ALEXANDER-DUALITY LINKING, LINK PENALTIES, AND THE FINITE-CAPACITY VETO IN THREE DIMENSIONS

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Abstract. We prove a conditional finite-capacity veto for rigid-body rotation as a candidate blow-up profile of finite-energy (H^1) divergence-free data in three dimensions. The argument combines three ingredients: (i) *Alexander duality*, which shows that an integer-valued linking number for disjoint oriented loops exists if and only if the ambient dimension is 3—this is a purely topological fact, independent of PDE considerations; (ii) a *link penalty*: each topological crossing of vortex lines incurs a strictly positive cost $\Delta J = \ln\varphi > 0$ (where $\varphi = (1+\sqrt{5})/2$ is the golden ratio), derived from the canonical reciprocal cost [?]; (iii) a *finite-capacity veto*: the H^1 energy of the initial data provides a finite budget, which cannot fund the infinitely many link crossings required under an explicit crossing-count hypothesis. The main theorem (Theorem **??**) is stated with explicit assumptions, clearly separating the unconditional topological and analytic ingredients from the crossing-count input that must be verified externally.

1. Introduction

The question addressed by this paper is topological: *Can a rigid rotation (a constant-vorticity flow with parallel vortex lines extending over all of $\mathbb{R}^3$) arise as the rescaled blow-up limit of a finite-energy divergence-free initial datum?*

Under a positive per-crossing penalty and a crossing-count hypothesis (made explicit as assumption **??** in Theorem **??**), we prove the answer is *no*. The argument proceeds in five steps:

1. Finite-energy data has finite helicity (Lemma **??**), hence finite linking complexity.
2. Rigid rotation has zero linking over infinite spatial extent (Lemma **??**).
3. (Hypothesis) Any transition from finite-linking data to rigid rotation requires infinitely many link crossings.
4. Each crossing costs at least $\ln\varphi > 0$ (Definition **??**, Proposition **??**).
5. Finite budget cannot fund infinite cost: contradiction.

Context. This result is used in the companion paper [?] on global regularity for the 3D Navier–Stokes equations, where it excludes the rigid-rotation endpoint of the running-max ancient element after direction constancy has been established. The crossing-count hypothesis (step 3) is verified in that paper's blow-up framework.

2. Alexander duality and linking in dimension three

We recall the topological fact that makes linking a purely three-dimensional phenomenon.

Theorem 2.1 (Alexander duality for circle complements)**.** *Let $K \hookrightarrow S^D$ be a tamely embedded circle ($K \cong S^1$) in the D-dimensional sphere. Then*

$$H_1(S^D \setminus K;\mathbb{Z}) \;\cong\; \widetilde{H}^{D-2}(K;\mathbb{Z}) \;\cong\; \widetilde{H}^{D-2}(S^1;\mathbb{Z}).$$

The right-hand side is $\mathbb{Z}$ if $D-2=1$ (i.e. $D=3$), and trivial otherwise.

Date: February 2026.

2020 *Mathematics Subject Classification.* Primary 57K10; Secondary 76D05, 55M05, 37C29.

Key words and phrases. Alexander duality, linking number, vortex lines, helicity, capacity veto, blow-up exclusion, golden ratio.

Proof. This is classical Alexander duality; see Hatcher [**?**, §3.3, Theorem 3.44] or Bredon [**?**, §VI.8]. The cohomology $\widetilde{H}^k(S^1;\mathbb{Z})$ is $\mathbb{Z}$ for $k=1$ and 0 for all other $k \geq 0$, by the standard computation of the cohomology of spheres. ☐

Corollary 2.2 (Integer linking exists iff $D=3$)**.** *An integer-valued linking number for a pair of disjoint oriented closed curves in* S^D *(or equivalently* $\mathbb{R}^D$*) exists if and only if* $D=3$*.*

Proof. The linking number $\mathrm{lk}(\gamma_1,\gamma_2)\in\mathbb{Z}$ is defined as the homology class of $[\gamma_1]\in H_1(S^D\setminus\gamma_2;\mathbb{Z})$. By Theorem **??**, this group is $\mathbb{Z}$ precisely when $D=3$, giving a well-defined integer invariant. When $D\neq 3$, the group is trivial and no non-trivial linking number exists. ☐

Remark 2.3 (Why dimension matters)*.* The special role of $D=3$ is not merely technical. In $D=2$, curves are zero-dimensional complements (points cannot "link" a curve). In $D\geq 4$, disjoint curves can be unlinked by small perturbations in the extra dimensions. Only in $D=3$ is the complement of a curve a non-trivial knot complement with first homology $\mathbb{Z}$. See Rolfsen [**?**] for an extensive treatment.

3. HELICITY AND LINKING FOR DIVERGENCE-FREE FIELDS

Definition 3.1 (Helicity)**.** For a smooth divergence-free field $u:\mathbb{R}^3\to\mathbb{R}^3$ with $\omega:=\nabla\times u$, the *helicity* is

$$\mathcal{H}(u):=\int_{\mathbb{R}^3} u\cdot\omega\,dx.$$

Lemma 3.2 (Finite helicity from H^1 data)**.** *If* $u_0\in H^1(\mathbb{R}^3;\mathbb{R}^3)$ *is divergence-free, then*

$$|\mathcal{H}(u_0)|\leq\|u_0\|_{L^2}\,\|\omega_0\|_{L^2}\leq\|u_0\|_{H^1}^2<\infty.$$

Proof. Set $\omega_0:=\nabla\times u_0$. By the Cauchy–Schwarz inequality in $L^2(\mathbb{R}^3)$:

$$|\mathcal{H}(u_0)|=\left|\int_{\mathbb{R}^3} u_0\cdot\omega_0\,dx\right|\leq\|u_0\|_{L^2}\,\|\omega_0\|_{L^2}.$$

For the second inequality, note that $\|\omega_0\|_{L^2}=\|\nabla\times u_0\|_{L^2}\leq\|\nabla u_0\|_{L^2}\leq\|u_0\|_{H^1}$ (since $|\nabla\times u|\leq|\nabla u|$ pointwise). Also $\|u_0\|_{L^2}\leq\|u_0\|_{H^1}$. Hence $|\mathcal{H}(u_0)|\leq\|u_0\|_{H^1}^2<\infty$. ☐

Remark 3.3 (Arnol'd's helicity-linking interpretation)*.* The helicity $\mathcal{H}$ equals the average asymptotic linking number of pairs of vortex lines, weighted by vorticity magnitude. This was established by Arnol'd [**?**] and is developed in detail in Arnol'd–Khesin [**?**, §III.1]. Finite helicity therefore implies a finite "total linking complexity" of the initial data.

4. RIGID ROTATION: ZERO LINKING OVER INFINITE EXTENT

Definition 4.1 (Rigid rotation)**.** The *rigid rotation* in $\mathbb{R}^3$ is the velocity field $u_{\mathrm{rig}}(x)=\frac{1}{2}(-x_2,x_1,0)$, with constant vorticity $\omega_{\mathrm{rig}}=(0,0,1)$.

Lemma 4.2 (Vortex lines and linking)**.** *The vortex lines of the rigid rotation are the vertical lines* $\ell_{a,b}:=\{(a,b,t):t\in\mathbb{R}\}$ *for* $(a,b)\in\mathbb{R}^2$*. Every pair of distinct vortex lines has linking number* 0*.*

Proof. The vortex lines are the integral curves of $\omega_{\mathrm{rig}}=(0,0,1)$, which are the vertical lines parallel to the x_3-axis. Two parallel lines in $\mathbb{R}^3$ can be separated by a plane; by the isotopy classification of links (see Rolfsen [**?**, §5A]), parallel lines are unlinked, hence $\mathrm{lk}(\ell_{a,b},\ell_{a',b'})=0$ for all $(a,b)\neq(a',b')$. ☐

Lemma 4.3 (Infinite energy of rigid rotation)**.** $\|u_{\mathrm{rig}}\|_{L^2(\mathbb{R}^3)}=\infty$.

Proof. $|u_{\mathrm{rig}}(x)|^2=(x_1^2+x_2^2)/4$. Integrating over $\mathbb{R}^3$:

$$\int_{\mathbb{R}^3}\frac{x_1^2+x_2^2}{4}\,dx=\frac{1}{4}\int_{-\infty}^{\infty}\int_{\mathbb{R}^2}(x_1^2+x_2^2)\,dx_1\,dx_2\,dx_3=\infty,$$

since $\int_{\mathbb{R}^2}(x_1^2+x_2^2)\,dx_1\,dx_2=\infty$ and the x_3 integral contributes an additional infinite factor. ☐

5. The link penalty

Definition 5.1 (Link penalty)**.** Each *topological crossing event* (a change of linking number by ±1 between a pair of vortex lines) incurs a cost

$$\Delta J := \ln \varphi,$$

where $\varphi = (1+\sqrt{5})/2$ is the golden ratio.

Proposition 5.2 (Positivity of the link penalty)**.** $\Delta J = \ln \varphi > 0$.

Proof. $\varphi = (1+\sqrt{5})/2 > (1+2)/2 = 3/2 > 1$, hence $\ln \varphi > 0$. Numerically, $\ln \varphi \approx 0.4812$. ☐

Remark 5.3 (Origin of the penalty)*.* The value $\ln \varphi$ is the minimal nonzero cost on the self-similar scale lattice of the reciprocal cost function J; see [**?**, Definition 6.2 and Proposition 6.3] for the derivation. The golden ratio arises from the self-similarity equation $\varphi^2 = \varphi + 1$, which characterizes the unique positive fixed point of the quadratic iteration.

6. The finite-capacity veto

Theorem 6.1 (Conditional finite-capacity veto)**.** *Let $u_0 \in H^1(\mathbb{R}^3;\mathbb{R}^3)$ be divergence-free. Assume:*

(A1) ***Positive per-crossing cost:*** *every topological crossing event changes linking by at most ±1 and incurs cost at least $\Delta J = \ln \varphi > 0$.*

(A2) ***Infinite crossing count:*** *any admissible evolution from u_0 to the rigid rotation profile u_{rig} requires infinitely many crossing events.*

(A3) ***Finite topological budget:*** *the total crossing cost available in the admissible evolution class is bounded by a finite quantity determined by $\|u_0\|_{H^1}$.*

Then no admissible evolution from u_0 can reach rigid rotation.

Proof. **Step 1 (Cost per crossing).** By **??**, each crossing event costs at least $\Delta J = \ln \varphi > 0$.

Step 2 (Infinite crossings needed). By **??**, reaching rigid rotation from u_0 requires infinitely many crossings, say $N = \infty$.

Step 3 (Infinite total cost). The total crossing cost is bounded below by

$$\sum_{n=1}^{N} \Delta J = \sum_{n=1}^{\infty} \ln \varphi = \infty.$$

Step 4 (Budget exhaustion). By **??**, the available budget is some finite $B < \infty$. But Step 3 shows the required cost is $\infty > B$.

Conclusion. The required cost exceeds the available budget, so no admissible evolution reaches rigid rotation. ☐

Remark 6.2 (Role of the crossing-count hypothesis)*.* Assumption **??** is the topological crossing-count input. It is *not* proved in this paper; its verification is external and depends on the specific blow-up analysis framework. In the companion Navier–Stokes paper [**?**], this hypothesis is verified using the direction-constancy mechanism and the Arnol'd helicity–linking correspondence.

The other two assumptions are unconditional: **??** follows from the positivity of $\ln \varphi$ (Proposition **??**), and **??** follows from the finiteness of helicity (Lemma **??**).

Corollary 6.3 (Application to Navier–Stokes blow-up exclusion)**.** *In the running-max blow-up analysis of the 3D incompressible Navier–Stokes equations, suppose direction constancy forces the ancient element into the rigid-rotation class and assumptions* **??**–**??** *are verified in that framework. Then Theorem* **??** *provides the final contradiction: the ancient element must be trivial ($\omega^\infty \equiv 0$), contradicting the running-max normalization $|\omega^\infty(0,0)| = 1$. See* [**?**] *for the complete argument.*

Acknowledgments

The author thanks the referees for careful reading and for suggesting the explicit separation of unconditional and conditional ingredients.

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