A Conditional Scalar Uniqueness Theorem for the Quadratic Rule in a Two-Outcome Cost-Functional Measurement Model

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Abstract

This paper proves a conditional scalar uniqueness theorem for the quadratic rule $P(c) = |c|^2$ within an adopted two-outcome cost-functional measurement model. It does not provide a general reconstruction of quantum probability or a derivation of the quadratic rule from cost axioms alone. More precisely, within an adopted two-outcome binary-split measurement model, given the scalar ansatz (SA), the 2-norm convention (NC), continuity (CS), product-amplitude composition (PA), and binary no-signaling $(\mathrm{P5})_2$, the quadratic rule $P(c) = |c|^2$ is the unique scalar solution. Importantly, Route B replaces the modulus-multiplicativity axiom (MA) with product-amplitude composition (PA) and binary no-signaling $(\mathrm{P5})_2$ rather than adopting (MA) as a primitive; once $P(c) = |c|^2$ is established from this package, (MA) follows as an algebraic corollary. It further shows that the reduced scalar chart T_{bin} admits inequivalent quantum and classical completions, so this fragment alone does not determine Hilbert-space structure, while the adopted two-outcome geometry is internally consistent. It remains open whether the full geometric package (G1)–(G4) is equivalent to, strictly weaker than, or incomparable with Hilbert-space structure, whether the assumption package is minimal, and whether the analysis extends beyond the present two-outcome model.

I. INTRODUCTION

The Born rule assigns to each outcome amplitude c the probability $P(c) = |c|^2$. Any rigorous claim about that rule must specify two things with precision: the exact theorem proved and the exact assumptions under which it is proved. Here the question is narrow: in an adopted two-outcome binary-split measurement model, do the scalar ansatz (SA), the 2-norm normalization convention (NC), continuity of the scalar rule (CS), product-amplitude composition (PA), and binary no-signaling $(\mathrm{P5})_2$ (all formally defined in Section IV) force the quadratic rule $P(c) = |c|^2$ uniquely? The paper answers yes within the adopted model, but does not claim a general reconstruction of quantum probability.

The broader literature approaches the Born-rule problem at several structural levels. Gleason-type theorems constrain probability assignments on Hilbert-space projection structure once that framework is already in place [1–7]. Operational reconstructions such as

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Hardy's and Chiribella–D'Ariano–Perinotti's derive the rule as part of a wider reconstruction of quantum theory [8–13]. Other programs relocate the probabilistic work by appealing to entanglement symmetry, decision-theoretic rationality, or normative coherence [14–21]. A 2025 result by Torres Alegre shows that, within finite-dimensional GPTs satisfying the purification postulate, no-signaling with respect to steering is sufficient to select $P = |c|^2$ as the unique causally consistent probability rule [22]. For the present paper, the relevant comparison is therefore not theorem strength but premise placement: where the probabilistic work is done, and whether additivity or composition structure has genuinely been reduced or merely relocated elsewhere in the setup [23–27]. Section VII B returns to this comparison in schematic form.

The RS framework starts from a different primitive: a cost functional on positive ratios together with ledger bookkeeping, path actions, and a d'Alembert-type functional-equation uniqueness chain for the cost itself [28–32]. It therefore contrasts both with Gleason-type approaches, which presuppose Hilbert-space measure structure, and with Hardy- or CDP-style reconstructions, which begin with operational axioms about states, composites, and transformations. The RS setup begins with neither. Instead it derives a unique cost geometry from a reciprocal composition law and builds path weights and complex amplitudes from that geometry. Those ingredients fix single-path amplitude moduli via the $C = 2A$ bridge and later support multiplicative modulus structure under subsystem composition, but they do not yet determine the probability law for outcomes reached by multiple paths. The live question is therefore which additional assumptions, beyond cost and amplitude bookkeeping, force the quadratic rule.

The argument has three layers. First, Route A is the benchmark calculation: if one assumes modulus-multiplicativity (MA), i.e. $P(c_1c_2) = P(c_1)P(c_2)$, then the quadratic rule follows by a standard scalar functional-equation argument. Second, Route B establishes the quadratic rule from the shared premises (SA), (NC), and (CS) together with subsystem composition (PA) and binary no-signaling $(\mathrm{P5})_2$, without taking (MA) as a primitive scalar axiom; once $P(c) = |c|^2$ is established, modulus-multiplicativity follows algebraically as a corollary (Remark V.8). Third, both scalar routes are separated from the more limited geometric consistency checks supplied by the adopted binary-split model (G1)–(G4). Proposition V.6 is the functional core, Theorem VI.6 packages the two-outcome result, and Theorem VI.11 shows that the reduced scalar chart T_{bin} does not determine Hilbert-space structure among its completions.

The main result is thus a precise identification of the assumption package within the adopted RS-inspired two-outcome model under which the scalar quadratic rule is uniquely selected, together with the explicit dependency map in Table I. Whether this package is minimal remains open.

Structure of the paper. Section II fixes the empirical target by recalling Born's original scattering rule. Sections III–VI then develop the core argument: the RS setup, the assumption inventory, the two scalar routes, and the adopted two-outcome geometry with its packaged theorem and scope limitation. Section VII records post-theorem checks and literature comparison, Section VIII gives the empirical meaning and conclusion, Section IX sketches a broader RS research program not used in the proofs, and Appendix A separates public Lean-verified upstream RS ingredients from manuscript-only Born-rule-facing claims.

Figure 1 summarizes this architecture: empirical target, RS setup, assumption inventory, Route B functional core, packaged two-outcome theorem, and scope limitation, with Route A and the T_{bin} scope limitation shown as side branches.

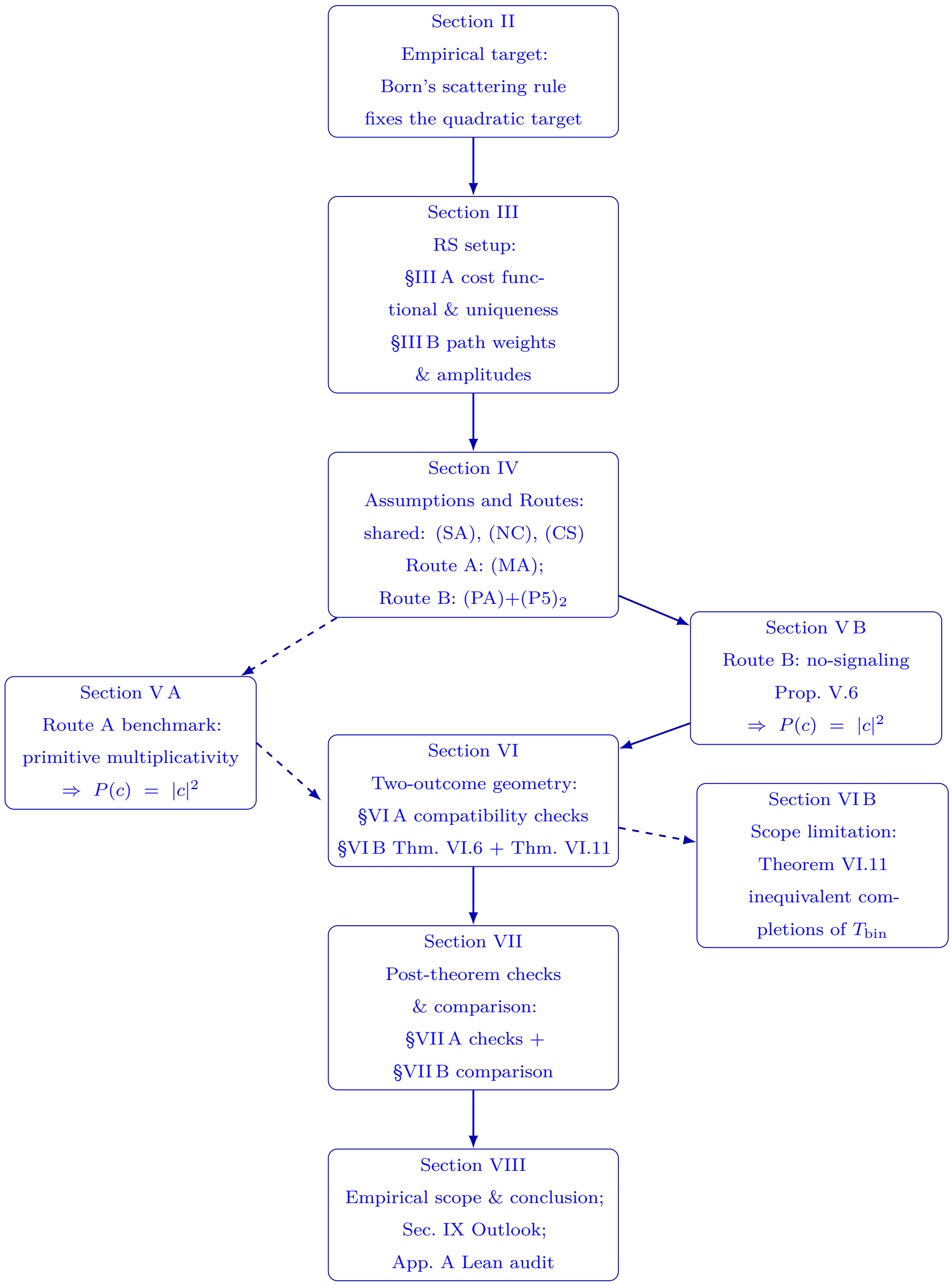

Figure 1. Roadmap of the argument. The solid central path tracks the main conditional route from the empirical target through the RS setup, the Route B functional core, and the packaged two-outcome theorem. Dashed arrows mark side branches that clarify scope rather than add a new derivational route: Route A is the benchmark calculation, and Theorem VI.11 limits what the reduced chart T_{bin} can establish.

II. BORN'S 1926 SCATTERING RULE AS THE EMPIRICAL TARGET

Born's rule first appeared in particle scattering [33, 34]. For an incident plane wave e^{ikz} scattered by a rotationally symmetric potential, the wave function at large radius $\rho = |\mathbf{r}|$ takes the form

$$\psi(\mathbf{r}) \xrightarrow{\rho\to\infty} e^{ikz} + f_{\rm sc}(\theta)\,\frac{e^{ik\rho}}{\rho}, \tag{1}$$

where $f_{\rm sc}(\theta)$ is the scattering amplitude. The differential cross-section satisfies

$$\frac{d\sigma}{d\Omega} = |f_{\rm sc}(\theta)|^2, \tag{2}$$

so the detection probability in direction θ is proportional to $|f_{\rm sc}(\theta)|^2\, d\Omega$. This square-modulus dependence is the *quadratic rule* [35–38].

The key point for this paper is that the RS framework assigns each path a positive weight $w = e^{-C}$ that is phase-blind, so the passage from path weight to outcome probability requires additional assumptions. Sections III–VI identify exactly which assumptions—scalar ansatz (SA), 2-norm convention (NC), continuity (CS), product-amplitude composition (PA), and binary no-signaling $(\mathrm{P5})_2$—uniquely force the quadratic dependence on outcome amplitudes in the adopted two-outcome model.

III. THE RS FRAMEWORK: COST FUNCTIONAL, PATHS, AND AMPLITUDES

This section sets up the two RS ingredients used throughout the paper. Section III A fixes the cost functional J and records its uniqueness theorem. Section III B defines path actions $C[\gamma]$, positive weights $w(\gamma) = e^{-C}$, and complex amplitudes $\psi(\gamma) = e^{-C/2}e^{i\phi}$. Neither component by itself determines a probability law; together they supply the objects used in Section V.

A. Cost Functional and Uniqueness

This subsection states the two RS axioms on the cost functional $J : (0, \infty) \to \mathbb{R}$, which assigns a mismatch cost to each positive ratio, and records the resulting uniqueness theorem. It does not yet supply a probability law or a measurement model; the phase variable introduced in Section III B is additional structure, not a consequence of the cost axioms.

Definition III.1 (Ledger)**.** A ledger is the RS bookkeeping device for recognition events. Formally it is a collection of triples $(t_i, r_i, J(r_i))$, where t_i indexes the event, $r_i > 0$ is the compared ratio, and $J(r_i)$ is its assigned mismatch cost. The specific form of J is derived below from axioms (A1)–(A2) in Theorem III.4. A ledger region is any subset of these events. Later sections use disjoint ledger regions to represent independent subsystems [31, 39]; that subsystem interpretation is part of the adopted RS setup.

Remark III.2 (Discrete ledger entries and continuous path actions)*.* The ledger picture is discrete, whereas the path action introduced in Definition III.6 is written as a continuum integral. Here a continuum idealization is adopted: when the event spacing is sufficiently fine relative to the scale of the path description, the discrete sum $\sum_i J(r_i)\,\Delta(\ln r_i)$ is replaced by $\int J(r)\,d(\ln r)$. All later path-action manipulations assume this regime. A genuinely discrete treatment is not developed here.

The guiding principle is reciprocal symmetry: exchanging overshoot and undershoot sends ratio x to $1/x$, so mismatch in opposite directions should be treated symmetrically. In the present framework that principle is encoded in two axioms [28–30, 32]:

(A1) $J(xy) + J(x/y) = 2J(x)J(y) + 2J(x) + 2J(y),$

for all $x, y > 0$ (Recognition Composition Law, RCL)

(A2) $\left.\dfrac{d^2}{dt^2} J(e^t)\right|_{t=0} = 1$ (Calibration)

Lemma III.3 (Normalization from the RCL)**.** *Any solution satisfying (A1)–(A2) obeys* $J(1) = 0$*. Continuity is not required for this conclusion; it enters only in Theorem III.4 for the d'Alembert classification.*

Proof. Setting $x = y = 1$ in (A1) gives $J(1) + J(1) = 2J(1)^2 + 4J(1)$, i.e. $2J(1)^2 + 2J(1) = 0$, so $J(1) \in \{0, -1\}$. Now set $y = 1$ in (A1). Then

$$J(x) + J(x) = 2J(x)J(1) + 2J(x) + 2J(1),$$

hence $J(1)(J(x) + 1) = 0$ for every $x > 0$. If $J(1) = -1$, it follows that $J(x) = -1$ for all $x > 0$, so J is constant. But a constant function has $\left.\frac{d^2}{dt^2} J(e^t)\right|_{t=0} = 0 \neq 1$, contradicting (A2). Therefore $J(1) = 0$. □

Theorem III.4 (Cost Uniqueness)**.** *The unique continuous real-valued solution to (A1)–(A2) is*

$$J(x) = \tfrac{1}{2}(x + x^{-1}) - 1, \tag{3}$$

satisfying $J(x) \geq 0$ *with equality iff* $x = 1$*.*

Proof sketch. By Lemma III.3, $J(1) = 0$. Setting $H(t) = 1 + J(e^t)$ reduces the RCL to the d'Alembert equation

$$H(s + t) + H(s - t) = 2H(s)H(t). \tag{4}$$

Continuity of J gives continuity of H. Standard classification results for continuous real solutions of d'Alembert's equation [40] then leave only the cosh-family, the cos-family, and the constant/zero degeneracies. Here $H(0) = 1$ and calibration gives $H''(0) = 1 > 0$, so the oscillatory, zero, and constant branches are excluded and one obtains uniquely $H(t) = \cosh t$. Hence

$$J(x) = \tfrac{1}{2}(x + x^{-1}) - 1.$$

The full derivation is given in [29]; a Lean 4 formalization of this uniqueness result appears in [32, 41]. □

Axiom (A1) encodes the composition rule for mismatch costs, Axiom (A2) fixes the local calibration at exact balance, and continuity then forces the unique closed form (3). This fixes the ratio-cost geometry used in Section III B, but not yet a probability law, phase structure, or measurement model.

Remark III.5 (Immediate structural consequences of the closed form)*.* Equation (3) yields three structural consequences, not additional postulates. First, $J(x) = J(1/x)$ (reciprocal symmetry): overshoot and undershoot carry the same cost. Second, $J(x) \geq 0$ with equality only at $x = 1$: exact balance is the unique zero-cost state; this ensures $C[\gamma] \geq 0$ and hence $w(\gamma) = e^{-C[\gamma]} \in (0, 1]$, which is used in the path-weight construction. Third, $J(x) \to \infty$ as $x \to 0^+$ or $x \to \infty$: extreme mismatch is infinitely costly. The first and third are structural background; the second is directly used in Section III B.

B. Path Weights and Amplitudes

This subsection defines the path-level objects used throughout the paper: the path action $C[\gamma]$, the path weight $w(\gamma)$, and the complex amplitude $\psi(\gamma)$. The construction assigns a cost-derived modulus and a free phase to each path, but it remains per-path bookkeeping only: it does not yet provide a normalized path measure, an outcome probability rule, or an outcome-level sum-over-paths postulate in the sense needed for Born-rule claims [42, 43]. Those questions are deferred to Sections IV–VI.

Definition III.6 (Path Action)**.** A recognition path γ is a continuous curve in the space of positive ratios $(0, \infty)$, parameterizing a sequence of scale-ratio comparisons on a ledger region. For such a path, the path action is

$$C[\gamma] = \int_\gamma J(r)\, d(\ln r), \tag{5}$$

where $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ is the unique cost functional of Theorem III.4, r is the local scale ratio along the path, and $d(\ln r) = dr/r$ is the adopted logarithmic integration measure.

Remark III.7 (Conventions on the path-action measure)*.* The choice of $d(\ln r) = dr/r$ is part of the RS formulation adopted here [28, 29]. Because positive ratios compose multiplicatively, this logarithmic measure is the natural left-invariant measure on $(\mathbb{R}_{>0}, \cdot)$; equivalently, it is the Haar measure on that group up to scale. Here that observation motivates the convention but does not derive it. The choice matters later because $\int J(r)\, d(\ln r)$ diverges on the semi-infinite measurement branches of Section VI; there the finite quantity is the induced half-action rate instead (Remark VI.3).

Definition III.8 (Path Weight and Amplitude)**.** Each path is assigned a positive weight $w(\gamma) = e^{-C[\gamma]}$ and a complex amplitude

$$\psi(\gamma) = e^{-C[\gamma]/2} \cdot e^{i\phi(\gamma)}, \tag{6}$$

where $\phi(\gamma) \in \mathbb{R}/2\pi\mathbb{Z}$ is an additional phase variable not fixed by the cost alone. With this definition, the single-path modulus is tied identically to the cost:

$$|\psi(\gamma)|^2 = e^{-C[\gamma]} = w(\gamma).$$

Remark III.9 (The $C = 2A$ bridge)*.* Definition III.8 gives $|\psi(\gamma)| = e^{-C[\gamma]/2}$ directly. Writing $A(\gamma) := -\ln|\psi(\gamma)|$ for the *half-action* of path γ, this is equivalent to $C[\gamma] = 2A(\gamma)$. This relation—referred to throughout Section VI as the $C = 2A$ bridge—is an algebraic consequence of Definition III.8 alone; it requires no additional assumption. In particular, it does not presuppose the Born rule: the modulus $|\psi|$ is assigned by the cost, not by an outcome probability. Section VI later uses the bridge to convert the convergent half-action integral of Proposition VI.2 into the adopted branch modulus.

Remark III.10 (Why the single-path identity is not yet the Born rule). Definition III.8 gives $|\psi(\gamma)|^2 = w(\gamma)$ by algebra, so no Born-rule theorem has yet been obtained. The crucial distinction is between a single-path amplitude and the amplitude for an outcome. If several paths lead to the same outcome and one writes $\psi_i = \sum_{\gamma \to i} \psi(\gamma)$, then in general

$$\left| \sum_{\gamma \to i} \psi(\gamma) \right|^2 \neq \sum_{\gamma \to i} |\psi(\gamma)|^2$$

because interference produces cross-terms. Probabilities therefore become an outcome-level question, not a path-by-path one. The positive path weight $w = e^{-C}$ is blind to the phase $\phi(\gamma)$, since the cost functional depends only on positive ratios; that phase blindness is the structural motivation for the later phase-independence assumption on probabilities. The actual Born-rule-facing question is therefore postponed to Sections IV–VI, where outcome amplitudes, subsystem composition, and scalar probability assignments are constrained by additional assumptions.

IV. ASSUMPTIONS AND ROUTES

This section presents and separates the assumptions underlying the conditional uniqueness claims, keeping three conceptual layers explicitly distinct: the framework conventions (P1)–(P7), the shared scalar-reduction premises (SA), (NC), (CS), and the two route-specific packages. Both routes require the scalar ansatz (SA), the 2-norm normalization convention (NC), and continuity (CS). Route A additionally assumes modulus multiplicativity (MA). Route B instead adopts product-amplitude composition (PA) and binary no-signaling $(\mathrm{P5})_2$, without assuming (MA) as a primitive axiom. The section is therefore not a flat list of premises but a structured logical map, summarized in Table I.

a. Complete list of postulates. The present analysis uses the following labeled assumptions and conventions:

- (P1) Cost-functional package: axioms (A1)–(A2), together with the continuity hypothesis on J used in Theorem III.4, fix the cost functional.
- (P2) Phase degree of freedom: each recognition path carries a phase $\phi(\gamma) \in \mathbb{R}/2\pi\mathbb{Z}$ independent of the cost magnitude. This is not derived from (A1)–(A2) here. The broader RS

forcing chain offers structural motivation via the 8-tick periodicity (Remark A.1), but that derivation is not reproduced here.

(P3) Log-ratio integration measure $d(\ln r)$ on path actions: the specific integration convention adopted in Definition III.6.

(P4) Geometric definitions (G1)–(G4): the adopted binary-split measurement model, including the angular parametrization, branch-modulus identification, and angular projection used in Section VI.

(P5) No-signaling for disjoint ledger regions: the subsystem-isolation condition stated in Definition IV.9. In the proofs, the binary restriction of this condition is used and denoted $\mathbf{(P5)}_2$.

(P6) Continuity of the scalar probability function: a regularity assumption used to rule out pathological Cauchy-equation solutions. This postulate is assigned the short label **(CS)** in the theorems and in Table I.

(P7) Product-amplitude composition: for independent subsystems, the joint outcome amplitude is modeled by the product rule of Remark IV.8. Corollary IV.7 supplies only the modulus part; additive phase is stipulated. This postulate is assigned the short label **(PA)** in the theorems and in Table I.

Three caveats are essential. Postulate (P2) is adopted independently of the cost axioms; the 2-norm normalization convention (NC) is not part of the (P1)–(P7) list and enters separately as a modeling choice; and whether the cost-functional package together with the geometric package (G1)–(G4) is genuinely weaker than Hilbert-space-equivalent structure remains unresolved (Remark VI.5). The exact premise dependencies of each result are therefore given in Table I, rather than by citing the full (P1)–(P7) list en bloc.

b. Labeled adopted conventions. The theorem statements use six short labels. Two are aliases for postulates already listed above—**(CS)**≡**(P6)** and **(PA)**≡**(P7)**. Three—**(SA)**, **(NC)**, and **(MA)**—are additional modeling choices not covered by (P1)–(P7). The last, $\mathbf{(P5)}_2$, is the binary restriction of postulate (P5). These identifications keep the theorem statements consistent with Table I.

(SA): *Scalar Ansatz.* $P(c_i) = f(|c_i|)$ for some function $f : [0, \infty) \to [0, \infty)$. Probability depends only on the amplitude modulus. This is adopted as a modeling hypothesis; it is not derived from (A1)–(A2).

(NC): *2-Norm Convention.* Outcome amplitudes satisfy $\sum_i |c_i|^2 = 1$. This normalization is adopted independently of the cost axioms; Section VI provides only an internal two-outcome compatibility check, not an upstream derivation from assumptions already shown to be weaker than Hilbert-space structure.

(PA)≡(P7): *Product-Amplitude Composition.* For independent subsystems, the joint outcome (i, j) is assigned amplitude $c_{1i} \cdot c_{2j}$. The modulus rule $|\psi_{12}| = |\psi_1| \cdot |\psi_2|$ follows from Corollary IV.7; the phase rule $\phi_{ij} = \phi_i + \phi_j$ is stipulated separately and is not derived from (A1)–(A2).

(CS)≡(P6): *Continuity of Scalar Rule.* The function f in (SA) is continuous on $[0, \infty)$. This is the regularity assumption used here to exclude pathological solutions of the additive Cauchy equation [40]. Stronger or weaker sufficient alternatives, such as monotonicity, measurability, local boundedness, or continuity at a point, are not used here; see Remark V.4.

Route A additionally requires:

(MA): *Modulus Multiplicativity.* $P(c_1 c_2) = P(c_1)P(c_2)$ for all independent amplitude pairs. This is the decisive assumption in Route A: once it is in place, the quadratic rule $P(c) = |c|^2$ follows by a purely classical functional-equation argument without any RS-specific geometric input (Theorem V.1).

Remark IV.1 (Two-layer role of the RS framework)*.* The RS framework plays two different roles in the later Born-rule analysis. First, it supplies the specifically RS objects used in this manuscript: the cost functional J, the ledger bookkeeping picture, the path-action convention, and the subsystem language of disjoint ledger regions. Second, it supplies motivation, but not full derivation, for several scalar probability choices, especially phase independence and the restriction to modulus-based probability laws. Once those extra choices are adopted, the later Route A and Route B arguments proceed by largely standard scalar functional-equation reasoning. The distinctive contribution of the paper is therefore not that every Born-rule-facing step is forced directly by the cost axioms alone, but that within the adopted RS measurement model one can isolate exactly which extra assumptions yield the quadratic law and which remain open.

Route A and Route B are distinct. Route A is the classical scalar uniqueness route: given (SA), (NC), (CS), and multiplicativity (MA), the quadratic rule $P = |c|^2$ follows by the standard Cauchy-equation argument (Theorem V.1). Route B does *not* use (MA) as a primitive scalar axiom; it replaces it with the binary subsystem package (PA)+$(\mathrm{P5})_2$ to derive $f(r) = r^2$ (Proposition V.6). This relocates the structural work from a primitive scalar product law to subsystem composition and no-signaling.

We now formalize the scalar probability function, then record the Route A requirements and the Route B subsystem package in that order.

Definition IV.2 (Probability Function)**.** A probability function P assigns to each outcome amplitude c_i in a fixed measurement context a real number $P(c_i)$ representing the probability of outcome i. The notation suppresses dependence on the full normalized state and measurement basis; the structural requirements below concern the functional dependence on the single amplitude argument.

This suppression carries a strong scalar-reduction assumption: $P(c_i)$ depends on c_i alone, not on the full normalized state, the measurement basis, or which other outcomes are jointly measurable. The setup is therefore narrower than Gleason-type frame-function analyses, which formulate probability assignments on full measurement contexts under a noncontextuality condition. Within the present RS framework the reduced form $P(c_i) = f(|c_i|)$ is adopted because the cost mechanism is defined on positive scale ratios and is blind to phase (Remark III.10). That alignment is motivational only. The paper therefore analyzes this restricted scalar class, not general contextual or state-dependent probability assignments.

Definition IV.3 (Scalar probability requirements used in Route A)**.** A scalar probability function in the class (SA) is required to satisfy all five of the following conditions. Conditions (i)–(iii) and (v) are shared with Route B; condition (iv) is the Route-A-specific hypothesis. Note: condition (i) follows immediately from (iii)/(SA), since $f : [0, \infty) \to [0, \infty)$ forces $P(c_i) = f(|c_i|) \geq 0$; it is listed explicitly for completeness and for direct comparison with standard Born-rule axiom sets in the literature.

(i) *Non-negativity:* $P(c_i) \geq 0$ for all c_i. [Subsumed by (SA); listed for completeness.]

(ii) *Normalization* [(NC)]: $\sum_i P(c_i) = 1$ whenever $\sum_i |c_i|^2 = 1$.

(iii) *Phase independence* [(SA)]: $P(c_i) = f(|c_i|)$ for some $f : [0, \infty) \to [0, \infty)$.

(iv) *Modulus multiplicativity* [(MA)]: $P(c_1 c_2) = P(c_1)P(c_2)$ for independent amplitude pairs.

(v) *Continuity* [(CS)]: f is continuous on $[0, \infty)$.

c. Motivation for the scalar requirements. Requirements (i)–(iii) and (v) are shared by both routes; multiplicativity (iv) is Route A only. Item (i) is subsumed by (SA). Item (ii) is the adopted (NC); Section VI records a two-outcome internal check, not a derivation of (NC) from the cost axioms. Item (iii)/(SA) is motivated by phase-blind costs but remains an adopted modeling restriction, not a general no-contextuality theorem.

Route B replaces multiplicativity (iv)/(MA) by the binary subsystem package. The *axioms* in that package are: (PA) (product-amplitude composition, Remark IV.8) and $(P5)_2$ (binary no-signaling, Definition IV.9). Cost additivity over disjoint regions (Definition IV.4) then yields weight multiplicativity as a *theorem* (Proposition IV.6), which in turn yields modulus multiplicativity (Corollary IV.7). Route B therefore does not posit weight multiplicativity as an axiom; it derives it from the disjoint-ledger structure and uses no-signaling to close the functional-equation argument. The premise load is thus relocated rather than simply reduced.

Definition IV.4 (Composite path in the composite continuum regime)**.** For two independent subsystems with paths γ_1 and γ_2 over disjoint ledger regions Λ_1 and Λ_2 (Definition III.1), assume that each subsystem and their union $\Lambda_1 \cup \Lambda_2$ lie in the continuum regime of Remark III.2. The composite path γ_{12} is then defined with integration domain $\Lambda_1 \cup \Lambda_2$. Since $\Lambda_1 \cap \Lambda_2 = \varnothing$, the continuum path-action integral over $\Lambda_1 \cup \Lambda_2$ splits by additivity of integration, yielding $C[\gamma_{12}] = C[\gamma_1] + C[\gamma_2]$.

Remark IV.5 (Composite continuum regime). The assumption that the composite system $\Lambda_1 \cup \Lambda_2$ lies in the continuum regime does not follow automatically from the individual regime conditions on Λ_1 and Λ_2 separately; it is an additional hypothesis of the product-amplitude composition model. In practice, it requires that the two subsystems' ledger regions do not interleave at a granularity that would invalidate the continuum approximation for the union. For subsystems that are spatially or temporally well-separated—the standard setting in which no-signaling is physically motivated—this condition is typically satisfied, because the disjoint regions can be ordered into two contiguous blocks, each already in the continuum regime. The case of subsystems whose ledger regions are finely interleaved at the inter-event scale falls outside the continuum regime assumed here and would require a discrete-sum treatment not developed in the present paper.

Proposition IV.6 (Cost-forced weight multiplicativity)**.** *If independent subsystems occupy disjoint ledger regions (Definition III.1), then the path weight* $w = e^{-C}$ *is multiplicative:* $w_{12} = w_1 \cdot w_2$.

Proof. By Definition IV.4, $C_{12} = C_1 + C_2$. Exponentiating: $w_{12} = e^{-C_{12}} = e^{-C_1} \cdot e^{-C_2} = w_1 w_2$. □

Corollary IV.7 (Tensor-product amplitude structure)**.** *For independent subsystems, the joint amplitude modulus is the product of the individual moduli:* $|\psi_{12}| = |\psi_1| \cdot |\psi_2|$.

Proof. By the $C = 2A$ bridge (Remark III.9), $|\psi| = e^{-C/2}$. Weight multiplicativity (Proposition IV.6) gives $|\psi_{12}| = e^{-C_{12}/2} = e^{-C_1/2} \cdot e^{-C_2/2} = |\psi_1| \cdot |\psi_2|$. □

Remark IV.8 (Operational composition of amplitudes). For independent subsystems we model joint outcomes by product amplitudes: if subsystem 1 has outcome amplitude c_{1i} and subsystem 2 has outcome amplitude c_{2j}, then the joint outcome (i, j) is assigned amplitude $c_{1i}c_{2j}$. Corollary IV.7 supplies the modulus part of this rule; the additive phase convention $\phi_{ij} = \phi_i + \phi_j$ is adopted and is not derived from (A1)–(A2). The product-amplitude rule for independent subsystems is the RS analogue of the tensor-product postulate in standard quantum mechanics; it is a substantive structural assumption. Proposition V.6 uses this product-amplitude representation as part of the subsystem model.

Definition IV.9 (No-signaling condition under the adopted 2-norm convention)**.** For subsystems occupying disjoint ledger regions and composed according to the product-amplitude model of Remark IV.8, the marginal probability of outcome i for subsystem 1 is invariant under changes of the measurement performed on subsystem 2. Here this condition is imposed only for subsystem-2 outcomes $\{j\}$ with amplitudes $\{c_{2j}\}$ satisfying the adopted normalization convention

$$\sum_j |c_{2j}|^2 = 1. \tag{7}$$

Thus

$$\sum_j P_{\text{joint}}(c_{1i} \cdot c_{2j}) = P(c_{1i}) \tag{8}$$

for every choice of measurement basis for subsystem 2. Here $P_{\text{joint}}(c_{1i} \cdot c_{2j}) := f(|c_{1i} \cdot c_{2j}|) = f(|c_{1i}||c_{2j}|)$ denotes the scalar rule (SA) applied to the product amplitude; this application of (SA) to joint outcomes is an explicit hypothesis of the model, not a consequence of

(SA) alone. The marginal is obtained by summing over subsystem-2 outcomes. This is the no-signaling condition specialized to the present scalar product-amplitude model; more general contextual or nonlinear probability assignments are outside the scope of the definition. For the uniqueness proof below, it suffices to impose this condition for binary measurements on subsystem 2; this binary restriction is denoted $(\mathrm{P5})_2$. Proposition V.6 should therefore be read as conditional on the adopted scalar/product-amplitude model and normalization convention, together with the binary no-signaling premise itself; it is not an independent derivation of those conventions.

This completes the assumption inventory. The two scalar derivations—Route A as benchmark and Route B as the no-signaling route—are given in Section V.

d. Assumption dependency summary. Table I maps each main result to the exact set of assumptions it uses. An entry "✓" means the assumption is directly invoked in the proof; "—" means it is not used. This table is the authoritative single-location reference for the premise load of each theorem.

Route A (Thm. V.1) and Route B (Prop. V.6) differ in route-specific hypotheses—(MA) versus $(\mathrm{PA})+(\mathrm{P5})_2$—while sharing (SA), (NC), and (CS). The geometric package (G1)–(G4) enters only in Section VI and in Thm. VI.6, not in the abstract scalar argument Prop. V.6.

V. CONDITIONAL SCALAR UNIQUENESS FOR THE QUADRATIC RULE

This section gives two independent scalar routes to $P(c) = |c|^2$ within the adopted model. Section V A records the benchmark calculation with primitive multiplicativity (MA): the substitution $g(x) = \ln f(e^x)$ turns $f(r_1 r_2) = f(r_1)f(r_2)$ into the additive Cauchy equation, and normalization then fixes the quadratic exponent. Section V B replaces (MA) by the no-signaling subsystem package of Section IV; there the substitution $h(x) = f(\sqrt{x})$ reaches the additive Cauchy equation directly from the binary no-signaling constraint.

A. Route A: Benchmark Under Primitive Multiplicativity

a. Benchmark status. Route A is the benchmark calculation: once one adopts the scalar ansatz (SA), the 2-norm normalization convention (NC), continuity (CS), and multiplicativity (MA), the quadratic rule follows by a standard functional-equation argument. This makes

TABLE I. Assumption dependency map. (A1)–(A2): RS cost axioms. (SA): scalar ansatz. (NC): 2-norm convention. (CS): continuity of scalar rule f—distinct from the continuity of J used in Thm. III.4. (MA): modulus multiplicativity (Route A only). (PA): product-amplitude composition. $(P5)_2$: binary no-signaling. (G1)–(G4): binary-split geometry.

Result	**(A1)–(A2)**	**(SA)**	**(NC)**	**(CS)**	**(MA)**	**(PA)**	**$(P5)_2$**	**(G1)–(G4)**
Lem. III.3 / Thm. III.4	✓	—	—	—*	—	—	—	—
Prop. IV.6 (weight mult.)	✓	—	—	—	—	—	—	—
Cor. IV.7 (modulus mult.)	✓	—	—	—	—	—†	—	—
Thm. V.1 (Route A)	—	✓	✓	✓	✓	—	—	—
Prop. V.6 (Route B core)	—	✓	✓	✓	—	✓	✓	—
Rem. VI.1 (normalization check)	—	—	—	—	—	—	—	✓
Prop. VI.2 (path consistency)	✓	—	—	—	—	—	—	✓
Thm. VI.11 (T_{bin} scope)	—	—	—	—	—	—	—	✓
Thm. VI.6 (packaged result)	—	✓	✓	✓	—	✓	✓	✓

*Thm. III.4 uses continuity of the cost functional $J : (0, \infty) \to \mathbb{R}$; the later non-negativity of J is a conclusion of that theorem, not part of its hypothesis. This continuity assumption is logically distinct from (CS), the continuity of the scalar probability function $f : [0, \infty) \to [0, \infty)$. †Cor. IV.7 establishes $|\psi_{12}| = |\psi_1| \cdot |\psi_2|$ for composite *path* amplitudes using cost additivity and the $C = 2A$ bridge alone; it does not use (PA). (PA) is the outcome amplitude product rule; the corollary provides motivation for (PA) but does not use it as input.

the classical premise load explicit before Proposition V.6 (Route B) replaces (MA) by the binary subsystem package of Section IV.

Theorem V.1 (Route A benchmark: scalar uniqueness under primitive multiplicativity)**.** *Let P be a scalar probability function satisfying the Route A requirements (i)–(v) of Definition IV.3. Then $P(c) = |c|^2$ is the unique scalar probability function satisfying those conditions.*

Remark V.2 (Independent conventions in Route A). The five Route A requirements (i)–

(v) invoke exactly four independent adopted conventions: (SA), (NC), (CS), and (MA). Condition (i) (non-negativity $P(c) \geq 0$) is subsumed by condition (iii)/(SA), since $f : [0, \infty) \to [0, \infty)$ is non-negative by domain. No subsystem-composition argument, no-signaling condition, or RS-specific geometric input enters the derivation beyond the scalar-reduction class.

Proof sketch. Step 1: Scalar reduction and regularity. By (SA), write $P(c) = f(|c|)$ for some $f : [0, \infty) \to [0, \infty)$. By (CS), f is continuous. For $r_1, r_2 > 0$, multiplicativity (MA) gives

$$f(r_1 r_2) = f(r_1) f(r_2) \qquad (r_1, r_2 > 0).$$

Step 2: Establish $f(1) = 1$ and $f(0) = 0$. Applying (MA) with $r_1 = r_2 = 1$ gives $f(1) = f(1)^2$, so $f(1) \in \{0, 1\}$. Normalization (NC) on the balanced two-outcome state with $|\alpha|^2 + |\beta|^2 = 1$ and $|\alpha| = |\beta| = 1/\sqrt{2}$ gives $2f(1/\sqrt{2}) = 1$, hence $f(1/\sqrt{2}) = 1/2 > 0$. Since $f(1)f(1/\sqrt{2}) = f(1/\sqrt{2}) > 0$, we must have $f(1) = 1$. Normalization (NC) on the degenerate state $(1, 0)$ with $1^2 + 0^2 = 1$ then gives $f(1) + f(0) = 1$, hence $f(0) = 0$.

Step 3: Positivity on $(0, \infty)$. If $f(r_0) = 0$ for some $r_0 > 0$, then $f(1) = f(r_0) f(r_0^{-1}) = 0$, contradicting $f(1) = 1$. Hence $f(r) > 0$ for all $r > 0$.

Step 4: Reduce to the additive Cauchy equation. Since $f(r) > 0$ for all $r > 0$ (Step 3), the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) := \ln f(e^x)$ is well defined. Continuity and positivity of f on $(0, \infty)$ together imply continuity of g. Multiplicativity of f becomes

$$g(x + y) = g(x) + g(y).$$

By the standard classification of continuous additive functions [4, 40], $g(x) = kx$ for some $k \in \mathbb{R}$, so $f(r) = r^k$ for $r > 0$.

Step 5: Pin $k = 2$ by normalization. Apply (NC) to a two-outcome state with amplitudes α, β satisfying $|\alpha|^2 + |\beta|^2 = 1$. Setting $p = |\alpha|^2 \in (0, 1)$ gives $p^{k/2} + (1 - p)^{k/2} = 1$ for all $p \in (0, 1)$. At $p = 1/2$: $2(1/2)^{k/2} = 1$, so $(k/2)\ln(1/2) = \ln(1/2)$. Since $\ln(1/2) \neq 0$, $k = 2$. Therefore $f(r) = r^2$ and $P(c) = |c|^2$. □

Remark V.3 (Scope limitation of Route A)*.* The decisive scalar product law (MA) is adopted rather than derived in Route A, so the route is diagnostic rather than foundational. Section VI does not upgrade it into an independent reconstruction: the two-outcome compatibility checks there are internal consistency statements for the adopted geometry, not new functional derivations. What Route A clarifies is the specific classical premise set it uses—namely (SA), (NC), (CS), (MA)—and thereby makes visible the exact additional work that Proposition V.6 (Route B) performs: replacing (MA) by the binary no-signaling subsystem package (PA)+$(\mathrm{P5})_2$.

B. Route B: A Scalar Uniqueness Lemma Induced by the Binary No-Signaling Subsystem Model

a. Regularity prerequisite. Continuity is the regularity condition imposed on the scalar function f in $P(c) = f(|c|)$ in Proposition V.6. It rules out pathological solutions of the additive Cauchy equation [40]; measurability or local boundedness would each suffice as weaker alternatives. This continuity hypothesis is separate from the continuity of J used in the d'Alembert classification of Theorem III.4. It is modest and technical, not a claim that every physical quantity in the model varies continuously under arbitrary parameter changes.

Remark V.4 (Monotonicity as a sufficient regularity condition)*.* Within the RS framework, one may replace continuity by a monotonicity hypothesis on f. If one strengthens the ordering intuition behind the cost functional to the statement that larger amplitude modulus should not correspond to smaller outcome probability, then f is non-decreasing on $[0, \infty)$. Monotone real functions are Lebesgue measurable, and measurability suffices to force additive Cauchy solutions to be linear [40]; the normalization step in Proposition V.6 then fixes the coefficient and hence the quadratic rule. For clarity, however, Proposition V.6 keeps continuity explicit and treats monotonicity only as an alternative sufficient condition.

Lemma V.5 (Binary subsystem model implies the scalar functional equation)**.** *Under the adopted binary subsystem model—product-amplitude composition (PA), scalar ansatz (SA),* 2*-norm convention (NC), and binary no-signaling* $(\mathrm{P5})_2$*—the scalar probability function* f *satisfies*

$$f(r \cdot s) + f\left(r\sqrt{1 - s^2}\right) = f(r) \tag{9}$$

for all $r > 0$ *and* $s \in (0, 1)$*. This is the weakest no-signaling requirement sufficient for Proposition V.6: only binary measurements on subsystem 2 are used. The full no-signaling condition of Definition IV.9 implies* (9)*, so the proposition remains valid under the stronger condition as well.*

Proof. By (PA), the joint outcome (i, j) has amplitude $c_{1i} \cdot c_{2j}$. By (SA), $P_{\text{joint}}(c_{1i} \cdot c_{2j}) = f(|c_{1i}||c_{2j}|)$. Set $r = |c_{1i}|$ and write the two subsystem-2 binary outcome moduli as $|c_{21}| = s$ and $|c_{22}| = \sqrt{1-s^2}$, so that $s^2+(1-s^2) = 1$ satisfies the (NC) constraint on subsystem 2. The no-signaling equation (8) of Definition IV.9 then becomes exactly $f(rs)+f(r\sqrt{1-s^2}) = f(r)$. Each of (PA), (SA), (NC), $(\text{P5})_2$ is needed; none alone suffices. □

Once the functional equation (9) is established, the mathematics of Proposition V.6 is classical. The substantive issue is therefore where the premises justifying (9) are placed, namely in the binary subsystem model of Lemma V.5.

Proposition V.6 (Conditional scalar uniqueness in the adopted binary subsystem model)**.** *Let $P(c) = f(|c|)$ be a scalar probability law with $f : [0, \infty) \to [0, \infty)$, and assume that*

- *(a)* Scalar reduction *(SA): $P(c) = f(|c|)$,*
- *(b)* Normalization *(NC): $\sum_i f(|c_i|) = 1$ whenever $\sum_i |c_i|^2 = 1$,*
- *(c)* Continuity *(CS): f is continuous on $[0, \infty)$,*
- *(d)* Functional equation*: $f(r \cdot s) + f\left(r\sqrt{1-s^2}\right) = f(r)$ for all $r > 0$, $s \in (0, 1)$.*

Then $f(r) = r^2$ for all $r \geq 0$, and hence $P(c) = |c|^2$, uniquely within this scalar class. Condition (d) is supplied physically by Lemma V.5 via the binary no-signaling subsystem model.

Proof. Step 0: Derive $f(0) = 0$. Condition (d) holds for all $r > 0$ and $s \in (0, 1)$. Fix any $r > 0$ and let $s \to 0^+$: by continuity (c) of f, $f(r \cdot s) \to f(0)$ and $f(r\sqrt{1-s^2}) \to f(r)$. Condition (d) then gives in the limit $f(0) + f(r) = f(r)$, hence $f(0) = 0$.

Define $h : [0, \infty) \to [0, \infty)$ by $h(x) := f(\sqrt{x})$. Since f is continuous, h is continuous; and $h(0) = f(\sqrt{0}) = f(0) = 0$ by Step 0.

Step 1: Reduce to the additive Cauchy equation. For any $r > 0$ and $s \in (0, 1)$, condition (d) gives $f(rs)+f(r\sqrt{1-s^2}) = f(r)$. Setting $R := r^2 > 0$ and $p := s^2 \in (0, 1)$, so that $rs = \sqrt{Rp}$ and $r\sqrt{1-s^2} = \sqrt{R(1-p)}$, this becomes $h(Rp) + h(R(1-p)) = h(R)$.

Step 2: All positive pairs. For any $x, y > 0$, set $R := x+y > 0$ and $p := x/(x+y) \in (0, 1)$. Then $Rp = x$ and $R(1-p) = y$, so the identity from Step 1 gives

$$h(x) + h(y) = h(x + y), \qquad \text{for all } x, y > 0. \tag{10}$$

Step 3: Identify h. Equation (10) is the additive Cauchy equation on $(0, \infty)$. Since h is continuous on $[0, \infty)$ and $h(0) = 0$, the equation extends to $[0, \infty)$ by continuity. From normalization (NC) applied to the degenerate two-outcome state with amplitudes $(1,0)$ satisfying $1^2+0^2=1$: $f(1)+f(0)=1$. Since $f(0)=0$ by Step 0, we get $f(1)=1$, and therefore $h(1)=f(\sqrt{1})=f(1)=1$. The unique continuous solution of $h(x)+h(y)=h(x+y)$ on $[0,\infty)$ with $h(0)=0$ and $h(1)=1$ is $h(x)=x$ (Theorem 1 of Ch. 2 in [40]; Theorem 5.2.1 in the 1989 Aczél–Dhombres edition).

Step 4: Conclude. $h(x)=x$ gives $f(r)=h(r^2)=r^2$ for all $r\geq 0$. Uniqueness follows from the uniqueness of the continuous solution of the Cauchy equation on $[0,\infty)$. Hence $P(c)=f(|c|)=|c|^2$. □

Remark V.7 (Partial minimality: individual necessity of conditions (a)–(d)). The four conditions (a)–(d) in Proposition V.6 are individually non-redundant within the adopted scalar subsystem class.

Necessity of (NC) [condition (b)]. Without the normalization condition, every function $f(r)=c\cdot r^2$ with constant $c>0$ satisfies conditions (a), (c), and (d): the functional equation (d) is verified by $c\cdot r^2(s^2+(1-s^2))=c\cdot r^2$, and continuity (c) holds trivially. Only (NC)—applied to the degenerate two-outcome state with amplitudes $(1,0)$ satisfying $1^2+0^2=1$—fixes $f(1)=1$, thereby pinning $c=1$ and uniquely selecting $f(r)=r^2$.

Necessity of (CS) [condition (c)]. Without continuity, the additive Cauchy equation $h(x)+h(y)=h(x+y)$ (derived in Steps 1–2) admits everywhere-dense, nowhere-linear pathological solutions [40]. Any weaker regularity condition—measurability, monotonicity, or local boundedness—excludes these pathological branches equally well. Continuity is the specific choice adopted here but can be replaced by any of these alternatives without affecting the conclusion.

Necessity of (d) [the functional equation]. Without the no-signaling functional equation, conditions (a)–(c) alone—scalar reduction, normalization, and continuity—place no constraint on the functional form of f beyond scalar reduction and continuity. No uniqueness conclusion is obtainable from (a)–(c) alone; the functional equation is the structural link that forces the additive Cauchy equation.

Status of (SA) [condition (a)]. Whether (SA) is necessary—i.e., whether any phase-dependent probability function could satisfy (b)–(d) and yet differ from $|c|^2$—depends on extending condition (d) to phase-dependent probability assignments, which lies outside the scalar class of the present analysis. This question is related to open problem 1 of

Section VIII B.

Together, these observations show that no condition in (b)–(d) can be dropped from the hypothesis of Proposition V.6. Full minimality of the package (SA)+(NC)+(CS)+(PA)+$(\mathrm{P5})_2$ in the sense that no proper subset suffices remains open; these remarks establish individual necessity only within the specific scalar subsystem class.

Remark V.8 (Multiplicativity as a corollary of Route B). Route B (Proposition V.6) derives $f(r) = r^2$ without ever invoking (MA). Once $f = r^2$ is established, modulus-multiplicativity $f(r_1 r_2) = f(r_1)f(r_2)$ is immediately verified: $(r_1 r_2)^2 = r_1^2 r_2^2$. In other words, (MA) is a *verified corollary of the conclusion* of Route B, not an intermediate step within its proof. This contrasts with Route A, in which (MA) is an input axiom.

The full constructive chain that motivates and connects the Route B hypotheses to each other spans Sections III–V:

$$\underbrace{\text{disjoint ledger regions} \to \text{additive costs} \to \text{mult. weights (Prop. IV.6)}}_{\text{Sec. III}}$$
$$\to \underbrace{\text{tensor-product moduli (Cor. IV.7)} \xrightarrow{+\,\text{phase convention}} \text{product amplitudes (Rem. IV.8)}}_{\text{Sec. IV}}$$
$$\to \underbrace{(\mathrm{PA}) + (\mathrm{P5})_2 + (\mathrm{SA}) + (\mathrm{NC}) + (\mathrm{CS}) \to f(r) = r^2 \text{ (Prop. V.6)}}_{\text{Sec. V}}.$$

The "+phase convention" label indicates that (PA) requires adopting $\phi_{ij} = \phi_i + \phi_j$ in addition to the modulus rule from Cor. IV.7; this phase stipulation is not derived from the cost axioms. What Route B establishes is that, within this premise package, (MA) need not be posited separately as a scalar probability axiom.

VI. TWO-OUTCOME GEOMETRY AND THE MAIN THEOREM

This section first records limited consistency checks for the adopted binary-split geometry (G1)–(G4), then states the positive packaging result (Theorem VI.6) and the companion scope-limitation result (Theorem VI.11). Neither part derives the 2-norm convention from the cost axioms alone or adds a new functional argument beyond Proposition V.6.

A. Limited Consistency Checks for the Adopted Two-Outcome Geometry

a. What this subsection establishes. This subsection records two limited consistency checks for the adopted two-outcome geometry (G1)–(G4). Remark VI.1 gives the normalization identity from (G4) alone. Proposition VI.2 checks compatibility with a finite auxiliary half-action accumulation (the raw path-action integral diverges on the semi-infinite branches). Neither result derives (NC) from the cost axioms or replaces the normalization hypothesis in Proposition V.6. The comparison of (G1)–(G4) with Hilbert-space structure is addressed separately in Section VI B (Remark VI.5).

b. Geometric definitions. The compatibility calculation uses four additional model clauses beyond the cost axioms (A1)–(A2) and their consequence $J(1) = 0$ (Lemma III.3). These clauses define the adopted two-outcome measurement geometry; they are not consequences of the cost axioms alone:

(G1) **Angular chart.** Positive scale ratios $r > 0$ are parametrized by an angle $\theta \in (0, \pi/2)$ via $r = \tan\theta$. The balance point $r = 1$ corresponds to $\theta = \pi/4$, while $r \to 0$ and $r \to \infty$ correspond to $\theta \to 0$ and $\theta \to \pi/2$. The normalized ratio vector $(r, 1)/\sqrt{1 + r^2}$ then has Cartesian components $(\sin\theta, \cos\theta)$; these are used to define the branch moduli in clause (G4).

(G2) **Two-branch structure.** A binary measurement partitions outcomes into two branches. The branch point is parametrized by an angle $\theta_s \in (0, \pi/2)$. The complementary branch is assigned the angle $\pi/2 - \theta_s$, motivated by the reciprocal symmetry $J(x) = J(1/x)$: since $\tan(\pi/2 - \theta) = 1/\tan\theta$, the two branches correspond to reciprocally related ratios.

(G3) **Path parametrization.** For branch 1, the path from the branch point to the boundary is described by the relative ratio $r(\theta) = \tan\theta/\tan\theta_s$ as θ ranges from θ_s to $\pi/2$. This is the monotone path used in the proof of Proposition VI.2.

(G4) **Angular projection.** The branch moduli are defined as the Cartesian components of the unit vector at angle θ_s in the angular chart (G1): $|\psi_1| = \sin\theta_s$ and $|\psi_2| = \cos\theta_s$. Whether this assignment is a logical consequence of (G1)–(G3) alone or independently encodes Hilbert-space content is left open.

Definitions (G1)–(G4) belong to the specified measurement model; they are not consequences of (A1)–(A2) alone. No Riemannian metric, geodesic extremality, or variational uniqueness claim is used here.

Remark VI.1 (Two-outcome normalization under (G4)). Under the adopted angular-projection clause (G4), $|\psi_1| = \sin\theta_s$ and $|\psi_2| = \cos\theta_s$. Branch-modulus normalization $|\psi_1|^2 + |\psi_2|^2 = 1$ is then immediate from $\sin^2\theta_s + \cos^2\theta_s = 1$. The clauses (G1)–(G3) play no role; this is a direct consequence of (G4) alone.

Proposition VI.2 (Half-action compatibility with the adopted branch moduli)**.** *Under (G1)–(G4) and the $C = 2A$ bridge (Remark III.9), the finite auxiliary half-action accumulation integral $A_i = \int q(u)\,du$ is compatible with the trigonometric moduli $|\psi_1| = \sin\theta_s$ and $|\psi_2| = \cos\theta_s$ defined by (G4). This proposition is a model-internal compatibility calculation: the raw path-action integral $C_i = \int J(r)\,d(\ln r)$ diverges on the semi-infinite branches, so the finite quantity established here is the auxiliary accumulation density $q(u) = \cos^2\theta(u)$ rather than the cost functional J directly.*

Proof. Step 1: Branch 1. By (G3), the relative ratio along branch 1 is $r(\theta) = \tan\theta/\tan\theta_s$ for $\theta \in [\theta_s, \pi/2)$. Pass to the log-ratio coordinate

$$u = \ln r = \ln(\tan\theta/\tan\theta_s),$$

so that $u = 0$ at $\theta = \theta_s$ and $u \to \infty$ as $\theta \to \pi/2$. Differentiating gives

$$\frac{d\theta}{du} = \sin\theta\cos\theta.$$

By (G4), along branch 1 the modulus is $|\psi(\theta)| = \sin\theta$. Hence

$$A(\theta) = -\ln(\sin\theta), \qquad \frac{dA}{d\theta} = -\cot\theta.$$

Therefore the half-action accumulation rate

$$q(u) := -\frac{dA}{du}$$

is

$$q(u) = \cot\theta\,\frac{d\theta}{du} = \cos^2\theta(u).$$

To express q explicitly in u: from $u = \ln(\tan\theta/\tan\theta_s)$ one has $\tan\theta = e^u\tan\theta_s$, so

$$q(u) = \cos^2\theta = \frac{1}{1+\tan^2\theta} = \frac{1}{1+e^{2u}\tan^2\theta_s}.$$

The half-action itself is then computed by changing variable from u back to θ. Since $u = \ln(\tan\theta/\tan\theta_s)$, differentiating gives $du = d\theta/(\sin\theta\cos\theta)$, so

$$q(u)\,du = \cos^2\theta \cdot \frac{d\theta}{\sin\theta\cos\theta} = \cot\theta\,d\theta.$$

Therefore

$$A_1 \;=\; \int_0^\infty q(u)\,du \;=\; \int_{\theta_s}^{\pi/2} \cot\theta\,d\theta \;=\; \Big[-\ln(\sin\theta)\Big]_{\theta_s}^{\pi/2} \;=\; -\ln(\sin\theta_s). \tag{11}$$

Thus the $C = 2A$ bridge gives $|\psi_1| = e^{-A_1} = \sin\theta_s$, confirming (G4).

Step 2: Complementary branch. For branch 2, the complementary angle is $\widetilde{\theta}_s = \pi/2 - \theta_s$ by (G2). The same calculation gives

$$A_2 = -\ln(\sin(\pi/2 - \theta_s)) = -\ln(\cos\theta_s),$$

so that $|\psi_2| = \cos\theta_s$, again confirming (G4). □

Remark VI.3 (Convergent vs. divergent integral)*.* The quantity evaluated in Proposition VI.2 is $A_1 = -\ln(\sin\theta_s)$, not $\frac{1}{2}C_1 = \frac{1}{2}\int J(r)\,d(\ln r)$, which diverges as $\theta \to \pi/2$; the two coincide only in the finite-interval regime. Specifically, $J(e^u) = \cosh(u) - 1$ gives $C_1 = \int_0^\infty (\cosh u - 1)\,du \to \infty$, while the auxiliary accumulation integral $A_1 = \int_0^\infty q(u)\,du$ with $q(u) = \cos^2\theta(u)$ is finite and equals $-\ln(\sin\theta_s)$ by the change of variable $\theta \mapsto u = \ln(\tan\theta/\tan\theta_s)$.

Remark VI.4 (Logistic structure of the accumulation density)*.* A brief calculation shows that the accumulation density $q(u) = \cos^2\theta(u)$ satisfies $q'(u) = -2q(u)(1 - q(u))$—the logistic equation with fixed points at $q = 0$ (boundary) and $q = 1$ (branch point). This is a derived side relation of the adopted angular parametrization; it is not an independent dynamical postulate, and the proof of Proposition VI.2 does not invoke it.

Remark VI.5 (Relationship to Hilbert-space structure)*.* The normalization analysis leaves one foundational question open: do the adopted geometric clauses (G1)–(G4) already encode Hilbert-space-equivalent content? Three observations matter.

Structural differences. The space of scale ratios $(\mathbb{R}_{>0}, \cdot)$ is a multiplicative group, not a vector space. At the level of the cost functional and ledger geometry, there is no primitive notion of state addition or scalar multiplication: the angular variable θ_s parametrizes cost comparisons rather than superpositions. Complex amplitudes and superposition enter only through the additional product-amplitude convention (PA) and phase postulate (P2) adopted in Sections IV–V, which lie beyond the cost-functional layer. The RS cost functional $J(e^t) = \cosh(t) - 1$ is also not a quadratic form in the natural coordinate $t = \ln x$: its Taylor expansion $t^2/2 + t^4/24 + \cdots$ contains higher-order terms. These differences show that the RS presentation is not trivially identical to Hilbert-space language, but by themselves they do not prove strict weakness.

Shared output. What the RS model shares with Hilbert-space geometry is the two-outcome trigonometric modulus chart and, once the scalar rule is packaged with it, the corresponding quadratic probability chart. Under the adopted geometry, (G4) alone assigns $|\psi_1| = \sin\theta_s$ and $|\psi_2| = \cos\theta_s$, and hence gives the normalization identity $\sum_i |\psi_i|^2 = 1$ by the Pythagorean identity. The probability identification $p_i = |\psi_i|^2$, however, is not supplied by (G4) alone; it enters through the conditional scalar uniqueness result packaged later in Theorem VI.6. This shared output is exactly why the comparison question between the RS framework and Hilbert-space quantum mechanics remains live.

Open question: three scenarios. The relationship between (A1)–(A2) together with (G1)–(G4) and Hilbert-space structure admits three possibilities:

(i) (G1)–(G4) are equivalent to the relevant Hilbert-space fragment. In this case, the paper is a reformulation in cost-functional language rather than an independent reconstruction.

(ii) (G1)–(G4) are strictly weaker than Hilbert-space structure. In this case, the paper is a genuine reconstruction from weaker assumptions. Establishing this would require a proof, which is absent.

(iii) (G1)–(G4) are *incomparable* with Hilbert-space structure: the two frameworks share some consequences (e.g., the two-outcome trigonometric modulus chart and the packaged quadratic probability chart) but neither implies the other. In this case, the paper derives the Born rule from a genuinely different axiomatic base, which is of foundational interest, but establishing incomparability requires explicit counter-models in both directions.

The structural differences above provide grounds for distinguishing the two frameworks but do not rule out a deeper equivalence. Resolving which scenario holds remains a central open problem for the RS program. Theorem VI.11 below resolves strict weakness only for the reduced chart T_{bin}, not for the original geometric package (G1)–(G4).

One piece of partial evidence is provided by the Lean 4 theorem `born_weight_forced` in `RS.Foundation.BornRuleForcing` (Appendix A), which proves within the Lean formalization that under the explicit two-branch calibration $w(\cos\theta) = \cos^2\theta$, the weight function is forced to be $w(r) = r^2$. This shows that the RS formalism can carry the scalar r^2 forcing step independently of Hilbert-space language, which is circumstantial evidence for scenario (ii) or (iii), but falls short of resolving which scenario holds.

B. Packaging: Main Theorem and Scope Limitation

Theorem VI.6 (Packaged conditional result in the adopted two-outcome model)**.** *Assume conventions* (SA)*,* (NC)*,* (CS)*,* (PA)*, no-signaling* $(\mathrm{P5})_2$*, and the angular-projection clause* (G4)*. The clauses (G1)–(G3) are part of the adopted measurement model but are not invoked in the proof. Concretely: within the adopted two-outcome RS model, let* $P(c) = f(|c|)$ *be a phase-independent probability assignment with continuous* $f : [0,\infty) \to [0,\infty)$*. Assume:*

- *(i)* Normalization *(NC):* $\sum_i f(|c_i|) = 1$ *whenever* $\sum_i |c_i|^2 = 1$*,*
- *(ii)* Product-amplitude composition *(PA): Remark IV.8,*
- *(iii)* Binary no-signaling *(P5)*$_2$*: Definition IV.9 restricted to binary subsystem-2 measurements.*

Then $P(c) = |c|^2$ *is the unique scalar rule satisfying these conditions in the adopted two-outcome binary-split model, and the adopted branch moduli satisfy* $|\psi_1|^2+|\psi_2|^2 = 1$ *under (G4).*

Proof. Assumptions (i)–(iii) satisfy the hypotheses of Proposition V.6 (via Lemma V.5 for condition (d)), which yields the unique functional result $f(r) = r^2$, hence $P(c) = |c|^2$. No further functional argument is needed. The normalization identity $|\psi_1|^2 + |\psi_2|^2 = 1$ is immediate under (G4) by Remark VI.1. Together, these give the packaged two-outcome result. □

Remark VI.7 (Theorem VI.6 is two-outcome only)*.* Theorem VI.6 applies only within the adopted two-outcome binary-split model. The abstract functional result of Proposition V.6 holds for any f satisfying the listed conditions and is not restricted to two outcomes. However, the geometric packaging via (G4) and the normalization identity of Remark VI.1 are specific to the binary-split model. Whether the binary-split geometry (G1)–(G4) extends to general n-outcome measurements remains open.

a. Scope limitation: the reduced scalar chart. The theorem above gives the positive conditional result. We now pass to the deliberately thinner scalar binary chart T_{bin}, which retains only the two-outcome probability assignments and complement involution. The result below is negative rather than reconstructive: it prevents overinterpretation of the reduced chart and does not by itself address the stronger question of whether the full geometric package (G1)–(G4) is weaker than Hilbert-space structure (Remark VI.5).

Definition VI.8 (Weak binary chart)**.** Let T_{bin} denote the weak binary chart: a family of two-outcome probability assignments indexed by a positive parameter $r \in (0, \infty)$, with

$$p_1(r) = \frac{r^2}{1+r^2}, \qquad p_2(r) = \frac{1}{1+r^2}, \tag{12}$$

together with the complement involution $\kappa(r) = 1/r$ exchanging the two outcomes. The structure T_{bin} contains only this scalar binary probability chart; it does not include amplitude vectors, superposition, phase, incompatible measurements, or tensor-product composition.

Definition VI.9 (Completion of the binary chart)**.** A completion of T_{bin} is an operational theory [10, 24] T with state space Ω_T, a distinguished binary measurement $E_T = \{e_T, u_T - e_T\}$, and an embedding $\iota_T : (0, \infty) \to \Omega_T$ such that $e_T(\iota_T(r)) = p_1(r)$ and $(u_T - e_T)(\iota_T(r)) = p_2(r)$ for every $r > 0$.

Definition VI.10 (Hilbert and classical completions)**.** T_H is the qubit theory with state space $\{\rho \in \mathbb{C}^{2\times 2} : \rho \geq 0, \operatorname{tr} \rho = 1\}$, measurement $E_Z = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, and embedding $\iota_H(r) = |\psi_r\rangle\langle\psi_r|$ where $|\psi_r\rangle = (r|0\rangle + |1\rangle)/\sqrt{1+r^2}$. A direct calculation gives $\langle\psi_r|0\rangle\langle 0|\psi_r\rangle = r^2/(1+r^2)$, so T_H is a completion of T_{bin}.

T_C is the classical bit theory with state space $\Delta_2 = \{(p, 1-p) : p \in [0,1]\}$ and the embedding $\iota_C(r) = (r^2/(1+r^2),\, 1/(1+r^2))$. By construction T_C is also a completion of T_{bin}.

Theorem VI.11 (Scope control: T_{bin} admits inequivalent completions)**.** *The completions T_H and T_C of T_{bin} are inequivalent as operational theories. Consequently, the reduced scalar chart T_{bin} alone does not uniquely select Hilbert-space structure among its completions. This is a scope-control result for the reduced chart, not a claim about the full geometric package (G1)–(G4).*

Proof. Inequivalence follows because simplex state-space structure and measurement compatibility are operational invariants [24]. T_C has simplex state space Δ_2, so all its measurements are jointly measurable [10]. T_H admits incompatible measurements (E_Z and $E_X = \{|+\rangle\langle +|, |-\rangle\langle -|\}$); these are not jointly measurable in qubit theory. Since T_H and T_C

differ on an operational invariant, they are inequivalent. Hence $T_{\rm bin}$ does not uniquely select Hilbert-space structure. $\square$

Remark VI.12 (Scope of Theorem VI.11). This result concerns $T_{\rm bin}$ alone. It prevents overinterpretation of the reduced chart, but it does not settle whether the full geometric package (G1)–(G4) is weaker than, equivalent to, or incomparable with Hilbert-space structure (Remark VI.5).

b. Transition. The main conditional result and its scope limitation are now complete. The following sections record downstream checks, literature comparison, empirical meaning, outlook, and formal-verification status; none adds a new functional derivation.

VII. POST-THEOREM CHECKS AND COMPARISON

This section records two downstream checks of the adopted model and then compares the conditional RS result with other Born-rule programs. Table II gives the schematic comparison, and Remark VII.2 expands the premise placement and scope contrast.

A. Post-Theorem Checks

The standard interference identity becomes physically relevant in the RS path picture only after adding the phase postulate (P2) and the adopted linear sum-over-paths representation, neither of which follows from (A1)–(A2).

a. Interference compatibility. Given the additional phase postulate (P2) of Section IV, the path-weight construction of Section III B assigns a complex amplitude $\psi(\gamma)$ to each path γ. Passing from single-path amplitudes to an outcome amplitude requires an additional representation choice not forced by the cost axioms: if paths $\gamma_1, \gamma_2, \ldots$ reach the same outcome i, one adopts the linear sum $\psi_i = \sum_{\gamma \to i} \psi(\gamma)$ (see Section III B). For two paths γ_1, γ_2 reaching outcome i, this becomes $\psi_i = \psi(\gamma_1) + \psi(\gamma_2)$. Writing $a = \psi(\gamma_1)$ and $b = \psi(\gamma_2)$, one has $\psi_i = a + b$ and hence $P(i) = |\psi_i|^2 = |a + b|^2$. The remark below records the corresponding interference expansion.

Remark VII.1 (Standard interference identity). Let $a, b \in \mathbb{C}$ be individual path amplitudes reaching the same outcome (where z^* denotes complex conjugate of $z \in \mathbb{C}$). Then

$$|a + b|^2 = |a|^2 + |b|^2 + 2\ \mathrm{Re}(a^* b).$$

Proof: $(a+b)(a+b)^* = |a|^2+ab^*+ba^*+|b|^2 = |a|^2+|b|^2+2\ \mathrm{Re}(a^*b)$, using $ab^*+ba^* = 2\ \mathrm{Re}(ab^*)$ and $\mathrm{Re}(ab^*) = \mathrm{Re}(\overline{ab^*}) = \mathrm{Re}(a^*b)$. The terms $|a|^2$ and $|b|^2$ are the diagonal (same-path) contributions to $P(i) = |a + b|^2$, and $2\ \mathrm{Re}(a^*b)$ is the standard off-diagonal interference cross-term. Note that $|a|^2$ and $|b|^2$ are not individually observable as single-path probabilities; only $P(i) = |a + b|^2$ is the observable outcome probability. Once complex phase and the adopted linear sum-over-paths representation are in place, the RS path picture is consistent with this identity; neither ingredient follows from the cost axioms alone.

B. Comparison with Other Born-Rule Derivations

This comparison concerns premise placement rather than theorem strength: the present result does not reconstruct a global state space, but isolates one scalar uniqueness route inside an adopted binary subsystem model. Table II is schematic and compressed rather than exhaustive; the rows indicate where each program locates its main probabilistic work, not a ranking of programs that address different questions at different structural levels. Torres Alegre [22] is included because it is the most directly comparable 2025 result using no-signaling as the decisive condition; its GPT-with-purification setting differs structurally from the present two-outcome scalar model. The incremental contribution is this premise map in the RS two-outcome setting, not a stronger general Born-rule theorem than the literature already provides.

Remark VII.2 (Literature comparison: programs, premises, and scope). Table II compares programs at different structural levels; it orients premise placement rather than ranking theorem strength. Gleason [1] uses *frame-function* regularity on the lattice of closed subspaces (dimension ≥ 3 for PVMs). Zurek [19] places the decisive work in entanglement symmetry (envariance) and branching. Deutsch–Wallace [14, 16] relocate it to decision-theoretic rationality constraints. Torres Alegre [22] uses no-signaling causal consistency in finite-dimensional GPTs with purification.

Hardy [8] and Chiribella–D'Ariano–Perinotti [11] take operational axioms on states, composites, and transformations and *derive* Hilbert-space structure and the Born rule;

TABLE II. Comparison of selected routes to the quadratic probability rule (schematic; orientation only). The RS column records a conditional scalar uniqueness result for the quadratic rule in an adopted two-outcome model, not a general Born-rule reconstruction.

Program	**Core premise placement and scope**
Gleason	Hilbert-space projection structure is assumed from the outset; the result constrains probability measures on that structure. For PVMs the standard theorem requires dimension ≥ 3.
Hardy	Five operational axioms on states, composites, and continuous reversible transformations reconstruct the full quantum formalism, with the Born rule following inside that broader reconstruction.
Zurek	The decisive work is done by envariance and Everettian branching inside standard quantum mechanics, yielding the Born rule from entanglement symmetry arguments.
Chiribella–D'Ariano–Perinotti	Informational and operational axioms reconstruct the full theory; the Born rule is part of that larger state-space and transformation-structure result.
Torres Alegre	In finite-dimensional GPTs with purification, no-signaling causal consistency is the decisive condition selecting $P = \lvert c \rvert^2$.
RS (this paper)	A ratio-cost functional is taken as primitive, then supplemented by the adopted scalar package. Route A assumes (MA); Route B replaces (MA) with (PA)+$(\text{P5})_2$. The result is a conditional scalar uniqueness theorem in a two-outcome model, with geometry packaged separately in Section VI.

Zurek assumes quantum mechanics including entanglement. The present argument takes a ratio-cost functional as primitive, then adds the adopted scalar and subsystem package and, in Section VI, the two-outcome geometry. Path weights and amplitudes are organized from there rather than from a globally reconstructed state space. That explains both the payoff—modulus multiplicativity (MA) is not a primitive scalar axiom in Route B—and the limitation: (SA), (NC), (CS), and (PA)+$(\text{P5})_2$ must be fixed explicitly before Proposition V.6, while Section VI supplies packaging beyond that abstract scalar step. The RS route trades a global state-space framework for cost-functional geometry plus an adopted scalar package.

Hardy's five axioms constrain global state space, composite structure, and continuous reversibility, and yield the full formalism; the present analysis constrains only a scalar probability law inside an adopted two-outcome model, so a direct axiom-count comparison would be misleading. In the conditional scalar route here, the probabilistic work sits in (SA), (NC), (CS), (PA), and $(P5)_2$, not in purification or a transformation-group reconstruction; the two-outcome geometry enters only in Theorem VI.6 and related checks, not in Proposition V.6 itself.

VIII. EMPIRICAL SCOPE AND CONCLUSION

Section VIII A identifies which assumptions would be called into question if the quadratic rule were empirically violated, and Section VIII B states the final conclusion and four principal open problems.

A. Empirical Scope of the Conditional Result

Because the paper adopts the same quadratic rule as ordinary quantum mechanics, it entails no empirical deviation from standard quantum theory and introduces no RS-specific prediction that would distinguish the two frameworks experimentally. Its empirical value is therefore diagnostic: it identifies which parts of the adopted premise package would be called into question if the broader quadratic phenomenology discussed here were empirically violated.

Given the additional phase postulate (P2) and the adopted complex path-sum representation, the quadratic rule implies the familiar second-order interference structure of ordinary quantum mechanics. The Sorkin third-order interference parameter I_3 (defined below) enters here only as a standard empirical consequence of the adopted quadratic rule, not as a direct test of the RS cost axioms by themselves. Under the quadratic rule, higher-order interference vanishes in the Sorkin classification [43, 44]. For three paths A, B, C to a detector, the third-order interference parameter is

$$I_3 = P(A \cup B \cup C) - P(A \cup B) - P(A \cup C) - P(B \cup C) + P(A) + P(B) + P(C). \quad (13)$$

When $P(i) = |\psi_i|^2$ and amplitudes add linearly, I_3 vanishes identically. To see this: substituting $P(A \cup B \cup C) = |\psi_A + \psi_B + \psi_C|^2$ and the corresponding bilinear expansions for two-path terms (Remark VII.1), each cross-term $2\,\mathrm{Re}(\psi_X^*\psi_Y)$ (with $\{X,Y\} \subset \{A,B,C\}$, $X \neq Y$) appears with coefficient $+1$ in $P(A \cup B \cup C)$ and coefficient -1 in the subtracted two-path term $P(X \cup Y)$, cancelling exactly; the diagonal terms $|\psi_A|^2$, $|\psi_B|^2$, $|\psi_C|^2$ cancel by inclusion–exclusion. Hence $I_3 = 0$ whenever the probability rule is the Born rule $P(c) = |c|^2$ and amplitudes superpose linearly. This is not a novel RS-specific prediction; it is a compatibility check with standard quantum behavior. Existing triple-slit experiments place strong bounds on third-order interference [45–47]. A detection of $I_3 \neq 0$ would require at least one element of the broader adopted package to fail—the quadratic rule, the linear superposition of path amplitudes, or the measurement model used to connect the scalar rule to multi-path interference phenomenology. It would therefore refute that broader package, though not by itself isolate whether the failure lies in the abstract two-outcome scalar theorem or in the extra structure used to extend it to this multi-path setting. Its non-detection, however, is evidentially one-sided here: it shows compatibility with the quadratic package but does not uniquely confirm the RS framework, since ordinary quantum mechanics and other quadratic frameworks imply the same null result.

a. Diagnostic decomposition. An empirical violation of the broader quadratic phenomenology discussed here would imply that at least one element of the adopted premise package fails, although the present analysis would not by itself determine which one. The diagnostic fault lines correspond to the separately identified premise groups:

- *Scalar reduction and phase independence.* A failure of the scalar ansatz $P(c) = f(|c|)$ would mean that outcome probabilities depend on more than amplitude modulus alone—for example on fuller state or measurement context. Note that phase independence (the condition $P(ce^{i\varphi}) = P(c)$ for all $\varphi \in \mathbb{R}$) is logically equivalent to the scalar ansatz: both express precisely the condition that $P(c)$ depends only on $|c|$, so they constitute a single empirical failure mode stated in two equivalent ways.
- *Adopted* 2*-norm convention.* A failure would break the normalization hypothesis used in both Route A and Route B.
- *Product-amplitude subsystem model or no-signaling.* A failure of either would invalidate the Route B functional argument leading to Proposition V.6.
- *Regularity.* A failure of continuity, or of any alternative regularity condition used in its place to exclude pathological Cauchy-type solutions, would reopen nonquadratic possibilities.

- *Adopted two-outcome geometry.* A failure here would undermine the packaging and compatibility statements tied to Section VI. By itself it would not overturn the abstract scalar uniqueness argument of Proposition V.6, which is geometry-free.

This decomposition has asymmetrical empirical force. A violation of the quadratic rule would show that the present premise package is not jointly adequate, but agreement with experiment would not by itself identify which premises are fundamentally right or which broader framework deserves credit for the observed quadratic law.

B. Conclusion and Open Problems

Within the adopted scalar binary subsystem model, Proposition V.6 shows that (SA), (NC), (CS), product-amplitude composition (PA), and binary no-signaling $(\mathrm{P5})_2$ uniquely select the quadratic scalar rule $P(c) = |c|^2$. Modulus-multiplicativity (MA) is not among those premises; it follows algebraically once the quadratic rule is established. Theorem VI.6 then packages this functional uniqueness with the geometric normalization $|\psi_1|^2 + |\psi_2|^2 = 1$ under (G4). Remark V.7 shows individual necessity of conditions (b)–(d) within the scalar class, while full minimality of the total premise package remains open.

Theorem VI.11 establishes the companion scope-control result: the reduced scalar chart T_{bin} admits inequivalent quantum and classical completions, so the reduced fragment alone does not determine Hilbert-space structure. The result is therefore a precise conditional uniqueness theorem within an adopted two-outcome model, not a general reconstruction of quantum probability.

a. Open problems. The following four items are the central open problems left by this manuscript.

1. *Scalar reduction.* The ansatz $P(c) = f(|c|)$ is adopted, not derived from the RS cost axioms. Deriving it from an RS-native symmetry or invariance argument is the most consequential open task.
2. *Geometric strength.* Strict weakness is proved only for the reduced chart T_{bin}, not for the full geometric package (G1)–(G4). Whether the adopted geometry is equivalent to, strictly weaker than, or incomparable with Hilbert-space structure remains unresolved (Remark VI.5).
3. *Multi-outcome extension.* The derivation is confined to the two-outcome binary-split

model. Whether the result extends to general n-outcome measurements or to iterated binary decompositions remains open (Remark VI.7).

4. *Unconditional chain.* The programmatic sketch in Section IX (which follows this section) outlines how four adopted elements—phase (P2), no-signaling (P5), continuity (P6), and the 2-norm convention (NC)—might be derived rather than assumed within a broader RS forcing chain. That program depends on several unproved links, particularly the no-signaling argument from ledger disjointness and the monotonicity step replacing continuity. Until those links are independently established and peer-reviewed, the conditional character of the paper is not reduced. Section IX should be read as a research program, not as supporting the proofs in this section.

IX. OUTLOOK: TOWARD AN UNCONDITIONAL RS DERIVATION

The conditional character of the paper rests on five adopted elements: the scalar ansatz (SA), the phase postulate (P2), no-signaling (P5), continuity (P6), and the 2-norm convention (NC). Of these, (SA) is open problem 1 (Section VIII B) and is not addressed here. This section therefore sketches four programmatic routes for the remaining four. Route 1: the RS 8-tick periodicity forces a period-8 shift operator T with $T^8 = \mathrm{id}$ on $\mathbb{R}^8$ [32]; the irreducible factor $x^2 + 1$ of its minimal polynomial forces complexification—Lean-verified as `no_real_root_x2_plus_1` and `complexification_forced` in `RS.Foundation.ComplexStructureForcing` (see Appendix A)—which would supply the phase degree of freedom (P2) if the extended cost functional can be shown to depend only on real count ratios and not on complex phases. Route 2: disjoint ledger regions share no recognition events, so, under an additional localization premise not yet derived from the RS axioms, a measurement on one region cannot affect statistics on a disjoint region, yielding no-signaling (P5) as a bookkeeping consequence. Route 3: if the cost ordering forces monotonicity of the scalar probability law f, then continuity (P6) could be replaced; monotonicity suffices to exclude pathological Cauchy solutions just as continuity does, but deriving monotonicity from the cost axioms has not been established. Route 4: the DFT-8 matrix diagonalizes T unitarily, Parseval's identity preserves the 2-norm [32], and if the RS dynamics generates a sufficiently large unitary group, the 2-norm convention (NC) would follow as a dynamical invariant rather than an adopted choice. None of these links is established here or used in any proof in the paper.

If Routes 1–4 were each established, the derivation chain would become:

$$\text{RS axioms} \rightarrow J \text{ unique} \rightarrow \text{8-tick} \rightarrow \mathbb{C} \xrightarrow{\text{DFT-8}} \text{unitarity}$$
$$\rightarrow \underbrace{\text{Parseval}}_{\Rightarrow \text{(NC)}} \xrightarrow{(SA)+\text{Route B}} P = |c|^2.$$

The scalar ansatz (SA) would remain the sole open gap; it is not addressed by any of the four routes. Route B (not Route A) is the endpoint of this chain because Route A still requires the independent multiplicativity axiom (MA).

Several upstream steps in this chain—complexification forcing, DFT-8 Parseval identity, and phase invariance—have Lean 4 build-verified counterparts in the public repository (see Appendix A). The critical remaining links—no-signaling from ledger disjointness and monotonicity from cost ordering—have not been formalized or peer-reviewed independently. This section should therefore be read as a structured research program identifying which adopted conditions are candidates for future elimination, not as a theorem here.

Appendix A: A Lean 4 Audit of the Conditional Derivation and Selected Upstream RS Ingredients

This appendix audits which upstream RS lemmas have Lean 4 build-verified counterparts in the repository `https://github.com/jonwashburn/recognition-science` (audited revision: commit `26851086105c105a9fd74098b2d39c25756d3624`), and which Born-rule-facing claims remain manuscript proofs only. The central conditional uniqueness results here—Proposition V.6, Theorem VI.6, and Theorem VI.11—are not currently public Lean theorems. *Build-verified* below means only that the declaration is present in the audited checkout and exposed through a successful Lean build; it does not mean that any manuscript Born-rule claim has been formalized.

a. Upstream RS ingredients with build-verified Lean counterparts. The following are reproducible from the audited public checkout:

- J uniqueness on $(0, \infty)$: `IndisputableMonolith.CostUniqueness.T5_uniqueness_complete`
- Log-coordinate cosh form and ODE uniqueness: `JcostCoshFormCert`; `ODECoshUniqueCert`

- Complexification forced (8-tick → $\mathbb{C}$): `complexification_forced`; `shift_period_8`; `no_real_root_x2_plus_1` (`RS.Foundation.ComplexStructureForcing`)
- DFT-8 Parseval (2-norm preservation): `dft8_preserves_norm`; `dft8_preserves_inner`
- Phase invariance of J-cost (upstream RS only; the extension argument for the manuscript's (P2) has not been independently verified): `jcost_phase_invariant`; `mode_cost_phase_invariant`
- Two-outcome Born probabilities ($\cos^2\theta$, $\sin^2\theta$): `IM.Verification.TwoOutcomeBornCert.P_cos_eq`; `P_sin_eq` — these formalize the two-branch normalized Born probability assignments of Section VI via the $C = 2A$ bridge.
- Born weight r^2 forcing under two-branch calibration: `RS.Foundation.BornRuleForcing.born_weight_forced` — proves that under the explicit calibration $w(\cos\theta) = \cos^2\theta$ for $\theta \in (0, \pi/2)$, one obtains $w(r) = r^2$ for $r \in (0, 1)$; closely related to Proposition V.6 (Route B).
- Sector measure normalization and phase invariance: `RS.Foundation.BornRuleForcing.sectorMeasure_total`; `sectorMeasure_phase_invariant`

b. Manuscript claims not covered by the audited public checkout. The following claims are proved only in the manuscript and are not currently exposed as Lean build targets: the no-signaling scalar uniqueness route (Prop. V.6 and Lemma V.5); the Route A classical uniqueness theorem (Thm. V.1); the scope-control theorem (Thm. VI.11); the packaged two-outcome result (Thm. VI.6); the measurement-geometry calculation (G1)–(G4) and path-action consistency check (Prop. VI.2); and the unconditional derivation chain (Section IX), for which the no-signaling and monotonicity links are not yet formalized.

Remark A.1 (8-tick periodicity and complex structure). Within the broader RS forcing chain [32], the period-8 shift operator forces complexification and supplies structural motivation for postulate (P2). That forcing chain is not derived here; it enters only as external motivation.

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AUTHOR CONTRIBUTIONS

J.W.: Conceptualization; Formal analysis; Methodology; Writing — original draft; Writing — review & editing. **M.S.:** Writing — review & editing; Project administration. **E.A.:** Formal analysis; Methodology; Writing — review & editing.

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DECLARATIONS

c. Conflict of interest. The authors declare that they have no conflict of interest.

DATA AVAILABILITY

This work is theoretical and contains no experimental data. The Lean 4 source code audited for this paper is publicly available at `https://github.com/jonwashburn/recognition-science`; Appendix A records the audited public revision and distinguishes direct public-checkout build evidence from manuscript-only steps not yet exposed as public build targets.

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