Rigidity and Compact Phase Emergence in Scale-Invariant Ratio-Based Energies

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We study a scale-invariant nearest-neighbor graph energy for positive vertex fields that penalizes adjacent mismatches through a reciprocal convex cost, $J(x) = \frac{1}{2}(x + x^{-1}) - 1$. The energy is nonnegative, vanishing only when neighboring ratios agree, and its exact invariance under global rescaling yields a continuous noncompact scale symmetry. We prove a finite-volume rigidity theorem: on any finite connected graph, every global minimizer is necessarily uniform—a static notion of "global coherence" that persists in local limits on $\mathbb{Z}^3$. We then establish how a compact $U(1)$ phase variable emerges by imposing an explicit discrete identification that treats rescalings by integer powers of the golden ratio ϕ as gauge-equivalent (the "ϕ-ladder gauge"). Within this quotient model, we compute the induced periodic phase-mismatch potential—which features an explicit harmonic stiffness of $\kappa = (\ln \phi)^2/2$—and prove that finite-volume minimizers remain phase-uniform. Together, these results establish a rigorous static formulation of the Global Co-Identity Constraint, defining the interplay between continuous scale symmetries and discrete gauge inputs.

I. INTRODUCTION

A common structural motif in lattice and graph models of statistical and condensed-matter physics is that a global energy can be written as a sum of local *mismatch penalties* on edges, and low-energy (or zero-temperature) states correspond to configurations in which neighboring degrees of freedom "agree" as much as possible. Ferromagnetic Ising and rotor models provide the canonical example: edge interactions favor alignment, and connectedness propagates local agreement into uniform ground states [1, 2]. A closely related viewpoint appears in convex gradient (height/interface) models, where energies depend on discrete gradients and constants minimize the associated Dirichlet-type terms [3, 4]. On graphs, the algebraic analogue is the discrete Dirichlet energy associated with the graph Laplacian, which is likewise minimized by constant fields on connected graphs [5, 6].

In this paper we analyze the same rigidity mechanism for a *multiplicative* (ratio-based) degree of freedom. We assign to each vertex v a positive real value $x_v \in \mathbb{R}_{>0}$ and penalize the ratio x_v/x_w across each edge using the reciprocal convex cost

$$J(x) = \frac{1}{2}\left(x + x^{-1}\right) - 1, \qquad x > 0. \tag{1}$$

In logarithmic variables $r_v = \ln x_v$, the edge cost becomes $J(e^{r_v - r_w}) = \cosh(r_v - r_w) - 1$ (Fig. 1): it is harmonic for small mismatch ($J(e^t) = \frac{1}{2}t^2 + O(t^4)$) and grows exponentially for large $|t|$. Thus, in log-coordinates the model is a convex gradient energy with potential $\cosh(\cdot) - 1$, placing it within the broad class of convex $\nabla\phi$ models studied in statistical mechanics and random surface theory [3, 7].

The particular form (1) is also singled out in Recognition Science (RS), where it is claimed to arise axiomatically from a multiplicative d'Alembert-type composition law [8, 9]. The rigidity results proved here, however, rely only on elementary properties of J: nonnegativity, a unique zero at $x = 1$, and inversion symmetry. We establish two fundamental results regarding this energy framework:

1. Because each edge term is nonnegative and vanishes only at perfect ratio agreement, the global minimum of the ratio energy on any finite connected graph is achieved if and only if every edge ratio equals 1. Connectedness then forces $x_v \equiv c$.

2. The exact invariance of the ratio energy is global rescaling $x_v \mapsto c\,x_v$ ($c > 0$), i.e. the noncompact group $(\mathbb{R}_{>0}, \times) \cong (\mathbb{R}, +)$. A compact, circle-valued phase $\Theta \in \mathbb{R}/\mathbb{Z}$ appears only after imposing an additional discrete identification $x \sim b^n x$. In RS one takes $b = \varphi = (1+\sqrt{5})/2$ and treats $\varphi^{\mathbb{Z}}$ rescaling as gauge-equivalent, leaving the fractional ladder coordinate $\Theta = \log_\varphi x \bmod 1$ as the residual "phase" label [9].

Under this quotient we compute the induced periodic phase-mismatch potential and show that finite-volume minimizer remain strictly Θ-uniform (a precise static version of the RS "Global Co-Identity Constraint"), and obtain an explicit small-gradient stiffness $\kappa = (\ln \varphi)^2/2$.

The remainder of this paper implements this framework systematically. In Sec. II we define the ratio energy and its non-compact symmetry. In Sec. III we prove finite-volume rigidity theorem and record a quantitative quadratic lower bound in log variables. In Sec. IV we translate rigidity into a thermodynamic-limit statement for local limits of finite-volume minimizers on $\mathbb{Z}^3$. In Sec. V we introduce the discrete scaling gauge, derive the reduced phase potential, and prove the corresponding phase rigidity result. In Sec. **??** and VI we discuss the main results and conclude. Technical proofs are collected in the appendices.

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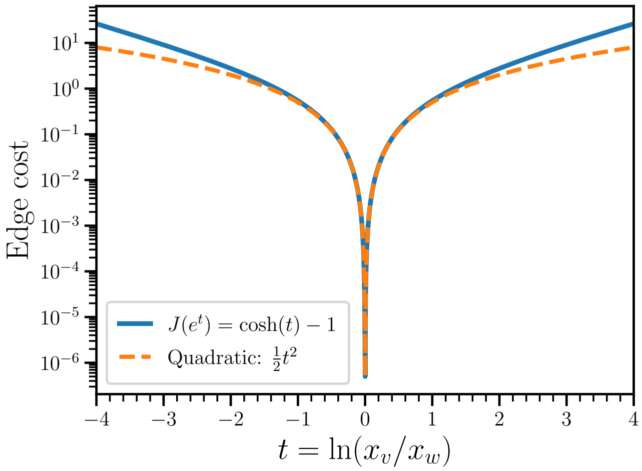

FIG. 1. Edge mismatch cost in logarithmic variables: $J(e^t) = \cosh t - 1$ (solid) together with the quadratic lower bound $\frac{1}{2}t^2$ (dashed), shown on a semilogarithmic vertical scale. Near $t = 0$ the curves coincide, motivating a "stiffness" interpretation for small gradients; the exponential growth for large $|t|$ penalizes strong mismatches far beyond the harmonic regime.

II. MODEL: A RECIPROCAL RATIO ENERGY AND ITS EXACT SYMMETRY

We work on a (simple) graph $G = (V, E)$, finite or countably infinite, where the vertex set V represents spatial sites (lattice points, nodes of a network, *etc.*) and the edge set E encodes nearest-neighbor adjacency. A *configuration* is a positive field $x : V \to \mathbb{R}_{>0}$, assigning $x_v > 0$ to each vertex. Throughout, x_v should be interpreted as a *local scale* variable: its absolute magnitude is not directly observable in the ratio-only model, while ratios x_v/x_w quantify mismatch between neighboring sites.

Given the reciprocal cost (1), we define the total ratio energy by summing the edge mismatch penalties

$$C_G[x] = \sum_{\langle v,w\rangle \in E} J\left(\frac{x_v}{x_w}\right), \tag{2}$$

where $\langle v, w\rangle$ denotes an unordered edge. For finite graphs, the sum (2) is well defined. For infinite graphs such as $\mathbb{Z}^3$, one typically works with finite-volume restrictions (boxes with free or periodic boundary conditions) and considers limits; this is the subject of Sec. IV.

A. Basic properties of the edge cost

All of the rigidity statements in this paper ultimately trace back to three elementary properties of J:

1. *Nonnegativity:* $J(x) \geq 0$ for all $x > 0$.
2. *Unique zero:* $J(x) = 0$ if and only if $x = 1$.
3. *Reciprocity:* $J(x) = J(x^{-1})$.

These follow immediately from the arithmetic–geometric mean inequality and standard inequalities for means (see, *e.g.*, Refs. [10–13]). Short self-contained proofs are provided in Appendix A.

Nonnegativity and the unique zero drive the connectedness rigidity theorem in Sec. III: since every edge contributes a nonnegative term, the only way to achieve the absolute minimum $C_G[x] = 0$ is to enforce perfect ratio agreement on every edge. Reciprocity ensures the mismatch cost is symmetric in the two endpoints, so the numerator/denominator convention is immaterial.

Two equivalent forms of (1) will be useful for intuition:

$$J(x) = \frac{(x-1)^2}{2x} = \frac{1}{2}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2. \tag{3}$$

These make the nonnegativity and the unique minimum at $x = 1$ completely transparent.

It is also natural to pass to logarithmic variables. Setting $t = \ln x$ gives

$$J(e^t) = \cosh t - 1 = 2\sinh^2\left(\frac{t}{2}\right), \tag{4}$$

so the ratio energy (2) becomes a *convex gradient energy*

$$C_G[x] = \sum_{\langle v,w\rangle \in E} \Big(\cosh(r_v - r_w) - 1\Big), \qquad r_v := \ln x_v. \tag{5}$$

The potential $V(t) = \cosh t - 1$ is *uniformly convex* since $V''(t) = \cosh t \geq 1$, and grows exponentially for large $|t|$. Such uniformly convex gradient models (at positive temperature) form a classical and well-developed class in the random-interface literature; see, for example, Refs. [3, 4, 7, 14] and references therein. The special choice $V(t) = \cosh t - 1$ (or close relatives) also appears in noncompact (hyperbolic) sigma models for disordered systems and reinforced random walks, where the underlying symmetry group is noncompact; see, *e.g.*, Refs. [15–19].[1]

A second key feature in log-coordinates is the global quadratic lower bound

$$\cosh t - 1 \geq \frac{1}{2}t^2 \qquad (t \in \mathbb{R}), \tag{6}$$

proved in Appendix A and recorded in many inequality compendia [13]. This estimate connects our ratio model to the standard discrete Dirichlet form

$$\mathcal{E}_G[r] \equiv \frac{1}{2}\sum_{\langle v,w\rangle \in E} (r_v - r_w)^2, \tag{7}$$

whose minimizers are constant fields on connected graphs [5, 6, 20–22]. At finite temperature, (7) is also the Gaussian free-field action, linking the harmonic approximation of (5) to classical random-surface theory [23].

[1] We cite this literature only as context: the present work uses only the elementary convexity and positivity properties of $V(t) = \cosh t - 1$.

B. Exact symmetry: global scaling

Because each edge term depends only on the *ratio* x_v/x_w, the total energy (2) is exactly invariant under global rescaling:

$$x_v \mapsto c\,x_v \quad (c>0) \qquad \Longrightarrow \qquad C_G[cx] = C_G[x]. \quad (8)$$

In log variables $r_v = \ln x_v$, this is the global additive shift symmetry $r_v \mapsto r_v + \ln c$. The exact symmetry group is therefore $(\mathbb{R}_{>0}, \times) \cong (\mathbb{R}, +)$, which is *noncompact.* As a result, the zero-energy manifold contains the full one-parameter family of constant configurations $x_v \equiv c$; no preferred scale exists in the bare ratio energy.

This structure differs sharply from compact phase models such as the classical XY (planar rotor) model, where the local degree of freedom is an angle $\theta_v \in [0, 2\pi)$ and the symmetry is the compact group $U(1)$ [24]. In two dimensions, the compact $U(1)$ symmetry underlies the Berezinskii–Kosterlitz–Thouless scenario [29–31] and is constrained by the Mermin–Wagner–Hohenberg theorem forbidding spontaneous breaking of continuous symmetries for a broad class of short-range models [32, 33].

In our ratio model, no compact phase exists at the level of (2). A circle-valued label $\Theta \in \mathbb{R}/\mathbb{Z}$ only appears after imposing an additional discrete identification that quotients the noncompact scaling symmetry by an integer subgroup; this is implemented in Sec. V (motivated in RS by a $\varphi^{\mathbb{Z}}$ "ladder gauge" [9, 34]).

III. FINITE-VOLUME RIGIDITY OF GLOBAL MINIMIZERS

We now establish the first main result of the paper: on any *finite* connected graph, the ratio energy (2) admits no nontrivial zero-energy states. The logical mechanism is the same one that underlies uniform ground states of ferromagnets and other nearest-neighbor mismatch models: if a Hamiltonian is a sum of *nonnegative* edge terms with a *unique* zero at perfect local agreement, then any global minimizer must enforce agreement on *every* edge, and connectedness propagates that edgewise constraint to the whole graph. This "edgewise-zero ⇒ global consensus" structure is standard in lattice statistical mechanics and graph-based energy minimization; see, *e.g.*, Refs. [1, 2, 35] for context.

A. The core rigidity mechanism

Because each edge contribution $J(x_v/x_w)$ is nonnegative (Sec. II and Appendix A), the energy satisfies $C_G[x] \ge 0$ for every configuration $x : V \to \mathbb{R}_{>0}$. A constant configuration $x_v \equiv c$ makes every ratio equal to 1 and therefore achieves $C_G[x] = 0$. The nontrivial content is that *nothing else* can: if $C_G[x] = 0$, then every summand must vanish individually, which by the unique-zero property forces $x_v = x_w$ on every edge. On a connected graph, equality on edges implies equality on all vertices by path propagation.

Result 1: Finite-volume rigidity. *Let G be a finite connected graph. Then $\inf_x C_G[x] = 0$, and $C_G[x] = 0$ if and only if x_v is constant on $V(G)$.* A self-contained proof is given in Appendix A 4.

Two remarks are worth making explicit: (i) *Connectedness is the only topological input:* On a disconnected graph, each connected component can independently choose its own constant value, and (ii) *The explicit form of J is not needed for Result 1:* The same argument applies verbatim to any edge penalty $V(y) \ge 0$ on $y > 0$ with $V(y) = 0$ if and only if $y = 1$. What the special choice (1) adds is quantitative control away from the minimizers.

B. Quantitative control of near-minimizers ("stiffness")

Finite-volume rigidity (Result 1) characterizes exact minimizers ($C_G = 0$). For applications, it is often just as important to understand *near*-minimizers: configurations with small but nonzero energy. Here the log-coordinate form of the cost, $J(e^t) = \cosh t - 1$, provides a natural "stiffness" lower bound.

Writing $r_v = \ln x_v$, the ratio energy becomes the convex gradient form (5), and the global inequality $\cosh t - 1 \ge t^2/2$ (see Appendix A) yields

$$C_G[x] = \sum_{\langle v,w\rangle \in E} \big(\cosh(r_v - r_w) - 1\big) \;\ge\; \mathcal{E}_G[r], \quad (9)$$

where $\mathcal{E}_G[r] = \frac{1}{2}\sum_{\langle v,w\rangle\in E}(r_v - r_w)^2$ is the discrete Dirichlet energy (7). For small variations of $\ln x$, the ratio model behaves like a harmonic elastic energy, while the full cosh potential penalizes large mismatches much more strongly (see Sec. II).

The Dirichlet lower bound (9) can be converted into an explicit control on how much the log-field can vary across the graph. Let $d_G(v, w)$ denote the graph distance (shortest-path length) between vertices v and w. Choose a shortest path $v = v_0, v_1, \ldots, v_m = w$ with $m = d_G(v, w)$. By Cauchy–Schwarz,

$$\begin{aligned} |r_v - r_w|^2 &= \Big|\sum_{i=0}^{m-1}(r_{v_i} - r_{v_{i+1}})\Big|^2 \\ &\le m \sum_{i=0}^{m-1}(r_{v_i} - r_{v_{i+1}})^2 \\ &\le m \sum_{\langle u,u'\rangle\in E}(r_u - r_{u'})^2 \\ &= 2m\,\mathcal{E}_G[r] \le 2m\,C_G[x]. \end{aligned} \quad (10)$$

Equivalently,

$$\Big|\ln\frac{x_v}{x_w}\Big| = |r_v - r_w| \le \sqrt{2\,d_G(v,w)\,C_G[x]}. \quad (11)$$

In particular, if $C_G[x]$ is small then all site-to-site ratios are close to unity, with a deviation controlled by graph distance.

A complementary "bulk" bound is obtained from the discrete Poincaré inequality for the graph Laplacian. Let $\bar{r} = |V|^{-1} \sum_{v \in V} r_v$ and let $\lambda_2(G) > 0$ denote the spectral gap (the second-smallest eigenvalue of the combinatorial Laplacian; also called the *algebraic connectivity* [36, 37]). Then, for any finite connected graph,

$$\sum_{v \in V} (r_v - \bar{r})^2 \le \frac{1}{\lambda_2(G)} \sum_{\langle v,w \rangle \in E} (r_v - r_w)^2 = \frac{2}{\lambda_2(G)} \mathcal{E}_G[r] \le \frac{2}{\lambda_2(G)} C_G[x], \quad (12)$$

see, *e.g.*, Refs. [5, 22, 38, 39]. Equation (12) makes precise the sense in which low energy forces the field to be nearly constant: after removing the unphysical global scale (subtracting $\bar{r}$), the mean-square deviation is controlled by the energy.

IV. THERMODYNAMIC-LIMIT CONSEQUENCE ON $\mathbb{Z}^3$

Result 1 establishes rigidity on any single finite connected graph. In lattice statistical mechanics one is typically interested in the *thermodynamic limit*: the behavior of minimizers or equilibrium states as the system size grows and boundary effects become negligible [2, 35]. We now show that the rigidity of finite-volume *global* minimizers survives this limit. Although we specialize to the cubic lattice $\mathbb{Z}^3$—the natural setting in Recognition Science, where the "voxel ledger" is modeled on a three-dimensional integer lattice—the argument is purely topological and applies verbatim to $\mathbb{Z}^d$ for any $d \ge 1$.

Let

$$\Lambda_n := [-n, n]^3 \cap \mathbb{Z}^3 \quad (13)$$

with its nearest-neighbor edge set, and let C_{Λ_n} denote the ratio energy (2) restricted to edges fully contained in Λ_n. We consider either free boundary conditions (no constraints outside Λ_n) or periodic boundary conditions (identifying opposite faces to obtain a discrete three-torus). By Result 1, every global minimizer $x^{(n)}$ of C_{Λ_n} is constant on Λ_n.

- Because the model has an exact global scaling symmetry $x \mapsto cx$ [Eq. (8)], finite-volume minimizers form a one-parameter family: if $x^{(n)}$ is a minimizer, so is $c\,x^{(n)}$. Thus a sequence of minimizers need not converge unless one fixes this redundancy. A convenient choice is to *gauge-fix the scale* by rescaling each minimizer so that

$$x_0^{(n)} = 1\,, \quad (14)$$

  equivalently $r_0^{(n)} = 0$ in log variables $r = \ln x$. With this convention, the finite-volume minimizer is unique and equals $x_v^{(n)} \equiv 1$. Without imposing (14), the statement below should be read as: *any locally convergent subsequence of minimizers has a constant limit.*

- **Result 2: Uniformity of minimizer limits.** *Let $x^{(n)}$ be a global minimizer of C_{Λ_n} under free or periodic boundary conditions, with the global scale fixed as in Eq.* (14). *Then any subsequential local limit of $\{x^{(n)}\}$ is constant on $\mathbb{Z}^3$ (in fact $x_v \equiv 1$).*

  The proof, given in Appendix A 5, is short: since each finite-volume minimizer is constant on Λ_n, any two sites $v, w \in \mathbb{Z}^3$ satisfy $x_v^{(n)} = x_w^{(n)}$ for all n large enough that both sites lie inside Λ_n. Passing to a locally convergent subsequence preserves this equality, so the limit is constant on all of $\mathbb{Z}^3$.

  The physical content of Result 2 is that uniform ratio agreement is not an artifact of finite boundaries: it persists in the bulk. In particular, *limits of finite-volume global minimizers* do not support domain walls, phase boundaries, or spatially modulated patterns. In RS language, this gives a precise static form of "global coherence" on the infinite lattice.

Result 2 concerns limits of *finite-volume global minimizers*. It does not classify all possible infinite-volume "ground states" under alternative statistical-mechanics notions (e.g., minimizers under *local* perturbations, or zero-temperature limits of Gibbs measures in the DLR sense) [2, 35]. The distinction is standard and becomes important when boundary conditions or topological constraints select nontrivial infinite-volume states; for related phenomena in gradient models see, e.g., Ref. [7].

V. FROM SCALING TO PHASE: A DISCRETE GAUGE QUOTIENT AND A COMPACT VARIABLE

Sections III and IV established that finite-volume *global* minimizers of the ratio energy are spatially uniform, and that this uniformity persists in local limits on $\mathbb{Z}^3$. Those statements live entirely in the noncompact symmetry of Eq. (8): the zero-energy manifold is the one-parameter family of constant configurations $x_v \equiv c$.

In this section we address a different (and optional) step. If one *declares* a discrete subgroup of the scaling symmetry to be physically redundant, then the noncompact shift symmetry of the log-field can be "compactified" into a circle-valued variable. This compactification mechanism is familiar in lattice field theory and statistical mechanics as the distinction between noncompact and compact formulations of $U(1)$ variables [25, 26], and in the XY/clock-model family where a real field is reduced modulo an integer lattice [24, 27, 28]. Here we implement it directly for the scale-invariant ratio energy.

The exact symmetry of the ratio energy is global rescaling $x_v \mapsto c\,x_v$ ($c > 0$), which becomes a global additive shift in logarithmic variables, $r_v = \ln x_v \mapsto r_v + \ln c$. Thus the symmetry group is $(\mathbb{R}_{>0}, \times) \cong (\mathbb{R}, +)$: it is noncompact, and periodicity is absent at the level of Eq. (2). A compact $U(1)$ phase can therefore only appear if we impose an *additional* discrete identification.

The underlying group-theory point is simple and can be stated explicitly. Fix a "compactification length" $\ell > 0$ and identify

$$r \sim r + n\ell, \qquad n \in \mathbb{Z}. \tag{15}$$

Then the quotient $\mathbb{R}/\ell\mathbb{Z}$ is naturally realized as a circle via the map

$$r \longmapsto \exp\left(\frac{2\pi i}{\ell}\, r\right) \in U(1), \tag{16}$$

which identifies r and $r + \ell$ and is surjective onto the unit circle.

In Recognition Science (RS) the discrete identification is implemented using the golden ratio

$$\varphi = \frac{1 + \sqrt{5}}{2}, \tag{17}$$

and the associated "φ-ladder" coordinate $r = \log_\varphi x$. The RS proposal is that rescalings by φ^n are gauge-equivalent labels [9, 34]. We emphasize that this is an *explicit model input*—it does not follow from ratio dependence alone. (Indeed, nothing in the construction below depends on φ specifically; any base $b > 1$ would lead to the same structure with $\ell = \ln b$.)

We fix $\varphi = (1 + \sqrt{5})/2$ and identify scales under

$$x \sim \varphi^n x, \qquad n \in \mathbb{Z}. \tag{18}$$

Equivalently, in ladder coordinates $r = \log_\varphi x$ we identify $r \sim r + n$. Under this assumption the residual label of a scale is the fractional part

$$\Theta \equiv r \bmod 1 \in \mathbb{R}/\mathbb{Z} \cong U(1). \tag{19}$$

A positive field x_v therefore induces a circle-valued phase field Θ_v.

A. Gauge-invariant phase mismatch and its potential

If the observable degree of freedom is $\Theta \in \mathbb{R}/\mathbb{Z}$, then an edge interaction must depend only on the phase difference $\delta = \Theta_v - \Theta_w$ modulo integers. There is still a modeling choice here: many even, 1-periodic functions could be used as a phase-mismatch penalty. The natural choice we adopt is the *induced zero-temperature potential* obtained by projecting the original cost onto the quotient: given a phase difference δ (defined modulo 1), we

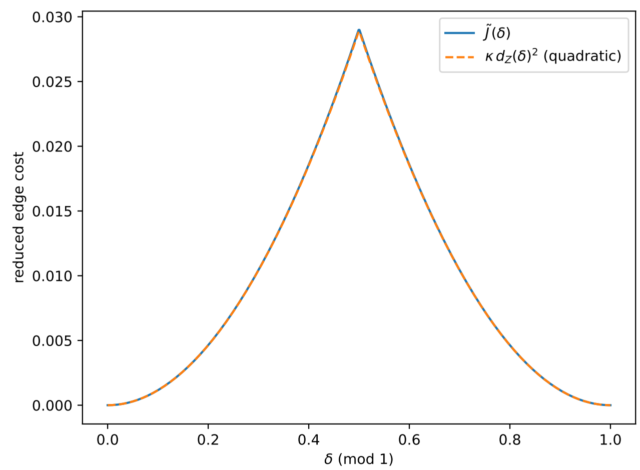

FIG. 2. Gauge-invariant phase-mismatch potential $\widetilde{J}(\delta)$ induced by the discrete scaling identification $x \sim \varphi^n x$. Over one period, $\widetilde{J}(\delta)$ depends only on the distance to the nearest integer $d_{\mathbb{Z}}(\delta)$. The dashed curve shows the quadratic small-mismatch approximation $\kappa\, d_{\mathbb{Z}}(\delta)^2$ with stiffness $\kappa = (\ln\varphi)^2/2$. need to update

choose the representative $\delta + n$ that minimizes the original ratio cost. This leads to the reduced potential

$$\widetilde{J}(\delta) \equiv \min_{n\in\mathbb{Z}} J\left(\varphi^{\,n+\delta}\right). \tag{20}$$

(Equivalently, one can view this as restricting to a fundamental domain for $\mathbb{R}/\mathbb{Z}$ and then extending periodically; compare the way periodic "Villain-type" potentials are built by summing or minimizing over integer shifts [27, 28].)

A short computation (Appendix B) yields a closed form. Write $\lambda = \ln\varphi$. Since $J(\varphi^t) = \cosh(\lambda t) - 1$, and $\cosh(\lambda\cdot) - 1$ is even and strictly increasing on $[0, \infty)$, the integer minimization reduces to choosing the nearest integer to δ:

$$\widetilde{J}(\delta) = \cosh\big(\lambda\, d_{\mathbb{Z}}(\delta)\big) - 1, \tag{21}$$

where

$$d_{\mathbb{Z}}(\delta) \equiv \min_{n\in\mathbb{Z}} |\delta - n| \in \left[0, \tfrac{1}{2}\right] \tag{22}$$

is the distance to the nearest integer. The reduced potential $\widetilde{J}$ is 1-periodic, even, and nonnegative, and it vanishes only at integer arguments (i.e., when the phases agree). Because $d_{\mathbb{Z}}$ has kinks at half-integers, $\widetilde{J}$ is continuous and piecewise analytic but not differentiable at $\delta \equiv \frac{1}{2} \pmod 1$; this nonanalyticity is simply the point where the minimizing integer branch switches. Figure 2 plots $\widetilde{J}$ over one period.

B. Phase rigidity: a static Global Co-Identity Constraint

We now define the phase-only analogue of the ratio energy. Given a phase field $\Theta_v \in \mathbb{R}/\mathbb{Z}$ on a finite graph, choose any lift $\vartheta_v \in \mathbb{R}$ with $\Theta_v = \vartheta_v \bmod 1$, and set

$$\widetilde{C}_G[\Theta] \equiv \sum_{\langle v,w\rangle \in E} \widetilde{J}(\vartheta_v - \vartheta_w). \qquad (23)$$

This is well defined (independent of the lift) because $\widetilde{J}$ is 1-periodic.

The ground-state structure is immediate: $\widetilde{J} \geq 0$ and $\widetilde{J}(\delta) = 0$ holds only when $\delta \in \mathbb{Z}$, i.e., when $\Theta_v = \Theta_w$. Thus the same connectedness argument used for Result 1 applies verbatim.

a. Result 3 (Phase rigidity / static GCIC). On any finite connected graph, $\widetilde{C}_G$ is minimized exactly by constant phase fields $\Theta_v \equiv \Theta_0$. A self-contained proof is given in Appendix A 6.

In the RS language, Result 3 is a precise *static* formulation of the Global Co-Identity Constraint (GCIC): within the quotient model where only $\Theta \in \mathbb{R}/\mathbb{Z}$ is retained, finite-volume ground states cannot support coexisting phase domains. As in Sec. IV, local limits of finite-volume phase minimizers on $\mathbb{Z}^3$ are therefore Θ-uniform.

It is worth stressing the logical dependency. The rigidity statement is forced by the same nonnegativity/unique-zero mechanism as before, but it applies to a *phase* variable only after the additional discrete identification (18). Without that input, there is no compact phase to constrain.

C. Small-gradient stiffness

Beyond the qualitative rigidity result, the explicit form (21) gives a quantitative penalty for small phase gradients. For $|\delta| \ll 1$ we have $d_{\mathbb{Z}}(\delta) = |\delta|$ and the Taylor expansion of cosh yields

$$\widetilde{J}(\delta) = \frac{\lambda^2}{2}\,\delta^2 + O(\delta^4), \qquad \lambda = \ln\varphi, \qquad (24)$$

so the harmonic stiffness is

$$\kappa = \frac{(\ln\varphi)^2}{2} \approx 0.1158. \qquad (25)$$

This coefficient sets the leading energy cost of long-wavelength phase variations: a small mismatch δ across an edge costs approximately $\kappa\,\delta^2$ per edge. For a general discrete gauge $x \sim b^n x$ with $b > 1$, the same computation gives $\kappa = (\ln b)^2/2$.

VI. DISCUSSION AND CONCLUSION

We analyzed a scale-invariant ratio-mismatch energy (2) built from the reciprocal convex cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ (1). The central message is a rigidity one: because each edge contribution is nonnegative and vanishes only at perfect agreement, global minimizers on any finite connected graph are necessarily uniform (Result 1). Specializing to boxes $\Lambda_n \subset \mathbb{Z}^3$, this implies that every subsequential local limit of finite-volume global minimizers is constant on $\mathbb{Z}^3$ (Result 2). In particular, within this zero-temperature setting the model does not support spatial domains or patterned ground states as limits of finite-volume minimizers.

A second message is conceptual rather than technical. The exact continuous symmetry of the ratio energy is *global scaling* $x_v \mapsto c\,x_v$ (8), equivalently a global shift $r_v \mapsto r_v + \ln c$ in log variables. This is the noncompact group $(\mathbb{R}_{>0}, \times) \cong (\mathbb{R}, +)$, so there is no intrinsic circle-valued "phase" at the level of (2). A compact phase $\Theta \in \mathbb{R}/\mathbb{Z}$ only appears after one imposes an additional discrete identification that quotients the scale symmetry by an integer subgroup. In Recognition Science this input is the φ-ladder gauge $x \sim \varphi^n x$ (18) [9, 34]. Once that quotient is adopted, the induced periodic edge potential $\widetilde{J}$ can be computed explicitly, and the same nonnegativity/unique-zero logic forces phase-uniform finite-volume minimizers (Result 3), providing a precise static formulation of the "Global Co-Identity Constraint" in this model. The closed form of $\widetilde{J}$ also yields an explicit small-gradient stiffness $\kappa = (\ln\varphi)^2/2$ (25).

It is worth emphasizing what is general and what is special. The uniformity of finite-volume minimizers is completely robust: it holds for *any* nearest-neighbor edge energy that is nonnegative and has a unique zero at equality, and it depends on no additional structure beyond connectedness. What is special to the reciprocal cost (1) is quantitative control: uniform convexity in log coordinates and the quadratic lower bound (9) connect the ratio energy to the discrete Dirichlet form (7) and motivate a stiffness interpretation for low-energy fluctuations.

Finally, our analysis is deliberately static and at zero temperature. The log-coordinate model (5) falls within the well-studied class of uniformly convex gradient (random interface) models, where existence/uniqueness of gradient Gibbs states and scaling limits have been developed in depth [2–4, 14]. Likewise, once a compact phase is introduced via a discrete quotient, the resulting phase field resembles a short-range $U(1)$-type lattice variable; at positive temperature one expects the usual questions about vortices, boundary-condition selection, and symmetry breaking to become relevant [24, 29, 31–33]. We do not pursue these extensions here. More generally, any added dynamics or auxiliary constraints (such as the "8-tick neutrality" condition discussed in Appendix **??**) should be checked explicitly for whether they preserve or break the scaling/phase symmetries identified above.

VII. ACKNOWLEDGMENT

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Appendix A: Proofs of cost properties and rigidity results

This appendix collects the short proofs referenced in the main text.

1. Nonnegativity and unique zero of J

For $x > 0$, the AM–GM inequality gives $\frac{1}{2}(x + x^{-1}) \geq \sqrt{x \cdot x^{-1}} = 1$, with equality iff $x = x^{-1}$, i.e. $x = 1$. Subtracting 1 yields $J(x) \geq 0$ with equality only at $x = 1$.

2. Log-coordinate form and convexity

Setting $t = \ln x$, one has $J(e^t) = \frac{1}{2}(e^t + e^{-t}) - 1 = \cosh t - 1$, which is strictly convex since $(\cosh t - 1)'' = \cosh t > 0$ for all t.

3. Global quadratic lower bound

Define $f(t) = \cosh t - 1 - \frac{1}{2}t^2$. Then $f(0) = f'(0) = 0$ and $f''(t) = \cosh t - 1 \geq 0$. Thus f' is nondecreasing with $f'(0) = 0$, so $f'(t) \geq 0$ for $t \geq 0$ and $f'(t) \leq 0$ for $t \leq 0$. Therefore $f(t) \geq 0$ for all t, i.e. $\cosh t - 1 \geq \frac{1}{2}t^2$.

4. Finite-volume rigidity (Result 1)

Let $G = (V, E)$ be finite and connected and let $x : V \to \mathbb{R}_{>0}$. Each term in (2) is nonnegative, so $C_G[x] \geq 0$. A constant configuration achieves $C_G = 0$, so the infimum is 0. Conversely, if $C_G[x] = 0$, then $J(x_v/x_w) = 0$ on every edge, forcing $x_v = x_w$. Connectedness implies x is constant on V.

5. Uniformity of minimizer limits on $\mathbb{Z}^3$ (Result 2)

Let $x^{(n)}$ be a global minimizer of C_{Λ_n} on $\Lambda_n = [-n, n]^3 \cap \mathbb{Z}^3$. By Result 1, each $x^{(n)}$ is constant on Λ_n. For any $v, w \in \mathbb{Z}^3$ and all n large enough that $v, w \in \Lambda_n$, we have $x_v^{(n)} = x_w^{(n)}$. Any locally convergent subsequence therefore has a constant limit on $\mathbb{Z}^3$.

6. Phase rigidity (Result 3)

Let $\Theta : V \to \mathbb{R}/\mathbb{Z}$ be a phase field on a finite connected graph G, with lift $\vartheta_v \in \mathbb{R}$. Since $\widetilde{J} \geq 0$ and $\widetilde{J}(\delta) = 0$ iff $\delta \in \mathbb{Z}$, we have $\widetilde{C}_G[\Theta] \geq 0$ with equality iff $\Theta_v = \Theta_w$ on every edge. Connectedness forces Θ to be constant.

Appendix B: Derivation of the reduced phase potential

Under the discrete identification $x \sim \varphi^n x$ with $\lambda = \ln \varphi$, the reduced potential is

$$\widetilde{J}(\delta) = \min_{n \in \mathbb{Z}} J(\varphi^{n+\delta}) = \min_{n \in \mathbb{Z}} \big(\cosh(\lambda(\delta + n)) - 1\big). \quad \text{(B1)}$$

Since $u \mapsto \cosh(\lambda u) - 1$ is even and strictly increasing on $[0, \infty)$, the minimizing n is the one that places $\delta + n$ closest to zero, giving $|\delta + n| = d_{\mathbb{Z}}(\delta)$. This yields Eq. (21).

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