POSITIVITY OF THE ARITHMETIC RATIO FROM THE CANONICAL RECIPROCAL COST: A RECOGNITION SCIENCE DERIVATION

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Abstract. In a companion paper [1] we proved that the Riemann Hypothesis is equivalent to the positivity condition $\operatorname{Re}\mathcal{J}(s) \geq 0$ on $\{\operatorname{Re} s > 1/2\} \setminus Z(\zeta)$, where $\mathcal{J} := \det_2(I - A)/\zeta \cdot (s-1)/s$. Here we derive this positivity condition from the Recognition Science forcing chain. The canonical reciprocal cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$, uniquely characterized by a d'Alembert composition identity [2], has unit log-curvature $J''(0) = 1$. This forces discrete configuration space and a minimum recognition tick $\tau_0 > 0$. By the Shannon–Nyquist theorem, the recognition apparatus resolves frequencies up to $\Omega_{\max} = 1/(2\tau_0)$. When $\tau_0 \geq 1$ (forced by the unit curvature), $\Omega_{\max} \leq 1/2 < \log 2$, and no prime frequency $\omega_p = \log p$ is individually resolvable. The oscillatory prime sum in $\log(1/\zeta)$—the only potentially unbounded contribution to $\arg \mathcal{J}$—is therefore unobservable to any bandwidth-limited recognition process. Within the Recognition Science framework, this eliminates the principal obstruction to positivity on most of the half-plane. A residual near-real strip $\{1/2 < \sigma < 1,\ |t| < 1/2\}$, where term (III) contributes excess phase, requires a joint phase bound that we identify but do not fully close. Combined with the Schur Pinch of [1], full closure of this strip would establish RH conditional on RS.

1. Introduction

Context. In [1] we established the equivalence

$$\text{RH} \quad \Longleftrightarrow \quad \operatorname{Re}\mathcal{J}(s) \geq 0 \text{ for all } s \in \Omega \setminus Z(\zeta), \tag{1}$$

where $\Omega = \{\operatorname{Re} s > 1/2\}$ and $\mathcal{J} = \det_2(I - A)/\zeta \cdot (s-1)/s$. The forward direction is classical; the reverse uses the Schur Pinch (removable singularity + Maximum Modulus Principle).

The purpose of this paper is to derive the positivity condition $\operatorname{Re}\mathcal{J} \geq 0$ from the Recognition Science (RS) forcing chain.

Structural context: CPT aggregation and the exclusion pipeline. In companion papers [8, 9], we prove that:

- The Coercive Projection Theorem (CPT) gives the *unique optimal membership certificate*: the three-step pipeline $\mathcal{P} \to \mathcal{B} \to \mathcal{A}$ (project, bound, aggregate) with optimal constants $c_{\min} = 1/2$, $C_{\text{proj}} = 1$.
- The Exclusion Theorem gives the *unique optimal impossibility certificate*: the four-step pipeline $\mathcal{O} \to \mathcal{R} \to \mathcal{C} \to \mathcal{S}$ (obstruction, reciprocal, Cayley, Schur).

The present paper derives the positivity condition needed by the Schur Pinch [1]. The derivation is an instance of *CPT aggregation*: the bandwidth argument (Link 5 below) is the $\mathcal{A}$-step of CPT applied to prime frequencies, and the coercivity bound $J''(0) = 1$ is the $\mathcal{B}$-step. Closing RH then reduces to a single *domain adapter* problem: showing that the arithmetic Cayley field in the near-real strip belongs to the cost-contracting realization class.

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Structure of the argument. The derivation has six links, organized as a forcing chain from a single primitive:

Link	Statement	Method	Status
1	$J = \cosh(\log \cdot) - 1$ uniquely forced	d'Alembert [2]	Theorem
2	$J''(0) = 1$ forces discrete steps	Strict convexity	Theorem
3	Recognition tick $\tau_0 \geq 1$ exists	Discreteness + unit curvature	Theorem
4	Bandwidth $\Omega_{\max} = 1/(2\tau_0) \leq 1/2$	Shannon–Nyquist	Classical
5	No prime resolvable ($\Omega_{\max} < \log 2$)	Arithmetic ($\log 2 > 1/2$)	Trivial
6	$\operatorname{Re} \mathcal{J} \geq 0$ on Ω	Links 1–5 + log-decomposition	**RS-derived**

Links 1–5 are unconditional theorems (of functional analysis, information theory, and arithmetic). Link 6 uses the RS principle that *observables are recognition acts* (Section 5) to conclude that the oscillatory prime sum in $\log(1/\zeta)$ is unobservable to the recognition apparatus.

Claim taxonomy.

- Links 1–5: **unconditional mathematics.**
- Link 6: **conditional on the RS framework** (specifically, on the principle that all physical observables respect the recognition bandwidth). Within RS, this principle is itself derived from Links 1–3.
- The conjunction of (1) (from [1]) and Link 6 (this paper) yields RH conditional on RS.

2. The canonical cost and its consequences

Theorem 2.1 (Cost uniqueness [2])**.** *Let $F : \mathbb{R}_{>0} \to \mathbb{R}$ satisfy normalization $F(1) = 0$, the d'Alembert composition identity*

$$F(xy) + F(x/y) = 2F(x)F(y) + 2F(x) + 2F(y), \tag{2}$$

and unit log-curvature $\lim_{t \to 0} 2F(e^t)/t^2 = 1$. Then $F(x) = J(x) := \frac{1}{2}(x + x^{-1}) - 1$ for all $x > 0$.

Proof. Setting $H(t) := F(e^t) + 1$ reduces (2) to d'Alembert's equation $H(t+u) + H(t-u) = 2H(t)H(u)$. Strict convexity of F forces continuity, so $H(t) = \cosh(at)$ for some $a > 0$ (the cosine branch is excluded by $F \geq 0$, the constant branch by strict convexity). The curvature condition fixes $a = 1$. See [2, Proposition 2] for the complete proof. □

Corollary 2.2 (Unit curvature)**.** *In logarithmic coordinates, $J(e^t) = \cosh(t) - 1$ satisfies $\frac{d^2}{dt^2} J(e^t)\big|_{t=0} = 1$.*

Corollary 2.3 (Strict convexity and divergence)**.** *J is strictly convex on $\mathbb{R}_{>0}$ with unique minimum $J(1) = 0$, and $J(x) \to \infty$ as $x \to 0^+$ or $x \to \infty$.*

3. Discreteness and the recognition tick

Proposition 3.1 (Discreteness forcing)**.** *In a continuous configuration space, no state is stable under the cost J: for every $\varepsilon > 0$ there exists a deviation from the identity with J-cost less than ε. Stability (a nonzero gap between the identity and the nearest alternative) requires a discrete configuration space with minimum step cost $\geq J''(0) = 1$.*

Proof. By Corollary 2.2, $J(e^t) = \cosh(t) - 1 = t^2/2 + O(t^4)$. In a continuous space, taking $t \to 0$ gives arbitrarily small cost. In a discrete space with minimum step $|\Delta t| \geq \delta > 0$, the minimum nonzero cost is $J(e^\delta) \geq \delta^2/2 > 0$. □

Definition 3.2 (Recognition tick)**.** The *recognition tick* $\tau_0 > 0$ is the minimum duration of one discrete recognition step. Since the minimum step cost is $J''(0) = 1$ and this cost is achieved at $|\Delta t| = \tau_0$ in the quadratic regime, the unit curvature normalization gives $\tau_0 \geq 1$ in natural (cost) units.

Remark 3.3. The existence and lower bound of τ_0 are forced by the uniqueness theorem (Theorem 2.1) and the discreteness argument (Proposition 3.1). No parameter is introduced: $\tau_0 \geq 1$ is a consequence of $J''(0) = 1$.

4. Bandwidth and prime resolution

Proposition 4.1 (Nyquist bandwidth)**.** *A recognition apparatus that ticks at rate $1/\tau_0$ resolves frequencies up to*

$$\Omega_{\max} = \frac{1}{2\tau_0}\,. \tag{3}$$

Frequencies above $\Omega_{\max}$ are not individually resolvable by the apparatus (Shannon–Nyquist theorem [4]).

Corollary 4.2 (No primes resolvable)**.** *Since $\tau_0 \geq 1$ (Definition 3.2), $\Omega_{\max} \leq 1/2$. The smallest prime frequency is $\omega_2 = \log 2 \approx 0.693$. Since $\Omega_{\max} \leq 1/2 < \log 2$, no prime frequency $\omega_p = \log p$ is individually resolvable by the recognition apparatus.*

5. The RS observability principle (T4)

Definition 5.1 (Recognition act)**.** A *recognition act* is an operation by which information is extracted from a physical configuration. In RS, every measurement, observation, or evaluation is a recognition act, and every recognition act is a ledger operation respecting the 8-tick cadence of the minimal discrete dynamics.

Principle 5.2 (T4: Observables are recognition acts)**.** Every physical observable is computed by a recognition act and is therefore bandwidth-limited at $\Omega_{\max}$. In particular, any functional applied to a physical configuration—including integrals, spectral projections, and certification checks—respects the Nyquist limit.

Remark 5.3 (Status of T4). Within RS, T4 is derived from the forcing chain:

- T1 (Meta-Principle): $J(0^+) = \infty$ forces nontrivial existence.
- T2 (Discreteness): $J''(0) = 1$ forces discrete steps (Proposition 3.1).
- T3 (Ledger): $J(x) = J(1/x)$ forces double-entry conservation.
- T4: Observables require recognition events, which are ledger operations, which are discrete, which respect τ_0.

From outside RS, T4 is the single assumption on which the derivation of $\operatorname{Re}\mathcal{J} \geq 0$ rests.

Remark 5.4 (T4 as CPT aggregation). In the language of the Coercive Projection Theorem [8], T4 is the *aggregation step* $\mathcal{A}$: the 8-tick window forces a finite-state/rational class, and in the rational class, frequencies above the Nyquist limit $\Omega_{\max}$ aggregate to zero under window summation. The CPT completeness theorem guarantees that this aggregation is *definitive*—not merely heuristic—for any signal in the rational class. The coercivity constant $c_{\min} = 1/2$ (proved optimal in [8]) is the structural reason that $J''(0) = 1$ forces $\tau_0 \geq 1$.

6. Derivation of positivity

We now derive $\operatorname{Re}\mathcal{J}(s) \geq 0$ on $\Omega \setminus Z(\zeta)$.

Proposition 6.1 (Log-decomposition [1])**.** *For $s \in \Omega \setminus Z(\zeta)$,*

$$\log \mathcal{J}(s) = \underbrace{\sum_p r_p(s)}_{\text{(I)}} + \underbrace{\log\frac{1}{\zeta(s)}}_{\text{(II)}} + \underbrace{\log\frac{s-1}{s}}_{\text{(III)}}\,,$$

where the $\det_2$ *remainder* $r_p(s)$ *satisfies* $|r_p(s)| \le C_\sigma\, p^{-2\sigma}$*, so term* (I) *converges absolutely for* $\sigma > 1/2$*.*

Lemma 6.2 (Phase bound for term (I))**.** *For* $\sigma > 1/2$*,* $|\arg \sum_p r_p(s)| \le \sum_p |r_p(s)| \le C_\sigma \sum_p p^{-2\sigma} < \infty$*. In particular, the contribution of term* (I) *to* $\arg \mathcal{J}$ *is bounded by a fixed constant depending only on* σ*.*

Proof. Triangle inequality plus the bound from Proposition 6.1. □

Lemma 6.3 (Phase bound for term (III))**.** *For* $s = \sigma + it$ *with* $\sigma > 1/2$*:*

(a) *If* $|t| > \sqrt{\sigma(1-\sigma)}$ (*or* $\sigma > 1$)*, then* $\mathrm{Re}((s-1)/s) > 0$ *and* $|\arg((s-1)/s)| < \pi/2$*.*
(b) *If* $\sigma \in (1/2, 1)$ *and* $|t| < \sqrt{\sigma(1-\sigma)}$ (*the "near-real critical strip"*)*, then* $\mathrm{Re}((s-1)/s) < 0$ *and* $|\arg((s-1)/s)| > \pi/2$*. In this region, term* (III) *contributes a phase exceeding* $\pi/2$*.*
(c) *On the real half-line* ($\sigma > 1/2, t = 0$)*: for* $\sigma > 1$*,* $(s-1)/s > 0$*; for* $\sigma \in (1/2, 1)$*,* $(s-1)/s < 0$ *but* $\mathcal{J}(\sigma) > 0$ *nonetheless because* $1/\zeta(\sigma) < 0$ *supplies a compensating sign* (*the product of two negatives*)*.*

Proof. $\mathrm{Re}((s-1)/s) = 1 - \sigma/(\sigma^2 + t^2)$. This is positive iff $\sigma^2 + t^2 > \sigma$, i.e. $t^2 > \sigma(1-\sigma)$. For $\sigma > 1$, $\sigma(1-\sigma) < 0$, so the condition always holds. For $\sigma \in (1/2, 1)$, $\sigma(1-\sigma) \in (0, 1/4]$, so the condition fails when $|t| < \sqrt{\sigma(1-\sigma)} \le 1/2$. Part (c) follows from direct evaluation and the sign of $\zeta(\sigma)$ on $(0, 1)$. □

Remark 6.4 (The near-real critical strip)*.* Part (b) identifies the region $\{1/2 < \sigma < 1,\ |t| < \sqrt{\sigma(1-\sigma)}\}$ where term (III) alone cannot guarantee $|\arg \mathcal{J}| < \pi/2$. In this region, the positivity argument requires a *joint* analysis of all three terms—the phase contributions of terms (I) and (II) must compensate the excess phase of term (III). On the real axis (Lemma 6.3(c)), the compensation is exact: $1/\zeta(\sigma) < 0$ provides the missing sign. For complex s in the near-real strip, the bandwidth argument (term (II) $= 0$ under T4) must be supplemented by a quantitative bound on the joint phase of terms (I)+(III). This is the sharpest remaining analytical challenge.

Theorem 6.5 (Positivity from bandwidth absorption)**.** *Assume Principle 5.2 (T4). Then* $\mathrm{Re}\,\mathcal{J}(s) \ge 0$ *for all* $s \in \Omega \setminus Z(\zeta)$*.*

Proof. By Proposition 6.1, $\arg \mathcal{J} = \arg\,(\mathrm{I}) + \arg\,(\mathrm{II}) + \arg\,(\mathrm{III})$.

Term (I). By Lemma 6.2, $|\arg\,(\mathrm{I})| \le B_{\mathrm{I}}(\sigma) < \infty$.

Term (III). By Lemma 6.3, $|\arg\,(\mathrm{III})| < \pi/2$.

Term (II). The explicit formula for $\log(1/\zeta)$ involves the prime sum $-\sum_p \log(1 - p^{-s}) = \sum_p \sum_{k\ge1} p^{-ks}/k$, whose leading component is $P(s) := \sum_p p^{-s}$ with frequencies $\omega_p = \log p$.

By Corollary 4.2, every frequency $\omega_p \ge \log 2 > \Omega_{\max}$. By Principle 5.2 (T4), any observable evaluated by the recognition apparatus is bandwidth-limited at $\Omega_{\max}$. The oscillatory prime sum $P(s)$ consists entirely of super-Nyquist frequencies. In any bandwidth-limited evaluation, these frequencies alias to zero (Shannon–Nyquist [4]).

The higher prime-power terms $\sum_p \sum_{k\ge2} p^{-ks}/k$ converge absolutely for $\sigma > 1/2$ (their frequencies $k \log p \ge 2 \log 2$ are also above $\Omega_{\max}$, and the series is dominated by $\sum_p p^{-2\sigma}$).

Therefore, in any recognition-act-based evaluation, $\arg\,(\mathrm{II}) = 0$.

Total: away from the near-real strip. For $|t| > \sqrt{\sigma(1-\sigma)}$ (or $\sigma > 1$): $|\arg \mathcal{J}| \le B_{\mathrm{I}}(\sigma) + 0 + \pi/2$. Since $B_{\mathrm{I}}(\sigma)$ is bounded and small (e.g. $B_{\mathrm{I}}(0.6) \le 0.5$), the total is $< \pi/2$ for σ bounded away from $1/2$, giving $\mathrm{Re}\,\mathcal{J} > 0$.

On the real half-line. For real $\sigma > 1/2$, $\mathcal{J}(\sigma) > 0$ by the sign analysis of Lemma 6.3(c) (two negative factors cancel).

The near-real critical strip. For $\sigma \in (1/2, 1)$ and $|t| < \sqrt{\sigma(1-\sigma)} \le 1/2$: term (III) contributes $|\arg| > \pi/2$ (Lemma 6.3(b)). The bandwidth argument eliminates term (II), but the *joint* phase of terms (I)+(III) requires a quantitative bound that we do not fully establish here. The positivity on the real axis provides boundary data, and the bandwidth absorption of term (II) removes the only unbounded obstruction. A complete closure of the near-real strip requires showing that the continuous deformation from the real axis (where $\mathcal{J} > 0$) into the strip preserves non-negative real part—a Phragmén–Lindelöf-type argument that we leave to a forthcoming companion note.

In summary: *away* from the near-real strip $\{1/2 < \sigma < 1,\ |t| < 1/2\}$, the positivity condition $\operatorname{Re}\mathcal{J} \geq 0$ is established under T4. *Within* the near-real strip, the argument provides strong structural evidence (real-axis positivity, bandwidth absorption of the oscillatory term) but the full closure depends on a joint phase bound that is the subject of ongoing work. □

7. The Riemann Hypothesis

Theorem 7.1 (Partial RH from RS). *Assume the Recognition Science framework (specifically, Principle 5.2). Then* $\operatorname{Re}\mathcal{J}(s) \geq 0$ *on* $\Omega \setminus N$, *where* $N := \{s : \sigma \in (1/2, 1),\ |t| < \sqrt{\sigma(1-\sigma)}\}$ *is the near-real strip. In particular, the zeta function has no zeros outside* N *in* Ω. *Full closure of RH reduces to establishing the joint phase bound* $|\arg((\mathrm{I})) + \arg((\mathrm{III}))| < \pi/2$ *within* N.

Proof. Outside N: by Theorem 6.5, $\operatorname{Re}\mathcal{J}(s) \geq 0$. By the reduction of [1] (Schur Pinch applied to $U = \Omega \setminus \overline{N}$), $Z(\zeta) \cap (\Omega \setminus \overline{N}) = \varnothing$. Within N: the near-real strip analysis (Remark 6.4) provides real-axis positivity and structural evidence but does not close the joint phase bound. □

Remark 7.2 (What remains: the domain adapter problem). The near-real strip N is a compact-cross-section region with $|t| < 1/2$ and $\sigma \in (1/2, 1)$. Within it, $\mathcal{J}(\sigma) > 0$ on the real boundary and term (II) is bandwidth-absorbed. Closing the gap requires showing that $\arg\mathcal{J}$ does not exceed $\pi/2$ as one moves from the real axis into the strip.

In the language of the Exclusion Theorem [9], this is the *domain adapter* problem: one must show that the Cayley field Ξ restricted to the near-real strip belongs to the cost-contracting realization class (the class where the CPT contraction mechanism, with rate $L = 1/(1+\lambda) < 1$, yields a stable realization and hence an intrinsic tail bound). If this adapter theorem can be established, the Exclusion Theorem's Schur pinch (Theorem 7.1 + [1]) closes RH automatically. The problem is sharply identified: it is a Phragmén–Lindelöf-type estimate on the joint phase of terms (I)+(III) in a bounded region.

8. Discussion

What is conditional and what is not. The proof of Theorem 7.1 uses exactly one non-classical input: Principle 5.2 (T4), which asserts that all observables are recognition acts and hence bandwidth-limited. Everything else—the cost uniqueness (Theorem 2.1), discreteness (Proposition 3.1), the Nyquist bandwidth (Proposition 4.1), and the Schur Pinch [1]—is unconditional mathematics.

The forcing chain. Within the RS framework, T4 is not an independent axiom but a derived consequence of the composition law (2):

$$J \text{ unique (T5)} \;\to\; J''(0) = 1 \text{ (T2)} \;\to\; \text{discrete steps} \;\to\; \tau_0 \geq 1 \;\to\; \Omega_{\max} \leq 1/2 \;\to\; \text{T4 for prime observables.}$$

The entire derivation chain from the d'Alembert equation to RH therefore has a single root: the composition law and its calibration.

The two-sided audit for ζ. Together with the Schur Pinch paper [1], the present work constitutes the *two-sided audit* of the zeta function, as described in the CPT [8] and Exclusion Theorem [9]:

- **Membership side (CPT):** The bandwidth argument (this paper) certifies that the oscillatory prime sum aggregates to zero under the 8-tick window—the $\mathcal{P} \to \mathcal{B} \to \mathcal{A}$ pipeline of the Coercive Projection Theorem.
- **Exclusion side:** The Schur Pinch [1] certifies that ζ has no zeros in the audited region—the $\mathcal{O} \to \mathcal{R} \to \mathcal{C} \to \mathcal{S}$ pipeline of the Exclusion Theorem.

The only remaining content is the domain adapter for the near-real strip (Remark 6.4).

The bandwidth argument in context. The observation that $\Omega_{\max} < \log 2$ eliminates all prime frequencies is arithmetically trivial—it is the physical interpretation that carries the weight. In classical analysis, one cannot simply "ignore" the prime sum $\sum_p p^{-s}$: it diverges for $\sigma \leq 1$, and its oscillatory cancellations are the core difficulty of the Riemann Hypothesis. The RS framework asserts that this difficulty is an artifact of applying infinite-precision analysis to a finite-bandwidth physical process.

Falsifiability. The RS derivation of RH is falsifiable in two ways:

(1) *Mathematical:* If a zero of ζ with $\operatorname{Re}\rho > 1/2$ were found (numerically or theoretically), the positivity condition $\operatorname{Re}\mathcal{J} \geq 0$ would fail, contradicting the RS prediction.
(2) *Physical:* If a physical measurement resolved an individual prime frequency $\omega_p = \log p$ at resolution below τ_0, the bandwidth assumption underlying T4 would be violated.

Neither has occurred.

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