Complement-Loop Selectors in Loop-Defect Recognition Science

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Abstract

We formulate loop-defect Recognition-Science (LD-RS) premodels: Recognition-Geometry objects equipped with explicit Boolean quotient, transition-cost, recurrence, smooth realization, complement-readout, and phase-readout predicates. For a smooth circle defect $K \subset M^d$ in a connected closed orientable smooth substrate, define

$$G_1(K, M) = \ker\big(H_1(M \setminus K; \mathbb{Z}) \to H_1(M; \mathbb{Z})\big).$$

We compute $G_1(K, M) = 0$ for $d = 2$ and $d \geq 4$, while for $d = 3$ one has $G_1(K, M) \cong \mathbb{Z}/m\mathbb{Z}$, with m the divisibility of the free part of $[K] \in H_1(M^3; \mathbb{Z})$. Hence a nontrivial defect-generated one-dimensional phase forces $d = 3$.

The finite recognition bridge is separate: BQ+CT+SR select a Hamiltonian circle in Q_n, SRL realizes it as the defect core, and LP supplies the complement-loop phase. With optional dimensional alignment DA, the theorem further gives $n = 3$ and $|V_n| = 8$. The contribution is an audited LD-RS extension architecture, not a new circle-complement computation or a derivation from the bare Recognition-Science core.

Keywords: Recognition Geometry; loop defects; complement homology; dimension selection; Gray codes; Aharonov–Bohm phase; algebraic topology.
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1 Introduction

Recognition Geometry supplies a language of recognizers, quotients, locality, and finite-resolution comparison. The reciprocal scalar cost law supplies local ratio penalties. Those ingredients by themselves do not select an ambient substrate dimension. This paper asks a more precise conditional question: what extra loop-defect readout structure would force dimension three once a finite recognition recurrence has been selected?

The main idea is simple. A smooth circle defect has a one-dimensional meridional complement class exactly in dimension three. In dimension two the boundary circles of an annular neighborhood are longitudinal and do not create defect-generated first homology. In dimensions $d \geq 4$, the local

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meridian has higher homological degree. Therefore a model whose admissible phases are loop-only and defect-generated can satisfy that phase condition only when $d = 3$.

The result is a theorem about the LD-RS extension package. We use the shorthand

$$\begin{aligned} \mathsf{RS}_0 &:= \mathsf{RG} + \text{reciprocal scalar cost input}, \\ \mathsf{RS}_{\mathrm{LD}} &:= \mathsf{RS}_0 + \mathrm{BQ} + \mathrm{CT} + \mathrm{SR} + \mathrm{SRL} + \mathrm{LP}, \\ \mathsf{RS}_{\mathrm{LD}}^{+} &:= \mathsf{RS}_{\mathrm{LD}} + \mathrm{DA}. \end{aligned}$$

The implications proved here are $\mathsf{RS}_{\mathrm{LD}} \Rightarrow d = 3$ and $\mathsf{RS}_{\mathrm{LD}}^{+} \Rightarrow n = 3$ with $|V_n| = 8$. Deriving $\mathsf{RS}_{\mathrm{LD}}$ from RS_0 is a separate problem recorded in Appendix C.

Reader's guide. The reader should separate three layers. Section 3 is pure topology: it computes defect-generated first homology for a smooth circle defect. Section 4 is finite recognition graph theory: it explains how BQ, CT, and SR produce a Hamiltonian recurrence circle. Sections 5 and 6 connect these layers through the LD-RS predicates SRL and LP. The final conclusion is conditional on those predicates, and the eight-state conclusion also requires DA.

Theorem A (defect-generated loop selector). *Let $K \cong S^1 \subset M^d$ be a smooth embedded circle in a connected closed orientable smooth d-manifold, with $d \geq 2$. If there is a nontrivial character*

$$G_1(K, M) \longrightarrow \mathrm{U}(1),$$

then $d = 3$. The proof is Theorem 3.10.

Theorem B (finite recognition circle). *In an LD-RS premodel satisfying BQ, CT, and SR, the selected recurrence is a Hamiltonian circle in Q_n using all 2^n recognized binary states. The proof is Theorem 6.2.*

Theorem C (LD-RS dimension selector). *Every LD-RS premodel satisfying BQ, CT, SR, SRL, and LP has ambient readout dimension $d = 3$. If DA is also assumed, then $n = 3$ and $|V_n| = 8$. The proof is Theorem 6.3 together with Corollary 6.4.*

The dependency chain is:

$$\begin{gathered} \mathrm{BQ} + \mathrm{CT} + \mathrm{SR} \Longrightarrow \text{Hamiltonian circle in } Q_n \\ \xrightarrow{\mathrm{SRL}} K \subset M^d \xrightarrow{\mathrm{LP}} G_1(K, M) \neq 0 \text{ or } G_{1,\mathrm{ad}} \neq 0 \\ \Longrightarrow d = 3 \xrightarrow{\mathrm{DA}} n = 3,\ |V_n| = 8. \end{gathered}$$

The proof should therefore be read from right to left: topology proves that a nontrivial defect-generated loop phase is possible only in dimension three; the finite recognition assumptions explain why the defect core is a Hamiltonian recurrence; DA is a separate alignment principle used only for the state-count conclusion.

A second separation is conceptual rather than algebraic. The finite recurrence circle is the defect core,

$$C_\gamma \xrightarrow{h} K \subset M,$$

Table 1: Main predicates and invariants.

Symbol	**Meaning**
BQ	full jointly realized Boolean quotient
CT	cost-to-transition bridge selecting the hypercube generator
SR	state-complete minimal recurrence
SRL	smooth realization of the selected recurrence as a defect core
LP	loop-only defect-generated phase readout with LP0-full or LP0-min coverage as specified
DA	optional alignment $n = d$
$G_1(K, M)$	defect-generated first homology $\ker(H_1(M \setminus K) \to H_1(M))$
$G_{1,\mathrm{ad}}$	compactified admissible quotient after end calibration

whereas LP probes complement loops in $M \setminus K$. The phase is not read on the recurrence core itself; it is read on one-dimensional histories encircling or otherwise detecting the defect-generated complement class.

The topology, the hypercube/Gray-code construction, and the general idea of holonomy around excluded cores are classical ingredients. The contribution is the model-class architecture: the premodel/predicate factorization, the use of G_1 and $G_{1,\mathrm{ad}}$ as readout targets, the core/complement separation, and the load-bearing analysis showing what each bridge predicate contributes.

Section 2 defines LD-RS premodels and the named predicates. Section 3 proves the defect-generated homology calculation. Section 4 proves the finite recognition bridge. Section 5 gives realization, compactification, and LP. Section 6 combines the pieces. The examples in Section 7 show why the hypotheses are load-bearing, and Section 9 locates the novelty relative to standard topology, defect phases, Gray codes, and dimension-selection literature.

2 LD-RS premodels and assumptions

An LD-RS premodel is candidate data over a Recognition-Geometry object. The model predicates are imposed separately. This separation prevents the selector from being manufactured by definition: the tuple supplies places where the bridge predicates can be evaluated, while the predicates themselves supply the mathematical content.

Definition 2.1 (LD-RS premodel over Recognition Geometry)**.** An *LD-RS premodel* over a Recognition-Geometry object is a tuple

$$\mathcal{E} = (\mathcal{R}, \beta, J_a, \rho, C_0, G_{\mathrm{prim}}, \gamma, h, \mathcal{M}, \Xi)$$

consisting of the following candidate data.

(D1) $\mathcal{R}$ is an object of the underlying Recognition-Geometry model class RG, with configuration set C.

(D2) $\beta : C \to \{0, 1\}^n$ is a candidate binary recognition map. Whether it is full jointly realized is the predicate BQ, not part of the word "premodel."

(D3) J_a, the local scale-ratio data ρ, and the transition functional C_0 are candidate reciprocal-cost transition data. Whether they define the primitive graph by additive product locality and cost indecomposability is the predicate CT.

(D4) G_{prim} is a candidate primitive transition graph on the finite recognized states, and γ is a candidate closed recurrence in that graph. Whether $G_{\text{prim}} = Q_n$ and whether γ is minimal and state-complete are consequences of CT and SR together with finite graph theory.

(D5) h is a candidate realization tie from the image subcomplex $C_\gamma \subset |G_{\text{prim}}|$ to a smooth defect circle K when that image is a circle. The predicate SRL requires the actual homeomorphism $h : C_\gamma \xrightarrow{\cong} K$ and the smooth substrate data needed to interpret K as a defect.

(D6) $\mathcal{M}$ is a candidate substrate-readout package. In the closed case it is expected to be a pair (M^d, K) with M^d connected, closed, orientable, and smooth and with $K \subset M^d$ a smooth embedded circle. In the noncompact case it is expected to be an admissible compactified readout package in the sense of Definition 5.7.

(D7) Ξ is a candidate phase datum. Whether it is a nontrivial one-dimensional complement-loop phase generated by the defect is the predicate LP.

Definition 2.2 (LD-RS model and named predicates)**.** An *LD-RS model* is an LD-RS premodel satisfying the named predicates BQ, CT, SR, SRL, and LP. A *dimensionally aligned LD-RS model* is an LD-RS model satisfying the additional predicate DA.

The predicates are as follows.

(A) **BQ** is full joint realization of the binary quotient, as in Definition 4.1.

(B) **CT** is the reciprocal cost-to-transition bridge of Definition 4.4; it asserts the local scale ratios, additive product locality, and the cost-indecomposability rule that make G_{prim} the primitive transition graph.

(C) **SR** is the state-complete recurrence principle: γ is a minimal state-complete closed recurrence in G_{prim}.

(D) **SRL** is smooth substrate realization, as in Definition 5.1: the selected recurrence circle is realized by a specified homeomorphism $h : C_\gamma \xrightarrow{\cong} K$ onto a smooth embedded defect circle in a closed or admissibly compactified substrate.

(E) **LP** is the loop-only defect phase axiom, Axiom 5.21: the selected defect carries a recognized-history-supported nontrivial character on defect-generated complement-loop charge, or on the corresponding admissible compactified quotient.

(F) **DA** is the optional dimensional-alignment principle $n = d$, or a theorem implying it, such as the sufficient smooth criterion in Appendix B.

For reference, the extension implications are

$$\mathsf{RS}_{\text{LD}} \Rightarrow d = 3, \qquad \mathsf{RS}^{+}_{\text{LD}} \Rightarrow n = 3 \text{ and } |V_n| = 8.$$

The first implication is the dimension selector. The second uses DA after the dimension has already been selected.

The dependency chain displayed in Section 1 is used as a road map for the formal predicates below. Section 4 proves the finite-recognition part; Section 3 proves the loop-selector part; and Section 6 composes the pieces.

The forgetful projection

$$\mathfrak{U} : \mathsf{LD\text{-}RS}_{\text{pre}} \longrightarrow \mathsf{RG}, \qquad \mathfrak{U}(\mathcal{E}) = \mathcal{R},$$

means only that the additional Boolean quotient, transition-cost, recurrence, realization, substrate, compactification, phase, and alignment data can be forgotten.

Remark 2.3 (Scope of the forgetful projection). The projection $\mathfrak{U}$ is structure-forgetting only. It is not asserted to be essentially surjective; in noncategorical language, not every Recognition-Geometry object is claimed to admit LD-RS data. It is not a conservativity theorem, not a theorem-lifting principle, and not a claim that the added predicates BQ, CT, SR, SRL, LP, or DA are theorems of bare Recognition Geometry. The LD-RS selector is therefore an extension theorem over Recognition Geometry, not a theorem internal to bare Recognition Geometry.

Remark 2.4 (Provenance of the assumptions). Recognition Geometry supplies the language of recognizers, quotients, locality, finite local resolution, and comparison. Reciprocal cost theory supplies the scalar reciprocal ratio penalty J_a used in local flip costs, but it does not by itself make semantic or readout interpretation a theorem. BQ, CT, SR, SRL, LP, the realization tie h, compactified end calibration, and DA are additional bridge principles unless they are separately proved from a more detailed recognition architecture. The sections below prove consequences of those structures rather than treating them as already contained in the background sources.

Remark 2.5 (Anti-circularity of LP). LP is not a definition of dimension three. It is a dimension-independent predicate on candidate circle-defect substrates: it asks for a nontrivial degree-one phase on defect-generated complement-loop charge. The topological theorem proves that this predicate is satisfiable for smooth circle defects only in ambient dimension three. Thus the selector classifies candidate substrates satisfying LP.

3 Defect-generated loop charge

This section contains the mathematical selector. It does not mention Recognition Science, hypercubes, or cost laws. It is included to make the selector self-contained. The tools and the normal-link/meridian intuition are classical; the use made of the calculation is model-theoretic, namely to identify exactly what LP must read in an LD-RS model.

Definition 3.1 (Defect-generated loop charge). Let $K \subset M$ be a closed defect in a path-connected space. The *defect-generated loop charge group* is

$$G_1(K, M) = \ker\big(H_1(M \setminus K; \mathbb{Z}) \longrightarrow H_1(M; \mathbb{Z})\big),$$

where the map is induced by inclusion. Thus $G_1(K, M)$ records complement loop classes that disappear when the defect is filled back in. If $H_1(M; \mathbb{Z}) = 0$, then $G_1(K, M) = H_1(M \setminus K; \mathbb{Z})$.

Proposition 3.2 (Universal defect-generated first-homology target). *Let $i : M \setminus K \hookrightarrow M$ and write $i_* : H_1(M \setminus K; \mathbb{Z}) \to H_1(M; \mathbb{Z})$. The inclusion*

$$G_1(K, M) \hookrightarrow H_1(M \setminus K; \mathbb{Z})$$

is universal among first-homology targets that become trivial after filling in the defect: for every abelian group A and homomorphism $f : A \to H_1(M \setminus K; \mathbb{Z})$ satisfying $i_* \circ f = 0$, there is a unique homomorphism $\bar{f} : A \to G_1(K, M)$ such that f is the composite $A \xrightarrow{\bar{f}} G_1(K, M) \hookrightarrow H_1(M \setminus K; \mathbb{Z})$. Thus $G_1(K, M)$ is not chosen merely because it makes the theorem work; it is the maximal ambient-blind first-homology subgroup created by deleting K.

Proof. The condition $i_* \circ f = 0$ says exactly that $\operatorname{im} f \subseteq \ker i_* = G_1(K, M)$. This inclusion of the image defines $\bar{f}$ uniquely because $G_1(K, M) \hookrightarrow H_1(M \setminus K; \mathbb{Z})$ is a monomorphism. □

Remark 3.3 (Role of the universal property). This universal property is elementary: it is the universal property of a kernel in the category of abelian groups. Its purpose here is not to claim deep new algebraic topology, but to identify the maximal ambient-blind first-homology subgroup and to prevent arbitrary choices of raw complement homology from being mistaken for the LD-RS defect signal.

Definition 3.4 (Defect-generated loop-only phase readout). A *defect-generated loop-only phase readout* of $K \subset M$ is a nontrivial character

$$\chi : G_1(K, M) \longrightarrow \mathrm{U}(1).$$

Equivalently, it is a stable phase assigned to one-dimensional closed histories in the complement whose homology class is created by the defect, not by ambient loop topology.

Proposition 3.5 (Characters detect nonzero finitely generated charge). *A finitely generated abelian group A admits a nontrivial character $A \to \mathrm{U}(1)$ if and only if $A \neq 0$. The homology groups used below are finitely generated because compact smooth manifolds and complements of smooth closed submanifolds have the homotopy type of finite CW complexes.*

Proof. A nontrivial character cannot have the zero group as its domain. Conversely, write $A \cong \mathbb{Z}^r \oplus T$ with T finite. If $r > 0$, send one free generator to an element of $\mathrm{U}(1)$ different from 1. If $r = 0$ and $T \neq 0$, choose a cyclic summand of order $m > 1$ and send its generator to an mth root of unity different from 1. Send all remaining summands to 1. □

Definition 3.6 (Divisibility in a lattice). Let L be a free abelian group and let $v \in L$. The divisibility $\mathrm{div}(v)$ is 0 if $v = 0$; otherwise it is the unique integer $m \geq 1$ such that $v = mv_0$ with v_0 primitive in L.

Lemma 3.7 (Surface Thom restriction). *Let $K \subset M^2$ be a two-sided smooth embedded circle in a connected closed orientable surface. Let N be an oriented annular tubular neighborhood of K and put $X = M^2 \setminus \operatorname{int} N$. Under excision and the Thom isomorphism*

$$H_2(M^2, X; \mathbb{Z}) \cong H_2(N, \partial N; \mathbb{Z}) \cong H_1(K; \mathbb{Z}),$$

the map

$$H_2(M^2; \mathbb{Z}) \longrightarrow H_2(M^2, X; \mathbb{Z})$$

sends the fundamental class $[M^2]$ to $\pm[K]$. Hence it is an isomorphism.

Proof. The map is restriction of the absolute fundamental class to the pair (M^2, X). By excision this is the relative fundamental class $[N, \partial N]$ of the oriented annulus, up to the sign determined by the chosen orientation of K. The Thom isomorphism for the oriented normal line bundle $N \to K$ sends $[N, \partial N]$ to the fundamental class of the zero section K. Since both $H_2(M^2; \mathbb{Z})$ and $H_1(K; \mathbb{Z})$ are infinite cyclic, the image of a generator is a generator, and the map is an isomorphism. □

The proof uses the long exact sequence of the pair (M, X), where $X = M \setminus \operatorname{int} N(K)$ is the compact complement of a tubular neighborhood, together with excision and the Thom isomorphism for the normal bundle of K.

Theorem 3.8 (Defect-generated circle charge in all substrate dimensions). *Let $d \geq 2$, let M^d be a connected closed orientable smooth d-manifold, and let $K \cong S^1 \subset M^d$ be a smooth embedded circle. Then:*

(a) *if $d = 2$, the inclusion $M^2 \setminus K \hookrightarrow M^2$ induces an injective map on first homology, and therefore*

$$G_1(K, M) = 0;$$

(b) *if $d \geq 4$, the inclusion $M^d \setminus K \hookrightarrow M^d$ induces an isomorphism on first homology, and therefore*

$$G_1(K, M) = 0;$$

(c) *if $d = 3$, then*

$$G_1(K, M) \cong \mathbb{Z}/m\mathbb{Z},$$

where $m = \operatorname{div}([K]_{\text{free}})$ and $[K]_{\text{free}}$ is the image of $[K] \in H_1(M^3; \mathbb{Z})$ in the free lattice $H_1(M^3; \mathbb{Z})/\operatorname{Tor} H_1(M^3; \mathbb{Z})$. The convention is $\mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$.

More explicitly, in dimension three the cases are

free class of K	$\operatorname{im}(\lambda_K)$	$G_1(K, M)$
0	0	$\mathbb{Z}$
primitive	$\mathbb{Z}$	0
divisibility $m > 1$	$m\mathbb{Z}$	$\mathbb{Z}/m\mathbb{Z}$,

where $\lambda_K : H_2(M^3; \mathbb{Z}) \to \mathbb{Z}$ is algebraic intersection with K.

Proof. Let N be a closed tubular neighborhood of K, and put $X = M^d \setminus \operatorname{int} N$. The complement $M^d \setminus K$ deformation retracts onto X, so $H_1(X; \mathbb{Z})$ computes the first homology of the complement. The maps to $H_1(M^d; \mathbb{Z})$ are compatible with this deformation retraction.

The normal bundle of K is orientable. Indeed, M^d and $K \cong S^1$ are orientable, and the Whitney formula gives

$$w_1(TM^d|_K) = w_1(TK) + w_1(\nu_K) = 0.$$

Excision and the integral Thom isomorphism for the oriented rank-$(d-1)$ disk bundle $N \to K$ give

$$H_q(M^d, X; \mathbb{Z}) \cong H_q(N, \partial N; \mathbb{Z}) \cong H_{q-(d-1)}(S^1; \mathbb{Z}). \tag{1}$$

For $d = 2$, the relevant part of the long exact sequence of the pair is

$$H_2(M^2; \mathbb{Z}) \longrightarrow H_2(M^2, X; \mathbb{Z}) \longrightarrow H_1(X; \mathbb{Z}) \longrightarrow H_1(M^2; \mathbb{Z}).$$

By Lemma 3.7, the first map is an isomorphism. By exactness, the connecting map $H_2(M^2, X; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ is zero, and $H_1(X; \mathbb{Z}) \to H_1(M^2; \mathbb{Z})$ is injective. Therefore $G_1(K, M) = 0$.

For $d \geq 4$, (1) gives $H_2(M^d, X; \mathbb{Z}) = H_{3-d}(S^1; \mathbb{Z}) = 0$ and $H_1(M^d, X; \mathbb{Z}) = H_{2-d}(S^1; \mathbb{Z}) = 0$. The long exact sequence of the pair (M^d, X) therefore contains

$$0 \longrightarrow H_1(X; \mathbb{Z}) \longrightarrow H_1(M^d; \mathbb{Z}) \longrightarrow 0,$$

so inclusion induces an isomorphism on first homology. Thus the kernel $G_1(K, M)$ is zero.

For $d = 3$, the same long exact sequence contains

$$H_2(M^3; \mathbb{Z}) \xrightarrow{\lambda_K} H_2(M^3, X; \mathbb{Z}) \cong H_0(S^1; \mathbb{Z}) \cong \mathbb{Z} \longrightarrow H_1(X; \mathbb{Z}) \longrightarrow H_1(M^3; \mathbb{Z}) \longrightarrow 0. \tag{2}$$

By exactness,

$$G_1(K, M) = \ker(H_1(X; \mathbb{Z}) \to H_1(M^3; \mathbb{Z})) \cong \operatorname{coker}(\lambda_K).$$

The map λ_K is algebraic intersection with K. Under Poincaré duality and the universal coefficient theorem, $H_2(M^3; \mathbb{Z})$ pairs with the free lattice $H_1(M^3; \mathbb{Z})/\operatorname{Tor} H_1(M^3; \mathbb{Z})$, and the image of the evaluation map on $[K]_{\text{free}}$ is exactly $m\mathbb{Z}$, where $m = \operatorname{div}([K]_{\text{free}})$. Therefore $\operatorname{coker}(\lambda_K) \cong \mathbb{Z}/m\mathbb{Z}$, with $m = 0$ giving $\mathbb{Z}$. □

Corollary 3.9 (Nonprimitive class criterion in dimension three)**.** *Let $K \cong S^1 \subset M^3$ be a smooth embedded circle in a connected closed orientable three-manifold. A nontrivial defect-generated loop-only character $G_1(K, M) \to \mathrm{U}(1)$ exists if and only if the free part of $[K] \in H_1(M^3; \mathbb{Z})$ is not primitive. Equivalently,*

$$G_1(K, M) \neq 0 \quad \Longleftrightarrow \quad \mathrm{div}([K]_{\mathrm{free}}) \neq 1,$$

where the zero free class has divisibility 0 *and gives $G_1(K, M) \cong \mathbb{Z}$.*

Proof. By Theorem 3.8, $G_1(K, M) \cong \mathbb{Z}/m\mathbb{Z}$ with $m = \mathrm{div}([K]_{\mathrm{free}})$ and the convention $\mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$. This group is zero exactly when $m = 1$, i.e. exactly when the free class is primitive. Since $G_1(K, M)$ is finitely generated, Proposition 3.5 gives the stated equivalence with the existence of a nontrivial $\mathrm{U}(1)$ character. □

Theorem 3.10 (Defect-generated loop selector)**.** *Let $d \geq 2$, let M^d be connected, closed, orientable, and smooth, and let $K \cong S^1 \subset M^d$ be a smooth embedded circle. If K admits a nontrivial defect-generated loop-only phase readout*

$$\chi : G_1(K, M) \to \mathrm{U}(1),$$

then $d = 3$.

Proof. By Proposition 3.5, the character implies $G_1(K, M) \neq 0$. Theorem 3.8 says that $G_1(K, M) = 0$ for $d = 2$ and for every $d \geq 4$. The only remaining substrate dimension with $d \geq 2$ is $d = 3$. □

Corollary 3.11 (Clean closed first-acyclic circle selector)**.** *Let $d \geq 2$, let M^d be a connected closed orientable smooth d-manifold with $H_1(M^d; \mathbb{Z}) = 0$, and let $K \cong S^1 \subset M^d$ be a smooth embedded circle. Then*

$$H_1(M^d \setminus K; \mathbb{Z}) \neq 0 \quad \Longleftrightarrow \quad d = 3.$$

In dimension three, $H_1(M^3 \setminus K; \mathbb{Z}) \cong \mathbb{Z}$, generated by a meridian.

Proof. The equality $H_1(M^d; \mathbb{Z}) = 0$ gives $G_1(K, M) = H_1(M^d \setminus K; \mathbb{Z})$. Theorem 3.8 then gives zero for $d = 2$ and for $d \geq 4$, and gives $\mathbb{Z}$ for $d = 3$ because $[K]_{\mathrm{free}} = 0$. □

Remark 3.12 (What the selector does and does not detect). The invariant $G_1(K, M)$ detects defect-generated loop charge. It does not detect knot type, it does not assert intrinsic linking of an entire graph, and it does not say that circle complements have homology only in dimension three. Higher-dimensional circle complements have meridional homology in higher degree; what is special about dimension three is that the meridian is a loop.

4 Finite recognition bridge

This section proves the finite bridge from binary recognizers to a circle. The arguments here are elementary, and the graph-theoretic ingredients are standard; the point is not to claim novelty for Gray codes or hypercube Hamiltonian cycles, but to identify their role in the recognition extension. The assumptions are load-bearing: binary surjectivity, additive local transition costs, and state-complete recurrence do not follow from topology. In particular, the hypercube is derived from CT, not from the reciprocal cost law alone. The cost law supplies the scalar local penalty; CT supplies locality, coordinate separability, and the primitive-transition criterion.

Definition 4.1 (BQ: full jointly realized binary recognition quotient)**.** Let C be a configuration set. A family of binary recognizers $\beta_1, \ldots, \beta_n : C \to \{0,1\}$ is *jointly surjective* if the product map

$$\beta : C \to \{0,1\}^n, \qquad \beta(c) = (\beta_1(c), \ldots, \beta_n(c))$$

is surjective. Under BQ, the finite recognition quotient is identified with $V_n = \{0,1\}^n$. No probabilistic, statistical, or causal independence is intended by this condition; it says only that every joint bit pattern is realized by at least one configuration.

Definition 4.2 (Reciprocal local flip penalty)**.** For $a > 0$, define

$$J_a(x) = \frac{1}{2}(x^a + x^{-a}) - 1, \qquad x \in \mathbb{R}_{>0}.$$

The calibrated reciprocal penalty is $J = J_1$.

Lemma 4.3 (Positivity of reciprocal penalties)**.** *For $x > 0$, $J_a(x) \geq 0$, with equality if and only if $x = 1$.*

Proof. Put $y = x^a > 0$. Then $J_a(x) = (y + y^{-1})/2 - 1 \geq 0$ by the arithmetic-geometric mean inequality, with equality exactly when $y = 1$, equivalently $x = 1$. □

Definition 4.4 (CT: reciprocal local transition cost)**.** Assume BQ. A transition cost on V_n satisfies CT if the following data are specified.

(a) For each coordinate i and directed flip $\epsilon \to 1 - \epsilon$, there is a local scale ratio $\rho_i(\epsilon, 1-\epsilon) > 0$ with $\rho_i(\epsilon, 1-\epsilon) \neq 1$.

(b) The scalar local flip penalty is the reciprocal cost $J_a(\rho_i(\epsilon, 1-\epsilon))$.

(c) Simultaneous coordinate changes are evaluated by additive product locality:

$$C_0(b, b') = \sum_{\{i:\, b_i \neq b'_i\}} J_a(\rho_i(b_i, b'_i)).$$

(d) A nontrivial directed transition $b \to b'$ is primitive when it is cost-indecomposable: there is no intermediate state $u \neq b, b'$ such that

$$C_0(b, b') = C_0(b, u) + C_0(u, b').$$

The primitive transition graph joins two states when at least one of the two directed transitions between them is cost-indecomposable.

Remark 4.5 (CT is the bridge, not the scalar formula)*.* The reciprocal function J_a assigns a penalty to a positive ratio. It does not by itself define a transition graph, forbid correlated multi-bit updates, or choose cost-indecomposability as the criterion for primitive adjacency. The hypercube follows from the CT bridge: local ratios, additive product locality, and the primitive-transition rule.

Remark 4.6 (Future CT derivation target)*.* A future derivation of CT from deeper Recognition-Science principles would need to start from reciprocal local ratio penalties plus a formal product-composition axiom for jointly realized recognizers, derive additive transition cost, and then prove that cost-indecomposable transitions are exactly single-coordinate flips. Until such a theorem is supplied, CT remains an explicit bridge principle.

Proposition 4.7 (CT derives hypercube adjacency). *Under CT, the cost-indecomposable nontrivial transitions are exactly the Hamming-distance-one transitions. Hence the primitive transition graph is the hypercube Q_n.*

Proof. By Lemma 4.3, each directed one-bit flip has positive local cost. Thus C_0 is a positive separable local bit-flip cost.

If b and b' differ in at least two coordinates, choose one differing coordinate j and let u be obtained from b by changing only coordinate j to its value in b'. The coordinates changed from b to u and from u to b' partition the coordinates changed from b to b', so additivity gives

$$C_0(b, b') = C_0(b, u) + C_0(u, b').$$

Thus a multi-bit transition is decomposable.

If b and b' differ in exactly one coordinate i, any intermediate $u \neq b, b'$ must leave and return along at least one coordinate $j \neq i$, in addition to making the required change in coordinate i. Since every local flip cost is positive,

$$C_0(b, u) + C_0(u, b') > C_0(b, b').$$

So no equality decomposition exists. The indecomposable transitions are exactly the one-bit transitions. □

The exact equality criterion is useful but idealized. The next result gives a robust version: if small interaction errors are present, the single-bit graph is recovered by a tolerance-gapped notion of primitiveness. The finite gap $\delta_{\min} > 0$ below is a uniform separation assumption; it is not automatic if local ratios are allowed to approach 1.

Definition 4.8 (Tolerance-indecomposable transition). Let C be a directed transition functional on V_n and let $\tau \geq 0$. A nontrivial transition $b \to b'$ is *τ-indecomposable* if there is no intermediate state $u \neq b, b'$ with

$$C(b, u) + C(u, b') \leq C(b, b') + \tau.$$

For $\tau = 0$, this excludes transitions that are no more expensive than a one-step decomposition. Positive τ treats near-decomposable transitions as nonprimitive.

Proposition 4.9 (Gap stability of the hypercube generator). *Let C_0 be a CT cost and put*

$$\delta_{\min} = \min_{i,\epsilon} J_a\big(\rho_i(\epsilon, 1-\epsilon)\big) > 0.$$

Suppose C is a perturbed directed transition functional satisfying $|C(b, b') - C_0(b, b')| \leq \varepsilon$ for all b, b', with $0 \leq \varepsilon < \delta_{\min}/3$. No nonnegativity of C is needed for this purely order-theoretic statement, although a literal nonnegative cost model should impose it separately. Then, for $\tau = 3\varepsilon$, the τ-indecomposable transitions of C are exactly the Hamming-distance-one transitions.

Proof. If b and b' differ in at least two coordinates, choose the one-coordinate intermediate u used in the proof of Proposition 4.7. For C_0 one has equality $C_0(b, b') = C_0(b, u) + C_0(u, b')$. Therefore

$$C(b, u) + C(u, b') \leq C_0(b, u) + C_0(u, b') + 2\varepsilon = C_0(b, b') + 2\varepsilon \leq C(b, b') + 3\varepsilon.$$

Thus the multi-bit transition is not 3ε-indecomposable.

If b and b' differ in exactly one coordinate, then for any intermediate $u \neq b, b'$ the unperturbed detour pays at least two extra local flips, so

$$C_0(b,u) + C_0(u,b') \geq C_0(b,b') + 2\delta_{\min}.$$

Hence

$$C(b,u) + C(u,b') \geq C_0(b,u) + C_0(u,b') - 2\varepsilon \geq C_0(b,b') + 2\delta_{\min} - 2\varepsilon \geq C(b,b') + 2\delta_{\min} - 3\varepsilon.$$

Since $\varepsilon < \delta_{\min}/3$, the right-hand side is strictly larger than $C(b,b') + 3\varepsilon$. No intermediate gives a 3ε-decomposition, so the one-bit transition is 3ε-indecomposable. □

Remark 4.10 (Cost terminology). In physical cost models one imposes nonnegativity on C. Proposition 4.9 isolates only the order comparison needed for generator stability, so its proof continues to hold for any perturbed functional satisfying the displayed uniform comparison bound.

Remark 4.11 (Uniform gap assumption). The quantity $\delta_{\min} > 0$ is a finite-model separation assumption. It is not automatic in a limiting model where some local ratio can approach 1, because then the reciprocal penalty can approach zero and the perturbative graph selection may lose its gap.

Definition 4.12 (Recognized-history category and SR)**.** For a primitive graph G on V_n, the finite recognized-history category $\mathcal{H}(G)$ has objects the states of V_n and morphisms finite directed edge paths, with composition by concatenation. A closed morphism is a *recurrence*. A recurrence is *state-complete* or *state-fair* if every recognized quotient state appears during one period. Axiom SR says that persistent recognized defects are represented by minimal-length state-complete recurrences in the primitive graph.

Remark 4.13 (What a future derivation of SR would need). SR is a dynamical, fairness, or least-action assumption. A derivation would need a no-starvation or ledger-completeness principle forcing every recognized state to occur at least once per period, followed by a minimal-action or minimal-length principle forbidding repetitions. The graph theorem below proves that such a recurrence is Hamiltonian in Q_n; it does not prove the dynamical principle that selects such recurrences.

Definition 4.14 (Reflected Gray code)**.** The reflected Gray code is defined recursively by

$$\mathcal{G}_1 = ((0),(1)), \qquad \mathcal{G}_{n+1} = (0 \cdot \mathcal{G}_n) \,\|\, (1 \cdot \mathcal{G}_n^{\mathrm{rev}}),$$

where $\|$ denotes concatenation and $\mathcal{G}_n^{\mathrm{rev}}$ denotes reversal.

Remark 4.15 (Graph-theoretic status). The existence of Hamiltonian cycles in hypercubes is standard. The reflected Gray code is used here only to keep the finite-recognition bridge self-contained: once CT gives Q_n and SR asks for a minimal state-complete recurrence, the selected recurrence circle is a familiar Hamiltonian cycle. No graph-theoretic novelty is claimed in Lemma 4.16; the novelty, if any, lies in the recognition interpretation of the selected recurrence.

Lemma 4.16 (Gray-code Hamiltonian cycle)**.** *For $n \geq 2$, $\mathcal{G}_n$ lists all 2^n vertices of Q_n exactly once, and successive entries, including the wraparound, differ in one coordinate. Hence it traces a Hamiltonian cycle in Q_n.*

Proof. The standard induction gives length 2^n, pairwise distinct entries, and one-coordinate changes between successive entries. In passing from n to $n+1$, the first half has leading bit 0, the second half has leading bit 1, the internal adjacency property is inherited, the middle transition changes only the leading bit, and the wraparound changes only the leading bit. For $n \geq 2$ the resulting closed walk has at least four vertices and is a simple cycle. □

Definition 4.17 (State-complete closed walk)**.** A closed walk in Q_n is a sequence $\gamma = (v_0, v_1, \ldots, v_L)$ with $v_L = v_0$ and each consecutive pair an edge. It is state-complete if $\{v_0, \ldots, v_{L-1}\} = V_n$. It is minimal if no shorter state-complete closed walk exists.

Proposition 4.18 (Minimal state-complete recurrences are circles)**.** *Let $n \geq 2$, and let γ be a minimal state-complete closed walk in Q_n. Then γ has length 2^n, visits every vertex exactly once before returning to its start, traverses 2^n pairwise distinct edges, and its image subcomplex $C_\gamma \subset |Q_n|$ is homeomorphic to S^1.*

Proof. A length-L closed walk lists at most L distinct vertices before the terminal return. State-completeness therefore forces $L \geq 2^n$. Lemma 4.16 attains $L = 2^n$, so any minimal state-complete closed walk has length 2^n and visits every vertex exactly once before returning.

It remains to exclude a repeated unoriented edge. Write indices modulo $L = 2^n$ and let $e_i = \{v_i, v_{i+1}\}$. If $e_i = e_j$ with $i \neq j$, then the cyclic uniqueness of vertices rules out $v_i = v_j$ and $v_{i+1} = v_{j+1}$, so one must have $v_i = v_{j+1}$ and $v_{i+1} = v_j$. Uniqueness of vertices in the cyclic list gives $i \equiv j + 1$ and $i + 1 \equiv j \pmod{L}$, hence $2 \equiv 0 \pmod{L}$. Since $L = 2^n \geq 4$, this is impossible. Thus the image is a finite graph with the vertices and edges attached in one cyclic order, hence a circle. $\square$

Remark 4.19 (Why the Hamiltonian condition matters)*.* The topological selector needs only a circle. A square face of a hypercube would also be a circle. The state-complete Hamiltonian condition matters only because it ties the selected circle to all 2^n recognized states. The reason for selecting that recurrence must be SR or a future dynamical theorem, not the algebraic topology of complements.

5 Substrate realization, compactification, and loop phases

The finite graph supplies only an abstract circle. To apply the topological selector, the circle must be realized as a smooth defect in a controlled ambient substrate, and the phase readout must detect defect-generated loop charge. LP is the decisive model bridge: without a nontrivial one-dimensional history phase on defect-generated loop charge, the topology has nothing to select.

The compactified readout maps have the form

$$H_1(Y \setminus K; \mathbb{Z}) \longrightarrow H_1(M \setminus K; \mathbb{Z}) \longrightarrow H_1(M; \mathbb{Z}), \qquad G_1(K, M) \twoheadrightarrow G_{1,\mathrm{ad}}(K, Y; M, E),$$

where the second display quotient kills the specified end-supported subgroup. This notation is fixed in Definitions 5.5–5.7.

Definition 5.1 (SRL: smooth substrate realization)**.** Let $C_\gamma \cong S^1$ be the image of a selected recurrence. The predicate SRL is a two-case realization predicate.

(SRL1) **Closed realization.** There is a connected closed orientable smooth d-manifold M^d, a smooth embedded circle $K \subset M^d$, and a specified realization map

$$h : C_\gamma \xrightarrow{\cong} K.$$

Equivalently, after identifying C_γ with a parametrized circle, one may write $K = \iota(S^1)$ for a smooth embedding $\iota : S^1 \hookrightarrow M^d$.

(SRL2) **Admissibly compactified realization.** There is a noncompact recognition substrate Y with defect $K \subset Y$, a connected closed orientable smooth compactification M^d with end set $E \subset M^d$ disjoint from K, an identification $Y = M^d \setminus E$, and an admissible compactified readout package in the sense of Definition 5.7. The same specified realization map $h : C_\gamma \xrightarrow{\cong} K$ ties the finite recurrence circle to the smooth defect core.

In both cases h is data, not notation: it is what ties the finite Hamiltonian recurrence to the smooth defect and transports the cyclic vertex labels to marked points on K. No global retraction $M^d \to C_\gamma$ is assumed.

5.1 Core recurrence versus complement readout

The realization map $h : C_\gamma \to K$ identifies the finite Hamiltonian recurrence with the *defect core*. LP does not evaluate a phase on that core loop. The core lies in K, while the LP phase is evaluated on closed recognized probe histories in $M^d \setminus K$. These probe histories are complement loops, typically supported on or near the boundary of a tubular neighborhood of K. This distinction is essential: the recurrence selects the circle defect; the complement readout measures how one-dimensional histories wind around the selected defect.

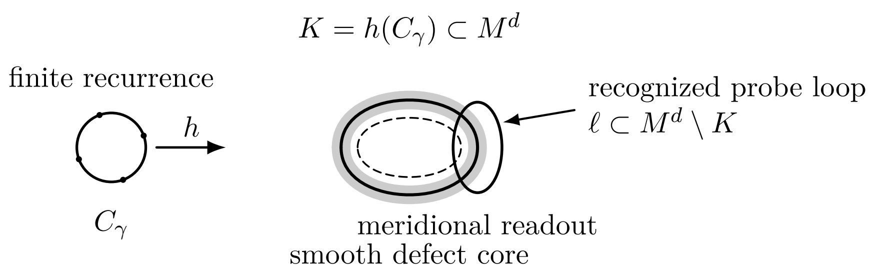

Figure 1: Core/complement separation. The Hamiltonian recurrence is tied to the smooth defect core by $h : C_\gamma \cong K$. LP is evaluated on recognized probe loops in $M^d \setminus K$, not on the core itself. The meridian is shown schematically.

Lemma 5.2 (Local meridional interface)**.** *Let $K \cong S^1$ be a smooth embedded circle in a connected orientable smooth d-manifold M^d, and let $N(K)$ be a tubular neighborhood. The boundary projection*

$$\partial N(K) \longrightarrow K$$

is a sphere bundle with fiber S^{d-2}. The fiber has nonzero first homology exactly when $d = 3$. In that case the fiber circle is the local meridian loop; after K is filled back in, it bounds the normal disk fiber in $N(K)$, so its homology class maps to zero in $H_1(M^d; \mathbb{Z})$ and is a candidate element of $G_1(K, M)$.

Proof. The normal bundle of a smooth circle in an orientable smooth manifold has rank $d-1$, so the boundary of its disk bundle has fiber S^{d-2}. Since $H_1(S^0; \mathbb{Z}) = 0$, $H_1(S^1; \mathbb{Z}) \cong \mathbb{Z}$, and $H_1(S^k; \mathbb{Z}) = 0$ for $k \geq 2$, the fiber has nonzero first homology exactly for $d - 2 = 1$, i.e. $d = 3$. When $d = 3$, the fiber is the usual meridian circle. It bounds the corresponding normal disk after the defect core is restored, so its image in ambient first homology is zero. Whether this local meridian survives as a nonzero global class in $G_1(K, M)$ is precisely the nontriviality content of LP, analyzed in Theorem 3.8 and Corollary 3.9. $\square$

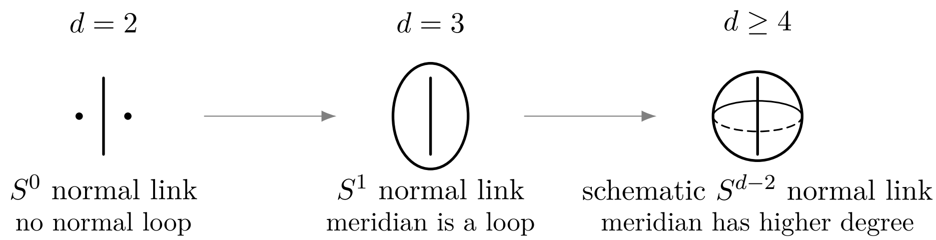

Figure 2: Normal-link degree for a circle defect. Locally the core looks like a line and the normal link is S^{d-2}. A defect-generated meridian is a one-dimensional loop exactly when $d = 3$; the $d \geq 4$ panel is schematic and is drawn like S^2 when $d = 4$.

Remark 5.3 (What LP reads). The Hamiltonian recurrence is not itself the meridional phase loop. It is the core whose removal creates the complement in which probe loops live. In $d = 2$, the normal fiber is S^0, so there is no local normal loop. In $d \geq 4$, the normal fiber has dimension at least two, so the local meridian is higher-dimensional rather than a loop. LP is therefore a one-dimensional complement-readout principle, not a phase assigned directly to the core recurrence.

Remark 5.4 (Surface boundary loops are longitudinal, not meridional). When $d = 2$, a tubular neighborhood of K is an annulus and $\partial N(K)$ has circle components. These boundary circles are longitudinal copies of the core direction, not normal meridians: the normal fiber is only S^0. The long exact sequence calculation in Theorem 3.8 shows that these longitudinal boundary classes inject into ambient $H_1(M^2; \mathbb{Z})$ and therefore do not define defect-generated loop charge in $G_1(K, M)$.

Readers interested only in closed substrates may skip the compactified readout definition on a first reading; the closed selector uses $G_1(K, M)$ directly.

Definition 5.5 (Tame compactified end basis)**.** Let Y be a noncompact recognition substrate with defect $K \subset Y$. A *tame compactified end basis* consists of a connected closed orientable smooth manifold M^d, a closed end set $E \subset M^d$ disjoint from K, an identification $Y = M^d \setminus E$, and a cofinal neighborhood basis

$$\mathcal{N}(E)$$

by compact codimension-zero smooth submanifolds with smooth boundary, each containing E and, after passing to sufficiently small neighborhoods, disjoint from K. Thus the phrase "supported arbitrarily close to E" is computed by ordinary smooth neighborhoods rather than by wild local pathologies. Finite sets of ideal points and compact smooth boundary-at-infinity pieces satisfy this condition. The compactified selector is not formulated for arbitrary wild closed end sets.

Definition 5.6 (End-supported subgroup)**.** Given a tame compactified end basis as in Definition 5.5, let

$$j : Y \setminus K \hookrightarrow M^d \setminus K$$

be inclusion. The *end-supported subgroup* is

$$E_1(Y, K; E) = \bigcap_{U \in \mathcal{N}(E)} \operatorname{im}\Big(H_1((U \cap Y) \setminus K; \mathbb{Z}) \longrightarrow H_1(Y \setminus K; \mathbb{Z})\Big).$$

For sufficiently small U, the condition $U \cap K = \varnothing$ makes the notation $(U \cap Y) \setminus K$ redundant, but it keeps the maps uniform. The right-hand side is an intersection of subgroups of $H_1(Y \setminus K; \mathbb{Z})$ and is therefore a subgroup. Lemma 5.11 records that a different cofinal tame basis gives the same subgroup.

Definition 5.7 (Admissible compactified quotient and phase)**.** An *admissible finite-recognition compactified readout* is a tame compactified end basis together with the following group-level calibration. Put

$$E_1^M(K,Y;E) = j_*(E_1(Y,K;E) \cap j_*^{-1}G_1(K,M)) \le G_1(K,M).$$

The compactified admissible defect group is the quotient

$$G_{1,\mathrm{ad}}(K,Y;M,E) = G_1(K,M)/E_1^M(K,Y;E),$$

with quotient projection

$$\pi_{\mathrm{ad}} : G_1(K,M) \twoheadrightarrow G_{1,\mathrm{ad}}(K,Y;M,E).$$

An admissible compactified phase is a nontrivial character

$$\bar{\chi} : G_{1,\mathrm{ad}}(K,Y;M,E) \to \mathrm{U}(1).$$

Equivalently,

$$\chi = \bar{\chi} \circ \pi_{\mathrm{ad}} : G_1(K,M) \to \mathrm{U}(1)$$

is a character that is trivial on $E_1^M(K,Y;E)$ and nontrivial on some non-end defect-generated class. A recognized loop in $Y \setminus K$ is evaluated only when its compactified class lies in $G_1(K,M)$, and its phase is computed after projection by π_{ad}.

Remark 5.8 (Compactified readouts are relative)*.* End-supported classes are not asserted to be absent; they are declared phase-trivial or inadmissible for this finite-recognition selector. This is a calibrated finite-readout convention, not a theorem that raw noncompact end modes vanish. In the noncompact case the dimension conclusion is relative to the chosen admissible compactification and end calibration. Different admissible compactifications or different end-calibration rules can define different finite-readout groups; the selector applies only to the specified compactified readout package. It is not a canonical theorem about all raw noncompact complements. The calibration is also not allowed to manufacture LP by arbitrary fiat: if all non-end defect-generated meridional classes are killed, then the compactified LP hypothesis simply fails.

Proposition 5.9 (Universal admissible compactified target)**.** *The quotient*

$$\pi_{\mathrm{ad}} : G_1(K,M) \twoheadrightarrow G_{1,\mathrm{ad}}(K,Y;M,E)$$

is universal among defect-generated loop targets on which end-supported classes are declared trivial. Namely, for every abelian group B and every homomorphism $\psi : G_1(K,M) \to B$ with $E_1^M(K,Y;E) \subseteq \ker\psi$, there is a unique homomorphism $\bar{\psi} : G_{1,\mathrm{ad}}(K,Y;M,E) \to B$ with $\psi = \bar{\psi} \circ \pi_{\mathrm{ad}}$. In particular,

$$\mathrm{Hom}(G_{1,\mathrm{ad}}(K,Y;M,E),\mathrm{U}(1)) \cong \{\chi : G_1(K,M) \to \mathrm{U}(1) \mid \chi|_{E_1^M(K,Y;E)} = 1\}.$$

Proof. This is the universal property of the quotient of an abelian group by a subgroup. The displayed bijection is obtained by taking $B = \mathrm{U}(1)$ and composing characters with π_{ad}. $\square$

Remark 5.10 (Elementary quotient hygiene)*.* This universal property is again elementary. Its role is to record exactly which end-supported defect classes the chosen finite-readout package declares phase-trivial, not to promote $G_{1,\mathrm{ad}}$ to a canonical invariant of every raw noncompact complement.

Lemma 5.11 (Basis independence of the end subgroup)**.** *If $\mathcal{N}(E)$ and $\mathcal{N}'(E)$ are cofinal tame neighborhood bases of E, then the intersection-of-images definition of $E_1(Y, K; E)$ gives the same subgroup of $H_1(Y \setminus K; \mathbb{Z})$ for the two bases.*

Proof. Let $E_1(\mathcal{N})$ and $E_1(\mathcal{N}')$ denote the two intersections. To show $E_1(\mathcal{N}') \subseteq E_1(\mathcal{N})$, fix $U \in \mathcal{N}(E)$. Cofinality gives $U' \in \mathcal{N}'(E)$ with $U' \subset U$. The image of $H_1((U' \cap Y) \setminus K; \mathbb{Z})$ in $H_1(Y \setminus K; \mathbb{Z})$ is contained in the corresponding image for U. Any class lying in every image for $\mathcal{N}'(E)$ therefore lies in the image for this U. Since U was arbitrary, it lies in $E_1(\mathcal{N})$. The reverse inclusion is identical with the two bases interchanged. □

Lemma 5.12 (Nontrivial admissible quotient implies non-end defect charge)**.** *If $G_{1,\mathrm{ad}}(K, Y; M, E) \neq 0$, then $G_1(K, M) \neq 0$. If $\bar{\chi} : G_{1,\mathrm{ad}}(K, Y; M, E) \to \mathrm{U}(1)$ is nontrivial, then its pullback $\chi = \bar{\chi} \circ \pi_{\mathrm{ad}}$ is nontrivial on a defect-generated class of $G_1(K, M)$ not lying in the end subgroup $E_1^M(K, Y; E)$.*

Proof. The admissible group is the quotient $G_1(K, M)/E_1^M(K, Y; E)$. If this quotient is nonzero, then the numerator is nonzero. If $\bar{\chi}$ is nontrivial, choose an admissible class $\bar{g}$ with $\bar{\chi}(\bar{g}) \neq 1$ and choose any lift $g \in G_1(K, M)$. The class g is not in $E_1^M(K, Y; E)$, because otherwise $\bar{g} = 0$. Moreover $\chi(g) = \bar{\chi}(\bar{g}) \neq 1$, so the pulled-back character is nontrivial on a non-end defect-generated class. □

Examples of admissible and inadmissible compactified readout

Example 5.13 (Good compactification: a bounded circle in $\mathbb{R}^3$)**.** Let $Y = \mathbb{R}^3$ and let $K \subset \mathbb{R}^3$ be a bounded smooth unknot. Compactify by $M = S^3$ and $E = \{\infty\}$, with the standard tame point-end neighborhood basis. The point at infinity is disjoint from K, and sufficiently small neighborhoods of ∞ are balls disjoint from the defect. Loops supported in those balls represent zero in $H_1(S^3 \setminus K; \mathbb{Z})$, so $E_1^M(K, Y; E) = 0$. Since $H_1(S^3; \mathbb{Z}) = 0$ and K is null-homologous, Theorem 3.8 gives

$$G_1(K, S^3) \cong \mathbb{Z}, \qquad G_{1,\mathrm{ad}}(K, \mathbb{R}^3; S^3, \{\infty\}) \cong \mathbb{Z}.$$

The ordinary meridian survives the compactified calibration, so a nontrivial character on this admissible group is a genuine LP datum rather than an artifact of the end.

Example 5.14 (The two-dimensional raw-complement trap)**.** The raw noncompact complement $\mathbb{R}^2 \setminus S^1$ has an exterior component with first homology $\mathbb{Z}$. If one treated this raw group as a phase domain, a loop signal would appear already in dimension two. That is not the LD-RS selector. Under one-point compactification, $\mathbb{R}^2$ becomes S^2 and the circle separates S^2 into two disks. The closed-pair calculation gives

$$G_1(S^1, S^2) = 0.$$

The exterior raw loop is therefore not a defect-generated compactified complement-loop charge of the LD-RS kind. This example is why closedness or an explicit admissible compactified/end-calibrated readout is a load-bearing hypothesis, not a technical convenience.

Example 5.15 (Degenerate calibration: LP fails)**.** Return to a null-homologous unknot $K \subset S^3$, where $G_1(K, S^3) \cong \mathbb{Z}$. If a proposed finite-readout convention declares the meridian to be end-supported or otherwise quotients it out, then either it violates the end-support admissibility rule above or it leaves the readout group zero:

$$G_{1,\mathrm{ad}} = 0.$$

There is then no nontrivial character $G_{1,\mathrm{ad}} \to \mathrm{U}(1)$. The correct conclusion is not a false dimension selection; the correct conclusion is that LP is not satisfied by that readout. In particular, the compactified formalism cannot manufacture the theorem by killing all non-end defect-generated classes and then pretending a nontrivial phase remains.

Proposition 5.16 (No compatible global retraction)**.** *Let $d \geq 2$, let M^d be connected, closed, orientable, and first-acyclic, and let Q_n be a hypercube with $n \geq 2$. There is no continuous map $r : M^d \to |Q_n|$ and embedding $i : |Q_n| \hookrightarrow M^d$ with $r \circ i = \mathrm{id}_{|Q_n|}$.*

Proof. If such maps existed, then on first homology $r_* \circ i_* = \mathrm{id}_{H_1(|Q_n|;\mathbb{Z})}$. Thus i_* would be injective. But $H_1(M^d;\mathbb{Z}) = 0$, while $H_1(|Q_n|;\mathbb{Z})$ is nonzero for $n \geq 2$. This is impossible. □

Definition 5.17 (Defect-relevant recognized 1-cycles)**.** Let $K \subset M$ be a selected defect. Let $Z_1^{\mathrm{rec}}(M \setminus K)$ be the subgroup of singular 1-cycles in $M \setminus K$ generated by closed recognized histories. Its defect-relevant part is

$$Z_{1,G}^{\mathrm{rec}}(M \setminus K) = \{z \in Z_1^{\mathrm{rec}}(M \setminus K) : [z] \in G_1(K, M)\}.$$

A *recognized boundary* is an element of $Z_{1,G}^{\mathrm{rec}}(M \setminus K) \cap B_1(M \setminus K;\mathbb{Z})$. The recognized cycles have *full coverage* of the closed defect-generated group if every class in $G_1(K, M)$ has a representative in $Z_{1,G}^{\mathrm{rec}}(M \setminus K)$. They have *minimal detected coverage* if at least one recognized cycle represents a nonzero class of $G_1(K, M)$ detected by the selected character. For an admissible compactified readout, full coverage means that recognized compactified classes project onto the readout target used in $G_{1,\mathrm{ad}}(K, Y; M, E)$, while minimal detected coverage means that at least one recognized compactified class projects to a nonzero admissible class detected by the selected character.

Proposition 5.18 (History phases descend to defect-generated characters)**.** *Suppose that recognized defect-relevant histories define a group homomorphism*

$$\Phi : Z_{1,G}^{\mathrm{rec}}(M \setminus K) \to \mathrm{U}(1)$$

with the following properties:

(i) *Φ is trivial on recognized boundaries, i.e. $\Phi(b) = 1$ for every $b \in Z_{1,G}^{\mathrm{rec}}(M \setminus K) \cap B_1(M \setminus K;\mathbb{Z})$;*

(ii) *every class in $G_1(K, M)$ has a representative in $Z_{1,G}^{\mathrm{rec}}(M \setminus K)$.*

Then Φ induces a character

$$\chi_\Phi : G_1(K, M) \to \mathrm{U}(1).$$

If $\Phi(z) \neq 1$ for some recognized cycle representing a class of $G_1(K, M)$, then χ_Φ is nontrivial.

Proof. For $g \in G_1(K, M)$, choose $z \in Z_{1,G}^{\mathrm{rec}}(M \setminus K)$ with $[z] = g$ and set $\chi_\Phi(g) = \Phi(z)$. If z' is another recognized cycle with $[z'] = g$, then $z - z' \in Z_{1,G}^{\mathrm{rec}}(M \setminus K) \cap B_1(M \setminus K;\mathbb{Z})$, so $\Phi(z - z') = 1$. Since Φ is a homomorphism, $\Phi(z) = \Phi(z')$. Thus χ_Φ is well defined. Additivity of homology classes and multiplicativity of Φ give the character property. Nontriviality follows from any recognized representative on which Φ is not 1. □

Remark 5.19 (Relation to holonomy and defect phases). LP is structurally analogous to familiar phase readouts around excluded flux or defect cores, including Aharonov–Bohm phases and topological-defect holonomies. The LD-RS target is narrower: the phase must be nontrivial on defect-generated complement-loop charge, not merely on raw complement homology, ambient first homology, or noncompact end modes. This is why the kernel $G_1(K, M)$ and, in the compactified setting, the admissible quotient $G_{1,\mathrm{ad}}$ are used.

Proposition 5.20 (Abelian holonomy form of LP)**.** *In the closed case, specifying an abelian flat phase readout on complement loops is equivalent to specifying a character*

$$\eta : H_1(M \setminus K; \mathbb{Z}) \to \mathrm{U}(1).$$

Such a readout satisfies the LD-RS defect criterion exactly when its restriction

$$\eta|_{G_1(K,M)} : G_1(K, M) \to \mathrm{U}(1)$$

is nontrivial and ambient modes are ignored by passing to the defect-generated subgroup. In an admissibly compactified readout, the corresponding criterion is a nontrivial character on $G_{1,\mathrm{ad}}(K, Y; M, E)$, equivalently a character on $G_1(K, M)$ that is trivial on $E_1^M(K, Y; E)$ and nontrivial after quotienting.

Proof. For abelian phase readouts, homology-invariant loop phases are precisely group homomorphisms from first homology to U(1). The closed LD-RS target is not all of $H_1(M \setminus K; \mathbb{Z})$ but the subgroup $G_1(K, M)$ of classes killed when the defect is filled back in. Thus defect sensitivity is exactly nontriviality of the restriction to $G_1(K, M)$. In the compactified case, Definition 5.7 defines the admissible group as the quotient of $G_1(K, M)$ by end-supported defect classes, so the same statement is Proposition 5.9, the universal property of the quotient projection π_{ad}. □

Axiom 5.21 (LP: loop-only defect phase)**.** For the selected circle defect, stable recognized phases are represented by homology-invariant one-dimensional complement-history phases on recognized cycles with a specified coverage standard and are nontrivial after projection to defect-generated loop charge. LP is used in two strengths. *LP-full* is the intended semantic readout version; *LP-min* is the weaker version sufficient for the dimension-selector theorem. Both share subclauses LP1–LP5.

(LP0-full) Full representative coverage. The recognized complement-history cycles represent the readout group on which the phase is evaluated. In the closed case, every class of $G_1(K, M)$ has a recognized representative. In the compactified case, recognized compactified classes project onto the admissible target in $G_{1,\mathrm{ad}}(K, Y; M, E)$ on which the selected character is evaluated. Statements that construct or use a character on all of $G_1(K, M)$, such as Proposition 5.18, use this full version unless explicitly restricted to a subgroup.

(LP0-min) Minimal detected coverage. For the selector theorem it is enough that at least one recognized complement-history cycle represents a nonzero defect-generated class detected by the selected character. In the compactified case, this means at least one recognized compactified class projects to a nonzero class of $G_{1,\mathrm{ad}}(K, Y; M, E)$ detected by the selected character. Theorem 6.1 and Theorem 6.3 require this minimal version.

(LP1) **Degree-one carriers.** The admissible phase carriers are recognized 1-cycles, or equivalently sequential closed histories, in the complement $M^d \setminus K$ or in the compactified complement under consideration.

(LP2) **Homology invariance.** The phase is invariant under recognized boundaries and therefore descends, as in Proposition 5.18, to a character on the relevant first-homology quotient; the recognized cycles used for this descent are part of the LP data, not an automatic consequence of topology.

(LP3) **Defect-generated target.** In the closed case the target is not raw ambient complement homology but

$$G_1(K, M) = \ker\big(H_1(M \setminus K; \mathbb{Z}) \to H_1(M; \mathbb{Z})\big).$$

Ambient first-homology modes are not substitutes for defect-generated charge.

(LP4) **Compactified end rule.** In an admissible compactified readout $(Y, E, M^d, K, \pi_{\mathrm{ad}})$, end-supported classes are phase-trivial or inadmissible, and the phase is a character on the specified admissible quotient

$$\bar{\chi} : G_{1,\mathrm{ad}}(K, Y; M, E) \to \mathrm{U}(1),$$

equivalently a pullback

$$\chi = \bar{\chi} \circ \pi_{\mathrm{ad}} : G_1(K, M) \to \mathrm{U}(1)$$

that is trivial on $E_1^M(K, Y; E)$.

(LP5) **Nontriviality.** The selected phase is nontrivial on at least one non-end defect-generated class; in the compactified case this means nontrivial after projection to $G_{1,\mathrm{ad}}(K, Y; M, E)$.

Thus both LP-min and LP-full are recognized-history phase assumptions, not merely the existence of an abstract homology character disconnected from recognized histories. Higher-degree homology readouts, ambient first-homology readouts, and end-supported classes are not admissible substitutes for this axiom.

LP is the central bridge assumption of the paper. It is not derived from BQ, CT, SR, SRL, Recognition Geometry, or the scalar reciprocal cost law in the present paper; it is the loop-readout bridge assumed by the LD-RS model class. The topology proves only that such a loop-only defect-generated phase can occur for a circle defect only in ambient dimension three. In non-first-acyclic three-dimensional substrates, nontriviality includes the assertion that the selected embedding class has nonzero defect-generated loop charge; by Corollary 3.9, this is equivalent to the free homology class of K not being primitive after the selector has forced $d = 3$.

Why loop-only phases are the selector

LP is not a definition of dimension three. It is a dimension-independent predicate on candidate circle-defect substrates: it asks for a nontrivial degree-one phase on defect-generated complement-loop charge. The topological theorem proves that this predicate is satisfiable for smooth circle defects only in ambient dimension three.

The topological theorem does not say that circle complements have homology only in dimension three. For a circle defect in a d-dimensional substrate, the normal link is S^{d-2}, so the local meridional class lives in degree $d - 2$. The selector chooses $d = 3$ because the admissible recognized phases in LP are sequential one-dimensional histories. If higher-degree readouts are allowed, the same normal-link mechanism selects different ambient dimensions; Section 8 records the general degree matching rule.

Remark 5.22 (Subgroup, not quotient). The group $G_1(K, M)$ is a subgroup of $H_1(M \setminus K; \mathbb{Z})$, not a quotient. Thus the correct statements are that a readout is defined on $G_1(K, M)$, or that a complement character has nontrivial restriction to $G_1(K, M)$. One should not say that a character "factors through G_1" unless an actual quotient has first been defined, as in the compactified admissible group above.

6 Recognition selector and state-count separation

We can now state the recognition results in separate layers. This prevents SR, LP, topology, and DA from doing each other's work.

Theorem 6.1 (Loop theorem: SRL plus LP)**.** *Let $d \geq 2$. Suppose a recognition model selects a persistent circle defect and SRL realizes it as a smooth embedded circle $K \subset M^d$ in a connected* *closed orientable smooth substrate, or in an admissible compactified substrate whose readout is evaluated in such a compact pair. If LP, with at least LP0-min coverage, supplies a nontrivial defect-generated loop-only phase, then $d = 3$. In the compactified alternative, this is a statement about the specified finite-readout package $(Y, E, M, K, \pi_{\mathrm{ad}})$, not about all possible compactifications of Y or the raw complement $Y \setminus K$.*

> **Compactified-readout warning.** Changing the compactification or the end-calibration rule may change $G_{1,\mathrm{ad}}$. The compactified conclusion of Theorem 6.1 applies to the specified admissible finite-readout package $(Y, E, M, K, \pi_{\mathrm{ad}})$ and is not asserted to be invariant under arbitrary compactifications of the same raw noncompact complement.

Proof. In the closed case, LP gives a nontrivial character $G_1(K, M) \to \mathrm{U}(1)$, so Theorem 3.10 gives $d = 3$. In the compactified case, LP gives a nontrivial character $\bar{\chi} : G_{1,\mathrm{ad}}(K, Y; M, E) \to \mathrm{U}(1)$. By Proposition 3.5, $G_{1,\mathrm{ad}}(K, Y; M, E) \neq 0$. By Lemma 5.12, the pulled-back character is nontrivial on a non-end class of $G_1(K, M)$; in particular $G_1(K, M) \neq 0$. The topological selector then applies to the compact pair (M, K) and forces $d = 3$. This conclusion is relative to the chosen admissible compactification and end calibration. □

This loop theorem is the dimension selector. BQ, CT, and SR are essential for tying the selected defect core to the finite recognition architecture, but they are not the source of the dimension-three conclusion.

Theorem 6.2 (State-complete theorem: BQ, CT, and SR)**.** *Let $n \geq 2$. Under BQ and CT, the primitive transition graph is Q_n. Under SR, the selected minimal state-complete recurrence is a Hamiltonian circle in Q_n and uses every one of the 2^n recognized binary states exactly once per period.*

Proof. BQ identifies the finite recognition quotient with $V_n = \{0, 1\}^n$. CT gives Q_n by Proposition 4.7, or by Proposition 4.9 in the robust tolerance-gapped setting. SR selects a minimal state-complete closed walk in this graph. By Proposition 4.18, that walk has length 2^n, visits every vertex exactly once before returning to its start, and has image homeomorphic to S^1. □

Theorem 6.3 (Loop-defect extension selector)**.** *Let $n \geq 2$ and $d \geq 2$, and let $\mathcal{E}$ be an LD-RS premodel over a Recognition-Geometry object. If $\mathcal{E}$ satisfies BQ, CT, SR, SRL, and LP, then its smooth closed or admissibly compactified substrate has ambient dimension $d = 3$. If, in addition, $\mathcal{E}$ satisfies DA, then Corollary 6.4 gives $n = 3$ and the full Boolean quotient has $2^3 = 8$ states.*

Proof. Theorem 6.2 supplies the selected Hamiltonian circle in the finite recognition quotient. SRL realizes the selected circle as a smooth embedded circle in the closed substrate, or in the compactified substrate where readouts are explicitly evaluated. LP supplies a nontrivial defect-generated loop-only phase with at least LP0-min coverage. The loop theorem, Theorem 6.1, then forces $d = 3$. □

Corollary 6.4 (Dimensional alignment and the eight-state conclusion)**.** *If the extension in Theorem 6.3 is also dimensionally aligned, or if DA is derived from the smooth faithful-minimality criterion of Theorem B.2, then $n = 3$ and the full binary quotient contains $2^3 = 8$ states.*

Proof. Theorem 6.3 gives $d = 3$. DA gives $n = d$, hence $n = 3$. BQ identifies the quotient with $\{0, 1\}^n$, so its cardinality is $2^3 = 8$. □

Table 2: Proof dependencies for the selector and state-count conclusions.

Conclusion	What proves it
$d = 3$ from a selected smooth circle	SRL, LP, and the topological loop selector.
State-complete circle using all binary states	BQ, CT, SR, and finite graph theory.
$d = 3$ for a loop-defect RS extension	BQ, CT, SR, SRL, LP, and the two previous layers.
$n = 3$	The dimension conclusion plus DA, or a theorem deriving DA.
$2^3 = 8$ states	BQ plus $n = 3$.

Corollary 6.5 (First-acyclic bridge selector: closed and compactified cases)**.** *In the closed case of Theorem 6.3, if the substrate is additionally first-acyclic, $H_1(M^d;\mathbb{Z}) = 0$, then*

$$G_1(K, M) = H_1(M^d \setminus K; \mathbb{Z}),$$

so LP may equivalently be specified as a nontrivial loop-only complement character

$$H_1(M^d \setminus K; \mathbb{Z}) \to \mathrm{U}(1).$$

In the admissibly compactified case, if the compactifying closed pair satisfies $H_1(M^d;\mathbb{Z}) = 0$, then the same equality $G_1(K, M) = H_1(M^d \setminus K; \mathbb{Z})$ holds at the compactified level, but LP is not an arbitrary character on the full complement group. It is a nontrivial character on the admissible quotient

$$H_1(M^d \setminus K; \mathbb{Z})/E_1^M(K, Y; E),$$

equivalently a character on $H_1(M^d \setminus K; \mathbb{Z})$ that is trivial on the end subgroup $E_1^M(K, Y; E)$ and nontrivial after projection.

Proof. In the closed case the identity follows immediately from Definition 3.1, since the target of $H_1(M^d \setminus K; \mathbb{Z}) \to H_1(M^d;\mathbb{Z})$ is zero. In the compactified case, the same identity holds for the compactified closed pair, but the admissible readout is defined after quotienting by the calibrated subgroup $E_1^M(K, Y; E)$ in Definition 5.7. Therefore admissibility requires triviality on end-supported classes and nontriviality only after projection to the admissible quotient. □

Example 6.6 (Single-substrate coherence witness)**.** Let the ambient substrate be $M = S^3$ and choose an active coordinate chart $U \subset S^3$ with coordinates x_1, x_2, x_3 on a cube $(-1, 1)^3 \subset U$. Let

$$C = (-1, 1)^3 \setminus \{x_i = 0 \text{ for some } i\}$$

be the active recognition region and define three threshold recognizers on this chart by

$$\beta_i(x) = \begin{cases} 1, & x_i > 0, \\ 0, & x_i < 0. \end{cases}$$

Every sign pattern occurs, so BQ holds with $n = 3$ inside the same smooth substrate used for the readout. Choose local coordinate scales $\iota_i(0) = 1$ and $\iota_i(1) = r_i > 1$, and set

$$\rho_i(\epsilon, 1 - \epsilon) = \frac{\iota_i(1 - \epsilon)}{\iota_i(\epsilon)}.$$

The reciprocal local penalties $J_a(\rho_i)$ satisfy CT when combined with additive product locality and the indecomposability criterion. Hence they give Q_3. Choose a reflected Gray-code Hamiltonian recurrence in Q_3 and impose SR for that period. Choose an unknot $K \subset S^3$ with eight marked points indexed in that Gray-code order, and take the SRL realization tie $h : C_\gamma \xrightarrow{\cong} K$ to send the finite recurrence circle to this marked smooth defect core. Since $H_1(S^3;\mathbb{Z}) = 0$, Theorem 3.8 gives $G_1(K, S^3) \cong \mathbb{Z}$. For any $\theta \in \mathbb{R} \setminus \mathbb{Z}$, the formula

$$\chi(k) = \exp(2\pi i\theta k), \qquad k \in \mathbb{Z},$$

defines a nontrivial LP character. The three recognizer differentials are the coordinate covectors on the active chart and span the chart cotangent spaces, so the sufficient alignment criterion in Theorem B.2 gives DA for this witness: $n = d = 3$. This example proves only that the extension predicates are mutually coherent in one smooth substrate. It is a consistency witness for the extension class, not a model-selection argument.

Example 6.7 (Dimensional alignment is not notation)**.** Take $n = 4$ and the hypercube Q_4. A Hamiltonian cycle in Q_4 can be realized as a smooth embedded circle in S^3. The defect-generated selector then gives $d = 3$ and a loop meridian, while the binary quotient still has $2^4 = 16$ states. Thus topology plus loop readout does not imply $n = 3$. The eight-state conclusion requires DA or a theorem implying DA.

A sufficient smooth criterion for dimensional alignment is recorded in Appendix B. The main text uses only the logical separation: the loop selector gives $d = 3$, while the eight-state conclusion requires DA or a separate theorem deriving DA.

7 Counterexamples and load-bearing assumptions

The bridge assumptions are not optional decoration. Dropping them changes the conclusion. The examples below are explicit witnesses for the load-bearing analysis, not informal caveats. They are ordered along the dependency chain BQ, CT, SR, SRL, LP, and DA.

Remark 7.1 (Failure-mode audit for the LD-RS proof architecture)*.* Within the typed proof architecture of this paper, each named predicate is load-bearing for the step assigned to it. Dropping a predicate, and interpreting the remaining data only where their types still make sense, yields the failure witnesses summarized in Table 3. This is an audit of the present proof architecture, not a theorem of global logical independence, uniqueness of the architecture, or impossibility of alternative future bridges.

The witnesses are direct constructions at the level of premodel data. Without BQ, the map β need not be surjective onto a Boolean cube, so neither the hypercube bridge nor the 2^n count has a defined target; Proposition 7.2 records the corresponding quotient-level underdetermination. Without CT, the reciprocal scalar penalty alone does not prevent declaring correlated multi-bit moves primitive, as in Example 7.4; hence the graph can be complete or generated by non-coordinate moves instead of being Q_n. Without SR, even a genuine Q_n may carry a selected history that is not Hamiltonian, for example a square subcycle in Q_3. Without SRL, Example 7.6 shows that an abstract cycle has no specified ambient complement, so $H_1(M \setminus K)$, $G_1(K, M)$, and $G_{1,\mathrm{ad}}$ are not defined for the selected recurrence. Without LP, the topological theorem has no nontrivial character on defect-generated loop charge as input; Examples 7.7, 7.8, and 7.9 show how the readout can miss the LP target. Finally, Example 6.7 satisfies the dimension conclusion while violating dimensional alignment, so DA is exactly what is needed for the state-count conclusion.

Table 3 summarizes the corresponding failure modes.

Table 3: Failure witnesses for the load-bearing bridge predicates in the present LD-RS proof architecture.

Dropped item	**Witness**	**Failed conclusion**
BQ	A finite recognition quotient omitting some bit pattern, as in Example 7.3, or a non-Boolean state set with the same ambient substrate data.	No full $\{0,1\}^n$ provenance and no justified 2^n state count.
CT	The same full Boolean state set equipped with a complete primitive graph or with correlated two-bit primitive generators, as in Example 7.4.	The primitive graph is not forced to be Q_n.
SR	Q_n with a selected proper subcycle, transient path, or closed recurrence that does not visit all vertices, as in Example 7.5.	No state-complete Hamiltonian circle using all recognized states.
SRL	An abstract Hamiltonian cycle in Q_n with no specified smooth substrate, no defect core K, and no realization tie $h : C_\gamma \cong K$, as in Example 7.6.	No complement homology target exists for LP to evaluate.
LP	A smooth circle defect in a candidate substrate with only the trivial character, or with phases restricted to ambient/end/higher-degree modes rather than G_1 or $G_{1,\mathrm{ad}}$, as in Examples 7.7 and 7.8.	No dimension selector is triggered.
DA	The explicit aligned-failure witness $n = 4, d = 3$ of Example 6.7.	The theorem still gives $d = 3$, but not $n = 3$ or $\|V_n\| = 8$.

Proposition 7.2 (Bare quotient data do not determine dimension)**.** *A recognition quotient by itself does not determine an ambient manifold dimension. In particular, the same finite quotient set can be treated as an abstract set, as a graph vertex set with many possible transition graphs, or as data realized in manifolds of different dimensions.*

Proof. A quotient set contains no intrinsic smooth structure, no transition generator, no closed recurrence, no embedding, and no complement homology. Any rule assigning a unique ambient dimension therefore uses additional structure beyond the quotient set. Example 6.7 also shows that even a full binary quotient plus a loop selector need not determine n without alignment. □

Example 7.3 (Dropping BQ: missing bit patterns)**.** Fix $n = 3$ and let a candidate recognizer map have image $\{000, 100, 010, 001\} \subsetneq \{0,1\}^3$. The data still look binary coordinatewise, but the full Boolean quotient is absent. There is then no state space $V_n = \{0,1\}^n$ on which the hypercube or a 2^n-state recurrence can be forced. This witnesses the BQ row of the audit in Remark 7.1.

Example 7.4 (Dropping CT: ratio costs alone do not specify a primitive graph)**.** The reciprocal function J_a assigns penalties to positive ratios. Without CT, it does not specify which binary states are connected by admissible transitions, whether transitions are all-to-all, whether only one-bit flips are allowed, or whether correlated multi-bit updates are primitive. The hypercube follows from local additive transition structure, not from the formula J_a by itself.

Example 7.5 (Dropping SR: a subcycle is not state-complete)**.** In Q_3, the closed walk

$$000 \to 100 \to 110 \to 010 \to 000$$

is a square subcycle. It is a perfectly valid closed loop in the hypercube, but it visits only four of the eight vertices. Without SR, a selected recurrence need not be Hamiltonian and need not encode all recognized states.

Example 7.6 (Dropping SRL: no smooth complement)**.** A Hamiltonian cycle in an abstract graph is a finite one-dimensional complex, not a defect in a smooth substrate. Before SRL is imposed there is no specified ambient manifold, no complement $M^d \setminus K$, no end calibration, and no map from complement loops to a defect-generated homology group. Thus the finite cycle alone cannot support the LP phase readout, even when it is a perfect state-complete Gray recurrence.

Example 7.7 (Dropping LP1: higher-degree readouts defeat the loop selector)**.** Embed S^1 in S^4. The loop group $H_1(S^4 \setminus S^1; \mathbb{Z})$ vanishes, but there is a meridional class in $H_2(S^4 \setminus S^1; \mathbb{Z})$. If stable phases were allowed to read higher-dimensional cycles, dimension four would not be excluded. The selector chooses $d = 3$ because LP restricts admissible phases to one-dimensional histories.

Example 7.8 (Dropping LP3: ambient homology can fake a raw complement signal)**.** If $H_1(M; \mathbb{Z}) \neq 0$, then $H_1(M \setminus K; \mathbb{Z})$ can contain loop classes already present in M. A raw complement character may detect those ambient modes rather than the defect. This is why the main theorem uses the subgroup $G_1(K, M)$ and a readout nontrivial on that subgroup.

Example 7.9 (Dropping LP5: a primitive class in dimension three has no charge)**.** Let $M = S^1 \times S^2$ and let $K = S^1 \times \{p\}$ be the standard circle factor. The free homology class $[K]_{\text{free}}$ is primitive, so Corollary 3.9 gives $G_1(K, M) = 0$. Thus dimension three alone does not guarantee LP in general substrates; first-acyclicity or a nonprimitive/null/torsion selected embedding class is part of the phase bridge.

Remark 7.10 (Dropping DA). Example 6.7 is the DA witness: a valid loop-defect readout can select $d = 3$ while the finite recognition quotient has $n = 4$ and 2^4 states. DA is therefore needed only for the state-count conclusion, not for the ambient dimension selector.

8 Degree variants and coefficient cautions

This section explains why the loop-only qualifier is not cosmetic. The selector is a degree-matching statement. For a circle defect in a d-dimensional substrate, the normal link is S^{d-2}, so the local meridional class is a loop exactly in dimension three. The same mechanism has a higher-dimensional analogue.

Definition 8.1 (Degree-r homology readout)**.** A degree-r homology readout of a defect $K \subset M$ with target abelian group A is a homomorphism

$$\chi_r : H_r(M \setminus K; \mathbb{Z}) \to A.$$

The loop-only phase readout is the case $r = 1$ and $A = \mathrm{U}(1)$.

Proposition 8.2 (Meridian degree for sphere defects)**.** *Let $p \geq 1$, let $d \geq p+2$, and let $K \cong S^p \subset S^d$ be a tame embedded sphere. Then the reduced homology of the complement is concentrated in degree $d - p - 1$:*

$$\widetilde{H}_i(S^d \setminus K; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & i = d - p - 1, \\ 0, & i \neq d - p - 1. \end{cases}$$

Thus a degree-r meridional readout of an S^p defect occurs when $r = d - p - 1$. If the readout degree is required to match the defect dimension, $r = p$, then $d = 2p + 1$.

Proof. Alexander duality gives

$$\widetilde{H}_i(S^d \setminus K; \mathbb{Z}) \cong \widetilde{H}^{d-i-1}(S^p; \mathbb{Z}).$$

For $p \geq 1$, the reduced cohomology of S^p is $\mathbb{Z}$ in degree p and zero in all other degrees. Hence the complement homology is $\mathbb{Z}$ exactly when $d - i - 1 = p$, i.e. $i = d - p - 1$. Setting $i = r = p$ gives $d = 2p + 1$. □

Remark 8.3 (Coefficient caution). The integral character formulation is stronger and cleaner than replacing the readout by homology over an arbitrary field. For example, a finite group $\mathbb{Z}/m\mathbb{Z}$ has nontrivial U(1) characters for every $m > 1$, while tensoring with a field sees it only in characteristics dividing m. The selector theorem therefore uses integral defect-generated charge and characters to U(1).

9 Related work and novelty

The homology calculation is classical; its role here is to identify the exact first-homology target used by the LD-RS readout. Tubular neighborhoods, excision, the Thom isomorphism, long exact sequences of pairs, Poincaré duality, Alexander duality, and the meridional description of circle complements are standard tools in algebraic topology and knot theory; see, for example, Bredon [3], Hatcher [7], Rolfsen [16], and Lickorish [11]. The topological role of the paper is self-containedness and the precise identification of the defect-generated first-homology target needed by the LD-RS readout principle.

What is new and what is not new

The paper uses several classical ingredients deliberately. Table 4 separates prior status from the contribution made here.

Table 4: Novelty audit. The contribution is the audited LD-RS assembly, not the classical ingredients in isolation.

Ingredient	**Prior status**	**What this paper contributes**
Circle-complement homology	Classical algebraic topology and knot theory.	A self-contained calculation of the exact defect-generated loop group used by LP.
Loop phases around defects	Standard in Aharonov–Bohm, vortex/string, and topological-defect settings.	A recognition-specific target: phases must be nontrivial on G_1 or $G_{1,\mathrm{ad}}$, not raw complement homology.
Hypercube/Gray-code recurrence	Standard combinatorics.	A finite recognition bridge: BQ+CT+SR selects a Hamiltonian state-complete circle using all 2^n recognized states.
Recognition Geometry	Existing quotient/locality framework.	Extension predicates that add transition, recurrence, smooth defect, phase, compactification, and optional alignment data.
Dimension selection	Existing physics literature uses stability, PDE, compactification, and anthropic arguments.	A conditional recognition-topology selector: loop-only defect-generated readout forces $d = 3$.
Eight-state conclusion	Not a topological consequence.	Isolated as $d = 3$ plus DA, preventing conflation of ambient dimension with recognizer count.

The contribution is therefore not an isolated new lemma but an assembly theorem: it identifies the precise Recognition-Science bridge predicates under which a finite recognition recurrence becomes a smooth circle defect whose admissible phase readout is forced into dimension three.

Loop phases around excluded cores are also not new. They are structurally related to the Aharonov–Bohm effect and to the broad literature on topological defects, vortices, strings, codimension-two defects, and higher-form symmetries [1, 9, 12, 5, 2]. The present paper does not claim to invent holonomy around a defect. Codimension-two defects, defect holonomies, and Aharonov–Bohm-style phases are standard. The distinct claim is that, in an LD-RS extension, the admissible phase domain is the defect-generated first-homology target $G_1(K, M)$, or the calibrated quotient $G_{1,\mathrm{ad}}$, after separating the finite recurrence core from complement probe loops. Thus LP does not use raw complement homology or arbitrary holonomy data. Ambient loop modes and end-supported modes are therefore not accepted as substitutes for defect-generated charge.

The finite graph ingredients are likewise standard. Hypercubes, reflected Gray codes, and Hamiltonian cycles are classical combinatorial objects; Savage's survey and Mütze's updated survey give broad entry points into this literature [17, 13]. Their role here is not graph-theoretic novelty. Their role is to connect BQ, CT, and SR to a state-complete recognition recurrence whose image is a circle using all 2^n recognized binary states.

The paper is also not meant to replace the broad dimension-selection literature in physics. Existing arguments use stability, field propagation, inverse-square laws, compactification, string-theoretic effective dimensions, and anthropic or observational constraints; see, for example, Ehrenfest [4], Tegmark [18], Kaluza [8], Klein [10], Green–Schwarz–Witten [6], and Polchinski [15]. The present argument is orthogonal to those programs. It proves a conditional recognition-topology selector inside the LD-RS extension package; it does not claim a dynamical compactification theory or a derivation of empirical physical space from the reciprocal cost law alone.

The novelty of the paper is therefore architectural. Recognition Geometry supplies a measurement-first quotient framework with recognizers, observable quotients, locality, and finite resolution [19]. Reciprocal cost theory supplies the scalar ratio-cost functional and leaves semantic or readout interpretation external to its bare mathematical theorems [20]. This paper defines an explicit LD-RS extension package over that background and separates premodel data from named predicates: BQ, CT, SR, SRL, LP, compactified end calibration, and optional DA. Under BQ+CT+SR, the recognition quotient supplies a Hamiltonian recurrence circle. Under SRL+LP, that circle is realized as a smooth defect with nontrivial defect-generated complement-loop phase, and classical topology forces $d = 3$. Under DA, and only under DA, the result becomes $n = 3$ and $|V_n| = 8$. Thus the contribution is a conditional recognition-topology selector with exact provenance accounting, not a theorem of bare RS_0 and not a new theorem of circle-complement topology.

10 Conclusion

The exact theorem proved in this paper is conditional and modular. It follows from Recognition Science only in the LD-RS extension sense defined here, not from bare published Recognition Geometry plus the reciprocal scalar cost law alone:

$$\text{LD-RS premodel satisfying BQ, CT, SR, SRL, and LP} \quad \Longrightarrow \quad d = 3.$$

The mathematical reason is the defect-generated loop selector:

$$\chi : G_1(K, M) \to \mathrm{U}(1) \text{ nontrivial for a smooth circle defect} \quad \Longrightarrow \quad d = 3$$

for connected closed orientable smooth substrates of dimension $d \geq 2$. The low-dimensional exclusion is included explicitly: in dimension two, inclusion of the circle complement into the surface is injective on first homology, so no *defect-generated* loop charge is created. In dimensions $d \geq 4$, the same conclusion follows from the Thom-isomorphism calculation. Dimension three is the unique dimension in which a circle defect can create one-dimensional meridional charge. The Hamiltonian recurrence is the defect core; the phase read by LP is a complement-loop phase around that core, not a phase on the core itself.

In dimension three the defect-generated group is the meridian quotient $\mathbb{Z}/m\mathbb{Z}$, where m is the divisibility of the free homology class of the circle. A nontrivial character exists exactly when that free class is not primitive; in first-acyclic substrates this condition is automatic and the meridian group is $\mathbb{Z}$.

The Recognition-Science content is a premodel-plus-predicates extension package, not an unstated entailment from the published core. Recognition Geometry supplies recognizers, quotients, locality, and finite-resolution language. Reciprocal cost theory supplies the scalar local ratio penalty and leaves semantic/readout interpretation outside its bare theorems. BQ supplies the Boolean quotient, CT supplies the hypercube generator, SR supplies the state-complete circle, SRL supplies the smooth closed or compactified readout substrate, and LP supplies the nontrivial one-dimensional defect phase. These assumptions force $d = 3$. They do not force $n = 3$; that requires DA or a separate theorem deriving DA. Consequently the eight-state conclusion is the output of the dimension selector plus dimensional alignment, never of the loop-topology selector alone.

The result isolates the exact bridge principles that a recognition-first framework must derive in order to turn the conditional selector into a full Recognition-Science derivation. In particular, LP must eventually be replaced by a theorem showing that recognized complement histories cover the relevant defect-generated readout classes and carry a nontrivial phase on them. The proved implications are

$$\mathsf{RS}_{\mathrm{LD}} \Rightarrow d = 3, \qquad \mathsf{RS}^{+}_{\mathrm{LD}} \Rightarrow n = 3,\ |V_n| = 8.$$

No theorem deriving $d = 3$ from RS_0 alone is proved in this paper. Thus the paper's contribution is not a new theorem of circle-complement topology and not a derivation from the bare quotient/cost core. It is a precise conditional theorem for LD-RS extensions, together with an audit of the bridge principles that a future full Recognition-Science derivation would have to prove. The immediate research agenda is therefore sharp: derive LP or an equivalent degree-one defect-generated phase principle, derive CT from reciprocal cost plus product composition, derive SR from ledger/no-starvation dynamics, and derive DA from a faithful minimal smooth realization. The next make-or-break theorem is not another complement calculation; it is a derivation of LP.

A Provenance audit and strengthening agenda

This appendix records the detailed provenance and future-strengthening material kept out of the opening flow of the paper.

The provenance audit is split into two smaller tables. Table 5 records inherited or proved ingredients; Table 6 records bridge predicates and future derivation targets. The boundary is deliberately strict: published Recognition Geometry contributes the recognizer, quotient, locality, and finite-resolution framework; published reciprocal cost theory contributes a scalar ratio-cost theory. Neither source alone supplies the loop-defect realization, the complement-loop phase, or dimensional alignment.

Table 5: Inherited or proved ingredients in the LD-RS selector.

Ingredient	**Status**	**Role here**
Recognition Geometry core	Inherited background: recognizers, quotients, locality, finite resolution, comparison, composition, and refinement.	Supplies the quotient language; not itself a dimension theorem.
Reciprocal scalar cost input	Inherited scalar ratio-cost functional J_a.	Supplies local cost input; not by itself a transition graph, phase law, or semantic readout theorem.
Robust CT consequence	Proved from CT plus a positive finite separation gap.	Shows that the hypercube generator conclusion is not merely an exact-equality artifact once the gap hypothesis is admitted.
Circle extraction	Elementary graph theory proved here from Q_n and SR.	Supplies the abstract Hamiltonian recurrence circle using all binary states.
Topological selector	Classical algebraic topology proved self-containedly in Section 3.	A nontrivial defect-generated loop character for a circle defect forces $d = 3$ for $d \geq 2$.
Compactified quotient hygiene	Elementary kernel/quotient and end-subgroup lemmas.	Records the chosen finite-readout package; not a canonical theorem about every raw noncompact complement.

Table 6: Bridge assumptions and future derivation targets.

Bridge or target	Status	Role here
LD-RS premodel	Additional candidate data over Recognition Geometry.	Separates data from predicates so the theorem is not manufactured by definition.
BQ	RG-compatible specialization, not forced by bare RG.	Identifies the finite quotient with $V_n = \{0, 1\}^n$.
CT	Bridge assumption adding product locality and cost-indecomposable primitive moves to the scalar cost input.	Produces the hypercube primitive graph Q_n.
SR	New dynamics/fairness/least-action bridge axiom.	Selects a state-complete Hamiltonian recurrence rather than an arbitrary subcycle.
Realization tie and SRL	New realization data and predicate for closed or admissibly compactified substrates.	Identifies the finite recurrence circle with a smooth defect core K so complement homology can be evaluated.
LP	Central phase bridge axiom, including recognized representative coverage, degree-one phases, homology invariance, defect-generated target, compactified end rule, and nontriviality.	Invokes the topological selector.
DA	Optional alignment principle or theorem.	Converts $d = 3$ into $n = 3$ and $\|V_n\| = 8$.
Full physical derivation	Future stronger theory, not this paper.	Would derive BQ, CT, SR, SRL, LP, compactified calibration, and DA from the published core.

Step-by-step strengthening ladder

The present theorem is intentionally not promoted as new topology, new hypercube combinatorics, new Aharonov–Bohm physics, or a derivation from bare RS_0. The stronger claims would require the following additional theorems.

Step 1: Topology-level strengthening: classify admissible compactified defect groups under controlled changes of end calibration, or prove a functorial uniqueness theorem for the defect-generated first-homology target beyond the elementary universal property proved below.

Step 2: Hypercube/Gray-code strengthening: derive a genuinely new optimality or classification theorem for reciprocal-cost constrained Gray recurrences, rather than using standard hypercube Hamiltonicity.

Step 3: Defect-phase strengthening: derive LP from recognition dynamics, showing that stable recognized histories provide representative coverage of the relevant defect-generated readout classes and produce degree-one homology-invariant phases trivial on ambient and end modes and nontrivial on the selected defect-generated class.

Step 4: Bare-RS strengthening: derive BQ, CT, SR, SRL, LP, compactified end calibration, and DA from the published recognition and scalar-cost core. Until those theorems are proved, the result is an LD-RS extension theorem.

B A sufficient smooth alignment criterion

This appendix records a precise sufficient criterion by which DA could be derived in a smooth realization. It is not a consequence of the preceding topology; it is a separate faithful-minimality principle. It also does not derive smooth faithful-minimality from Recognition Geometry. It only shows that, if a smooth threshold realization has linearly independent primitive recognizer differentials and no silent substrate directions, then dimensional alignment follows. This is a sufficient criterion, not a theorem that such realizations exist for every Recognition-Science model.

Definition B.1 (Smooth threshold differential recognition realization)**.** Let $U \subset M^d$ be an active recognition region. A family of binary recognizers is differentially realized on U if each recognizer is represented locally by a sign or threshold rule for a smooth decision function $f_i : U \to \mathbb{R}$; for example, after choosing a threshold t_i, the two local outcomes are $f_i > t_i$ and $f_i < t_i$ away from the threshold wall. The theorem uses only the differentials df_i on the active region where the binary states are being compared. The realization is:

(a) *differentially independent* if $df_1(x), \ldots, df_n(x)$ are linearly independent in T_x^*M for each x in the active region considered;

(b) *faithful-minimal* if there are no silent substrate directions, i.e. $df_1(x), \ldots, df_n(x)$ span T_x^*M at each such x.

Theorem B.2 (Sufficient smooth alignment criterion)**.** *In a smooth threshold differential recognition realization on a d-dimensional smooth substrate, differential independence gives $n \leq d$, faithful-minimality gives $d \leq n$, and both together give $n = d$.*

Proof. At each active point, the cotangent space T_x^*M has dimension d. A linearly independent family of n covectors in it has size at most d, so $n \leq d$. If the same n covectors span T_x^*M, then a d-dimensional vector space is spanned by n elements, so $d \leq n$. Together these inequalities give $n = d$. □

Remark B.3 (Status of the eight-state result). Theorem B.2 shows what an alignment proof would need to look like. It does not prove that every Recognition-Science model has such a smooth threshold realization, nor that every faithful realization is minimal in this sense. Until those facts are proved or adopted as axioms, $|V_n| = 8$ is a conditional alignment conclusion, not a topological conclusion.

C Targets for a future full Recognition-Science derivation

A future paper claiming a full Recognition-Science derivation should prove, or adopt as named axioms, the following package. None of the targets in this appendix is used as a proved result in the present paper. These targets are the precise open bridge obligations exposed by the provenance audit, not results already obtained here. The highest-payoff target is the loop-phase theorem: without LP or an equivalent one-dimensional defect-generated phase principle, the selector remains a conditional extension theorem rather than a bare-RS derivation.

Theorem A: **Binary quotient theorem.** Starting from recognizers and recognition quotients, construct a jointly surjective binary family $\beta_1, \ldots, \beta_n : C \to \{0, 1\}$ and prove that the product image is all of $\{0, 1\}^n$.

Theorem B: **Cost-to-transition theorem.** Starting from reciprocal local ratio penalties and product recognizer composition, derive additive product locality and prove that cost-indecomposability selects the one-coordinate generators.

Theorem C: **State-complete recurrence theorem.** Formalize a no-starvation, ledger-completeness, or state-fairness principle forcing every quotient state to occur in a closed period, and combine it with minimal action to obtain a Hamiltonian recurrence in Q_n.

Theorem D: **Substrate realization theorem.** Construct a closed smooth finite-recognition substrate, or give an admissible compactification theorem with an explicit end subgroup E_1, a compactified defect group, and phases that are trivial on end-supported modes.

Theorem E: **Loop-phase theorem.** Derive one-dimensional homology-invariant phase characters from recognized histories, prove recognized representative coverage of the defect-generated readout target, and prove nontriviality on $G_1(K, M)$ or on the specified admissible compactified quotient.

Theorem F: **Dimensional alignment theorem.** Derive $n = d$ from a faithful minimal smooth realization, for example by proving that linearly independent primitive recognizers provide linearly independent local covectors and that no silent substrate directions remain.

Without these layers, the selector remains a correct conditional theorem for loop-defect extensions. With these layers, it would become a genuine derivation inside a stronger Recognition-Science theory.

D Formalization roadmap

The finite components are suitable for near-term formalization: binary quotient surjectivity, reciprocal local costs, positivity of J_a, hypercube adjacency, the perturbative generator theorem, Gray-code cycles, and minimal state-complete recurrence. The topological bridge should be imported as an external theorem unless the proof-assistant library already supplies the needed homology of pairs, excision, Thom isomorphism for oriented disk bundles, Poincaré duality, the universal coefficient theorem, Alexander duality, and the Jordan curve theorem.

A safe formal dependency architecture is:

1. formalize the model class LD-RS and the forgetful projection to RG;
2. formalize BQ, CT, SR, SRL, LP, and DA as explicit hypotheses;
3. prove Proposition 4.7 and Proposition 4.9 for finite Boolean states;
4. prove Lemma 4.16 and Proposition 4.18;
5. import Theorem 3.8 or Theorem 3.10 as the topological bridge, including the $d = 2$ vanishing case;
6. derive Theorem 6.3 by composition of the named hypotheses and finite lemmas.

This prevents a formal assumption predicate from being mistaken for a proof of Alexander duality, the Thom isomorphism, or Poincaré duality.

Declarations

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