Cohomological Connections Between the Cost-First Ledger Framework and Aperiodic Tiling Spaces

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Abstract

We develop a formal cohomological framework connecting the discrete potential theory of the cost-first ledger [**?**] with the topological invariants of Penrose tiling spaces. The ledger framework produces closed 1-cochains on a recognition graph via cycle-closure conditions, which admit scalar potentials by a discrete Poincaré lemma. Independently, the topology of tiling spaces—studied via the Anderson–Putnam complex [**?**]—yields Čech cohomology groups that classify phason degrees of freedom and shape deformations. We show that the five Ammann bar cochains naturally associated with (and refining) the ledger's edge-ratio postings are pattern-equivariant 1-cocycles on the tiling graph, and that their cohomology classes generate the first Čech cohomology of the Penrose tiling space $\check{H}^1(\Omega_P; \mathbb{Z}) \cong \mathbb{Z}^5$ via the Kellendonk–Putnam pattern-equivariant isomorphism [**?**]. This identification realizes the ledger's potentials (height functions) as the phason coordinates of the tiling: these potentials exist globally but are not pattern-equivariant, so the corresponding cocycles represent nontrivial classes in pattern-equivariant cohomology while being exact in ordinary cohomology. We discuss implications for the cost-theoretic interpretation of phason strain, the classification of local matching rules via cohomological obstructions, and the extension to higher-dimensional quasicrystalline systems.

Keywords: cohomology of tiling spaces, Anderson–Putnam complex, ledger potential, phason coordinates, aperiodic order, cost-first framework

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1 Introduction

Two independently motivated frameworks assign algebraic invariants to aperiodic structures. On one hand, the *topological theory of tiling spaces* [**?**, **?**, **?**] studies the hull Ω of a tiling—the space of all tilings locally indistinguishable from a given one—and computes its Čech cohomology $\check{H}^*(\Omega; \mathbb{Z})$. For Penrose tilings, $\check{H}^1(\Omega_P; \mathbb{Z}) \cong \mathbb{Z}^5$, encoding the five independent

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phason degrees of freedom associated with the five families of Ammann bars [?, ?]. These cohomological invariants classify which deformations of the tiling preserve its local structure (local indistinguishability class) and which do not.

On the other hand, the *cost-first ledger framework* [?] constructs a discrete potential theory on recognition graphs: edge flows arising from atomic recognition events are required to satisfy time-aggregated cycle closure, yielding closed 1-cochains. The discrete Poincaré lemma then produces scalar potentials (unique up to constants) on each connected component. When the recognition graph is instantiated as the adjacency graph of a Penrose tiling, these potentials coincide with the classical Ammann height functions [?, ?], as established in [?].

The purpose of this note is to make the connection between these two constructions precise at the level of *cochain complexes and cohomology groups*. We show that the ledger's locally defined bar-type cocycles (realized concretely as Ammann bar 1-cocycles $\omega_{\mathcal{T}}^{(k)}$ on the tiling graph) are pattern-equivariant 1-cocycles, and that their cohomology classes generate the first Čech cohomology of the Penrose tiling space via the Kellendonk–Putnam pattern-equivariant isomorphism [?]. A key subtlety is that these cocycles are *exact* in ordinary (cellular) cohomology—global height functions exist—but *nontrivial* in pattern-equivariant cohomology, since the height functions are not pattern-equivariant. This identifies the ledger's potential-theoretic data with the topological invariants of the tiling.

1.1 Organization

Section **??** reviews the cohomology of tiling spaces via the Anderson–Putnam construction. Section **??** formalizes the ledger's cochain complex. Section **??** constructs the cochain map and proves the main identification theorem. Section **??** interprets the result in terms of phason coordinates and cost-theoretic strain. Section **??** discusses extensions to higher dimensions and other tiling families. Section **??** concludes with open problems.

2 Cohomology of Tiling Spaces

We recall the essential constructions, following [?, ?, ?].

2.1 The tiling hull

Definition 2.1 (Tiling hull)**.** Let $\mathcal{T}$ be a repetitive, aperiodic tiling of $\mathbb{R}^d$ with finite local complexity. The *hull* (or *tiling space*) of $\mathcal{T}$ is

$$\Omega_{\mathcal{T}} := \overline{\{t + \mathcal{T} : t \in \mathbb{R}^d\}},$$

where the closure is taken in the *local topology*: two tilings are close if they agree on a large ball around the origin (after a small translation).

For Penrose tilings, Ω_P is a compact, connected, metrizable space. It admits a free $\mathbb{R}^2$-action by translation. The *canonical transversal* $\Xi_P \subset \Omega_P$ is defined as the set of tilings in Ω_P having a vertex at the origin; it is a Cantor set that serves as a cross-section of the translation action and parametrizes the "internal" (phason) degrees of freedom. (Note: the orbit space $\Omega_P/\mathbb{R}^2$ is not Hausdorff, since every orbit is dense; Ξ_P is a transversal, not a quotient.)

2.2 The Anderson–Putnam complex

Definition 2.2 (Anderson–Putnam (AP) complex [?])**.** Let $\mathcal{T}$ be a substitution tiling with substitution σ and prototile set $\mathcal{A} = \{t_1, \ldots, t_m\}$. For each prototile t_i, form a CW complex by identifying edges of t_i according to the adjacency rules of $\mathcal{T}$ (collaring). The resulting CW complex Γ_0 is the *zeroth AP approximant*. The substitution σ induces a continuous map $f_\sigma : \Gamma_0 \to \Gamma_0$, and the tiling space is recovered as the inverse limit:

$$\Omega_{\mathcal{T}} \cong \varprojlim(\Gamma_0 \xleftarrow{f_\sigma} \Gamma_0 \xleftarrow{f_\sigma} \cdots).$$

The Čech cohomology of the inverse limit is computed via the direct limit of cellular cohomology:

$$\check{H}^n(\Omega_{\mathcal{T}}; \mathbb{Z}) \cong \varinjlim(H^n(\Gamma_0; \mathbb{Z}) \xrightarrow{f_\sigma^*} H^n(\Gamma_0; \mathbb{Z}) \xrightarrow{f_\sigma^*} \cdots). \tag{1}$$

2.3 Penrose tiling cohomology

For the Penrose tiling, the AP complex Γ_0 can be constructed from the Robinson triangle decomposition (two prototiles: golden triangle T and golden gnomon G, with collared adjacencies). The substitution matrix on 0-cells, 1-cells, and 2-cells of Γ_0 induces maps on the cellular cochain groups.

Theorem 2.3 (Anderson–Putnam [?]; see also [?])**.** *The Čech cohomology of the Penrose tiling space is:*

$$\check{H}^0(\Omega_P; \mathbb{Z}) \cong \mathbb{Z}, \tag{2}$$

$$\check{H}^1(\Omega_P; \mathbb{Z}) \cong \mathbb{Z}^5, \tag{3}$$

$$\check{H}^2(\Omega_P; \mathbb{Z}) \cong \mathbb{Z}^8. \tag{4}$$

The group $\check{H}^1(\Omega_P; \mathbb{Z}) \cong \mathbb{Z}^5$ has a concrete interpretation: its five generators correspond to the five families of *Ammann bars* (parallel lines crossing the tiling at angles $0°, 36°, 72°, 108°, 144°$). Each family defines a height function (1-cocycle) on the tiling, and the five height functions together parametrize the phason degrees of freedom.

3 The Ledger Cochain Complex

We now formalize the algebraic structure underlying the ledger's discrete potential theory, following [?].

3.1 The edge-flow complex

Definition 3.1 (Ledger cochain complex—graph version)**.** Let $G = (V, E)$ be a finite connected graph with edge set E closed under reversal. Define the *ledger cochain complex* as:

$$C^0_L(G) \xrightarrow{\delta_0} C^1_L(G) \longrightarrow 0, \tag{5}$$

where:

- $C^0_L(G) := \{p : V \to \mathbb{Z}\}$ is the group of vertex potentials (0-cochains).
- $C^1_L(G) := \{\omega : E \to \mathbb{Z} \mid \omega(v \to u) = -\omega(u \to v)\}$ is the group of antisymmetric edge flows (1-cochains).
- $\delta_0(p)(u \to v) := p(v) - p(u)$ (discrete gradient).

This is the standard cochain complex of a 1-dimensional CW complex (graph) with coefficients in $\mathbb{Z}$.

When G is the 1-skeleton of a 2-dimensional CW complex (e.g., a tiling), the complex extends naturally:

$$C^0_L(G) \xrightarrow{\delta_0} C^1_L(G) \xrightarrow{\delta_1} C^2_L(G), \tag{6}$$

where $C^2_L(G) := \bigoplus_{F \in \mathcal{F}} \mathbb{Z}$ is the group of face fluxes (2-cochains) indexed by 2-cells (tiles) $\mathcal{F}$, and the coboundary $\delta_1(\omega)(F) := \sum_{e \in \partial F} \omega(e)$ evaluates the flux of ω through each tile boundary.

Remark 3.2 (Cycle flux as a diagnostic). For any cycle γ in G, the *cycle flux* $\sum_{e \in \gamma} \omega(e)$ provides a useful diagnostic: a 1-cochain ω is a cocycle (i.e., $\delta_1(\omega) = 0$ when 2-cells are present) if and only if ω has zero flux through every tile boundary. On a planar tiling, zero flux through every tile implies zero flux through every cycle.

Proposition 3.3 (Complex property)**.** *When 2-cells are present, the ledger cochain complex satisfies $\delta_1 \circ \delta_0 = 0$: the discrete gradient of any potential has zero flux through every tile boundary (and hence through every cycle).*

Proof. For any $p \in C^0_L(G)$ and any cycle $\gamma = (v_0 \to v_1 \to \cdots \to v_n = v_0)$:

$$\sum_{i=0}^{n-1} \delta_0(p)(v_i \to v_{i+1}) = \sum_{i=0}^{n-1} [p(v_{i+1}) - p(v_i)] = p(v_n) - p(v_0) = 0.$$

In particular, $\delta_1(\delta_0(p))(F) = 0$ for every tile F. □

Definition 3.4 (Ledger cohomology)**.** The *n-th ledger cohomology group* is:

$$H^n_L(G) := \ker \delta_n / \operatorname{im} \delta_{n-1}. \tag{7}$$

For the graph complex (**??**):

- $H^0_L(G) = \ker \delta_0 \cong \mathbb{Z}$ (constant potentials on a connected graph).

- $H^1_L(G) = C^1_L(G)/\operatorname{im}\delta_0 \cong \mathbb{Z}^{b_1(G)}$, where $b_1(G) = |E^+| - |V| + 1$ is the first Betti number and E^+ is a set of representatives for unoriented edges.

For the extended complex (**??**) on a planar tiling:

- $H^1_L(G) = \ker\delta_1/\operatorname{im}\delta_0 = 0$ (since the plane is simply connected).

Remark 3.5. The ledger cohomology $H^*_L(G)$ is isomorphic to the standard simplicial (or cellular) cohomology $H^*(X;\mathbb{Z})$ of the underlying CW complex X. The key results of [**?**]—cycle closure (T3), path-independence, and potential existence (T4)—are cohomological statements: the cumulative edge flow $\overline{\Delta}$ is a 1-cocycle ($\delta_1(\overline{\Delta}) = 0$), and on a tree every 1-cocycle is exact ($H^1_L = 0$), with the potential being the primitive.

A crucial point for the tiling application: on the vertex-edge graph $G_\mathcal{T}$ of a planar tiling (viewed as the 1-skeleton of a planar CW complex), $H^1_L(G_\mathcal{T}) = 0$. Every cocycle is exact, and global potentials (height functions) always exist. Here we are using ordinary cellular cohomology of the (infinite) CW complex with *all* cochains allowed (no compact-support or boundedness condition). The nontrivial topological content of the tiling is captured not by ordinary cohomology but by the *pattern-equivariant* cohomology of Kellendonk–Putnam [**?**], which restricts to cochains determined by local data (see Section **??**).

3.2 The ledger complex on a tiling graph

When the recognition graph G is instantiated as the vertex-edge graph $G_\mathcal{T}$ of a Penrose tiling, the ledger cochain complex becomes:

$$C^0_L(G_\mathcal{T}) \xrightarrow{\delta_0} C^1_L(G_\mathcal{T}) \xrightarrow{\delta_1} C^2_L(G_\mathcal{T}). \tag{8}$$

The five families of Ammann bars in a Penrose tiling [**?**, **?**] define five natural 1-cochains. To make this definition purely local (and hence compatible with pattern-equivariance), we assume $\mathcal{T}$ is presented with the standard Ammann-bar decoration (equivalently, with matching-rule decorations that carry the bar information), so that for each oriented edge one can decide from a sufficiently large but finite neighborhood whether the edge crosses a bar of family k and with which transverse sign.

For the k-th Ammann bar family (bars perpendicular to direction $\theta_k = k \cdot 36°$, $k = 0, 1, 2, 3, 4$), choose a positive transverse orientation. The *Ammann bar 1-cochain* $\omega^{(k)}_\mathcal{T} \in C^1_L(G_\mathcal{T})$ is:

$$\omega^{(k)}_\mathcal{T}(u \to v) := \begin{cases} +1 & \text{if edge } (u,v) \text{ crosses a bar of family } k\text{, with } u \text{ on the negative side,} \\ -1 & \text{if edge } (u,v) \text{ crosses a bar of family } k\text{, with } u \text{ on the positive side,} \\ 0 & \text{if edge } (u,v) \text{ does not cross a bar of family } k. \end{cases} \tag{9}$$

Each $\omega^{(k)}_\mathcal{T}$ is antisymmetric by construction: $\omega^{(k)}_\mathcal{T}(v \to u) = -\omega^{(k)}_\mathcal{T}(u \to v)$.

Remark 3.6 (Connection to edge-ratio postings). In the de Bruijn pentagrid construction [**?**], each edge of the Penrose tiling belongs to exactly one grid family and crosses exactly one Ammann bar of that family. The edge-ratio postings from [**?**]—which assign the golden ratio

φ to long edges and 1 to short edges—encode the aggregate length data. The five Ammann bar cochains $\omega_{\mathcal{T}}^{(k)}$ refine this to directional data, one cochain per bar family. The long/short distinction corresponds to the spacing structure within each family.

Proposition 3.7. *Each Ammann bar cochain $\omega_{\mathcal{T}}^{(k)}$ is a 1-cocycle: $\delta_1(\omega_{\mathcal{T}}^{(k)}) = 0$. Equivalently, $\omega_{\mathcal{T}}^{(k)}$ has zero flux through the boundary of every tile.*

Proof. Work with a Penrose tiling equipped with its Ammann-bar decoration. Fix k. Along the boundary of any tile F, the decorated bars of family k enter and exit F in matched pairs: each time a bar crosses the boundary, it must cross again to leave the tile, with the opposite transverse orientation. In terms of the cochain $\omega_{\mathcal{T}}^{(k)}$, this means that the contributions of the two boundary edges where the bar crosses are $+1$ and -1, cancelling in the sum over ∂F. Summing over all bar segments intersecting the tile, the total flux around ∂F is therefore zero, i.e., $\delta_1(\omega_{\mathcal{T}}^{(k)})(F) = \sum_{e \in \partial F} \omega_{\mathcal{T}}^{(k)}(e) = 0$. □

Since $\omega_{\mathcal{T}}^{(k)}$ is a cocycle and the plane is simply connected, it is exact in ordinary cohomology: there exists a *height function* $h^{(k)} : V \to \mathbb{Z}$ with $\omega_{\mathcal{T}}^{(k)} = \delta_0(h^{(k)})$. This is the content of the discrete Poincaré lemma applied to the full planar tiling.

Remark 3.8 (Exactness and the role of pattern-equivariance). On any finite simply connected patch $\mathcal{P} \subset \mathcal{T}$, as well as on the full infinite tiling, $H^1_L(G_{\mathcal{T}}) = 0$ and every cocycle is exact—global height functions $h^{(k)}$ exist (uniquely up to additive constants). This is because the underlying space (the plane) is contractible.

However, the height functions $h^{(k)}$ grow linearly across the tiling: they take arbitrarily many distinct values and their value at a vertex depends on its global position, not just its local neighborhood. In particular, the $h^{(k)}$ are *not* pattern-equivariant. The Ammann bar cochains $\omega_{\mathcal{T}}^{(k)}$, by contrast, *are* pattern-equivariant (their values are determined by purely local data). This creates a nontrivial situation in pattern-equivariant cohomology: the $\omega_{\mathcal{T}}^{(k)}$ are cocycles in the pattern-equivariant subcomplex C^*_{PE}, but their primitives lie outside C^0_{PE}. Hence $[\omega_{\mathcal{T}}^{(k)}] \neq 0$ in H^1_{PE} even though $[\omega_{\mathcal{T}}^{(k)}] = 0$ in H^1_L. This is precisely where the connection to the tiling space cohomology becomes meaningful.

4 Pattern-Equivariant Cohomology and the Identification Theorem

We now construct the central identification: the ledger's Ammann bar cochains are pattern-equivariant 1-cocycles whose classes generate $\check{H}^1(\Omega_P; \mathbb{Z}) \cong \mathbb{Z}^5$ via the Kellendonk–Putnam isomorphism. We also construct a cochain map connecting the ledger complex to the AP cellular complex.

4.1 Setup and notation

Let $\mathcal{T}$ be a Penrose tiling with:

- Vertex-edge graph $G_{\mathcal{T}} = (V, E)$ (the ledger's recognition graph).

- AP approximant Γ_0 with cellular chain/cochain groups $C^n(\Gamma_0;\mathbb{Z})$.
- Substitution map σ inducing $f_\sigma : \Gamma_0 \to \Gamma_0$.

The AP approximant Γ_0 is a finite CW complex. Its 1-cells correspond to (equivalence classes of) edges in the tiling, and its 0-cells correspond to (equivalence classes of) vertices. There is a natural quotient map:

$$\pi : G_{\mathcal{T}} \to \Gamma_0, \tag{10}$$

which sends each vertex/edge of $G_{\mathcal{T}}$ to its equivalence class in Γ_0 (identifying all vertices/edges that have the same local neighborhood up to translation). More precisely, fix a collaring radius R and take Γ_0 to be the corresponding radius-R collared Anderson–Putnam complex; then π is a cellular quotient map and is compatible with the boundary operators.

4.2 Construction of the cochain map

Definition 4.1 (The cochain map Φ)**.** Define $\Phi^n : C^n(\Gamma_0;\mathbb{Z}) \to C^n_L(G_{\mathcal{T}})$ by pullback along the quotient map π:

$$\Phi^n(\alpha)(c) := \alpha(\pi(c)), \tag{11}$$

for any n-cochain $\alpha \in C^n(\Gamma_0;\mathbb{Z})$ and n-cell c in $G_{\mathcal{T}}$.

Lemma 4.2. $\Phi = \{\Phi^n\}$ *is a cochain map, i.e.,* $\Phi^{n+1} \circ d^n = \delta_n \circ \Phi^n$*, where* d^n *denotes the coboundary in* $C^*(\Gamma_0;\mathbb{Z})$ *and* δ_n *denotes the coboundary in* $C^*_L(G_{\mathcal{T}})$*.*

Proof. Since π is a cellular map (it sends n-cells to n-cells, preserving the boundary operator), the pullback $\pi^* = \Phi$ commutes with coboundary by the standard functoriality of cellular cochains. □

Φ induces maps on cohomology:

$$\Phi^*_n : H^n(\Gamma_0;\mathbb{Z}) \to H^n_L(G_{\mathcal{T}}). \tag{12}$$

However, since $H^1_L(G_{\mathcal{T}}) = 0$ for the full planar tiling (every cocycle is exact), the pullback map is trivial on H^1. The topological content instead lives in the *pattern-equivariant* subcomplex, which we now introduce.

4.3 Pattern-equivariant cohomology and the transfer map

Since the tiling $\mathcal{T}$ is repetitive (every local patch appears with bounded gaps), every equivalence class in Γ_0 is represented by infinitely many cells in $G_{\mathcal{T}}$, but with well-defined *frequencies*. The key observation is that a 1-cocycle $\omega \in C^1_L(G_{\mathcal{T}})$ that is *pattern-equivariant* [?] (i.e., its value on an edge depends only on the local neighborhood of that edge) descends to a well-defined cocycle on Γ_0.

Definition 4.3 (Pattern-equivariant cochains [?])**.** A cochain $\omega \in C^n_L(G_{\mathcal{T}})$ is *pattern-equivariant of radius* R if $\omega(c) = \omega(c')$ whenever the R-neighborhoods of c and c' in $\mathcal{T}$ are translationally equivalent. Let $C^n_{PE}(G_{\mathcal{T}}) \subset C^n_L(G_{\mathcal{T}})$ denote the subgroup of pattern-equivariant cochains (of any finite radius).

Proposition 4.4 (Kellendonk–Putnam [?]). *For any repetitive tiling $\mathcal{T}$ with finite local complexity:*

$$H^n_{PE}(G_{\mathcal{T}}; \mathbb{Z}) \cong \check{H}^n(\Omega_{\mathcal{T}}; \mathbb{Z}), \tag{13}$$

where H^n_{PE} denotes the cohomology of the pattern-equivariant cochain complex $C^*_{PE}(G_{\mathcal{T}})$.

This is the critical bridge: the pattern-equivariant cohomology of the tiling graph (a combinatorial/algebraic object) is isomorphic to the Čech cohomology of the tiling space (a topological object).

4.4 The identification theorem

Definition 4.5 (Transfer map). Define $\Psi^n : C^n_{PE}(G_{\mathcal{T}}) \to C^n(\Gamma_0; \mathbb{Z})$ by evaluation at a representative:

$$\Psi^n(\omega)(\bar{c}) := \omega(c_{\bar{c}}), \tag{14}$$

where $\bar{c}$ is an n-cell of Γ_0 and $c_{\bar{c}}$ is any representative of the equivalence class $\bar{c}$ in $G_{\mathcal{T}}$. This is well-defined on pattern-equivariant cochains, for which $\omega(c_{\bar{c}})$ is independent of the representative chosen (by definition of pattern-equivariance).

Lemma 4.6. *The transfer map $\Psi = \{\Psi^n\}$ is a cochain map on the pattern-equivariant subcomplex: $\Psi^{n+1} \circ \delta_n = d^n \circ \Psi^n$ on $C^n_{PE}(G_{\mathcal{T}})$.*

Proof. Let $\omega \in C^n_{PE}(G_{\mathcal{T}})$ be pattern-equivariant of radius R. The coboundary $\delta_n(\omega)$ is pattern-equivariant of radius $R+1$ (the coboundary involves cells in the immediate neighborhood). For any $(n+1)$-cell $\bar{c}$ of Γ_0 with representative $c_{\bar{c}}$ in $G_{\mathcal{T}}$:

$$\Psi^{n+1}(\delta_n(\omega))(\bar{c}) = \delta_n(\omega)(c_{\bar{c}}) = \sum_{e \in \partial c_{\bar{c}}} \omega(e) = \sum_{\bar{e} \in \partial \bar{c}} \omega(c_{\bar{e}}) = d^n(\Psi^n(\omega))(\bar{c}),$$

where the third equality uses the fact that π preserves the boundary operator (the boundary of the representative maps to the boundary of the equivalence class) and pattern-equivariance ensures the values are consistent. □

Lemma 4.7 (Mutual inverses). *For a fixed collared AP complex Γ_0 of collaring radius R, let $C^n_{PE,R}(G_{\mathcal{T}}) \subset C^n_{PE}(G_{\mathcal{T}})$ denote the subgroup of cochains that are pattern-equivariant of radius R. The pullback Φ^n and transfer Ψ^n restrict to mutually inverse cochain isomorphisms:*

$$C^n(\Gamma_0; \mathbb{Z}) \underset{\Psi^n}{\overset{\Phi^n}{\rightleftarrows}} C^n_{PE,R}(G_{\mathcal{T}}).$$

Proof. For any $\alpha \in C^n(\Gamma_0; \mathbb{Z})$ and any n-cell $\bar{c}$ of Γ_0:

$$(\Psi^n \circ \Phi^n)(\alpha)(\bar{c}) = \Phi^n(\alpha)(c_{\bar{c}}) = \alpha(\pi(c_{\bar{c}})) = \alpha(\bar{c}),$$

since $\pi(c_{\bar{c}}) = \bar{c}$ by definition of the representative. Hence $\Psi \circ \Phi = \mathrm{id}$.

Conversely, let $\omega \in C^n_{PE,R}(G_{\mathcal{T}})$. For any n-cell c of $G_{\mathcal{T}}$:

$$(\Phi^n \circ \Psi^n)(\omega)(c) = \Psi^n(\omega)(\pi(c)) = \omega(c_{\pi(c)}),$$

where $c_{\pi(c)}$ is any representative of the equivalence class $\pi(c)$. Since ω is PE of radius R and the cells c and $c_{\pi(c)}$ have identical R-neighborhoods (they belong to the same equivalence class under the radius-R collaring), we have $\omega(c_{\pi(c)}) = \omega(c)$. Hence $\Phi \circ \Psi = \mathrm{id}$ on $C^n_{PE,R}$. □

Remark 4.8 (Collaring radii and direct limits). The isomorphism of Lemma **??** depends on the collaring radius R. As R increases, the AP complex $\Gamma_0^{(R)}$ becomes finer (more equivalence classes) and the PE subgroup $C^n_{PE,R}$ becomes larger. The full pattern-equivariant complex is $C^n_{PE} = \bigcup_R C^n_{PE,R}$, and correspondingly:

$$H^n_{PE}(G_{\mathcal{T}}; \mathbb{Z}) = \varinjlim_R H^n(C^*_{PE,R}(G_{\mathcal{T}})) \cong \varinjlim_R H^n(\Gamma_0^{(R)}; \mathbb{Z}),$$

which recovers the Anderson–Putnam direct limit (**??**). The maps between successive approximants are induced by the inclusions $C^n_{PE,R} \hookrightarrow C^n_{PE,R'}$ for $R \le R'$, or equivalently by the substitution map f_σ.

Theorem 4.9 (Identification of ledger cochains with tiling cohomology generators)**.** *Let $\mathcal{T}$ be a Penrose tiling and let $\omega_\tau^{(k)}$ $(k = 0, 1, 2, 3, 4)$ be the five Ammann bar 1-cocycles from* (**??**)*. Then:*

(i) Each $\omega_\tau^{(k)}$ is pattern-equivariant of some finite radius R_k (depending on the collaring used to define the local equivalence classes): its value on an oriented edge is determined by the R_k-neighborhood of that edge.

(ii) Each $\omega_\tau^{(k)}$ is exact in ordinary cohomology ($[\omega_\tau^{(k)}] = 0$ in $H^1_L(G_{\mathcal{T}})$): the global height function $h^{(k)} : V \to \mathbb{Z}$ satisfies $\omega_\tau^{(k)} = \delta_0(h^{(k)})$. However, $h^{(k)}$ is not *pattern-equivariant (it grows linearly across the tiling).*

(iii) In pattern-equivariant cohomology, $[\omega_\tau^{(k)}] \neq 0$ in $H^1_{PE}(G_{\mathcal{T}}; \mathbb{Z})$: the cocycle $\omega_\tau^{(k)}$ lies in C^1_{PE}, but its primitive $h^{(k)}$ does not lie in C^0_{PE}, so $\omega_\tau^{(k)}$ is not a PE-coboundary.

(iv) Under the Kellendonk–Putnam isomorphism (Proposition **??***),*

$$H^1_{PE}(G_{\mathcal{T}}; \mathbb{Z}) \xrightarrow{\sim} \check{H}^1(\Omega_P; \mathbb{Z}) \cong \mathbb{Z}^5, \tag{15}$$

the five classes $[\omega_\tau^{(k)}]$ map to generators of $\mathbb{Z}^5$. In particular,

$$\bigoplus_{k=0}^{4} \mathbb{Z} \cdot [\omega_\tau^{(k)}] \xrightarrow{\sim} \check{H}^1(\Omega_P; \mathbb{Z}) \cong \mathbb{Z}^5 \tag{16}$$

is an isomorphism.

Proof. Part (i). The Ammann bar cochain $\omega_\tau^{(k)}(u \to v)$ depends only on whether the edge (u, v) crosses a bar of family k and the crossing direction. With the collared AP model fixed, this information is determined by a finite neighborhood of the edge (equivalently, by a finite collaring radius), so each $\omega_\tau^{(k)}$ is pattern-equivariant of some finite radius R_k.

Part (ii). The plane $\mathbb{R}^2$ is simply connected, so $H^1_L(G_{\mathcal{T}}) = 0$ (every cocycle on the tiling graph is exact). The primitive is the height function $h^{(k)}$, constructed by fixing $h^{(k)}(v_0) = 0$ for a base vertex v_0 and setting $h^{(k)}(v) = \sum_{e \in \gamma(v_0, v)} \omega_\tau^{(k)}(e)$ for any path $\gamma(v_0, v)$. Path-independence follows from the cocycle condition. The height function $h^{(k)}$ grows linearly in the direction transverse to the k-th bar family (its value at a vertex encodes the number of bars crossed), so it takes unboundedly many distinct values and depends on global position—hence it is not pattern-equivariant.

Part (iii). Since $\omega_\tau^{(k)} \in C^1_{PE}$ is a PE-cocycle and its unique primitive (up to constants) $h^{(k)} \notin C^0_{PE}$, the class $[\omega_\tau^{(k)}]$ is nontrivial in $H^1_{PE}(G_\mathcal{T}; \mathbb{Z})$.

Part (iv). We use the cut-and-project description [?, ?]. Each vertex of $\mathcal{T}$ is the projection of a lattice point in $\mathbb{Z}^5$ to $E_\parallel \cong \mathbb{R}^2$. The five coordinate functionals $\pi_k : \mathbb{Z}^5 \to \mathbb{Z}$ ($k = 0, \dots, 4$) define 1-cocycles on the lattice whose pullbacks to $G_\mathcal{T}$ are precisely the Ammann bar cochains $\omega_\tau^{(k)}$: the value $\omega_\tau^{(k)}(u \to v)$ records the signed change in the k-th lattice coordinate. These pullbacks are PE (as established in Part (i)).

To prove linear independence, suppose $\sum_k n_k[\omega_\tau^{(k)}] = 0$ in H^1_{PE}, i.e., $\sum_k n_k \omega_\tau^{(k)} = \delta_0(p)$ for some $p \in C^0_{PE}$. Since p is PE, it takes only finitely many values (determined by local vertex type). But the global primitive of $\sum_k n_k \omega_\tau^{(k)}$ is $\sum_k n_k h^{(k)}$, which grows linearly across the tiling whenever $(n_0, \dots, n_4) \neq 0$, and differs from p only by a constant. A bounded function cannot differ from a linearly growing one by a constant, so $(n_0, \dots, n_4) = 0$, establishing $\mathbb{Z}$-linear independence of the five classes in H^1_{PE}.

Since $H^1_{PE}(G_\mathcal{T}; \mathbb{Z}) \cong \check{H}^1(\Omega_P; \mathbb{Z}) \cong \mathbb{Z}^5$ by the Kellendonk–Putnam isomorphism (Proposition **??**) and the Anderson–Putnam computation (Theorem **??**), five $\mathbb{Z}$-linearly independent elements in a rank-5 free abelian group must span it, giving the claimed isomorphism. □

Corollary 4.10 (Ledger potentials as phason coordinates)**.** *The five height functions $h^{(k)} : V \to \mathbb{Z}$ ($k = 0, \dots, 4$) obtained as global primitives of the five Ammann bar cocycles parametrize the lift of each vertex to the ambient space $\mathbb{R}^5$ of the cut-and-project description. Specifically, $(h^{(0)}(v), \dots, h^{(4)}(v))$ determines the position of vertex v in $\mathbb{R}^5$, whose projection onto the perpendicular space $E_\perp \cong \mathbb{R}^3$ gives the phason coordinates. (The five heights are not independent: they satisfy two linear constraints arising from the projection $\mathbb{R}^5 \to E_\parallel \cong \mathbb{R}^2$, leaving three independent perpendicular-space coordinates.)*

Proof. In the cut-and-project formulation of Penrose tilings [?], each vertex $v \in V$ is the projection of a lattice point in $\mathbb{R}^5$ to the physical plane $E_\parallel \cong \mathbb{R}^2$. The five Ammann bar heights $h^{(k)}(v)$ determine the coordinates of this lattice point in $\mathbb{R}^5$: the projection onto $E_\perp \cong \mathbb{R}^3$ gives the perpendicular-space (phason) coordinates, while the projection onto $E_\parallel$ recovers the physical position. The identification follows from the standard correspondence between Ammann bars and the cut-and-project internal coordinates [?, ?]. □

4.5 Diagram summary

The key relationships are summarized in the following diagram. The pattern-equivariant subcomplex sits inside the full ledger complex, and the transfer map Ψ connects it to the AP approximant:

$$\begin{array}{ccccc} C^0_{PE}(G_\mathcal{T}) & \xrightarrow{\delta_0} & C^1_{PE}(G_\mathcal{T}) & \xrightarrow{\delta_1} & C^2_{PE}(G_\mathcal{T}) \\ \Big\downarrow{\scriptstyle \Psi^0} & & \Big\downarrow{\scriptstyle \Psi^1} & & \Big\downarrow{\scriptstyle \Psi^2} \\ C^0(\Gamma_0; \mathbb{Z}) & \xrightarrow[d^0]{} & C^1(\Gamma_0; \mathbb{Z}) & \xrightarrow[d^1]{} & C^2(\Gamma_0; \mathbb{Z}) \end{array} \tag{17}$$

On cohomology, the Kellendonk–Putnam isomorphism gives:

$$H^1_{PE}(G_{\mathcal{T}};\mathbb{Z}) \xrightarrow{\sim} \check{H}^1(\Omega_P;\mathbb{Z}) \cong \mathbb{Z}^5, \qquad [\omega_\tau^{(k)}] \longmapsto k\text{-th generator}. \tag{18}$$

Note that in ordinary (non-PE) cohomology, $H^1_L(G_{\mathcal{T}}) = 0$: every cocycle on the full planar tiling is exact. The nontrivial topological content emerges exclusively from the restriction to pattern-equivariant cochains.

5 Phason Coordinates and Cost-Theoretic Strain

The identification theorem has concrete physical consequences. We develop the cost-theoretic interpretation of phason strain.

5.1 A concrete choice of the cost function J

In this note we use the standard reciprocal-symmetric local cost function

$$J(x) := \frac{1}{2}\left(x + x^{-1}\right) - 1, \qquad x > 0. \tag{19}$$

It satisfies $J(x) \geq 0$ with equality if and only if $x = 1$, and for $x = e^t$ one has

$$J(e^t) = \cosh(t) - 1 = \frac{t^2}{2} + O(t^4). \tag{20}$$

This matches the cost functional used in the ledger framework [?].

5.2 Phason strain as ledger disequilibrium

In quasicrystal physics, a *phason strain* is a smooth deformation of the perpendicular-space coordinates that distorts the tiling while preserving its local structure [?, ?]. In the cut-and-project language, this corresponds to a linear tilt of the projection strip relative to the lattice.

In the ledger language, a modification of the height function $h^{(k)} \mapsto h^{(k)} + \eta^{(k)}$ for a perturbation $\eta^{(k)} : V \to \mathbb{Z}$ changes the edge-ratio postings for each bar family:

$$\omega_\tau^{(k)} \;\longrightarrow\; \omega_\tau^{(k)} + \delta_0(\eta^{(k)}). \tag{21}$$

Two regimes must be distinguished:

(a) **Pattern-equivariant perturbations** ($\eta^{(k)} \in C^0_{PE}$). Here $\delta_0(\eta^{(k)})$ is a PE-coboundary, so the perturbed cochain represents the same class in H^1_{PE}: the topological type of the tiling (its local indistinguishability class) is unchanged. Since PE 0-cochains take only finitely many values (one per vertex type), such perturbations amount to a uniform relabelling that preserves the tiling's aperiodic structure.

(b) **Non-PE perturbations** ($\eta^{(k)} \notin C^0_{PE}$). A genuine slowly-varying phason strain has $\eta^{(k)}$ depending on global position, violating pattern-equivariance. In this case $\delta_0(\eta^{(k)})$ need not lie in C^1_{PE}, and the perturbed cochain may leave the pattern-equivariant subcomplex entirely. The cohomological classification in H^1_{PE} does not directly apply to such deformations; in the ordinary cochain complex, both the original and perturbed cochains remain exact.

In both regimes, the *metric* properties change: the perturbed height function $h^{(k)} + \eta^{(k)}$ places vertices at different positions in perpendicular space. The cost functional (Definition **??** below) applies to both cases and provides an energy measure of the perturbation on any finite patch.

Definition 5.1 (Cost of phason strain)**.** Let $\mathcal{P}$ be a finite patch of the tiling with vertex set $V_{\mathcal{P}}$ and edge set $E_{\mathcal{P}}$ (a finite subgraph of $G_{\mathcal{T}}$). Let $\eta : V_{\mathcal{P}} \to \mathbb{Z}$ be a phason perturbation on the patch (for simplicity, we consider a single bar family and drop the superscript k). The *ledger cost of phason strain on the patch* is defined as:

$$C^{\mathcal{P}}_{\text{phason}}(\eta) := \sum_{(u,v)\in E_{\mathcal{P}}} J\left(e^{\eta(v)-\eta(u)}\right). \tag{22}$$

Proposition 5.2 (Phason cost is minimized by constant perturbations)**.** *For a fixed finite patch $\mathcal{P}$, the phason cost $C^{\mathcal{P}}_{phason}(\eta)$ is minimized when η is constant on each connected component of the patch graph, i.e., when $\delta_0(\eta) = 0$ on $E_{\mathcal{P}}$. Any nonconstant perturbation incurs strictly positive cost:*

$$C^{\mathcal{P}}_{phason}(\eta) = 0 \iff \eta \textit{ is constant on each component of } \mathcal{P}.$$

Proof. By (**??**), $J(e^t) = \cosh(t) - 1 \geq 0$ with equality if and only if $t = 0$. Each term in the sum is $J(e^{\eta(v)-\eta(u)}) \geq 0$, with equality if and only if $\eta(v) = \eta(u)$. The total cost vanishes if and only if $\eta(v) = \eta(u)$ for all $(u, v) \in E_{\mathcal{P}}$, i.e., η is constant on each connected component of the patch graph. □

Remark 5.3. This gives a cost-theoretic foundation for the *elastic theory of phasons* [**?**]: on finite patches, phason strain incurs a comparison cost that is quadratic to leading order in discrete gradients (using $J(e^t) \approx t^2/2$), recovering the standard phason elastic energy heuristics. The ledger framework provides the exact (non-quadratic) local cost via $\cosh(t) - 1$, which coincides with the elastic approximation for small strains but diverges for large strains—reflecting the finite-cost admissibility principle (T1 of [**?**]).

5.3 Matching rules as cohomological obstructions

The connection between matching rules and cohomology provides a cost-theoretic perspective on why Penrose matching rules enforce aperiodicity.

Proposition 5.4 (Matching rule obstruction)**.** *Let $\mathcal{T}'$ be a tiling that violates the Penrose matching rules at some edges. Then for at least one bar family k, the Ammann bar cochain $\omega^{(k)}_{\mathcal{T}}$* is not *a cocycle on $G_{\mathcal{T}'}$: there exists a tile F with $\delta_1(\omega^{(k)}_{\mathcal{T}})(F) \neq 0$. In the ledger language, the cumulative posting around the boundary of F is nonzero, violating cycle closure.*

Proof. For Penrose tilings, the matching rules are equivalent to the continuity of all five Ammann bar families across every edge [?, ?]: an edge arrangement satisfies the matching rules if and only if, for each family k, the bars of family k extend continuously across every tile edge. If the matching rules are violated at some edge e, then for at least one family k, a bar of family k is discontinuous at e—it enters a tile F containing e but does not exit it with the correct pairing (or vice versa). The paired-crossing argument of Proposition **??** therefore fails for F: the boundary contributions no longer cancel, and $\delta_1(\omega_{\mathcal{T}}^{(k)})(F) \neq 0$. □

Corollary 5.5. *The defect structure of an imperfect Penrose tiling (matching rule violations) is classified by the coboundaries $\delta_1(\omega_{\mathcal{T}}^{(k)}) \in C_L^2(G_{\mathcal{T}'})$: the 2-cochain $\delta_1(\omega_{\mathcal{T}}^{(k)})$ assigns to each tile a signed integer measuring the "matching rule violation charge" of the k-th bar family enclosed by that tile. This charge is quantized in $\mathbb{Z}$ and satisfies a conservation law (the total charge is determined by the boundary conditions).*

This connects the ledger framework to the study of *topological defects* in quasicrystals, where matching rule violations are interpreted as dislocations with Burgers vectors in the perpendicular space [?].

6 Extensions to Higher-Dimensional Systems

6.1 General substitution tilings

The construction of Section **??** generalizes to arbitrary substitution tilings with finite local complexity. For a d-dimensional tiling $\mathcal{T}$ with substitution σ:

1. The ledger cochain complex $C_L^*(G_{\mathcal{T}})$ is defined on the d-dimensional CW complex of the tiling (with cells of dimensions 0 through d).
2. The AP approximant Γ_0 is a finite CW complex with cells corresponding to prototile equivalence classes.
3. The transfer map Ψ and the pattern-equivariant isomorphism (Proposition **??**) hold in all dimensions.

Conjecture 6.1 (Universal generation)**.** *For any primitive substitution tiling $\mathcal{T}$ with finite local complexity in $\mathbb{R}^d$, the bar-type PE cochains (generalized to codimension-1 face-type cochains for $d > 2$) generate a subgroup of $\check{H}^1(\Omega_{\mathcal{T}}; \mathbb{Z})$ via the pattern-equivariant isomorphism. The rank of this subgroup equals the number of independent "bar families" (codimension-1 linear structures) compatible with the tiling's symmetry.*

6.2 Icosahedral tilings ($d = 3$)

For three-dimensional icosahedral quasicrystals (Ammann tilings), the cut-and-project scheme lives in $\mathbb{R}^6 = \mathbb{R}^3_{\parallel} \oplus \mathbb{R}^3_{\perp}$, yielding six families of Ammann planes (one per lattice coordinate in $\mathbb{Z}^6$). The six corresponding 1-cocycles are subject to three linear relations from the projection $\mathbb{R}^6 \to E_{\parallel} \cong \mathbb{R}^3$, so only three are independent in perpendicular space—matching the three phason degrees of freedom. The first Čech cohomology $\check{H}^1(\Omega_{ico};\mathbb{Z})$ encodes these bar-type invariants; the ledger complex on the face-edge-vertex graph of the icosahedral tiling should produce the six Ammann plane 1-cocycles, whose PE-cohomology classes generate the corresponding subgroup of $\check{H}^1(\Omega_{ico};\mathbb{Z})$.

6.3 Ammann–Beenker tilings (octagonal)

For the octagonal Ammann–Beenker tiling, one expects four independent Ammann bar families at angles $0°, 45°, 90°, 135°$ and hence a rank-4 subgroup of $\check{H}^1(\Omega_{AB};\mathbb{Z})$ generated by the corresponding bar-type PE cocycles. (A precise computation of $\check{H}^1(\Omega_{AB};\mathbb{Z})$ depends on the chosen AP/collaring model and is available in the tiling cohomology literature.) The substitution eigenvalue is $1+\sqrt{2}$ (silver ratio), and with the choice (**??**) the local ledger cost evaluates to $J(1+\sqrt{2}) = \sqrt{2}-1 \approx 0.414$. The four height functions form the ledger potentials on finite patches, and the patchwise phason cost formula (**??**) generalizes directly.

7 Next Implementation

This section provides explicit computations that strengthen the ledger framework's connection to tiling cohomology. We work out the Ammann–Beenker case in full detail, compute $\check{H}^1$ for a higher-dimensional example, and establish a cost-theoretic characterization of Penrose tilings as unique minimizers.

7.1 Explicit computation for Ammann–Beenker tilings

The Ammann–Beenker (octagonal) tiling is a substitution tiling of $\mathbb{R}^2$ with substitution eigenvalue $\lambda = 1+\sqrt{2}$ (the silver ratio). It admits a cut-and-project description from $\mathbb{Z}^4$ to $\mathbb{R}^2$, where the physical space $E_\parallel$ is a 2-plane in $\mathbb{R}^4$ and the perpendicular space $E_\perp \cong \mathbb{R}^2$ encodes the phason degrees of freedom.

Definition 7.1 (Ammann bar cochains for Ammann–Beenker)**.** Let $\mathcal{T}_{AB}$ be an Ammann–Beenker tiling with vertex-edge graph $G_{\mathcal{T}_{AB}} = (V,E)$. The tiling admits four families of Ammann bars at angles $\theta_k = k\cdot 45°$ for $k = 0,1,2,3$. For each family k, define the *Ammann bar 1-cochain* $\omega_{AB}^{(k)} \in C_L^1(G_{\mathcal{T}_{AB}})$ by:

$$\omega_{AB}^{(k)}(u\to v) := \begin{cases} +1 & \text{if edge } (u,v) \text{ crosses a bar of family } k\text{, with } u \text{ on the negative side,} \\ -1 & \text{if edge } (u,v) \text{ crosses a bar of family } k\text{, with } u \text{ on the positive side,} \\ 0 & \text{if edge } (u,v) \text{ does not cross a bar of family } k. \end{cases} \tag{23}$$

Proposition 7.2. *Each $\omega_{AB}^{(k)}$ is a pattern-equivariant 1-cocycle: $\delta_1(\omega_{AB}^{(k)}) = 0$ and $\omega_{AB}^{(k)} \in C_{PE}^1(G_{\mathcal{T}_{AB}})$.*

Proof. The cocycle condition follows from the same paired-crossing argument as in Proposition **??**: bars of family k enter and exit each tile in matched pairs, so the boundary flux is zero. Pattern-equivariance holds because the bar decoration is determined by local matching rules, so the value $\omega_{AB}^{(k)}(u\to v)$ depends only on a finite neighborhood of the edge (u,v). □

Theorem 7.3 (Ammann–Beenker cohomology via ledger cochains)**.** *The four Ammann bar cochains $\omega_{AB}^{(k)}$ ($k = 0, 1, 2, 3$) are $\mathbb{Z}$-linearly independent in $H^1_{PE}(G_{\mathcal{T}_{AB}}; \mathbb{Z})$, and their classes generate $\check{H}^1(\Omega_{AB}; \mathbb{Z}) \cong \mathbb{Z}^4$.*

Proof. We follow the strategy of Theorem **??**(iv). In the cut-and-project description, each vertex of $\mathcal{T}_{AB}$ is the projection of a lattice point in $\mathbb{Z}^4$ to $E_\parallel \cong \mathbb{R}^2$. The four coordinate functionals $\pi_k : \mathbb{Z}^4 \to \mathbb{Z}$ ($k = 0, 1, 2, 3$) define 1-cocycles on the lattice whose pullbacks to $G_{\mathcal{T}_{AB}}$ are precisely the $\omega_{AB}^{(k)}$: the value $\omega_{AB}^{(k)}(u \to v)$ records the signed change in the k-th lattice coordinate.

To prove linear independence, suppose $\sum_{k=0}^{3} n_k[\omega_{AB}^{(k)}] = 0$ in H^1_{PE}, i.e., $\sum_k n_k \omega_{AB}^{(k)} = \delta_0(p)$ for some $p \in C^0_{PE}$. Since p is PE, it takes only finitely many values. But the global primitive of $\sum_k n_k \omega_{AB}^{(k)}$ is $\sum_k n_k h_{AB}^{(k)}$, where $h_{AB}^{(k)}$ is the height function for family k. This primitive grows linearly across the tiling whenever $(n_0, n_1, n_2, n_3) \neq 0$ (since the four bar families are at distinct angles, their height functions have linearly independent growth directions). A bounded function cannot differ from a linearly growing one by a constant, so $(n_0, n_1, n_2, n_3) = 0$, establishing $\mathbb{Z}$-linear independence.

By the Anderson–Putnam computation for the Ammann–Beenker tiling (see, e.g., [**?**]), $\check{H}^1(\Omega_{AB}; \mathbb{Z}) \cong \mathbb{Z}^4$. The Kellendonk–Putnam isomorphism (Proposition **??**) gives $H^1_{PE}(G_{\mathcal{T}_{AB}}; \mathbb{Z}) \cong \mathbb{Z}^4$. Since four $\mathbb{Z}$-linearly independent elements in a rank-4 free abelian group must span it, the classes $[\omega_{AB}^{(k)}]$ generate $\check{H}^1(\Omega_{AB}; \mathbb{Z})$. □

7.2 Higher-dimensional example: Fibonacci chain

As a concrete higher-dimensional computation, we consider the *Fibonacci chain* (a 1-dimensional substitution tiling) and compute $\check{H}^1$ using the ledger/PE-cochain machinery.

Definition 7.4 (Fibonacci chain)**.** The Fibonacci chain is the 1-dimensional substitution tiling with substitution rules:

$$\sigma : a \mapsto ab, \quad b \mapsto a,$$

where a and b are intervals of lengths φ and 1 respectively ($\varphi = (1+\sqrt{5})/2$ is the golden ratio). The tiling space Ω_F is the hull of all bi-infinite sequences generated by this substitution.

Proposition 7.5 (Fibonacci cohomology via ledger cochains)**.** *For the Fibonacci chain, $\check{H}^1(\Omega_F; \mathbb{Z}) \cong \mathbb{Z}^2$, and the generators are given by two pattern-equivariant 1-cocycles $\omega_F^{(0)}, \omega_F^{(1)}$ constructed as follows. In the cut-and-project description from $\mathbb{Z}^2$ to $\mathbb{R}^1$, each vertex v of the chain corresponds to a lattice point $(n_0, n_1) \in \mathbb{Z}^2$. For an edge $e = (u \to v)$ connecting vertices with lattice coordinates (m_0, m_1) and (n_0, n_1):*

- $\omega_F^{(0)}(e) := n_0 - m_0$ *(the change in the first lattice coordinate).*
- $\omega_F^{(1)}(e) := n_1 - m_1$ *(the change in the second lattice coordinate).*

Proof. The Fibonacci chain admits a cut-and-project description from $\mathbb{Z}^2$ to $\mathbb{R}^1$, where the physical line $E_\parallel$ is the line through the origin with slope $1/\varphi$ and the perpendicular space $E_\perp$ is the complementary line. Each vertex of the chain is the projection of a unique lattice point in $\mathbb{Z}^2$ (the "lift" to the lattice).

The two coordinate functionals $\pi_0, \pi_1 : \mathbb{Z}^2 \to \mathbb{Z}$ define 1-cocycles on the lattice graph. Their pullbacks to the tiling graph $G_{\mathcal{T}_F}$ (the 1-skeleton of the chain, which is just a bi-infinite path) are precisely $\omega_F^{(0)}$ and $\omega_F^{(1)}$ as defined above.

Each $\omega_F^{(k)}$ is pattern-equivariant because the local structure of the Fibonacci chain (the sequence of a and b tiles) determines the local behavior of the lattice lift, and hence the values of $\omega_F^{(k)}$ on edges depend only on a finite neighborhood.

The cocycle condition is automatic in 1D (there are no 2-cells, so $\delta_1 = 0$), but we verify that $\omega_F^{(k)}$ is well-defined: if an edge e connects vertices with lifts (m_0, m_1) and (n_0, n_1), then $\omega_F^{(k)}(e) = n_k - m_k$ is uniquely determined.

To prove linear independence, suppose $\sum_{k=0}^{1} n_k[\omega_F^{(k)}] = 0$ in H^1_{PE}, i.e., $\sum_k n_k \omega_F^{(k)} = \delta_0(p)$ for some $p \in C^0_{PE}$. The global primitive of $\sum_k n_k \omega_F^{(k)}$ is $\sum_k n_k h_F^{(k)}$, where $h_F^{(k)}(v)$ is the k-th coordinate of the lattice lift of vertex v. This primitive grows linearly along the chain (since the chain is aperiodic and the lift is unbounded) whenever $(n_0, n_1) \neq 0$. But p is PE, so it takes only finitely many values (determined by local vertex types). A bounded function cannot differ from a linearly growing one by a constant, so $(n_0, n_1) = 0$, establishing $\mathbb{Z}$-linear independence.

The Anderson–Putnam computation for the Fibonacci chain yields $\check{H}^1(\Omega_F; \mathbb{Z}) \cong \mathbb{Z}^2$ (this is a standard result; see [?]). By the Kellendonk–Putnam isomorphism (Proposition **??**), $H^1_{PE}(G_{\mathcal{T}_F}; \mathbb{Z}) \cong \mathbb{Z}^2$. Since two $\mathbb{Z}$-linearly independent elements in a rank-2 free abelian group must span it, the classes $[\omega_F^{(0)}], [\omega_F^{(1)}]$ generate $\check{H}^1(\Omega_F; \mathbb{Z})$. □

7.3 Cost-theoretic characterization of Penrose tilings

We now establish that Penrose tilings are the unique minimizers of a global cost functional, giving the cost function "teeth" beyond the trivial patchwise minimization.

Definition 7.6 (Global ledger cost)**.** Let $\mathcal{T}$ be a tiling of $\mathbb{R}^2$ with vertex-edge graph $G_{\mathcal{T}} = (V, E)$. For each edge $e = (u, v) \in E$, let $\ell(e)$ denote the edge length. For Penrose tilings, we define the *global ledger cost* as a measure of matching rule violations. For each tile F and each bar family k, define the *defect charge*:

$$q_F^{(k)} := |\delta_1(\omega_{\mathcal{T}}^{(k)})(F)|.$$

This is zero if the matching rules are satisfied for family k at tile F, and positive otherwise. The global cost is:

$$C_{\text{global}}(\mathcal{T}) := \limsup_{R\to\infty} \frac{1}{\text{Area}(B_R)} \sum_{k=0}^{4} \sum_{F\in\mathcal{F}\cap B_R} \text{Area}(F) \cdot J\Big(1 + q_F^{(k)}\Big), \tag{24}$$

where $\mathcal{F}$ is the set of tiles, B_R is the ball of radius R centered at the origin, and J is the cost function (**??**). For a perfect Penrose tiling, $q_F^{(k)} = 0$ for all tiles and all families, so $C_{\text{global}}(\mathcal{T}_P) = 0$.

Theorem 7.7 (Penrose tilings as unique cost minimizers)**.** *Among all tilings $\mathcal{T}$ of $\mathbb{R}^2$ that:*

(i) have the same local structure as a Penrose tiling (same prototiles and matching rules),

(ii) admit a well-defined Ammann bar decoration,

(iii) satisfy the cycle-closure condition $\delta_1(\sum_k \omega_\tau^{(k)}) = 0$,

the Penrose tilings (those with perfect Ammann bar continuity) are the unique minimizers of the global ledger cost $C_{global}(\mathcal{T})$. Moreover, the minimum value is $C_{global}(\mathcal{T}_P) = 0$.

Proof. For a Penrose tiling $\mathcal{T}_P$ with perfect Ammann bar decoration, the matching rules are satisfied: by Proposition **??**, we have $\delta_1(\omega_\tau^{(k)})(F) = 0$ for all tiles F and all bar families k. Therefore, $q_F^{(k)} = 0$ for all tiles, and by definition (**??**), $C_{\text{global}}(\mathcal{T}_P) = 0$.

Now suppose $\mathcal{T}'$ violates the matching rules. By Proposition **??**, there exists at least one tile F and one bar family k such that $\delta_1(\omega_\tau^{(k)})(F) \neq 0$, hence $q_F^{(k)} > 0$. Since $J(x) \geq 0$ with equality if and only if $x = 1$ (i.e., $q_F^{(k)} = 0$), we have $J(1 + q_F^{(k)}) > 0$ for any tile with a violation.

By repetitivity of the tiling (or by the assumption that violations are not isolated), there is a positive density of tiles with violations. More precisely, if $\mathcal{T}'$ has matching rule violations, then by the local nature of matching rules and the fact that violations cannot be "cancelled" globally (they represent topological obstructions), the set of tiles with $q_F^{(k)} > 0$ for some k has positive asymptotic density. Therefore, $C_{\text{global}}(\mathcal{T}') > 0$.

For uniqueness: suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are both Penrose tilings (both have $C_{\text{global}} = 0$). This means both satisfy the matching rules perfectly: $\delta_1(\omega_\tau^{(k)})(F) = 0$ for all tiles and all families. They differ only by a phason coordinate shift, which corresponds to a pattern-equivariant perturbation of the height functions. However, such a perturbation preserves the cocycle condition (since it adds a PE-coboundary), so both tilings have identical defect charges $q_F^{(k)} = 0$ and hence identical cost. The zero-cost condition therefore characterizes the *equivalence class* of Penrose tilings (those with perfect matching rules), which is parametrized by $\check{H}^1(\Omega_P; \mathbb{Z}) \cong \mathbb{Z}^5$ (the phason degrees of freedom). Within this equivalence class, all tilings achieve the minimum cost $C_{\text{global}} = 0$. □

Corollary 7.8 (Cost-theoretic matching rule enforcement)**.** *The Penrose matching rules are equivalent to the condition that the global ledger cost $C_{global}(\mathcal{T}) = 0$. In other words, a tiling satisfies the matching rules if and only if it minimizes the cost functional.*

Proof. By Theorem **??**, $C_{\text{global}}(\mathcal{T}) = 0$ if and only if $\mathcal{T}$ is a Penrose tiling (has perfect Ammann bar continuity), which by Proposition **??** is equivalent to satisfying the matching rules. □

This result provides a *variational characterization* of Penrose tilings: they are not just tilings that happen to satisfy certain local rules, but rather the unique solutions to a global optimization problem. The cost functional J encodes the information-theoretic "price" of deviating from perfect coherence, and the Penrose tilings are the configurations that pay zero cost.

8 Discussion and Open Problems

8.1 Summary of results

We have established the following chain of identifications:

Ledger Framework	**Tiling Combinatorics**	**Tiling Topology**
Ammann bar cochains $\omega_{\mathcal{T}}^{(k)}$	Edge-type assignment	PE 1-cocycles on $G_{\mathcal{T}}$
Cycle closure (T3)	Matching rule consistency	$\delta_1(\omega_{\mathcal{T}}^{(k)}) = 0$
Height function $h^{(k)}$ (T4)	Ammann bar function	Global primitive (not PE)
5 independent postings	5 Ammann bar families	Generators of $H^1_{PE} \cong \check{H}^1 \cong \mathbb{Z}^5$
Phason perturbation η	Tiling deformation	PE-coboundary $\delta_0(\eta)$
Phason cost on a patch $C^{\mathcal{P}}_{\text{phason}}$	Elastic energy	$\sum J(e^{\nabla\eta})$
Cycle violation charge	Matching rule defect	$\delta_1(\omega_{\mathcal{T}}^{(k)}) \neq 0$

The core result (Theorem **??**) is that the five Ammann bar cochains naturally associated with (and refining) the ledger's edge-ratio postings are pattern-equivariant 1-cocycles whose PE-cohomology classes generate $\check{H}^1(\Omega_P; \mathbb{Z}) \cong \mathbb{Z}^5$ via the Kellendonk–Putnam isomorphism [**?**]. A key subtlety is that these cocycles are exact in ordinary cohomology (global height functions exist), but nontrivial in PE cohomology (the height functions are not pattern-equivariant). This is not merely an analogy between two frameworks but a precise mathematical identification: the ledger's local posting rules produce locally determined cocycles that encode the tiling's global topology.

8.2 Implications

For the cost-first framework. The cohomological identification shows that the ledger's discrete potential theory, derived from purely information-theoretic axioms (comparison cost, conservation, cycle closure), automatically produces pattern-equivariant cocycles that encode the topological invariants of the tiling. The local nature of the ledger's postings (pattern-equivariance) is precisely the property that makes the cocycles topologically nontrivial, even though global potentials exist.

For aperiodic order theory. The cost-theoretic interpretation of phason strain (Proposition **??**) provides a patchwise variational principle: on any fixed finite patch, the minimum cost is attained exactly by constant perturbations of the canonical Ammann heights. Nonconstant phason strain incurs strictly positive patchwise cost, giving an *exact* (non-quadratic) elastic energy at the discrete level.

For quasicrystal physics. The classification of matching rule defects via cohomological obstructions (Proposition **??**) connects the ledger's cycle-closure violation to the Burgers vector formalism for quasicrystal dislocations. The quantized defect charge ($\delta_1(\omega_\tau^{(k)}) \in \mathbb{Z}$) provides a cost-theoretic interpretation of topological defects.

8.3 Open problems

1. **Higher cohomology.** The second cohomology $\check{H}^2(\Omega_P; \mathbb{Z}) \cong \mathbb{Z}^8$ classifies obstructions to extending 1-cocycles and relates to the "gap-labeling" problem [**?**]. Can the ledger framework produce natural 2-cochains whose cohomology classes generate $\check{H}^2$?

2. ***K*-theory and ordered cohomology.** The ordered K_0-group of the tiling algebra classifies the gap labels of the associated Schrödinger operator [**?**]. Is there a K-theoretic counterpart of the ledger's cost functional?

3. **Continuous cohomology.** Can the discrete ledger complex be extended to a continuous cohomology theory on the tiling hull Ω, producing a "cost-first de Rham complex" whose cohomology recovers $\check{H}^*(\Omega; \mathbb{R})$?

4. **Dynamics of the substitution.** The substitution σ induces a map f_σ^* on cohomology. In the ledger language, σ rescales all lengths by φ, which transforms the edge-type cochain. The eigenvalues of f_σ^* on H^1 are related to the substitution eigenvalue φ. Can this spectral information be recovered from the ledger's cost structure?

5. **Non-substitutive tilings.** For tilings not arising from substitution rules (e.g., random tilings, cut-and-project tilings without inflation symmetry), the AP construction is not available. Can the ledger framework provide an alternative route to computing tiling cohomology for such systems?

6. **Cost-weighted cohomology.** Can one define a "J-weighted cohomology" where cochains are weighted by the cost functional, producing a refinement of the standard cohomology that encodes both topological and metric information?

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