

# WINDOW NEUTRALITY, FUTURE-BALANCE PROJECTION, AND THE PHANTOM BALANCE CONSTRAINT

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ABSTRACT. We study integer-valued signals on  $\mathbb{Z}$  subject to a *window neutrality constraint*: the sum over every aligned window of width  $W$  must vanish. (We fix  $W = 8$  throughout, though the results hold for any  $W \geq 2$ .) We prove three groups of results. First, any nonzero contribution at a single tick forces a compensating balance over the remaining ticks in its window (the *future-balance theorem*), with an explicit backward projection bound  $|d| \leq B \cdot R$  relating the accumulated debt  $d$ , the remaining ticks  $R$ , and the per-tick bound  $B$ . Second, we formalize the *boundary-balance rule* (the last tick of each window absorbs the residual) and prove it implies full window neutrality by a telescoping argument. Third, we prove that every 8-tick neutral schedule admits a unique phase-correcting rotation offset that re-aligns it to any desired boundary phase. These results provide the windowed-cancellation and phantom-debt tools used in companion papers on arithmetic phase bounds [?] and gauge-field defect annihilation.

## 1. SETUP AND DEFINITIONS

Throughout, we fix the window width  $W := 8$ . (Every result below holds with  $W$  replaced by any integer  $\geq 2$ ; only the specific numerics change.)

**Definition 1.1** (Window signal). A *window signal* is a function  $w : \{0, 1, \dots, W-1\} \rightarrow \mathbb{Z}$ . We say  $w$  is *neutral* if  $\sum_{k=0}^{W-1} w(k) = 0$ .

**Definition 1.2** (Lock event). A *lock event* at position  $p \in \{0, \dots, W-1\}$  with contribution  $\delta \in \mathbb{Z} \setminus \{0\}$  is an assignment  $w(p) = \delta$  for one tick of the window.

**Definition 1.3** (Balance debt and remaining ticks). Given a partial signal with contributions at positions  $0, \dots, k$  ( $0 \leq k \leq W-1$ ), the *balance debt* is  $d := \sum_{j=0}^k w(j)$  and the *remaining ticks* is  $R := W-1-k$ .

## 2. THE FUTURE-BALANCE THEOREM

The first result says that in a neutral window, any single tick's contribution determines an equal-and-opposite obligation on the rest.

**Theorem 2.1** (Lock forces future balance). *Let  $w$  be a neutral window signal. If  $w(p) = \delta$  for some  $p \in \{0, \dots, W-1\}$  with  $\delta \neq 0$ , then*

$$\sum_{k \neq p} w(k) = -\delta.$$

*Proof.* By neutrality,  $\sum_{k=0}^{W-1} w(k) = 0$ . Split the sum into  $w(p)$  and the rest:  $w(p) + \sum_{k \neq p} w(k) = 0$ . Hence  $\sum_{k \neq p} w(k) = -w(p) = -\delta$ .  $\square$

**Corollary 2.2** (Nonzero contribution creates debt). *After a lock at position  $p$  with value  $\delta \neq 0$ , the remaining  $R = W-1-p$  ticks must collectively contribute  $-\delta$  to achieve window neutrality.*

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*Proof.* By Theorem ??, the total over all other positions is  $-\delta$ . Positions  $0, \dots, p-1$  are already fixed (their values are determined). Hence the unfilled positions  $p+1, \dots, W-1$  (there are  $R = W-1-p$  of them) must supply the remaining  $-\delta$  minus whatever the earlier positions contributed. In the worst case (earlier sum zero), the full obligation  $-\delta$  falls on the remaining  $R$  ticks.  $\square$

**Example 2.3.** With  $W = 8$  and a lock  $w(2) = 5$ , positions 3, 4, 5, 6, 7 (five remaining ticks) must collectively contribute  $-5$ .

### 3. PHANTOM MAGNITUDE AND BACKWARD PROJECTION

We now quantify how “urgent” the outstanding debt is.

**Definition 3.1** (Phantom magnitude). For balance debt  $d \in \mathbb{Z}$  and remaining ticks  $R \geq 0$ :

$$\Phi(d, R) := \frac{|d|}{R+1}.$$

(The denominator  $R+1$  rather than  $R$  avoids division by zero when  $R = 0$  and captures the interpretation that  $R+1$  is the total number of future decisions, including the current one.)

**Proposition 3.2** (Phantom properties). (a)  $\Phi(d, R) \geq 0$  for all  $d, R$ .

(b)  $\Phi$  is monotone increasing in  $|d|$  for fixed  $R$ .

(c)  $\Phi$  is monotone decreasing in  $R$  for fixed  $d \neq 0$ .

(d)  $\Phi(0, R) = 0$ : zero debt implies zero phantom.

(e)  $\Phi(d, 0) = |d|$ : when no ticks remain, the phantom equals the full outstanding debt.

*Proof.* All five properties follow directly from the formula  $\Phi(d, R) = |d|/(R+1)$  and the fact that  $R+1 \geq 1 > 0$ .  $\square$

**Theorem 3.3** (Backward projection principle). Suppose each of the  $R$  remaining positions is bounded in magnitude by  $B \geq 0$  (i.e.  $|w(k)| \leq B$  for each remaining position  $k$ ). Then any feasible completion satisfying window neutrality requires

$$|d| \leq B \cdot R.$$

Equivalently: if  $|d| > BR$ , no feasible completion exists.

*Proof.* Neutrality requires  $\sum_{\text{remaining}} w(k) = -d$ . By the triangle inequality:

$$|d| = \left| \sum_{\text{remaining}} w(k) \right| \leq \sum_{\text{remaining}} |w(k)| \leq B \cdot R. \quad \square$$

**Corollary 3.4** (Urgency bound). Under the hypotheses of Theorem ??,  $\Phi(d, R) \leq BR/(R+1) < B$ .

*Proof.* By Theorem ??,  $|d| \leq BR$ . Hence  $\Phi(d, R) = |d|/(R+1) \leq BR/(R+1)$ . Since  $R/(R+1) < 1$  for all  $R \geq 0$ , we get  $\Phi(d, R) < B$ .  $\square$

*Remark 3.5* (Interpretation). The urgency bound says the average debt per remaining decision is strictly less than the per-tick capacity. This means there is always “room” to service the debt—the phantom never exceeds the local budget.

### 4. SCHEDULE NEUTRALITY AND PHASE-CORRECTING ROTATION

We now extend from a single window to an infinite schedule on  $\mathbb{Z}$ .

**Definition 4.1** (Aligned  $W$ -tick schedule). An integer signal  $s : \mathbb{Z} \rightarrow \mathbb{Z}$  satisfies  $W$ -tick neutrality if  $\sum_{k=0}^{W-1} s(t_0 + k) = 0$  for every  $t_0 \equiv 0 \pmod{W}$ .

**Definition 4.2** (Window index process). Fix an initial phase  $\iota_0 \in \{0, \dots, W-1\}$ . Define  $\text{idx} : \mathbb{Z}_{\geq 0} \rightarrow \{0, \dots, W-1\}$  by

$$\text{idx}(0) := \iota_0, \quad \text{idx}(n+1) := (\text{idx}(n) + 1) \bmod W.$$

This is a deterministic cyclic counter with period  $W$ .

**Theorem 4.3** (Window index tracking). *For the process in Definition ??,  $\text{idx}(n) = (\iota_0 + n) \bmod W$  for every  $n \geq 0$ .*

*Proof.* By induction on  $n$ . *Base case:*  $\text{idx}(0) = \iota_0 = (\iota_0 + 0) \bmod W$ .  $\checkmark$

*Inductive step:* assume  $\text{idx}(n) = (\iota_0 + n) \bmod W$ . Then by Definition ??:

$$\text{idx}(n+1) = (\text{idx}(n) + 1) \bmod W = ((\iota_0 + n) \bmod W + 1) \bmod W = (\iota_0 + n + 1) \bmod W. \quad \square$$

**Definition 4.4** (Boundary-balance rule). An integer signal  $s : \mathbb{Z} \rightarrow \mathbb{Z}$  satisfies the *boundary-balance rule* if for every block index  $j \geq 0$ :

$$s(Wj + W-1) = - \sum_{k=0}^{W-2} s(Wj + k).$$

That is, the last tick of each aligned block absorbs the negative of the partial sum of the preceding  $W-1$  ticks.

**Theorem 4.5** (Neutrality from boundary balance). *If  $s$  satisfies the boundary-balance rule, then  $s$  satisfies  $W$ -tick neutrality: for every  $j \geq 0$ ,*

$$\sum_{k=0}^{W-1} s(Wj + k) = 0.$$

*Proof.* Fix  $j \geq 0$ . Split the block sum:

$$\sum_{k=0}^{W-1} s(Wj + k) = \underbrace{\sum_{k=0}^{W-2} s(Wj + k)}_{=:P} + s(Wj + W-1).$$

By the boundary-balance rule,  $s(Wj + W-1) = -P$ . Hence the total is  $P + (-P) = 0$ .  $\square$

**Theorem 4.6** (Schedule neutrality rotation). *Let  $s$  satisfy  $W$ -tick neutrality. Then there exists a unique offset  $r \in \{0, \dots, W-1\}$  such that for every  $k \geq 0$ :*

$$\sum_{j=0}^{W-1} s(r + Wk + j) = 0 \quad \text{and} \quad \text{idx}(r + Wk) = 0.$$

*In particular, after reindexing by the offset  $r$ , neutrality holds on every aligned block. The offset is  $r = (W - \iota_0) \bmod W$ .*

*Proof.* Set  $r := (W - \iota_0) \bmod W$ . By Theorem ??,  $\text{idx}(r) = (\iota_0 + r) \bmod W = (\iota_0 + W - \iota_0) \bmod W = 0$ . Since  $\text{idx}(r) = 0$ , the  $W$  consecutive ticks  $r, r+1, \dots, r+W-1$  form a complete aligned block starting at a multiple of  $W$  in the reindexed coordinate. By  $W$ -tick neutrality, their sum is zero. Repeating with  $r + Wk$  for every  $k \geq 0$  gives the full statement.

Uniqueness: if  $r'$  also satisfies  $\text{idx}(r') = 0$ , then  $r' \equiv r \pmod{W}$ , so  $r' = r$  in  $\{0, \dots, W-1\}$ .  $\square$

## 5. PHANTOM COST AUGMENTATION

When the window neutrality constraint is composed with a cost functional (such as the  $J$ -cost of [?]), the phantom balance acts as an additive penalty that lifts the effective cost.

**Definition 5.1** (Augmented cost). For a configuration  $x > 0$ , phantom data  $(d, R)$ , and penalty scale  $\lambda \geq 0$ :

$$J_{\text{PL}}(x; d, R) := J(x) + \lambda \cdot \Phi(d, R).$$

**Proposition 5.2** (Augmented cost dominates base cost).  $J_{\text{PL}}(x; d, R) \geq J(x)$  for all  $x, d, R, \lambda \geq 0$ . Equality holds iff  $d = 0$  or  $\lambda = 0$ .

*Proof.*  $\Phi(d, R) = |d|/(R+1) \geq 0$  and  $\lambda \geq 0$ , so  $\lambda\Phi \geq 0$ . If  $d \neq 0$  and  $\lambda > 0$ , then  $\lambda\Phi > 0$  and the inequality is strict.  $\square$

*Remark 5.3* (Phase-winding obstruction). In applications to arithmetic phase dynamics (see [?]), the phantom mechanism forbids sustained one-sided phase winding: any phase debt accumulated at tick  $k$  must be balanced by tick  $k+W-1$ . The backward projection bound (Theorem ??) quantifies the maximum sustainable debt at each intermediate tick. Together, the future-balance theorem and backward projection provide a discrete analog of the no-drift property: the windowed neutrality constraint prevents unbounded accumulation.

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## REFERENCES

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