GOLDEN AND METALLIC STRUCTURES ON HESSIAN MANIFOLDS

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Abstract. We consider the reciprocal cost function $J(x) = \frac{1}{2}(x + x^{-1}) - 1$ and its n-dimensional extension

$$J(x_1, \dots, x_n) = \frac{1}{2}(R + R^{-1}) - 1, \qquad R = \prod_{i=1}^{n} x_i^{\alpha_i}, \qquad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \setminus \{0\}.$$

In logarithmic coordinates $t_i = \log x_i$, the Hessian of J has rank one at every point. The associated Hessian geometry is degenerate and does not define a Riemannian metric.

To obtain a nondegenerate geometric structure, we introduce a family of Hessian metrics h_λ. Combining the rank-one tensor with the Hessian metric h_λ, we construct a $(1,1)$-tensor field A_λ. Its trace normalization defines a projector P_λ, which induces an almost product structure and the corresponding golden and metallic structures.

We study several properties of the projector P_λ and the induced structures, including eigendistributions, parallelism, integrability, and curvature. The construction is given in arbitrary dimension, and explicit formulas are obtained in the two-dimensional case. In particular, we show that the projector P_λ is generally not parallel with respect to either the canonical flat affine connection or the Levi-Civita connection ∇^λ of the Hessian metric h_λ.

Keywords: Hessian geometry, golden structures, metallic structures, projector, reciprocal cost function.

MSC (2020): 53A15, 53C15, 53B20.

1. Motivation

The golden ratio is known since Euclid and appears under different names, such as the golden section, divine ratio, golden mean or golden proportion. It occurs in nature, especially in patterns related to Fibonacci numbers, like phyllotaxis and certain flowers.

It also appears in music, in harmonic relations, and in proportions of the human body. From ancient times, it has played an important role in architecture and art, for example in the proportions of temples, sculptures, and paintings. The golden ratio can be defined geometrically by dividing a segment into two parts such that the ratio of the whole to the larger part equals the ratio of the larger part to the smaller one. This ratio is the positive solution of the equation $x^2 - x - 1 = 0$. It appears in geometric figures such as the pentagon, decagon and dodecagon. On the other hand, let us consider the general quadratic equation

$$x^2 - \alpha x - \beta = 0,$$

where α and β are positive integers. Its positive solution is

$$\sigma_{\alpha,\beta} = \frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2},$$

which defines the *metallic means family*. This family includes, for instance, the golden mean, the silver mean, the subtle mean, etc., and was introduced by Spinadel [17, 18].

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The golden ratio appears in quasicrystals, dynamical systems, and certain models in mathematical physics (see e.g. [4, 6, 13, 15, 21] and references therein).

Following [22, 23], we consider the canonical reciprocal cost function in one dimension

$$J(x) = \frac{1}{2}\big(x + x^{-1}\big) - 1, \qquad x > 0, \tag{1.1}$$

Cost functions are ubiquitous in optimization problems, and different cost functions can have different motivations. In [23], it is proved that this particular function appears as a unique solution of the polynomial composition law together with the curvature calibration. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \setminus \{0\}$ the n-dimensional extension obtained by composing (1.1) with $R = \prod_i x_i^{\alpha_i}$ is

$$J(x_1, \ldots, x_n) = \frac{1}{2}(R + R^{-1}) - 1,$$

so $J(t) = \cosh(\alpha \cdot t) - 1$ in logarithmic coordinates. The Hessian of J is then $\cosh(\alpha \cdot t)\, \alpha \otimes \alpha$, which is positive semidefinite of rank one and does not define a Riemannian metric.

Let M be a smooth manifold and I the identity endomorphism of the tangent bundle TM. A $(1,1)$-tensor field Q on M is called a *polynomial structure* if it satisfies a polynomial identity $P(Q) = 0$. The two quadratic cases studied in this paper are the *golden structure*

$$Q^2 = Q + I, \tag{1.2}$$

which is motivated by the classical golden ratio equation. More generally, for positive integers p, q, the (p, q)-*metallic structure* (see e.g. [12]) is defined by

$$Q^2 = p\,Q + q\,I. \tag{1.3}$$

Both golden and metallic structures belong to a broader class of polynomial structures introduced by Goldberg and Yano [9]. Golden structures on differentiable manifolds were first introduced by Hrețcanu and Crășmareanu [10], who further developed their properties in [5]. Using an approach similar to the one developed for golden structures, Hrețcanu and Crășmareanu studied the metallic structures on Riemannian manifolds in [12].

We pair the rank-one tensor $\tilde{g}$ with a one-parameter family of Hessian metrics

$$h_\lambda = \nabla_x^2 \Phi_\lambda, \qquad \Phi_\lambda(x) = \sum_{i=1}^{n} J(x_i) + \lambda J(R), \qquad \lambda \in \mathbb{R}.$$

The associated $(1,1)$-tensor A_λ is defined by

$$h_\lambda(A_\lambda X, Y) = \tilde{g}(X, Y),$$

and its normalization gives the projector P_λ. The corresponding almost product, golden, and metallic structures are obtained from P_λ.

We study several properties of these structures, including eigendistributions, parallelism, integrability, and curvature. The parameter λ deforms the Hessian metric by the term $J(R)$, producing a family associated with the reciprocal cost function.

The projector construction is general and can be applied to other rank-one tensors and nondegenerate metrics. In this paper, we consider the projector induced by the rank-one tensor $\tilde{g}$ and the Hessian metrics h_λ arising from reciprocal cost geometry.

The paper is organized as follows. Section 2 introduces the reciprocal cost geometry and the associated Hessian structures. In Section 3 we construct the cost-induced projector. Section 4 is devoted to the associated almost product, golden, and metallic structures. In

Section 5 we consider the two-dimensional case and derive explicit formulas for the tensors A_λ and P_λ. Section 6 contains the n-dimensional construction. In Section 6.1, we study properties of the induced structures, including eigendistributions, parallelism, integrability, and curvature.

2. Definitions and basic properties

Let (M, g) be a Riemannian manifold, let I denote the identity on TM, and let $Q\colon TM \to TM$ be a $(1,1)$-tensor field.

Definition 2.1. *A $(1,1)$-tensor field Q on M is called a* polynomial structure *if it satisfies a polynomial relation of the form*

$$Q^n + a_{n-1}Q^{n-1} + \cdots + a_1 Q + a_0 I = 0, \tag{2.1}$$

where I is the identity operator on TM and $a_i \in \mathbb{R}$.

Golden and metallic structures are special cases of polynomial structures. In particular, $Q^2 = -I$ defines an almost complex structure, $Q^2 = I$ defines an almost product structure, and $Q^2 = 0$ defines an almost tangent structure (see, e.g., [25]).

Definition 2.2. *For integers p, q, a $(1,1)$-tensor field Q is called a* (p,q)-metallic structure *if*

$$Q^2 = p\,Q + q\,I. \tag{2.2}$$

A Riemannian metric g is called *Q-compatible* if

$$g(X, QY) = g(QX, Y), \qquad X, Y \in \Gamma(TM). \tag{2.3}$$

When Q is a (p,q)-metallic structure and g is Q-compatible, the pair (g, Q) is called a *metallic Riemannian structure*. In the particular case $p = q = 1$, the pair (g, Q) is called a *golden Riemannian structure* [5, 10].

Replacing X by QX in (2.3) and using (2.2), we obtain

$$g(QX, QY) = p\,g(X, QY) + q\,g(X, Y).$$

It is known that a decomposition of the tangent bundle of a differentiable manifold M into complementary distributions can be described in terms of projector operators. For instance, let $T_1, \ldots, T_k$ be differentiable distributions on M such that for every point $p \in M$ one has

$$T_pM = T_1(p) \oplus \cdots \oplus T_k(p).$$

This decomposition can be equivalently expressed by a family of $(1,1)$-tensor fields π_i, $i = 1, \ldots, k$, called *projectors*, satisfying

$$\sum_{i=1}^{k} \pi_i = I, \qquad \pi_i \pi_j = \delta^i_j\,\pi_i,$$

where δ^i_j are the Kronecker symbols. In this case $T_i = \operatorname{Im}(\pi_i)$.

In the case $k = 2$, such a decomposition determines an almost product structure. Indeed, if π is one of the projectors, then define

$$F = 2\pi - I,$$

and obtain a $(1,1)$-tensor field satisfying $F^2 = I$.

Conversely, any almost product structure F, induces the complementary projectors

$$\pi^+ = \frac{1}{2}(I+F), \qquad \pi^- = \frac{1}{2}(I-F),$$

and the decomposition

$$T_pM = T^+(p) \oplus T^-(p),$$

where

$$T^\pm(p) = \{v \in T_pM : Fv = \pm v\}.$$

Theorem 2.1 ([5])**.** *Let (M, g, Q) be a golden Riemannian manifold. Then*

$$Q^n = f_n Q + f_{n-1} I \tag{2.4}$$

for every integer $n > 0$, where $(f_n)_n$ is the Fibonacci sequence.

Using Binet's formula, relation (2.4) can be written as

$$Q^n = f_n Q + f_{n-1} I = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} Q + \frac{\varphi^{n-1} - (1-\varphi)^{n-1}}{\sqrt{5}} I,$$

for every $n > 1$.

Some structures in this paper are related to generalized secondary Fibonacci sequences (GSFS) (see [19, 20]) given by

$$G(n+1) = pG(n) + qG(n-1), \qquad n \geq 1,$$

with $G(0) = a \in \mathbb{R}$, $G(1) = b \in \mathbb{R}$ and $p, q \in \mathbb{R}$.

The ratio $G(n+1)/G(n)$ of two consecutive terms of GSFS converges to:

- the golden mean $\varphi = \frac{1+\sqrt{5}}{2}$, for $p = q = 1$, determined by the ratio of two consecutive classical Fibonacci numbers;
- the silver mean $\sigma_{2,1} = 1 + \sqrt{2}$, for $p = 2$ and $q = 1$, determined by the ratio of two consecutive Pell numbers;
- the bronze mean $\sigma_{3,1} = \frac{3+\sqrt{13}}{2}$, for $p = 3$ and $q = 1$;
- the subtle mean $\sigma_{4,1} = 2 + \sqrt{5} = \varphi^3$, for $p = 4$ and $q = 1$;
- the copper mean $\sigma_{1,2} = 2$, for $p = 1$ and $q = 2$;
- the nickel mean $\sigma_{1,3} = \frac{1+\sqrt{13}}{2}$, for $p = 1$ and $q = 3$.

In the case $q = 1$ and $p = k$, one gets the k-Fibonacci sequence

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \qquad F_{k,0} = 0,\ F_{k,1} = 1,$$

which generalizes the classical Fibonacci sequence.

2.1. **Reciprocal cost geometry.** The main point of our construction is related to the canonical reciprocal cost function $J : \mathbb{R}_{>0} \to \mathbb{R}$,

$$J(x) = \frac{1}{2}(x + x^{-1}) - 1, \tag{2.5}$$

which is the unique solution of the polynomial composition law together with the curvature calibration (for more details see [22]). The function J is reciprocal, $J(x) = J(x^{-1})$, nonnegative, with a minimum at $x = 1$. In logarithmic coordinates,

$$J(e^t) = \cosh(t) - 1.$$

Near $t = 0$, one has

$$J(e^t) = \frac{t^2}{2} + O(t^4).$$

Among many possible multidimensional extensions, the form considered here is motivated by the multiplicative structure of the one-dimensional reciprocal cost and by the logarithmic representation $J(e^t) = \cosh(t) - 1$.

We study the family of reciprocal cost functions (see, e.g., [24])

$$J(x_1, \dots, x_n) = \frac{1}{2}(R + R^{-1}) - 1, \qquad R = \prod_{i=1}^{n} x_i^{\alpha_i}, \tag{2.6}$$

where $x_i > 0$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \setminus \{0\}$. In logarithmic coordinates $t_i = \log x_i$, the function (2.6) takes the form

$$J(t) = \cosh(\alpha \cdot t) - 1.$$

Therefore, the function J depends only on the scalar $S(t) = \alpha \cdot t = \sum_{i=1}^{n} \alpha_i t_i$. Its Hessian is the rank-one tensor

$$\nabla^2 J = \cosh(\alpha \cdot t) \left(\sum_{i=1}^{n} \alpha_i \, dt_i \right) \otimes \left(\sum_{i=1}^{n} \alpha_i \, dt_i \right).$$

Since $\nabla^2 J$ has rank one, the tensor $\tilde{g} := \nabla^2 J$ is degenerate for $n \geq 2$.

The induced geometry is degenerate, with a distinguished direction generated by α and an integrable $(n-1)$-dimensional null distribution. In particular, the ambient space is n-dimensional and the associated Hessian structure in logarithmic coordinates reduces to a one-dimensional geometry. Hessian geometry and Hessian manifolds play an important role in affine differential geometry and information geometry (see, e.g., [1, 16]).

The rank-one property of this Hessian tensor motivates the construction of additional geometric structures. To obtain a nondegenerate geometric structure, we combine the rank-one tensor associated with the reciprocal cost geometry with a family of Hessian metrics. This construction produces an associated $(1,1)$-tensor field whose normalization defines a projector. The projector then induces an almost product structure and the corresponding golden and metallic structures.

More precisely, the rank-one tensor $\tilde{g}$ is defined by (6.2). By combining $\tilde{g}$ with the nondegenerate Hessian metric h_λ defined by (6.1), we obtain the associated $(1,1)$-tensor A_λ, given by (6.3). Its normalization gives the projector P_λ, (6.4). The corresponding almost product, golden, and metallic structures are introduced in (6.6), (6.7) and (6.8), respectively.

3. The Cost Induced Projector

Let M be a smooth manifold, let g be a nondegenerate metric on M, and let $\tilde{g}$ be a positive semidefinite symmetric $(0,2)$-tensor field of rank one. On an open set $U \subseteq M$ where $\tilde{g} \neq 0$ there exists a vector field V such that

$$\tilde{g}(X, Y) = g(V, X)\, g(V, Y), \qquad X, Y \in \Gamma(TU). \tag{3.1}$$

The associated $(1,1)$-tensor field A is defined by

$$g(AX, Y) = \tilde{g}(X, Y) \qquad X, Y \in \Gamma(TU). \tag{3.2}$$

On U, using (3.1), we obtain

$$AX = g(V, X)\, V, \qquad X \in \Gamma(TU). \tag{3.3}$$

We have

$$g(AX,Y) = \tilde{g}(X,Y) = \tilde{g}(Y,X) = g(AY,X) = g(X,AY)$$

Lemma 3.1. *The tensor A defined by (3.2) satisfies*

$$A^2 = \mu A,$$

where $\mu = g(V,V)$. Moreover, $\mu = \operatorname{tr}(A)$.

Proof. Using (3.3), we compute

$$A^2X = A(AX) = A\big(g(V,X)V\big) = g(V,X)\,AV = g(V,X)\,g(V,V)\,V.$$

Hence

$$A^2X = g(V,V)\,AX,$$

so $A^2 = \mu A$ with $\mu = g(V,V)$. From $AX = g(V,X)V$, we obtain

$$\operatorname{tr}(A) = g(V,V),$$

which completes the proof. □

Corollary 3.1. *On the open subset $U \subseteq M$ where $\mu = g(V,V) \neq 0$, the tensor*

$$P := \frac{1}{\mu}A \tag{3.4}$$

is a projector. Moreover,

$$\operatorname{im}(P) = \operatorname{span}\{V\}, \qquad \ker(P) = \{X \in TM|_U : g(V,X) = 0\}.$$

Hence

$$TM|_U = \operatorname{im}(P) \oplus \ker(P).$$

Proof. Since $A^2 = \mu A$ and $\mu \neq 0$, we have

$$P^2 = \frac{1}{\mu^2}A^2 = \frac{1}{\mu}A = P.$$

Thus P is a projector.

Using $AX = g(V,X)V$, we get

$$PX = \frac{g(V,X)}{g(V,V)}\,V.$$

Hence $\operatorname{im}(P) = \operatorname{span}\{V\}$. Also,

$$PX = 0 \iff g(V,X) = 0,$$

so

$$\ker(P) = \{X \in TM|_U : g(V,X) = 0\}.$$

Finally, every vector field $X \in \Gamma(TM|_U)$ decomposes as

$$X = PX + (X - PX),$$

where $PX \in \operatorname{im}(P)$ and $X - PX \in \ker(P)$. Therefore

$$TM|_U = \operatorname{im}(P) \oplus \ker(P).$$

□

4. Golden and Metallic Operators

In this section, starting from the projector P given by (3.4) and induced splitting

$$TM|_U = \mathrm{im}(P) \oplus \ker(P),$$

we construct the almost-product, golden, and metallic structures. The constructions below follow from the identity $P^2 = P$ and hold for any projector. The reciprocal cost geometry provide a particular projector to which this construction is applied. Let us define

$$F := 2P - I.$$

Proposition 4.1. *The tensor F satisfies*

$$F^2 = I.$$

Moreover, $F|_{\mathrm{im}(P)} = I$ and $F|_{\ker(P)} = -I$.

Proof. Since $F = 2P - I$ and P is a projector i.e. $P^2 = P$, we have

$$F^2 = (2P - I)^2 = 4P^2 - 4P + I = I.$$

Moreover, for $X \in \mathrm{im}(P)$ one has $PX = X$, so

$$FX = 2X - X = X.$$

For $X \in \ker(P)$ one has $PX = 0$, so

$$FX = -X.$$

This completes the proof. □

Let us now consider an operator of the form

$$G = \alpha P + \beta(I - P), \qquad \alpha, \beta \in \mathbb{R}.$$

Since $P^2 = P$ and $P(I - P) = 0$, we have

$$G^2 = \alpha^2 P + \beta^2 (I - P).$$

We require that G satisfies the golden equation $G^2 = G + I$, then

$$\alpha^2 = \alpha + 1, \qquad \beta^2 = \beta + 1.$$

Thus α and β are roots of the equation

$$x^2 = x + 1,$$

which has two solutions

$$\varphi = \frac{1 + \sqrt{5}}{2}, \qquad 1 - \varphi = \frac{1 - \sqrt{5}}{2}.$$

Taking the two roots, we obtain

$$\alpha = \varphi, \qquad \beta = 1 - \varphi = -\varphi^{-1}.$$

Therefore

$$G = \varphi P + (1 - \varphi)(I - P).$$

or equivalently, since $F = 2P - I$,

$$G = \frac{1}{2}(I + \sqrt{5}\, F). \tag{4.1}$$

A direct computation shows that G satisfies

$$G^2 = G + I.$$

Thus the golden structure is induced by the projector.

Corollary 4.1. *The operator G has eigenvalues φ on $\operatorname{im}(P)$ and $-\varphi^{-1}$ on $\ker(P)$.*

Proof. On $\operatorname{im}(P)$ one has $F = I$, hence $G = \frac{1}{2}(1+\sqrt{5})I = \varphi I$. On $\ker(P)$ one has $F = -I$, hence $G = \frac{1}{2}(1-\sqrt{5})I = -\varphi^{-1}I$. □

Let us now, for $p, q \in \mathbb{N}$, define

$$M_{p,q} := \frac{p}{2}I + \frac{1}{2}\sqrt{p^2+4q}\,F. \tag{4.2}$$

Theorem 4.1. *The operator $M_{p,q}$ given by (4.2) satisfies*

$$M_{p,q}^2 = p\,M_{p,q} + q\,I.$$

Proof. Using $F^2 = I$, from (4.2) we obtain

$$M_{p,q}^2 = \left(\frac{p}{2}I + \frac{1}{2}\sqrt{p^2+4q}\,F\right)^2 = \left(\frac{p^2}{4} + \frac{p^2+4q}{4}\right)I + \frac{p}{2}\sqrt{p^2+4q}\,F.$$

Therefore

$$M_{p,q}^2 = \left(\frac{p^2}{2} + q\right)I + \frac{p}{2}\sqrt{p^2+4q}\,F.$$

On the other hand,

$$p\,M_{p,q} + qI = p\left(\frac{p}{2}I + \frac{1}{2}\sqrt{p^2+4q}\,F\right) + qI = \left(\frac{p^2}{2} + q\right)I + \frac{p}{2}\sqrt{p^2+4q}\,F.$$

which coincides with $M_{p,q}^2$. □

By construction, each operator in the family $\{M_{p,q}\}_{p,q\in\mathbb{N}}$ is a polynomial expression in the projector P, and the golden structure $G = M_{1,1}$ corresponds to the case $p = q = 1$.

The following proposition gives the main properties of a general metallic structure.

Proposition 4.2 (Hreţcanu-Crăşmăreanu [12])**.** *Let $M_{p,q}$ be defined by*

$$M_{p,q} = \frac{p}{2}I + \frac{1}{2}\sqrt{p^2+4q}\,F, \qquad p, q \in \mathbb{N},$$

where $F^2 = I$. Then the following properties hold:

(1) For every integer $n \geq 1$,

$$M_{p,q}^n = G(n)\,M_{p,q} + q\,G(n-1)\,I,$$

where $(G(n))_{n\geq 0}$ is the generalized secondary Fibonacci sequence defined by

$$G(n+1) = pG(n) + qG(n-1), \qquad G(0) = 0, \quad G(1) = 1.$$

(2) *The operator $M_{p,q}$ is an isomorphism on each tangent space T_xM, hence invertible. Its inverse is polynomial (of quadratic type, but not metallic) and is given by*

$$\bar{M}_{p,q} = M_{p,q}^{-1} = \frac{1}{q}M_{p,q} - \frac{p}{q}I.$$

It satisfies

$$q\bar{M}_{p,q}^2 + p\bar{M}_{p,q} - I = 0.$$

(3) *The eigenvalues of $M_{p,q}$ are*

$$\frac{p+\sqrt{p^2+4q}}{2}, \qquad \frac{p-\sqrt{p^2+4q}}{2}.$$

5. The two-dimensional case

We now illustrate the construction in the two-dimensional case. Consider the canonical reciprocal cost function (2.5). Following the n-dimensional case (2.6), define on $\mathbb{R}^2_{>0}$:

$$R(x,y) := \frac{x}{y}, \qquad J(x,y) := \frac{1}{2}\left(R + R^{-1}\right) - 1. \tag{5.1}$$

This corresponds to the choice $\alpha = (1,-1)$ in the n-dimensional model.

5.1. **Logarithmic coordinates.**

Introduce logarithmic coordinates $u = \log x$, $v = \log y$. Then

$$R = e^{u-v}, \qquad J(u,v) = \cosh(u-v) - 1. \tag{5.2}$$

In this case, the function $J(u,v)$ depends only on the quantity $u-v$. The Hessian tensor $\tilde{g} = \nabla^2 J$ of $J(u,v) = \cosh(u-v) - 1$ is

$$\tilde{g} = \cosh(u-v)\,(du-dv)\otimes(du-dv) = \cosh(u-v)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}. \tag{5.3}$$

Since $\cosh(u-v) > 0$, the tensor $\tilde{g}$ is positive semidefinite of rank one, with distinguished direction $V = \partial_u - \partial_v$.

5.2. (x,y)**-coordinates.**

Passing to the original (x,y)-coordinates, and using $u = \log x$, $v = \log y$, we obtain

$$V = x\partial_x - y\partial_y.$$

Using (5.3), $du = \frac{dx}{x}, dv = \frac{dy}{y}$, and

$$\cosh\left(\log\left(\frac{x}{y}\right)\right) = \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x}\right) = \frac{x^2+y^2}{2xy}$$

we obtain

$$\tilde{g} = \frac{x^2+y^2}{2xy}\left(\frac{dx}{x} - \frac{dy}{y}\right)\otimes\left(\frac{dx}{x} - \frac{dy}{y}\right).$$

Hence

$$\tilde{g} = \frac{x^2+y^2}{2xy}\begin{pmatrix}x^{-2} & -\frac{1}{xy}\\ -\frac{1}{xy} & y^{-2}\end{pmatrix}. \tag{5.4}$$

So, the tensor $\tilde{g}$ is a rank-one tensor.

On the other hand, we can also consider the full Hessian of the cost function directly in the original (x, y)-coordinates, i.e.

$$\tilde{g}_x = \nabla_x^2 J(x, y).$$

A direct computation gives

$$\det(\tilde{g}_x) = -\frac{(x^2 - y^2)^2}{4x^4 y^4}.$$

Thus $\tilde{g}_x$ is generically nondegenerate and indefinite, while it becomes singular on the locus $x = y$. The geometry determined by the Hessian metric $\tilde{g}_x$ was studied in [24].

Remark 5.1. *The rank-one tensor $\tilde{g}$ obtained from the logarithmic representation of the cost function and the full Hessian metric $\tilde{g}_x$ in the original (x, y)-coordinates carry different geometry. In particular, $\tilde{g}$ is degenerate of constant rank one and determines the distinguished comparison direction generated by $V = x\partial_x - y\partial_y$, while $\tilde{g}_x$ is generically nondegenerate and becomes singular on the locus $x = y$.*

By Lemma 3.1 and Corollary 3.1, the rank-one property of $\tilde{g}$, when paired with a nondegenerate metric, leads to the construction of a projector. The tensor $\tilde{g}_x$ does not have this property and does not produce the projector-type structures. For this reason, the projector construction developed in the following part is associated with $\tilde{g}$.

5.3. **The induced projector.** We introduce a one-parameter family of nondegenerate Hessian metrics on $\mathbb{R}^2_{>0}$ by

$$\Phi_\lambda(x, y) = J(x) + J(y) + \lambda J\left(\frac{x}{y}\right), \qquad \lambda \in \mathbb{R}. \tag{5.5}$$

For $\lambda = 0$ this reduces to $\Phi_0(x, y) = J(x) + J(y)$, whose Hessian

$$h_0 = \nabla^2 \Phi_0 = \begin{pmatrix} x^{-3} & 0 \\ 0 & y^{-3} \end{pmatrix} \tag{5.6}$$

is positive definite on $\mathbb{R}^2_{>0}$ and it is used as a separable reference metric.

The associated $(1, 1)$-tensor field is defined by

$$A_0 = h_0^{-1} \tilde{g}. \tag{5.7}$$

The metric h_0 provides a nondegenerate reference metric for the distinguished comparison direction generated by $V = x\partial_x - y\partial_y$.

The term $\lambda J(x/y)$ introduces mixed second derivatives along ω, where

$$\omega = d \log(x/y) = \frac{dx}{x} - \frac{dy}{y},$$

so the Hessian metric $h_\lambda = \nabla_x^2 \Phi_\lambda$ is nondiagonal for $\lambda \neq 0$. The associated $(1, 1)$-tensor is

$$A_\lambda = h_\lambda^{-1} \tilde{g},$$

and the projector is obtained by normalization, as given in Section 3.

Remark 5.2. *The construction involves two natural affine structures associated with the reciprocal cost geometry.*

In the two-dimensional case the rank-one tensor $\tilde{g} = \cosh(u - v)\,(du - dv) \otimes (du - dv)$ is derived with respect to the flat structure in the logarithmic coordinates (u, v), while the

Hessian metric $h_\lambda = \nabla_x^2 \Phi_\lambda$ is used with respect to the flat structure in the original (x, y)-coordinates. The n-dimensional construction considered in Section 6 is based on the same choice of affine structures.

Let

$$h_\lambda = \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \qquad a = \frac{1+\lambda y}{x^3}, \quad b = -\lambda \frac{x^2+y^2}{2x^2y^2}, \quad d = \frac{1+\lambda x}{y^3}, \tag{5.8}$$

Then A_λ has the form given in the next proposition.

Proposition 5.1. *Let h_λ is defined by*

$$h_\lambda = \nabla_x^2 \Phi_\lambda, \qquad \Phi_\lambda(x, y) = J(x) + J(y) + \lambda J\left(\frac{x}{y}\right), \tag{5.9}$$

and let $\tilde{g}$ is given by (5.4). Assume that $\det(h_\lambda) = ad - b^2 \neq 0$, *then*

$$A_\lambda = \frac{x^2+y^2}{2xy} \frac{1}{ad-b^2} \begin{pmatrix} \dfrac{d}{x^2} + \dfrac{b}{xy} & -\dfrac{d}{xy} - \dfrac{b}{y^2} \\ -\dfrac{b}{x^2} - \dfrac{a}{xy} & \dfrac{b}{xy} + \dfrac{a}{y^2} \end{pmatrix}. \tag{5.10}$$

Proof. From the definition of Φ_λ, we have

$$h_\lambda = \nabla_x^2 \Phi_\lambda = \begin{pmatrix} a & b \\ b & d \end{pmatrix},$$

where

$$a = \frac{1+\lambda y}{x^3}, \qquad b = -\lambda \frac{x^2+y^2}{2x^2y^2}, \qquad d = \frac{1+\lambda x}{y^3}.$$

Thus

$$h_\lambda^{-1} = \frac{1}{ad-b^2} \begin{pmatrix} d & -b \\ -b & a \end{pmatrix}.$$

Multiplying h_λ^{-1} with the matrix of $\tilde{g}$ given by (5.4), we obtain (5.10). □

Theorem 5.1. *The tensor A_λ given by (5.10) satisfies $A_\lambda^2 = \mu_\lambda A_\lambda$, where $\mu_\lambda = \operatorname{tr}(A_\lambda)$. The tensor*

$$P_\lambda := \frac{1}{\mu_\lambda} A_\lambda$$

is a projector on the open subset of $\mathbb{R}^2$ where $ad - b^2 \neq 0$ and $\mu_\lambda \neq 0$.

Proof. Since h_λ is nondegenerate and $A_\lambda = h_\lambda^{-1}\tilde{g}$, the tensors A_λ and $\tilde{g}$ have same rank. Since $\tilde{g}$ has rank one, then A_λ also has rank one. Therefore

$$A_\lambda^2 = \mu_\lambda A_\lambda,$$

where $\mu_\lambda = \operatorname{tr}(A_\lambda)$. If $\mu_\lambda \neq 0$, then

$$P_\lambda := \frac{1}{\mu_\lambda} A_\lambda$$

is well defined and satisfies

$$P_\lambda^2 = P_\lambda.$$

□

Corollary 5.1. *The corresponding projector is*

$$P_\lambda(X) = \frac{\omega(X)}{\omega(V_\lambda)}\, V_\lambda, \qquad V_\lambda = h_\lambda^{-1}\omega,$$

where

$$\omega = \frac{dx}{x} - \frac{dy}{y}.$$

Moreover,

$$\operatorname{im}(P_\lambda) = \operatorname{span}\{V_\lambda\}, \qquad \ker(P_\lambda) = \ker\omega.$$

In particular, $\ker(P_\lambda)$ *is generated by* $x\partial_x + y\partial_y$.

Proof. Since

$$\tilde{g} = \frac{x^2+y^2}{2xy}\,\omega\otimes\omega$$

and $V_\lambda = h_\lambda^{-1}\omega$, the tensor A_λ has the form

$$A_\lambda X = \frac{x^2+y^2}{2xy}\,\omega(X)V_\lambda.$$

After normalization, we have

$$P_\lambda(X) = \frac{\omega(X)}{\omega(V_\lambda)}V_\lambda.$$

Hence $\operatorname{im}(P_\lambda) = \operatorname{span}\{V_\lambda\}$. Also,

$$P_\lambda X = 0 \iff \omega(X) = 0,$$

so $\ker(P_\lambda) = \ker\omega$. Since

$$\omega(x\partial_x + y\partial_y) = \frac{dx}{x}(x\partial_x + y\partial_y) - \frac{dy}{y}(x\partial_x + y\partial_y) = 1 - 1 = 0,$$

and $\ker\omega$ is one-dimensional, we get

$$\ker(P_\lambda) = \operatorname{span}\{x\partial_x + y\partial_y\}.$$

□

6. The n-dimensional Hessian construction

In this part, we will describe the construction of a projector in the n-dimensional case. For $\alpha = (\alpha_1,\dots,\alpha_n) \neq 0$, $\lambda \in \mathbb{R}$, and $x \in \mathbb{R}^n_{>0}$, we have

$$\Phi_\lambda(x_1,\dots,x_n) = \sum_{i=1}^n J(x_i) + \lambda J(R), \qquad R = \prod_{i=1}^n x_i^{\alpha_i}.$$

The associated Hessian metric is

$$h_\lambda = \nabla_x^2\Phi_\lambda \tag{6.1}$$

defines the associated Hessian metric. The parameter λ can be viewed as a deformation parameter. For $\lambda = 0$, we obtain

$$h_0 = \operatorname{diag}(x_1^{-3},\dots,x_n^{-3}),$$

which is positive definite on $\mathbb{R}^n_{>0}$.

For $\lambda \neq 0$, the term $\lambda J(R)$ introduces mixed second derivatives, so h_λ is generally nondiagonal. Its positive definite locus depends on (x,λ). Moreover, for every fixed point (x,y), positive definiteness is preserved for sufficiently small values of $|\lambda|$. An analysis of the signature and singular loci of h_λ in n-dimensional lies beyond the scope of the present paper. We work on open subsets where h_λ is positive definite.

In the two-dimensional case, explicit conditions for positive definiteness are given in the following example.

Example 6.1. *Let us consider the two-dimensional case, with h_λ given by (5.8). We have*

$$\det(h_\lambda) = ad - b^2 = \frac{4xy\big(1+\lambda(x+y)\big) - \lambda^2(x^2-y^2)^2}{4x^4y^4}.$$

By Sylvester's criterion, h_λ is positive definite at $(x,y) \in \mathbb{R}^2_{>0}$ if and only if $a > 0$ and $\det(h_\lambda) > 0$ or equivalently,

$$1+\lambda y > 0 \quad \text{and} \quad 4xy\big(1+\lambda(x+y)\big) - \lambda^2(x^2-y^2)^2 > 0.$$

Remark 6.1. *For $\lambda \neq 0$, the metric h_λ is not positive definite on $\mathbb{R}^2_{>0}$. For fixed $x > 0$ and sufficiently large y, $\det(h_\lambda) < 0$ because the term $-\lambda^2 y^4$ dominates the numerator. Therefore, we restrict the construction to open subsets of $\mathbb{R}^2_{>0}$ where h_λ is positive definite.*

Remark 6.2. *For $\lambda = 0$, the metric h_0 is positive definite on $\mathbb{R}^n_{>0}$. Since the coefficients of h_λ depend on λ and on variables x_i, this property is preserved for sufficiently small values of $|\lambda|$ at every fixed point $(x_1,\dots,x_n)$. We consider open subsets where h_λ is positive definite.*

Recall that, in logarithmic coordinates $t_i = \log x_i$, the reciprocal cost function (2.6) has the form

$$J(t) = \cosh(\alpha\cdot t) - 1, \qquad \alpha\cdot t = \sum_{i=1}^n \alpha_i t_i.$$

with Hessian

$$\nabla^2 J = \cosh(\alpha\cdot t)\,\alpha\otimes\alpha.$$

In the original x-coordinates, let

$$\omega = \sum_{i=1}^n \alpha_i \frac{dx_i}{x_i}.$$

Then the rank-one tensor is

$$\tilde{g} = \cosh(\alpha\cdot t)\,\omega\otimes\omega. \tag{6.2}$$

The associated $(1,1)$-tensor field $A_\lambda := h_\lambda^{-1}\tilde{g}$ is defined by

$$h_\lambda(A_\lambda X, Y) = \tilde{g}(X,Y), \qquad X, Y \in \Gamma(T\mathbb{R}^n_{>0}).$$

Let $V_\lambda := h_\lambda^{-1}(\omega)$ defined by $h_\lambda(V_\lambda, X) = \omega(X)$ for all $X \in \Gamma(T\mathbb{R}^n_{>0})$. Then, by (6.2), we obtain

$$\begin{aligned}\tilde{g}(X,Y) &= \cosh(\alpha\cdot t)\,\omega(X)\omega(Y)\\ &= \cosh(\alpha\cdot t)\,\omega(X)\,h_\lambda(V_\lambda, Y).\end{aligned}$$

Applying Lemma 3.1 and Corollary 3.1 with $g = h_\lambda$, $V = V_\lambda$, and $\tilde{g}$ as in (6.2), we get

$$A_\lambda X = \cosh(\alpha\cdot t)\,\omega(X)V_\lambda, \qquad A_\lambda = \cosh(\alpha\cdot t)\,V_\lambda\otimes\omega, \tag{6.3}$$

and

$$A_\lambda^2 = \mu_\lambda A_\lambda, \qquad \mu_\lambda = \cosh(\alpha\cdot t)\,\omega(V_\lambda) = \operatorname{tr}(A_\lambda).$$

We consider open subset of $\mathbb{R}^n_{>0}$ where $\mu_\lambda \neq 0$. On this set the tensor

$$P_\lambda = \frac{1}{\mu_\lambda} A_\lambda \tag{6.4}$$

is a projector. More precisely,

$$P_\lambda X = \frac{\omega(X)}{\omega(V_\lambda)} V_\lambda. \tag{6.5}$$

Since $\tilde{g}$ is symmetric,

$$h_\lambda(A_\lambda X, Y) = h_\lambda(X, A_\lambda Y),$$

hence A_λ, and therefore P_λ, are self-adjoint with respect to h_λ. Applying the construction given in Section 4, we obtain the induced almost product, golden, and metallic structures

$$F_\lambda = 2P_\lambda - I, \qquad F_\lambda^2 = I, \tag{6.6}$$

$$G_\lambda = \frac{1}{2}(I + \sqrt{5}\, F_\lambda), \qquad G_\lambda^2 = G_\lambda + I, \tag{6.7}$$

and, for $p, q \in \mathbb{N}$,

$$M^\lambda_{p,q} = \frac{p}{2} I + \frac{1}{2}\sqrt{p^2 + 4q}\, F_\lambda, \qquad (M^\lambda_{p,q})^2 = pM^\lambda_{p,q} + qI. \tag{6.8}$$

Consequently, F_λ, G_λ, and $M^\lambda_{p,q}$ are also self-adjoint with respect to h_λ, since they are polynomial expressions in the self-adjoint operator P_λ with real coefficients.

6.1. **Properties of the induced projector.** We now study properties of the projector P_λ constructed in the previous sections. Let ∇^λ denote the Levi-Civita connection of the Hessian metric $h_\lambda = \nabla^2_x \Phi_\lambda$ on the open subset of $\mathbb{R}^n_{>0}$ where h_λ is positive definite. We denote by D the canonical flat affine connection associated with the chosen affine structure. In affine coordinates, its Christoffel symbols vanish identically.

We first check whether P_λ is parallel with respect to the canonical flat affine connection in logarithmic coordinates, using the two-dimensional case.

Example 6.2. *Let $n = 2$, $\alpha = (1, -1)$, and $\lambda = 0$. In logarithmic coordinates $u = \log x$, $v = \log y$, we have*

$$\omega = du - dv.$$

For the metric h_0, we have

$$V_0 = h_0^{-1}(\omega) = e^u \partial_u - e^v \partial_v,$$

and

$$\omega(V_0) = e^u + e^v.$$

Therefore

$$P_0 = \frac{1}{e^u + e^v} \begin{pmatrix} e^u & -e^u \\ -e^v & e^v \end{pmatrix}.$$

Hence

$$\partial_u (P_0)^1{}_1 = \frac{e^{u+v}}{(e^u + e^v)^2} \neq 0.$$

Since the canonical flat affine connection D satisfies

$$D_{\partial_u}\partial_u = D_{\partial_u}\partial_v = D_{\partial_v}\partial_u = D_{\partial_v}\partial_v = 0,$$

we obtain

$$(D_{\partial_u} P_0)^1{}_1 = \partial_u (P_0)^1{}_1 \neq 0.$$

Therefore $DP_0 \neq 0$, *i.e. the projector* P_0 *is not parallel with respect to the canonical flat affine connection in logarithmic coordinates.*

Remark 6.3. *The induced tensor* G_λ *satisfies*

$$G_\lambda^2 = G_\lambda + I,$$

and is self-adjoint with respect to the Hessian metric h_λ. *On every open subset of* $\mathbb{R}^n_{>0}$ *where* h_λ *is positive definite,* $(\mathbb{R}^n_{>0}, h_\lambda, G_\lambda)$ *defines a golden Riemannian manifold.*

The following theorem describes the family of linear connections preserving the induced golden structure G_λ.

Theorem 6.1 (Theorem 5.1, [5])**.** *Let* $F_\lambda = 2P_\lambda - I$ *be the induced almost product structure associated with the golden structure* G_λ. *Then the set of linear connections* ∇ *satisfying* $\nabla G_\lambda = 0$ *is given by*

$$\nabla_X Y = \frac{1}{5}\Big[3\widetilde{\nabla}_X Y + 2G_\lambda(\widetilde{\nabla}_X G_\lambda Y) - G_\lambda(\widetilde{\nabla}_X Y) - \widetilde{\nabla}_X G_\lambda Y\Big] + \mathcal{O}_{F_\lambda} Q(X,Y),$$

where $\widetilde{\nabla}$ *is an arbitrary fixed linear connection and* Q *is a* $(1,2)$*-tensor field for which* $\mathcal{O}_{F_\lambda} Q$ *is an associated Obata operator*

$$\mathcal{O}_{F_\lambda} Q(X,Y) = \frac{1}{2}\Big[Q(X,Y) + F_\lambda Q(X, F_\lambda Y)\Big].$$

We now study the parallelism properties of P_λ with respect to the Levi-Civita connection ∇^λ of h_λ.

Example 6.3. *In the two-dimensional case, the projector* P_0 *is not parallel with respect to the Levi-Civita connection* ∇^0 *of* h_0. *We have*

$$h_0 = \begin{pmatrix} x^{-3} & 0 \\ 0 & y^{-3} \end{pmatrix},$$

and

$$P_0 = \frac{1}{x+y}\begin{pmatrix} x & -x^2/y \\ -y^2/x & y \end{pmatrix}.$$

A direct computation gives

$$(\nabla^0_{\partial_x} P_0)^x{}_x = \partial_x\left(\frac{x}{x+y}\right) = \frac{y}{(x+y)^2} \neq 0.$$

Therefore, $\nabla^0 P_0 \neq 0$.

Example 6.4. *For* $\lambda \neq 0$, *the projector* P_λ *is not parallel with respect to* ∇^λ.

The coefficients of P_λ *and the Christoffel symbols of* ∇^λ *contain nontrivial mixed terms from* $J(x/y)$. *For example, at the point* $(x,y) = (1,1)$, *we obtain*

$$\left(\nabla^\lambda_{\partial_x} P_\lambda\right)^x{}_x = \frac{1+\lambda}{4(1+2\lambda)}.$$

At the point $(1,1)$, *the metric* h_λ *is positive definite only for* $1 + 2\lambda > 0$, *and consequently* $\left(\nabla^\lambda_{\partial_x} P_\lambda\right)^x{}_x \neq 0$. *Hence* P_λ *is not parallel with respect to the Levi-Civita connection* ∇^λ.

The two-dimensional case represents the simplest nontrivial realization of the general construction. The above examples show that the induced almost product, golden, and metallic structures are non-parallel in general.

6.2. **Integrability of the eigendistributions.** Motivated by Proposition 5.3 of [5], we study the integrability of the distributions induced by the projector P_λ.

Since the almost product, golden, and metallic structures $F_\lambda = 2P_\lambda - I$, G_λ, $M^\lambda_{p,q}$ are obtained from P_λ, they have the same eigendistributions, $\mathrm{Im}(P_\lambda)$ and $\ker(P_\lambda)$.

The distribution $\mathrm{Im}(P_\lambda) = \mathrm{span}\{V_\lambda\}$ is one-dimensional and therefore integrable. For $\ker(P_\lambda)$, recall from (6.5) that

$$\ker(P_\lambda) = \ker(\omega),$$

where

$$\omega = \sum_{i=1}^{n} \alpha_i \frac{dx_i}{x_i} = d(\log R).$$

Since ω is exact, it is closed. Hence, by the Frobenius theorem, the distribution $\ker(P_\lambda)$ is integrable.

Therefore, both eigendistributions of P_λ are integrable. Consequently, the induced almost product, golden, and metallic structures are integrable.

6.3. **A rank-one representation of the cost-induced projector.** Let P_λ be the cost induced projector. Since

$$P_\lambda X = \frac{h_\lambda(V_\lambda, X)}{h_\lambda(V_\lambda, V_\lambda)} V_\lambda,$$

on the open set where $h_\lambda(V_\lambda, V_\lambda) \neq 0$, we define

$$\xi_\lambda := V_\lambda, \qquad \eta_\lambda(X) := \frac{h_\lambda(V_\lambda, X)}{h_\lambda(V_\lambda, V_\lambda)}.$$

Then $\eta_\lambda(\xi_\lambda) = 1$ and $P_\lambda X = \eta_\lambda(X)\xi_\lambda$. Therefore,

$$P_\lambda = \eta_\lambda \otimes \xi_\lambda.$$

Consequently, the cost induced golden structure G_λ given by (6.7) and the metallic structures $M^\lambda_{p,q}$ given by (6.8) can be written in the form

$$G_\lambda = (1-\varphi)I + \sqrt{5}\,\eta_\lambda \otimes \xi_\lambda,$$

and

$$M^\lambda_{p,q} = \left(\frac{p}{2} - \frac{1}{2}\sqrt{p^2+4q}\right) I + \sqrt{p^2+4q}\,\eta_\lambda \otimes \xi_\lambda.$$

Thus the reciprocal cost geometry together with the Hessian metric h_λ determines the projector P_λ and the associated polynomial structures. The vector field V_λ generates $\mathrm{im}(P_\lambda)$, while $\ker(P_\lambda)$ gives the complementary distribution.

6.4. **Curvature properties of the Hessian metric h_λ.** We now consider curvature properties of the Hessian metric $h_\lambda = Dd\Phi_\lambda$, where D denotes the canonical flat affine connection on $\mathbb{R}^n_{>0}$.

Let ∇^λ be the Levi-Civita connection of h_λ. The difference tensor K^λ of the Levi-Civita connection ∇^λ of h_λ and D is defined by

$$K^\lambda_X Y := \nabla^\lambda_X Y - D_X Y. \tag{6.9}$$

Since both ∇^λ and D are symmetric connections, the tensor K^λ is symmetric, i.e. $K^\lambda_X Y = K^\lambda_Y X$. Starting from

$$X(h_\lambda(Y,Z)) = h_\lambda(\nabla^\lambda_X Y, Z) + h_\lambda(Y, \nabla^\lambda_X Z).$$

and substituting

$$\nabla^\lambda_X Y = D_X Y + K^\lambda_X Y,$$

we have

$$X(h_\lambda(Y,Z)) = h_\lambda(D_X Y, Z) + h_\lambda(K^\lambda_X Y, Z) + h_\lambda(Y, D_X Z) + h_\lambda(Y, K^\lambda_X Z).$$

Hence

$$(D_X h_\lambda)(Y,Z) = h_\lambda(K^\lambda_X Y, Z) + h_\lambda(Y, K^\lambda_X Z).$$

Since h_λ is Hessian metric, the cubic tensor

$$C_\lambda(X,Y,Z) := (D_X h_\lambda)(Y,Z) \tag{6.10}$$

is totally symmetric. Therefore, $h_\lambda(K^\lambda_X Y, Z) = h_\lambda(Y, K^\lambda_X Z)$, and consequently

$$h_\lambda(K^\lambda_X Y, Z) = \frac{1}{2}(D_X h_\lambda)(Y,Z). \tag{6.11}$$

Therefore K^λ is completely determined by the tensor $C_\lambda(X,Y,Z)$.

Since the connection D is flat, the curvature tensor of ∇^λ is given by

$$R^\lambda(X,Y)Z = K^\lambda_X K^\lambda_Y Z - K^\lambda_Y K^\lambda_X Z + (D_X K^\lambda)(Y,Z) - (D_Y K^\lambda)(X,Z).$$

Using (6.10), (6.11), and the Codazzi-type identity $(D_X K^\lambda)(Y,Z) = (D_Y K^\lambda)(X,Z)$, we obtain

$$R^\lambda(X,Y,Z,W) = \frac{1}{4}\Big(h_\lambda^{-1}\big(C_\lambda(X,Z,\cdot), C_\lambda(Y,W,\cdot)\big) - h_\lambda^{-1}\big(C_\lambda(Y,Z,\cdot), C_\lambda(X,W,\cdot)\big)\Big),$$

where $R^\lambda(X,Y,Z,W) := h_\lambda(R^\lambda(X,Y)Z, W)$. Using $C_\lambda(X,Y,Z) = 2h_\lambda(K^\lambda_X Y, Z)$, the last formula becomes

$$R^\lambda(X,Y,Z,W) = \frac{1}{4}\Big(h_\lambda(K^\lambda_X Z, K^\lambda_Y W) - h_\lambda(K^\lambda_Y Z, K^\lambda_X W)\Big). \tag{6.12}$$

Thus the Riemannian curvature of h_λ is completely determined by the cubic tensor C_λ. If $\{E_i\}_{i=1}^n$ is a local h_λ-orthonormal frame, then the coresponding Ricci tensor is

$$\mathrm{Ric}_{h_\lambda}(Y,Z) = \sum_{i=1}^n R^\lambda(E_i, Y, Z, E_i).$$

Using (6.12), we obtain

$$\mathrm{Ric}_{h_\lambda}(Y,Z) = \frac{1}{4}\sum_{i=1}^n \Big(h_\lambda(K^\lambda_Y Z, K^\lambda_{E_i} E_i) - h_\lambda(K^\lambda_{E_i} Z, K^\lambda_Y E_i)\Big).$$

Therefore the Ricci tensor is also completely determined by the cubic tensor C_λ. The scalar curvature of h_λ is defined by $\mathrm{Scal}_{h_\lambda} = \mathrm{tr}_{h_\lambda}(\mathrm{Ric}_{h_\lambda})$.

In the two-dimensional case, the Ricci tensor is completely determined by the scalar curvature. For $\lambda = 0$, the Hessian metric is $h_0 = \mathrm{diag}(x^{-3}, y^{-3})$, and the corresponding curvature vanishes, and $\mathrm{Scal}_{h_0} = 0$.

For $\lambda \neq 0$, the scalar curvature is nonzero in general and depends on λ. By direct calculation, we have

$$\mathrm{Scal}_{h_\lambda} = -\frac{4\lambda x^2 y^2\Big(\lambda(x+y)^3 + 3(x^2+y^2)\Big)}{\Big(\lambda^2(x^2-y^2)^2 - 4\lambda xy(x+y) - 4xy\Big)^2}.$$

Since the expression is not identically zero for $\lambda \neq 0$, the Hessian metric h_λ is non-flat in general. Thus the interaction term $\lambda J(x/y)$ produces nontrivial curvature of h_λ.

7. Conclusion

In this paper, we studied a family of projector induced polynomial structures associated with the reciprocal cost geometry. Starting from the rank-one tensor $\tilde{g}$ determined by the reciprocal cost function in logarithmic coordinates and a family of Hessian metrics h_λ, we constructed $(1,1)$-tensor field A_λ and the projector P_λ. The projector P_λ determines a splitting

$$TU = \mathrm{im}(P_\lambda) \oplus \ker(P_\lambda),$$

which induces the almost product structure $F_\lambda = 2P_\lambda - I$, and the corresponding golden and metallic structures. The Hessian structure is determined by the metric $h_\lambda = \nabla_x^2 \Phi_\lambda$ with respect to the x-coordinates, while the rank-one tensor $\tilde{g}$ is obtained from the logarithmic reciprocal cost geometry.

We studied several properties of the projector P_λ and the induced structures, including eigendistributions, parallelism, integrability, and curvature. In particular, we showed that P_λ is generally not parallel with respect to either the canonical flat affine connection in logarithmic coordinates or the Levi-Civita connection of the Hessian metric h_λ. The eigendistributions of P_λ are integrable and are determined by the vector field V_λ and the one-form $\omega = d(\log R)$. The polynomial structures are induced by the projector associated with the rank-one tensor $\tilde{g}$ and the Hessian metric h_λ. The construction is developed in arbitrary dimension, while the two-dimensional case is considered in detail.

The obtained structures depend on the choice of the Hessian metric h_λ. In particular, the deformation parameter λ changes the corresponding projector, the induced polynomial structures, and their geometric properties.

The construction in the paper shows that a rank-one tensor obtained from reciprocal cost geometry, together with a nondegenerate Hessian metric, determines a projector and the associated almost product, golden, and metallic structures. In this way, reciprocal cost geometry leads to polynomial structures on Hessian manifolds.

Further directions will include the study of the positive-definite locus and curvature properties of h_λ and possible relations with Hessian [16] and information geometry [1].

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