0}, J)$ [21–23] deserves explicit treatment: the two functionals differ in connection rather than in carrier.

These questions preserve our main theme: how much variational structure is already forced by the algebra of the underlying cost.

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Appendix A Hessian Riemannian path-energy on $(\mathbb{R}_{>0}, g_J)$

We collect here the structural facts about the Hessian path-energy

$$\mathcal{E}_{\mathrm{Hess}}[\gamma] \;=\; \int_a^b \tfrac{1}{2}\,\frac{\dot\gamma(t)^2}{\gamma(t)^3}\,dt$$

of §5.2 that we did not need for the free-sector convexity story of Section 4. The material here is included for completeness and to make the $\mathcal{A}$-vs-$\mathcal{E}_{\mathrm{Hess}}$ distinction unambiguous; we use none of it in the main theorems of this work. A fuller variational analysis of $\mathcal{E}_{\mathrm{Hess}}$, including any connection to Otto-type Wasserstein-gradient-flow geometry [21,23] on a related space of probability measures, is left to future work; we do not invoke Otto calculus here.

Appendix A.1 Explicit geodesic family

Theorem A1 (Explicit geodesic family)**.** *The family* $\gamma(t) = (at+b)^{-2}$ *with* $a \neq 0$ *and* $at+b > 0$ *satisfies the geodesic equation* (11)*.*

Proof. With $u = at+b$: $\gamma = u^{-2}$, $\dot\gamma = -2au^{-3}$, $\ddot\gamma = 6a^2u^{-4}$, $\Gamma(\gamma) = -\frac{3}{2}u^2$, and $\ddot\gamma + \Gamma(\gamma)\dot\gamma^2 = 6a^2u^{-4} - 6a^2u^{-4} = 0$. □

Appendix A.2 Geodesic completeness

Remark A2 (Geodesic completeness of $(\mathbb{R}_{>0}, g)$ at 0 but not at ∞)**.** The Riemannian distance element of $g(x) = x^{-3}$ is $ds = \sqrt{g(x)}\,dx = x^{-3/2}\,dx$. Direct integration gives

$$\int_0^1 x^{-3/2}\,dx \;=\; +\infty, \qquad \int_1^\infty x^{-3/2}\,dx \;=\; 2.$$

Thus the boundary $x = 0$ is at infinite Riemannian distance from any interior point (the metric is complete at 0), while $x = \infty$ is at finite Riemannian distance (the metric is incomplete at ∞). Consistently, the geodesic family $\gamma(t) = (at+b)^{-2}$ with $a < 0$ reaches $\gamma = +\infty$ at the finite parameter $t = -b/a$, whereas $\gamma \to 0^+$ requires $|t| \to \infty$.

This (in)completeness is a property of the Hessian path-energy $\mathcal{E}_{\mathrm{Hess}}$ and its underlying metric g, *not* of the kinetic action $\mathcal{A}$, which uses the log-affine (e-)connection rather than the Riemannian connection of g. Our free-sector theorems of Section 4 do not depend on completeness of g. A geodesically complete continuation of $(\mathbb{R}_{>0}, g)$ at ∞ is left to future work.

Appendix A.3 Structural comparison of $\mathcal{A}$ and $\mathcal{E}_{\mathrm{Hess}}$

Remark A3 ($\mathcal{A}$ and $\mathcal{E}_{\mathrm{Hess}}$ are distinct functionals)**.** The kinetic Euler–Lagrange equation (Theorem 5.3) selects uniform-log-velocity paths $\gamma(t) = x_a\,e^{\dot\xi_0(t-a)}$. The Hessian-metric geodesic equation (11) selects the power-law family $\gamma(t) = (at+b)^{-2}$ together with the limiting positive constant geodesics. As *parameterized* curves $t \mapsto \gamma(t)$ these are different families on $(0,\infty)$, though both include every positive constant path as a trivial solution. We stress that the distinction is one of parameterization, not of trajectory: on the 1-manifold $\mathbb{R}_{>0}$ any two fixed-endpoint geodesics share the same image – the arc between the endpoints – since a 1-dimensional manifold admits no choice of route, and a connection on it carries no curvature and only fixes which time-law $t \mapsto \gamma(t)$ is affine/constant-speed. Thus the exponential and power-law solutions above are reparameterizations of one another as curves; what differs is the time-law each connection selects as its geodesic parameterization. The additional static cost-rate condition for $\mathcal{J}$ selects only $\gamma \equiv 1$, so all three critical conditions agree precisely at the normalized ground state. Accordingly $\mathcal{A}$ and $\mathcal{E}_{\text{Hess}}$ are genuinely different functionals – they score parameterizations differently and pick out different extremal time-laws – rather than geometrically distinct trajectories in $\mathbb{R}_{>0}$.

The kinetic action $\mathcal{A}$ is the d'Alembert-calibrated object – J applied to the infinitesimal *step* – and the free-sector convexity theorems of Section 4 apply to it. The Hessian path-energy $\mathcal{E}_{\text{Hess}}$ is a separate Riemannian path-energy whose natural notion of convexity is *geodesic convexity* along the geodesics of the underlying metric $g(x) = x^{-3}$ – a different notion from the geometric-interpolation (log-affine) convexity of $\mathcal{A}$. The two pictures are complementary rather than equivalent: $\mathcal{A}$ formalizes the action principle from the d'Alembert side, while $\mathcal{E}_{\text{Hess}}$ formalizes the Riemannian geometry of the choice manifold from the Hessian side. We use neither $\mathcal{E}_{\text{Hess}}$ nor its geodesic structure in our main theorems; a fuller development of $\mathcal{E}_{\text{Hess}}$'s geodesic and Otto-Wasserstein structure is left to future work.

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