TOPOLOGICAL FRUSTRATION AND RESIDUAL COST: EXACT LOCAL MINIMIZERS FOR RECIPROCAL BOND GEOMETRY

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ABSTRACT. Let $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ for $x > 0$. For fixed positive neighbor data $n_1, \dots, n_d$, we study the local bond cost $\mathcal{C}(v) = \sum_{i=1}^{d} J(v/n_i)$. We prove an exact closed form for the unique minimizer:

$$v^\star = \sqrt{\frac{\sum_i n_i}{\sum_i n_i^{-1}}} =: \mathbb{JM}(n_1, \dots, n_d),$$

and an exact minimal residual $\min_{v>0} \mathcal{C}(v) = \sqrt{(\sum_i n_i)(\sum_i n_i^{-1})} - d$. This residual is strictly positive unless all neighbors are equal (by Cauchy–Schwarz). For two incompatible neighbors $n_1 \neq n_2$, we obtain the explicit frustration formula $\min_{v>0}(J(v/n_1) + J(v/n_2)) = \sqrt{n_1/n_2} + \sqrt{n_2/n_1} - 2 > 0$. At graph level, if local neighborhoods produce different J-means at two vertices, then any simultaneous local minimizer field is necessarily non-uniform. We call this forced non-uniformity *topological frustration* and prove it in the precise form: the optimizing field at each node is uniquely determined by its neighborhood (strict convexity), so distinct neighborhoods force distinct node values. We record application-interface statements for fluid-direction smoothing, constraint relaxation, and lattice embedding.

1. Introduction

The reciprocal cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ is nonnegative, reciprocal-symmetric ($J(x) = J(1/x)$), and strictly convex on $(0, \infty)$. These one-variable facts are developed in detail in the companion paper [?], where it is shown that J is the unique such function satisfying the d'Alembert functional identity with unit curvature.

This paper addresses the *multi-bond local problem*: given fixed bond neighbor values $n_1, \dots, n_d > 0$, what value $v > 0$ minimizes $\sum_{i=1}^{d} J(v/n_i)$? The answer is an explicit closed form. This immediately yields precise frustration inequalities: when neighbors disagree, the residual cost is provably positive, with an exact formula.

The paper is organized as follows. Section **??** derives the closed form and the J-mean. Section **??** proves the strict frustration inequalities with explicit lower bounds. Section **??** lifts the local result to a graph-level non-collapse theorem. Section **??** records application interfaces used by companion papers.

2. Local optimization against fixed neighbors

Definition 2.1 (Local bond cost)**.** For fixed $n_1, \dots, n_d > 0$ ($d \geq 1$), define

$$\mathcal{C}(v) := \sum_{i=1}^{d} J(v/n_i), \qquad v > 0.$$

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Lemma 2.2 (Closed form)**.** *With* $S_1 := \sum_{i=1}^{d} n_i$ *and* $S_{-1} := \sum_{i=1}^{d} n_i^{-1}$*,*

$$\mathcal{C}(v) = \frac{1}{2}\left(S_{-1}\, v + \frac{S_1}{v}\right) - d. \tag{1}$$

Proof. Each summand expands as $J(v/n_i) = \frac{1}{2}(v/n_i + n_i/v) - 1$. Summing over i:

$$\mathcal{C}(v) = \frac{v}{2}\sum_i \frac{1}{n_i} + \frac{1}{2v}\sum_i n_i - d = \frac{1}{2}\left(S_{-1}\, v + \frac{S_1}{v}\right) - d. \qquad \square$$

Theorem 2.3 (Exact minimizer and strict convexity)**.** $\mathcal{C}$ *is strictly convex on* $(0,\infty)$ *with unique minimizer*

$$v^\star = \sqrt{\frac{S_1}{S_{-1}}} = \sqrt{\frac{\sum_i n_i}{\sum_i n_i^{-1}}}. \tag{2}$$

Proof. Differentiate (**??**):

$$\mathcal{C}'(v) = \frac{1}{2}\left(S_{-1} - \frac{S_1}{v^2}\right), \qquad \mathcal{C}''(v) = \frac{S_1}{v^3} > 0.$$

Strict positivity of $\mathcal{C}''$ gives strict convexity. Since $\mathcal{C}(v) \to +\infty$ as $v \to 0^+$ or $v \to +\infty$ (from (**??**)), $\mathcal{C}$ is coercive. A strictly convex coercive function has exactly one critical point, which is the global minimum. Setting $\mathcal{C}'(v) = 0$: $S_{-1} = S_1/v^2 \implies v^2 = S_1/S_{-1} \implies v^\star = \sqrt{S_1/S_{-1}}$. $\square$

Definition 2.4 (J-mean)**.** For positive data $n_1, \dots, n_d$, define the *J-mean* as

$$\mathbb{JM}(n_1, \dots, n_d) := \sqrt{\frac{\sum_i n_i}{\sum_i n_i^{-1}}}.$$

Corollary 2.5 (AM/HM interpretation)**.** *With* $\mathrm{AM} := \frac{1}{d}\sum_i n_i$ *and* $\mathrm{HM} := d/\sum_i n_i^{-1}$*,*

$$\mathbb{JM}(n_1, \dots, n_d) = \sqrt{\mathrm{AM}\cdot\mathrm{HM}}.$$

For $d = 2$: $\mathbb{JM}(n_1, n_2) = \sqrt{n_1 n_2}$ (the geometric mean).

Proof. $S_1/S_{-1} = d\cdot\mathrm{AM}\cdot\mathrm{HM}/d = \mathrm{AM}\cdot\mathrm{HM}$. For $d = 2$: $(n_1 + n_2)/(n_1^{-1} + n_2^{-1}) = (n_1 + n_2)\cdot n_1 n_2/(n_1 + n_2) = n_1 n_2$. $\square$

Remark 2.6 (Relation to other means). The J-mean satisfies $\mathrm{HM} \le \mathbb{JM} \le \mathrm{AM}$ by the AM-HM inequality, with equality throughout iff all n_i are equal. It coincides with the geometric mean for $d = 2$ but differs for $d \ge 3$ (for instance, $\mathbb{JM}(1,1,4) = \sqrt{6/(1+1+1/4)} = \sqrt{6/(9/4)} = \sqrt{8/3} \approx 1.633$, while $\mathrm{GM}(1,1,4) = 4^{1/3} \approx 1.587$). See Bullen [**?**] for a comprehensive discussion of mean inequalities.

Proposition 2.7 (Exact minimum value)**.**

$$\min_{v>0} \mathcal{C}(v) = \sqrt{S_1 S_{-1}} - d \ge 0.$$

Equality holds iff $n_1 = \cdots = n_d$.

Proof. Evaluate (**??**) at $v^\star = \sqrt{S_1/S_{-1}}$:

$$\begin{aligned}\mathcal{C}(v^\star) &= \frac{1}{2}\left(S_{-1}\sqrt{S_1/S_{-1}} + \frac{S_1}{\sqrt{S_1/S_{-1}}}\right) - d \\ &= \frac{1}{2}(\sqrt{S_1 S_{-1}} + \sqrt{S_1 S_{-1}}) - d = \sqrt{S_1 S_{-1}} - d.\end{aligned}$$

By the Cauchy–Schwarz inequality in $\mathbb{R}^d$ (applied to the vectors $(n_1^{1/2}, \dots, n_d^{1/2})$ and $(n_1^{-1/2}, \dots, n_d^{-1/2})$):

$$S_1 S_{-1} = \Big(\sum_i n_i\Big)\Big(\sum_i n_i^{-1}\Big) \ge \Big(\sum_i 1\Big)^2 = d^2.$$

Hence $\sqrt{S_1 S_{-1}} \geq d$, giving $\mathcal{C}(v^\star) \geq 0$. Equality in Cauchy–Schwarz occurs iff $n_i^{1/2}$ and $n_i^{-1/2}$ are proportional for all i, i.e. iff all n_i are equal. □

Example 2.8. For $d = 3$, $n_1 = 1$, $n_2 = 2$, $n_3 = 8$: $S_1 = 11$, $S_{-1} = 1 + 1/2 + 1/8 = 13/8$, $v^\star = \sqrt{88/13} \approx 2.602$, $\min \mathcal{C} = \sqrt{143/8} - 3 \approx 1.227$.

3. Strict frustration inequalities

The central message of this section: when neighbor data disagree, no choice of v can bring the total bond cost to zero.

Theorem 3.1 (No simultaneous annihilation)**.** *If $n_1, n_2 > 0$ and $n_1 \neq n_2$, there is no $v > 0$ with $J(v/n_1) = J(v/n_2) = 0$.*

Proof. By [?, Corollary 2.5], $J(v/n_i) = 0 \iff v = n_i$. Simultaneous vanishing forces $v = n_1$ and $v = n_2$, contradicting $n_1 \neq n_2$. □

Corollary 3.2 (Strict two-bond frustration)**.** *If $n_1 \neq n_2$ ($n_1, n_2 > 0$), then for every $v > 0$: $J(v/n_1) + J(v/n_2) > 0$.*

Proof. Each summand is ≥ 0 (since $J \geq 0$). If the sum were zero, both summands would vanish, contradicting Theorem **??**. □

Proposition 3.3 (Exact two-neighbor residual)**.** *For $n_1, n_2 > 0$:*

$$\min_{v>0} \big(J(v/n_1) + J(v/n_2)\big) = \sqrt{\frac{n_1}{n_2}} + \sqrt{\frac{n_2}{n_1}} - 2. \tag{3}$$

If $n_1 \neq n_2$, this minimum is strictly positive.

Proof. By Corollary **??**, the minimizer is $v^\star = \sqrt{n_1 n_2}$. Set $r := \sqrt{n_2/n_1}$, so $v^\star/n_1 = r$ and $v^\star/n_2 = 1/r$. Then:

$$\begin{aligned} J(r) + J(1/r) &= \frac{1}{2}(r + r^{-1}) - 1 + \frac{1}{2}(r^{-1} + r) - 1 \\ &= (r + r^{-1}) - 2 \\ &= \sqrt{\frac{n_2}{n_1}} + \sqrt{\frac{n_1}{n_2}} - 2. \end{aligned}$$

By the AM-GM inequality, $r + r^{-1} \geq 2\sqrt{r \cdot r^{-1}} = 2$, with equality iff $r = 1$ (i.e. $n_1 = n_2$). □

Corollary 3.4 (d-bond positivity from one incompatible pair)**.** *Let $n_1, \ldots, n_d > 0$ with $n_i \neq n_j$ for some pair $i \neq j$. Then for every $v > 0$:*

$$\sum_{\ell=1}^{d} J(v/n_\ell) > 0.$$

Moreover,

$$\sum_{\ell=1}^{d} J(v/n_\ell) \geq \sqrt{\frac{n_i}{n_j}} + \sqrt{\frac{n_j}{n_i}} - 2 > 0.$$

Proof. All d terms are ≥ 0. Drop all except the (i, j) pair: $\sum_\ell J(v/n_\ell) \geq J(v/n_i) + J(v/n_j)$. Apply Corollary **??** for strict positivity and Proposition **??** for the explicit bound (noting that the bound holds for every v, hence also at the d-bond optimum). □

Example 3.5. With $n_1 = 2$, $n_2 = 8$: residual $= \sqrt{1/4} + \sqrt{4} - 2 = 1/2 + 2 - 2 = 1/2$. With $n_1 = 1$, $n_2 = 100$: residual $= 1/10 + 10 - 2 = 8.1$. Wide disparity produces large frustration.

4. Graph-level non-collapse

We now lift the local frustration to a global statement on graphs.

Definition 4.1 (Prescribed neighborhood data)**.** Let $G = (V, E)$ be a finite connected graph. For each vertex $u \in V$, fix a positive multiset $\mathcal{N}(u) = \{n_{u,1}, \dots, n_{u,d_u}\}$. The *local J-mean* is $m_u := \mathbb{JM}(\mathcal{N}(u))$.

Theorem 4.2 (Distinct local means force distinct optimal states)**.** *Suppose there exist vertices $u, w \in V$ with $m_u \neq m_w$. If a field $x = (x_v)_{v \in V}$ satisfies*

$$x_u \in \arg\min_{t>0} \sum_{i=1}^{d_u} J(t/n_{u,i}) \quad \text{for every } u \in V,$$

then $x_u \neq x_w$. In particular, no such field is spatially uniform.

Proof. By Theorem **??**, each local arg min is a singleton: $\arg\min_{t>0} \sum_i J(t/n_{u,i}) = \{m_u\}$. Hence $x_u = m_u$ for every u. If $m_u \neq m_w$, then $x_u \neq x_w$. A uniform field $x_u \equiv c$ would require $m_u = c$ for all u, contradicting $m_u \neq m_w$. □

Remark 4.3 (Scope of the non-collapse statement)*.* Theorem **??** concerns optimization against *prescribed* (externally fixed) neighborhood data. For the unconstrained edge-energy functional $\sum_{\{u,w\} \in E} J(x_u/x_w)$, constant fields achieve zero energy and are trivially optimal. Therefore the topological frustration mechanism requires either incompatible prescribed boundary conditions, external forcing, or additional constraint structure (such as window neutrality [**?**]). This distinction is important in applications.

5. Application interfaces

We record the precise forms used by companion papers.

Remark 5.1 (Fluid-direction smoothing interface)*.* If direction updates near the vorticity zero set $\{\omega = 0\}$ are modeled by local reciprocal cost minimization against neighboring directions, then incompatible local neighborhoods generate strictly positive residual cost (Corollary **??**), preventing collapse to a single linearized average in defect regions. See [**?**] for the full blow-up exclusion argument.

Remark 5.2 (Constraint-relaxation interface)*.* In graph encodings of combinatorial constraints, if a relaxed local state must satisfy incompatible neighboring targets, the strict residual cost of Corollary **??** forbids a zero-cost fractional survivor. See the companion paper on combinatorial feasibility for the complete reduction.

Remark 5.3 (Lattice embedding interface)*.* In reciprocal-cost lattice updates, distinct local prescribed neighborhoods force differentiated local optima by Theorem **??**, providing a direct non-collapse criterion for gauge-field lattice embeddings.

Acknowledgments

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References

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