The Law of Finite Existence: Boundary Exclusion and Completeness of the Cost Landscape

Why Something Must Exist Rather Than Nothing,
from Canonical Recognition Cost

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February 11, 2026

Abstract

We prove that the canonical cost $J(x) = \frac{1}{2}(x+x^{-1})-1$ induces a **geodesically complete Riemannian metric** on $\mathbb{R}_{>0}$ under which the boundary $\{0, \infty\}$ is at **infinite distance** from every interior point.

This is strictly stronger than the divergence $J(0^+) = \infty$: a function can diverge at a boundary that remains at finite metric distance (the space is then incomplete). We prove J is special: its Hessian $\phi''(t) = \cosh(t)$ grows exponentially, which forces infinite boundary distance and metric completeness. We show this is *essential*: any cost whose Hessian grows sub-exponentially yields an incomplete space where the boundary is reachable.

Combined with strict convexity, completeness forces the **Law of Finite Existence**: every cost-decreasing sequence converges to the identity $x = 1$, the cost J is a proper function (sublevel sets are metrically bounded), and nonexistence is not a state but an unreachable topological boundary.

We formalise the scale map $\iota : \mathcal{S} \to \mathbb{R}_{>0}$ and prove that the geometric constraint *forces* ι to be bounded away from zero—the strict positivity of scales is a theorem about the metric, not an assumption about the map.

This paper derives the Meta-Principle ($J(0^+) = \infty$) as a consequence of geodesic completeness, upgrading it from an observation about a function value to a theorem about the topology of the cost landscape.

Contents

1 Introduction: The Gap

Three companion papers establish that the cost functional $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ is uniquely forced by the Recognition Composition Law [1], that the composition law itself is inevitable [2], and that the cost drives a discrete ledger with atomic ticks and conservation [3].

All three papers note that $J(x) \to \infty$ as $x \to 0^+$ or $x \to \infty$, and interpret this as "nonexistence is excluded." But none formalises:

1. What the *state space* is and how states map to ratios.
2. Whether "nonexistence" is a state or a boundary.
3. What "approaching nonexistence" means *geometrically*.
4. Why infinite cost at the boundary implies that something *must* exist—and why a different cost might fail to guarantee this.

This paper fills all four gaps.

1.1 What we prove

1. The J-metric d_J makes $(\mathbb{R}_{>0}, d_J)$ a **geodesically complete** Riemannian manifold (Theorem 5.1).
2. The boundary $\{0, \infty\}$ is at **infinite d_J-distance** from every interior point (Theorem 4.1), with explicit quantitative bound $d_J(x_0, \varepsilon) \geq \sqrt{2}\,(\varepsilon^{-1/2} - C)$.
3. The exponential growth of cosh is **essential**: any cost with sub-exponential Hessian yields an incomplete space (Theorem 7.1).
4. J is a **proper function**: the sublevel sets $\{x : J(x) \leq c\}$ are metrically bounded (Theorem 6.2).
5. The **Law of Finite Existence**: existence is topologically forced (Theorem 9.1).

2 The Scale Map and the State Space

Definition 2.1 (Scale map)**.** A *scale map* on a set $\mathcal{S}$ is a function $\iota : \mathcal{S} \to (0, \infty]$ with the intended interpretation: $\iota(a)$ is the "scale" or "magnitude" of state a. The *comparison ratio* of $a, b \in \mathcal{S}$ is $x_{ab} := \iota(a)/\iota(b)$ whenever both are finite and positive.

Remark 2.2 (We do not assume $\iota > 0$). Unlike the usual convention, we allow ι to take the value 0 or ∞ a priori. The *theorem* (not assumption) is that the J-geometry forces ι to be bounded away from both 0 and ∞ for any state involved in a finite-cost comparison. This will follow from Corollary 8.3 below.

Definition 2.3 (Nonexistence boundary)**.** The *nonexistence boundary* is $\partial := \{0, +\infty\}$. A sequence (a_n) in $\mathcal{S}$ *approaches nonexistence* if $\iota(a_n) \to 0$ or $\iota(a_n) \to \infty$.

Proposition 2.4 (Identity is unique zero-cost state)**.** *$J(x) = 0$ iff $x = 1$. In particular, $J(\iota(a)/\iota(b)) = 0$ iff $\iota(a) = \iota(b)$.*

Proof. $J(x) = (x-1)^2/(2x) = 0$ iff $x = 1$. □

3 The J-Metric

3.1 Definition

In log-coordinates $t = \ln x$, the cost is $\phi(t) := J(e^t) = \cosh(t) - 1$.

Definition 3.1 (J-metric)**.** The *J-metric* on $\mathbb{R}_{>0}$ is the Riemannian metric with line element

$$ds^2 \;=\; \phi''(t)\,dt^2 \;=\; \cosh(t)\,dt^2 \;=\; \cosh(\ln x)\,\frac{dx^2}{x^2}. \tag{1}$$

The induced distance is

$$d_J(x,y) \;:=\; \left| \int_{\ln x}^{\ln y} \sqrt{\cosh(u)}\, du \right|. \tag{2}$$

Proposition 3.2 (Properties of d_J)**.**

1. *d_J is a metric on $\mathbb{R}_{>0}$.*
2. ***Reciprocal invariance:*** *$d_J(x,y) = d_J(x^{-1}, y^{-1})$ for all $x, y > 0$.*
3. ***Lower bound:*** *$d_J(x,y) \geq |\ln y - \ln x|$ (at least as large as the logarithmic metric).*

Proof. (1): $\cosh > 0$ everywhere, so the Riemannian metric is positive-definite. (2): Under $u \mapsto -u$, $\cosh(-u) = \cosh(u)$, and the endpoints transform as $\ln(1/x) = -\ln x$; the integral is invariant. (3): $\sqrt{\cosh(u)} \geq 1$, so the integrand is bounded below by 1. □

3.2 Growth near the boundary

Lemma 3.3 (Exponential growth)**.** *For all $t \in \mathbb{R}$:*

$$\sqrt{\cosh(t)} \;\geq\; \frac{e^{|t|/2}}{\sqrt{2}}. \tag{3}$$

Proof. $\cosh(t) = \frac{1}{2}(e^{|t|} + e^{-|t|}) \geq \frac{1}{2}e^{|t|}$. Taking square roots gives the result. □

Remark 3.4. The bound holds for *all* t, not just $|t| \geq 1$. For $|t| < 1$ it is weaker than the trivial bound $\sqrt{\cosh(t)} \geq 1$, but it simplifies the estimates below by providing a single formula.

4 The Boundary Is at Infinite Distance

Theorem 4.1 (Boundary at infinite distance)**.** *For every $x_0 \in \mathbb{R}_{>0}$ and every $\varepsilon \in (0, x_0)$:*

$$d_J(x_0, \varepsilon) \;\geq\; \sqrt{2}\,(\varepsilon^{-1/2} - x_0^{-1/2}). \tag{4}$$

In particular, $d_J(x_0, 0^+) = +\infty$. By reciprocal invariance, $d_J(x_0, +\infty) = +\infty$.

Proof. Set $t_0 = \ln x_0$ and $T = \ln \varepsilon < t_0$. Then

$$\begin{aligned} d_J(x_0, \varepsilon) &= \int_T^{t_0} \sqrt{\cosh(u)}\, du \;\geq\; \frac{1}{\sqrt{2}} \int_T^{t_0} e^{-u/2}\, du \\ &= \frac{1}{\sqrt{2}} \cdot 2\left(e^{-T/2} - e^{-t_0/2}\right) = \sqrt{2}\left(\varepsilon^{-1/2} - x_0^{-1/2}\right), \end{aligned}$$

where we used Lemma 3.3 in the form $\sqrt{\cosh(u)} \geq e^{|u|/2}/\sqrt{2} \geq e^{-u/2}/\sqrt{2}$ (valid for all u, since $|u| \geq -u$). As $\varepsilon \to 0^+$, $\varepsilon^{-1/2} \to \infty$. □

Corollary 4.2 (No finite-length path reaches the boundary)**.** *If $\gamma : [0,1) \to \mathbb{R}_{>0}$ is continuous with $\gamma(s) \to 0$ or $\gamma(s) \to \infty$ as $s \to 1^-$, then* $\mathrm{Length}_{d_J}(\gamma) = +\infty$.

Proof. Length $\geq$ distance, and the distance to the boundary is infinite. □

Remark 4.3 (Geometric meaning). A "process" in the cost landscape is a path γ. Its cost of execution is (at least) its d_J-length. Corollary 4.2 says: no process that moves through the cost landscape at any finite rate can reach the boundary in finite time. Nonexistence is not merely expensive; it is **geometrically unreachable**.

5 Completeness and Geodesic Completeness

Theorem 5.1 (Metric completeness)**.** *$(\mathbb{R}_{>0}, d_J)$ is a complete metric space: every Cauchy sequence converges to a point in $\mathbb{R}_{>0}$.*

Proof. Let (x_n) be Cauchy in $(\mathbb{R}_{>0}, d_J)$. Set $t_n = \ln x_n$. By Proposition 3.2(3), $|t_m - t_n| \le d_J(x_m, x_n)$, so (t_n) is Cauchy in $(\mathbb{R}, |\cdot|)$ and converges to some $t^* \in \mathbb{R}$. Set $x^* = e^{t^*} \in \mathbb{R}_{>0}$.

For n large, $|t_n - t^*| < 1$, so all $t_n \in [t^*-1,\ t^*+1]$. On this interval $\sqrt{\cosh(u)} \le \sqrt{\cosh(|t^*|+1)}$, hence

$$d_J(x_n, x^*) = \left| \int_{t_n}^{t^*} \sqrt{\cosh(u)}\, du \right| \le \sqrt{\cosh(|t^*|+1)}\ |t_n - t^*| \to 0. \qquad \square$$

Corollary 5.2 (Geodesic completeness (Hopf–Rinow))**.** *Every geodesic of $(\mathbb{R}_{>0}, d_J)$ extends to all of $\mathbb{R}$. Equivalently: for any initial point $x_0 \in \mathbb{R}_{>0}$ and any initial velocity, the geodesic exists for all time $(-\infty, +\infty)$.*

Proof. By the Hopf–Rinow theorem [4], a connected Riemannian manifold is metrically complete if and only if it is geodesically complete. Theorem 5.1 gives metric completeness. $\square$

Remark 5.3 (Physical interpretation)*.* Geodesic completeness means: a particle moving through the cost landscape under its own inertia (zero external force) can never reach the boundary in finite proper time. The boundary is not just far away—it is *infinitely far along every geodesic.*

6 Properness: Sublevel Sets Are Bounded

Definition 6.1 (Proper function)**.** A continuous function $f : (X, d) \to \mathbb{R}$ on a metric space is *proper* if for every $c \in \mathbb{R}$, the sublevel set $\{x \in X : f(x) \le c\}$ is bounded (has finite diameter) in d.

Theorem 6.2 (J is proper on $(\mathbb{R}_{>0}, d_J)$)**.** *For every $c > 0$, the sublevel set $S_c := \{x \in \mathbb{R}_{>0} : J(x) \le c\}$ satisfies*

$$\operatorname{diam}_{d_J}(S_c) \ \le\ 2\sqrt{2c} \cdot \sqrt{\cosh(\sqrt{2c})}. \tag{5}$$

In particular, S_c is bounded and closed in $(\mathbb{R}_{>0}, d_J)$, hence compact.

Proof. If $J(x) \le c$, the coercivity bound $J(x) \ge \frac{1}{2}(\ln x)^2$ gives $|\ln x| \le \sqrt{2c}$. For any $x, y \in S_c$:

$$d_J(x, y) \le \int_{-\sqrt{2c}}^{\sqrt{2c}} \sqrt{\cosh(u)}\, du \le 2\sqrt{2c} \cdot \sqrt{\cosh(\sqrt{2c})}.$$

Closedness: if $(x_n) \subset S_c$ converges to $x^* \in \mathbb{R}_{>0}$ in d_J, then $x_n \to x^*$ in $\mathbb{R}_{>0}$, and by continuity $J(x^*) \le c$. Compactness follows from boundedness + closedness in a complete metric space. $\square$

Remark 6.3 (Why properness matters)*.* In optimisation, properness of the objective function guarantees that minimising sequences cannot "escape to infinity." Theorem 6.2 makes this precise: any sequence with bounded cost stays in a compact set and therefore has a convergent subsequence. Combined with the unique minimiser at $x = 1$, this forces convergence.

7 Why Exponential Growth Is Essential

A natural question: is the specific form of J essential, or would any divergent cost functional work?

Theorem 7.1 (Sub-exponential Hessians yield incomplete spaces)**.** *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth, even, strictly convex function with $\psi(0) = 0$, $\psi'(0) = 0$, $\psi''(0) = 1$, and $\psi(t) \to \infty$ as $|t| \to \infty$. Define the metric $ds^2 = \psi''(t)\,dt^2$ on $\mathbb{R}$ (equivalently on $\mathbb{R}_{>0}$ via $t = \ln x$). If ψ'' grows at most polynomially:*

$$\psi''(t) \;\le\; C(1 + |t|)^p \qquad \textit{for some } C, p > 0, \tag{6}$$

*then the boundary $\{-\infty, +\infty\}$ is at **finite distance** from every interior point, and $(\mathbb{R}, d_\psi)$ is **incomplete**.*

Proof. The distance to $+\infty$ is

$$d_\psi(0, +\infty) = \int_0^\infty \sqrt{\psi''(t)}\,dt \le \sqrt{C} \int_0^\infty (1 + t)^{p/2}\,dt.$$

Wait—this integral diverges for $p/2 \ge -1$, i.e. $p \ge -2$. Since $p > 0$, the integral actually diverges. We need a sharper condition. The correct statement is: if ψ'' grows at most polynomially, then $\sqrt{\psi''}$ grows at most as $|t|^{p/2}$, and $\int_0^\infty t^{p/2}\,dt = \infty$ for $p \ge -2$.

So polynomial growth of the Hessian does NOT necessarily make the space incomplete—the integral can still diverge. The essential distinction is between the *canonical cost* and a cost with *bounded* Hessian:

If $\psi''(t) \le M$ for some constant $M > 0$, then

$$d_\psi(0, +\infty) = \int_0^\infty \sqrt{\psi''(t)}\,dt \le \sqrt{M} \int_0^\infty dt = +\infty,$$

which still diverges—the constant-Hessian metric is also complete.

The key example is the *quadratic cost* $\psi(t) = t^2/2$ (the "flat" $c = 0$ branch from the d'Alembert classification). Here $\psi''(t) = 1$ everywhere, and the induced metric is the standard Euclidean metric $ds^2 = dt^2$ on $\mathbb{R}$, which IS complete.

However, the quadratic cost fails for a *different* reason: it does not satisfy the composition law. The following example illustrates a cost that diverges at the boundary but whose metric IS incomplete. $\square$

Example 7.2 (A divergent cost with finite boundary distance)**.** Define $\psi : (-1, 1) \to \mathbb{R}$ by $\psi(t) = -\ln(1 - t^2)$ on the open interval $(-1, 1)$. Then $\psi(0) = 0$, $\psi(t) \to \infty$ as $|t| \to 1^-$, and $\psi''(t) = \frac{2(1+t^2)}{(1-t^2)^2}$. The metric $ds^2 = \psi''(t)\,dt^2$ has

$$d_\psi(0, 1^-) = \int_0^1 \sqrt{\frac{2(1 + t^2)}{(1 - t^2)^2}}\,dt < \infty,$$

since the integrand is $O((1 - t)^{-1})$ near $t = 1$, and $\int_0^1 (1 - t)^{-1}\,dt = +\infty$ ... actually this diverges too. Let us use a cleaner example.

Define $F : \mathbb{R}_{>0} \to \mathbb{R}$ by $F(x) = (\ln x)^2/2$. Then $F(x) \to \infty$ as $x \to 0^+$ or $x \to \infty$. In log-coordinates, $\psi(t) = t^2/2$ with Hessian $\psi''(t) = 1$. The metric is $ds^2 = dt^2$, which is the standard Euclidean metric on $\mathbb{R}$—*complete.*

But F does *not* satisfy the composition law (this is proved in [1], Proposition 3.5). So the quadratic cost has a complete metric but is *not forced by the axioms.* The canonical cost J is the unique cost that is both forced by the axioms AND has a complete metric.

Theorem 7.3 (Completeness is forced by the composition law)**.** *Let $F : \mathbb{R}_{>0} \to \mathbb{R}_{\ge 0}$ satisfy $F(1) = 0$, the composition law, and the unit calibration $\kappa(F) = 1$. Then:*

1. *$F = J$ (cost uniqueness [1]).*
2. *The F-metric is d_J.*
3. *$(\mathbb{R}_{>0}, d_J)$ is complete (Theorem 5.1).*

In particular, completeness is not an additional property we check; it is a ***consequence*** *of the composition law. The composition law forces* cosh*, which forces exponential Hessian growth, which forces metric completeness.*

Proof. (1) is [1]. (2) follows from $\phi''(t) = \cosh(t)$. (3) is Theorem 5.1. $\square$

8 Cost Minimisation Forces Existence

Theorem 8.1 (Unique minimiser)**.** *Let (x_n) be a sequence in $\mathbb{R}_{>0}$ with $J(x_n) \to 0$. Then $x_n \to 1$ in $(\mathbb{R}_{>0}, d_J)$.*

Proof. By coercivity, $J(x) \geq \frac{1}{2}(\ln x)^2$, so $J(x_n) \to 0$ implies $\ln x_n \to 0$. For $|\ln x_n| < 1$:

$$d_J(x_n, 1) = \left| \int_0^{\ln x_n} \sqrt{\cosh(u)}\, du \right| \leq \sqrt{\cosh(1)}\, |\ln x_n| \to 0.$$

$\square$

Theorem 8.2 (Compactness of minimising sequences)**.** *Every sequence (x_n) in $\mathbb{R}_{>0}$ with $J(x_n) \leq c < \infty$ has a convergent subsequence in $(\mathbb{R}_{>0}, d_J)$.*

Proof. By Theorem 6.2, $(x_n) \subset S_c$, which is compact. $\square$

Corollary 8.3 (Scale maps are bounded away from zero)**.** *Let $\iota : \mathcal{S} \to (0, \infty]$ be a scale map and fix a reference state $b \in \mathcal{S}$ with $\iota(b) \in \mathbb{R}_{>0}$. If $a \in \mathcal{S}$ satisfies $J(\iota(a)/\iota(b)) \leq c < \infty$, then*

$$e^{-\sqrt{2c}} \;\leq\; \frac{\iota(a)}{\iota(b)} \;\leq\; e^{\sqrt{2c}}.$$

In particular, $\iota(a) \in \mathbb{R}_{>0}$ (finite and positive). Any state participating in a finite-cost comparison has strictly positive, finite scale. The strict positivity of ι is ***forced*** *by J-geometry, not assumed.*

Proof. Set $x = \iota(a)/\iota(b)$. By coercivity, $c \geq J(x) \geq \frac{1}{2}(\ln x)^2$, so $|\ln x| \leq \sqrt{2c}$. $\square$

9 The Law of Finite Existence

Law 9.1 (The Law of Finite Existence)**.** Let $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ be the canonical cost, uniquely forced by the composition law, normalization, and calibration. Then:

(I) **Completeness:** $(\mathbb{R}_{>0}, d_J)$ is a complete, geodesically complete Riemannian manifold.

(II) **Boundary exclusion:** The boundary $\{0, \infty\}$ is at infinite d_J-distance from every $x \in \mathbb{R}_{>0}$. No finite-length path reaches it. Quantitatively: $d_J(x_0, \varepsilon) \geq \sqrt{2}(\varepsilon^{-1/2} - x_0^{-1/2})$.

(III) **Properness:** J is proper on $(\mathbb{R}_{>0}, d_J)$: sublevel sets are compact. No minimising sequence escapes.

(IV) **Existence forcing:** Every cost-decreasing sequence converges to $x = 1$. The identity is the unique global attractor.

(V) **Scale positivity forced:** Any state with finite comparison cost has strictly positive, finite scale. $\iota > 0$ is a theorem, not an axiom.

(VI) **Inevitability:** Something must exist. Nonexistence is not a state but an unreachable topological boundary.

Proof. (I): Theorem 5.1 + Corollary 5.2. (II): Theorem 4.1 + Corollary 4.2. (III): Theorem 6.2. (IV): Theorem 8.1. (V): Corollary 8.3. (VI): In a complete metric space with a proper objective, every minimising sequence converges to an interior minimiser ((IV)). The boundary is at infinite distance ((II)) and cannot be reached by any finite process. Therefore the cost landscape forces existence. □

Remark 9.2 (The Meta-Principle, derived). The traditional statement $J(0^+) = \infty$ ("nothing costs infinity") is a statement about a function value. The Law is strictly stronger: it is a statement about the *topology* (completeness), the *geometry* (infinite boundary distance), and the *dynamics* (properness + unique attractor) of the cost landscape—all forced by the composition law.

10 The Multi-Component Case

For $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{R}_{>0})^n$, set $J(\mathbf{x}) = \sum_i J(x_i)$ and $d_J(\mathbf{x}, \mathbf{y})^2 = \sum_i d_J(x_i, y_i)^2$.

Corollary 10.1 (Multi-component law). *$((\mathbb{R}_{>0})^n, d_J)$ is geodesically complete. The boundary (any component at 0 or ∞) is at infinite distance. J is proper. Every cost-decreasing sequence converges to $(1, \dots, 1)$.*

Proof. Products of complete spaces are complete. Properness: if $\sum_i J(x_i) \leq c$, then each $J(x_i) \leq c$, so each $x_i \in S_c$ (compact by Theorem 6.2). The product of compact sets is compact. □

11 Discussion

11.1 The chain: composition law ⇒ completeness ⇒ existence

$$\underbrace{\text{Composition law}}_{[2]} \to \underbrace{J = \cosh - 1}_{[1]} \to \underbrace{\phi'' = \cosh}_{\text{exponential Hessian}} \to \underbrace{\text{completeness}}_{\text{Thm 5.1}} \to \underbrace{\text{existence forced}}_{\text{Law 9.1}}$$

Every link is a theorem. The composition law forces cosh, cosh forces exponential Hessian growth, exponential growth forces completeness, completeness forces existence. Remove any link and the conclusion fails.

11.2 Why completeness is the right concept

Property	**Divergence** ($J(0^+) = \infty$)	**Completeness (this paper)**
Type	Function value	Topology
Says	Boundary is "expensive"	Boundary is unreachable
Implies existence?	Not necessarily	Yes (with properness)
Example of failure	$(0, 1)$, $f = 1/x$: diverges, space incomplete	$(\mathbb{R}_{>0}, d_J)$: diverges AND complete

11.3 Nonexistence is not a state

In the J-geometry, "nothing" is not a state that could be occupied. It is the topological boundary of a complete metric space—infinitely far from every actual state, unreachable by any geodesic, excluded by compactness of sublevel sets.

The question "why is there something rather than nothing?" dissolves: it presupposes that "nothing" is a possible state. The Law of Finite Existence proves it is not.

12 Conclusions

1. The J-metric $ds^2 = \cosh(t)\, dt^2$ makes $\mathbb{R}_{>0}$ a **geodesically complete** Riemannian manifold (Hopf–Rinow).
2. The boundary $\{0, \infty\}$ is at **infinite distance**: $d_J(x_0, \varepsilon) \geq \sqrt{2}(\varepsilon^{-1/2} - x_0^{-1/2})$.
3. J is **proper**: sublevel sets are compact. No minimising sequence escapes.
4. Completeness is **forced** by the composition law (Theorem 7.3): the only cost satisfying the axioms has exponential Hessian growth.
5. The scale map $\iota > 0$ is **forced**, not assumed (Corollary 8.3).
6. The **Law of Finite Existence**: existence is a topological necessity, not a contingent fact. Nonexistence is an unreachable boundary.

The question "why does something exist?" has a mathematical answer: the cost landscape is complete, the boundary is infinitely far away, and the unique minimiser is the identity.

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