D'Alembert Inevitability: Polynomial Consistency Forces the Canonical Composition Law on $\mathbb{R}_{>0}$

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Abstract

Let $F : \mathbb{R}_{>0} \to \mathbb{R}$ be a real-valued functional on multiplicative ratios. A common modeling move in functional-equation based theories is to *postulate* a specific composition identity relating $F(xy) + F(x/y)$ to the pair $(F(x), F(y))$. Even in "zero-parameter" settings, the *choice of composition law* is itself an implicit parameter choice. In this paper we prove an *inevitability theorem* for the law. Assume only that there exists a *symmetric quadratic polynomial* $P \in \mathbb{R}[u, v]$ such that for all $x, y > 0$,

$$F(xy) + F(x/y) = P\big(F(x), F(y)\big), \qquad P(u, v) = P(v, u), \qquad F(1) = 0,$$

and impose a minimal nondegeneracy/regularity hypothesis ensuring that the range of F contains an interval near 0. Our first result shows that symmetry of the law forces *reciprocity* of the functional: $F(z) = F(1/z)$ for all $z > 0$. Our main classification theorem then proves that the polynomial law is forced to lie in the unique bilinear family

$$P(u, v) = 2u + 2v + c\,uv \qquad (c \in \mathbb{R}),$$

so that the original consistency equation reduces, after an affine change of variables, to the classical d'Alembert equation $H(t + u) + H(t - u) = 2H(t)H(u)$ in logarithmic coordinates. As an optional strengthening, a natural curvature calibration at the identity fixes the remaining scalar to the canonical value $c = 2$, yielding the exact composition law used in companion work on uniqueness of F. Selected algebraic steps in the classification are formalized in Lean 4 within the `IndisputableMonolith` library.

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1 Introduction

1.1 Motivation: "equation choice" is an implicit parameter

Functional equations on the multiplicative group $\mathbb{R}_{>0}$ arise whenever one models a real-valued quantity attached to *ratios* and demands compositional compatibility under multiplication. In applications one often interprets $F(x)$ as a *cost*, *discrepancy*, or *penalty* associated to a ratio $x > 0$, with $F(1) = 0$ reflecting that perfect agreement has no cost. One then seeks a composition identity relating the combined comparison $(x, y) \mapsto (xy, x/y)$ to the separate comparisons x and y.

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The standard move is to *postulate* a specific two-variable identity. For instance, the multiplicative d'Alembert (or *cosh-addition*) law

$$F(xy) + F(x/y) = 2F(x)F(y) + 2F(x) + 2F(y) \tag{1}$$

is ubiquitous: in logarithmic coordinates it becomes the classical d'Alembert equation

$$H(t+u) + H(t-u) = 2H(t)H(u), \qquad H(t) := 1 + F(e^t),$$

whose regular solutions are hyperbolic cosine (or cosine) families.

However, even in settings advertised as "zero-parameter," the *functional form* of the composition law is itself a hidden modeling choice. Changing (1) to another identity generally changes the theory qualitatively. At the simplest extreme, the additive law $F(xy) + F(x/y) = 2F(x) + 2F(y)$ corresponds in log-coordinates to a quadratic branch $G(t) = F(e^t)$ with constant second derivative (a "flat" regime), rather than the hyperbolic regime associated to (1). Thus, *uniqueness of solutions* to a postulated equation does not address the deeper question: why that equation rather than another?

This paper addresses that question in a mathematically clean way. We do *not* assume the specific form (1). Instead, we assume only that there exists some symmetric polynomial mechanism combining the values $F(x)$ and $F(y)$, and we show that this forces the d'Alembert/cosh-type structure (up to a single scalar, pinned by a natural calibration). In this sense, the composition law is *inevitable* under minimal structural constraints.

1.2 What is new in this paper

We work with the following axiom package. Let $F : \mathbb{R}_{>0} \to \mathbb{R}$ satisfy $F(1) = 0$. Assume there exists a polynomial $P \in \mathbb{R}[u, v]$ such that for all $x, y > 0$,

$$F(xy) + F(x/y) = P\big(F(x), F(y)\big), \tag{2}$$

and that P is symmetric: $P(u, v) = P(v, u)$. Finally, impose a minimal regularity/nondegeneracy condition ensuring that the image of F contains an interval around 0 (so that identities holding on range(F) force global polynomial identities).

Within this framework, the paper makes three contributions.

(i) Reciprocity is derived from symmetry of the law. Although reciprocity $F(z) = F(1/z)$ is often assumed as a property of "comparison," we show that it follows formally from the symmetry of the combiner P together with (2). This is conceptually useful: *symmetry is imposed on the law, not on the solution.*

(ii) Polynomial forcing: classification of admissible laws. Our main technical result proves that the only symmetric polynomial law compatible with (2) is the bilinear family

$$P(u, v) = 2u + 2v + c\,uv \tag{3}$$

for some constant $c \in \mathbb{R}$. This is the "meta" classification statement: it classifies *equations*, not solutions.

(iii) Reduction to classical d'Alembert; optional canonical coefficient. Passing to log-coordinates $G(t) = F(e^t)$ and applying the affine change $H(t) := 1 + \frac{c}{2}G(t)$ (for $c \neq 0$), (3) becomes the classical d'Alembert equation for H. Thus, under standard regularity assumptions, the solution theory reduces to the classical cosine/cosh classification. Optionally, a curvature calibration at the identity selects the canonical coefficient $c = 2$, matching the specific law (1).

The present paper is designed to pair with a companion "uniqueness of F" paper (Paper 1.1 in the project outline): once the law is forced to the canonical d'Alembert form and a normalization is fixed, one may invoke standard uniqueness results (or reprove them directly) to identify the unique regular solution as the canonical reciprocal cost $J(x) = \frac{1}{2}(x + x^{-1}) - 1$. Here we deliberately keep the focus on *inevitability of the law*.

1.3 Roadmap

Section 2 records the axiom package and the minimal regularity hypothesis used to promote identities from range(F) to polynomial identities on $\mathbb{R}^2$. In Section 3 we derive reciprocity and pin down the boundary constraints $P(u, 0) = 2u$ and $P(0, v) = 2v$. Section 4 contains the polynomial classification and proves that P must be of the bilinear form (3). In Section 5 we pass to log-coordinates and reduce to the classical d'Alembert equation by an affine change of variables. Section 6 (optional) shows how a natural curvature normalization fixes the remaining scalar to the canonical value $c = 2$. We conclude with examples and counterexamples clarifying the necessity of the hypotheses, and include a brief formalization note pointing to the Lean 4 implementation of the coefficient-forcing steps in `IndisputableMonolith` (module `Foundation/DAlembert/Inevitability`).

2 Preliminaries and axiom package

2.1 Domain and notation

Throughout, $\mathbb{R}_{>0}$ denotes the multiplicative group of positive real numbers. We consider a function

$$F : \mathbb{R}_{>0} \to \mathbb{R},$$

and we write $F(x)$ for the value of F at a ratio $x > 0$. It is convenient to pass to logarithmic coordinates: for $t \in \mathbb{R}$ set

$$G(t) := F(e^t). \tag{4}$$

Then multiplication and division of ratios become addition and subtraction in $\mathbb{R}$: for $x = e^t$ and $y = e^u$ one has $xy = e^{t+u}$ and $x/y = e^{t-u}$. Accordingly, any multiplicative consistency law for F becomes an additive functional equation for G.

2.2 Polynomial consistency: the core hypothesis

The main structural assumption is that F admits a *polynomial combiner* in the values $F(x)$ and $F(y)$.

Definition 1 (Polynomial composition law)**.** We say that $F : \mathbb{R}_{>0} \to \mathbb{R}$ admits a *polynomial composition law* if there exists a polynomial $P \in \mathbb{R}[u, v]$ such that

$$F(xy) + F(x/y) = P\big(F(x), F(y)\big) \qquad \text{for all } x, y > 0. \tag{5}$$

We call P a *combiner* for F.

Definition 2 (Symmetric law and normalization)**.** We say the polynomial law (5) is *symmetric* if

$$P(u, v) = P(v, u) \qquad \text{for all } u, v \in \mathbb{R}, \tag{6}$$

and we impose the normalization

$$F(1) = 0. \tag{7}$$

Passing to log-coordinates, (5) becomes

$$G(t+u) + G(t-u) = P\big(G(t), G(u)\big), \qquad t, u \in \mathbb{R}, \tag{8}$$

where G is defined by (4).

2.3 Regularity / nontriviality

Two technical issues arise immediately. First, (5) only constrains P *on the range of* F (or G), since its arguments are of the form $(F(x), F(y))$. Second, polynomial identities are rigid: if a polynomial vanishes on an interval then it vanishes identically. To leverage this rigidity we require that the range of F contains an interval near 0.

We isolate a minimal hypothesis that suffices for this purpose and is standard in the theory of functional equations.

Assumption 3 (Minimal regularity)**.** We assume:

- F is continuous on $\mathbb{R}_{>0}$,
- F is nontrivial: there exists $y_0 > 0$ with $F(y_0) \neq 0$.

Lemma 4 (Range contains an interval)**.** *Assume* (7) *and Assumption 3. Then there exists a nondegenerate closed interval $I \subset \mathbb{R}$ with $0 \in I$ such that $I \subset F(\mathbb{R}_{>0})$.*

Proof. Let $y_0 > 0$ with $F(y_0) \neq 0$ and set $k := F(y_0)$. Since $F(1) = 0$ and F is continuous on the connected set $\mathbb{R}_{>0}$, the intermediate value theorem implies that $F(\mathbb{R}_{>0})$ contains every value between 0 and k. Thus $[0, k] \subset F(\mathbb{R}_{>0})$ if $k > 0$, and $[k, 0] \subset F(\mathbb{R}_{>0})$ if $k < 0$. □

We will repeatedly use the following elementary rigidity principle.

Lemma 5 (Polynomial identity from an interval)**.** *Let $q \in \mathbb{R}[z]$ be a univariate polynomial. If $q(z) = 0$ for all z in a nondegenerate interval $J \subset \mathbb{R}$, then q is the zero polynomial. Equivalently, if two polynomials $q_1, q_2 \in \mathbb{R}[z]$ agree on a nondegenerate interval, then $q_1 = q_2$ in $\mathbb{R}[z]$.*

Proof. This is standard: a nonzero polynomial has only finitely many roots, whereas a nondegenerate interval contains infinitely many points. □

Remark 6 (Alternative regularity packages)**.** Assumption 3 is not the only convenient way to guarantee Lemma 4. Any hypothesis implying that F is not locally constant near 1 and has the intermediate value property suffices; for example, continuity at 1 plus non-constancy on every neighborhood of 1, or stronger smoothness hypotheses (e.g. existence of a nonzero quadratic term for $G(t) = F(e^t)$ at $t = 0$).

3 First forcing layer: reciprocity and boundary constraints

In this section we extract the first rigid consequences of polynomial consistency. The key point is that *symmetry of the law* $P(u, v) = P(v, u)$ forces *reciprocity* of F, and together with normalization it forces P to act like doubling on the coordinate axes. These constraints are the starting point for the coefficient elimination in the polynomial classification.

3.1 Reciprocity is derived

Lemma 7 (Swapping quotients)**.** *Assume that* $F : \mathbb{R}_{>0} \to \mathbb{R}$ *satisfies the consistency law* (5) *with a symmetric combiner* P *in the sense of* (6)*. Then for all* $x, y > 0$ *one has*

$$F(x/y) = F(y/x). \tag{9}$$

Proof. Write (5) for (x, y) and for (y, x):

$$F(xy) + F(x/y) = P(F(x), F(y)), \qquad F(yx) + F(y/x) = P(F(y), F(x)).$$

Since $xy = yx$ and P is symmetric, the right-hand sides coincide, hence the left-hand sides coincide. Cancelling the common term $F(xy)$ yields (9). □

Lemma 8 (Reciprocity)**.** *Under the hypotheses of Lemma 7, the functional* F *is reciprocal-symmetric:*

$$F(z) = F(1/z) \qquad \text{for all } z > 0. \tag{10}$$

Proof. Apply (9) with $y = 1$ to obtain $F(z/1) = F(1/z)$, i.e. $F(z) = F(1/z)$. □

3.2 Boundary values: $P(u, 0) = 2u$ and $P(0, v) = 2v$

Lemma 9 (Boundary constraint on the range)**.** *Assume* (5)*,* (6)*, and* (7)*. Then for every* $y > 0$*,*

$$P(0, F(y)) = 2F(y), \tag{11}$$

and for every $x > 0$*,*

$$P(F(x), 0) = 2F(x). \tag{12}$$

Proof. Set $x = 1$ in (5). Using $F(1) = 0$ and Lemma 8 we obtain

$$F(y) + F(1/y) = P(F(1), F(y)) = P(0, F(y)) \quad \Longrightarrow \quad 2F(y) = P(0, F(y)),$$

which is (11). The identity (12) follows by symmetry of P. □

Lemma 10 (Doubling on the axes)**.** *Assume Definition 1, the symmetry condition* (6)*, normalization* (7)*, and the regularity Assumption 3. Then the polynomial identity*

$$P(0, v) = 2v \tag{13}$$

holds for all $v \in \mathbb{R}$*. By symmetry, also* $P(u, 0) = 2u$ *for all* $u \in \mathbb{R}$*.*

Proof. Define the univariate polynomial $q(v) := P(0, v) - 2v \in \mathbb{R}[v]$. By Lemma 9 we have $q(F(y)) = 0$ for all $y > 0$. By Lemma 4, the range $F(\mathbb{R}_{>0})$ contains a nondegenerate interval I with $0 \in I$, hence $q(v) = 0$ for all $v \in I$. By Lemma 5, q is the zero polynomial, which is exactly (13). The identity $P(u, 0) = 2u$ follows from symmetry of P. □

Remark 11 (Interpretation)**.** Lemma 10 says that any admissible polynomial law must act as "doubling" on the axes. In particular, any higher-order terms in $P(0, v)$ or $P(u, 0)$ must vanish. This will be the first algebraic lever in the classification of P.

4 Polynomial classification: only the bilinear family survives

We now classify the admissible polynomial combiners P. The axis constraints from Section 3 already force strong divisibility properties; to finish the classification we adopt a standard and practically motivated restriction: the law is *at most quadratic* in its arguments. This is exactly the polynomial class used in the Lean 4 formalization of the coefficient forcing.

4.1 Quadratic polynomial hypothesis

Assumption 12 (Quadratic law)**.** In addition to the hypotheses of Definition 1 and (6), assume the combiner P has total degree at most 2, i.e. there exist real constants $a, b, c, d, e, f \in \mathbb{R}$ such that for all $u, v \in \mathbb{R}$,

$$P(u, v) = a + bu + cv + d\,uv + e\,u^2 + f\,v^2. \tag{14}$$

Remark 13 (Beyond quadratic)**.** Lemma 10 already implies that for a *general* polynomial combiner one may factor

$$P(u, v) = 2u + 2v + uv\,R(u, v)$$

for some symmetric polynomial $R \in \mathbb{R}[u, v]$. Showing that R must be constant (hence recovering the bilinear family) requires additional input beyond the axis constraints; one natural approach is to add a compatibility condition coming from composing (5) along three factors, or to assume sufficient smoothness of $G(t) = F(e^t)$ near 0 and compare Taylor data. For clarity we present the clean quadratic classification, which already yields the d'Alembert inevitability and matches the current formal development.

4.2 Eliminating coefficients using symmetry and axis constraints

Lemma 14 (Symmetry reduces parameters)**.** *Assume* (6) *and the quadratic form* (14)*. Then $b = c$ and $e = f$, and P can be rewritten as*

$$P(u, v) = a + b(u + v) + d\,uv + e(u^2 + v^2). \tag{15}$$

Proof. The identity $P(u, v) = P(v, u)$ forces equality of coefficients in (14) after swapping u and v. In particular, comparing the coefficients of u and v gives $b = c$, and comparing the coefficients of u^2 and v^2 gives $e = f$. □

Lemma 15 (Axis constraints kill constant and square terms)**.** *Assume* (13) *(equivalently, $P(0, v) = 2v$ for all v) and the symmetric quadratic form* (15)*. Then $a = 0$, $b = 2$, and $e = 0$.*

Proof. Set $u = 0$ in (15) to obtain

$$P(0, v) = a + bv + e\,v^2.$$

By (13), this must equal $2v$ for all $v \in \mathbb{R}$. Comparing coefficients yields $a = 0$, $b = 2$, and $e = 0$. □

4.3 Main classification theorem

Theorem 16 (Bilinear family is forced)**.** *Assume Definition 1, symmetry* (6)*, normalization* (7)*, the regularity Assumption 3, and the quadratic hypothesis Assumption 12. Then there exists a constant $c \in \mathbb{R}$ such that*

$$P(u, v) = 2u + 2v + c\,uv \qquad \text{for all } u, v \in \mathbb{R}. \tag{16}$$

Equivalently, the original consistency law (5) *must take the form*

$$F(xy) + F(x/y) = 2F(x) + 2F(y) + c\,F(x)F(y) \qquad \text{for all } x, y > 0. \tag{17}$$

Proof. By Lemma 10, we have $P(0, v) = 2v$ for all v. By Lemma 14 we may write P in the symmetric quadratic form (15). Applying Lemma 15 yields $a = 0$, $b = 2$, and $e = 0$, so

$$P(u, v) = 2(u + v) + d\,uv.$$

Setting $c := d$ gives (16). Substituting (16) into (5) yields (17). □

Remark 17 (Connection to d'Alembert)**.** Theorem 16 is the key "equation inevitability" statement. In the next section we pass to log-coordinates and apply an affine change of variables to convert (17) into the classical d'Alembert equation, completing the reduction to the cosine/cosh families.

5 Reduction to classical d'Alembert

In this section we show that the bilinear family (17) is, after a simple change of variables, exactly the classical d'Alembert equation. This reduces the analytic part of the theory to the well-developed literature on d'Alembert-type functional equations.

5.1 Log-coordinates

Lemma 18 (Bilinear law in log-coordinates)**.** *Assume* (17) *and define* G *by* (4)*. Then for all* $t, u \in \mathbb{R}$*,*

$$G(t + u) + G(t - u) = 2G(t) + 2G(u) + c\,G(t)G(u). \tag{18}$$

Proof. Let $x = e^t$ and $y = e^u$ in (17). Using $xy = e^{t+u}$ and $x/y = e^{t-u}$ and the definition $G(t) = F(e^t)$ gives (18). □

Remark 19 (Evenness)**.** Reciprocity $F(z) = F(1/z)$ (Lemma 8) implies that G is even: $G(-t) = G(t)$ for all $t \in \mathbb{R}$. This parity is compatible with (18) and is often assumed a priori in analytic treatments.

5.2 Affine normalization to d'Alembert

Lemma 20 (Affine reduction)**.** *Assume* (18) *for some constant* $c \in \mathbb{R}$*.*

- *If* $c \neq 0$ *and we define*

$$H(t) := 1 + \frac{c}{2}\,G(t), \tag{19}$$

  then H *satisfies the classical d'Alembert equation*

$$H(t + u) + H(t - u) = 2H(t)H(u) \qquad \textit{for all } t, u \in \mathbb{R}. \tag{20}$$

- *If* $c = 0$*, then* (18) *reduces to the quadratic functional equation*

$$G(t + u) + G(t - u) = 2G(t) + 2G(u) \qquad \textit{for all } t, u \in \mathbb{R}. \tag{21}$$

Proof. If $c \neq 0$, substitute (19) into the left-hand side of (20) and expand:

$$H(t + u) + H(t - u) = 2 + \frac{c}{2}\big(G(t + u) + G(t - u)\big).$$

Use (18) to rewrite the bracketed term:

$$2+\frac{c}{2}\big(2G(t)+2G(u)+c\,G(t)G(u)\big)=2+cG(t)+cG(u)+\frac{c^2}{2}G(t)G(u).$$

On the other hand,

$$2H(t)H(u)=2\Big(1+\frac{c}{2}G(t)\Big)\Big(1+\frac{c}{2}G(u)\Big)=2+cG(t)+cG(u)+\frac{c^2}{2}G(t)G(u),$$

so (20) holds. If $c=0$, (18) is exactly (21). □

5.3 Interpretation and standard solution families

Under mild regularity assumptions (e.g. continuity, measurability with boundedness on an interval, or local boundedness), the d'Alembert equation (20) is classically classified: one obtains cosine and hyperbolic cosine families (and degenerate constant solutions), with the parameter determined by initial curvature data. In particular, if $F(1)=0$ then $G(0)=0$ and hence $H(0)=1$. When combined with evenness (Remark 19) and mild regularity, the typical branches take the form

$$H(t)=\cosh(kt)\quad\text{or}\quad H(t)=\cos(kt)$$

for some $k\in\mathbb{R}$. The remaining coefficient c in (17) is then selected by a normalization/calibration principle; in the canonical case $c=2$ and $H(t)=\cosh(t)$ one recovers $F(x)=\frac{1}{2}(x+x^{-1})-1$.

6 Canonical coefficient selection $c=2$ (optional)

The preceding sections show that polynomial consistency and symmetry force the composition law into the one-parameter family (17). The remaining scalar c is not fixed by the purely algebraic argument: it reflects the *choice of scale* for the cost functional. In many applications (including the Recognition Science setting that motivates this work) one imposes an additional *calibration* at the identity that selects a canonical scale and hence fixes c. We record one convenient formulation.

6.1 Log-curvature at the identity

Assume $F(1)=0$ and define $G(t)=F(e^t)$ as in (4). Whenever the following limit exists, we call it the *log-curvature* of F at the identity:

$$\kappa(F):=\lim_{t\to 0}\frac{2G(t)}{t^2}=\lim_{t\to 0}\frac{2F(e^t)}{t^2}. \tag{22}$$

If G is twice differentiable at 0, then $\kappa(F)=G''(0)$.

6.2 Calibration forces $c=2$ after normalization

Under the d'Alembert reduction, the solution parameter appearing in the standard family $H(t)=\cosh(kt)$ may be absorbed by a harmless rescaling of the additive coordinate on $\mathbb{R}_{>0}$. Concretely, replacing $F(x)$ by $F(x^\alpha)$ (equivalently, replacing $t=\log x$ by αt) preserves the bilinear law (17) but rescales the parameter k by α. Thus one may adopt the normalization $k=1$ without changing the structural content of the theory.

Theorem 21 (Calibration selects the canonical coefficient)**.** *Assume the hypotheses of Theorem 16, with $c \neq 0$, and assume moreover that the associated function H defined by* (19) *is a regular d'Alembert solution of the hyperbolic type*

$$H(t) = \cosh(kt) \qquad (k > 0).$$

Fix the additive coordinate on $\mathbb{R}_{>0}$ so that $k = 1$, and impose the curvature calibration

$$\kappa(F) = 1. \tag{23}$$

Then the coefficient in (17) *is forced to be the canonical value*

$$c = 2.$$

Proof. Under the stated hypotheses,

$$G(t) = \frac{2}{c}(H(t) - 1) = \frac{2}{c}\big(\cosh(kt) - 1\big).$$

Using $\cosh(kt) - 1 = \frac{1}{2}k^2t^2 + o(t^2)$ as $t \to 0$, we obtain

$$\kappa(F) = \lim_{t\to 0} \frac{2G(t)}{t^2} = \lim_{t\to 0} \frac{4}{c} \cdot \frac{\cosh(kt) - 1}{t^2} = \frac{4}{c} \cdot \frac{k^2}{2} = \frac{2k^2}{c}.$$

After the normalization $k = 1$, the calibration $\kappa(F) = 1$ yields $1 = 2/c$, hence $c = 2$. $\square$

Remark 22 (Resulting canonical law)**.** With $c = 2$, the bilinear law (17) becomes the canonical composition rule

$$F(xy) + F(x/y) = 2F(x)F(y) + 2F(x) + 2F(y),$$

and the corresponding d'Alembert lift is simply $H(t) = 1 + G(t)$. Under the standard regularity assumptions used to classify d'Alembert solutions, the calibrated hyperbolic branch yields $H(t) = \cosh(t)$ and therefore $F(x) = \frac{1}{2}(x + x^{-1}) - 1$.

7 Companion corollary: unique calibrated cost

The goal of this paper is the *classification of admissible composition laws.* Nevertheless, once the law has been forced to the canonical form and a calibration has fixed its remaining scalar, the associated cost functional is also pinned down under standard regularity assumptions. We record this as a companion corollary.

Corollary 23 (Unique calibrated solution)**.** *Assume the hypotheses of Theorem 16 and suppose the regularity assumptions used in the classical classification of the d'Alembert equation* (20) *(e.g. continuity of H, or measurability plus local boundedness). Assume further that $c \neq 0$ and that the curvature calibration* (23) *holds. Then necessarily $c = 2$ and*

$$F(x) = \frac{1}{2}(x + x^{-1}) - 1 \qquad \textit{for all } x > 0. \tag{24}$$

Proof. By Theorem 21, the calibration forces $c = 2$ (after the harmless normalization of the additive coordinate on $\mathbb{R}_{>0}$ described in Section 6). With $c = 2$ the d'Alembert lift satisfies $H(t) = 1 + G(t)$. By the classical regularity theory for (20), the calibrated hyperbolic branch yields $H(t) = \cosh(t)$. Thus $G(t) = \cosh(t) - 1 = \frac{1}{2}(e^t + e^{-t}) - 1$, and returning to multiplicative coordinates ($x = e^t$) gives (24). $\square$

Remark 24 (Relation to formal development)**.** In the Lean 4 development accompanying this project, the "inevitability of the law" (Sections 3–4) is isolated from the analytic classification of d'Alembert solutions. This separation mirrors the conceptual split between *uniqueness of the equation* (the present paper) and *uniqueness of the solution* (companion work).

8 Examples, counterexamples, and necessity of hypotheses

This section collects representative examples illustrating the theory and clarifies which assumptions are used where.

8.1 The canonical reciprocal cost

Define

$$J(x) := \frac{1}{2}(x + x^{-1}) - 1, \qquad x > 0. \tag{25}$$

Then $J(1) = 0$ and $J(x) = J(1/x)$. In logarithmic coordinates, $J(e^t) = \cosh(t) - 1$. Consequently, the d'Alembert lift $H(t) := 1 + J(e^t)$ is exactly $H(t) = \cosh(t)$ and therefore satisfies (20). Translating back to J yields the canonical composition rule (the $c = 2$ specialization of (17)):

$$J(xy) + J(x/y) = 2J(x)J(y) + 2J(x) + 2J(y) \qquad (x, y > 0).$$

8.2 The degenerate $c = 0$ branch: quadratic log-costs

When $c = 0$, the bilinear family (17) becomes

$$F(xy) + F(x/y) = 2F(x) + 2F(y),$$

equivalently (21) for $G(t) = F(e^t)$. Under mild regularity (e.g. continuity), (21) forces G to be a quadratic form on $\mathbb{R}$, and evenness then yields

$$G(t) = \alpha t^2 \qquad (\alpha \in \mathbb{R}).$$

Thus

$$F(x) = \alpha(\log x)^2.$$

This branch is consistent with the algebraic forcing up to Section 4, but it is qualitatively different from the hyperbolic/cosh regime. In applications one typically rules it out by an interaction/nonflatness gate (in the project codebase this role is played by `HasInteraction`), or by imposing a calibration that is incompatible with $c = 0$.

8.3 Why symmetry matters

If the symmetry condition (6) is dropped, reciprocity of F is no longer forced, and non-reciprocal solutions appear. For example, let $F(x) = \log x$ on $\mathbb{R}_{>0}$. Then for all $x, y > 0$,

$$F(xy) + F(x/y) = \log(xy) + \log(x/y) = 2\log x = 2F(x),$$

so (5) holds with the polynomial combiner $P(u, v) = 2u$, which is *not* symmetric. Here $F(1/x) = -F(x)$, so reciprocity fails. This simple example explains why symmetry of the law is a natural structural axiom: it is precisely what enforces the quotient-swap identity (Lemma 7) and hence reciprocity (Lemma 8).

8.4 Why polynomiality and regularity matter

The polynomial hypothesis is not a technical convenience but the core "inevitability" lever. Without it, one can always define a combiner on the range by

$$P\big(F(x), F(y)\big) := F(xy) + F(x/y),$$

and extend it arbitrarily to all of $\mathbb{R}^2$; in that generality, there is no meaningful classification of admissible laws.

Regularity enters at the step where we promote identities holding on range(F) to polynomial identities on all of $\mathbb{R}$. Lemma 4 guarantees that $F(\mathbb{R}_{>0})$ contains an interval, and Lemma 5 then forces a univariate polynomial determined on that interval to be determined everywhere. If $F(\mathbb{R}_{>0})$ were thin (e.g. discrete), then many distinct polynomials could agree on the range, and coefficient forcing would fail.

8.5 Why the quadratic restriction is used here

Theorem 16 currently assumes that P is quadratic. As noted in Remark 13, the axis constraints alone imply a factorization $P(u, v) = 2u + 2v + uv\, R(u, v)$ with R symmetric polynomial, but do not by themselves force R to be constant. Eliminating higher-order terms requires additional structure (for example, a compatibility axiom arising from composing (5) across three factors, or a smoothness hypothesis enabling Taylor comparison in log-coordinates). We leave these strengthenings to future work; the quadratic classification already yields the d'Alembert inevitability and matches the current formalization.

9 Formalization plan (Lean 4)

This section briefly records how the main arguments align with the current Lean 4 development in the `IndisputableMonolith` library, and what remains to be formalized for a fully end-to-end certificate.

9.1 What is already formalized

The algebraic "inevitability of the law" portion of the paper is the primary formal target and is already largely present. In particular, the key steps appear in the module `IndisputableMonolith/Foundation/DAlembert/`

- **Derived reciprocity from symmetry of the law.** The paper's Lemmas 7 and 8 correspond to Lean lemmas deriving $F(x/y) = F(y/x)$ and $F(z) = F(1/z)$ from the symmetry of P together with multiplicative consistency.
- **Axis constraints.** Lemma 10 (promotion of $P(0, v) = 2v$ from the range to a polynomial identity) is implemented using the intermediate value theorem to obtain an interval in the range of F, mirroring Lemma 4.
- **Quadratic coefficient elimination.** Theorem 16 is formalized as a coefficient-forcing statement for a symmetric quadratic ansatz for P, yielding the bilinear family $P(u, v) = 2u + 2v + cuv$.

This decomposition is intentional: the Lean development isolates the *algebraic classification of the law* from the analytic classification of d'Alembert solutions.

9.2 What remains (and how to proceed)

There are two natural directions for strengthening the formal certificate.

(i) Beyond quadratic: ruling out higher-degree symmetric polynomials. To remove Assumption 12, one needs additional structure beyond the axis constraints (Remark 13). A promising formal path is to introduce a *compatibility axiom* encoding associativity/consistency under composing three ratios, derive a polynomial identity for $R(u,v)$ in the factorization $P(u,v) = 2u+2v+uvR(u,v)$, and then use polynomial rigidity to show R is constant.

(ii) Analytic end-to-end: from d'Alembert to cosh **and the unique cost.** Sections 5–7 appeal to classical solution theory for (20). A full Lean certificate would either:

- import a suitable d'Alembert classification lemma from Mathlib (under continuity/measurability hypotheses), or
- develop a project-local classification for the even, normalized hyperbolic branch needed here.

With this in place, Theorem 21 and Corollary 23 become straightforward formal consequences of the already-formal algebraic reduction plus elementary calculus at the origin.

A A polynomial factorization lemma

In Sections 3–4 we repeatedly use that the axis constraints $P(0,v) = 2v$ and $P(u,0) = 2u$ force strong algebraic divisibility. For convenience we record the basic factorization statement used in Remark 13.

Lemma 25 (Axis constraints imply an uv factor)**.** *Let $P \in \mathbb{R}[u,v]$ satisfy*

$$P(0,v) = 2v \quad \text{for all } v \in \mathbb{R}, \qquad P(u,0) = 2u \quad \text{for all } u \in \mathbb{R}.$$

Then there exists a polynomial $R \in \mathbb{R}[u,v]$ such that for all $u, v \in \mathbb{R}$,

$$P(u,v) = 2u + 2v + uv\,R(u,v). \tag{26}$$

If moreover P is symmetric (i.e. $P(u,v) = P(v,u)$), then R is symmetric as well.

Proof. Define $Q(u,v) := P(u,v) - 2u - 2v \in \mathbb{R}[u,v]$. The hypotheses give $Q(0,v) = 0$ for all v and $Q(u,0) = 0$ for all u. Write

$$Q(u,v) = \sum_{i,j \geq 0} a_{ij} u^i v^j.$$

The identity $Q(0,v) = 0$ forces $a_{0j} = 0$ for all j, hence every monomial in Q has $i \geq 1$, i.e. u divides Q. Similarly, $Q(u,0) = 0$ forces $a_{i0} = 0$ for all i, hence every monomial has $j \geq 1$, i.e. v divides Q. Therefore uv divides Q, so $Q(u,v) = uv\,R(u,v)$ for some $R \in \mathbb{R}[u,v]$, proving (26). If P is symmetric, then Q is symmetric; since uv is symmetric, the quotient R must be symmetric as well. $\square$

B Coefficient elimination in the symmetric quadratic case

For completeness we spell out the coefficient forcing used in Section 4. Nothing deep occurs here: the work is to reduce the functional constraints to polynomial identities on the axes (Section 3), after which one compares coefficients.

Lemma 26 (Symmetry and axis constraints force the bilinear family)**.** *Let $P \in \mathbb{R}[u, v]$ be a polynomial of the quadratic form*

$$P(u, v) = a + bu + cv + d\,uv + e\,u^2 + f\,v^2 \qquad (a, b, c, d, e, f \in \mathbb{R}),$$

and assume P is symmetric: $P(u, v) = P(v, u)$ for all $u, v \in \mathbb{R}$. If additionally

$$P(0, v) = 2v \quad \textit{for all } v \in \mathbb{R},$$

then necessarily $a = 0$, $b = c = 2$, and $e = f = 0$, and therefore

$$P(u, v) = 2u + 2v + d\,uv.$$

Proof. Symmetry forces equality of coefficients after swapping u and v, hence $b = c$ and $e = f$. Setting $u = 0$ gives

$$P(0, v) = a + cv + f\,v^2.$$

By hypothesis this equals $2v$ for all $v \in \mathbb{R}$, so coefficient comparison yields $a = 0$, $c = 2$, and $f = 0$. Using $b = c$ and $e = f$ we obtain $b = 2$ and $e = 0$. Substituting into the quadratic form yields $P(u, v) = 2u + 2v + d\,uv$. □

Remark 27 (How this connects to the functional equation)**.** In the functional setting, the equality $P(0, v) = 2v$ is first obtained *on the range* via Lemma 9. Regularity (Lemma 4) then promotes it to an identity for all $v \in \mathbb{R}$ by polynomial rigidity (Lemma 5). The algebra in Lemma 26 is exactly the final step in the proof of Theorem 16.

C Classical d'Alembert classification (statement)

For completeness we record a standard classification statement for the d'Alembert equation. This is the analytic input used implicitly in Sections 5–7 (together with the curvature calibration in Section 6).

Theorem 28 (Classical classification of d'Alembert solutions (informal))**.** *Let $H : \mathbb{R} \to \mathbb{R}$ satisfy the d'Alembert equation*

$$H(t + u) + H(t - u) = 2H(t)H(u) \qquad (t, u \in \mathbb{R}),$$

and assume H is not identically zero. Then $H(0) = 1$ and there exists an additive function $a : \mathbb{R} \to \mathbb{R}$ such that

$$H(t) = \cosh(a(t)) = \frac{e^{a(t)} + e^{-a(t)}}{2}.$$

If, in addition, H is continuous (or measurable and locally bounded), then $a(t) = kt$ for some constant $k \in \mathbb{R}$, and hence

$$H(t) = \cosh(kt) \quad \textit{or} \quad H(t) = \cos(kt),$$

with the trigonometric case arising from purely imaginary "k" in the additive representation.

Remark 29 (Selecting the hyperbolic branch)**.** In the present paper, the lift H is obtained from a real-valued cost by (19) and satisfies $H(0) = 1$. A curvature calibration such as (23) together with nonflatness/interaction assumptions typically selects the hyperbolic branch $H(t) = \cosh(kt)$ with $k > 0$.