The Law of Mathematical Inevitability: Numbers, Proofs, and Universal Reference Forced by the d'Alembert Cost Equation

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February 2026

Abstract. The d'Alembert functional equation on $(\mathbb{R}_{>0}, \cdot)$, under normalization and non-degeneracy, uniquely determines the cost functional $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ [4, 1]. We derive three consequences for the foundations of inference: **(1)** the logarithm is the unique continuous additive invariant vanishing on product-one sequences (via Cauchy's theorem); **(2)** the golden ratio φ is the unique base whose geometric lattice admits a Fibonacci-type recursion; **(3)** a space is a universal referent for all positive-cost objects if and only if its infimal cost is zero. We compute explicit costs for a propositional-resolution example, characterize the generators of the balanced-proof monoid, and pose open problems connecting the framework to proof complexity.

MSC 2020: 39B22 (functional equations), 03F20 (complexity of proofs), 03A05 (philosophical and critical aspects of logic).

Keywords: d'Alembert functional equation, Cauchy equation, cost functional, golden ratio, proof complexity, universal reference, Wigner effectiveness.

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1 Introduction

The *d'Alembert functional equation*

$$g(s+t)+g(s-t) \;=\; 2\,g(s)\,g(t), \qquad s,t \in \mathbb{R}, \tag{1}$$

has been studied since d'Alembert (1769) and Cauchy; its continuous solutions are $g(t) = \cosh(\lambda t)$ for $\lambda \in \mathbb{R}$ [1, 3]. The *multiplicative form*, via $x = e^s$, $y = e^t$, $F(x) = g(\ln x) - 1$, yields

$$F(xy)+F(x/y) \;=\; 2F(x)F(y)+2F(x)+2F(y), \qquad x,y>0. \tag{2}$$

Under normalization $F(1) = 0$, non-degeneracy $F(x) > 0$ for $x \neq 1$, and calibration $F''_{\log}(0) = 1$, the unique solution is [4]

$$\boxed{J(x) \;:=\; \tfrac{1}{2}(x+x^{-1})-1 \;=\; \cosh(\ln x)-1, \qquad x>0.} \tag{3}$$

This functional arises as the "recognition cost" in [5, 6, 7] and as the unique symmetric non-negative solution to (2) [4].

In this paper we derive three consequences of J for the *foundations of inference*: a uniqueness theorem for balance conditions (Section 4), a characterization of the self-similar lattice (Section 3), and an if-and-only-if characterization of universal reference (Section 8). We state what is proved, what is conjectured, and what remains open.

2 The Cost Functional

Definition 2.1 (Axioms)**.** A *d'Alembert cost functional* is a continuous function $F : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ satisfying:
(A1) $F(1) = 0$ and $F(x) > 0$ for $x \neq 1$;
(A2) $F(x) = F(x^{-1})$ for all $x > 0$;
(A3) $F(xy) + F(x/y) = 2F(x)F(y) + 2F(x) + 2F(y)$ for all $x, y > 0$.

Theorem 2.2 (Uniqueness; [4, Thm. 1], cf. [1, Ch. 3])**.** *The unique d'Alembert cost functional is $J(x) = \frac{1}{2}(x + x^{-1}) - 1$.*

Proof sketch. Set $G(t) := F(e^t) + 1$. Then (A3) becomes the classical $G(s + t) + G(s - t) = 2G(s)G(t)$. By [1, Thm. 3.1.3], $G = \cosh(\lambda\,\cdot)$. (A1) forces $\lambda \neq 0$; calibration fixes $\lambda = 1$. Hence $F = \cosh \circ \ln -1 = \frac{1}{2}(x + x^{-1}) - 1$. □

Remark 2.3 (Heritage). The additive form (1) was solved by d'Alembert (1769); Aczél [1] and Aczél–Dhombres [2] provide the modern theory. Ref. [4] adds the calibrated multiplicative form with (A1), selecting the cosh branch.

Key properties (all *derived*, not assumed): $J \geq 0$ with $J = 0 \Leftrightarrow x = 1$ (AM–GM); $J(x) = J(x^{-1})$; $J''(x) = x^{-3} > 0$ (strict convexity).

Definition 2.4. $\varphi := (1 + \sqrt{5})/2$ (the golden ratio); $J_{\text{bit}} := \ln \varphi \approx 0.4812$ (the ledger bit cost).

3 The Self-Similar Lattice

Definition 3.1 (φ-Ladder)**.** $L : \mathbb{Z} \to \mathbb{R}_{>0}$, $L(n) = \varphi^n$.

Theorem 3.2 (Ladder)**.** *L is positive, strictly monotone, satisfies $L(0) = 1$, and obeys the Fibonacci recursion $L(n{+}2) = L(n{+}1) + L(n)$.*

Proof. $\varphi > 1$ gives positivity and monotonicity. $\varphi^{n+2} = \varphi^n\varphi^2 = \varphi^n(\varphi+1) = \varphi^{n+1} + \varphi^n$. □

Theorem 3.3 (Uniqueness)**.** *If $\alpha > 1$ satisfies $\alpha^2 = \alpha + 1$, then $\alpha = \varphi$.*

Proof. $\alpha = (1 \pm \sqrt{5})/2$; the constraint $\alpha > 1$ selects φ. □

Remark 3.4. The condition $\alpha^2 = \alpha + 1$ arises precisely when $\alpha^{n+2} = \alpha^{n+1} + \alpha^n$ (Fibonacci recursion on the lattice). No other base $\alpha > 1$ satisfies this; e.g., $2^{n+2} = 4 \cdot 2^n \neq 3 \cdot 2^n$.

Definition 3.5 (Metric)**.** $d(m, n) := |m - n| \cdot \ln \varphi$.

Proposition 3.6. *d is a metric on $\mathbb{Z}$. ($\ln \varphi$ times the standard metric.)*

Remark 3.7. The cost $J(\varphi^n) = \cosh(d(n, 0)) - 1$ is a strictly monotone convex function of d—not itself a metric, but an equivalent measure of distance from unity.

Corollary 3.8 (Zeckendorf [15])**.** *Every positive integer has a unique sum of non-consecutive Fibonacci numbers.*

4 Balanced Ledger Sequences

Definition 4.1 (Proof)**.** A *recognition proof* $p = (r_1, \ldots, r_N)$ with $r_k > 0$ has cost $C(p) = \sum J(r_k)$ and log-balance $\beta(p) = \sum \ln r_k$. It is *balanced* when $\beta(p) = 0$.

Uniqueness of the balance condition

Theorem 4.2 (Cauchy on $(\mathbb{R}_{>0}, \cdot)$; [1, Ch. 2])**.** *Every continuous homomorphism $\phi : (\mathbb{R}_{>0}, \cdot) \to (\mathbb{R}, +)$ has the form $\phi = \lambda \ln$ for some $\lambda \in \mathbb{R}$.*

Proof. Set $f(u) := \phi(e^u)$; then $f(u + v) = f(u) + f(v)$ continuously, so $f = \lambda\,\mathrm{id}$ [1, Thm. 2.1.1]. □

Theorem 4.3 (Unique Balance)**.** *If Φ is continuous, additive on sequences, and vanishes on every product-one chain ($\prod r_k = 1$), then $\Phi = \lambda \sum \ln r_k$.*

Proof. Additivity: $\Phi = \sum \phi(r_k)$. Pair (t, t^{-1}) gives $\phi(t) + \phi(t^{-1}) = 0$. Triple $(r, s, (rs)^{-1})$ gives $\phi(rs) = \phi(r) + \phi(s)$. Apply Theorem 4.2. □

Corollary 4.4. *Log-balance is the unique continuous additive closed-chain invariant on $(\mathbb{R}_{>0}, \cdot)$.*

Theorem 4.5 (Monoid)**.** *Balanced proofs form a monoid; C is a homomorphism to $(\mathbb{R}_{\geq 0}, +)$.*

Example 4.6 (Minimal balanced proof)**.** $p = (\varphi, \varphi^{-1})$: $\beta = 0$, $C = 2J(\varphi) = \sqrt{5} - 2 \approx 0.236$.

Remark 4.7 (Semantics). The ratio r of a proof step models an "informational exchange rate." Theorem 4.3 holds *regardless* of the concrete assignment $r \mapsto$ inference-rule. The mapping to specific formal systems is an open problem (Section 6).

5 Further Consequences

Proposition 5.1 (Canonical ordering)**.** *C induces a total preorder on balanced proofs, independent of the choice of decreasing beauty functional. A cost-minimizer $p^* = \arg\min C$ exists for any provable theorem (cf. Erdős's "Proof from the Book" [14]).*

Conjecture 5.2 (Gödel saddle)**.** If a sentence requires unbounded self-reference depth, both proof and refutation have unbounded J.

The self-reference cost $S(n) \geq nJ_{\mathrm{bit}} = n \ln \varphi$ diverges (Archimedean property). The contribution is the interpretation: J_{bit} is uniquely fixed by (A1)–(A3), giving a machine-independent analogue of Chaitin's [13] algorithmic incompleteness.

6 Toward Formal Proof Theory

Definition 6.1 (Complexity ratio)**.** For a complexity measure κ : Formulas $\to \mathbb{N}$,

$$r(A \to B) \;:=\; \frac{\kappa(B) + 1}{\kappa(A) + 1}. \tag{4}$$

Proposition 6.2. $A = B \implies r = 1,\ J = 0.$ *A cyclic derivation $A \to \cdots \to A$ has $\prod r_k = 1$ (balanced).*

Worked example: propositional resolution

Example 6.3. $(A \vee B) \wedge (\neg A \vee B) \implies B$ by resolution on A, with $\kappa =$clause width.

Step	From	To	r	$J(r)$
1	$A \vee B$ (κ=2)	B (κ=1)	2/3	1/12
2	$\neg A \vee B$ (κ=2)	B (κ=1)	2/3	1/12
Balance to cycle: add steps $(3/2, 3/2)$				

Derivation: $\beta = 2\ln(2/3) \approx -0.81 \neq 0$ (not balanced).
Balanced cycle $p' = (2/3,\ 2/3,\ 3/2,\ 3/2)$: $\beta = 0,\ \ C = 4/12 = 1/3$.

Proof monoid generators

Proposition 6.4 (Generators)**.** *The balanced-proof monoid over the φ-lattice is generated by the pairs $(\varphi^k, \varphi^{-k})$, $k \geq 1$. The cheapest generator is (φ, φ^{-1}) with $C = \sqrt{5} - 2$.*

Proof. Any balanced sequence $\sum n_k = 0$ decomposes into matched $\pm$-pairs. The cost $2J(\varphi^k) = \varphi^k + \varphi^{-k} - 2$ is minimized at $k = 1$. □

Open problems

(Q1) Does C correlate with Frege proof length or resolution width?
(Q2) Is C-minimality equivalent to a known proof optimality?
(Q3) Does the generator structure correspond to an algebraic invariant of proof systems?

7 Axiom of Choice: A Cost Interpretation

Theorem 7.1. *$J(x) < \infty$ for all $x > 0$.*

Theorem 7.2. *$J(x) \to +\infty$ as $x \to 0^+$.*

Remark 7.3. We do not derive AC from weaker axioms. In any cost landscape where $J < \infty$ on $\mathbb{R}_{>0}$ and $J(0^+) = \infty$, selection from nonempty sets is always finitely accessible. In compact settings, $\arg\min_A J$ gives a constructive choice.

8 Zero-Cost Universal Reference

Theorem 8.1 (Backbone)**.** *For any (P, J_P) with $J_P(o) > 0$, there exists $(M, 0)$ with $J_M(s) = 0 < J_P(o)$.*

Theorem 8.2 (Necessity)**.** *If M is a universal referent (i.e., for every (P, J_P) and o with $J_P(o) > 0$ there exists $s \in M$ with $J_M(s) < J_P(o)$), then $\inf_M J_M = 0$.*

Proof. If $\inf = c > 0$, take $P = \{o\}$ with $J_P(o) = c/2$: contradiction. □

Theorem 8.3 (Sufficiency)**.** *If $\inf_M J_M = 0$, then M is a universal referent.*

Proof. Given $\varepsilon = J_P(o) > 0$, choose $s \in M$ with $J_M(s) < \varepsilon$. □

Corollary 8.4 (Characterization)**.** *M is a universal referent $\iff$ $\inf_M J_M = 0$.*

Corollary 8.5 (Existence)**.** $\ker(J) = \{1\}$ *satisfies* $\inf J = 0$.

Remark 8.6 (Wigner). Theorem 8.4 is analogous to Shannon's source coding theorem [12]: zero intrinsic cost $\Leftrightarrow$ optimal compression. Wigner's puzzle [9] becomes a theorem about cost asymmetry.

9 Summary

Theorem 9.1 (Main)**.** *Any d'Alembert cost functional (Theorem 2.1) determines:*

1. *$F = J$ uniquely* *(Theorem 2.2);*
2. *φ as the unique self-similar base* *(Theorem 3.3);*
3. *the metric $d = |m - n| \ln \varphi$* *(Theorem 3.6);*
4. *$\ln$ as the unique balance invariant* *(Theorem 4.3);*
5. *the zero-cost subspace as the unique universal referent* *(Theorem 8.4).*

The φ-lattice proof monoid is generated by $\{(\varphi^k, \varphi^{-k}) : k \geq 1\}$ (Theorem 6.4).

Remark 9.2 (Scope). These are necessary *infrastructure* for mathematical reasoning, not a derivation of ZFC or second-order arithmetic. The contribution is that this infrastructure is uniquely forced by a single functional equation, and that concrete open problems (Section 6) connect it to proof complexity.

10 Discussion

Claims and non-claims

Three structures (ordered lattice, balance condition, zero-cost reference) are uniquely forced. We do *not* derive ZFC, refute Gödel, or derive AC from weaker axioms.

Prior work

Reference	Relation to this work
Aczél [1]	Classical d'Alembert theory; we apply it to cost and derive mathematical structures.
Tegmark [10]	Postulates math is fundamental; we derive *specific* structures from one equation.
Wheeler [11]	"It from Bit"; we quantify the bit: $J_{\text{bit}} = \ln \varphi$.
Chaitin [13]	Algorithmic incompleteness; J replaces Kolmogorov complexity with a *unique* measure.
Shannon [12]	Source coding; zero-cost reference ≈ optimal compression (structural analogy).
Cauchy	$f(x+y) = f(x)+f(y)$; Theorem 4.3 is a direct application to $(\mathbb{R}_{>0}, \cdot)$.

Open questions

(Q4) Completion of $(\mathbb{Z}, d)$ to a φ-adic space?
(Q5) Topos structure on balanced proofs?
(Q6) Constructive selection via J-minimization in non-compact settings?

11 Conclusion

We have derived three consequences of the d'Alembert cost equation.

Balance uniqueness (Theorem 4.3): log-balance is the unique continuous additive closed-chain invariant, by Cauchy's classical theorem. The worked resolution example (Theorem 6.3) and monoid generators (Theorem 6.4) give a first algebraic picture.

Lattice uniqueness (Theorem 3.3): φ is the unique base admitting a Fibonacci recursion.

Universality (Theorem 8.4): a space is a universal referent iff its infimal cost is zero—both necessary and sufficient.

The most promising direction is the connection to proof complexity (Section 6): does J-cost correlate with standard proof-length measures?

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