Dimensional Rigidity: D=3 from Linking of Loops, Kepler Stability, and Minimal Dyadic Synchronization

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Abstract

We give three mathematically precise constraints that each single out the spatial dimension $D = 3$. First, we show that an integer-valued linking invariant for disjoint oriented loops (embedded copies of S^1) exists only in $D = 3$; this follows from Alexander duality, since for an embedded circle $K \subset \mathbb{S}^D$ the group $H_1(\mathbb{S}^D \setminus K)$ is isomorphic to $\mathbb{Z}$ if and only if $D = 3$. Second, for the D-dimensional Kepler potential determined by the Green's function of the Laplacian, $V_D(r) \propto -r^{2-D}$ (for $D \geq 3$), we derive the Binet equation and show that near-circular bound orbits are stable only for $D < 4$, and are non-precessing (hence closed in the linearized regime) only for $D = 3$. Third, given a fixed odd "gap" period 45, we show that the synchronization length $\mathrm{lcm}(2^D, 45)$ is minimized over $D \geq 3$ uniquely at $D = 3$, where it equals 360. We also point to a Lean 4 formalization in the `IndisputableMonolith` library certifying the discrete (hypercube/Gray-cycle) and arithmetic (lcm) parts.

Keywords: dimension, linking number, Alexander duality, hypercube, Gray code, Kepler problem, apsidal precession
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1 Introduction

Why is physical space three-dimensional? There are many proposed answers (anthropic, dynamical, field-theoretic). In this note we record a compact rigidity phenomenon: several independent, cleanly stated mathematical requirements each force $D = 3$.

To keep the argument modular, we isolate three constraints.

- **(T) Topological loop-linking.** There should exist an integer-valued invariant measuring how two disjoint oriented loops link. We show this is possible only in $D = 3$.
- **(K) Kepler stability from a point-source potential.** If one insists that the natural point-source potential in $\mathbb{R}^D$ (Green's function of the Laplacian) supports stable, non-precessing near-circular bound orbits, then $D = 3$ is forced.
- **(S) Minimal dyadic synchronization with a fixed odd period.** If a system carries a dyadic cycle of length 2^D (e.g. a Gray-cycle on a D-cube) and an independent odd cycle of length 45, then the synchronization length $\mathrm{lcm}(2^D, 45)$ grows exponentially in D. Among $D \geq 3$, its unique minimizer is $D = 3$ (yielding 360).

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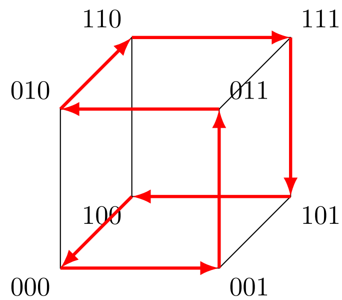

Figure 1: A standard Gray 8-cycle on Q_3: $000 \to 001 \to 011 \to 010 \to 110 \to 111 \to 101 \to 100 \to 000$.

The proofs use standard tools (Alexander duality; Binet equation and linearization; elementary number theory), and we include a diagram of a canonical Gray 8-cycle on the cube.

Motivating context. In the broader Recognition Science program, the dyadic period 2^D arises from a ledger traversal on a D-dimensional binary register, while the odd period 45 appears as a "gap" index; the accepted Recognition Geometry paper provides a measurement-first axiomatic setting for this viewpoint (see [1]). Nothing in the proofs below depends on that framework.

2 Preliminaries: the hypercube and Gray cycles

Definition 2.1 (Hypercube graph)**.** *For $D \in \mathbb{N}$, the D-dimensional hypercube graph Q_D has vertex set $\{0,1\}^D$ with an edge between two vertices iff they differ in exactly one coordinate.*

Definition 2.2 (Gray cycle)**.** *A* Gray cycle *on Q_D is a Hamiltonian cycle on Q_D, equivalently a cyclic ordering of the 2^D binary strings where consecutive strings differ in one bit.*

Proposition 2.3 (Existence of Gray cycles)**.** *For every $D \geq 1$, Q_D admits a Gray cycle.*

Construction sketch. Let $\mathrm{Gray}_1 = (0,1)$. Given a Gray cycle Gray_D on $\{0,1\}^D$, form a Gray cycle on $\{0,1\}^{D+1}$ by

$$\mathrm{Gray}_{D+1} := (0\,\mathrm{Gray}_D,\ 1\overline{\mathrm{Gray}_D}),$$

where $0\,\mathrm{Gray}_D$ means prepend a 0 to each word in Gray_D, and $\overline{\mathrm{Gray}_D}$ denotes the reverse traversal. This yields a Hamiltonian cycle on Q_{D+1} with one-bit steps. □

Remark 2.4. *The existence of Gray codes is classical (see [4, 5]). We only need that the natural dyadic period associated to a full traversal is 2^D.*

Definition 2.5 (Dyadic period)**.** *We define the dyadic period of dimension D as* $\mathrm{per}(D) := 2^D$.

3 Constraint (T): an integer linking invariant for loops forces $D = 3$

We record the clean algebraic-topological fact behind "linking of loops is special to three dimensions."

Definition 3.1 (Linking number via homology)**.** *Let $K \subset \mathbb{S}^3$ be an oriented embedded circle, and let $L \subset \mathbb{S}^3 \setminus K$ be another oriented embedded circle disjoint from K. By Alexander duality, $H_1(\mathbb{S}^3 \setminus K) \cong \mathbb{Z}$. The* linking number $\mathrm{lk}(L,K) \in \mathbb{Z}$ *is the image of $[L] \in H_1(\mathbb{S}^3 \setminus K)$ under this identification.*

Theorem 3.2 (Alexander duality computation). *Let $D \geq 2$ and let $K \subset \mathbb{S}^D$ be an embedded circle. Then*

$$H_1(\mathbb{S}^D \setminus K) \cong \widetilde{H}^{D-2}(S^1).$$

In particular,

$$H_1(\mathbb{S}^D \setminus K) \cong \begin{cases} \mathbb{Z} & D = 3, \\ 0 & D \neq 3. \end{cases}$$

Proof. Alexander duality gives $\widetilde{H}_i(\mathbb{S}^D \setminus K) \cong \widetilde{H}^{D-i-1}(K)$ for compact, locally contractible $K \subset \mathbb{S}^D$ (see [2, Ch. 3]). Taking $i = 1$ and $K \simeq S^1$, we obtain $\widetilde{H}_1(\mathbb{S}^D \setminus K) \cong \widetilde{H}^{D-2}(S^1)$. Since $\widetilde{H}^j(S^1)$ is $\mathbb{Z}$ only for $j = 1$ and is 0 otherwise, the claim follows. □

Corollary 3.3 (Topological rigidity for loop-linking). *An integer-valued linking number for disjoint oriented loops, defined via $H_1(\mathbb{S}^D \setminus K)$, can exist only when $D = 3$.*

Remark 3.4 (General intersection dimension). *From the perspective of geometric topology, linking numbers arise as intersection numbers of a bounding chain. If M^p and N^q are disjoint closed oriented submanifolds in $\mathbb{R}^D$, a generic chain C^{p+1} with $\partial C = M$ will intersect N^q stably at isolated points if and only if:*

$$(p+1) + q = D \implies p + q + 1 = D.$$

For 1D observables (worldlines or loops), we have $p = q = 1$, which forces $1 + 1 + 1 = D$, uniquely selecting $D = 3$. This places the result in the standard hierarchy of intersection invariants (e.g., points link loops in $D = 2$, surfaces link surfaces in $D = 5$).

Example 3.5 (Hopf link). *In $\mathbb{S}^3$ there exist disjoint loops with $\mathrm{lk} = 1$, e.g. the Hopf link. This shows the invariant is nontrivial in $D = 3$ (see [3]).*

4 Constraint (K): Kepler stability from a point-source potential

We next record a dimension-sensitive computation showing that $D = 3$ is singled out by stability and non-precession for the Kepler problem induced by the Laplacian Green's function.

4.1 The D-dimensional Kepler potential

For $D \geq 3$, the fundamental solution of the Laplacian implies a point-source potential of the form

$$V_D(r) = -\mu\, r^{2-D}, \qquad r > 0, \tag{1}$$

for a constant $\mu > 0$ (we only use the power $2 - D$).

4.2 Binet equation and linearization

Motion in a central potential is confined to a plane, so we may use polar coordinates (r, θ) and set $u(\theta) = 1/r(\theta)$. For a radial force $F(r) = -V_D'(r)$, the Binet equation reads

$$u''(\theta) + u(\theta) = -\frac{m}{\ell^2 u(\theta)^2}\, F\big(1/u(\theta)\big), \tag{2}$$

where m is the mass and ℓ is the (scalar) angular momentum.

Lemma 4.1 (Binet form for V_D)**.** *For the potential* (1) *with* $D \geq 3$*, the Binet equation becomes*

$$u'' + u = \beta\, u^{D-3}, \tag{3}$$

for some constant $\beta > 0$*.*

Proof. Compute $V_D'(r) = -\mu(2-D)r^{1-D} = \mu(D-2)r^{1-D}$, so the attractive radial force is $F(r) = -V_D'(r) = -\mu(D-2)r^{1-D}$. Substituting $r = 1/u$ gives $F(1/u) = -\mu(D-2)u^{D-1}$. Plugging into (2) yields $u'' + u = \frac{m\mu(D-2)}{\ell^2}\, u^{D-3}$. Set $\beta = \frac{m\mu(D-2)}{\ell^2} > 0$. □

Proposition 4.2 (Stability and apsidal angle)**.** *Assume* $D \geq 3$ *and consider a circular orbit solution* $u \equiv u_0 > 0$ *of* (3)*. Linearizing* $u = u_0 + \varepsilon$ *around* u_0 *yields*

$$\varepsilon'' + (4-D)\,\varepsilon = 0.$$

Consequently:

1. *For* $D \geq 5$*, circular orbits are linearly unstable.*
2. *For* $D = 4$*, circular orbits are marginal* ($\varepsilon'' = 0$)*.*
3. *For* $D = 3$*, circular orbits are stable with angular frequency* 1 *in the* θ *variable; the apsidal angle is exactly* π*, so near-circular bound orbits are non-precessing.*

Proof. A circular orbit $u \equiv u_0$ satisfies $u_0 = \beta u_0^{D-3}$, hence $\beta = u_0^{4-D}$. Write $u = u_0 + \varepsilon$ with $|\varepsilon| \ll 1$ and expand:

$$\beta(u_0+\varepsilon)^{D-3} = \beta u_0^{D-3} + \beta(D-3)u_0^{D-4}\varepsilon + O(\varepsilon^2).$$

Using $\beta u_0^{D-3} = u_0$ and $\beta u_0^{D-4} = 1$, the linearized equation becomes $\varepsilon'' + \varepsilon = (D-3)\varepsilon$, i.e. $\varepsilon'' + (4-D)\varepsilon = 0$. The conclusions follow by inspecting the sign of $4-D$. For $D = 3$, solutions are periodic with frequency 1, so the radial oscillation period equals 2π in θ, giving apsidal angle π (no precession in the linear regime). □

Remark 4.3. *In* $D = 3$ *the Kepler problem is exactly integrable and yields closed conics; Bertrand's theorem characterizes the Kepler and harmonic oscillator potentials as the only central potentials for which all bounded orbits are closed [6].*

Remark 4.4 (The case $D = 2$)**.** *In* $D = 2$ *the Laplacian Green's function yields a logarithmic potential* $V_2(r) \propto \log r$ *rather than a power law. The Binet analysis above is stated for* $D \geq 3$*; in particular, the no-precession phenomenon recovered in Proposition 4.2 is specific to* $D = 3$ *within the Laplacian point-source family.*

5 Constraint (S): minimal synchronization with an odd period

Fix an odd integer N (in the RS-motivated application, $N = 45$). Define the synchronization length

$$S(D) := \operatorname{lcm}(2^D, N).$$

When N is odd, $\gcd(2^D, N) = 1$, so $S(D) = N \cdot 2^D$.

Lemma 5.1. *For every* $D \geq 0$*,* $\gcd(2^D, 45) = 1$ *and hence*

$$\operatorname{lcm}(2^D, 45) = 45 \cdot 2^D.$$

Proof. Since 45 is odd, it shares no prime factor with 2^D. Thus $\gcd(2^D, 45) = 1$ and $\mathrm{lcm}(2^D, 45) = 2^D \cdot 45$. $\square$

Proposition 5.2 (Unique minimizer over $D \geq 3$). *Let $S(D) := \mathrm{lcm}(2^D, 45)$. Then $S(D)$ is strictly increasing in D and, among integers $D \geq 3$, the unique minimizer is $D = 3$ with $S(3) = 360$.*

Proof. By Lemma 5.1, $S(D) = 45 \cdot 2^D$. Hence $S(D+1) = 2S(D)$, so S is strictly increasing. In particular, for $D \geq 3$ we have $S(D) \geq S(3) = 45 \cdot 8 = 360$, with equality iff $D = 3$. $\square$

Remark 5.3. *This "minimal synchronization" constraint is only meaningful when paired with an independent lower bound $D \geq 3$ (e.g. from Constraint (T) or (K)). In the RS documents, 45 is additionally motivated as the triangular number $T_9 = 1 + \cdots + 9$; the Lean module `IndisputableMonolith.Gap45.PhysicalMotivation` records that viewpoint.*

6 Combined rigidity theorem

Theorem 6.1 (Dimensional rigidity at $D = 3$). *Each of the constraints (T), (K), and (S) forces $D = 3$. In particular, if any two of these conditions are required simultaneously, they are compatible only in dimension three.*

Proof. Constraint (T) gives $D = 3$ by Corollary 3.3. Constraint (K) gives $D = 3$ by Proposition 4.2. Constraint (S) gives that among $D \geq 3$ the unique minimizer is $D = 3$ by Proposition 5.2. $\square$

7 Lean 4 formalization (discrete and arithmetic components)

Parts of the discrete and arithmetic content of this note are mechanized in Lean 4. The code lives in the `IndisputableMonolith` library.

Verified items

- **Gray 8-cycle on Q_3.** In `IndisputableMonolith/Patterns/GrayCycle.lean`:

```
def grayCycle3Path : Fin 8 -> Pattern 3 := ...
theorem grayCycle3_bijective : Function.Bijective grayCycle3Path := by ...
theorem grayCycle3_oneBit_step : forall i : Fin 8, OneBitDiff ... := by ...
```

- **Synchronization identity and lcm forcing.** In `IndisputableMonolith/RecogSpec/Bands.lean`:

```
theorem lcm_pow2_45_eq_iff (D : Nat) :
  Nat.lcm (2^D) 45 = 360 <-> D = 3 := by ...
```

- **Dimension forced from "cover + sync" predicate.** In `IndisputableMonolith/Verification/Dimension.lean`:

```
theorem rs_counting_gap45_absolute_iff_dim3 :
  RSCounting_Gap45_Absolute D <-> D = 3 := by ...
```

Remark 7.1. *The topological and dynamical constraints (Alexander duality; Kepler stability) are standard mathematics and are presented here as conventional proofs; they are not currently the focus of the Lean mechanization.*

8 Scope and limitations

- **Topology.** Constraint (T) is about *loop-loop* integer linking; other dimensions support other linking phenomena (e.g. S^2–S^2 linking in $D = 5$). The claim here is that loops enjoy an integer linking invariant only in $D = 3$.
- **Dynamics.** Constraint (K) assumes the point-source potential is the Laplacian Green's function. Other potentials may behave differently; the point is that the most canonical "no free scale" potential family is dimension-sensitive, and the Kepler/non-precession behavior is uniquely $D = 3$.
- **Arithmetic.** Constraint (S) treats the odd period 45 as an external input. The rigorous statement is a minimality fact: among $D \geq 3$, synchronization overhead grows like 2^D, and the unique minimizer is $D = 3$.

9 Conclusion

We highlighted three independent mechanisms that single out $D = 3$: loop-linking via H_1 of the complement, stability and non-precession for the Laplacian point-source Kepler potential, and minimal dyadic synchronization with an odd period.

Acknowledgments

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