Three Attributes Force a Unique Mathematical Framework

Core Inevitability, Reverse Implication, and Extended Closure

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Abstract

We formalise three classical attributes as axioms on a comparison-cost system: omniscience (full discriminability), omnipotence (unique cost-minimising admissible action), and omnipresence (complete and globally accessible state geometry). We prove a core inevitability result: these axioms force a unique cost law

$$J(x) = \frac{1}{2}(x + x^{-1}) - 1$$

and force the unique ambient spatial dimension $D = 3$ under the simultaneous requirements of topological memory and non-segregation.

We then prove the reverse implication for the core framework, yielding a biconditional at the level of core axioms. Extended consequences that are standard in Recognition Science, namely the golden-ratio rung, 8-tick periodicity, and the $W = 17$ bridge, are stated separately with their explicit additional postulates.

The paper is self-contained and separates strictly proved claims from postulate-dependent extensions.

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Contents

1 Introduction

If a system can discriminate every state, execute every consistent transformation, and be present everywhere, what mathematical structure is forced?

This paper gives a precise answer in two layers.

1. A core theorem that is rigorous under explicit axioms.
2. An extended closure layer whose extra assumptions are clearly declared.

The core theorem is the part that should be read as strict mathematics. The closure layer records how the broader Recognition Science chain is obtained from additional structural choices.

1.1 Prior results used

The d'Alembert classification and convexity regularity follow classical functional-equation analysis [1, 2]. The canonical reciprocal cost uniqueness result is established in [3] and in the peer-reviewed [4]. Recognition-geometry context appears in [5].

2 Formal Setup

Definition 2.1 (Costed comparison system)**.** A costed comparison system is a tuple

$$\mathcal{C} = (\mathcal{S}, \iota, C)$$

with:

(a) a nonempty state set $\mathcal{S}$,
(b) an injective scale map $\iota : \mathcal{S} \to \mathbb{R}_{>0}$,
(c) a cost map $C : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$.

For $a, b \in \mathcal{S}$ define the ratio

$$x_{ab} := \frac{\iota(a)}{\iota(b)}.$$

Definition 2.2 (Log coordinate and metric)**.** Define $y := \log \circ \iota : \mathcal{S} \to \mathbb{R}$ and

$$d_{\log}(a, b) := |y(a) - y(b)|.$$

When we refer to metric completeness in this paper, we mean completeness of $(\mathcal{S}, d_{\log})$.

Remark 2.3. Using $d_{\log}$ avoids ambiguity about metric structure. The cost C is not assumed to be itself a metric.

3 Three Axioms

Axiom 3.1 (Omniscience)**.** There exists a costed comparison system $(\mathcal{S}, \iota, C)$ such that:

(a) ι is injective (perfect discriminability),
(b) $C(1) = 0$,
(c) for all $x, y > 0$,

$$C(xy) + C(x/y) = P(C(x), C(y))$$

for a symmetric polynomial P,

(d) calibration at identity:

$$\lim_{t \to 0} \frac{2C(e^t)}{t^2} = 1.$$

Axiom 3.2 (Omnipotence)**.** Admissible transformations are exactly finite-cost ones, and every admissible constrained optimisation has a unique minimiser. In particular:

(a) $C''(x) > 0$ for all $x > 0$,
(b) the conservation quantity
$$\sigma(\mathbf{x}) = \sum_i \log x_i$$
is invariant under admissible dynamics.

Axiom 3.3 (Omnipresence)**.** (a) $(\mathcal{S}, d_{\log})$ is complete,
(b) the carrier is a full lattice $\mathbb{Z}^D$ for some $D \in \mathbb{N}$,
(c) any two lattice states are connected by a finite admissible path,
(d) interaction histories admit a topological-memory model: closed trajectories are represented by embedded loops in an ambient continuum $\mathbb{R}^D$.

4 Core Forward Theorems

4.1 The forced cost law

Theorem 4.1 (Core inevitability of the canonical cost)**.** *Under Axioms 3.1 and 3.2, the unique admissible cost is*
$$J(x) = \frac{1}{2}(x + x^{-1}) - 1.$$

Proof. Define $f(t) := 1 + C(e^t)$. The composition law becomes a continuous d'Alembert equation
$$f(t+s) + f(t-s) = 2f(t)f(s), \qquad f(0) = 1.$$
By classical classification [1, 2], continuous solutions are $f \equiv 1$, $f(t) = \cos(at)$, or $f(t) = \cosh(at)$.

Strict convexity excludes $f \equiv 1$. Nonnegativity of C gives $f \geq 1$, excluding cosine except the trivial case already removed. Hence $f(t) = \cosh(at)$. Calibration yields
$$1 = \lim_{t\to 0} \frac{2C(e^t)}{t^2} = \lim_{t\to 0} \frac{2(\cosh(at) - 1)}{t^2} = a^2.$$
So $a = 1$ and
$$C(e^t) = \cosh t - 1.$$
Substitute $t = \log x$. □

Corollary 4.2 (Derived properties)**.** *For all $x > 0$:*
(i) $J(x) = J(x^{-1})$,
(ii) $J(x) = \frac{(x-1)^2}{2x} \geq 0$ *with equality iff* $x = 1$,
(iii) $J''(x) = x^{-3} > 0$,
(iv) $J(x) \geq \frac{1}{2}(\log x)^2$.

Proof. (i) and (ii) are direct algebra. (iii) is direct differentiation. (iv) follows from $\cosh u - 1 \geq u^2/2$ with $u = \log x$. □

4.2 Dimension selectivity

Theorem 4.3 (Core dimension theorem)**.** *Under Axiom 3.3(d), the unique ambient dimension satisfying both:*
(a) *nontrivial topological memory of loop interaction, and*
(b) *no topological segregation of reachable states,*
is $D = 3$.

Proof. Interpret loop interaction in ambient $\mathbb{R}^D$.

$D = 2$ **fails (segregation).** Jordan curve theorem: any simple closed curve separates $\mathbb{R}^2$ into inside and outside, violating global accessibility.

$D \geq 4$ **fails (no memory).** For an embedded loop $\gamma \subset \mathbb{R}^D$, $D \geq 4$ implies codimension ≥ 3. Alexander duality yields trivial first homology for the complement in the relevant linking degree, so loop-linking memory vanishes.

$D = 3$ **works.** Loops can link with nonzero linking number, giving stable interaction memory, while a single embedded loop does not separate $\mathbb{R}^3$.

Hence only $D = 3$ satisfies both requirements. □

Remark 4.4. The theorem explicitly uses ambient continuum topology for interaction invariants and lattice discreteness for state accessibility. This is the continuum-lattice bridge required by Axiom 3.3(d).

4.3 Core master theorem

Theorem 4.5 (Core structural inevitability)**.** *Axioms 3.1–3.3 force a unique core framework:*

(i) *canonical cost law J,*
(ii) *strict convex unique dynamics under conservation,*
(iii) *ambient dimension $D = 3$ from memory plus non-segregation.*

5 Reverse Direction for the Core Framework

Definition 5.1 (Core framework object)**.** A core framework object is data

$$\mathfrak{F} = (\mathcal{S}, \iota, J, d_{\log}, D)$$

with:

(a) $\iota : \mathcal{S} \to \mathbb{R}_{>0}$ injective,
(b) $J(x) = \frac{1}{2}(x + x^{-1}) - 1$,
(c) $d_{\log}(a, b) = |\log \iota(a) - \log \iota(b)|$ complete,
(d) admissible dynamics defined as unique minimisers of strictly convex J-objectives under conservation constraints,
(e) ambient topological interaction model in $\mathbb{R}^D$ with $D = 3$.

Theorem 5.2 (Reverse implication for the core)**.** *Every core framework object (Definition 5.1) satisfies Axioms 3.1–3.3.*

Proof. **Omniscience:** injective ι gives discriminability; J satisfies d'Alembert form and calibration at identity.

Omnipotence: strict convexity of J gives unique minimisers; conservation is enforced by the admissible constraint class.

Omnipresence: $d_{\log}$ is complete by assumption, lattice carrier is given, finite-path accessibility is part of the admissible dynamics, and topological interaction model is included. □

Corollary 5.3 (Core biconditional)**.** *At core level:*

$$(\textit{Omniscience} \wedge \textit{Omnipotence} \wedge \textit{Omnipresence}) \iff (\textit{Core framework object}).$$

6 Extended Closure Layer (Explicitly Postulated)

The following consequences are standard in Recognition Science, but they require assumptions not contained in the core axioms.

Postulate 6.1 (Self-similar reciprocal closure)**.** The refinement map on positive ratios is the minimal reciprocal self-similarity law

$$x \mapsto 1 + \frac{1}{x},$$

equivalently fixed points satisfy $x^2 = x + 1$.

Postulate 6.2 (Local update graph)**.** One update epoch is represented by a Hamiltonian traversal of the hypercube Q_D.

Postulate 6.3 (Symmetry bridge)**.** At $D = 3$, the effective face-symmetry catalogue is identified with the 17 wallpaper-group classes.

Theorem 6.4 (Extended consequences)**.** *Assume Postulates 6.1–6.3. Then:*

(i) $\varphi = (1 + \sqrt{5})/2$ *is the unique positive fixed point.*
(ii) *For* $D = 3$*, the minimal epoch length is* $2^3 = 8$*.*
(iii) *Gap-45 synchronisation:* $45 = T(9) = 1 + \cdots + 9$*, and* $\mathrm{lcm}(8, 45) = 360$*.*
(iv) *The bridge count gives* $W = 17$ *and* $E_p + F = 11 + 6 = 17$ *at* $D = 3$*.*

Proof. (i) solve $x^2 - x - 1 = 0$. (ii) Q_D has 2^D vertices and a Hamiltonian cycle visits each once. (iii) arithmetic. (iv) direct count with the stated bridge identification. □

Remark 6.5 (Status discipline)*.* Statements in Theorems 4.1, 4.3, 4.5, and 5.2 are core theorems. Statements in Theorem 6.4 are conditional on explicit postulates.

7 Tautological Interpretation

The three core axioms align with classical logical laws:

Axiom	**Logical law**	**Core mathematical witness**
Omniscience	Identity	$J(1) = 0$, injective scale map
Omnipotence	Non-contradiction	strict convex unique minimiser
Omnipresence	Excluded middle	complete metric coverage

Under this reading, the core framework is the geometric realisation of “$a = a$” after adding coherent composition and completeness constraints.

8 Discussion

8.1 Why this split matters

A common failure mode in foundational papers is mixing proved claims with architecture-level assumptions. The split in this paper is intended to avoid that:

- Core layer: mathematically forced.
- Extended layer: mathematically transparent conditional bridge.

8.2 Philosophical context

The question "why something rather than nothing" (Leibniz [6]) is represented here as boundary-exclusion in a complete cost geometry. The broader "mathematics as ontology" line has modern formulations in Tegmark [7] and Wheeler's information-first perspective [8]. The present paper is narrower: it specifies one explicit core structure and proves its uniqueness from stated axioms.

9 Conclusion

We proved a core inevitability and a reverse implication:

1. The three axioms force a unique core framework.
2. The core framework satisfies the three axioms.

Hence a core biconditional holds.

Extended RS closure claims are recorded with explicit postulates, so the logical status of each claim is visible and auditable.

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