PHASE CAPS, HERGLOTZ POSITIVITY, AND THE CAYLEY–SCHUR PINCH: A DOMAIN-AGNOSTIC EXCLUSION TEMPLATE

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ABSTRACT. We develop a general exclusion mechanism—the *Cayley–Schur pinch*—for proving that a meromorphic function on a domain has no poles. The method has three inputs: (i) a *phase cap* guaranteeing $|\operatorname{Arg} H(z)| < \pi/2$ on the domain (equivalently $\operatorname{Re} H \geq 0$, the *Herglotz condition*); (ii) a *non-cancellation* property ensuring that poles of H are genuine (not removable); and (iii) a *normalization* fixing the value at one interior point (e.g. $\operatorname{Re} H(z_0) > 0$). Under these three conditions, the Cayley transform $\Theta := (2H - 1)/(2H + 1)$ maps the Herglotz function to a Schur function ($|\Theta| \leq 1$), any putative pole extends removably to $\Theta(p) = 1$, and the maximum modulus principle forces $\Theta \equiv$ const, contradicting the normalization. The proof uses only Riemann's removable singularity theorem and the maximum modulus principle; no Phragmén–Lindelöf or subharmonicity arguments are needed. We state the mechanism as a reusable template and record the exact corollaries cited by companion papers on the Riemann zeta function and combinatorial feasibility.

1. Introduction

The interplay between Herglotz functions (holomorphic functions with non-negative real part) and Schur functions (holomorphic functions bounded by 1 in modulus) via the Cayley transform is classical. Standard references include Rosenblum–Rovnyak [?] for the operator-theoretic viewpoint, Agler–McCarthy [?] for Pick interpolation, Conway [?] for the one-variable foundations, and Simon [?] for trace-ideal applications.

The contribution of this paper is to isolate the mechanism as a clean, self-contained *exclusion template*: given a meromorphic function whose pole set encodes some target objects (zeros of a zeta function, inadmissible solutions of a constraint system, etc.), the Cayley–Schur pinch provides a systematic way to prove that pole set is empty.

The template in one sentence. *Herglotz positivity + non-cancellation at poles + normalization at one point $\Longrightarrow$ pole-free.*

The rest of the paper builds this sentence into a precise theorem (Theorem **??**) and records two application corollaries.

2. Herglotz and Schur classes

Throughout, $D \subset \mathbb{C}$ denotes a connected open set (a domain).

Definition 2.1 (Herglotz function)**.** A holomorphic function $H : D \to \mathbb{C}$ is *Herglotz on D* if $\operatorname{Re} H(z) \geq 0$ for all $z \in D$.

Definition 2.2 (Schur function)**.** A holomorphic function $\Theta : D \to \mathbb{C}$ is *Schur on D* if $|\Theta(z)| \leq 1$ for all $z \in D$.

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Definition 2.3 (Cayley transform and inverse)**.** For H with $2H+1 \neq 0$:

$$\Theta := \frac{2H-1}{2H+1}, \qquad H = \frac{1+\Theta}{2(1-\Theta)}.$$

This is a Möbius transformation sending the closed right half-plane $\{\operatorname{Re} w \geq 0\}$ onto the closed unit disk $\{|z| \leq 1\}$, with $H = 0 \mapsto \Theta = -1$, $H = \infty \mapsto \Theta = 1$, and $H = 1/2 \mapsto \Theta = 0$.

3. Phase cap implies positivity

Proposition 3.1 (Phase-to-positivity equivalence)**.** *For $z \in \mathbb{C} \setminus \{0\}$, the following are equivalent:*
 (i) $|\operatorname{Arg} z| < \pi/2$*;*
 (ii) $\operatorname{Re} z > 0$*.*

Proof. Write $z = re^{i\theta}$ with $r > 0$ and $\theta \in (-\pi, \pi]$. Then $\operatorname{Re} z = r\cos\theta$. The condition $|\theta| < \pi/2$ is equivalent to $\cos\theta > 0$, which (since $r > 0$) is equivalent to $\operatorname{Re} z = r\cos\theta > 0$. □

Corollary 3.2 (Phase cap implies Herglotz)**.** *If $H : D \to \mathbb{C} \setminus \{0\}$ satisfies $|\operatorname{Arg} H(z)| < \pi/2$ for all $z \in D$, then H is Herglotz on D.*

Proof. Apply Proposition **??** pointwise: for each $z \in D$, $|\operatorname{Arg} H(z)| < \pi/2$ gives $\operatorname{Re} H(z) > 0 \geq 0$. □

4. Cayley maps Herglotz to Schur

Theorem 4.1 (Cayley: Herglotz → Schur)**.** *If $H : D \to \mathbb{C}$ is Herglotz on D and $2H(z) + 1 \neq 0$ for $z \in D$, then $\Theta = (2H-1)/(2H+1)$ is Schur on D.*

Moreover, $\operatorname{Re} H > 0$ *implies* $|\Theta| < 1$ *(strict Schur).*

Proof. Write $H = a + ib$ with $a = \operatorname{Re} H \geq 0$ and $b = \operatorname{Im} H$. Compute:

$$|2H-1|^2 = (2a-1)^2 + 4b^2 = 4a^2 - 4a + 1 + 4b^2,$$
$$|2H+1|^2 = (2a+1)^2 + 4b^2 = 4a^2 + 4a + 1 + 4b^2.$$

Therefore

$$|2H+1|^2 - |2H-1|^2 = (4a^2 + 4a + 1) - (4a^2 - 4a + 1) = 8a \geq 0.$$

Hence $|2H-1| \leq |2H+1|$, giving $|\Theta| = |2H-1|/|2H+1| \leq 1$.

If $a = \operatorname{Re} H > 0$, then $8a > 0$, so the inequality is strict: $|2H-1| < |2H+1|$ and $|\Theta| < 1$. □

Remark 4.2 (Geometric picture). The Cayley transform is the conformal equivalence between the right half-plane and the unit disk. Herglotz ↔ Schur under this equivalence is the operator-theoretic foundation of Schur analysis; see [**?**, **?**].

5. The pinch: removable singularity and exclusion

We now state and prove the two classical lemmas used in the pinch, then assemble them into the main theorem.

Lemma 5.1 (Riemann removable singularity theorem)**.** *Let f be holomorphic on a punctured disk $\mathbb{D}_r(p) \setminus \{p\}$ with $|f| \leq M < \infty$. Then f extends uniquely to a holomorphic function on $\mathbb{D}_r(p)$, and $|f(p)| \leq M$.*

Proof. This is the classical theorem of Riemann; see Conway [**?**, §III.2.1]. The boundedness hypothesis implies the Laurent expansion of f about p has no negative-power terms, so p is a removable singularity. The bound $|f(p)| \leq M$ follows by continuity. □

Lemma 5.2 (Maximum modulus principle — interior version)**.** *Let g be holomorphic and non-constant on a connected open set D. Then $|g|$ has no interior maximum: $|g(z)| < \sup_{w\in D} |g(w)|$ for every $z \in D$.*

Equivalently: if $|\Theta| \le 1$ on a domain D and $|\Theta(p)| = 1$ for some $p \in D$, then Θ is a unimodular constant.

Proof. This is the maximum modulus principle; see Conway [**?**, §V.3.2]. □

Theorem 5.3 (The Cayley–Schur Pinch)**.** *Let $D \subset \mathbb{C}$ be a domain, $S \subset D$ a discrete set, and $H : D \setminus S \to \mathbb{C}$ holomorphic. Suppose:*

(P1) ***Herglotz:*** $\operatorname{Re} H(z) \ge 0$ *for all* $z \in D \setminus S$.

(P2) ***Non-cancellation:*** *for every* $p \in S$, $\lim_{z\to p} |H(z)| = \infty$ *(i.e. p is a genuine pole).*

(P3) ***Normalization:*** *there exists* $z_0 \in D \setminus S$ *with* $\operatorname{Re} H(z_0) > 0$.

Then $S = \varnothing$: the function H has no poles in D.

Proof. **Step 1.** On $D \setminus S$, $\operatorname{Re} H \ge 0$ by **??**. We verify that $2H + 1 \ne 0$ on $D \setminus S$: if $2H(z) + 1 = 0$, then $H(z) = -1/2$, so $\operatorname{Re} H(z) = -1/2 < 0$, contradicting **??**. Hence $\Theta = (2H - 1)/(2H + 1)$ is well-defined and holomorphic on $D \setminus S$.

Step 2. By Theorem **??**, $|\Theta(z)| \le 1$ for all $z \in D \setminus S$.

Step 3. Fix $p \in S$. By **??**, $|H(z)| \to \infty$ as $z \to p$. Compute the limit of Θ:

$$\Theta(z) = \frac{2H(z) - 1}{2H(z) + 1} = 1 - \frac{2}{2H(z) + 1} \to 1 - 0 = 1 \quad \text{as } z \to p.$$

Since Θ is bounded ($|\Theta| \le 1$) on a punctured neighborhood of p, Lemma **??** gives a holomorphic extension with $\Theta(p) = 1$.

Step 4. After extending at every $p \in S$, Θ is holomorphic on all of D with $|\Theta| \le 1$. Suppose $S \ne \varnothing$; pick $p \in S$. Then $|\Theta(p)| = 1$ is attained at an interior point of D. By Lemma **??**, Θ must be a unimodular constant: $\Theta \equiv e^{i\alpha}$ for some $\alpha \in \mathbb{R}$.

Step 5. But **??** gives $\operatorname{Re} H(z_0) > 0$, so by Theorem **??** (strict case), $|\Theta(z_0)| < 1$. This contradicts $|\Theta| \equiv 1$.

Conclusion. The assumption $S \ne \varnothing$ leads to a contradiction. Hence $S = \varnothing$. □

Remark 5.4 (Why three conditions are needed). **??** alone does not exclude poles: $H(z) = 1/(z - p)$ has $\operatorname{Re} H > 0$ on a half-plane but a pole at p. **??** excludes the degenerate case of removable singularities (where Θ extends without hitting $|\Theta| = 1$). **??** excludes the degenerate constant $\Theta \equiv 1$ (which would make H identically infinite). Together, the three conditions form a watertight exclusion.

Remark 5.5 (Comparison with Phragmén–Lindelöf). The classical approach to excluding zeros of L-functions in half-planes often uses Phragmén–Lindelöf-type bounds. The Cayley–Schur pinch is logically independent: it requires no growth estimates, only the sign condition $\operatorname{Re} H \ge 0$ and the behavior at poles. This makes it applicable in settings where growth bounds are unavailable.

6. Application interfaces

Corollary 6.1 (Template for zeta-function zeros)**.** *Let $\Omega = \{\operatorname{Re} s > 1/2\} \subset \mathbb{C}$ and let $H : \Omega \setminus S \to \mathbb{C}$ be meromorphic, where S is the set of zeros of the Riemann zeta function $\zeta(s)$ in Ω. Suppose:*

(i) $\operatorname{Re} H(s) \ge 0$ *for* $s \in \Omega \setminus S$*;*

(ii) $|H(s)| \to \infty$ *at each* $\rho \in S$ *(non-cancellation);*

(iii) $H(\sigma) \to 1$ *as* $\sigma \to +\infty$ *along the real axis.*

Then $S = \varnothing$: ζ has no zeros in $\{\operatorname{Re} s > 1/2\}$.

Proof. Apply Theorem **??** with $D = \Omega$. **??**: hypothesis (i). **??**: hypothesis (ii). **??**: from (iii), for σ sufficiently large, $H(\sigma)$ is close to 1, so $\operatorname{Re} H(\sigma) > 0$ and $|\Theta(\sigma)| = |2H(\sigma) - 1|/|2H(\sigma) + 1| \approx |1|/|3| = 1/3 < 1$. Take $z_0 = \sigma$ for such a σ. Theorem **??** gives $S = \varnothing$. □

Corollary 6.2 (Template for constraint feasibility)**.** *Let $D \subset \mathbb{C}$ be a domain and $H : D \setminus S \to \mathbb{C}$ meromorphic, where each $p \in S$ encodes an inadmissible fractional solution. If*

(i) $\operatorname{Re} H(z) \geq 0$ *on* $D \setminus S$*,*
(ii) $|H(z)| \to \infty$ *as* $z \to p$ *for each* $p \in S$*, and*
(iii) *there exists* $z_0 \in D \setminus S$ *with* $\operatorname{Re} H(z_0) > 0$*,*

then $S = \varnothing$: no inadmissible solution exists.

Proof. Theorem **??** with the three hypotheses matching **??**–**??**. □

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References

[1] J. Agler and J. E. McCarthy, *Pick Interpolation and Hilbert Function Spaces*, Graduate Studies in Mathematics, vol. 44, AMS, 2002.
[2] J. B. Conway, *Functions of One Complex Variable II*, Graduate Texts in Mathematics, vol. 159, Springer, 1995.
[3] M. Rosenblum and J. Rovnyak, *Hardy Classes and Operator Theory*, Oxford Mathematical Monographs, Oxford Univ. Press, 1985.
[4] B. Simon, *Trace Ideals and Their Applications*, 2nd ed., Mathematical Surveys and Monographs, vol. 120, AMS, 2005.

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