

The Golden Ratio as a Universal Coherence Eigenvalue: Bridging Penrose Aperiodic Order and Information-Theoretic Comparison

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Abstract

The golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ occupies a distinguished position in mathematics, appearing across diverse domains from number theory and dynamical systems to geometric tilings and quasicrystal physics. This paper establishes a formal bridge between two independently motivated occurrences of φ : as the forced inflation eigenvalue in Penrose aperiodic tilings, and as the unique self-similar fixed point in a cost-first framework for coherent ratio comparison. We demonstrate that the log-ratio transformation $t = \ln x$ provides a precise structural isomorphism, converting Penrose’s multiplicative geometric scaling $x \mapsto \varphi x$ into the ledger framework’s additive information-theoretic shift $t \mapsto t + \ln \varphi$. This correspondence reveals φ not as an arbitrary constant but as a universal *coherence eigenvalue*: the unique scale at which local reciprocal symmetry constraints and global self-similarity lock together. We derive explicit formulas connecting the Penrose substitution matrix eigenstructure to the ledger’s reciprocal cost functional $J(x) = \frac{1}{2}(x + x^{-1}) - 1$, and interpret $J(\varphi) \approx 0.118$ as the minimal “coherence cost of aperiodicity.” This bridge suggests that aperiodic geometric order and information-theoretic comparison dynamics are unified manifestations of the same underlying mathematical principle.

1 Introduction

1.1 Motivation: the ubiquity and uniqueness of φ

The golden ratio $\varphi = (1 + \sqrt{5})/2 \approx 1.618$ is distinguished among real numbers by its self-reciprocal algebraic property:

$$\varphi^2 = \varphi + 1 \quad \iff \quad \varphi = 1 + \frac{1}{\varphi}. \quad (1)$$

This simple identity has profound consequences across mathematics and physics. In classical geometry, φ governs the proportions of the regular pentagon and the golden rectangle’s self-similar subdivision. In modern mathematics, it appears as:

- the Perron–Frobenius eigenvalue of Fibonacci substitution systems and the asymptotic ratio of consecutive Fibonacci numbers;

- the inflation/deflation scaling factor in Penrose aperiodic tilings and other quasicrystalline structures;
- a fundamental spectral constant in the theory of quasicrystals with five-fold symmetry;
- the most irrational number (in the sense of continued fractions), making it the “last” rotation number to develop periodic orbits in circle maps;
- a critical parameter in KAM theory and the stability of quasiperiodic motion in Hamiltonian systems.

What unifies these diverse appearances? A common structural theme emerges: φ appears when local coherence constraints produce global rigidity, and self-similarity is imposed as a consistency requirement. Unlike other distinguished constants (e.g., π , e), the golden ratio is not defined analytically but algebraically: it is the unique positive solution to a minimal quadratic self-consistency equation. This paper argues that this algebraic minimality is precisely what makes φ a universal *coherence eigenvalue* across disparate mathematical structures.

1.2 Two independent frameworks, one shared scale

This paper concerns two independently developed mathematical frameworks, each of which produces φ from first principles with no *a priori* connection:

Framework I: Penrose tilings and aperiodic geometric order. Penrose tilings are planar tessellations by a finite set of prototiles (e.g., dart and kite, or thick and thin rhombi) that tile the plane *aperiodically*—without translational symmetry—when subject to local matching rules. Discovered by Roger Penrose in the 1970s [1], these tilings provided the first mathematical model of quasicrystals, aperiodic atomic structures with sharp diffraction patterns discovered experimentally in 1984 and recognized with the 2011 Nobel Prize in Chemistry.

The appearance of φ in Penrose tilings is *forced*, not chosen:

1. **Geometric forcing.** The prototiles are built from isosceles triangles with angles 36° , 72° , and 108° (multiples of $\pi/5$, the fundamental angle of pentagonal symmetry). Elementary trigonometry shows that the diagonal-to-edge ratio in a regular pentagon is φ , which forces the long/short edge-length ratio in Penrose tiles to be φ .
2. **Substitution eigenvalue.** Penrose tilings admit inflation/deflation substitution rules: each tile is replaced by a finite patch of tiles, and the geometry is rescaled by φ to return to the original scale. The associated 2×2 substitution matrix has Perron–Frobenius eigenvalue φ , which governs tile-frequency growth and establishes self-similarity.

Thus φ is the unique scale at which local matching rules (pentagonal geometry) and global substitution hierarchy (inflation eigenvalue) are mutually consistent.

Framework II: Cost-first ledger and coherent ratio comparison. In a recently developed information-theoretic framework [2], the primitive question is: *What is the information cost of asserting that one quantity exceeds another by a given ratio?* Starting from axiomatic coherence requirements—that comparison costs compose consistently under multiplicative chaining and respect reciprocal symmetry $J(x) = J(x^{-1})$ —the framework derives a unique cost function:

$$J(x) = \frac{1}{2} (x + x^{-1}) - 1, \quad (2)$$

where $x = a/b$ is the ratio of two positive quantities. This cost is minimal at perfect balance ($J(1) = 0$) and grows symmetrically as x deviates from unity in either direction.

The appearance of φ in this framework is more subtle: it arises when one introduces a self-similar coarse-graining hypothesis on comparison updates. Specifically, if ratios evolve via the minimal reciprocal self-correction rule

$$x_{n+1} = 1 + \frac{1}{x_n}, \quad (3)$$

then the unique positive fixed point is φ , satisfying equation (1). In the natural log-ratio coordinate $t = \ln x$, the cost becomes $J(e^t) = \cosh(t) - 1$, quadratic near equilibrium, and the scale $\ln \varphi \approx 0.481$ emerges as a fundamental additive information-theoretic unit.

1.3 The central question and main results

The central question motivating this work is:

Is the shared appearance of φ in Penrose tilings and cost-first comparison a numerical coincidence, or does it reflect a deeper structural correspondence?

We argue for the latter. Our main contributions are:

1. **The log-ratio isomorphism** (Section 4): We show that the coordinate transformation $t = \ln x$ provides a precise group isomorphism between multiplicative scaling (Penrose) and additive translation (ledger). Under this isomorphism,

$$x \mapsto \varphi x \quad \iff \quad t \mapsto t + \ln \varphi,$$

making $\ln \varphi$ a universal additive scale for coherent ratio dynamics.

2. **Geometric interpretation of the cost functional** (Section 5): We prove that φ is the unique positive ratio satisfying $x - 1 = 1/x$, meaning its additive deviation from unity exactly equals its reciprocal. Evaluating $J(\varphi) = \varphi - 3/2 \approx 0.118$ provides a quantitative measure of the “coherence cost of aperiodicity.”
3. **Unified interpretation** (Section 6): We demonstrate that both frameworks exhibit the same abstract structure: local reciprocal/pentagonal constraints plus global self-similarity requirements force a unique scale φ . The logarithm is the key mathematical mechanism that converts geometric (multiplicative) inflation into information-theoretic (additive) translation.

1.4 Outline and scope

The remainder of this paper proceeds as follows. Section 2 reviews the mathematical structure of Penrose tilings, emphasizing the geometric and spectral origins of φ . Section 3 summarizes the cost-first ledger framework, deriving the reciprocal cost (2) and the self-similar fixed point. Section 4 develops the central bridge via log-ratio coordinates. Section 5 provides a detailed geometric interpretation of $J(\varphi)$ in terms of Penrose edge-length ratios. Section 6 synthesizes the correspondence and discusses broader implications. Section 7 concludes with open questions and potential applications.

This is an expository and conceptual paper aimed at clarifying a mathematical correspondence. We do not develop new theorems about Penrose tilings or ledger dynamics per se, but rather establish a formal bridge that reveals their shared mathematical essence. All results cited from the individual frameworks are taken from existing literature; our contribution is the synthesis and the identification of the log-ratio transformation as the unifying mechanism.

2 Penrose tilings: φ as forced geometry and inflation eigenvalue

Penrose tilings are planar tessellations constructed from a finite alphabet of prototiles that tile the plane *aperiodically*—without translational symmetry—when subject to local matching rules. This section establishes two independent but related facts: φ is forced by the pentagonal geometry of the prototiles, and φ emerges as the Perron–Frobenius eigenvalue of the substitution operator.

2.1 Dart–kite geometry and pentagonal forcing

The dart–kite Penrose tiling uses two quadrilateral prototiles: a concave “dart” and a convex “kite.” Both tiles are built from isosceles triangles with apex angles that are integer multiples of $36^\circ = \pi/5$, the fundamental angle of pentagonal symmetry. Specifically:

- The *golden gnomon* is an isosceles triangle with apex angle 36° and base-to-leg ratio $1 : \varphi$.
- The *golden triangle* is an isosceles triangle with apex angle 108° and leg-to-base ratio $1 : \varphi$.

The key trigonometric identity linking 36° to φ is:

$$\cos(36^\circ) = \frac{\varphi}{2} = \frac{1 + \sqrt{5}}{4}. \quad (4)$$

This can be derived from the double-angle formula and the fact that $72^\circ = 2 \cdot 36^\circ$ satisfies $\cos(72^\circ) = (\varphi - 1)/2 = 1/(2\varphi)$.

In a regular pentagon with unit side length, the diagonal length is precisely φ . Since dart and kite tiles are constructed from these triangular building blocks, each prototile has edges of two distinct lengths. Normalizing the short edge to length 1, the long edge necessarily has length φ . This is not a design choice but a *forced outcome* of the pentagonal angle structure.

Matching rules enforce aperiodicity. Without additional constraints, one could combine a dart and a kite to form a rhombus, allowing periodic tilings. To prevent this, Penrose introduced *matching rules*: edges are decorated with markers (or equivalently, Conway arcs) such that only certain adjacencies are permitted. These local constraints forbid periodic recombinations and thereby enforce global aperiodicity. Crucially, the matching rules are *compatible* with the φ edge-length ratio and with the substitution structure described below. Any other ratio would break this consistency.

2.2 Inflation/deflation as substitution with eigenvalue φ

Penrose tilings admit *substitution rules* (also called inflation and deflation) that define a hierarchical self-similar structure. The inflation operator \mathcal{I} acts on a tiling as follows:

1. **Combinatorial substitution:** Replace each prototile (dart or kite) by a finite patch of smaller tiles according to a fixed rule.
2. **Geometric rescaling:** Scale the entire configuration by the factor φ so that the new tiles have the same physical size as the originals.

The combinatorial replacement is encoded in a 2×2 substitution matrix M :

$$M = \begin{pmatrix} n_{DD} & n_{DK} \\ n_{KD} & n_{KK} \end{pmatrix}, \quad (5)$$

where n_{ij} counts the number of tiles of type j (Dart or Kite) produced when inflating a tile of type i . For the dart–kite system, explicit calculation yields:

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{with eigenvalues } \lambda_1 = \varphi, \quad \lambda_2 = -1/\varphi. \quad (6)$$

The leading eigenvalue $\lambda_1 = \varphi$ is the Perron–Frobenius eigenvalue (the unique largest positive eigenvalue of a nonnegative primitive matrix). It governs:

- The asymptotic tile-count growth under repeated inflation: after n inflations, tile counts scale as φ^n .
- The asymptotic frequency ratio of darts to kites, given by the associated positive eigenvector.
- The geometric scaling factor needed to return the tiling to its original scale after substitution.

Self-similarity and fixed-point structure. The deflation operator $\mathcal{D} = \mathcal{I}^{-1}$ reverses inflation, subdividing each tile into a patch of φ -smaller tiles. The key result is that Penrose tilings are *fixed points* of the inflation/deflation cycle up to rescaling by φ :

$$\mathcal{I}(T) = (\text{rescaled copy of } T),$$

meaning the tiling “looks the same” at all scales related by powers of φ . This is the precise mathematical sense of *aperiodic self-similarity*: the structure repeats at hierarchically scaled resolutions, with φ as the universal scaling constant.

Summary: φ is forced by coherence. In both the geometric (edge lengths) and dynamical (substitution eigenvalue) perspectives, φ is not a free parameter. It is the unique value that makes the Penrose system *coherent*: local matching rules, tile geometry, and global substitution hierarchy all mutually reinforce one another, and φ is the scale at which they lock together.

3 Ledger comparison: reciprocal cost, log-ratio coordinates, and a φ fixed point

This section summarizes a recently developed information-theoretic framework for coherent ratio comparison. We begin with the axiomatic derivation of the reciprocal cost functional, introduce the natural log-ratio coordinate, and show how a self-similar update rule produces φ as a fixed point.

3.1 The comparison primitive and reciprocal cost

In many contexts, the fundamental observable is not an absolute quantity but a *ratio*:

$$x = \frac{a}{b} \in \mathbb{R}_{>0}, \quad (7)$$

where $a, b > 0$ are positive quantities (prices, probabilities, energy levels, etc.). The primitive question is: *What is the information cost of asserting that a exceeds b by ratio x ?*

Three axiomatic requirements constrain the cost functional $J : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{\geq 0}$:

1. **Multiplicative composition.** Comparing a to b and then b to c should compose to comparing a to c : if $x_1 = a/b$ and $x_2 = b/c$, then $x = x_1 x_2 = a/c$.
2. **Coherence (d’Alembert equation).** The total cost of a composition should equal the sum of the individual costs:

$$J(x_1 x_2) = J(x_1) + J(x_2).$$

This is the defining property of “coherent” comparison—costs are additive under ratio multiplication.

3. **Reciprocal symmetry.** The cost of comparing a to b should equal the cost of comparing b to a :

$$J(x^{-1}) = J(x). \quad (8)$$

Derivation of the unique reciprocal cost. Under mild regularity (continuity), the general solution to the d’Alembert functional equation is $J(x) = c \ln x$ for some constant c . However, imposing reciprocal symmetry (8) forces $c = 0$, yielding only the trivial solution $J \equiv 0$.

To obtain a nontrivial cost, one relaxes the d’Alembert equation to hold *on average* and imposes a normalization: small deviations from balance ($x \approx 1$) should cost quadratically

in the log-ratio $t = \ln x$, i.e., $J(e^t) \approx t^2/2$ for $|t| \ll 1$. This calibration (motivated by the principle that costs near equilibrium should be Euclidean in the natural coordinate) uniquely determines:

$$J(x) = \frac{1}{2} (x + x^{-1}) - 1. \quad (9)$$

Properties of the reciprocal cost. The function J satisfies:

- **Reciprocal symmetry:** $J(x^{-1}) = J(x)$ by construction.
- **Minimum at balance:** $J(x) \geq 0$ with equality if and only if $x = 1$; the unique global minimum is at perfect balance.
- **Quadratic near equilibrium:** In log coordinates $x = e^t$,

$$J(e^t) = \cosh(t) - 1 = \frac{t^2}{2} + \frac{t^4}{24} + O(t^6), \quad (10)$$

so near $t = 0$ the cost is Euclidean in log-ratio space.

- **Divergence at extremes:** $J(x) \rightarrow \infty$ as $x \rightarrow 0^+$ or $x \rightarrow \infty$, penalizing large imbalances.

3.2 Log-ratio coordinates: the natural linearizing transformation

Because comparisons are multiplicative, the logarithm provides the natural linearizing coordinate:

$$t := \ln x. \quad (11)$$

Mathematically, $t = \ln x$ is the unique (up to scaling) group isomorphism from $(\mathbb{R}_{>0}, \times)$ to $(\mathbb{R}, +)$. Under this transformation:

- Multiplicative rescaling becomes additive translation: $x \mapsto cx$ corresponds to $t \mapsto t + \ln c$.
- Perfect balance $x = 1$ corresponds to $t = 0$.
- Reciprocal inversion $x \mapsto x^{-1}$ corresponds to reflection $t \mapsto -t$.
- The cost becomes $J(e^t) = \cosh(t) - 1$, manifestly even in t .

This coordinate choice is not arbitrary but forced by the requirement that the cost be locally Euclidean (quadratic) near equilibrium. Any other coordinate would destroy this property.

3.3 Self-similar reciprocal updates and the φ fixed point

Beyond the static cost structure, suppose ratios evolve via a dynamical update rule. A minimal self-similar reciprocal update is:

$$x_{n+1} = 1 + \frac{1}{x_n}. \quad (12)$$

This can be interpreted as: “the next ratio is formed by adding unity to the reciprocal of the current ratio.” The map $f(x) = 1 + 1/x$ satisfies:

- **Reciprocal balance:** $f(x^{-1}) = f(x)$, so it respects the symmetry (8).
- **Minimality:** It is the simplest nontrivial combination of additive (+1) and reciprocal (+1/x) corrections.

The positive fixed point x^* satisfies:

$$x^* = 1 + \frac{1}{x^*} \iff (x^*)^2 - x^* - 1 = 0 \iff x^* = \varphi. \quad (13)$$

Thus, under the self-similar recursion (12), φ emerges as the unique positive attractor.

In log coordinates $t = \ln x$, the update becomes:

$$t_{n+1} = \ln(1 + e^{-t_n}), \quad (14)$$

with fixed point $t^* = \ln \varphi \approx 0.4812$. This motivates defining the ledger’s natural information-theoretic scale:

$$J_{\text{bit}} := \ln \varphi, \quad (15)$$

as the fundamental additive unit for self-similar ratio dynamics.

Important distinction. We emphasize that φ does *not* follow from the reciprocal symmetry or coherence axioms alone. It arises *only* when one adds the self-similar update rule (12). This is the exact parallel to Penrose tilings: φ is not assumed but emerges as the unique fixed point of a consistency requirement (substitution self-similarity in Penrose, reciprocal update self-similarity in ledger).

4 The bridge: logarithms turn Penrose inflation into ledger translations

Having established the independent origins of φ in Penrose tilings (geometric forcing and substitution eigenvalue) and in the ledger framework (reciprocal cost and self-similar fixed point), we now develop the central structural correspondence.

4.1 The log-ratio isomorphism

Penrose inflation acts multiplicatively on geometric lengths: a patch of linear size ℓ is rescaled to size $\varphi \ell$. In the ledger framework, comparisons are expressed as ratios $x = a/b$, which are also multiplicative objects. The logarithm provides the canonical transformation that converts multiplication into addition:

Mathematical statement. The log-ratio coordinate $t = \ln x$ is the unique (up to scaling) group isomorphism

$$\ln : (\mathbb{R}_{>0}, \times) \rightarrow (\mathbb{R}, +).$$

Under this isomorphism, Penrose inflation and ledger comparison share the same additive structure:

$$\boxed{x \mapsto \varphi x \iff t \mapsto t + \ln \varphi.} \tag{16}$$

Interpretation. This identity has several immediate consequences:

1. **Universal additive scale.** The quantity $\ln \varphi \approx 0.4812$ is the universal additive “step size” for any system whose ratios scale by φ , regardless of whether those ratios represent geometric lengths, tile frequencies, energy levels, or abstract comparison costs.
2. **Self-similarity in both languages.** Penrose tilings are self-similar under multiplication by φ ; ledger comparisons (under the recursion (12)) are self-similar under translation by $\ln \varphi$. These are not two different structures but one structure expressed in two equivalent languages (multiplicative vs. additive).
3. **Inflation as information-theoretic translation.** If one interprets Penrose inflation as a “coarse-graining” operation (grouping micro-tiles into macro-tiles), then in log-ratio coordinates this corresponds to shifting the observational scale by $\ln \varphi$. Conversely, deflation (refinement) corresponds to shifting by $-\ln \varphi$.

4.2 Compatibility with the coherence-forced cost

The reciprocal cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ transforms under $x = e^t$ to:

$$J(e^t) = \cosh(t) - 1 = \frac{t^2}{2} + \frac{t^4}{24} + O(t^6). \tag{17}$$

This is the key to understanding why the log-ratio coordinate is “natural”: it is the unique coordinate in which the coherence-forced cost is locally Euclidean (quadratic) near equilibrium.

The inflation shift (16) moves one between “nearby” or “farther” regimes of recognizability in a controlled way:

- Starting from balance $t = 0$, one inflation step moves to $t = \ln \varphi \approx 0.481$, incurring cost

$$J(\varphi) = \cosh(\ln \varphi) - 1 \approx \frac{(\ln \varphi)^2}{2} \approx 0.116.$$

(The exact value is $J(\varphi) = \varphi - 3/2 \approx 0.118$; the quadratic approximation is already quite accurate.)

- Multiple inflation steps correspond to translations by integer multiples of $\ln \varphi$, moving through a discrete hierarchy of scales analogous to Penrose’s inflation/deflation hierarchy.

4.3 Operational analogy: inflation as coarse-graining

We can now make precise the operational correspondence between Penrose and ledger dynamics:

Operation	Penrose (geometric)	Ledger (comparison)
Primitive scale	length ℓ	ratio $x = a/b$
Rescaling	$\ell \mapsto \varphi \ell$	$x \mapsto \varphi x$
Natural coordinate	$t = \ln \ell$ (log-length)	$t = \ln x$ (log-ratio)
Rescaling in coord.	$t \mapsto t + \ln \varphi$	$t \mapsto t + \ln \varphi$
Self-similarity	inflation/deflation	update rule (12)
Coherence eigenvalue	Perron–Frobenius $\lambda_1 = \varphi$	fixed point $x^* = \varphi$

Both systems exhibit the same abstract structure: a multiplicative primitive (length/ratio) with a distinguished rescaling factor (φ), which becomes an additive translation ($\ln \varphi$) in the natural logarithmic coordinate. The logarithm is not an external tool but the intrinsic linearizing map for any multiplicatively-structured system.

5 Geometric interpretation: $J(\varphi)$ as the coherence cost of aperiodicity

The reciprocal cost functional J has a particular algebraic form: it is a shifted arithmetic mean of a ratio and its reciprocal. This section explores the deeper geometric meaning of evaluating J at the golden ratio scale forced by Penrose tilings.

5.1 The golden ratio’s self-reciprocal property

The defining equation $\varphi^2 = \varphi + 1$ can be rewritten as:

$$\varphi = 1 + \frac{1}{\varphi}, \tag{18}$$

which equivalently states:

$$\varphi - 1 = \frac{1}{\varphi}. \tag{19}$$

This identity is the key to understanding why φ appears in both Penrose tilings and cost-first comparison. It says that φ ’s *additive deficit from unity* (left-hand side: how much φ exceeds 1) exactly equals its *multiplicative reciprocal* (right-hand side: the inverse ratio). In other words:

At $x = \varphi$, the additive deviation $x - 1$ and the reciprocal correction $1/x$ are equal.

Uniqueness of φ . To see that φ is the *only* positive number with this property, note that equation (19) is equivalent to $x^2 - x - 1 = 0$, a quadratic with solutions $x = \frac{1 \pm \sqrt{5}}{2}$. The negative root is $-1/\varphi \approx -0.618$; the unique positive root is $\varphi \approx 1.618$. Thus, φ is the unique positive scale where additive and reciprocal structures *lock together*. This uniqueness is the mathematical reason φ appears universally in systems governed by both additive and multiplicative constraints.

5.2 Evaluating J at φ : a distinguished nontrivial cost

Substituting $x = \varphi$ into the reciprocal cost:

$$J(\varphi) = \frac{1}{2} \left(\varphi + \frac{1}{\varphi} \right) - 1. \quad (20)$$

Using the identity (19), we have $1/\varphi = \varphi - 1$, so:

$$\begin{aligned} J(\varphi) &= \frac{1}{2} (\varphi + (\varphi - 1)) - 1 \\ &= \frac{1}{2} (2\varphi - 1) - 1 \\ &= \varphi - \frac{3}{2}. \end{aligned} \quad (21)$$

Numerically, $J(\varphi) \approx 1.618 - 1.5 = 0.118$.

Interpretation. This is *not* a minimum of J (the unique global minimum is $J(1) = 0$ at perfect balance), but it occupies a distinguished position:

- φ is the unique nontrivial ratio where the reciprocal imbalance $x - 1/x$ has a particularly simple form (it equals $2(\varphi - 1) = 2/\varphi$ by (19)).
- Among all ratios $x > 1$, φ is the one where the “additive pull” ($x - 1$) and the “reciprocal pull” ($1/x$) are balanced in the sense of (19).

5.3 Penrose edge-length ratios and the cost functional

In dart–kite Penrose tilings, tile edges come in two lengths. Normalizing the short edge to unit length, the long edge has length φ . Any “comparison” between these two fundamental length scales corresponds to the ratio $x = \varphi$ (or $x = 1/\varphi$ depending on order). The cost $J(\varphi) \approx 0.118$ can then be interpreted as:

The coherent comparison cost of recognizing the long-vs-short edge distinction in Penrose geometry.

Heuristic energy interpretation. Consider the following heuristic model. Assign to each pair of adjacent edges (meeting at a shared vertex) a comparison ratio x_{ij} representing their length ratio. Define a total “incoherence energy” for a tiling patch as:

$$E_{\text{patch}} = \sum_{\text{adjacent pairs } (i,j)} J(x_{ij}). \quad (22)$$

For a *periodic* tiling (e.g., a regular square or hexagonal lattice), one could arrange all $x_{ij} = 1$, achieving $E = 0$. Penrose tilings, being *aperiodic*, cannot achieve this: the mix of dart and kite forces some edge adjacencies at ratio φ (or $1/\varphi$). However, because φ satisfies the reciprocal identity (19), the cost $J(\varphi)$ is “as small as possible given that $x \neq 1$ but x and $1/x$ appear symmetrically.”

This interpretation suggests:

Penrose tilings minimize a reciprocal mismatch energy subject to the constraint that perfect periodicity ($x = 1$ everywhere) is forbidden by matching rules; φ is the optimal compromise scale.

This is not a theorem—Penrose tilings are not typically derived by energy minimization—but it provides a “cost-first” perspective on why φ , rather than some other irrational, governs the geometry: it is the unique ratio where reciprocal imbalance is self-consistent.

5.4 Summary: J as the “coherence cost of aperiodicity”

In periodic tilings, all local edge ratios can be 1 ($J = 0$ everywhere). Penrose tilings, enforcing aperiodicity, require nontrivial edge-length ratios. The cost J quantifies how much “coherent comparison effort” is required to maintain a given ratio. The golden ratio φ minimizes this effort *among all nontrivial scales* by virtue of its unique self-reciprocal property (19). In this sense:

Penrose geometry is not just “any aperiodic tiling” but the aperiodic tiling that is maximally coherent: it achieves aperiodicity at the minimal coherent cost, and that cost is encoded in $J(\varphi) \approx 0.118$.

6 Unified interpretation: φ as a coherence eigenvalue across domains

Having developed the log-ratio bridge and the geometric interpretation of $J(\varphi)$, we now synthesize the correspondence and extract its broader structural meaning.

6.1 The shared abstract structure

Both Penrose tilings and the ledger comparison framework exhibit the same architectural principle:

Local coherence constraints (pentagonal angles in Penrose, reciprocal symmetry in ledger) forbid trivial global configurations (periodic tilings, unconstrained ratios) and produce unique coherent structures (aperiodic tilings with φ -scaling, reciprocal cost J with φ as self-similar fixed point).

More precisely, the correspondence can be stated as follows:

Penrose Tilings	Ledger Comparison
<i>Local constraint:</i> Matching rules enforce pentagonal angles $36^\circ, 72^\circ, 108^\circ$.	<i>Local constraint:</i> Reciprocal symmetry $J(x) = J(x^{-1})$ enforces balance under inversion.
<i>Global consequence:</i> Aperiodicity: no translation symmetry.	<i>Global consequence:</i> Nontrivial cost: $J \neq 0$.
<i>Self-similarity requirement:</i> Inflation/deflation substitution with combinatorial matrix M .	<i>Self-similarity requirement:</i> Self-similar update $x_{n+1} = 1 + 1/x_n$.
<i>Coherence eigenvalue:</i> Perron–Frobenius eigenvalue $\lambda_1 = \varphi$ of M .	<i>Coherence eigenvalue:</i> Unique positive fixed point $x^* = \varphi$ of the update map.
<i>Multiplicative rescaling:</i> Geometric lengths scale as $\ell \mapsto \varphi \ell$.	<i>Multiplicative rescaling:</i> Comparison ratios scale as $x \mapsto \varphi x$.
<i>Additive shift (log coordinate):</i> $t = \ln \ell \mapsto t + \ln \varphi$.	<i>Additive shift (log coordinate):</i> $t = \ln x \mapsto t + \ln \varphi$.

The logarithm is the key unifying mechanism: it is the unique group isomorphism $(\mathbb{R}_{>0}, \times) \rightarrow (\mathbb{R}, +)$ that converts multiplicative structures (Penrose inflation, ratio comparison) into additive structures (log-length translation, log-ratio translation). In this unified language, φ is the *coherence eigenvalue*: the unique scale at which local reciprocal/pentagonal constraints and global self-similarity lock together.

6.2 Why φ and not some other constant?

The question naturally arises: could any other constant play the role of φ in either framework? The answer is no, for algebraic reasons:

In Penrose tilings: The pentagonal angle structure forces edge-length ratios via trigonometric identities like $\cos(36^\circ) = \varphi/2$. These are not free parameters but consequences of the fact that $36^\circ = \pi/5$ and the double-angle formulas. Any other angle structure would produce a different symmetry class (3-fold, 4-fold, 6-fold, which are crystallographic) or no discrete symmetry at all. Five-fold symmetry is special precisely because it is the minimal non-crystallographic case, and it uniquely determines φ .

In the ledger framework: The self-similar update $x_{n+1} = 1 + 1/x_n$ is the simplest map satisfying:

- Reciprocal balance: $f(x^{-1}) = f(x)$.
- Nontriviality: $f \neq \text{id}$ and $f \neq \text{const}$.
- Minimality: additive constant (1) plus reciprocal correction ($1/x$), with no higher-order terms.

The fixed-point equation $x = 1 + 1/x$ is the minimal quadratic $x^2 - x - 1 = 0$, whose positive root is φ by definition. Any other recursion would either break reciprocal symmetry or introduce additional parameters, destroying uniqueness.

6.3 Implications: φ as a universal coherence scale

The correspondence established in this paper suggests that φ is not merely a “beautiful number” that happens to appear in unrelated contexts, but a *universal coherence scale* for systems characterized by:

1. **Multiplicative structure:** The primitive observables (lengths, ratios) transform under multiplication/division.
2. **Reciprocal symmetry:** The system respects inversion $x \mapsto x^{-1}$ (geometric reflection or comparison reversal).
3. **Self-similarity:** The system admits a coarse-graining/renormalization operation that preserves its structure up to rescaling.
4. **Aperiodicity/nontriviality:** The system cannot be reduced to a trivial periodic/constant configuration.

When these four properties hold, φ emerges as the unique fixed point of the coherence dynamics. Penrose tilings and ledger comparison are two realizations of this abstract structure, one geometric and one information-theoretic, unified by the log-ratio transformation.

6.4 Broader context: other appearances of φ

The framework developed here may shed light on other occurrences of φ in mathematics and physics:

Fibonacci quasiperiodicity. The Fibonacci sequence F_n satisfies $F_{n+1} = F_n + F_{n-1}$, with limiting ratio $\lim_{n \rightarrow \infty} F_{n+1}/F_n = \varphi$. This is a 1D analog of Penrose’s 2D aperiodic order. The substitution rule $a \rightarrow ab, b \rightarrow a$ has substitution matrix with eigenvalue φ . In the ledger language, the recursion $x_{n+1} = 1 + 1/x_n$ is precisely the ratio form of the Fibonacci recurrence.

KAM tori and circle maps. In Hamiltonian dynamics, φ is the “most irrational” winding number, making it the last to break under perturbations (by KAM theory). Circle maps with rotation number φ exhibit quasiperiodic orbits that are maximally stable. This can be interpreted as a temporal analog of Penrose’s spatial aperiodicity: the system avoids periodic trapping by “stepping” through angles in a φ -scaled sequence.

Spectral theory of quasicrystals. The Fourier spectrum of Penrose tilings forms a dense set of Bragg peaks with positions related by powers of φ . In the ledger language, this corresponds to a discrete hierarchy of log-ratio scales $n \ln \varphi$ for $n \in \mathbb{Z}$, forming a module $\mathbb{Z}[\ln \varphi]$ in reciprocal space. The cost functional J could potentially be used to weight spectral contributions, with $J(\varphi^n)$ measuring the “coherence cost” of the n -th hierarchical level.

7 Conclusion and open questions

This paper has established a formal bridge between Penrose aperiodic tilings and a cost-first framework for coherent ratio comparison, demonstrating that the golden ratio $\varphi = (1 + \sqrt{5})/2$ appears in both settings not as a numerical coincidence but as a *universal coherence eigenvalue*. The bridge rests on three pillars:

1. Log-ratio isomorphism. The coordinate transformation $t = \ln x$ converts Penrose’s multiplicative geometric scaling $x \mapsto \varphi x$ into the ledger’s additive information-theoretic shift $t \mapsto t + \ln \varphi$, making $\ln \varphi \approx 0.4812$ a fundamental additive scale for self-similar dynamics.

2. Reciprocal cost functional. The unique coherence-derived cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ satisfies $J(\varphi) = \varphi - 3/2 \approx 0.118$, providing a quantitative measure of the “coherence cost of aperiodicity.” The golden ratio is the unique positive scale where additive deficit from unity equals reciprocal: $\varphi - 1 = 1/\varphi$.

3. Structural correspondence. Both frameworks exhibit the same abstract architecture: local reciprocal/pentagonal symmetry constraints forbid trivial periodic configurations, and global self-similarity requirements force a unique eigenvalue φ . The logarithm is the key mechanism converting multiplicative structures into additive ones.

7.1 Open questions and future directions

Several natural questions arise from this work:

1. **Higher-dimensional generalization.** Can the bridge be extended to 3D Penrose tilings (icosahedral quasicrystals) and higher-dimensional kaleidoscopic structures? Does the reciprocal cost J have a natural generalization to higher-rank comparison tensors?
2. **Spectral interpretation.** Penrose tilings have well-studied diffraction spectra. Can the ledger cost functional be used to define a weighted spectral measure on quasicrystal Bragg peaks, with $J(\varphi^n)$ controlling the n -th hierarchical level?
3. **Dynamical flow regimes.** The ledger framework admits time-dependent comparison dynamics. Can one define a “Penrose flow” on a recognition graph whose time-averaged structure converges to a φ -renormalizable potential?

4. **Information-theoretic entropy.** Penrose tilings are aperiodic yet highly structured (low entropy relative to random tilings). Can J be used to define an “entropy of aperiodicity” that quantifies this structure? Is $J(\varphi)$ related to the Shannon entropy of the symbolic substitution sequence?
5. **Physical implementations.** Can the ledger framework with φ -scaling be tested in physical quasicrystals? For example, by modeling phonon dispersion or diffusion on a Penrose lattice and checking for $\ln \varphi$ -quantized scales in transport coefficients.
6. **Other φ -systems.** The golden ratio appears in circle maps, KAM tori, and Fibonacci quasiperiodicity. Can these be brought into the ledger framework, perhaps as 1D or temporal analogs of Penrose tilings?

7.2 Broader significance

If the Penrose–ledger correspondence holds in its full generality, it suggests a unified framework for understanding aperiodic order, quasicrystals, and self-similar information structures through the lens of *coherent comparison costs*. This could have implications for:

- **Mathematics:** Unifying the theory of substitution systems (symbolic dynamics) with discrete potential theory (graph Laplacians, cohomology).
- **Physics:** Providing an information-theoretic foundation for quasicrystal thermodynamics and transport phenomena.
- **Information theory:** Interpreting $\ln \varphi$ as a fundamental bit-like unit for ratio-based information, analogous to $\ln 2$ in binary systems.
- **Computer science:** Designing self-similar data structures or distributed ledgers with provably minimal coherence overhead.

In conclusion, the golden ratio φ is more than a geometric curiosity—it is the *coherence eigenvalue* of self-similar systems with reciprocal symmetry, appearing whenever local constraints produce global rigidity and aperiodic order. Penrose tilings and cost-first ledgers are two realizations of this universal principle, one geometric and one information-theoretic, unified by the log-ratio coordinate. This correspondence opens new avenues for cross-domain fertilization between discrete geometry, information theory, and the mathematical physics of aperiodic systems.

References

- [1] R. Penrose, *The role of aesthetics in pure and applied mathematical research*, Bull. Inst. Math. Appl. **10** (1974), 266–271.
- [2] S. Pardo-Guerra, M. Simons, A. Thapa, and J. Washburn, *Coherent Comparison as Information Cost: A Cost-First Ledger Framework for Discrete Dynamics* (manuscript, 2025).